Optimal Capital Requirements in Financial Networks with Fire Sales

by

Jong Soo Hong

Business Administration
Duke University

Date: ______________________

Approved:

____
David Brown, Supervisor

____
Aleksandar Pekec

____
Peng Sun

____
S. Viswanathan

Dissertation submitted in partial fulfillment of the requirements for the degree of
Doctor of Philosophy in the Department of Business Administration
in the Graduate School of Duke University
2016

Abstract

Optimal Capital Requirements in Financial Networks with Fire Sales

by

Jong Soo Hong

Department of Business Administration
Duke University

Date: ______________________

Approved:

____
David Brown, Supervisor

____
Aleksandar Pekec

____
Peng Sun

____
S. Viswanathan

An abstract of a dissertation submitted in partial fulfillment of the requirements for the degree of Doctor of Philosophy in the Department of Business Administration in the Graduate School of Duke University
2016

Abstract

I explore and analyze a problem of finding the socially optimal capital requirements for financial institutions considering two distinct channels of contagion: direct exposures among the institutions, as represented by a network and fire sales externalities, which reflect the negative price impact of massive liquidation of assets. These two channels amplify shocks from individual financial institutions to the financial system as a whole and thus increase the risk of joint defaults amongst the interconnected financial institutions; this is often referred to as systemic risk. In the model, there is a trade-off between reducing systemic risk and raising the capital requirements of the financial institutions. The policymaker considers this trade-off and determines the optimal capital requirements for individual financial institutions. I provide a method for finding and analyzing the optimal capital requirements that can be applied to arbitrary network structures and arbitrary distributions of investment returns.

In particular, I first consider a network model consisting only of direct exposures and show that the optimal capital requirements can be found by solving a stochastic linear programming problem. I then extend the analysis to financial networks with default costs and show the optimal capital requirements can be found by solving a stochastic mixed integer programming problem. The computational complexity of this problem poses a challenge, and I develop an iterative algorithm that can be efficiently executed. I show that the iterative algorithm leads to solutions that are nearly optimal by comparing it with lower bounds based on a dual approach. I also

show that the iterative algorithm converges to the optimal solution.

Finally, I incorporate fire sales externalities into the model. In particular, I am able to extend the analysis of systemic risk and the optimal capital requirements with a single illiquid asset to a model with multiple illiquid assets. The model with multiple illiquid assets incorporates liquidation rules used by the banks. I provide an optimization formulation whose solution provides the equilibrium payments for a given liquidation rule. I further show that the socially optimal capital problem using the "socially optimal liquidation" and prioritized liquidation rules can be formulated as a convex and convex mixed integer problem, respectively. Finally, I illustrate the results of the methodology on numerical examples and discuss some implications for capital regulation policy and stress testing.

To my family

Contents

List of figures

List of symbols

Symbols

N	total number of banks
M	total number of illiquid assets
K	size of sample space of returns
L	liability matrix
G	adjacency matrix
π	relative liability matrix
$\bar{p}_i$	total liability of bank i
$\bar{p}_i^I$	total interbank liability of bank i
$\bar{p}_i^O$	total non-bank liability of bank i
$\bar{y}$	initial value of outside assets
c_i	capital of bank i
p_i	realized debt payment of bank i
y_i	realized value of outside asset of bank i
a_i	realized total asset value of bank i
l_i	loss of asset value relative to total debts of bank i
p_i^*	equilibrium debt payment of bank i
a_i^*	equilibrium asset value of bank i
l_i^*	loss of equilibrium asset value relative to total debts of bank i
c^*	optimal capital

ϕ^{EN}	operator for Eisenberg and Noe (2001) model
ϕ^{RV}	operator for Rogers and Veraart (2013) model
ϕ^{RV2}	operator for cross default model
α	probability level of CVaR_α
γ	targeted value of systemic risk in the constrained form of the socially optimal capital requirements problem
λ	systemic risk tolerance parameter in the unconstrained form of the socially optimal capital requirements problem
β	default costs parameter
$\mathbb{Q}^*$	optimal probability measure
$\mathbb{Q}^{EN}$	Eisenberg and Noe probability measure
ϕ^{FS}	operator for fire sales model
$f_m(x)$	inverse demand function on asset m
θ	price impact parameter in the inverse demand function
$Q_m(z)$	equilibrium price function of asset m on liquidation amount z under linear $f_m(x)$
τ	targeted value ratio of debt to equity in the leverage targeting rule
$s_{im}(p, q)$	liquidation function of bank i on asset m
$t_i(p, q)$	total liquidation amount bank i for leverage targeting

Acknowledgements

I am sincerely thankful to my advisor Professor David Brown for his endless support and invaluable advice throughout my PhD studies at Duke University. Without his support, I could not be at the point of writing this acknowledgement for my dissertation.

Looking back, I have experienced a number of occasions that could have forced me to give up my PhD studies. In particular, the final one-and-a-half years of my period at Fuqua were one of the most challenging times in my life. Despite such difficulties, Professor Brown has supported me until the very end.

I started my project on systemic risk and financial network during the third year of my PhD studies. If Professor Brown had not guided me as his student, I could not have pursued my research project. I am deeply indebted to him for sharing his wisdom, knowledge and precious time. In particular, I learned from him the importance of intuition in academic research. His intuition was always right and guided me towards a constructive way of conducting research.

I also would like to express my gratitude to Professors Sasa Pekec, Peng Sung and S. "Vish" Viswanathan for becoming the members of the committee for my defense, especially despite a very late decision of my defense date. Additionally, I would like to thank Professor Jim Bettman, the formal Director of Graduate Studies, and Ms. Bobbie D. Clinkscales, the PhD Program Assistant, for providing me with various supports and care.

I would like to thank my family for their endless support and encouragement towards me, without which, I would not be who I am today. My parents have prayed for me and given me all kinds of support throughout my PhD years. I also would like to thank my wife Bona and my 21-month-old son Jehyun. Especially in my final year at Fuqua, Bona took care of Jehyun alone in a distant state because of her own study there, for which I am very grateful. Also, I would like to thank Jehyun because he is always a source of my happiness.

Lastly, I would like to express my thankfulness to all others that I do not list here who have helped me and supported me during my years of study. If someone reads this writing, I would like to convey one message to that person - please do not give up your hope; if you have a goal that you really want to achieve, keep challenging yourself until you achieve it, no matter how you feel frustrated in the process due to your lack of ability or your limitations. I would like to tell them that there are some people in the world who initially might not be perfect candidates in reaching their goals but who nevertheless persevere and keep going - just like me.

1

Introduction

1.1 Introduction

One of the most important lessons from the financial crisis of 2007-2009 is that the failure of individual financial institutions can significantly increase the risk of collapse of the entire financial system. This is often called systemic risk. There are two distinct channels of propagation of shocks from individual institutions to the financial system as a whole. As stressed by Yellen (2013) and Caruana (2015), the interconnectedness of financial institutions is one of the critical channels. Another critical channel is through fire sales, where asset values may be significantly decreased due to massive liquidation. In response, policymakers and regulators such as the Federal Reserve and the Basel Committee have been seeking new regulatory frameworks to manage possible spillover effects due to the interconnectedness of financial institutions as well as fire sales.

Enhancing capital requirements to account for those phenomena is one of the major issues of recent regulatory frameworks. Since 2011, the Fed has annually assessed the capital of 19 large bank holding companies to ensure the adequacy of certain levels of capital (Bernanke 2013 and The Board of Governors of the Federal Reserve System 2016). The Basel Committee is working on a new regulatory framework, called Basel III, which requires higher capital requirements for "globally systemic important banks (GSIBs)". However, as mentioned by Glasserman and Young (2015a) and Haldane (2009), the interconnectedness of financial institutions is the least well understood factor among the many factors that contributed to the financial crisis. Hence, there is still a need for methods to assess systemic risk considering interconnectedness and fire sales and to determine banks' capital requirements based on their impact on systemic risk.

In this dissertation, I develop and analyze a model of finding socially optimal capital requirements for interconnected financial institutions with fire sales externalities. In this model, a policymaker faces a trade-off between reducing systemic risk, which is defined as the risk of joint defaults of interconnected financial institutions, and raising capital requirements of the financial institutions. The policymaker considers this trade-off when determining optimal capital requirements of the institutions. I provide a framework for finding and analyzing the optimal capital requirements given an arbitrary network structure and distributions of investment returns.

The basic structure of the model can be described as follows. There are multiple financial institutions (henceforth banks) and an outside investor representing depositors and non-bank creditors in an economy with two dates. On the initial date, each bank has an investment opportunity and finances the investment with equity capital, interbank loans, and loans from the outside investor. The interbank debts can be represented as a network, where a node represents a bank and a directed arc represents the amount of debt which one bank should pay to another bank. I assume the network structure is given. At the final date, the returns on investments are realized and banks pay their debts with the realized investment and debt payments from other banks. If a bank's total asset value is below its total debt, the bank defaults. Due to the cross exposures between banks, the failure of one bank can lead to a cascade of failure of other banks.

Given the financial network, a policymaker is interested in finding the capital requirements to reduce the overall risk of defaults in the economy. Since I focus on the policymaker's perspective, I assume banks do not hold additional capital beyond the regulatory capital requirements. Higher capital requirements can decrease the likelihood of defaults by banks, but there are potential costs associated with higher capital requirements, such as a credit crunch. Considering this trade-off, the policymaker is assumed to seek capital requirements which minimize both the aggregate capital and systemic risk. Similar to a mean-variance framework for portfolio selection problems, there is a systemic risk tolerance parameter in the policymaker's objective function, which captures the policymaker's preference between aggregate capital and systemic risk.

In this setting, the combined effects of the network structure of interbank liabilities and the dependence structure among investment returns make the analysis of the optimal capital requirements complex. I first show that the optimal capital requirements in a model with interconnected banks without fire sales can be found by solving a stochastic linear programming problem if there are no other fictions in the model.

Within the basic financial network model, I consider two distinctive frictions that are critical in enhancing contagion risk of the financial network. One is default costs and the other is fire sales externalities. In practice, when a bank defaults, there is a reduction of asset value due to additional costs, such as legal and administrative costs. I follow the model of Rogers and Veraart (2013) where there is a discrete change in asset value when a bank defaults. In this framework, I show that the optimal capital requirements can be found by solving a stochastic mixed integer programming problem. This problem is challenging to solve, mainly because the formulation involves "big-M" constraints that represent whether a bank defaults or not in a realized scenario. As a result, the formulation has weak LP relaxations.

Due to the computational difficulty of solving the optimal capital requirements problem with default costs, I develop an iterative algorithm that finds an approximate solution and lower bounds that evaluate the performance of the approximate solution. The lower bound is obtained by using the representation theorem of coherent risk measures. According to this representation theorem, the coherent risk measure can be expressed as the worst case expectation over a family of probability measures. Moreover, I develop an iterative algorithm that improves the lower bound and eventually converges to optimal solution.

As mentioned earlier, fire sales externalities are also an important channel for the propagation of shocks to the assets of banks. Here, a fire sales externality is defined as the negative price impact associated with the liquidation of assets of banks on the market. When a bank sells some of its assets, this may reduce prices because of insufficient demand for the liquidated assets. This results in distress of other banks that hold the same assets. Moreover, the affected banks might sell other assets for various reasons such as leverage targeting or risk management. This liquidation spiral can amplify shocks to the whole market.

In addition, these two channels are likely to affect each other. If a defaulted bank cannot honor its debt contract, the creditor banks as well as the defaulted bank might also liquidate their assets to honor their contracts and this liquidation can trigger distress of other banks. Thus, I study the combined effect of interconnected liabilities and fire sales by developing a model capturing both channels in the problem of socially optimal capital requirements.

When modeling fire sales externalities, I adopt the Cifuentes, Ferrucci, and Shin (2005) approach. In Cifuentes, Ferrucci, and Shin (2005), there is a single illiquid asset and the fire sales mechanism is captured by a price function of banks asset. I extend Cifuentes, Ferrucci, and Shin (2005) to a multiple illiquid asset model. I show that the equilibrium payments are a solution of an optimization problem with the assumption of certain liquidation rules. The optimization problems under socially optimal and prioritized liquidation rules can be formulated as a convex and convex mixed integer problem, respectively. I then provide an equivalent optimization formulation for computing the systemic risk and the optimal capital requirements.

I further investigate the implications of these methods by applying them to a data set collected by a central bank and as well as randomly generated network structures. In models with default costs, the numerical experiments show that the iterative algorithm quickly finds an approximate solution that is close to an optimal solution. In models with fire sales, by comparing the optimal capital requirements with and without fire sales, I observe that the effects of fire sales externalities on bank capital are different across individual banks. For example, with a fixed network structure, the changes in the optimal capital levels of individual banks due to the fire sales externalities are significantly different. This suggests that regulators should consider fire sales externalities as well as the interbank network structure when setting capital requirements.

2

Literature review

2.1 Literature review

The framework is related to the growing literature that studies financial networks and the implications for systemic risk. In particular, after the financial crisis of 2007-2009, there has been a considerable increase of papers on this topic (see, e.g., Yellen 2013 and Glasserman and Young 2015a). Moreover, as pointed out by Bernanke (2014), monitoring financial stability has become an important role of policymakers, especially the Federal Reserve, after the financial crisis. To address this, many methodologies for monitoring financial stability have been developed and implemented by policymakers and academic researchers (see, e.g., Adrian, Covitz and Liang 2013). The framework in this dissertation has several advantages over the existing approaches, as I now discuss.

2.1.1 Interconnectedness of direct exposures

As mentioned earlier, there are two sets of papers, each focusing on two distinct channels of propagating shocks. A first set of papers focuses on financial networks with direct exposures. Eisenberg and Noe (2001) provided a pioneering and practical framework that analyzes contagion through the direct debt payments among financial institutions. Subsequently, there have been several papers that generalized the Eisenberg and Noe (2001) framework. Rogers and Veraart (2013) show that financial institutions in the network have an incentive to merge when considering default costs. Glasserman and Young (2015b) provide a bound on the probability of contagion. Elliott, Golub and Jackson (2014) study the relationship between dependence on other banks and diversification considering the financial network with equity shares only. Acemoglu, Ozdaglar and Tahbaz-Salehi (2015) study the relationship between the structure of financial networks and the likelihood of contagion.

Cifuentes, Ferrucci and Shin (2005) and Shin (2008) are the first papers to incorporate price reduction of a single illiquid asset into the Eisenberg and Noe (2001) framework. Amini, Filipović and Minca (2015 and 2016) extend this work by formalizing the uniqueness and existence of equilibrium payments and also analyzing the effects of full versus partial multilateral netting. However, these papers do not incorporate fire sales of multiple illiquid assets into their models of financial networks.

Recently, Feinstein (2015) considered fire sales of multiple illiquid assets into a model of a financial network. In contrast to Feinstein (2015), I extend my analysis to finding the optimal capital requirements and computing systemic risk. The method of finding the optimal capital requirements derived in this desertion is the first work that considers a model of financial network with both direct exposures as well as fire sales of multiple assets

2.1.2 Fire sales

There is a set of recent papers that focus on fire sales of multiple illiquid assets and the resulting impact on systemic risk. Greenwood, Landier and Thesmar (2015) first provided a framework that quantifies the aggregate vulnerability of a financial system by fire sales of multiple illiquid assets using each bank's portfolio data. Extending this, Duarte and Eisenbach (2015) use more detailed balance sheet information to track the aggregate vulnerability over time. Capponi and Larsson (2015) consider higher order effects of fire sales in a nonbanking sector that does not target a leverage ratio. Chen, Iyengar and Moallemi (2014) analyze the interplay between the asset holding structure and asset price contagion. In a different model, Caballero and Simsek (2013) provide a model of fire sales to explain the cause of the financial crisis. In contrast to these papers, the framework here considers the combined effects of fire sales and interbank lending to analyze the optimal capital requirements and the resulting systemic risk at equilibrium.

Moreover, one of the main differences between the framework here and the papers mentioned above is that the framework here focuses on an ex ante analysis on systemic risk and capital requirements. All of the above papers analyze the contagion by taking an adverse shock to asset returns or a risk factor as given. In contrast, the capital requirements here are preemptive since a policy maker considers the risk of contagion for a given distribution of returns instead of a given adverse shock.

Finally, the model in this dissertation reflects the feedback and interaction among the financial institutions, which are not captured by many approaches and the current stress testing process (see, e.g., Bookstaber et al. 2014, Glasserman and Tangirala 2015 and Pritsker 2012). This allows one to monitor the financial stability more accurately.

2.1.3 Systemic risk

Even though it is challenging to measure systemic risk (Hansen 2013), there is a growing literature on systemic risk measures that attempt to quantify the risk of joint distress of financial institutions in the economy. Bisias, Flood, Lo and Valavanis (2012) conducted a survey of 31 systemic risk measures. Among them, Adrian and Brunnermeier (2016)'s CoVaR measure and Acharya, Pederson, Philippon and

Richardson (2010)'s systemic expected shortfall are the most well-known. Giesecke and Kim (2011) develop a systemic risk measure based on a statistical model for timing of bank defaulting. However, these measures are based on market data that does not explicitly consider propagation mechanisms behind adverse shocks in the economy. In contrast, the systemic risk measure used here captures relationships among asset values of banks through interbank lending and commonly held assets.

Chen, Iyengar and Moallemi (2013) provide an axiomatic approach on systemic risk and a framework for risk attribution among individual agents. A distinctive feature of the framework here relative to Chen, Iyengar and Moallemi (2013) is that I consider how changes in capital affect risk and, in particular, the optimal capital levels.

2.1.4 Dual approaches

In solving the model with default costs as in Rogers and Veraart (2013), I develop two types of dual approach to find lower bounds on the optimal capital requirements problem. First, I use the representation theorem of coherent risk measures and CVaR (Artzner et al. (1999)). According to this representation theorem, a coherent risk measure can be equivalently written as a worst-case expected value of a set of probability measures. Using this, I find a lower bound, which is obtained by a relaxation of the set of probability measures.

On the other hand, I provide another type of lower bound based on information relaxations and gradient penalty method in Brown and Smith (2014). I provide an approximate solution and show it is nearly optimal by comparing it with the lower bounds obtained by information relaxations and gradient penalty method.

3

Basic framework of optimal capital requirements in financial networks

3.1 Basic model

3.1.1 Financial network

There are N financial institutions, called banks, and an outside investor, which represents depositors and non-bank creditors. There exist two dates, $t = 0$ and 1, in the economy. Banks are indexed by i. The outside investor is denoted by o. I assume banks' balance sheets are given at date 0. In particular, bank i's balance sheet at date 0 is expressed as follows.

Assets	Liabilities
Outside assets: $\bar{y}_i$	Non-interbank debt: $\bar{p}_i^o$
Bank lending: $\sum_{k=1}^{N} \bar{p}_{ki}$	Bank borrowings: $\sum_{j=1}^{N} \bar{p}_{ij} = \bar{p}_i^I$
	Equity: c_i

In liabilities, bank i has capital c_i and debts to an outside investor $\bar{p}_i^o$. The outside debts are held by the non-banking sector. $\bar{p}_{ij}$ is the face value of the bank i's debt to bank j. Hence, $\bar{p}_i^I = \sum_j \bar{p}_{ij}$ is the face value of the total debt of bank i to other banks.

