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Elusive Worldsheet Instantons in Heterotic String Compactifications

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Abstract

We compute the spectrum of massless gauge singlets in some heterotic string compactifications using Landau–Ginzburg, orbifold and non-linear σ -model methods. This probes the worldsheet instanton corrections to the quadratic terms in the spacetime superpotential. Previous results predict that some of these states remain massless when instanton effects are included. We find vanishing masses in many cases not covered by these predictions. However, we discover that in the case of the Z -manifold the corrections do not vanish. Despite this, in all the examples studied, we find that the massless spectrum in the orbifold limit agrees with the nonlinear σ -model computation.

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1 Introduction

The oldest approach to phenomenology in string theory is to compactify the heterotic string on a Calabi–Yau manifold X together with a choice of vector bundle $E \rightarrow X$. Among compactifications with $\mathcal{N} = 1$ spacetime supersymmetry, such models are unique in having a relatively straightforward worldsheet formulation in terms of a $(0, 2)$ superconformal field theory, and can be studied beyond the supergravity approximation. The effects of worldsheet instantons in these models present some formidable technical challenges and are far from completely understood.

In this paper we consider the moduli space of $(0, 2)$ -theories. Unlike $(2, 2)$ -theories we may have obstructions to first order deformations and, correspondingly, we have a notion of a superpotential in would-be moduli fields. The superpotential is subject to instanton corrections at any degree. A linear correction destabilizes the vacuum completely while a quadratic correction removes a first-order deformation. Higher terms in the superpotential affect the obstructions to the first-order deformations.

One may approach the computation of worldsheet instantons in two ways. The direct way is to explicitly attack the geometry of the rational curves in the target space. This has been pursued in works such as [1–6]. In this method one needs to compute Pfaffians associated to each curve and then sum over all curves while correctly taking into account multiple covers, etc. In principle this allows a complete computation of the full superpotential; in practice even the quadratic terms present a daunting challenge.

The second approach, which avoids such formidable computations, is to compare a perturbative non-linear σ -model count of massless states with an exact conformal field theory method such as Landau–Ginzburg theories. The difference will yield precisely the instanton corrections to the quadratic terms in the superpotential. Surprisingly, this difference is frequently zero.

The history of worldsheet instantons destabilizing the vacuum is interesting. One knows that $(2, 2)$ vacua are stable and so the “standard embedding” where E is the tangent bundle $T(X)$ yields a zero superpotential. Beyond that it was expected that a generic CY 3-fold would have rational curves giving generic instanton contributions leading to a nonzero superpotential, indicating that the compactification specified by E is not in fact a supersymmetric vacuum [1]. Thus, anything other than a very specially chosen vector bundle on a particular Calabi–Yau manifold would fail to give a good heterotic string compactification.

It was then realized [2, 3] that the contribution to worldsheet instanton corrections from a single rational curve would be zero if the bundle E split nontrivially over the curve. While trivial splitting is generic, one could imagine finding special examples where every rational curve had a nontrivial splitting and thus the total instanton correction would be zero.

It was further realized that, in many cases which were understood as an exact conformal field theory, even if single instantons contributed nontrivially to the superpotential, there must a cancellation to produce a zero net result [7]. This was further explored for the quintic threefold in [4]. This apparently miraculous cancellation was explained in work by Beasley and Witten [8]. This latter paper also indicated that such a cancellation must happen in general when X is a complete intersection in a toric variety V and either

1. E is a pullback of a vector bundle on V or
2. E is in the form of a monad naturally realized by the gauged linear σ -model.

This accounts for a very wide range of possibilities and naturally one might ask the question: *Can we ever measure nonzero instanton contributions to the superpotential from the exact conformal field theory?*

We will address this question in the easiest of settings, namely where E is the tangent bundle $T(X)$. The conformal field theory exhibits $(2,2)$ worldsheet supersymmetry. The model has an E_6 spacetime gauge symmetry, and the spectrum includes massless E_6 singlet chiral multiplets corresponding to first-order deformations of T . In the supergravity approximation, these are classically counted by $h^1(\text{End}(T))$, and perturbative corrections in α' to the spacetime superpotential are prohibited by supersymmetry. We further restrict our attention to terms in the superpotential quadratic in these fields, which represent instanton-induced mass terms.

If worldsheet instantons contribute to these terms in the superpotential then the number of these massless gauge singlet chiral superfields, computed in the conformal field theory, will be smaller than $h^1(\text{End}(T))$. Thus we may look for worldsheet instantons by comparing classical (i.e., perturbative nonlinear σ -model) $h^1(\text{End}(T))$ computations with exact results from the worldsheet. In this paper we will use mirror symmetry to compute the exact result. In particular, we will compute the spectrum from a Landau–Ginzburg realization of the mirror.

Any deformation of E that can be associated to a simple deformation of the gauged linear σ -model Lagrangian is a truly marginal deformation of the conformal field theory, protected from worldsheet instanton-induced obstruction at any degree by the Beasley–Witten result. Thus we need to focus on more complicated deformations.

We will see that the cancellation miracle persists in some cases beyond Beasley–Witten (as had previously been observed in [9]). We will also find an example where the number of massless singlets is drastically reduced from $h^1(\text{End}(T))$ and thus we do indeed have instanton corrections. This example is given by the Z -manifold, i.e., a crepant resolution of T^6/\mathbb{Z}_3 .

Even though the Z -manifold suffers from instanton correction we will see that it surprisingly obeys the “heterotic McKay correspondence”. That is, we may correctly compute $h^1(\text{End}(T))$ by counting untwisted singlets from a 6-torus and then adding in 27 copies of the twisted singlets from a \mathbb{Z}_3 -quotient singularity as explained in [10]. This is unexpected from the point of view of string theory. Unlike the McKay correspondence which, from the point of view of string theory, relates two computations of an invariant quantity valid in different regions of the moduli space, the quantity we are computing here – the number of massless E_6 singlet fields – is not invariant and changes as we move about the moduli space. The calculation of [10] is valid on the orbifold locus, a nine-dimensional subspace describing the space with quotient singularities (unresolved). As we shall see, upon resolving the singularities some of these modes acquire a mass. $h^1(\text{End}(T))$, on the other hand, computes the number of fields whose masses vanish exponentially as the volume of the space increases to

infinity (in such a way that the sizes of all holomorphic curves grow). It is interesting that these numbers agree.

2 Deformations of the Tangent Bundle

Let V be a toric variety and let $X \subset V$ be the desired Calabi–Yau threefold given as a complete intersection given by s equations $f^a = 0$, for $a = 0, \dots, s - 1$.

We review the construction of V to fix notation. Let x_0, \dots, x_{N-1} be the homogeneous coordinates on V . This is the homogeneous coordinate ring in the sense of Cox [11]

$$R = \mathbb{C}[x_0, \dots, x_{N-1}]. \quad (1)$$

We have a short exact sequence

$$0 \longrightarrow M \longrightarrow \mathbb{Z}^{\oplus N} \xrightarrow{\Phi} D \longrightarrow 0, \quad (2)$$

where D is a lattice¹ of rank r . Each column of the matrix Φ can be thought of as a $U(1)^r$ charge vector of the coordinates x_i . That is, R has the structure of an r -multigraded ring.

The toric variety is given as

$$V = \frac{\text{Spec}(R) - Z(B)}{(\mathbb{C}^*)^r}, \quad (3)$$

where B is the “irrelevant ideal” in R and $Z(B)$ is the associated subvariety of \mathbb{C}^N . B is determined combinatorially from the fan describing V .

Let \mathbf{v} denote an element of the lattice D , i.e., an r -vector. If M is a multigraded R -module then we may shift multi-gradings to form $M(\mathbf{v})$ in the usual way. Correspondingly, if \mathcal{O}_V is the structure sheaf of V , then we may denote by $\mathcal{O}_V(\mathbf{v})$ the twisted sheaf associated to the module $R(\mathbf{v})$. Line bundles on V correspond to $\mathcal{O}_V(\mathbf{v})$ for various $\mathbf{v} \in D$. If V is smooth then every element of D defines a line bundle.

Let \mathbf{q}_i denote the row vectors of the transpose of Φ . That is, \mathbf{q}_i represents the multi-grading of the homogeneous coordinate x_i . Let T_V be the tangent sheaf of V . Assuming V is smooth, we have the generalization of the Euler exact sequence for a toric variety [12]

$$0 \longrightarrow \mathcal{O}_V^{\oplus r} \xrightarrow{x_i \mathbf{q}_i} \bigoplus_{i=0}^{N-1} \mathcal{O}_V(\mathbf{q}_i) \longrightarrow T_V \longrightarrow 0. \quad (4)$$

Let

$$\mathbf{Q} = \sum_{i=0}^{N-1} \mathbf{q}_i. \quad (5)$$

¹Assumed to be torsion-free.

For the complete intersection X we have the adjunction exact sequence:

$$0 \longrightarrow T_X \longrightarrow T_{V|X} \longrightarrow \bigoplus_a \mathcal{O}_X(\mathbf{Q}_a) \longrightarrow 0, \quad (6)$$

where \mathbf{Q}_a is the multi-degree of the equation f^a . The Calabi–Yau condition is $\sum_a \mathbf{Q}_a = \mathbf{Q}$.

Since all the sheaves in (4) are locally-free, we may restrict to X and the sequence will remain exact. Combining this with the sequence (6) yields the following fact. The tangent sheaf is given by the cohomology at the middle term of

$$0 \longrightarrow \mathcal{O}_X^{\oplus r} \xrightarrow{x_i \mathbf{q}_i} \bigoplus_i \mathcal{O}_X(\mathbf{q}_i) \xrightarrow{\partial_i W_a} \bigoplus_a \mathcal{O}_X(\mathbf{Q}_a) \longrightarrow 0. \quad (7)$$

Obviously we may deform this complex to

$$0 \longrightarrow \mathcal{O}_X^{\oplus r} \xrightarrow{E} \bigoplus_i \mathcal{O}_X(\mathbf{q}_i) \xrightarrow{J} \bigoplus_a \mathcal{O}_X(\mathbf{Q}_a) \longrightarrow 0, \quad (8)$$

for generic matrices E and J of the correct multi-degree. By varying E and J we produce a family of sheaves containing the tangent sheaf. This family of sheaves can be understood in terms of the gauged linear σ -model [13, 14]. As such these are deformations of the $(0, 2)$ -model protected by Beasley–Witten. In general this is a subspace of the moduli space of $(0, 2)$ deformations. Correspondingly, there are deformations of T_X that this description does not capture. These deformations are not protected and worldsheet instanton effects can be nontrivial.

We thus need a more complete description of the space of deformations of T_X . The first order deformations of T_X are given by the vector space

$$\mathrm{Ext}^1(T_X, T_X) = H^1(X, \mathrm{End}(T)). \quad (9)$$

We may follow [9] and compute this as follows. The sheaf $\mathrm{End}(T)$ can be written as the cohomology of (7) tensored with its dual. That is,

$$\begin{array}{c} \bigoplus_a \mathcal{O}_X(-\mathbf{Q}_a)^{\oplus r} \longrightarrow \begin{array}{c} \bigoplus_{i,a} \mathcal{O}_X(\mathbf{q}_i - \mathbf{Q}_a) \\ \oplus \\ \bigoplus_i \mathcal{O}_X(-\mathbf{q}_i)^{\oplus r} \end{array} \longrightarrow \begin{array}{c} \mathcal{O}_X^{\oplus r^2} \\ \oplus \\ \bigoplus_{i,j} \mathcal{O}_X(\mathbf{q}_i - \mathbf{q}_j) \\ \oplus \\ \mathcal{O}_X \end{array} \longrightarrow \\ \dots \end{array}$$

$$\begin{array}{c} \bigoplus_i \mathcal{O}_X(\mathbf{q}_i)^{\oplus r} \\ \oplus \\ \bigoplus_{i,a} \mathcal{O}_X(\mathbf{Q}_a - \mathbf{q}_i) \end{array} \longrightarrow \bigoplus_a \mathcal{O}_X(\mathbf{Q}_a)^{\oplus r} \quad (10)$$

The cohomology is at the term we have underlined. We will consider the underlined term position “zero” in the complex. If a sheaf is presented as the cohomology (at position zero) of a complex

$$\dots \longrightarrow \mathcal{E}^{-1} \longrightarrow \mathcal{E}^0 \longrightarrow \mathcal{E}^1 \longrightarrow \dots, \quad (11)$$

then there is a spectral sequence [9] converging to the cohomology of the sheaf whose E_1 term is given by

$$E_1^{p,q} = H^q(X, \mathcal{E}^p). \quad (12)$$

As explained in [9], there is a strong resemblance between row 0 (i.e., $q = 0$) and the first order deformations of the linear σ -model (8). The space $H^0(V, \mathcal{O}_V(\mathbf{q}))$ is the space of global sections of $\mathcal{O}_V(\mathbf{q})$ and its dimension is counted by the number of monomials in R with multi-degree \mathbf{q} . We can count the deformations of (8) by considering all possible matrices of polynomials E and J such that $J.E = 0$ and then subtracting the number of reparametrizations induced by changes of homogeneous coordinates. This amounts to computing the cohomology in the middle term of the sequence:

$$\begin{array}{ccc} H^0(\mathcal{O}_V^{\oplus r^2}) & & \\ \oplus & \oplus_i H^0(\mathcal{O}_V(\mathbf{q}_i)^{\oplus r}) & \\ \oplus_{i,j} H^0(\mathcal{O}_V(\mathbf{q}_i - \mathbf{q}_j)) \longrightarrow & \oplus & \longrightarrow H^0(\mathcal{O}_V(\mathbf{Q})^{\oplus r}). \\ \oplus & \oplus_i H^0(\mathcal{O}_V(\mathbf{Q} - \mathbf{q}_i)) & \\ H^0(\mathcal{O}_V) & & \end{array} \quad (13)$$

Comparing this to the zeroth cohomology of (10) we see two obvious differences:

1. The computation is on V rather than X and
2. The left two terms of (10) are missing.

In simple cases these differences have no effect and the zeroth row of the spectral sequence accurately represents the deformations as seen by the linear σ -model.

For the quintic threefold in \mathbb{P}^4 this is the complete story. There is no contribution to $h^1(\text{End}(T))$ from any rows other than the zeroth row. Thus all deformations of T are understood in terms of the linear σ -model. They are unobstructed and cannot be spoiled by worldsheet instantons.

3 The Octic

Let us summarize the results of [9] where X is the resolved octic hypersurface in $\mathbb{P}^4_{\{2,2,2,1,1\}}$.

- There are 179 deformations of T that are seen by the gauged linear σ -model and are given by the bottom row of the spectral sequence.
- The total count of $h^1(\text{End}(T))$ depends on the complex structure. Generically it is 188 while for the Fermat hypersurface it is 200. All even values between these extremes can be obtained by choosing suitable octic defining equations.

- For generic values of the map E the number of deformations is 188. Thus some, if not all, of the extra 12 deformations associated to the Fermat complex structure are obstructed.

This jumping of the value of $h^1(\text{End}(T))$ shows that we have a nontrivial superpotential for the singlets.