Let $\bar{p}_i$ denote the face value of the total debt of bank i, i.e., $\bar{p}_i = \bar{p}_i^o + \bar{p}_i^I$. Thus, the total liability is equal to $c_i + \bar{p}_i$. Within assets, $\bar{y}_i$ represents the value of assets that bank i can invest outside the banking system. Bank i holds capital c_i due to regulatory requirements or risk management purposes. The total asset is equal to $\bar{y}_i + \sum_k \bar{p}_{ki}$. Since the typical role of banks is to borrow from households to invest in the economy, I assume all banks are net borrowers, i.e., $\bar{p}_i > \sum_k \bar{p}_{ki}$, where $\sum_k \bar{p}_{ki}$ is the face value of the claim that bank i has against other banks in the economy.

Given the balance sheet of all banks, I follow the Eisenberg and Noe (2001) approach and describe the structure of interbank liabilities as a directed graph. Node i represents bank i. The directed edge from node i to node j represents bank i's liability to bank j. The capacity of edge $i - j$ is $\bar{p}_{ij}$, i.e., the face value of bank i's debt to bank j.

At date 1, the outside asset of each bank yields a random return. Given a realization of outside investments, there is a flow of cash from outside the network coming into each node. Each bank needs to pay debts to other banks based on the realized value of the outside investment. The debt payments of a bank are affected by the payments of other banks due to the interbank claims, as I now explain.

Formally, let y_i denote the realized value of the outside investment for a bank i. I denote a realized total debt payment by bank i by p_i, which will depend on y_i and the debt payments of other banks. Due to limited liability, p_i should be in $[0, \bar{p}_i]$. Assuming all banks and the outside investor are of equal seniority, the realized asset value of bank i is then equal to $y_i + \sum_k \pi_{ki} p_k$, where π_{ij} denotes proportion of bank i's debt to bank j to its total debt, i.e. $\pi_{ij} = ((\bar{p}_{ij})/\bar{p}_i)$. Due to the seniority of debts to equity, if $y_i + \sum_k \pi_{ki} p_k$ is larger than bank i's total debt $\bar{p}_i$, bank i pays its debts in full, i.e. $p_i = \bar{p}_i$. If not, bank i defaults and pays the full asset value, which is less than $\bar{p}_i$. Furthermore, as in Elsinger, Lehar and Summer (2006), one can consider seniorities among debts as well. I consider the seniority between the short term and long term debts in section 5.3.

This leads to the question concerning the existence of a payment vector $p = (p_1, ..., p_N)$ which simultaneously satisfies the above conditions. Eisenberg and Noe (2001) (hereafter EN) show the existence of the payment vector without default costs and analyze its properties.

3.1.2 Equilibrium payments

In this subsection, I discuss the equilibrium payments without default costs. Even though I later incorporate default costs and fire sales effects into the analysis of the optimal capital requirements, the EN model serves as an important foundation for the rest of the dissertation. Specifically, the results presented in this subsection are useful in the LP formulation of the optimal capital requirements problem and in the evaluation of approximate solutions that consider default costs and fire sales externalities.

Following EN, I define the equilibrium payment, sometimes called a clearing vector, as follows.

Definition 3.1.1. *If there are no default costs or fire sales externalities in the economy, the equilibrium payment $p^{*(EN)}$ for given the realized outside asset value y and liability matrix $\bar{p}$ is a fixed point of the following operator ϕ^{EN}:*

$$\phi^{EN}(p)_i = \begin{cases} \bar{p}_i, & \text{if } y_i + \sum_k \pi_{ki} p_k \geq \bar{p}_i \\ y_i + \sum_k \pi_{ki} p_k, & \text{otherwise} \end{cases} \tag{3.1}$$

According to the above definition, the equilibrium payment $p^{*(EN)}$ is a payment vector that mutually satisfies the conditions in (3.1). As I later show and discuss, the equilibrium payments with default costs or fire sales externalities will be a fixed point of other operators analogous to ϕ^{EN}.

There are two noteworthy properties of $p^{*(EN)}$. First, due to the assumption of no default costs, there is no additional loss of asset values for a defaulted bank. Thus, the total debt payment of a defaulted bank i is equal to the value of its total asset, i.e., $y_i + \sum_k \pi_{ki}\, p_k$. In addition, the asset value of a bank i, whether it is defaulted or not, is affected only by the interbank network channel and the value of outside investments. The effects of the network channel are captured by the value of interbank claims held by bank i, i.e., $\sum_k \pi_{ki}\, p_k$, which depends only on the payments of other banks p_k. In reality, bank i's asset may decrease further due to the negative price impact of liquidation of commonly held outside investments by other banks.

EN show the existence of $p^{*(EN)}$ and provide sufficient conditions for the uniqueness of $p^{*(EN)}$. Moreover, they provide an iterative algorithm that efficiently computes $p^{*(EN)}$. The following lemma summarizes the results of EN.

Lemma 3.1.1 (Eisenberg and Noe 2001). *The equilibrium payment $p^{*(EN)}$ exists and is unique if $y_i > 0$ for all i. Moreover, $p^{*(EN)}$ can be computed at most N steps of the iterative algorithm as follows.*

i) Set $i = 0$ and $p^i = \bar{p}_i$.

ii) Compute $p^{i+1} = \phi^{EN}(p^i)$.

iii) If $p^{i+1} = p^i$, then $p^{*(EN)} = p^i$. Otherwise set $i \to i+1$ and go back to step ii).

Note that in Lemma 3.1.1, I mention only one sufficient condition for uniqueness, i.e., $y_i > 0$ for all i. As I will describe shortly, y_i will be modeled as random; it is very unlikely that the realized asset value of a bank becomes zero. As long as the total returns are always above zero in every scenarios, there will be a unique equilibrium payment in every scenarios.

The iterative algorithm that EN provide is useful in evaluating payments for given vectors y and c (i.e., outside investments and capital) since it computes the equilibrium payment efficiently. However, it is difficult to use the iterative algorithm to find optimal capital requirements. In this sense, another lemma from EN is useful for the formulation of the optimal capital requirements problem.

Lemma 3.1.2 (Eisenberg and Noe 2001)**.** *If f is a strictly increasing function $f : [0, \bar{p}] \to \mathbb{R}$, then the payment equilibrium $p^{*(EN)}$ is a solution of the following optimization problem.*

$$\begin{aligned} \underset{p}{\textit{maximize}} \quad & f(p) \\ & p_i \leq \bar{p}_i \\ & p_i \leq y_i + \sum_k \pi_{ki}\, p_k \end{aligned} \tag{3.2}$$

Lemma 3.1.2 provides another method to find the equilibrium payment $p^{*(EN)}$. If I take $f(p) = \sum_i p_i$, which I will later use to measure systemic risk, $p^{*(EN)}$ is an optimal solution of a linear programming problem. I will use this linear programming problem shortly to formulate the optimal capital requirements problem. In the following section, I will formulate the optimal capital requirements problem in detail.

3.1.3 Realized asset value with stochastic return

In reality, the value of outside investments y is random at time 1. To capture this uncertainty, I assume that the returns of the outside assets are stochastic, and denoted by $R_\omega := (R_{\omega 1}, \ldots, R_{\omega N})$. $R_{\omega i}$ represents the gross return on the outside investment of bank i in scenario ω. Moreover, I assume $R_{\omega i}\bar{y}_i > 0$ for all ω and

i almost surely. Let $p^{*(EN)}_{\omega i}$ denote the equilibrium payment of bank i in scenario ω. With $R_{\omega i}$ and $p^{*(EN)}_{\omega i}$, I can write the equilibrium value of bank i' total asset in scenario ω as $a^{*(EN)}_{\omega i} = R_{\omega i}\bar{y}_i + \sum_k \pi_{ki}\, p^{*(EN)}_{\omega k}$.

Due to the balance sheet identity, the value of asset and liability should be equal, i.e., $\bar{y}_i + \sum_k \bar{p}_{ki} = c_i + \bar{p}_i$. This leads to the following expression for the realized asset value:

$$a^{*(EN)}_{\omega i} = R_{\omega i}(c_i + \bar{p}_i - \sum_k \bar{p}_{ki}) + \sum_k \pi_{ki}\, p^{*(EN)}_{\omega k} \tag{3.3}$$

Note that the asset value $a^{*(EN)}_{\omega i}$ depends on initial capital c_i and increases in c_i. This corresponds to the role of bank capital as a loss absorbing layer that protects creditors, e.g., household depositors and other banks in the network. Since a default occurs when $a^{*(EN)}_{\omega i} < \bar{p}_i$, holding more capital reduces the likelihood of defaults in the economy. Moreover, for a given defaulted bank i, its total debt payments without default costs $p^{*(EN)}$ is equal to its total asset value $a^{*(EN)}_{\omega i}$. Thus, the severity of default, which can be measured by shortfall of the payment, i.e. $\bar{p}_i - p^{*(EN)}_i$, is reduced by reserving more bank capital.

The results of the optimal capital requirements problem do not require any assumptions on the return distributions. Hence, the framework can capture correlations between asset returns, which can also be an important source of contagion (Elsinger, Lehar and Summer 2006 and Allen, Babus and Carletti 2012). I will explain how the return distribution is incorporated into the framework in detail in the next section.

3.2 Systemic risk measure

The basic goal of the various systemic risk measures proposed in the literature is to measure the risk of the joint distress of multiple banks caused by financial contagion. Following this motivation, in this dissertation, I take the expected shortfall on aggregate losses as a measure of systemic risk. I first introduce a function $\rho(X)$, which maps a random vector X to a real value, i.e., $\rho : X \to \mathbb{R}$. In the framework here, X is a random vector $a^*(c) \in \mathbb{R}_+^N$ representing the equilibrium asset values of all banks as a function of the capital levels $c \in \mathbb{R}^N$. With this, $\rho(a^*(c))$ represents systemic risk, which measures the distress of the financial system as a whole at payment equilibrium. $\rho(a^*(c))$ potentially captures the joint distress of all banks in the economy.

In this dissertation, I will choose ρ with the following form:

$$\begin{aligned} \rho(a^*(c)) &= \mathrm{CVaR}_\alpha\left(-\sum_i (\bar{p}_i - a_i^*(c))^+\right) \\ &= \mathrm{CVaR}_\alpha\left(-\sum_i l_i^*(c)\right) \end{aligned} \tag{3.4}$$

where CVaR denotes the conditional value-at-risk (or expected shortfall). CVaR is a well-known coherent risk measure (Artzner et al. 1999) and is defined for random variables Y as $\mathrm{CVaR}_\alpha(Y) = -\mathbb{E}[\,Y|Y \leq -\mathrm{VaR}_\alpha(Y)\,]$ and VaR is the so-called value-at-risk, defined as the $\alpha\%$ quantile, i.e., $Pr(Y < -\mathrm{VaR}_\alpha(Y)) = \alpha$. It should be noted that I will use $a^*(c)$ to represent not only the equilibrium payments in the EN model but also in other models that I will discuss in later chapters.

The systemic risk measure (3.4) is defined as the expected value of aggregate losses of banks over the worst $\alpha\%$ scenarios. This measure is closely related to the other proposed systemic risk measures such as the "systemic expected shortfall (SES)" measure proposed by Acharya, Pedersen, Philippon, Richardson (2010) and the axiomatic approach of Chen, Iyengar and Moallemi (2013). Acharya, Pedersen, Philippon, Richardson (2010) model a crisis in which the aggregate bank capital falls below a fraction of the aggregate assets and estimate SES by assuming that the crisis corresponds to the worst 5% of market output. The systemic risk measure (3.4) is closely related with the approach of Chen, Iyengar and Moallemi (2013) by considering the equilibrium asset value for a given interbank liability structure.

Note that the systemic risk measure (3.4) considers only loss of a defaulted bank at the payment equilibrium, i.e., the loss of a bank's equilibrium asset value, $a^*_{\omega i}(c)$, relative to its total debts, $\bar{p}_i$. Generally, regulators cannot use the profits of some banks to subsidize the losses of other banks. Thus, it is natural to assume that regulators focus on losses. I consider losses relative to total debts since bank defaults directly propagate adverse shocks to the banking system as a whole.

There are several advantages to the systemic risk measure (3.4). First, I can easily incorporate other realistic features by slightly modifying (3.4). For example, one can consider other distressed situations by looking at losses compared to a threshold such as $(1+\eta)\,\bar{p}_i$, where $\eta \geq 0$. Also, risk measures other than CVaR can be used to measure systemic risk. Moreover, unlike Greenwood, Landier and Thesmar (2015), I consider losses associated with the equilibrium asset values.

3.2.1 Optimization formulation for computing systemic risk

I use the following equivalent definition of CVaR :

$$\mathrm{CVaR}_\alpha(Y) \ = \min_{\nu\in\mathbb{R}} \left\{\nu + \tfrac{1}{\alpha}\mathbb{E}(-\nu - Y)^+\right\} \tag{3.5}$$

It is well known (e.g. Rockafeller and Uryaserv 2000 and Pflug 2000) that (A.1) is equal to $-\mathbb{E}[\,Y|Y \leq -\mathrm{VaR}_\alpha(Y)\,]$ for continuous distributions. The aggregate equilibrium loss $\sum_i (\bar{p}_i - a^*_{\omega i}(c))^+$ could have a discontinuous distribution due to fire sales or default costs. However, one can nonetheless roughly interpret (A.1) as the expected loss over the α% worst scenarios.

One of advantages of the equivalent definition (A.1) is that it is in a form that is convenient to evaluate through simulation since it is expressed as an unconditional expected value. Using this property, I will solve the optimal capital problem with IID samples of R_ω. I will assume from now on that one has the K samples of R_ω, denoted by $R_1, ..., R_K$, where $R_\omega \in \mathbb{R}^N$ denotes the returns for all banks in scenario $\omega \in \{1, ...K\}$.

Proposition 3.2.1. *The systemic risk* $\mathrm{CVaR}_\alpha \left(-\sum_i l^*_{\omega i}(c) \right)$ *is the optimal value of the following LP:*

$$
\begin{array}{ll}
\underset{p,u,\nu}{\textit{minimize}} & \nu + \frac{1}{\alpha K} \sum_\omega u_\omega \\
\textit{subject to} & u_\omega \geq -\nu + \sum_i (\bar{p}_i - p_{\omega i}) \\
& p_{\omega i} \leq \bar{p}_i \\
& p_{\omega i} \leq R_{\omega i}(c_i + \bar{p}_i - \sum_k \bar{p}_{ki}) + \sum_k \pi_{ki} p_{\omega k} \\
& p, c, u \geq 0
\end{array}
\tag{3.6}
$$

Note that the term $\nu + \frac{1}{\alpha K} \sum_\omega u_\omega$ in the objective function and the constraint $u_\omega \geq -\nu + \sum_i (\bar{p}_i - p_{\omega i})$ and $u \geq 0$ come from the equivalent formulation of CVaR in (A.1) and the assumption of IID samples of R_ω. There are K samples of R_ω so I evaluate the expected value in (A.1) with $\frac{1}{K} \sum_\omega u_\omega$, where $u_\omega = \left(-\nu + \sum_i (\bar{p}_i - p^*_{\omega i}(c)) \right)^+$.

Based on Lemma 3.1.2, the two constraints on $p_{\omega i}$ reflect the fact that payments should be smaller than both the total liability and the realized asset values. Any vector p that simply satisfies these two constraints is not necessarily the equilibrium payment: the payment vector needs to be maximal as well. The key property which leads to p_ω being the equilibrium payments is that the Eisenberg and Noe payment vector is a fixed point of $\phi^{EN}(p_\omega)$ for given capital levels and a given return scenario.

3.3 Socially optimal capital requirements

As discussed in the previous section, the loss of bank i's asset value can reduce the asset values of other banks through the network of interbank liabilities. In this sense, bank capital serves as an important tool in maintaining not only the solvency of an individual bank but also the soundness of financial system as a whole. Thus, I study capital requirements that minimize a weighted combination of aggregate capital while maintaining a desired level of systemic risk. I call this the socially optimal capital requirements and define it for a general systemic risk measure.

Given the interbank network model in section 3.1.1, I consider a policymaker who is interested in maximizing social welfare that depends on aggregate level of capital and the joint distress of multiple banks. The policymaker knows the interbank network and the return distribution of outside investments. Based on this information, the regulatory capital requirements of each bank is set by the policymaker. Since I restrict attention to the policymaker's perspective, I assume that banks do not hold additional capital in excess of regulatory capital requirements.

I define the socially optimal capital requirements as follows.

Definition 3.3.1 (Socially Optimal Capital Requirements)**.** *For a given interbank network and return distribution, the socially optimal capital requirements of banks, denoted by* c^**, is an optimal solution of the following problem:*

$$\begin{array}{ll} \underset{c\geq 0}{\textit{minimize}} & \sum\limits_i c_i \\ \textit{subject to} & \rho(a^*_\omega(c)) \leq \gamma \end{array} \tag{3.7}$$

The objective function in (3.7), $\sum\limits_i c_i$, represents the aggregate level of capital. The important feature of the optimal capital requirements problem (3.7) is that the policymaker wants to minimize the aggregate capital level while targeting the systemic risk. Higher capital requirements can decrease systemic risk. However, there can be potential costs associated with higher capital requirements such as a credit crunch (Bolton and Freixas 2006). In general, the systemic risk measure $\rho(a^*_\omega(c))$ decreases with the capital c since systemic risk is decreasing in the asset value and the equilibrium asset value $a^*_\omega(c)$ increases with the capital c. Thus, by minimizing the aggregate capital and controlling systemic risk, a regulator balances a trade-off between raising capital requirements and mitigating the risk of a financial system collapse.

The regulator chooses a target value γ for the systemic risk. I refer to γ as the systemic risk tolerance parameter in the constrained form, which is similar to risk tolerance parameter in mean-variance framework for portfolio choice problems.

With the systemic risk measure defined in (3.4), we can write (3.7) as

$$\begin{aligned} \underset{c \geq 0}{\text{minimize}} \quad & \sum_i c_i \\ \text{subject to} \quad & \mathrm{CVaR}_\alpha \left(-\sum_i (\bar{p}_i - a^*_{\omega i}(c))^+ \right) \leq \gamma. \end{aligned} \tag{3.8}$$

I also use the properties of Eisenberg and Noe payments in Lemma 3.1.2, which shows that the payment vector without any frictions is a solution of a linear programming problem. Using these properties, I show that the formulation of the optimal capital problem (3.4) without fire sales is a linear programming problem.