At the Gepner point we have 206 singlets associated with the bundle data. (That is 206 singlets aside from deformations of complex structure, deformations of Kähler form and partners of the $U(1)^4$ gauge symmetry.) By deforming the superpotential of the *mirror* Landau–Ginzburg theory we lose 6 of these 206. Thus 6 singlets are an artifact of being at the Gepner radius.

On deforming the superpotential of the Landau–Ginzburg theory we can lose up to 12 more singlets. In fact we get *perfect* agreement between the Landau–Ginzburg theory and the geometrical result. For any defining equation, i.e., superpotential, the number of computed singlets associated to $h^1(\text{End}(T))$ is between 188 and 200 and the Landau–Ginzburg matches the geometry.²

One should be able to, in principle, compute the precise obstruction theory for the bundle and compute all correlation functions between the singlets in the Landau–Ginzburg model. This would allow a precise comparison of the superpotential between geometry and the exact result. This, in turn, would show if there were any instanton corrections to the superpotential. We have not done this but the agreement above does show that there are no corrections that would affect the *masses* of the singlets.

This shows that there is some “miracle” that kills instanton corrections to these $(0, 2)$ -models that goes beyond [7, 8]. Indeed we have checked many examples of Calabi–Yau *hypersurfaces* in toric varieties and we always find agreement of singlet counting between classical and exact methods.

It is natural to ask, therefore, whether this unnatural agreement persists for all $(0, 2)$ -models with the standard embedding. We will see that it is not the case.

4 The Z -Manifold

4.1 Geometry

Let T be the 2-torus given by $z \in \mathbb{C}$ under the identification $z \mapsto z + 1$ and $z \mapsto z + \omega$, where $\omega = \exp(2\pi i/3)$. Let Z_0 be the orbifold $(T \times T \times T)/\mathbb{Z}_3$ where the \mathbb{Z}_3 is generated by the action

$$g : (z_0, z_1, z_2) \mapsto (\omega z_0, \omega z_1, \omega z_2). \quad (14)$$

This is the well-known Z -orbifold introduced in [10] in which the associated conformal field theory was written in terms of free fields and the spectrum computed exactly.

²Note that this Landau–Ginzburg computation is stuck at small radius while the classical computation is stuck at large radius. It is just conceivable that this perfect agreement at large and small radius is spoiled at intermediate radii. We will assume this is not the case.

The \mathbb{Z}_3 action has 27 fixed points yielding 27 singularities in Z_0 . These can be resolved by blowing up each point with a \mathbb{P}^2 exceptional set to yield the Z -manifold.

It is very easy to compute the Hodge numbers of Z by using homology. Each blow-up introduces a 4-cycle in terms of the exceptional \mathbb{P}^2 . Adding this 27 to the 9 invariant cycles from the covering 6-torus yields $b_2 = h^{1,1} = 36$. Similarly one can argue that $h^{2,1} = 0$ and the Z -manifold is rigid.

It is considerably harder to compute $h^1(\text{End}(T))$. String theory implies there is a way to add local contributions of the blow-ups to some global contribution of the 6-torus but we do not know how to *rigorously* formulate this geometrically.³ Instead we use another construction of the Z -manifold from which we do know how to extract $h^1(\text{End}(T))$.

Since toric varieties offer a tractable path, we embed Z into a toric variety. Unfortunately, it cannot be embedded as a hypersurface since all such hypersurfaces have a mirror [15]. We can, however write it as the complete intersection of 3 equations.

The computation of $h^1(\text{End}(T))$ for the Z -manifold is very lengthy and we confine the details to an appendix. The result is that

$$h^1(\text{End}(T)) = 208. \tag{15}$$

It turns out that, of these 208, only 6 come from the bottom row of the spectral sequence and are thus protected by Beasley–Witten from instanton corrections.

4.2 The Landau–Ginzburg Picture

At a point in the $(2, 2)$ moduli space, the superconformal nonlinear sigma model on the Z -manifold is equivalent to a Gepner model [16]. This is most directly seen by recalling that the sigma model on an elliptic curve, at precisely the complex structure exhibiting a \mathbb{Z}_3 symmetry mentioned in the previous section, is equivalent at one point in its Kähler moduli space to a Gepner model, a \mathbb{Z}_3 quotient of the product $A_1^{\otimes 3}$. The construction of the Z as an orbifold then shows that at a point in the nine-dimensional moduli space of the orbifold theory (before blowing up) the model is equivalent to a $(\mathbb{Z}_3)^4$ quotient of the product of superconformal minimal models $A_1^{\otimes 9}$.

In this form, it is straightforward to count the 270 massless E_6 singlet fields at this point in the moduli space [17]. Of these, of course, 36 are the $(2, 2)$ moduli, but this still leaves 234 singlets, more than the 208 found at large radius above. One part of the discrepancy is clear. At the Gepner point the theory exhibits a $U(1)^6 \times SU(3)$ gauge symmetry, which is broken by generic (Kähler) deformations. This leads to D -term masses (via the Higgs mechanism) for 14 of the singlet fields. Masses for any of the remaining 220 fields are generated by the spacetime superpotential.

In the Gepner model mentioned (and the associated Landau–Ginzburg model) the $(2, 2)$ moduli are all twisted fields under the orbifold projection, making it difficult to study the model away from this one point. To get around this we study instead the mirror of the

³An ALE space must be deformed before it is glued in and the torus metric must be deformed away from being flat. Such deformations may *a priori* affect $h^1(\text{End}(T))$.

Z -manifold. At a point in its moduli space, the superconformal theory on this is equivalent to a quotient of the same Gepner model [18]. The construction leads to a \mathbb{Z}_3^5 quotient of the product of minimal models. It is simpler, in this case, to note that since an elliptic curve is its own mirror, up to a relabeling of the fields this is equivalent to a \mathbb{Z}_3^2 quotient. As expected, the mirror model has a 36-dimensional space of $(2, 2)$ deformations. Of these, 30 are untwisted under the quotient and can be represented in the associated Landau–Ginzburg model as deformations of the worldsheet superpotential. The methods of [19] then enable a computation of the number of massless singlet fields at any point in this 30-dimensional subspace of the full moduli space. The orbifold locus intersects this subspace along a three-dimensional subspace.

The Landau–Ginzburg model contains nine $(2, 2)$ chiral superfields which we denote X^i, Y^i, Z^i , $i = 1 \dots 3$ interacting via a cubic superpotential. The model has a $U(1)$ R-symmetry under which the fields all have charge $1/3$. In the infrared this model flows to a $(2, 2)$ superconformal field theory with a $U(1) \times U(1)$ R-symmetry. The Gepner model (after GSO projection) is a \mathbb{Z}_6 orbifold of this, and our mirror model is a further quotient by a \mathbb{Z}_3 generated by $(X^i, Y^j, Z^k) \mapsto (\omega X^i, \omega^2 Y^j, Z^k)$. The most general invariant superpotential can be written as

$$W = A_{ijk} X^i X^j X^k + B_{ijk} Y^i Y^j Y^k + C_{ijk} Z^i Z^j Z^k + D_{ijk} X^i Y^j Z^k . \quad (16)$$

The 57 parameters in this superpotential are subject to an action of $GL(3)^3$, which allows us to bring it to the form

$$W = \sum (X^i)^3 - 3A_x X^1 X^2 X^3 + \sum (Y^i)^3 - 3A_y Y^1 Y^2 Y^3 + \sum (Z^i)^3 - 3A_z Z^1 Z^2 Z^3 + D_{ijk} X^i Y^j Z^k . \quad (17)$$

This is our explicit representation of the 30-dimensional family. The orbifold locus corresponds to $D_{ijk} = 0$.