Proposition 3.3.1. *If there are no fire sales externalities, the socially optimal capital requirements c^{EN} can be found by solving the following linear programming problem and the corresponding optimal solution p^{EN}_ω are the equilibrium payments (i.e., Eisenberg and Noe) in each scenario ω in given c^{EN}:*

$$\begin{aligned} \underset{c,p,u,\nu}{\textit{minimize}} \quad & \sum_i c_i \\ \textit{subject to} \quad & \nu + \frac{1}{\alpha K} \sum_\omega u_\omega \leq \gamma \\ & u_\omega \geq -\nu + \sum_i (\bar{p}_i - p_{\omega i}) \\ & p_{\omega i} \leq \bar{p}_i \\ & p_{\omega i} \leq R_{\omega i}(c_i + \bar{p}_i - \sum_k \bar{p}_{ki}) + \sum_k \pi_{ki} p_{\omega k} \\ & p, c, u \geq 0. \end{aligned} \tag{3.9}$$

3.3.1 Unconstrained form

One can also consider the socially optimal capital requirements problem where a policy maker minimizes both the aggregate capital and systemic risk as follows:

$$\underset{c\geq 0}{\text{minimize}} \quad \sum_i c_i + \lambda \, \text{CVaR}_\alpha \left(-\sum_i (\bar{p}_i - a^*_{\omega i}(c))^+ \right). \tag{3.10}$$

I refer to this problem as an unconstrained capital requirements problem and λ as the systemic risk tolerance parameter in this unconstrained form. Following similar reasoning as in the previous section, I show that an unconstrained problem (3.10) has an equivalent LP formulation. The following proposition formalizes this.

Proposition 3.3.2. *(Unconstrained form). If there are no fire sales externalities, the socially optimal capital requirements c^{EN} in (3.10) can be found by solving the following linear programming problem and the corresponding optimal solution p^{EN}_ω are the equilibrium payments (i.e., Eisenberg and Noe) in each scenario ω in given c^{EN}:*

$$\begin{array}{ll} \underset{c,p,u,\nu}{\textit{minimize}} & \sum_i c_i + \lambda(\nu + \frac{1}{\alpha K}\sum_\omega l_\omega) \\ \textit{subject to} & u_\omega \geq -\nu + \sum_i (\bar{p}_i - p_{\omega i}) \\ & p_{\omega i} \leq \bar{p}_i \\ & p_{\omega i} \leq R_{\omega i}(c_i + \bar{p}_i - \sum_k \bar{p}_{ki}) + \sum_k \pi_{ki} p_{\omega k} \\ & p, c, u \geq 0. \end{array} \tag{3.11}$$

4

Financial networks with default costs

4.1 Financial network model with default costs

4.1.1 Equilibrium payment with default costs

To incorporate default costs into the model, I follow Rogers and Veraart (2013) (hereafter RV). The RV model is a framework that models the default costs as a discrete reduction of a bank's asset value as in Acemoglu, Ozdaglar, and Tahbaz-Salehi (2015) and Elliott, Golub, and Jackson (2014).

Following RV, I assume that only the $(1-\beta)\%$ of the asset value is retained when a defaulted bank is liquidated. Slightly abusing the notation of equilibrium payments p^*, the asset value of the defaulted bank is

$$(1-\beta)(y_i + \sum_k \pi_{ki} p_k^*). \tag{4.1}$$

The parameter β models the costs of a defaulted bank that includes senior commitments, such as taxes and wages, and administrative expenses, such as legal fees. Many empirical papers suggest that β is not negligible. James (1991) shows that the cost of default is about 10 percent of the total asset. In a more recent paper, Davydenko, Strebulaev, and Zhao (2012) estimate that β is 21.7 percent on average.

The equilibrium payments p^* are affected by default costs β as well. The additional loss of asset value of a defaulted bank further reduces the asset values of the creditor banks and their debt payments can decrease as well. It should be noted that the equilibrium payment p^* in (4.1) has yet to be defined. RV show the following.

Lemma 4.1.1 (Rogers and Veraart 2013)**.** *Under the assumption of the default costs model in (4.1), there exists a largest equilibrium payment, denoted by $p^{*(RV)}$ and $p^{*(RV)}$ as a fixed point of the following operator Φ^{RV}.*

$$\phi^{RV}(p)_i = \begin{cases} \bar{p}_i & \text{if } y_i + \sum_k \pi_{ki} p_k \geq \bar{p}_i, \\ (1-\beta)\left(y_i + \sum_k \pi_{ki} p_k^*\right) & \text{otherwise.} \end{cases} \tag{4.2}$$

As in the previous section of the Eisenberg and Noe (2001) framework, I denote by $p^{*(RV)}$ the equilibrium payment in the RV framework. As a fixed point of Φ^{RV}, $p^{*(RV)}$ is a payment vector that mutually satisfies the equations in (4.2). To find $p^{*(RV)}$, RV provide an iterative algorithm that is similar to the EN algorithm in Lemma 3.1.1 with the operator Φ^{RV} in place of Φ^{EN}. As in Lemma 3.1.2, one can compute the equilibrium payments for a given realized outside asset value y at most N steps. I refer to this algorithm as the RV iterative algorithm.

As in the analysis without frictions such as default costs and fire sales in the previous section, the RV iterative algorithm is not sufficient to find the optimal capital requirements. Thus, I provide an optimization formulation to find $p^{*(RV)}$, which can be used to compute the optimal capital requirements.

The following proposition shows how to find the equilibrium payments using a mixed integer programming problem.

Proposition 4.1.1. *With the default cost model in (4.1), an optimal solution p^{RV} of the following mixed integer programming problem is the largest equilibrium payment*

$p^{*(RV)}$ *for the financial network.*

$$
\begin{array}{ll}
\underset{p,x}{\textit{maximize}} & \sum_i p_i \\
\textit{subject to} & p_i \le \bar{p}_i \\
& p_i \le (1-\beta)\left(y_i + \sum_k \pi_{ki}\, p_i\right) + M_1(1-x_i) \\
& -M_2 x_i \le y_i + \sum_k \pi_{ki} p_i - \bar{p}_i \\
& M_2(1-x_i) \ge y_i + \sum_k \pi_{ki} p_i - \bar{p}_i \\
& x_i \in \{0,1\},
\end{array}
\tag{4.3}
$$

for sufficiently large M_1 *and* M_2.

The key idea of the above MIP formulation is using the "big-M" constraints to express the default conditions of banks. The last two constraints before $x_i \in \{0,1\}$ represent whether bank i defaults or not at equilibrium. At equilibrium, $y_i + \sum_k \pi_{ki}\, p_i - \bar{p}_i$ is negative if bank i defaults and positive if not. Thus, in order to find the optimal solution p^{RV} that is equivalent to $p^{*(RV)}$, I need to choose an M_2 such that $-M_2 \le y_i + \sum_k \pi_{ki}\, p_i - \bar{p}_i$ if bank i defaults and $M_2 \ge y_i + \sum_k \pi_{ki}\, p_i - \bar{p}_i$ if bank i does not default.

Fortunately, it is not difficult to find such an M_2 when solving (4.3) with the reasonable upper and lower bounds of the asset value of a bank by examining the return data. One can set $M_2 = \max\left(\bar{R}_i \bar{y}_i + \sum_k \bar{p}_{ki} - \bar{p}_i, \bar{p}_i\right)$ where $\bar{R}_i$ is the largest possible return of outside asset of bank i. The realized asset value do not become more than $\bar{R}_i$ times the initial outside asset value or less than zero with the realized returns. With this M_2, the integer variable x_i represents bank i's default status. At an optimal solution, x_i will be one if $y_i + \sum_k \pi_{ki}\, p_i - \bar{p}_i$ is negative and zero if not.

Similarly, M_1 is a large number that makes $(1-\beta)\left(y_i + \sum_k \pi_{ki} p_i\right) + M_1$ greater than $\bar{p}_i$ if bank i is solvent at equilibrium. For example, $M_1 = \bar{p}_i$ is sufficient.

Proposition 4.1.1 provides an optimization formulation for finding the payment equilibrium $p^{*(RV)}$. The main idea is introducing binary variables with appropriate constraints. With (4.3), I follow the same reasoning as Proposition 3.3.1 to formulate the optimal capital problem as a mixed-integer programming problem.

4.2 Socially optimal capital requirements with default costs

Using the results of Proposition 4.1.1, I now show the optimization formulation of finding the socially optimal capital requirements considering both interbank liabilities and default costs. Moreover, I provide an iterative algorithm to solve the optimization problem. In this chapter, I consider the unconstrained capital requirements problem (3.10). If one uses the expectation of the aggregate losses to measure systemic risk instead of CVaR, the iterative algorithm that I provide is not helpful to find the optimal capital requirements. In this case, I develop another method based on information relaxations and describe this in Section 4.5.

Proposition 4.2.1. *Under the default costs model in (4.1), an optimal solution c^{RV} of the following mixed integer linear programming problem is optimal to the socially optimal capital requirements problem and a corresponding optimal solution p_ω^{RV} is the equilibrium payment for a given R_ω and c^{RV}.*

$$\begin{array}{ll} \underset{c,p,x,u,\nu}{\textit{minimize}} & \sum_i c_i + \lambda(\nu + \frac{1}{\alpha K}\sum\limits_\omega u_\omega) \\ \textit{subject to} & u_\omega \geq -\nu + \sum\limits_i (\bar{p}_i - p_{\omega i}) \\ & c, u_\omega \geq 0 \\ & (p_\omega, x_\omega) \in \Gamma_\omega(c), \end{array} \tag{4.4}$$

where $\Gamma_\omega(c)$ is the set of feasible (p_ω, x_ω) in MIP (4.3) for a given vector of returns R_ω and capital levels c.

As in Proposition 3.3.1, $\nu + \frac{1}{\alpha K}\sum\limits_\omega u_\omega$ in the objective function and the constraints

$u_\omega \geq -\nu + \sum_i (\bar{p}_i - p_{\omega i})$ and $u_\omega \geq 0$ comes from the equivalent formulation of CVaR in (A.1) and the assumption of i.i.d samples of R_ω. In addition, the constraint $(p_\omega, x_\omega) \in \Gamma_\omega(c)$ reflect the fact that the equilibrium payments $p^{*(RV)}$ can be obtained by solving (4.3) and the systemic risk measure on losses are minimizing in the unconstrained capital problem (3.10).

Even though I can formulate the optimal capital requirements problem as a mixed-integer programming problem, (4.4) has "big-M" constraints. In general, mixed-integer programming problems with such constraints are notoriously difficult to solve. Due to these coefficients, the linear programming (LP) relaxations are typically weak. Moreover, there are $N+K+2\times N\times K+1$ continuous variables, $N\times K$ binary variables, $N+2K+7N\times K+1$ linear constraints and $2\times N\times K$ big-M constraints. To address the difficulty of solving this MIP, I develop an approximation algorithm that efficiently finds good approximate solutions and converges to the optimal solution.

4.3 Dual bounds and iterative algorithm

I now describe an algorithm that provides approximate solutions to (4.4) and lower bounds. I first obtain the upper and lower bounds using a well-known representation theorem (Artzner et al. 1999) for coherent risk measures and c^{EN}, the optimal capital requirements without default costs. In particular, I use this representation result for CVaR since this dissertation focuses on the systemic risk measure based on CVaR.

4.3.1 Representation theorem for coherent risk measure

Artzner et al. (1999) define coherent risk measures and prove a representation theorem that shows that any coherent risk measures can be represented as the worst-case expected value over a family of "generalized scenarios". For detailed explanation of coherent risk measure and the representation theorem, I refer the reader to Artzner et al. (1999), Rockafellar and Uryasev (2000) and Bertsimas and Brown (2009).

I introduce some notation to explain the representation theorem for CVaR. Consider a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and random variable $Y : \Omega \to \mathbb{R}$. I assume Y is a discrete random variable with support of cardinality K, i.e., $Y = Y_i$ with probability $\mathbb{P}_i$ for $i = 1, ..., K$.

With above notation, I present the representation theorem for CVaR as follows.

Theorem 4.3.1 (Representation Theorem for CVaR). *For* CVaR *defined by (A.1) and a discrete random variable Y,*

$$\text{CVaR}(Y) = \sup_{\mathbb{Q} \in \mathcal{Q}_\alpha} \mathbb{E}_{\mathbb{Q}}[-Y] \tag{4.5}$$

where $\mathcal{Q}_\alpha = \{\mathbb{Q} \in \Delta^K | \mathbb{Q}_i \leq \mathbb{P}_i/\alpha, i = 1, ..., K\}$ and $\mathbb{E}_{\mathbb{Q}}[Y]$ and Δ^K denote the expectation of the random variable Y under the measure $\mathbb{Q}$ and the probability simplex in K dimensions, respectively.

The representation theorem shows that CVaR can be represented as the worst-case expected value over a family of "generalized scenarios". In particular, under the assumption of a uniform distribution of a random variable Y with K scenarios, i.e., $\mathbb{P}_i = \frac{1}{K}$ for all i, and αK is an integer, I have

$$\text{CVaR}(Y) = -\tfrac{1}{\alpha K} \sum_{i=1}^{\alpha K} Y_{(i)}, \tag{4.6}$$

where $Y_{(i)}$ are the increasing order statistics of Y_i, i.e., $Y_{(1)} \leq Y_{(2)} \leq ... \leq Y_{(\alpha K)}$. Hence, one can easily compute CVaR once one knows the αK smallest values of Y.

Moreover, since I use IID samples of outside asset value returns to find the optimal capital requirements, the assumptions on the uniform distribution and αK being an integer are not restrictive in solving (4.4).

4.3.2 Lower bounds using the representation theorem

Note that the representation theorem 4.3.1 asserts that for a fixed $\mathbb{Q} \in \mathscr{Q}_\alpha$, $-\mathbb{E}_{\mathbb{Q}}[Y]$ is a lower bound on CVaR(Y). If one picks a $\mathbb{Q}$ such that $\mathbb{Q}_i = 1/\alpha K$ at the smallest αK scenarios of Y_i and $\mathbb{Q}_i = 0$ elsewhere, the lower bound equals CVaR(Y). I will use this property to obtain a tight lower bound on (4.4).

Before describing the approach, I introduce some notation. In particular, let V denote the optimal value of (4.4) with default costs, that is

$$V = \min_{c \geq 0} \left\{ \sum_i c_i + \lambda \, \mathrm{CVaR}_\alpha \left(-l_\omega^{*(RV)}(c) \right) \right\} \tag{4.7}$$

where $l_\omega^{*(RV)}(c) = \sum_i (\bar{p} - a_{\omega i}^{*(RV)}(c))^+$ and $a_\omega^{*(RV)}(c)$ is the equilibrium payment with default costs given c in scenario ω.

Also, motivated by the representation theorem, for a given probability measure $\mathbb{Q}$ and c, I consider a relaxation of (4.7), defined as follows:

$$\widetilde{V}_{\mathbb{Q}} = \min_{c} \left\{ \sum_i c_i + \lambda \, \mathbb{E}_{\mathbb{Q}} \left[l_\omega^{*(RV)}(c) \right] \right\}. \tag{4.8}$$

$\widetilde{V}_{\mathbb{Q}}$ also can be found by solving a mixed programming problem similar to (4.4). By the representation theorem, for any $\mathbb{Q} \in \mathscr{Q}_\alpha$, I know CVaR$(-l_\omega^{*(RV)}(c)) \geq \mathbb{E}_{\mathbb{Q}}[l_\omega^{*(RV)}(c)]$. This implies that $\widetilde{V}_{\mathbb{Q}}$ is a lower bound on V. The following lemma formalizes this.

Lemma 4.3.1. *For any probability measure $\mathbb{Q} \in \mathscr{Q}_\alpha$,*

$$V \geq \widetilde{V}_{\mathbb{Q}}. \tag{4.9}$$

Moreover, if $V = \widetilde{V}_{\mathbb{Q}}$, then $\mathrm{CVaR}_\alpha \left(-l_\omega^{*(RV)}(c^{RV}) \right) = \mathbb{E}_{\mathbb{Q}}[l_\omega^{*(RV)}(c^{RV})]$.

Let $\mathbb{Q}^*$ denote a probability measure where $\text{CVaR}(-l_\omega^{*(RV)}(c^{RV}))$ is equal to $\mathbb{E}_{\mathbb{Q}^*}[l_\omega^{*(RV)}(c^{RV})]$ and c^{RV} is an optimal solution of (4.7). I call $\mathbb{Q}^*$ an optimal probability measure. It should be noticed that if the equality holds at (4.9), then $\mathbb{Q}$ is an optimal probability measure. This implies that if $\mathbb{Q}$ is not an optimal probability measure, then the equality does not hold at (4.9). Hence, finding such a $\mathbb{Q}$ is necessary to obtain a tight lower bound.

One of the advantages of formulation (4.8) is that I can solve for $\widetilde{V}_{\mathbb{Q}}$ with relative ease if $\mathbb{Q}$ has much smaller support than the original probability measure $\mathbb{P}$. Indeed, the support of the optimal probability measure $\mathbb{Q}^*$ is smaller than the original probability measure $\mathbb{P}$. By (4.6), $\mathbb{Q}_i^* = 1/\alpha K$ at the worst αK scenarios of $l_i^{*(RV)}(c^{RV})$ and $\mathbb{Q}_i^* = 0$ elsewhere. Thus, I can obtain a possibly tight lower bound $\widetilde{V}_{\mathbb{Q}^*}$ relatively easily compared to the original problem (4.7).

In general, one does not know $\mathbb{Q}^*$ since the optimal capital c^{RV} with default costs is unknown. Given this, it is natural to ask which $\mathbb{Q}$ should be used to get a good lower bound.

4.3.3 Measure using frictionless solution

Now I consider a method of finding a probability measure that gives a lower bound and that can be easily obtained. I use the optimal capital requirements without default costs c^{EN} to construct an initial $\mathbb{Q}$. In particular, I focus on a probability measure $\mathbb{Q}^{EN}$ for which the lower bound of $\text{CVaR}(-l_\omega^{*(EN)}(c^{EN}))$ is tight.

Definition 4.3.1. *For a given c^{EN}, $\mathbb{Q}^{EN}$ is a probability measure where*

$$\text{CVaR}(-l_\omega^{*(EN)}(c^{EN})) = \mathbb{E}_{\mathbb{Q}^{EN}}[l_\omega^{*(EN)}(c^{EN})]. \tag{4.10}$$

If $\mathbb{Q}^{EN}$ is not unique, I choose any measure that satisfies (4.10). Following (4.6), I know

$$\mathbb{Q}_i^{EN} = \begin{cases} \frac{1}{\alpha K} & \text{if } l_i^{*(EN)}(c^{EN}) \geq l_{(\alpha K)}^{*(EN)}(c^{EN}) \\ 0 & \text{otherwise} \end{cases} \tag{4.11}$$

The above definition provides an intuitive method of computing $\mathbb{Q}^{EN}$. According to Proposition 3.3.1, I can compute the optimal capital requirements c^{EN} and the equilibrium losses $l^{*(EN)}(c^{EN})$ by solving LP (3.11). The dual representation of CVaR (4.6) implies that $\mathbb{Q}^{EN} = 1/\alpha K$ at the largest αK values of $l_\omega^{*(EN)}(c^{EN})$. I can then find the largest αK values of $l_\omega^{*(EN)}(c^{EN})$ since the c^{EN} and $l_\omega^{*(EN)}(c^{EN})$, precisely $p_\omega^{*(EN)}(c^{EN})$, are part of the solution of LP (3.11).

The main intuition for choosing $\mathbb{Q}^{EN}$ as a lower bound $\widetilde{V}_{\mathbb{Q}}$ is that the largest αK losses without default costs $l_\omega^{*(EN)}(c^{EN})$ may not be much different from one with default costs $l_\omega^{*(RV)}(c^{RV})$. It should be noted however that the optimal capital requirements with (i.e., c^{RV}) and without default costs, (i.e., c^{EN}), could be quite different.