We can consider $(0, 2)$ deformations of this as a Landau–Ginzburg model; this is a degenerate version ($E = 0$) of the gauged linear sigma model counting (8) and we find 82 such deformations, providing a lower bound on the number of massless singlets at any point in the moduli space.

Supersymmetric ground states are found using the left-moving $N = 2$ superconformal algebra in the cohomology of the right-moving supercharge \bar{Q} and classified by their charges (q, \bar{q}) under the $U(1)$ symmetry contained in this algebra and under the right-moving $U(1)$ symmetry inherited from the $(2, 2)$ structure. States are described as excitations by the lowest oscillator modes of free bosonic fields ϕ_a^i (labeled by $a = x, y, z$ and $i = 1, 2, 3$) with charges $(1/3, 1/3)$ and left-moving fermionic fields γ_a^i with charges $(-2/3, 1/3)$.

The generator of the Gepner quotient acts as $e^{\pi i q}$ while the additional \mathbb{Z}_3 quotient acts as $\phi_a^i \rightarrow \omega^{w_a} \phi_a^i$; $\gamma_a^i \rightarrow \omega^{w_a} \gamma_a^i$ with $w = (1, -1, 0)$. Twisted vacua are labeled by $(k; l)$ with $k = 0, \dots, 5$ and $l = 0, 1, 2$. It should be emphasized that this orbifold construction of the mirror of the Z -manifold is distinct from the orbifold construction of the Z -orbifold itself as

in [10]. In particular twisted states in one orbifold need not correspond to twisted states in the other. In the $(k; l)$ sector the fields have a twisted moding

$$\begin{aligned}\phi_a^i(z) &= \sum_{s \in \mathbb{Z} - \nu_a} x_{a(s)}^i z^{-s-1/6}, & \gamma_a^i(z) &= \sum_{s \in \mathbb{Z} - \tilde{\nu}_a} \gamma_{a(s)}^i z^{-s-2/3} \\ 2\partial\bar{\phi}_{ai}(z) &= \sum_{s \in \mathbb{Z} + \nu_a} \rho_{ai(s)} z^{-s-5/6}, & \bar{\gamma}_{ai}(z) &= \sum_{s \in \mathbb{Z} + \tilde{\nu}_a} \bar{\gamma}_{ai(s)} z^{-s-5/6},\end{aligned}\quad (18)$$

where

$$\begin{aligned}\nu_a(k; l) &= \frac{k}{6} + \frac{lw_a}{3} \pmod{1} & 0 \leq \nu_a < 1 \\ \tilde{\nu}_a(k; l) &= \frac{-k}{3} + \frac{lw_a}{3} \pmod{1} & -1 < \tilde{\nu}_a \leq 0\end{aligned}\quad (19)$$

The ground state energy and charges of the twisted vacua are

$$\begin{aligned}E(k; l) &= -\frac{5}{8} + \frac{3}{2} \sum_a \left(\nu_a(1 - \nu_a) + \tilde{\nu}_a(1 + \tilde{\nu}_a) \right) \quad k \text{ odd} \\ q(k; l) &= -\sum_a \left(2\tilde{\nu}_a + \nu_a + \frac{1}{2} \right) \\ \bar{q}(k; l) &= \sum_a \left(\tilde{\nu}_a + 2\nu_a - \frac{1}{2} \right).\end{aligned}\quad (20)$$

$$(21)$$

For k even $E(k; l) = 0$.

Massless fermions arise in \mathbb{R} (odd k) sectors and correspond to excitations with $E = 0$. E_6 singlets are characterized by $q = 0$. The right-moving charge \bar{q} of a state determines the spacetime multiplet to which the fermion belongs: $\bar{q} = \pm\frac{3}{2}$ are fermions in vector multiplets while $\bar{q} = \pm\frac{1}{2}$ are fermions in chiral multiplets. We construct states with $E = q = 0$ by acting on $|k; l\rangle$ with the lowest excited modes, which we denote

$$x_a^i \equiv x_{a(-\nu_a)}^i, \quad \rho_{ai} \equiv \rho_{ai(\nu_a-1)}, \quad \gamma_a^i \equiv \gamma_{a(-1-\tilde{\nu}_a)}^i, \quad \bar{\gamma}_{ai} \equiv \bar{\gamma}_{ai(\tilde{\nu}_a)}.\quad (22)$$

In describing the \bar{Q} cohomology we will also need the conjugate modes

$$x^{\dagger}_{ai} \equiv \rho_{ai(\nu_a)}, \quad \rho^{\dagger}_a \equiv x_{a(1-\nu_a)}^i, \quad \gamma^{\dagger}_{ai} \equiv \bar{\gamma}_{ai(1+\tilde{\nu}_a)}, \quad \bar{\gamma}^{\dagger}_a \equiv \gamma_{a(-\tilde{\nu}_a)}^i.\quad (23)$$

Acting on these states, the \bar{Q} operator takes the general form

$$\bar{Q} = \sum_{a,i} \left\{ \gamma_a^i \partial_{ai} W_{1+\tilde{\nu}_i} + \bar{\gamma}^{\dagger}_{ai} \partial_{ai} W_{\tilde{\nu}_i} \right\}.\quad (24)$$

The quotient breaks the S_9 permutation symmetry of the product to a subgroup. Of interest to us is an unbroken S_3 subgroup permuting the a indices of the fields. This relabels the generators of the quotient group and correspondingly permutes the twisted sectors.

4.2.1 Untwisted States

In the untwisted R sector $(k; l) = (1; 0)$ we have $\nu_a = \frac{1}{6}$, $\tilde{\nu}_a = -\frac{1}{3}$ so that the ground state has $E(1; 0) = -1$, $q(1; 0) = 0$, $\bar{q}(1; 0) = -\frac{3}{2}$. The complex of $E = q = 0$ states upon which \bar{Q} acts is

$$\bar{q} = -\frac{3}{2} \qquad \bar{q} = -\frac{1}{2} \qquad (25)$$

$$\begin{array}{ccc} \phi_a^i \rho_{aj} |1; 0\rangle_{27} & \xrightarrow{\bar{Q}} & \phi_a^i \phi_a^j \gamma_a^k |1; 0\rangle_{54} \\ \oplus & & \oplus \\ \bar{\gamma}_{ai} \gamma_a^j |1; 0\rangle_{27} & & \phi_a^i \phi_b^j \gamma_c^k |1; 0\rangle_{81} \end{array} \quad (a \neq b \neq c)$$

where subscripts on kets indicate the dimension of the space of states of a given form; repeated indices on fields are not summed. We write \bar{Q} as a sum of three terms

$$\bar{Q} = \bar{Q}_G + \bar{Q}_O + \bar{Q}_D . \qquad (26)$$

\bar{Q}_G , the supercharge at the Gepner point, is in this sector given by

$$\bar{Q}_G = 6 \sum_{a,i} \gamma_a^i x_a^i \rho_a^{\dagger i} + 3 \bar{\gamma}_a^{\dagger i} (x_a^i)^2 . \qquad (27)$$

This has a nine-dimensional kernel spanned by $J_a^i = \frac{1}{3} (x_a^i \rho_{ai} - 2 \bar{\gamma}_{ai} \gamma_a^i) |1; 0\rangle$, indicating that the enhanced gauge group at the Gepner point has rank eight (one generator is the $U(1) \subset E_6$), and that the untwisted sector gives rise to 90 chiral singlets. \bar{Q}_O , the additional charge at a generic point on the orbifold locus $D = 0$ is in this sector given by

$$\bar{Q}_O = -3 \sum_a A_a \sum_{i \neq j \neq k} [\gamma_a^i (x_a^j \rho_a^{\dagger k} + x_a^k \rho_a^{\dagger j}) + \bar{\gamma}_a^{\dagger i} x_a^j x_a^k] . \qquad (28)$$

Adding this to \bar{Q}_G with generic A_a reduces the dimension of the kernel to three, spanned by $J_a = \sum_i J_a^i$, indicating an enhanced gauge group of rank two, and a total of 84 chiral singlets. Adding

$$\begin{aligned} \bar{Q}_D = \sum_{ijk} D_{ijk} & [\gamma_x^i (y^j \rho_z^{\dagger k} + z^k \rho_y^{\dagger j}) + \bar{\gamma}_x^{\dagger i} y^j z^k \\ & + \gamma_y^j (x^i \rho_z^{\dagger k} + z^k \rho_x^{\dagger i}) + \bar{\gamma}_y^{\dagger j} x^i z^k + \gamma_z^k (y^j \rho_x^{\dagger i} + x^i \rho_y^{\dagger j}) + \bar{\gamma}_z^{\dagger k} x^i y^j] \end{aligned} \qquad (29)$$

leaves a one-dimensional kernel (guaranteed by the quasihomogeneity of W) generated by $J = \sum_a J_a$, indicating the gauge symmetry is reduced to E_6 and leaving 82 neutral chiral multiplets.

4.2.2 Twisted States

Since the S_3 symmetry permutes the twisted sectors, a calculation in any one is sufficient to produce the entire spectrum. In the $(k; l) = (1; 1)$ sector we have $\nu(1; 1) = (\frac{1}{2}, \frac{5}{6}, \frac{1}{6})$, $\tilde{\nu}(1; 1) =$

$(0, -\frac{2}{3}, -\frac{1}{3})$ and hence $E(1; 1) = -\frac{1}{2}$, $q(1; 1) = -1$, $\bar{q}(1; 1) = \frac{1}{2}$. The complex of $E = q = 0$ states is

$$\begin{array}{ccccccc}
\bar{q} = -\frac{3}{2} & & \bar{q} = -\frac{1}{2} & & \bar{q} = \frac{1}{2} & & \bar{q} = \frac{3}{2} & (30) \\
& & \rho_x \bar{\gamma}_x^2 |1; 1\rangle_9 & & x \bar{\gamma}_x |1; 1\rangle_9 & & & \\
& & \oplus & & \oplus & & & \\
\rho_y^3 \bar{\gamma}_x^3 |1; 1\rangle_{10} & \xrightarrow{\bar{Q}} & z \rho_y^2 \bar{\gamma}_x^2 |1; 1\rangle_{54} & \xrightarrow{\bar{Q}} & z^2 \rho_y \bar{\gamma}_x |1; 1\rangle_{54} & \xrightarrow{\bar{Q}} & z^3 |1; 1\rangle_{10} & \\
& & \oplus & & \oplus & & & \\
& & \rho_y \bar{\gamma}_x \bar{\gamma}_z |1; 1\rangle_{27} & & z \bar{\gamma}_z |1; 1\rangle_9 & & & \\
& & \oplus & & \oplus & & & \\
& & \rho_y \bar{\gamma}_x^3 \gamma_y |1; 1\rangle_9 & & z \bar{\gamma}_x^2 \gamma_y |1; 1\rangle_{27} & & &
\end{array}$$

In this sector we have

$$\bar{Q}_G = 3 \sum_i \left[\gamma_x^i (\rho_{xi}^\dagger)^2 + 2 \bar{\gamma}_x^{\dagger i} x^i \rho_x^{\dagger i} + \gamma_y^i (\rho_{yi}^\dagger)^2 + 2 \bar{\gamma}_y^{\dagger i} y^i \rho_y^{\dagger i} + 2 \gamma_z^i z^i \rho_{zi}^\dagger + \bar{\gamma}_y^{\dagger i} (z^i)^2 \right]. \quad (31)$$

Acting on the $\bar{q} = \frac{3}{2}$ space this has a one-dimensional kernel generated by $\rho_{y1} \rho_{y2} \rho_{y3} \bar{\gamma}_x^3 |1; 1\rangle$; the six vector multiplets arising from twisted sectors fill out the enhanced gauge group $U(1)^6 \times SU(3)$ with the Cartan subgroup generated by the charges associated to $J_a - \frac{1}{3} J$.

Acting on the $\bar{q} = \frac{1}{2}$ space we find a 39-dimensional kernel, so that each twisted sector contributes 30 massless E_6 singlet chiral multiplets; when added to the 90 we found in the untwisted sector this reproduces the expected 270 singlets at the Gepner point.

The analysis is made easier by organizing the zero-energy states into $SU(3)$ multiplets and recalling that \bar{Q}_G as well as \bar{Q}_O are $SU(3)$ invariant and that \bar{Q} acts within a given twist sector. We find

$$\begin{array}{ccccccc}
\bar{q} = -\frac{3}{2} & & \bar{q} = -\frac{1}{2} & & \bar{q} = \frac{1}{2} & & \bar{q} = \frac{3}{2} & (32) \\
\mathbf{1}^{34} & & \mathbf{1}^{36} \oplus \mathbf{3}^{27} & & \mathbf{1}^{36} \oplus \bar{\mathbf{3}}^{27} & & \mathbf{1}^{34} & \\
\oplus & & \oplus & & \oplus & & \oplus & \\
\mathbf{8}^{10} & \xrightarrow{\bar{Q}} & \mathbf{8}^9 & \xrightarrow{\bar{Q}} & \mathbf{8}^9 & \xrightarrow{\bar{Q}} & \mathbf{8}^{10} & \\
& & \oplus & & \oplus & & & \\
& & \mathbf{1}^{54} \oplus \mathbf{3}^{108} \oplus \bar{\mathbf{3}}^{54} & & \mathbf{1}^{54} \oplus \mathbf{3}^{54} \oplus \bar{\mathbf{3}}^{108} & & &
\end{array}$$

where the first row represents the contribution of the untwisted sectors and the last the contributions of the twisted sectors. Adjoints, in the second row, are counted separately as they are assembled from twisted and untwisted states. In the untwisted sector we computed above that \bar{Q}_G has rank 27 when acting on the space of $SU(3)$ singlets. Acting on the adjoints it has the maximum rank possible. Since untwisted states with $\bar{q} > 0$ arise in the conjugate $(5; 0)$ sector the map $\bar{q} = -\frac{1}{2} \rightarrow \bar{q} = \frac{1}{2}$ vanishes for untwisted states. In the twisted sector, acting on the $SU(3)$ singlet states \bar{Q}_G has rank 36. In the $(1; 1)$ sector the three-dimensional

kernel at $\bar{q} = -\frac{1}{2}$ is spanned by $\rho_{xi}\bar{\gamma}_{xj}\bar{\gamma}_{xk}|1;1\rangle$ with $i \neq j \neq k$. The rank on the SU(3) charged fields is maximal and we find that the physical states are vector multiplets in the representation

$$\mathbf{1}^7 \oplus \mathbf{8}, \quad (33)$$

of which one of the singlets is a Cartan generator of E_6 , and chiral multiplets in the representation

$$\underbrace{\mathbf{1}^9 \oplus \mathbf{3}^{27}}_{\text{untwisted}} \oplus \underbrace{\mathbf{1}^{18} \oplus \mathbf{3}^{54}}_{\text{twisted}}. \quad (34)$$

The (2,2) moduli are $\mathbf{1}^3 \oplus \mathbf{3}^9 \oplus \mathbf{1}^6$ as expected.