In addition to the lower bound $\widetilde{V}_{\mathbb{Q}}$ based on $\mathbb{Q}^{EN}$, I can also obtain an upper bound on V by using $\mathbb{Q}^{EN}$. Let c_{Q}^{EN} denote an optimal solution in (4.8) for a given $\mathbb{Q}^{EN}$. I obtain c_{Q}^{EN} when I find the lower bound $V_{\mathbb{Q}}$ with $\mathbb{Q}^{EN}$. I consider c_{Q}^{EN} as an approximate solution of (4.7). Let $V(c_{\mathrm{Q}}^{EN})$ denote the objective function in (4.7) for c_{Q}^{EN}, that is

$$V(c_{\mathrm{Q}}^{EN}) = \sum_i c_{\mathrm{Q}i}^{EN} + \lambda \,\mathrm{CVaR}_\alpha \left(-l^{*(RV)}(c_{\mathrm{Q}}^{EN})\right) \tag{4.12}$$

$V(c_{\mathrm{Q}}^{EN})$ is an upper bound on V since c_{Q}^{EN} is feasible in (4.7). Moreover, $V(c_{\mathrm{Q}}^{EN})$ is easy to compute. One can first obtain the $p_\omega^{*(RV)}(c_{\mathrm{Q}}^{EN})$ by the RV iterative algorithm and then compute $\mathrm{CVaR}_\alpha \left(-l^{*(RV)}(c_{\mathrm{Q}}^{EN})\right)$ by taking the average of the largest αK losses, i.e., $l_\omega^{*(RV)}(c_{\mathrm{Q}}^{EN})$, as in (4.6).

4.3.4 Iterative procedure to improve lower bound

Despite the good performance of $V(c_{\mathrm{Q}}^{EN})$ and $\widetilde{V}_{\mathbb{Q}^{EN}}$ in the examples I later discuss, the gap between $V(c_{\mathrm{Q}}^{EN})$ and $\widetilde{V}_{\mathbb{Q}^{EN}}$ may be relatively large for some risk tolerance parameters λ. Hence, one might attempt to improve $V(c_{\mathrm{Q}}^{EN})$ and $\widetilde{V}_{\mathbb{Q}^{EN}}$. Here, I provide an iterative algorithm that improves the initial lower and upper bounds and eventually solves the optimal capital requirements problem with default costs.

To better describe the iterative algorithm, let $\Omega^{EN} = \{\omega \in \Omega \,|\, l_{\omega}^{*(EN)}(c^{EN}) \leq l_{(\alpha\mathrm{K})}^{*(EN)}(c^{EN})\}$, i.e., the set corresponding to the largest αK losses without default costs given the optimal capital without default costs c^{EN}. Similarly, let $\Omega^{RV} = \{\omega \in \Omega \,|\, l_{\omega}^{*(RV)}(c^{RV}) \leq l_{\alpha K}^{*(RV)}(c^{RV})\}(c^{RV})\}$, i.e., the set of the largest αK losses with default costs given the optimal αK losses with default costs given the optimal capital with default costs c^{RV}.

The lower bound of the iterative algorithm is based on solving a MIP with fewer constraints than the original MIP (4.4). In particular, I consider constraints for only some scenarios. I call this MIP the *restricted problem* and define it as follows.

Definition 4.3.2. *For a given subset $\Omega' \subset \Omega$, the restricted problem of (4.4) is a mixed integer programming problem that has the form of (4.4) but only considers variables and constraints in Ω'. That is, the restricted problem is*

$$\begin{array}{ll} \underset{c,p,x,u,\nu}{\textit{minimize}} & \sum_i c_i + \lambda(\nu + \frac{1}{\alpha K}\sum\limits_{\omega} u_{\omega}) \\ \textit{subject to} & u_{\omega} \geq -\nu + \sum\limits_{i}(\bar{p}_i - p_{\omega i}) \\ & (p_{\omega}, x_{\omega}) \in \Gamma_{\omega}(c) \\ & \omega \in \Omega'. \end{array} \tag{4.13}$$

Let $\mathcal{V}(\Omega')$ denote the optimal value of the above problem (4.13). The restricted problem is a relaxed version of (4.4). (4.13) considers only constraints of scenarios in Ω', a subset of Ω. Note that $\tilde{V}_{\mathbb{Q}^{EN}} = \mathcal{V}(\Omega^{EN})$ since both of them use the same set of scenarios based on c^{EN} and $l_{\omega}^{EN}(c^{EN})$. It is easy show that $\mathcal{V}(\Omega') \leq V$.

Lemma 4.3.2. *For any $\Omega' \subset \Omega$,*

$$V \geq \mathcal{V}(\Omega'). \tag{4.14}$$

By Lemma 4.3.2, one can get a lower bound $\mathcal{V}(\Omega')$ by solving a restricted problem for a given set of scenarios Ω'. It is important to note that if Ω' is a proper subset of Ω, i.e., $\Omega' \subset \Omega$ and $\Omega' \neq \Omega$, then the restricted problem (4.13) has a smaller number of constraints than (4.4). Thus, solving the restricted problem (4.13) may be easier than solving (4.4). Based on these observations, I develop an iterative algorithm that starts with solving the restricted problem (4.13) with a small size of Ω' and then proceeds to increase the size of Ω'. The iterative algorithm is described as follows:

Algorithm 1 (Iterative algorithm)**.**

0) Initialization. Let $k = 0$ and $\Omega_0 = \Omega^{EN}$.

1) Solve the restricted problem (4.13) with a set of scenarios Ω_k. Denote the optimal capital requirements by c^{Ω_k}.

2) Compute the equilibrium loss with default costs of c^{Ω_k} for all scenarios Ω, i.e., $l^{*(RV)}_{\omega}(c^{\Omega_k})$ for all $\omega \in \Omega$, using the RV iterative algorithm and find the largest αK losses. Let $\Omega^{(k)}$ be the set corresponding to these losses, i.e., $\Omega^{(k)} = \{\omega \in \Omega \,|\, l^{*(RV)}_{\omega}(c^{\Omega_k}) \leq l^{*(RV)}_{(\alpha\mathrm{K})}(c^{\Omega_k})\}$.

3) If $\Omega^{(k)} \subset \Omega_k$, stop.

4) If not, increase k to $k+1$ and $\Omega_{k+1} = \Omega_k \bigcup \Omega^{(k)}$.

In each iteration, Algorithm 1 involves increasing the set of the scenarios Ω_k and solving the restricted problem given Ω_k. In the algorithm, one constructs Ω_k using the optimal capital requirements of the restricted problem c^{Ω_k}. One first evaluates the equilibrium losses with c^{Ω_k} for all ω, and then examines the largest αK among them. If all of the worst αK scenarios are included in Ω_k, the procedure ends. If not, Ω_k is then increased by taking the union with the set of the largest αK scenarios. This procedure repeats until it stops. Termination of Algorithm 1 is guaranteed since Ω_k increases up to Ω and the size of Ω is bounded, i.e., $|\Omega| = K$.

The basic intuition as to why I check the largest αK scenarios and merge them to Ω_k is that at the optimal capital solution c^{RV}, only the worst αK scenarios affect CVaR in the objective function of (4.7). Of course, a priori, one needs losses in all scenarios to find the optimal capital c^{RV}. However, once the largest αK scenarios of losses are included in Ω_k when $c = c^{RV}$, the gap between the lower and upper bounds becomes zero. Using this property, one can show that Algorithm 1 converges to the optimal solution.

Proposition 4.3.1 (Convergence to optimal capital)**.** *Algorithm 1 terminates in most K iterations and when it terminates, the capital requirements c^{Ω_k} from step 1 are equal to the optimal capital requirements c^{RV} of problem (4.7).*

Solving the restricted problem (4.13) with Ω_k becomes more challenging as the size of Ω_k becomes larger. In my numerical experiments, the runtime for solving a restricted problem under some feasible parameters becomes much longer as more scenarios are added in Ω_k (about 1 hour with 50 scenarios and about 24 hours with 100 scenarios) in these examples. In practice, one can always terminate the iterative algorithm earlier and obtain lower and upper bounds with $\mathcal{V}(\Omega_k)$ and $V(c^{\Omega_k})$, respectively, where $V(c^{\Omega_k})$ denotes the value of the objective function in (4.7) for c^{Ω_k}.

In this sense, it is natural to ask how to initialize Algorithm 1 so that it converges as quickly as possible. Proposition 4.3.1 implies that the restricted problem (4.13) provides the optimal capital c^{RV} when the scenarios corresponding to the αK largest losses with $c = c^{RV}$ of losses are included in Ω'. Thus, the efficiency of the algorithm depends on how quickly I find the worst αK scenarios associated with the optimal capital requirements c^{RV}.

To this end, I initialize Ω_0 to be Ω^{EN}. In the examples I later study, I observe a small gap between $V(c_{\mathbb{Q}}^{EN})$ and $\widetilde{V}_{\mathbb{Q}^{EN}}$, which implies that Ω^{EN}, the support of $\mathbb{Q}^{EN}$, approximates Ω^{RV} quite well. Hence it may be reasonable in many cases to use Ω^{EN} as an initial set to ensure fast convergence of the Algorithm 1.

4.4 Extended default model of Rogers and Veraart

Here I incorporate a fire sales mechanism into the model through an extension of Rogers and Veraart (2013). Specifically, I assume that the default of a bank directly decreases the asset values of other banks. To distinguish this approach from well-known fire sales models based on Cifuentes, Ferrucci, and Shin (2005), which uses an inverse demand function to capture the price impact of liquidating assets and which will be studied in the next chapter, I refer to the model in this section as the "cross default" model. This expresses the direct impact of a defaulted bank on other banks' assets value. To capture this, I define the equilibrium payment under the cross default model as follows:

Definition 4.4.1. *The equilibrium payment under the cross default model, denoted by $p^{*(RV2)}$, is defined as the largest fixed point of the following operator ϕ^{RV2}*

$$\phi^{RV2}(p)_i = \begin{cases} \bar{p}_i & \text{if} \left(1 - \sum_j \beta_{ij} x_j\right)\left(y_i + \sum_k \pi_{ki}\, p_k\right) \geq \bar{p}_i, \\ \left(1 - \sum_j \beta_{ij} x_j\right)\left(y_i + \sum_k \pi_{ki}\, p_k\right) & \text{otherwise,} \end{cases} \tag{4.15}$$

where $x_i = 1$ if $\left(1 - \sum_{j \neq i} \beta_{ij} x_j\right) \left(y_i + \sum_k \pi_{ki}\, p_k\right) < \bar{p}_i$ and $x_i = 0$ otherwise.

The implicit default effect on other banks' assets in the above definition is intuitive. Bank i's asset value is decreased by the factor of β_{ij} if bank j defaults. Hence, the asset value of bank i can be affected by the defaults of other banks even though there is no interbank liability between them. This is an extension of the RV model in the previous section. The RV model is a special case where there are no cross default terms, i.e., $\beta_{ij} = 0$ if $j = i$.

As in the MIP formulation of the RV model, x_i can be interpreted as the variable representing the default status of bank i. Hence, I exclude x_i in its own default condition of $(1 - \sum_{j \neq i} \beta_{ij} x_j)(y_i + \sum_k \pi_{ki} p_k) < \bar{p}_i$. On the other hand, in the term for $(1 - \sum_j \beta_{ij} x_j)$ in (4.15), I include x_i since it represents the discrete reduction of asset value of a defaulted bank i.

One of the advantages of the above framework is that it can be easily extended to more general cross default effects. Here I implicitly consider the negative price impact caused by defaulted banks only. Price impact induced by liquidation decisions other than default can be considered by adding additional variables. For instance, I can introduce additional variables $\tilde{x}_i$, representing that bank i's asset value is below some threshold, and consider the price impact function $(1 - \sum_j \beta_{ij} x_j - \sum_j \tilde{\beta}_{ij} \tilde{x}_j)$. This can represent the price impact of liquidation decisions of solvent banks, which is modeled in Cifuentes et al. (2005).

The existence of payment equilibrium $p^{*(RV2)}$ defined in Definition 4.4.1 can be shown by the same reasoning as RV. This follows from the fact that the operator ϕ^{RV2} is bounded and monotone.

Lemma 4.4.1. *There exists an equilibrium payment, $p^{*(RV2)}$, that is the largest fixed point of the operator ϕ^{RV2}.*

Also, $p^{*(RV2)}$ can be computed using the iterative algorithm similar to RV. As in the previous RV framework, to find the optimal capital requirements, I show an optimization formulation to find $p^{*(RV2)}$, which extends Proposition 4.1.1.

Proposition 4.4.1. *With the cross default model in (4.15), an optimal solution* p^{RV2} *of the following mixed integer programming problem is the equilibrium payment for the financial network.*

$$\begin{aligned}
\underset{p,x,w}{\textit{maximize}} \quad & \sum_i p_i \\
\textit{subject to} \quad & p_i \le \bar{p}_i \\
& p_i \le \left(y_i + \sum_k \pi_{ki}\, p_k \right) - \sum_j \beta_{ij} w_{ij} \\
& -M_2 x_i \le y_i + \sum_k \pi_{ki}\, p_k - \sum_{j \ne i} \beta_{ij} w_{ij} - \bar{p}_i \\
& M_2(1 - x_i) \ge y_i + \sum_k \pi_{ki}\, p_k - \sum_{j \ne i} \beta_{ij} w_{ij} - \bar{p}_i \\
& w_{ij} \le \bar{a}_i x_j \\
& w_{ij} \le y_i + \sum_k \pi_{ki}\, p_k \\
& w_{ij} \ge \bar{a}_i (x_j - 1) + y_i + \sum_k \pi_{ki}\, p_k \\
& w_{ij} \ge 0 \\
& x_j \in \{0, 1\}
\end{aligned} \tag{4.16}$$

Similar to formulation (4.3), there are integer variables x_i and two "big M"" constraints, which check bank i's default status. Compared to MIP (4.3), there are four new constraints with w_{ij} in (4.16). w_{ij} represents $x_j(y_i + \sum_k \pi_{ki}\, p_k)$, which includes a product term between x_j and the asset value $y_i + \sum_k \pi_{ki}\, p_k$. Those four constraints come from linearizing the product term. By setting $\bar{a}_i$ to be greater than $y_i + \sum_k \pi_{ki}\, p_k$ at the equilibrium payment $p^{*(RV2)}$, $w_{ij} = y_i + \sum_k \pi_{ki}\, p_k$ if $x_i = 1$ and $w_{ij} = 0$ otherwise. Similar to the constant M_2, I can find the proper $\bar{a}_i$ by examining feasible realized asset values. In the numerical examples in this dissertation, I set $\bar{a}_i = \bar{R}_i \bar{y}_i + \sum_k \bar{p}_{ki}$ where $\bar{R}_i$ is the largest possible return of outside assets for bank i.

With (4.16), I can use the same methodology to compute the optimal capital requirements as in the RV model in the previous sections. Since the equilibrium payments are an optimal solution of (4.16), the socially optimal capital requirements problem can be found by solving the same MIP as (4.4) with the constraints of (4.16). Here $\Gamma_\omega(c)$ is defined as a feasible set in (4.16). Moreover, I can calculate analogous lower bounds using c^{EN} and $\mathbb{Q}^{EN}$ and Algorithm 1.

4.5 Expected loss and penalty method

4.5.1 Expected loss

Here I consider a simple risk measure, which has the following form:

$$\begin{aligned} \rho(a^*_\omega(c)) &= \mathbb{E}\left[\sum_i (\bar{p}_i - a^*_{\omega i}(c))^+\right] \\ &= \mathbb{E}\left[\sum_i l^*_{\omega i}(c)\right] \end{aligned} \tag{4.17}$$

This risk measure averages over losses in all scenarios instead of the worst $\alpha\%$ scenarios. Also, note that when $\alpha = 100\%$, $\text{CVaR}_\alpha(Y) = \mathbb{E}(Y)$. Hence, (4.17) is a systemic risk measure that is a special case of (3.4).

However, Algorithm 1 would not help to find the optimal capital requirements with (4.17). The iterative algorithm uses the property that CVaR_α focuses on only the worst case $\alpha\%$ scenarios of the aggregate losses. In contrast, (4.17) considers the losses in all scenarios, so the benefit of considering only a subset of scenarios as done in Algorithm 1 cannot be achieved.

Following Brown and Smith (2014), however, I provide a method of finding an approximate solution and lower bounds that evaluates the performance of the approximate solution. Moreover, I provide an approximation solution for this problem, which is different from the approach in the previous section.

Similar to Proposition 4.2.1, I first rewrite the socially optimal capital requirements problem with (4.17) as a mixed integer programming problem as follows:

$$\begin{array}{ll} \underset{c\geq 0,p,x}{\text{minimize}} & \sum_i c_i + \lambda \left(\frac{1}{K} \sum_{\omega} \sum_{i} (\bar{p}_i - p_{\omega i})^+ \right) \\ \text{subject to} & (p_\omega, x_\omega) \in \Gamma_\omega(c) \end{array} \tag{4.18}$$

where $\Gamma_\omega(c)$ is a set of feasible (p_ω, x_ω) in MIP (4.3) for a given return R_ω and capital c.

In (4.18), there is no variable ν that comes from the equivalent formulation of CVaR. However, this MIP is still difficult to solve since it still has "big M" constraints as in MIP (4.4).

4.5.2 Approximate solution

I use the EN optimal capital with $\hat{\lambda}$, denoted by $c^{\hat{\lambda}(EN)}$, which can be different from the λ in the problem (4.18) as an approximate solution. I choose $\hat{\lambda}$ so that $c^{\hat{\lambda}(EN)}$ gives the smallest objective value in (4.18). The following definition formalizes the definition of the approximate solution.

Definition 4.5.1. *Given λ, an approximation solution $\hat{c}^\lambda$ is defined as follows:*

$$\hat{c}^\lambda := \arg\min_{\hat{\lambda}\geq 0} \left\{ \sum_i c_i^{\hat{\lambda}(EN)} + \lambda \mathbb{E}\left[\sum_i l^*_{\omega i}(c^{\hat{\lambda}(EN)}) \right] \right\} \tag{4.19}$$

where $c^{\hat{\lambda}(EN)}$ is an optimal capital in EN LP problem (3.11) with systemic risk tolerance parameter $\hat{\lambda}$ and losses are calculated according to the RV iterative algorithm.

The basic intuition of using $\hat{c}^\lambda$ as an approximate solution is that I approximate the increased losses with default costs with losses in the frictionless model, but scaled by a constant. In particular, I first solve for the optimal capital requirements in the frictionless model for various risk tolerance parameters and then uses the solution that gives the lowest objective value with the default costs as an approximate solution.

I can easily find $c^{\hat{\lambda}(EN)}$ by solving EN LP given $\hat{\lambda}$ or solving parametric LP. The objective value corresponding to each $c^{\hat{\lambda}(EN)}$ is easily computed by using the RV iterative algorithm.

4.5.3 Lower bounds using information relaxations

Following Brown and Smith (2014), I now provide lower bounds on the optimal value of (4.18). With this, I can assess the quality of the approximate solution $\hat{c}^{\lambda}$ in the previous section.

Let $\mathbb{E}V_{\omega}(c^*)$ and $\mathbb{E}\hat{V}_{\omega}(c^{EN})$ denote the optimal value of the socially optimal capital requirements problem using expected loss (4.17) with and without default costs, respectively. It is easy to show that $\mathbb{E}\hat{V}_{\omega}(c^{EN})$ is a lower bound of $\mathbb{E}V_{\omega}(c^*)$.

Lemma 4.5.1.

$$\mathbb{E}V_{\omega}(c^*) \geq \mathbb{E}\hat{V}_{\omega}(c^{EN}) \tag{4.20}$$

From the above lemma, I can evaluate the approximate solution $\hat{c}^{\lambda}$ by comparing $\mathbb{E}V_{\omega}(\hat{c}^{\lambda})$ and $\mathbb{E}\hat{V}_{\omega}(c^{EN})$, where $\mathbb{E}V_{\omega}(\hat{c}^{\lambda})$ denotes the value of the optimal capital problem with default costs given $\hat{c}^{\lambda}$. In the numerical examples I study, the gap between $\mathbb{E}V_{\omega}(\hat{c}^{\lambda})$ and $\mathbb{E}\hat{V}_{\omega}(c^{EN})$ is large.

I consider the lower bounds based on information relaxation where one can choose different capital levels in each scenario. To improve the performance of these lower bounds, I find another lower bound by solving the information relaxation problem by adding a penalty as in Brown and Smith (2014).

Lemma 4.5.2 (Brown and Smith 2014)**.** *If* $\mathbb{E}[z_{\omega}(c)] \leq 0$ *for any* c*, then*

$$\mathbb{E}V_{\omega}(c^*) \geq \mathbb{E}[\min_{c \geq 0} V_{\omega}(c) + z_{\omega}(c)]. \tag{4.21}$$

Lemma 4.5.2 implies that by solving the inner problem $\min_{c \geq 0} V_{\omega}(c) + z_{\omega}(c)$ for each scenario ω and taking the mean of optimal values of $\min_{c \geq 0} V_{\omega}(c) + z_{\omega}(c)$, one obtains a lower bound to the original problem. Clearly, adding the term $\mathbb{E}[z_\omega(c)]$ to $\mathbb{E}V_\omega(c^*)$ preserve the fact that this is a lower bound.

Importantly, following Brown and Smith (2014), if $z_\omega(c) = \mathbb{E}\hat{V}_\omega(c) - \hat{V}_\omega(c)$, then the information relaxation with the penalty gives a better lower bound than EN LP $\mathbb{E}\hat{V}_\omega(c^{EN})$. I formalize this in the following lemma.

Lemma 4.5.3. $z_\omega(c) = \mathbb{E}\hat{V}_\omega(c) - \hat{V}_\omega(c)$, *then*

$$\mathbb{E}[\min_{c\geq 0} V_\omega(c) + z_\omega(c)] \geq \mathbb{E}\hat{V}_\omega(c^{EN}). \tag{4.22}$$

This follows from the fact that $V_\omega(c) - \hat{V}_\omega(c) \leq 0$ since the losses with default costs are larger than the losses without default costs. However, it is not obvious how to solve the "inner problem" $\min_{c\geq 0} V_\omega(c) + z_\omega(c)$ with $z_\omega(c) = \mathbb{E}\hat{V}_\omega(c) - \hat{V}_\omega(c)$ since it is not necessary a convex optimization problem.

4.5.4 Gradient penalty

To overcome the difficulty of solving the inner problem with with $z_\omega(c) = \mathbb{E}\hat{V}_\omega(c) - \hat{V}_\omega(c)$, I adopt the gradient penalty method in Brown and Smith (2014) and use it as a lower bound. Following Brown and Smith (2014), I define the gradient as follows:

Definition 4.5.1.

$$z_{\nabla\omega}(c) = [\nabla_c \mathbb{E}\hat{V}_\omega(c^{EN}) - \nabla_c \hat{V}_\omega(c^{EN})]^\top (c - c^{EN}) + \mathbb{E}\hat{V}_\omega(c^{EN}) - \hat{V}_\omega(c^{EN}) \tag{4.23}$$

Hence, $z_{\nabla\omega}(c)$ is a first-order Taylor series expansion of $z_\omega(c)$ around $c = c^{EN}$. Notice that $\mathbb{E}\hat{V}_\omega(c)$ might not be differentiable at c^{EN}. In this case, I use a subgradient of $\mathbb{E}\hat{V}_\omega(c)$ at c^{EN} (see Brown and Smith 2014 for a more detailed discussion of choosing a subgradient).

Lemma 4.5.4. *With* $z_{\nabla\omega}(c)$ *defined as (4.23),*

$$\mathbb{E}[\min_{c\geq 0} V_\omega(c) + z_{\nabla\omega}(c)] \geq \mathbb{E}\hat{V}_\omega(c^{EN}) \tag{4.24}$$

Lemma 4.5.4 shows that the information relaxation with gradient penalty $z_{\nabla\omega}(c)$ gives a lower bound that is better than the EN LP solution. Notice that the inner problem $\min_{c\geq 0} V_\omega(c)+z_{\nabla\omega}(c)$ is a mixed integer programming problem with different capital requirements c in each scenario. This problem is relatively easy to solve compared to MIP (4.18) for the socially optimal problem since it has constraints only for a single return scenario. By solving this small MIP for each scenario and computing the mean of $V_\omega(c) + z_{\nabla\omega}(c)$, I find a lower bound that improves the EN LP problem. This approaches can be generalized to an "imperfect" information relaxation where the inner problem is defined over subset of scenarios instead of a single scenario. Hence, the inner problem is a mixed integer programming problem with different capital requirements c in each subset. I show the performance of this lower bound in the later section in detail.

4.6 Numerical results

I now apply my framework to a model based on a real data set consisting of five banks. I will compare the gaps between the lower bounds and the upper bounds with varying systemic risk tolerance parameter λ. I also study how the cross default mechanism affects each bank's optimal capital requirements.

4.6.1 Data description

In the data, the basic information on balance sheets for my model - the amount of interbank debts, outside debts and outside investments for five major banks - are collected from 2004 to 2009. The more detailed information of the data can be found in Webber and Willison (2011). In particular, I focus on data from 2007, since it is the start of the financial crisis. All the results presented in this section are based on data from 2007.

I assume the return on the outside asset follows a multivariate normal distribution. The correlation of the returns captures the dependence structure of returns between banks. The data set contains an estimated mean and covariance. With the return distribution, I generate 1,000 sample returns on outside assets, i.e., $K = 1000$, and use the samples to solve the optimal capital requirements problem. Since I use the simulation to approximate the return distribution, it is easy to consider other distribution models in this framework.

In the examples, I assume that $\beta_{ii} = 0.06$ and $\beta_{ij} = 0.01$. This implies that the cross-default mechanism induces at least 6% and 1% losses of asset value to a bank itself and other banks, respectively. Hence, if all 5 banks default, there is at least 10% loss in each bank's asset value.

Finally, I use 5% CVaR, i.e., $\alpha = 0.05$.

4.6.2 Results

I use both CPLEX and Gurobi to solve MIP (4.4) with varying λ from 1 to 5. With 1000 samples, it is computationally intractable to solve MIP (4.4). However, with 50 samples, I can solve MIP (4.4) with those solvers. Thus, the bounds using Algorithm 1 can be applied since $\alpha K = 50$.

Figure 4.1 shows the upper and lower bounds, $V(c_{\mathbb{Q}}^{EN})$ and $\widetilde{V}_{\mathbb{Q}^{EN}}$ respectively, using the measure $\mathbb{Q}^{EN}$. The upper and lower bounds are evaluated at 10 different λ varying from 1 to 5. The red line represents the upper bound $V(c_{\mathbb{Q}}^{EN})$ and the green line represents the lower bound $\widetilde{V}_{\mathbb{Q}^{EN}}$. The blue line represents the optimal value without frictions. I focus on λ between 1 and 5 since the optimal capital requirements are trivial outside this range. When λ is less than one, the optimal capital requirements are zero since the capital becomes more important than systemic risk in the objective value. On the other hand, the systemic risk term dominates when λ becomes larger than 5 and the optimal capital requirements protect all of the possible return scenarios.

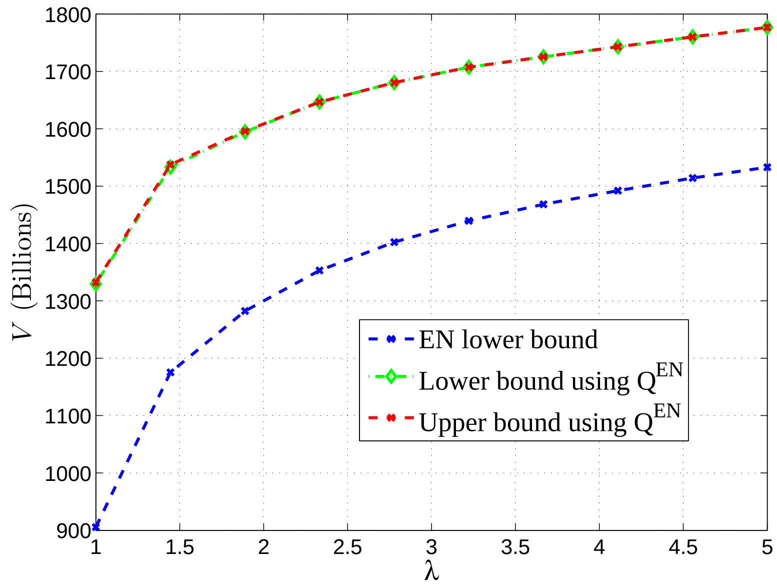


FIGURE 4.1: Upper and lower bounds using frictionless measure

As explained in the previous section, the fictionless lower bound is computed by solving LP (3.11). Based on the solution of that LP, $\mathbb{Q}^{EN}$ takes the worst αK scenarios of losses at the optimal solution without cross default mechanism, i.e., $l_\omega^{*(EN)}(c^{EN})$. With $\mathbb{Q}^{EN}$, the lower bound $\widetilde{V}_{\mathbb{Q}^{EN}}$ is computed by solving MIP (4.8), which has constraints only for the αK scenarios. Finally, the upper bound $V(c_{\mathbb{Q}}^{EN})$ is computed by evaluating loss with cross default effects, which is efficiently computed with the iteration algorithm with the operator ϕ^{RV2}.

From this, one can see that the gaps between upper and lower bounds for different λ are quite small. Actually, the gaps are zero except $\lambda = 1.444$ and 1.888. This implies that the approximate solution $c_{\mathbb{Q}}^{EN}$ is close to optimal and $\mathbb{Q}^{EN}$ is also close to $\mathbb{Q}^*$. Hence, the intuition that the worst αK scenarios of the optimal capital requirements with and without defaults seems reasonably correct in these examples.

Since the gap between $V(c_{\mathbb{Q}}^{EN})$ and $\widetilde{V}_{\mathbb{Q}^{EN}}$ is not zero when $\lambda = 1.444$ and 1.888, I apply Algorithm 1 to improve the bounds.

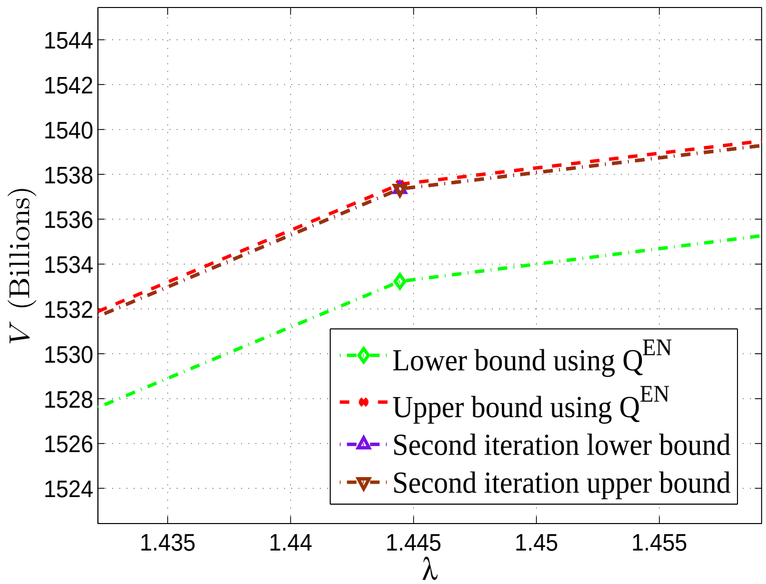


FIGURE 4.2: Iterative bounds

Figure 4.2 shows the upper and lower bounds based on Algorithm 1. The results are based only on one iteration of Algorithm 1. The purple and brown lines indicate the improved lower and upper bounds using Algorithm 1, respectively. The gap between the improved lower and upper bounds is zero.

4.6.3 Cross-default effects on optimal capital requirements

Finally, I examine how the cross default mechanism affects an individual bank's optimal capital requirements. As a comparison and an illustrative example, I search for the largest λ where the approximate solution c_{Q}^{EN} gives $\mathrm{CVaR}_{\alpha}(l^{*(EN)}\left(c_{\mathrm{Q}}^{EN}\right)) \leq 0.05(\sum_i \bar{p}_i)$ and check how the individual bank's optimal capital is changed.

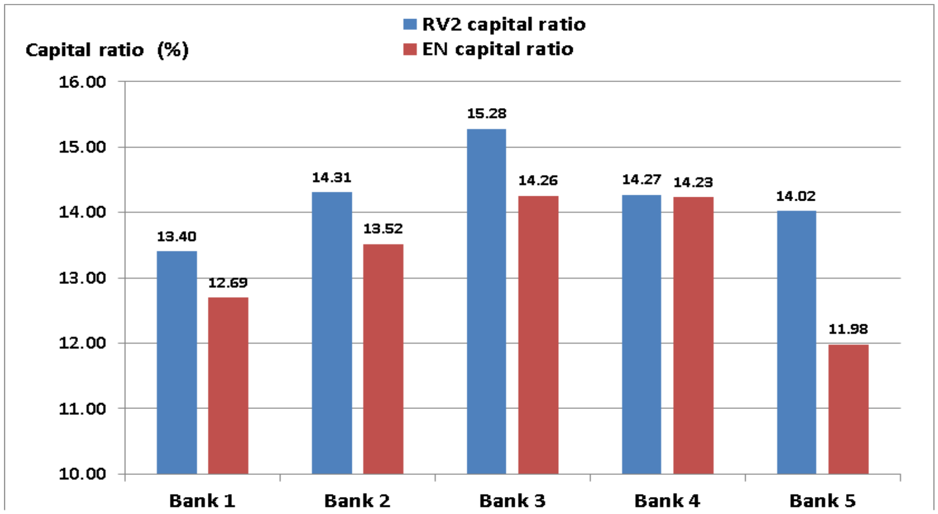


FIGURE 4.3: Individual bank capital

Figure 4.3 shows that all banks' level of capital requirements increase when cross default effects are considered. However, even though β_{ij} is the same across the banks, the cross-default effect is not uniform to individual banks. For instance, the optimal capital level of bank 5 increases by more than 2% compared to only a 0.04% increase of bank 4's optimal capital level. Thus the impact of default costs as well as the structure of interbank liabilities can have a substantial impact on the optimal capital requirements.

4.6.4 Numerical results on the gradient penalty

Here I present the results of the penalty method in Section 4.5. I consider the same underlying model. MIP problem (4.18) with 1000 samples is computationally intractable. However, with 100 samples, I can solve MIP (4.18). Thus, I can construct the penalty bounds by solving 10 MIP problems, each with 100 subsamples, which are the inner problems of (A.49).

First, Figure 4.4 plots the value of systemic risk tolerance parameter of the approximate solution $\hat{c}^{\lambda}$ calculated in (4.19), which is denoted by $\hat{\lambda}^{*}$, for different λ. As explained in Section 4.5, $\hat{\lambda}^{*}$ is typically larger than λ since the approximate solution $\hat{c}^{\lambda}$ solves an EN LP problem that approximates the large losses with a higher risk tolerance parameter.

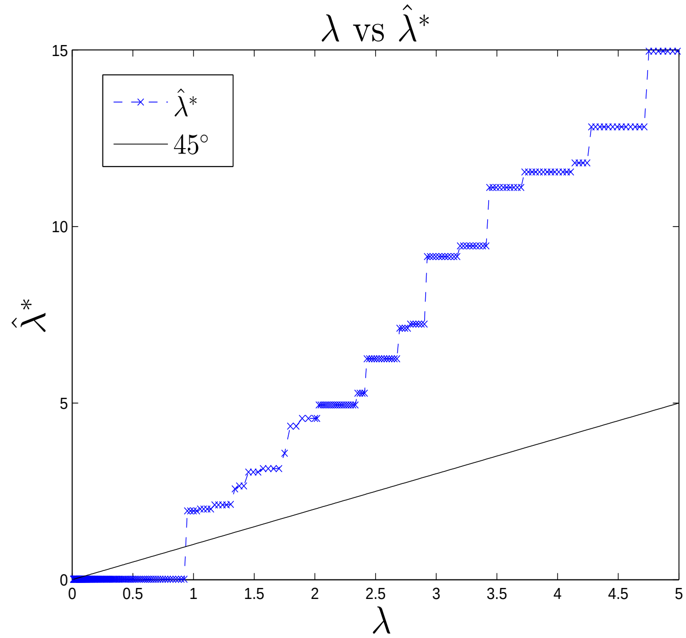
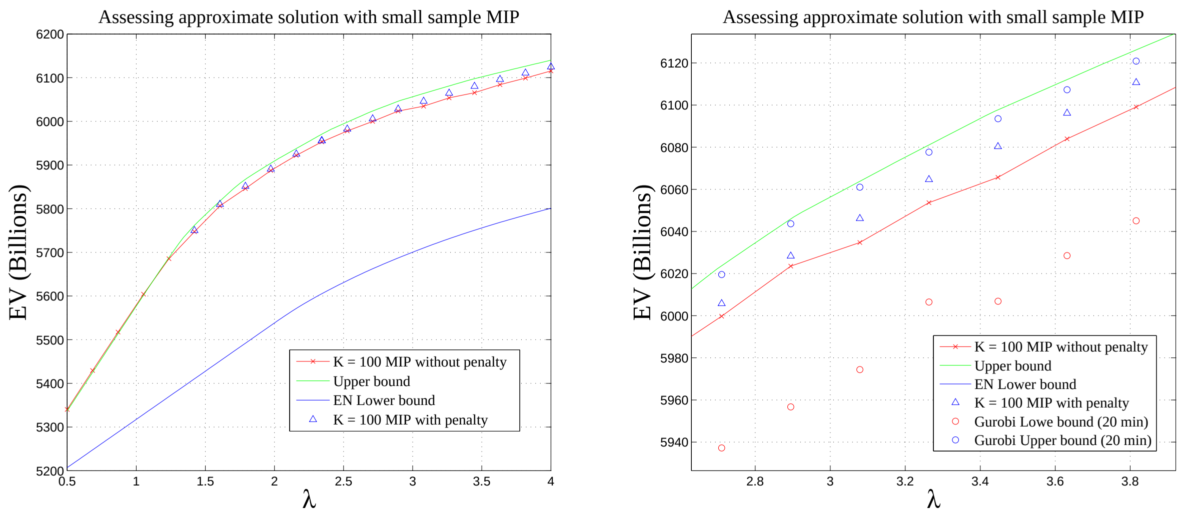


FIGURE 4.4: Approximate risk tolerance parameter $\hat{\lambda}$ from (4.19)

Figure 4.5 shows the upper bound based on the approximate solution $\hat{c}^{\lambda}$ and the lower bounds with and without the gradient penalty. The green lines represent



(a) Approximate solutions and bounds

(b) Closer view of penalty bounds

FIGURE 4.5: Approximate solutions and bounds

the upper bound based on the approximate solution $\hat{c}^{\lambda}$ and the blue line represents the optimal value of the EN LP. The red line represents the information relaxation lower bound without a gradient penalty. Finally, the blue triangles represent the information relaxation bounds with the gradient penalty as in (4.24).