Now we move away from the Gepner point but remain mirror to the *orbifold* by adding \bar{Q}_O . In the untwisted sector, as noted above, the rank of the map between SU(3) singlet states increases from 27 to 33. This is the Higgs mechanism breaking the enhanced gauge symmetry $SU(3) \times U(1)^6$ to SU(3) by removing a vector multiplet and a chiral multiplet for each of the six broken generators of the gauge group. In the twisted sector we find that the rank of the map on singlets increases from 36 to 48. In the (1;1) sector the one-dimensional kernel at $\bar{q} = \frac{1}{2}$ is now spanned by $\sum_{ijk} \epsilon^{ijk} \rho_{xi}\bar{\gamma}_{xj}\bar{\gamma}_{xk}|1;1\rangle$. The cohomology thus contains 9 vector multiplets in the representation $\mathbf{1} \oplus \mathbf{8}$ and 252 chiral multiplets in the representation

$$\underbrace{\mathbf{1}^3 \oplus \mathbf{3}^{27}}_{\text{untwisted}} \oplus \underbrace{\mathbf{1}^6 \oplus \mathbf{3}^{54}}_{\text{twisted}}, \quad (35)$$

as was found in [10].

Finally we blow-up the orbifold in the mirror by adding \bar{Q}_D . Now the enhanced gauge symmetry is completely broken. In the untwisted sector we saw that the rank of \bar{Q} grows by two. This is again the Higgs mechanism for the Cartan elements of the SU(3) gauge symmetry, and it is accompanied by a corresponding change in rank by one in the map $\bar{q} = -\frac{3}{2} \rightarrow \bar{q} = -\frac{1}{2}$ in each twisted sector. In addition, we find that the rank of the map $\bar{q} = -\frac{1}{2} \rightarrow \bar{q} = \frac{1}{2}$ increases from 62 to 80, indicating that 108 chiral multiplets are lifted by spacetime superpotential interactions (F -terms). This leaves a total of 136 E_6 neutral chiral multiplets at generic points in the moduli space.

Comparing these numbers to the results in the previous section we find that a worldsheet instanton generated spacetime superpotential leads to mass terms for 108 of the perturbatively massless scalars found by the large-radius analysis.

5 Summary and Discussion

It is surprising that $h^1(\text{End}(T))$ is so resilient to instanton corrections. It would seem that there are mechanisms beyond those of [7,8] that protect these states from acquiring masses away from the large radius limit. From the examples we have considered, it is tempting to conjecture that a good candidate for a class of models in which there are no corrections is that of hypersurfaces in toric varieties. Typically these have contributions to $h^1(\text{End}(T))$

above the zeroth row of the spectral sequence and, in all cases we have considered, their masses are not affected by instantons.

The fact that there seems to be a large class of models without corrections should lead to some interesting mathematics. The agreement between the cohomology of the complexes of the Landau–Ginzburg picture and the spectral sequence of the σ -model is not yet understood. One may also be able to make curious statements such as $h^1(\text{End}(T))$ being equal for mirror pairs of Calabi–Yau threefolds.

On the other hand, we now have an example where instantons *do* give mass to $H^1(\text{End}(T))$ modes. An interesting next step will be to enumerate these instantons more precisely from the conformal field theory computation.

In the case of the Z -manifold, we find that of the 252 massless E_6 singlet fields in the Z -orbifold spectrum, 8 acquire D -term masses via the Higgs mechanism and 108 acquire F -term masses via a spacetime superpotential as we blow up the quotient singularities. Removing the 36 $(2, 2)$ moduli, this leaves precisely 100 massless singlets corresponding to bundle deformations at a generic point in the (30-dimensional) moduli space. Comparing this to the geometric computation in the large-radius limit we find again that of the 208 singlet fields counted by $h^1(\text{End}(T))$ 108 acquire masses via instanton-induced superpotential terms. It is natural to conjecture that one can identify the full 208-dimensional space of massless states at the orbifold locus with $H^1(\text{End}(T))$ and that the two sets of F -terms are associated to the same superpotential.

This counting argument suggests that even in this case where instanton contributions are nonzero there is a McKay-like correspondence for local contributions to $h^1(\text{End}(T))$ from an orbifold resolution. This would imply that there is some local picture for noncompact Calabi–Yau threefolds where instantons can be shown to cancel. Clearly further investigation of this would be interesting.

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A $h^1(\text{End}(T))$ for the Z -manifold

Z is the resolution of T^6/\mathbb{Z}_3 at the 27 fixed points of the orbifold action. We may write the 2-torus as a cubic in \mathbb{P}^2 . Hence Z can be written as a complete intersection of three “cubics” in V where V is the crepant desingularization of

$$V' = \frac{\mathbb{P}^2 \times \mathbb{P}^2 \times \mathbb{P}^2}{\mathbb{Z}_3}. \quad (36)$$

V is a toric variety as follows. The 1-dimensional rays of the associated fan are given by the rows of the matrix

$$\begin{pmatrix} 1 & 1 & -2 & 1 & 1 & 0 \\ -1 & -1 & 2 & -1 & -1 & -1 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & -1 & -1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ -1 & -1 & -1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 \end{pmatrix} \quad (37)$$

The homogeneous coordinates (x_0, \dots, x_9) are associated to these rays. The Stanley-Reisner ideal is given by

$$I = \langle x_0x_1x_2, x_3x_4x_5, x_0x_3x_6, x_6x_7x_8, x_1x_2x_9, x_4x_5x_9, x_7x_8x_9 \rangle. \quad (38)$$

Z is now the complete intersection in V associated to the ideal $\langle f^0, f^1, f^2 \rangle$ where

$$\begin{aligned} f^0 &= x_1^3 + x_2^3 + x_0^3x_9 \\ f^1 &= x_4^3 + x_5^3 + x_3^3x_9 \\ f^2 &= x_7^3 + x_8^3 + x_6^3x_9. \end{aligned} \quad (39)$$

We have a representation of the sheaf $\text{End}(T_Z)$ in terms of a complex of sums of line bundles on Z given by (10). It is more convenient for computational purposes if everything is written in terms of line bundles over the ambient toric variety V .

To see how to do this let us consider the much simpler case of the tangent bundle of the quintic 3-fold. This is given as the complex

$$\mathcal{O}_X \xrightarrow{x_i} \mathcal{O}_X(1)^{\oplus 5} \xrightarrow{\partial_i f} \mathcal{O}_X(5). \quad (40)$$

Let \mathcal{O} denote the structure sheaf of the ambient \mathbb{P}^4 . \mathcal{O}_X is then equivalent, in the derived category, to

$$\mathcal{O}(-5) \xrightarrow{f} \mathcal{O}. \quad (41)$$

Thus, by mapping cones, the complex given by the last two terms of (40) is given by

$$\mathcal{O}(-4)^{\oplus 5} \xrightarrow{\begin{pmatrix} -fI \\ \partial_i f \end{pmatrix}} \begin{array}{c} \mathcal{O}(1)^{\oplus 5} \\ \oplus \\ \mathcal{O} \end{array} \xrightarrow{(\partial_i f f)} \mathcal{O}(5). \quad (42)$$

The complete complex (40) is then given by the mapping cone of a map from complex representing \mathcal{O}_X to the complex (42). This morphism is given by the chain map

$$\begin{array}{ccc} \mathcal{O}(-5) & \xrightarrow{f} & \mathcal{O} \\ \downarrow x_i & & \downarrow \begin{pmatrix} -x_i \\ 5 \end{pmatrix} \\ \mathcal{O}(-4)^{\oplus 5} & \xrightarrow{\begin{pmatrix} -fI \\ \partial_i f \end{pmatrix}} & \begin{array}{c} \mathcal{O}(1)^{\oplus 5} \\ \oplus \\ \mathcal{O} \end{array} \xrightarrow{(\partial_i f f)} \mathcal{O}(5). \end{array} \quad (43)$$

Note, in particular, the need for the “5” to produce a chain map. The tangent sheaf of the quintic can thus be represented by the chain complex

$$\mathcal{O}(-5) \xrightarrow{\begin{pmatrix} -f \\ x_i \end{pmatrix}} \begin{array}{c} \mathcal{O} \\ \oplus \\ \mathcal{O}(-4)^{\oplus 5} \end{array} \xrightarrow{\begin{pmatrix} -x_i & -fI \\ 5 & \partial_i f \end{pmatrix}} \begin{array}{c} \mathcal{O}(1)^{\oplus 5} \\ \oplus \\ \mathcal{O} \end{array} \xrightarrow{(\partial_i f f)} \mathcal{O}(5). \quad (44)$$

We need to extend this construction to complete intersections. To this end we may prove the following theorem analogously to the above.