From Figure 4.5, one can see that even the information relaxation lower bound without a penalty improves the EN LP lower bound significantly. However, the gaps between the upper bounds and information relaxation lower bounds without penalty are not negligible. Figure 4.5 shows that the differences between the upper and lower bounds decrease significantly by using the gradient penalty (4.24).

5

Financial networks with fire sales

In this chapter, I incorporate fire sales by extending the framework of Cifuentes, Ferrucci, and Shin (2005) (hereafter CFS). In CFS, the price impact of liquidation is captured by an inverse demand function $f(x)$, where x represents the number of shares of liquidated assets. I take $f(0) = q_0$, where q_0 is the price without any sales of the assets. CFS assumes that an inverse demand function $f(x)$ is exogenously given. In this setting, CFS defines and computes the equilibrium price and the equilibrium payment similar to EN and RV.

I first consider a fire sales model of a single illiquid asset as in Amini, Filipović and Minca (2015 and 2016) and then extend the analysis to a multiple illiquid asset model as in Feinstein (2015). Similar to the RV model with default costs, I present an optimization formulation to compute the equilibrium payment and the optimal capital. Moreover, I show that the optimization formulation can incorporate general liquidation rules such as the socially optimal rule, the prioritized rule and the leverage targeting rule. I first analyze the payment equilibrium with the equivalent optimization formulation and then present the optimization formulation for the optimal capital problem. Finally, I apply the method here to examples with randomly generated networks and see how network structures affect the systemic risk and the optimal capital requirements.

5.1 Single illiquid asset model

I begin the analysis by incorporating a fire sales model of single illiquid asset into an interbank network. Recently, Amini, Filipović and Minca (2015 and 2016) (hereafter AFM) extend CFS framework to the liquidation model of debt payment and provide a rigorous proof of the uniqueness of equilibrium price and payments, and an algorithm to find the equilibrium price and payments that is similar to the EN algorithm.

Following AFM, I define the equilibrium price and equilibrium payment, as follows:

Definition 5.1.1. *Given the inverse demand function $f(x)$, the equilibrium price $q^{*(FS)}$ and equilibrium payment $p^{*(FS)}$ for realized outside asset value y is a fixed point of the following operator ϕ^{FS} on $[Q_{min}, Q] \times [0, \bar{p}]$:*

$$\phi^{FS}(q,p) = \begin{cases} \phi_0^{FS}(q,p) & = f\left(\sum_i \min\left(\frac{(\bar{p}_i - c_i - \sum_k \pi_{ki}\, p_k)^+}{q}, y_i\right)\right) \\ \phi_i^{FS}(q,p) & = \min\left(q y_i + c_i + \sum_k \pi_{ki}\, p_k, \bar{p}_i\right), i = 1, \ldots, N \end{cases} \tag{5.1}$$

Notice that $\min\left(\frac{(\bar{p}_i - c_i - \sum_k \pi_{ki}\, p_k)^+}{q}, y_i\right)$ represents the value of the liquidated asset that is not affected by q. One can interpret this as the number of shares of liquidated assets or the real value of liquidated assets. The inverse demand function $f(x)$ is defined on the number of shares of liquidated assets, i.e., the real value of liquidated assets, instead of the nominal value of liquidated assets. Let y_{tot} denote the total number of shares of illiquid assets in the economy, which represents the maximum number of shares of illiquid assets that can be liquidated in the economy. Generally, $y_{tot} = \sum_i y_i$. Hence the domain of the inverse demand function $f(x)$ is

$[0, y_{tot}]$.

In addition, AFM assumes the following conditions of $f(x)$ to guarantee uniqueness of $p^{*(FS)}$ and $q^{*(FS)}$.

Assumption 1. *AFM assumptions on the inverse demand function:*

(i) $f(0) = q_0$*;*

(ii) $f(x)$ *is non-increasing on* $[0, y_{tot}]$*;*

(iii) $xf(x)$ *is increasing on* $[0, y_{tot}]$*.*

The first condition implies that the price of illiquid asset is equal to the initial price q_0 if there are no sales in the market. The second condition states that the price of the illiquid asset is non-increasing with liquidation amount in the market. The third condition states that the cash generated from liquidation increases as more of the illiquid asset is liquidated. With only conditions (i) and (ii), one cannot guarantee the uniqueness of the equilibrium payment and price. The third condition is critical to guarantee the uniqueness of $p^{*(FS)}$ and $q^{*(FS)}$. As in AFM, the optimization formulation I use will need this condition to guarantee the uniqueness of $p^{*(FS)}$ and $q^{*(FS)}$.

Moreover, the operator $\phi_0^{FS}(q, p)$ captures a distinctive property of the AFM model compared to the EN and RV models. The value $\phi_0^{FS}(q, p)$ captures the negative price impact of liquidation and the purpose of the liquidation, i.e., paying debts, as well. Moreover, it should be noticed that the only purpose of a bank's liquidation is to pay its total debts at $t = 1$. There is no additional liquidation for other purposes such as leverage targeting. In reality, a bank also has long term liabilities, which do not need to be paid in the short term period. Hence, it is natural to consider the total debt $\bar{p}_i$ in the model as short term debts. I extend this to the model with long term debts later in the leverage targeting rule.

The expression $(\bar{p}_i - c_i - \sum_k \pi_{ki}\, p_k)^+$ reflects the fact that bank i would sell liquid assets prior to selling illiquid assets. A bank would not liquidate any illiquid asset if its liquid asset value $c_i - \sum_k \pi_{ki}\, p_k$ exceeds its total debt $\bar{p}_i$. If the liquid asset value is less than the total debt, the amount that is needed to be liquidated by bank i is equal to the shortfall of the liquid asset value relative to the total debt, i.e. $\bar{p}_i - c_i - \sum_k \pi_{ki}\, p_k$.

Finally, notice that the inverse demand function $f(x)$ depends on the number of shares of the liquidated assets, i.e., the value of the liquidated assets is divided by current market price q, instead of the nominal value of asset sales.

As with EN and RV, I consider an equivalent optimization formulation whose solution is equal to $p^{*(FS)}$ and $q^{*(FS)}$ by changing the equality of the operator $\phi^{FS}(q,p)$ in (5.1) to an inequality and maximizing a function that is increasing in p and q. This leads to the following result:

Lemma 5.1.1. *If there are fire sales externalities as in the AFM model, an optimal solution p^{FS}, q^{FS} of the following optimization problem is the equilibrium payment and the equilibrium price for the financial network.*

$$\begin{aligned} \underset{p,q}{\text{maximize}} \quad & \sum_i \left(q y_i + c_i + \textstyle\sum_k \pi_{ki}\, p_k\right) \\ \text{subject to} \quad & q \le f\left(\sum_i \min\left(\frac{(\bar{p}_i - c_i - \sum_k \pi_{ki}\, p_k)^+}{q}, y_i\right)\right) \\ & p_i \le \bar{p}_i \\ & p_i \le q y_i + c_i + \sum_k \pi_{ki}\, p_k \\ & p, q \ge 0. \end{aligned} \tag{5.2}$$

Unlike the EN and RV models, it is not easy to solve the problem (5.2), mainly due to the first constraint. In the first constraint, since $f(x)$ is decreasing, I cannot simply replace the min operator with two inequalities for payments as in the EN and RV model. To find a tractable optimization formulation, I use the following proposition.

Proposition 5.1.1. *The real asset sales at equilibrium payment* $p^{*(FS)}$ *and price* $q^{*(FS)}$ *satisfy the following equation:*

$$\min\left(\frac{(\bar{p}_i - c_i - \sum_k \pi_{ki}\, p_k^{*(FS)})^+}{q^{*(FS)}}, y_i\right) = \frac{(p_i^{*(FS)} - c_i - \sum_k \pi_{ki}\, p^{*(FS)})^+}{q^{*(FS)}}. \qquad (5.3)$$

The above proposition can be shown by a simple argument using the properties of the equilibrium payment. When bank i pays debts in full, its equilibrium payment is $p_i^{*(FS)} = \bar{p}_i$ and the sales amount should be no greater than y_i since it has not defaulted. On the other hand, if bank i has defaulted, its real sales amount is y_i and its equilibrium payment $p_i^{*(FS)} = q^{*(FS)} y_i + c_i + \sum\limits_k \pi_{ki}\, p_k^{*(FS)}$. Hence, equality holds in (5.3).

Proposition 5.1.1 is straightforward but critical to formulating a simple optimization problem whose solutions are the equilibrium payment and price. I refer to this optimization problem as an equivalent optimization problem. For many well-known types of $f(x)$ such as linear and exponential, the corresponding equilibrium optimization problem is a convex optimization problem. Hence, as in the EN model, the socially optimal capital problem can be formulated as a convex optimization problem as well. Moreover, the equivalent optimization problem for AFM payment has an important implication in extension to models with other liquidation rules and multiple illiquid assets, which I will show in the later section.

Notice that $\min\left(\frac{(\bar{p}_i - c_i - \sum_k \pi_{ki}\, p_k^{*(FS)})^+}{q^{*(FS)}}, y_i\right)$ and $\min\left((\bar{p}_i - c_i - \sum_k \pi_{ki}\, p_k^{*(FS)})^+, q^{*(FS)} y_i\right)$ represent the number of shares, i.e., the value of the liquidated assets is divided by current market price q and the nominal value of assets that are liquidated by bank i, respectively. Proposition 5.1.1 shows that both the real and nominal asset sales can be expressed as a function of the equilibrium payment $p^{*(FS)}$ and the price $q^{*(FS)}$ without a minimum operator.

In particular, I let z_i^* denote the nominal asset sales by bank i at equilibrium, i.e.,

$$z_i^* = \min\left((\bar{p}_i - c_i - \sum\nolimits_k \pi_{ki}\, p_k^{*(FS)})^+, q^{*(FS)} y_i\right). \tag{5.4}$$

By Proposition 5.1.1, z_i^* can be expressed as

$$z_i^* = (p_i^{*(FS)} - c_i - \sum\nolimits_k \pi_{ki}\, p^{*(FS)})^+. \tag{5.5}$$

Using Proposition 5.1.1 and the above expression for z^*, I can formulate an equivalent optimization problem as follows.

Proposition 5.1.2. *Under Assumption 1, the equilibrium payment $p^{*(FS)}$, price $q^{*(FS)}$ and asset sales $z^{*(FS)}$ are optimal solutions to the following optimization problem:*

$$\begin{aligned}
\underset{p,q,z}{\textit{maximize}} \quad & \sum_i \left(y_i q + c_i + \sum_k \pi_{ki}\, p_k \right) \\
\textit{subject to} \quad & p_i \leq \bar{p}_i \\
& p_i \leq z_i + c_i + \sum_k \pi_{ki}\, p_k \\
& z_i \leq \; q y_i \\
& \sum_i z_i \leq \; q f^{-1}(q) \\
& p, z \geq 0.
\end{aligned} \tag{5.6}$$

Moreover, if $x f^{-1}(x)$ is concave, 5.6 is a convex optimization problem.

The constraints in (5.6) have an intuitive interpretation. Recall that z_i represents the nominal value of asset liquidated by bank i. It then follows that the amount of cash available to each bank i is equal to $z_i + c_i + \sum_k \pi_{ki}\, p_k$. With this, the second constraint imposing limited liability, i.e., $p_i \leq q y_i + c_i + \sum_k \pi_{ki}\, p_k$ is changed to

$p_i \leq z_i + c_i + \sum_k \pi_{ki}\, p_k$, which represents the fact that each bank i's debt payment is smaller than its total amount of cash.

The right hand side of the third constraint, qy_i represents the ex-post value of the illiquid asset. Hence, the third constraint represents the fact that the nominal value of the asset liquidated by bank i cannot exceed the value of the illiquid asset held by bank i .

Using Proposition 5.1.1, I rewrite the constraint of $q \leq f(z/q)$ as in the last constraint $z_i \leq \; qf^{-1}(q)$. With this form, I can check whether the optimization problem (5.6) is convex or not. As long as $xf^{-1}(x)$ is concave, (5.6) is a convex optimization problem. For example, if $f(x) = exp(-\theta x)$ for some $\theta > 0$ as in CFS, then $xf^{-1}(x) = -\dfrac{x}{\theta} ln(x)$ is concave.

Interestingly, the third assumption on the inverse demand function is critical to guarantee the optimality of $p^{*(FS)}$, $q^{*(FS)}$ and $z^{*(FS)}$ to (5.6). It is not obvious that the constraint $z_i \leq \; qy_i$ in (5.6) holds with equality when each bank i defaults. There can exist other liquidation rules where a defaulted bank i's assets are not fully liquidated. The third assumption guarantees that full liquidation is the best liquidation strategy to maximize the asset value of a defaulted bank.

Among the many possible forms of $f(x)$, I focus on a linear price impact function in this dissertation. The linear model has been widely used in the study of fire sales (see e.g., Greenwood, Landier and Thesmar 2015 and Capponi and Larsson 2015). In particular, if $f(x)$ is linear, the equilibrium price can be expressed as a square root function. Moreover, the corresponding optimization problem (5.6) is convex. From now on I assume the linear price impact function.

Assumption 2. *The price decreases linearly with the sales amount, i.e.,*

$$f(x) = q_0 - \theta x, \tag{5.7}$$

for some $\theta \geq 0$.

Given Assumption 2, the equilibrium price given the total liquidation amounts z can be expressed as follows:

$$Q(z) = \left(\frac{q_0 + \sqrt{q_0^2 - 4\theta \sum_i z_i}}{2} \right). \tag{5.8}$$

This corresponds to the largest root of the quadratic equation $q^2 - q_0 q^2 - \theta \sum_i z_i^* = 0$, which is obtained by solving $f\left(\frac{\sum_i z_i^*}{q}\right) = q$. Notice that $Q(z)$ is decreasing in z and is concave. With this, I formulate an equivalent convex optimization problem for linear price impact function as follows.

Proposition 5.1.3. *If $f(x) = q_0 - \theta x$, then the equilibrium payment $p^{*(FS)}$ and asset sales $z^{*(FS)}$ are optimal to the following convex optimization problem:*

$$\begin{aligned} \underset{p,z}{\text{maximize}} \quad & \sum_i \left(y_i Q(z) + c_i + \sum_k \pi_{ki}\, p_k \right) \\ \text{subject to} \quad & p_i \leq \bar{p}_i \\ & p_i \leq z_i + c_i + \sum_k \pi_{ki}\, p_k \\ & z_i \leq \; y_i Q(z) \\ & p, z \geq 0. \end{aligned} \tag{5.9}$$

This simply results from Proposition 5.1.2 and the closed-form expression of (5.8). It should be notice that the constraint $\sum_i z_i \leq \; q f^{-1}(q)$ is not needed anymore since $Q(z)$ captures $f\left(\frac{\sum_i z_i^*}{q}\right) = q$. I now have an equivalent convex optimization formulation as in EN. Moreover, (5.9) is also the equivalent optimization formulation for other liquidation functions if $Q(z)$ represents the price given the liquidation amount z. Hence, I can extend the analysis to the socially optimal capital problem under the AFM model, where fire sales occur with a single illiquid asset and the purpose of liquidation of each bank is to pay its debts. I first extend the results to a fire sales model with multiple illiquid assets and then present the optimization formulation for the optimal capital problem.

5.2 Fire sale model with multiple illiquid assets

In reality, a bank holds multiple types of illiquid assets. Under the current regulatory stress test, the projected losses are computed in five to seven asset categories (see Glasserman and Tangirala 2015). Greenwood, Landier, and Thesmar (2015) and Duarte and Eisenbach (2015) empirically measure the fire sales effect with multiple illiquid asset classes.

To address this, I consider and analyze a model with multiple illiquid assets. In contrast to the single illiquid asset model of AFM, a bank that holds multiple assets needs to decide which and what proportion of its illiquid assets are to be liquidated to pay its debts. This complicates the problem of finding the corresponding equilibrium payment and price. To address this issue, I introduce the concept of a liquidation function as in Feinstein (2015). Recently, Feinstein (2015) shows the existence and uniqueness of the equilibrium payment and price with multiple illiquid assets and provides an recursive algorithm similar to EN and RV. In contrast, I provide an equivalent optimization formulation that is useful to solve the optimal capital problem.

5.2.1 General liquidation rule

In the baseline setting of the multi-asset model, there are M illiquid assets indexed by $m = 1, \ldots, M$ in the economy. Bank i holds y_{im} units of asset m. The price of asset m is denoted by q_m. Note that q_m does not necessarily represent the equilibrium price. Each illiquid asset has its own price impact function $f_m(x)$.

Moreover, in the single illiquid asset model, I learned that the nominal asset sales z is useful to express price and derive the equivalent optimization problem. Motivated by this, I let z_{im} denote the nominal amount of asset m liquidated by bank i for given payment p and price vector q. Hence, $z_{i1}, \ldots, z_{iM}$ are mutually defined by the following equations:

$$z_{im} = \min\left((\bar{p}_i - c_i - \sum\nolimits_k \pi_{ki}\, p_k - \sum\nolimits_{l \neq m} z_{il})^+, q_m y_i\right). \tag{5.10}$$

This equation implies that each bank i now has the freedom to choose among $z_{i1}, \ldots, z_{iM}$ to pay its debt $\bar{p}_i$. Different liquidation rules can lead to the same equilibrium payments, whereas in the single asset model, there is only one liquidation rule that leads to the unique equilibrium payment. Thus, I need to define another concept of the equilibrium payment and price that captures the different liquidation rules.

To define the equilibrium payment and price in the multi-asset model, I first introduce a liquidation function, denoted by $s_{im}(p, q)$, which represents the liquidation amount of asset m that bank i wishes to sell for a given payment p and price q to pay their debts.

Moreover, it is natural to assume that banks liquidate their assets only to pay their debts. Selling illiquid assets decreases a bank's own asset value due to negative price impact. Hence, banks would not liquidate more assets than are necessary to pay their debts. As in Feinstein(2015), the following minimal liquidation condition on the liquidation function $s_{im}(p, q)$ expresses this property.

Assumption 3. *The liquidation function $s(p, q)$ satisfies the minimal liquidation condition:*

$$\sum_m \min\left(q_m y_{im}, s_{im}(p, q)\right) = \min\left(\sum_m q_m y_{im}, (\bar{p}_i - c_i - \sum\nolimits_k \pi_{ki}\, p_k)^+\right) \tag{5.11}$$

With this liquidation function $s(p,q)$, I define the payment equilibrium and price as follows:

Definition 5.2.1. *The equilibrium payment $p^{*(FS)}$ and price $q^{*(FS)}$ for a given realized outside asset value y is a fixed point of the following operator ϕ^{FS} on $[0,\bar{p}] \times [Q_{min}, Q]$:*

$$\phi^{FS}(q,p) = \begin{cases} \phi_i^{FS}(q,p) & = \min\left(\sum_m q_m y_{im} + c_i + \sum_k \pi_{ki}\, p_k, \bar{p}_i\right), i = 1,\dots,N \\ \phi_i^{FS}(q,p) & = f_m\left(\sum_i \min\left(\frac{s_{im}(p,q)}{q_m}, y_{im}\right)\right) i = N+1,\dots,N+M \end{cases} \tag{5.12}$$

I slightly abuse the notation of $p^{*(FS)}$, $q^{*(FS)}$ and ϕ^{FS} by using the same notation in the single asset model in Definition 5.1.1 in the previous section. I can consider Definition 5.1.1 as a special case of definition 5.2.1 with $M = 1$.

Similarly, Feinstein(2015) considers a multiple assets model as above. In contrast to Feinstein(2015), the liquidation function $s_{im}(p,q)$ is defined as the *nominal value* of a liquidated asset m instead of the number of shares of asset m that bank i wants to sell. As in the single asset model, defining a liquidation function in terms of asset values is helpful in finding a tractable optimization formulation that is useful to solve the optimal capital problem.

I can also express the liquidation function in terms of the number of *units* of asset as in Feinstein (2015). To distinguish it from $s(p,q)$, I call this the unit liquidation function and denote it by $\tilde{s}_{im}(p,q)$. Obviously,

$$\tilde{s}_{im}(p,q) = \frac{s_{im}(p,q)}{q_m}. \tag{5.13}$$

With this, I summarize Feinstein (2015)'s result of the existence of equilibrium payment $p^{*(FS)}$ and price $q^{*(FS)}$.

Proposition 5.2.1 (Feinstein 2015)**.** *Consider a financial network with a liability matrix L with given liquid capital c and illiquid asset y with the continuous price impact function f_m.*

1) If $\tilde{s}_{im}(p,q)$ is continuous, then there exists an equilibrium price and payment, q^* and p^*.

2) If $\tilde{s}_{im}(p,q)$ is nonincreasing, then there exists a greatest and least equilibrium price and payment, $(p^+,q^+) \geq (p^-,q^-)$.

3) With a strictly positive y, if $\tilde{s}_{im}(p,q)$ is nonincreasing and $\sum_m \beta_m f_m(\beta)$ is strictly increasing on $\beta \in \mathbb{R}^m_+$, then there exists a unique equilibrium payment and price.

As in RV, Feinstein (2015) provides a modified fictitious default algorithm of EN to compute the equilibrium payment and price. As mentioned earlier, this has the limitation of only providing the equilibrium payment and price for a single fixed return scenario and given capital levels.

I now present the optimization problem where the equilibrium p^* and q^* are optimal. For the optimization formulation, let z^*_{im} denote the equilibrium liquidation amount of asset m by bank i, i.e., $z^* = \min(q^*_m y_{im}, s_{im}(p^*,q^*))$. As in the single illiquid asset model, I will focus on a linear price impact function in the multi-asset model and assume that the initial price equals one , i.e., $f_m(x) = 1 - \theta_m x$.

Proposition 5.2.2. *(p^*,q^*,z^*) is an optimal solution of the following optimization* *problem:*

$$\begin{aligned}
\underset{p,q,z}{\textit{maximize}} \quad & \sum_i \left(\sum_m y_{im} q_m + c_i + \sum_k \pi_{ki}\, p_k \right) \\
\textit{subject to} \quad & p_i \leq \bar{p}_i \\
& p_i \leq \sum_m z_{im} + c_i + \sum_k \pi_{ki}\, p_k \\
& z_{im} \leq \; y_{im} q_m \\
& z_{im} \leq \; s_{im}(p,q) \\
& q_m \leq Q_m(z_m) \\
& p, q, z \geq 0.
\end{aligned} \tag{5.14}$$

The first and second constraints of (5.14) represent the debt payment property reflecting the fact that each bank's payment cannot exceed its total debts and the amount of total available cash. The third and fourth constraints represent the upper limit of the liquidation amount z_{im}. As in the single asset model (5.6), the nominal value of the liquidation amount z_{im} cannot exceed the initial holdings of bank i. The last constraint also represents the upper limit of the price of asset m, a square root form due to the assumption of a linear price impact function.

The key property to derive the above formulation is the minimal liquidation property of $s(p,q)$ and the decreasing property of $Q_m(z_m)$. Since the objective function is increasing in p, one of the first two constraints should be satisfied with equality. One of the third and fourth constraints should be satisfied with equality as well due to the minimal liquidation condition. The last constraint should be satisfied since the objective function is increasing in q. On the other hand, it is clear that the equilibrium payment, price and liquidation are feasible in (5.14). If bank i defaults at equilibrium, $p_i^* = \sum_m y_{im} q_m^* + c_i + \sum_k \pi_{ki}\, p_k^* \leq \bar{p}_i$ and $z_{im}^* = \; y_{im} q_m^* \leq s_{im}(p^*, q^*)$ for all m. If bank i does not default, $p_i^* = \bar{p}_i \leq \sum_m y_{im} q_m^* + c_i + \sum_k \pi_{ki}\, p_k^*$ and $z_{im}^* = s_{im}(p^*, q^*) \leq y_{im} q_m^*$ for all m. Therefore, the above formulation gives the correct amount of the aggregate wealth in the economy at equilibrium. So this enables us to formulate an optimization problem with any systemic risk measure that is based on the aggregate wealth in the economy.

Recall that $Q_m(z_m)$ is the square root price function (5.8) corresponding to each f_m and is concave. Hence, if $s_{im}(p,q)$ is concave, then (5.14) is a convex optimization problem. The most intuitive liquidation functions, such as the market value proportional liquidation rule where the proportion of sales of assets are equal to the proportion of the market value and the prioritized liquidation rule where the assets are liquidated following an order, as in Feinstein (2015), are not concave. However, the prioritized liquidation rule can be formulated as a convex mixed programming problem.

As in the EN and RV optimization formulations, any function $F(p,q)$ that is non-decreasing in p and q can be used as an objective function in (5.2) where the equilibrium payment $p^{*(FS)}$ and price $q^{*(FS)}$ are optimal. As in the previous section on EN and RV, I choose the aggregate asset value since I focus on the systemic risk measure based on the aggregate asset value. If the systemic risk is measured in other ways, such as by the number of defaulting banks or the asset value of the external debts, I can use the corresponding function, e.g., $F(p,q) = \sum_i 1\left\{\sum_m y_{im}q_m + c_i + \sum_k \pi_{ki}\, p_k \geq \bar{p}_i\right\}$ or $F(p,q) = \sum_i p_i^o$ as an objective function in (5.2). I formalize this in the following lemma.

Lemma 5.2.1. *The equilibrium payment, price and sales* (p^*,q^*,z^*) *are optimal to the optimization problem with the same constraints as (5.14) if the objective function* $F(p,q)$ *is nondecreasing in each component of* (p,q).

Note that when $M = 1$, (5.14) reduces to (5.9). The sales function $s_{i1}(p,q) = (\bar{p}_i - c_i - \sum_k \pi_{ki}\, p_k)^+$. Due to the net borrowing assumption, $(\bar{p}_i - c_i - \sum_k \pi_{ki}\, p_k)^+ =$

$\bar{p}_i - c_i - \sum_k \pi_{ki}\, p_k$. Hence, the constraint $z_{i1} \leq s_{i1}(p, q)$ is redundant in this case since the constraint $p_i \leq z_{i1} + c_i + \sum_k \pi_{ki}\, p_k$ exists and $Q_1(z_1)$ is decreasing in z_{i1}.

5.2.2 Socially optimal liquidation

I first consider a liquidation rule that maximizes the aggregate asset value of banks in the economy. This is in line with the optimization formulation of Proposition 5.1.3 in the single asset model. In the single asset case, I already learned that the equilibrium payment and price maximizes the aggregate asset value of the economy. I call this policy "socially optimal" since it maximizes the total asset value of all investors, i.e., both equity and debt holders in the economy. I extend this concept to the multi-asset model. Interestingly, the optimization formulation for the socially optimal liquidation rule is a baseline model that can be used as a benchmark for other liquidation rules.

I first define the socially optimal liquidation policy that maximizes the total asset value of the economy.

Definition 5.2.2. *The socially optimal liquidation rule is defined by the liquidation function $\tilde{s}(p, q)$, where the aggregate asset value of the corresponding equilibrium payment and price $\tilde{p}^*$ and $\tilde{q}^*$ is larger than with any other liquidation rule $s(p, q)$.*

Obviously, it is difficult to find a liquidation function $\tilde{s}(p, q)$ that corresponds to the socially optimal rule. However, I can find an equilibrium payment and price of the socially optimal policy by using the optimization formulation (5.14) and Proposition 5.2.2. The following proposition shows an optimization formulation for the socially optimal liquidation rule.

Proposition 5.2.3. *The socially optimal equilibrium payment $\tilde{p}^*$, price $\tilde{q}^*$ and liqui-*

dation amount $\tilde{z}^*$ are optimal solutions to the following convex optimization problem:

$$
\begin{aligned}
\underset{p,q,z}{\textit{maximize}} \quad & \sum_i \left(\sum_m y_{im} Q_m(z_m) + c_i + \sum_k \pi_{ki}\, p_k \right) \\
\textit{subject to} \quad & p_i \le \bar{p}_i \\
& p_i \le \sum_m z_{im} + c_i + \sum_k \pi_{ki}\, p_k \\
& z_{im} \le \ y_{im} Q_m(z_m) \\
& p, q, z \ge 0.
\end{aligned}
\tag{5.15}
$$

First of all, Proposition 5.2.3 implies the existence of the socially optimal equilibrium ($\tilde{p}^*$, $\tilde{q}^*$). The feasible set of (5.15) is bounded since $0 \le p_i \le \bar{p}_i$ and $0 \le z_{im} \le y_{im}$, and nonempty since $z_{im} = 0$ and $p_i = \min(\bar{p}_i, c_i + \sum_k \pi_{ki}\, p_k)$ satisfy the constraints. Finally, (5.15) is a convex optimization problem. This leads to the following result.

Lemma 5.2.2. *The socially optimal equilibrium ($\tilde{p}^*$, $\tilde{q}^*$) exists.*

Note that (5.15) is derived simply by removing the constraints $z_{im} \le s_{im}(p,q)$ of (5.14). It is intuitive that removing constraints $z_{im} \le s_{im}(p,q)$ corresponds to the socially optimal rule. I can consider that the value of the liquidation function given p and q, $\tilde{s}_{im}(p,q)$ is equal to z. However, I need to check the minimal liquidation condition on this sales function. In the appendix, I show that $\tilde{s}_{im}(p,q)$ satisfies the minimal liquidation condition.

Moreover, (5.15) shows the optimization formulation of Proposition 5.2.2 is useful in defining other types of the socially optimal liquidation. Following this, I can consider a liquidation policy that maximizes other measures such as the number of defaults and the total payments to outside creditors by removing the constraints $z_{im} \le s_{im}(p,q)$.

5.2.3 Prioritized liquidation

Various liquidation strategies can be expressed with $s(p, q)$. Hence I consider the liquidation strategy where a bank sequentially liquidates assets in the order that they are labeled. An asset with a high priority would be fully and sequentially liquidated. I call this the prioritized liquidation rule. This can be expressed with the following liquidation function:

$$\widehat{s}_{im}(p, q) = \left(\overline{p}_i - c_i - \sum_k \pi_{ki} p_k - \sum_{l=1}^{m-1} q_l y_{il} \right)^+ . \tag{5.16}$$

Notice that in (5.16) the orders of liquidations are identical across the banks. However, one can incorporate the different orderings of individual banks by assuming that the index variable l in $q_l y_{il}$ is different across the banks.

Without loss of generality, I assume that the asset with a smaller index has a higher priority in liquidation. Hence, if the total liquidated value of the first $m - 1$ asset is enough to cover the shortfall, i.e., $\sum_{l=1}^{m-1} z_{lm} = \left(\overline{p}_i - c_i - \sum_k \pi_{ki} p_k \right)^+$, the right hand side of (5.16) equals zero since $\sum_{l=1}^{m-1} z_{lm} \leq \sum_{l=1}^{m-1} q_l y_{il}$. Hence, $\widehat{s}_{im}(p, q)$ equals zero. Furthermore, $\widehat{s}_{im}(p, q)$ with a larger m equals zero as well. This represents the fact that the remaining assets whose priority is lower than the mth asset would not be liquidated at all.

Considering this, I can formulate the following convex mixed integer programming problem that gives us the equilibrium total payments with a prioritized rule.

Proposition 5.2.4. *The equilibrium payments, price and liquidation amount under the prioritized liquidation rule* ($\widehat{p}^{*(FS)}$,$\widehat{q}^{*(FS)}$,$\widehat{z}^{*(FS)}$) *are optimal to the following* *optimization problem:*

$$
\begin{aligned}
\underset{p,q,z,x}{\text{maximize}} \quad & \sum_i \left(\sum_m y_{im} q_m + c_i + \sum_k \pi_{ki}\, p_k \right) \\
\text{subject to} \quad & p_i \leq \bar{p}_i \\
& p_i \leq \sum_m z_{im} + c_i + \sum_k \pi_{ki}\, p_k \\
& z_{im} \leq \; y_{im} q_m \\
& q_m \leq Q_m(z_m) \\
& z_{im} \leq \; \overline{p}_i - c_i - \sum_k \pi_{ki} p_k - \sum_{l=1}^{m-1} q_l y_{il} + \mathcal{M}_{im}(1 - x_{im}) \\
& z_{im} \leq \; \mathcal{M}_{im} x_{im} \\
& \overline{p}_i - c_i - \sum_k \pi_{ki} p_k - \sum_{l=1}^{m-1} q_l y_{il} \leq \mathcal{M}_{im} x_{im} \\
& \overline{p}_i - c_i - \sum_k \pi_{ki} p_k - \sum_{l=1}^{m-1} q_l y_{il} \geq -\mathcal{M}_{im}(1 - x_{im}) \\
& p, q, z \geq 0 \\
& x_{im} \in \{0, 1\}
\end{aligned}
\tag{5.17}
$$

for sufficiently large $\mathcal{M}_{im}$.

(5.17) is a straightforward extension of (5.15) by replacing $q_m = Q_m(z)$ with $q_m \leq Q_m(z)$ and adding logical constraints representing the prioritized rule. The first four constraints are the same as the constraints in the socially optimal formulation (5.15) with $q_m \leq Q_m(z)$. Adding the price variables q_m and adding the constraints $q_m \leq Q_m(z)$ is essential in this formulation. By setting $q_m = Q_m(z)$, the inequality constraint $\overline{p}_i - c_i - \sum_k \pi_{ki} p_k - \sum_{l=1}^{m-1} q_l y_{il} \geq -\mathcal{M}_{im}(1 - x_{im})$ is not convex anymore. One can set $\mathcal{M}_{im} = \max \left(\overline{p}_i, \overline{p}_i - c_i - \sum_k \bar{p}_{ki} - \sum_{l=1}^{m-1} \bar{R}_{il} \bar{y}_{il} \right)$ where $\bar{R}_{il}$ is the largest possible return of illiquid asset l of bank i.

As in the RV formulation, I use the MIP formulation to express the logical constraints of the prioritized rule. The integer variable $x_{im} = 1$ if $\overline{p}_i - c_i - \sum_k \pi_{ki} p_k -$

$\sum_{l=1}^{m-1} q_l y_{il}$ is positive and $x_{im} = 0$ otherwise. Hence, x_{im} captures whether bank i needs to liquidate asset m or not. If $x_{im} = 1$, $z_{im} \leq \min\left(y_{im} q_m, \overline{p}_i - c_i - \sum_k \pi_{ki} p_k - \sum_{l=1}^{m-1} q_l y_{il} \right)$ and if $x_{im} = 0$, $z_{im} = 0$. This corresponds to the prioritized liquidation function.

If the asset with mth priority is not fully liquidated, this implies that $z_{im} = \overline{p}_i - c_i - \sum_k \pi_{ki} p_k - \sum_{l=1}^{m-1} q_l y_{il}$. Hence, for any $m' > m$, $\overline{p}_i - c_i - \sum_k \pi_{ki} p_k - \sum_{l=1}^{m'-1} q_l y_{il} \leq 0$. Following this, assets $m+1, ..., M$ are not liquidated if the mth asset is not fully liquidated. Given this, I can add the constraints $x_{im} \geq x_{im+1}$ to make all integer variables whose priorities are lower than m, i.e., $x_{im+1}, \ldots, x_{iM}$, become zero once x_{im} becomes zero.

Finally, it should be noted that (5.17) is a convex mixed integer programming problem since the objective and the constraints are defined by convex functions. Of course, it may not be easy to compute the equilibrium payment and price by solving this convex MIP. To compute the payment and price equilibrium, the recursive algorithm in Feinstein (2015) would be much faster. However, I propose an optimization formulation since the purpose is to use it to solve the optimal capital requirements problem and to compute systemic risk.

5.3 Leverage targeting

A bank may liquidate more assets than necessary to pay its debt (Greenwood, Landier and Thesmar 2015). A bank liquidates assets not only to pay its debt but also to maintain a certain leverage ratio. In this section, I extend the model to incorporate the leverage target rule.

To consider this, I need to distinguish between short-term and long-term liabilities. Unlike the model so far examined, there are two time periods $t \in \{0, 1, 2\}$ in the economy. As in Acemoglu et al. (2015), I assume that all interbank liabilities are short-term. The total amount of short-term and long-term liabilities of bank i is denoted by $\bar{p}_i^s$ and $\bar{p}_i^l$, respectively. I will use variables p_i^s and p_i^l representing short and long term liabilities. Hence, I simply change the previous model with total liability $\bar{p}_i$ with $\bar{p}_i^s$ and add additional variables for long term liabilities $\bar{p}_i^l$. Let $\bar{p}_i$ denote the total liabilities, i.e., $\bar{p}_i = \bar{p}_i^s + \bar{p}_i^l$.

Bank i would liquidate its asset in order to pay its short-term debts $\bar{p}_i^s$ and additional amounts to meet the leverage target. I assume that the liquidation for leverage targeting occurs only at time $t = 0$. To make this formal, let $t_i(p, q)$ denote the total amount of liquidation that is needed for bank i's leverage targeting. The debt-to-equity ratio is the measure for leverage targeting. Hence, $t_i(p, q)$ satisfies

$$\frac{\bar{p}_i - t_i(p, q)}{\sum\limits_l q_l y_{il} + c_i + \sum\limits_k \pi_{ki} p_k - \bar{p}_i} \leq \tau. \tag{5.18}$$

The numerator of (5.18) is the total debt after deleveraging t amount of debts. The denominator is the realized equity of bank i. τ is the targeted ratio of debt to equity. Let $a_i(p, q) = \sum\limits_l q_l y_{il} + c_i + \sum\limits_k \pi_{ki} p_k$ denote the realized asset value of bank i. By rewriting (5.18), I have

$$t_i(p, q) \geq (\tau + 1)\bar{p}_i - \tau a_i(p, q). \tag{5.19}$$

The right hand side of (5.19) is the minimum amount of liquidation for bank i necessary to meet the leverage ratio τ. I let $\psi_i(p, q)$ denote this, i.e.,

$$\psi_i(p, q) = (\tau + 1)\bar{p}_i - \tau a_i(p, q). \tag{5.20}$$

Notice that $\psi_i(p, q)$ changes with the realized asset value. To characterize $t_i(p, q)$, I need to consider how the value of $\psi_i(p, q)$ changes.