Theorem 1 *Consider a commutative diagram of sheaves on V of the following form:*

$$\begin{array}{ccccccc} \dots & \longrightarrow & A^{n-1} & \xrightarrow{a_{n-1}} & A^n & \xrightarrow{a_n} & A^{n+1} \longrightarrow \dots \\ & & \downarrow h_{n-1} & & \downarrow h_n & & \downarrow h_{n+1} \\ \dots & \longrightarrow & B^{n-1} & \xrightarrow{b_{n-1}} & B^n & \xrightarrow{b_n} & B^{n+1} \longrightarrow \dots \\ & & \downarrow g_{n-1} & & \downarrow g_n & & \downarrow g_{n+1} \\ \dots & \longrightarrow & C^{n-1} & \xrightarrow{c_{n-1}} & C^n & \xrightarrow{c_n} & C^{n+1} \longrightarrow \dots \\ & & \downarrow f_{n-1} & & \downarrow f_n & & \downarrow f_{n+1} \\ \dots & \longrightarrow & D^{n-1} & \xrightarrow{d_{n-1}} & D^n & \xrightarrow{d_n} & D^{n+1} \longrightarrow \dots \end{array} \quad (45)$$

satisfying

1. Each column is exact except it has cohomology L^n supported on Z in the last position.
2. the composition of two horizontal maps is zero when restricted to Z .

The complex L^\bullet is quasi-isomorphic to the complex with terms $A^{n+3} \oplus B^{n+2} \oplus C^{n+1} \oplus D^n$ and differentials

$$\begin{pmatrix} a_{n+3} & (-1)^n \alpha_{n+2} & \beta_{n+1} & (-1)^n \gamma_n \\ (-1)^n h_{n+3} & b_{n+2} & (-1)^n \delta_{n+1} & \epsilon_n \\ 0 & (-1)^n g_{n+2} & c_{n+1} & (-1)^n \zeta_n \\ 0 & 0 & (-1)^n f_{n+1} & d_n \end{pmatrix} : \begin{pmatrix} A^{n+3} \\ B^{n+2} \\ C^{n+1} \\ D^n \end{pmatrix} \longrightarrow \begin{pmatrix} A^{n+4} \\ B^{n+3} \\ C^{n+2} \\ D^{n+1} \end{pmatrix} \quad (46)$$

where

$$\begin{aligned} \zeta_n &= f_{n+2}^{-1} d_{n+1} d_n \\ \delta_n &= g_{n+2}^{-1} (c_{n+1} c_n - \zeta_n f_n) \\ \epsilon_n &= g_{n+3}^{-1} (c_{n+2} \zeta_n - \zeta_{n+1} d_n) \\ \alpha_n &= h_{n+2}^{-1} (b_{n+1} b_n - \delta_n g_n) \\ \beta_n &= h_{n+3}^{-1} (b_{n+2} \delta_n - \delta_{n+1} c_n + \epsilon_n f_n) \\ \gamma_n &= h_{n+4}^{-1} (b_{n+3} \epsilon_n - \delta_{n+2} \zeta_n + \epsilon_{n+1} d_n) \end{aligned} \quad (47)$$

Let the multi-degree of the equation f_i be denoted \mathbf{Q}_i . The sheaf \mathcal{O}_Z can be resolved in terms of the Koszul complex:

$$\mathcal{O}(-\sum_i \mathbf{Q}_i) \xrightarrow{h} \bigoplus_i \mathcal{O}(-\sum_{j \neq i} \mathbf{Q}_j) \xrightarrow{g} \bigoplus_i \mathcal{O}(-\mathbf{Q}_i) \xrightarrow{f} \mathcal{O} \longrightarrow \mathcal{O}_Z, \quad (48)$$

where \mathcal{O} is the structure sheaf of V .

Applying the theorem to this and the complex (10) yields a rather messy complex of line bundles on V representing $\text{End}(T)$ for the Z -manifold. The explicit form is too large to give here and we will write it more concisely as

$$\mathcal{F}_{(12)}^{-5} \longrightarrow \mathcal{F}_{(106)}^{-4} \longrightarrow \mathcal{F}_{(371)}^{-3} \longrightarrow \mathcal{F}_{(667)}^{-2} \longrightarrow \mathcal{F}_{(667)}^{-1} \longrightarrow \mathcal{F}_{(371)}^0 \longrightarrow \mathcal{F}_{(106)}^1 \longrightarrow \mathcal{F}_{(12)}^2 \quad (49)$$

where each term is a sum of line bundles on V where the rank is denoted by the subscript.

Now, to compute the cohomology we have a spectral sequence $E_1^{p,q} = H^q(\mathcal{F}^p)$. This yields

$$\begin{array}{ccccccc}
 & & & & & & q \\
 & & & & & & \uparrow \\
 & & & & & & 0 \\
 & & & & & & \vdots \\
 & & & & & & 24 \\
 & & & & & & \vdots \\
 & & & & & & 48 \\
 & & & & & & \vdots \\
 & & & & & & 432 \longrightarrow 104 \\
 & & & & & & \vdots \\
 & & & & & & 0 \\
 & & & & & & \vdots \\
 & & & & & & 35 \longrightarrow 100 \longrightarrow 60 \\
 & & & & & & \downarrow \\
 & & & & & & p
 \end{array}
 \quad (50)$$

$E_1^{p,q} :$

where each number represents the dimension of the space at each position. The dotted line shows the terms that will ultimately contribute to $h^1(\text{End}(T))$. Now we need to compute the maps above and take cohomology to proceed to $E_2^{p,q}$.

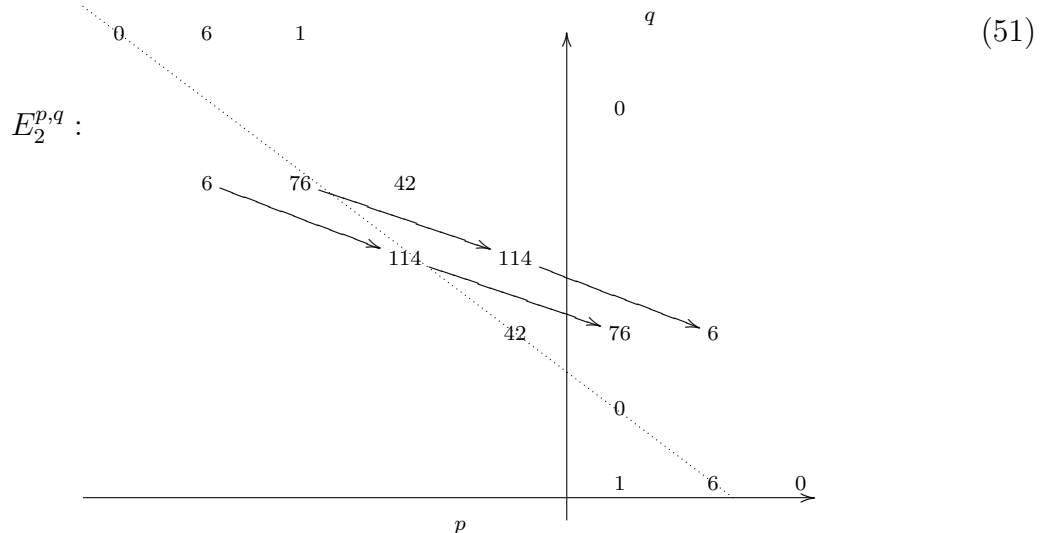
To do this we follow the method given in [9]. We may regard the spectral sequence as having arisen from a double complex with the given complex maps in the horizontal direction and Čech cohomology in the vertical direction.

Let $B = (m_1, m_2, \dots, m_l)$, where m_i are monomials, be the irrelevant ideal which is the Alexander dual [20] of the Stanley–Reisner ideal (38). An open cover of V that can be used to compute Čech cohomology is then given by the set of $U_i = V - Z(m_i)$. This is equivalent to a local cohomology computation as explained in [21]. This local cohomology approach to computing the cohomology of line bundles has also been explored in the context of string theory in [22–24]. We refer to the appendix of [9] for a review of the ideas required here.

An element of the underlying double complex is given by a Čech cochain which is a collection of Laurent monomials $\{c_{j_1, j_2, \dots}\}$ each of which takes values in the localization $R_{m_{j_1}, m_{j_2}, \dots}$. This structure simplifies a little once we go to the E_1 stage by taking vertical cohomology. Each cochain must be coclosed and so we require exact cancellations on certain multiple overlaps of open patches U_i . Suppose we fix one of the $c_{j_1, j_2, \dots}$'s to be a fixed monomial of homogeneous coordinates. In straight-forward cases, such a cancellation tends to require that the other $c_{j_1, j_2, \dots}$'s are given (perhaps up to some fixed constant) by the exact same monomial. Thus we specify a given dimension of the space of cohomology by specifying a particular monomial. In other words, computing the dimension of the cohomology amounts to counting Laurent monomials of a certain form.

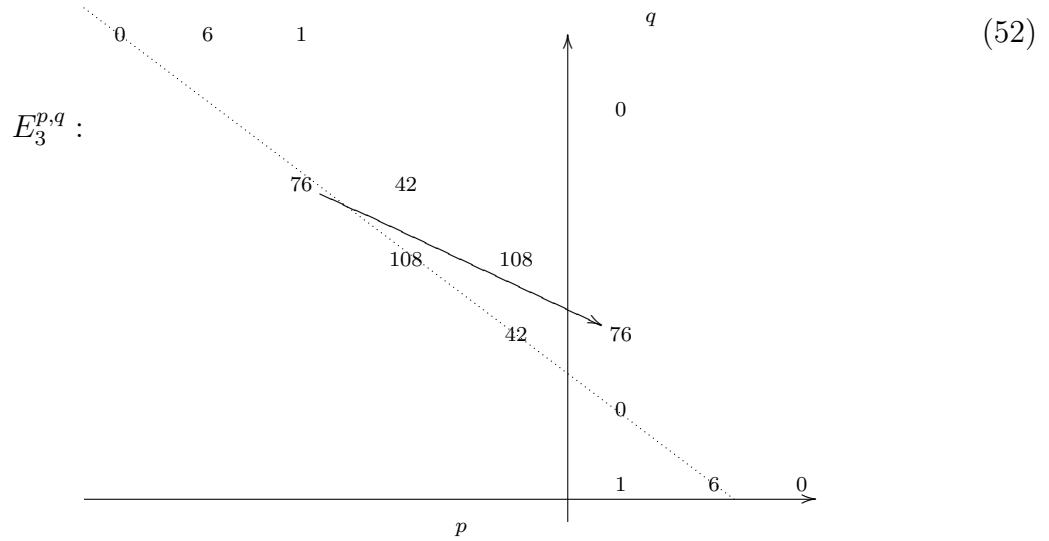
Whether this simple counting method works depends on the finely-graded Betti numbers $\text{Tor}_i^R(I, \mathbb{C})_\alpha$ as in Corollary 3.1 of [25]. They are required all be 0 or 1. While there are certainly counterexamples where these Betti numbers are bigger than one, such examples are combinatorially quite complicated and, fortunately, the Z -manifold does not fall into this class.

The upshot of all this is that we can represent entries in $E_1^{p,q}$ simply by Laurent monomials (rather than collections of monomials associated to intersections of patches). The combinatorics of counting such monomials has been discussed in [21, 24]. What is even nicer is that it easily follows that the horizontal maps in the $E_1^{p,q}$ stage of the spectral sequence are given by the obvious maps between monomials inherited from the original complex (49). An exercise in Macaulay 2 programming then yields:

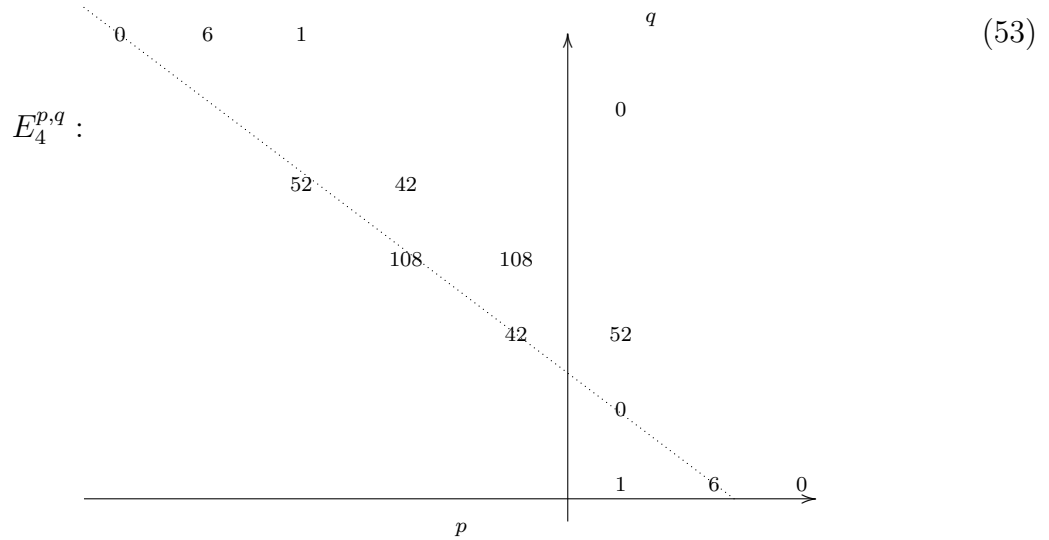


We now need to compute the d_2 maps shown in the above diagram. These are computed following the staircases of maps as described in [26]. We need to go right-down-right. An example of this was described completely explicitly in [9] and so we will be brief here. The composition of two right maps obviously gives zero in the underlying complex (49). However, the downwards map serves to scramble some of the signs in the map because of

the sign choices in Čech cohomology. Thus, the d_2 maps need not be zero. Fortunately most of the entries in the matrices representing d_2 can be shown to be zero because of the large number of zeroes in the horizontal maps. The result is:



Mercilessly, the spectral sequence is not done with us yet and we need to compute one d_3 map. This can be tackled using the same method as we used for d_2 . The result is:



Finally the spectral sequence degenerates and so

$$h^1(\text{End}(T)) = 208. \tag{54}$$

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