Clearly, $t_i(p, q) = 0$ if $\psi_i(p, q)$, is negative. Bank i does not need to deleverage. On the other hand, due to the negative price impact of liquidations, $t_i(p, q) = \psi_i(p, q)$

if $\psi_i(p,q)$ is positive. I have the following condition on $a_i(p,q)$, which is equivalent to $\psi_i(p,q) \geq 0$:

$$\psi_i(p,q) \geq 0 \Leftrightarrow a_i(p,q) \leq \left(\frac{\tau+1}{\tau}\right)\bar{p}_i. \tag{5.21}$$

Thus, I have

$$t_i(p,q) = \psi_i(p,q) \text{ if } a_i(p,q) \leq \left(\frac{\tau+1}{\tau}\right)\bar{p}_i \tag{5.22}$$

and $t_i(p,q) = 0$ otherwise. Hence, I have an upper threshold on the realized asset value $\left(\frac{\tau+1}{\tau}\right)\bar{p}_i$ to check whether $t_i(p,q) = \psi_i(p,q)$ or $t_i(p,q) = 0$.

Since $t_i(p,q) = 0$ if $a_i(p,q) > \left(\frac{\tau+1}{\tau}\right)\bar{p}_i$, I need to consider the liquidation for short-term debt payment $\bar{p}_i^s$. Thus, the total amount of liquidation, considering both leverage targeting and short term debt payments, is

$$\max\left((\tau+1)\bar{p}_i - \tau a_i(p,q), \bar{p}_i^s\right). \tag{5.23}$$

It should be noticed that (5.23) implies that a bank could liquidate its illiquid assets more than the amount that is needed for leverage targeting. This is an important distinction from the models in Greenwood, Landier and Thesmar (2015) and Feinstein and El-Masri (2015), as these models do not consider the liquidation amount needed for debt payments.

The total amount of liquidation needed when bank i does not default is

$$\left(\max\left((\tau+1)\bar{p}_i - \tau a_i(p,q), \bar{p}_i^s\right) - c_i - \sum_k \pi_{ki}\, p_k\right)^+. \tag{5.24}$$

Therefore, as in the debt payment model, I can consider the sales function $s^\tau(p,q)$ given as follows:

$$\begin{aligned} &\sum_m \min\left(q_m y_{im}, s_{im}^\tau(p,q)\right) \\ &= \min\left(\sum_m q_m y_{im}, \left(\max\left((\tau+1)\bar{p}_i - \tau a_i(p,q), \bar{p}_i^s\right) - c_i - \sum_k \pi_{ki}\, p_k\right)^+\right). \end{aligned} \tag{5.25}$$

I can apply the same logic as in Feinstein (2015) and show that the equilibrium payment and price exist under this sales function. Note that the total liquidation amount $\max\left((\tau+1)\bar{p}_i - \tau a_i(p,q), \bar{p}_i^s\right)$ is continuous and componentwise nonincreasing in (p,q). Hence, the equilibrium payments and prices exist.

Proposition 5.3.1. *Given a sales function $s^\tau(p,q)$ in (5.25), which represents the leverage targeting and short term debt payments, there exists an equilibrium payment $p^{*(LT)}$ and price $q^{*(LT)}$.*

5.3.1 Optimization formulation for leverage targeting

As in the previous models of EN, RV and AFM, I find an equivalent optimization formulation as follows:

Proposition 5.3.2. *Let* $(p^{*(LT)}, q^{*(LT)}, z^{*(LT)})$ *denote the equilibrium payment, price and liquidation amount, given the sales function* $s^{\tau}(p,q)$*. Then* $(p^{*(LT)}, q^{*(LT)}, z^{*(LT)})$ *is an optimal solution of the following optimization problem:*

$$
\begin{aligned}
\underset{p,q,z,w}{\text{maximize}} \quad & \sum_i \left(\sum_m y_{im} q_m + c_i + \sum_k \pi_{ki}\, p_k^s \right) \\
\text{subject to} \quad & p_i^s \leq \bar{p}_i^s, p_i^l \leq \bar{p}_i^l \\
& p_i^s + p_i^l \leq \sum_m y_{im} q_m + c_i + \sum_k \pi_{ki}\, p_k^s \\
& p_i^s \leq \sum_m z_{im} + c_i + \sum_k \pi_{ki}\, p_k^s \\
& (\tau+1)(p_i^s + p_i^l) - \tau a_i(p,q) \leq \sum_m z_{im} + c_i + \sum_k \pi_{ki}\, p_k^s \\
& z_{im} \leq \ y_{im} q_m, z_{im} \leq \ s_{im}^{\tau}(p,q) \\
& q_m \leq Q_m(z_m) \\
& \frac{p_i^l}{\bar{p}_i^l} \leq w_i \leq \frac{p_i^s}{\bar{p}_i^s} \\
& p, z, q \geq 0 \\
& w_i \in \{0,1\}\,.
\end{aligned}
\tag{5.26}
$$

In (5.26), as in the previous equivalent optimization formulations, the first, second and third constraints represent the property of short and long term debts payments. Payments cannot exceed the initial debts and the total asset value of each bank. The fourth and fifth constraints are distinctive to the model of debt payment only and reflect that the total amount of cash should be used for either short term debt payments or leverage targeting. Note that the left hand sides of the fourth and fifth constraints are the liquidation amounts needed for short term debt payments and leverage targeting, respectively. The sixth, seventh and eighth constraints are the same as in (5.14).

The last inequality constraint $\frac{p_i^l}{\bar{p}_i^l} \leq w_i \leq \frac{p_i^s}{\bar{p}_i^s}$ with the integer variable w_i expresses that the short term debts are senior to the long term debts, i.e., the short term debts are to be paid prior to the long term debts. The integer variable $w_i = 0$ if $p_i^s < \bar{p}_i^s$ and it results in $p_i^l = 0$. On the other hand, w_i can be either 1 or 0 if $p_i^s = \bar{p}_i^s$. With this, p_i^l can be positive. Putting all of this together, $p_i^l = 0$ if $p_i^s < \bar{p}_i^s$ and $p_i^l \geq 0$ if $p_i^s = \bar{p}_i^s$ by introducing the integer variable w_i. Hence, the last inequality captures the seniority of the short term debts relative to the long term debts.

As in the model for debt payments (5.14), the constraint $z_{im} \leq \ s_{im}^{\tau}(p, q)$ is the key constraint that determines whether (5.26) is a convex MIP or not. I leave the problem of representing various liquidation rules with $s_{im}^{\tau}(p, q)$ and verifying whether or not (5.26) is a convex MIP as a future research project.

5.4 Optimal capital problem with fire sales

Using the results of Proposition 5.2.2, I finally show the optimization formulation of finding the socially optimal capital requirements, considering both interbank liabilities and fire sales externalities.

I focus on the constrained form of the optimal capital problem. Recall that the socially optimal capital problem in the constrained form is as follows:

$$\begin{array}{ll} \underset{c \geq 0}{\text{minimize}} & \sum_i c_i \\ \text{subject to} & \text{CVaR}_{\alpha}\left(- \sum_i (\bar{p}_i - a_{\omega i}^*(c))^+ \right) \leq \gamma. \end{array} \tag{5.27}$$

Proposition 5.4.1. *Under the fire sales model of definition 5.2.1 with a liquidation function $s(p, q)$, an optimal solution c^{FS} of the following optimization problem is optimal to the socially optimal capital requirements problem and a corresponding* *optimal solution $p^\sharp$ is the equilibrium payment given R_ω and c^{FS}:*

$$\begin{array}{ll} \underset{c,p,q,z,u,\nu}{\text{minimize}} & \sum_i c_i \\ \text{subject to} & \nu + \frac{1}{\alpha K}\sum\limits_{\omega} u_\omega \leq \gamma \\ & u_\omega \geq -\nu + \sum\limits_{i}(\bar{p}_i - p_{\omega i}) \\ & (p_\omega, q_\omega, z_\omega, x_\omega) \in \Gamma^{FS}_\omega(c), \end{array} \tag{5.28}$$

where $\Gamma^{FS}_\omega(c)$ is a set of feasible $(p_\omega, q_\omega, z_\omega)$ in (5.14) for a given return R_ω and capital c.

This follows the same logic of the optimization formulation as Proposition 4.2.1 for the RV model.

5.5 Numerical examples

I now apply the model of a single illiquid asset to the same data set with five banks as in Section 4.6. I set the illiquid asset price impact parameter to $\theta = 1.5 \times 10^{-5}$ (0.15 bps) and $\alpha = 0.05$. Since the equivalent optimization problem (5.6) is a convex optimization problem, I can solve the optimal capital problem efficiently. I plot the aggregate value of the optimal capital for each target of systemic risk γ in Figure 5.1.

Moreover, I investigate the fire sales effects on the optimal capital of the individual banks, as in the previous example using the RV model. As an illustrative example, I find the optimal capital that targets 5% of the aggregate debts in the economy, i.e., $\text{CVaR}_\alpha(l^{*(FS)}(c^{\text{FS}})) \leq 0.05\left(\sum_i \bar{p}_i\right)$. Figure 5.2 shows the optimal capitals of individual bank in the EN (red bar) and AFM (blue bar) models, respectively. It shows that fire sales effects on the individual capital are significant and different across the banks. The optimal capital of bank 5 increases by about 1 % compared to a 2% increase for bank 1. These results suggest that policymakers should consider

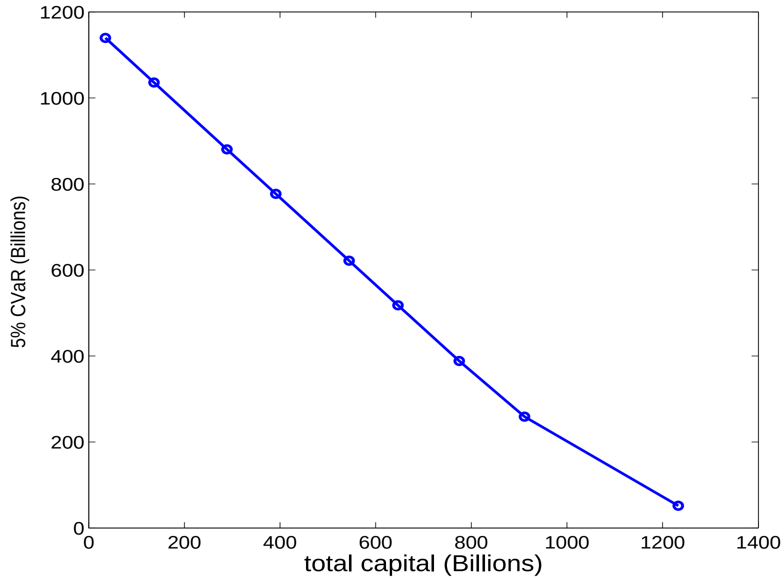


FIGURE 5.1: Capital vs. 5 % CVaR

not only the network of direct exposures but also effects of the fires sales in setting capital requirements.

5.6 Numerical analysis with different network structures

I now apply the proposed method to randomly generated networks and conduct a comparative static analysis to see how the network structure affects the optimal capital requirements. As many papers have suggested, I find that the network structure of interbank liabilities and illiquid asset holdings can affect systemic risk considerably. In particular, the numerical examples show that the joint effects of both interconnected liabilities and illiquid asset holdings are significant factors contributing to systemic risk and the optimal capital requirements.

I consider two types of randomly generated networks as in Elliott et al. (2014)(EGJ): homogeneous random networks (Erdös and Rényi 1959) and a core-periphery random

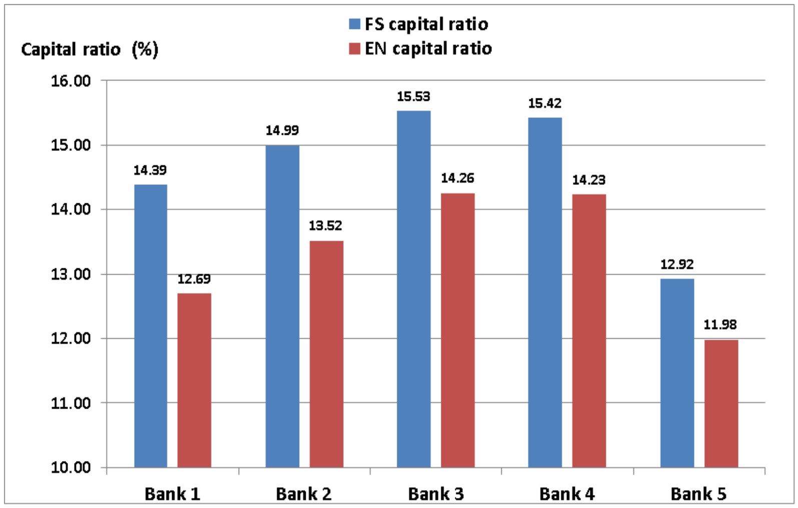


FIGURE 5.2: Individual bank capital

network. Even though EGJ considers only a cross-holding equity matrix instead of a liabilities matrix, their numerical examples show how to conduct the comparative statistics in liability matrix as well. I incorporate fire sales mechanisms of single and multiple illiquid assets with the prioritized rule into each type of random network. In the examples of multiple illiquid assets, I assume that all banks follow the same liquidation ordering where a bank liquidates a more liquid asset, i.e., with smaller θ, prior to a less liquid asset, i.e., with larger θ.

The first examples are homogenous random networks with a single illiquid asset. I vary the the parameters of expected number of creditors of a bank and the proportion of the aggregate interbank debt to total debt of each model. The last two examples are based on the core-periphery model, which is known to capture the real interbank network by recent empirical papers. I first consider core-periphery networks with a single asset and then consider the model with multiple illiquid assets.

5.6.1 Single asset homogenous model

Similar to EGJ, I first generate an adjacency matrix G and then compute the liability matrix L. Specifically, there are two parameters that characterize the financial network : $d_g \in [0, N-1]$, the expected number of creditors of a bank, and $a_g \in [0\ 1]$, the proportion of the aggregate interbank debt to total debt. d_g and a_g reflect the level of diversification and integration in the financial network, respectively. As d_g gets larger, one bank's total interbank debt gets more distributed to other banks. On the other hand, as a_g gets larger, a bank would rely more on their interbank liabilities.

Based on these parameters, I first generate an adjacency and liability matrix, G and L respectively, as follows:

1. Simulate IID Bernoulli random variables taking $G_{ij} = 1$ with probability $d_g/(N-1)$ and $G_{ij} = 0$ with probability $1 - d_g/(N-1)$. $G_{ii} = 0$ for all i.

2. For each bank i, let $Ł_{ij} = (a_g\, \overline{p}_i) \,/ \sum\limits_j G_{ij}$ if $G_{ij} = 1$, otherwise $L_{ij} = 0$.

3. For $j = N+1$, let $Ł_{iN+1} = \overline{p}_i - \sum\limits_{j=1}^{N} L_{ij}$.

Following the above procedure, the matrix L represents a randomly generated network, where the expected number of connected banks is d_G and all of the liabilities to creditors are equal to $(a_g\, \overline{p}_i) \,/ \sum\limits_j G_{ij}$. Note that $\sum\limits_j G_{ij}$ equals the number of interbank creditors of bank i and $a_g\, \overline{p}_i$ corresponds to the total amount of interbank debts, which depends on G_{ij}. However, the total debt $\overline{p}_i$ is fixed across the simulated network. Hence, I can compare the effects of diversification and integration in the network structure on the optimal capital requirements. Entity $N+1$ represents the creditor outside the interbank system, and L_{iN+1} represents bank ith's debt to an outside creditor.

I simulate the returns on the illiquid assets, which is denoted by $R_y = (R_{y1}, \ldots, R_{yN})$ and assume that R_y follows a multivariate lognormal distribution with parameters (μ, Σ).

To focus on the effects of network structure, I assume that parameters other than d_g and a_g are the same across banks. All banks have the same value of total debt $\bar{p}_i$ and initial capital c_0. The marginal return distribution is also the same across the banks with the same μ_i and σ_i. The correlation between two returns, denoted by ρ_{ij}, are the same as well. I set the correlation not perfectly correlated since I assume the assets are in the same class. Moreover, I use the same simulated returns R_y and solve the optimal capital problem using this as the R_y. The parameters that are fixed across the simulated networks are as follows:

Table 5.1: Fixed Parameters

Parameters	Description	Value
N	number of banks	20
M	number of return scenarios	1000
$\bar{p}_i$	total debt	100
c_0	initial capital	0
α	risk level of $\text{CVaR}_\alpha(X)$	0.1
γ	targeted value of systemic risk	20
θ	illiquidity coefficient	2.778×10^{-5}
μ_i	mean of lognormal return	0.03
σ_i	std of lognormal return	0.03
ρ_{ij}	correlation between returns of bank i and j	0.5

γ is set to 20 percent of total debts in the economy, i.e., $\gamma = 0.2\left(\sum_i \bar{p}_i\right)$. Roughly speaking, a regulator targets an average 20 percent loss of aggregate debts in an economy in the worst 5 percent scenarios. The coefficient for the linear price impact function θ is set to $\dfrac{1}{3\left(\sum_i \bar{p}_i\right)}$, which is equal to 2.778×10^{-5}.

5.6.2 Effects of diversification

Figure 5.3 displays how systemic risk changes with the level of diversification d_g. In order to better understand the change in systemic risk, I plot the relative value of $\mathrm{CVaR}_\alpha(-l^*_\omega)$ to the total liabilities in the economy $\sum_i \bar{p}_i$. One can see that the effect of diversification d_g on systemic risk is not monotone. However, it has a decreasing trend. Since the network is simulated, there is a simulation error in measuring the effect of d_g on $\mathrm{CVaR}_\alpha(-l^*_\omega)$. Considering this, beyond $d_g = 11$, it seems that diversification does not affect systemic risk significantly.

Figure 5.4 illustrates how the aggregate optimal capital $\sum_i c^*_i$ changes as the level of diversification d is varied. As in Figure 5.3, I plot the relative value of $\sum_i c^*_i$ to the total liabilities of the economy $\sum_i \bar{p}_i$ on the y axis. As d increases, the aggregate optimal capital decreases. In particular, one sees a substantial decrease of $\sum_i c^*_i$ between $d = 1$ and $d = 3$, and $\sum_i c^*_i$ remains almost the same beyond $d = 3$. It seems that the effect of diversification is significant only at small values of d.

It should be noticed that examples in Figure 5.3 and Figure 5.4 are the numerical results from the same set of simulated returns R. I conducted several experiments with another set of simulated returns R and liabilities L and found similar patterns to those in Figure 5.3 and 5.4.

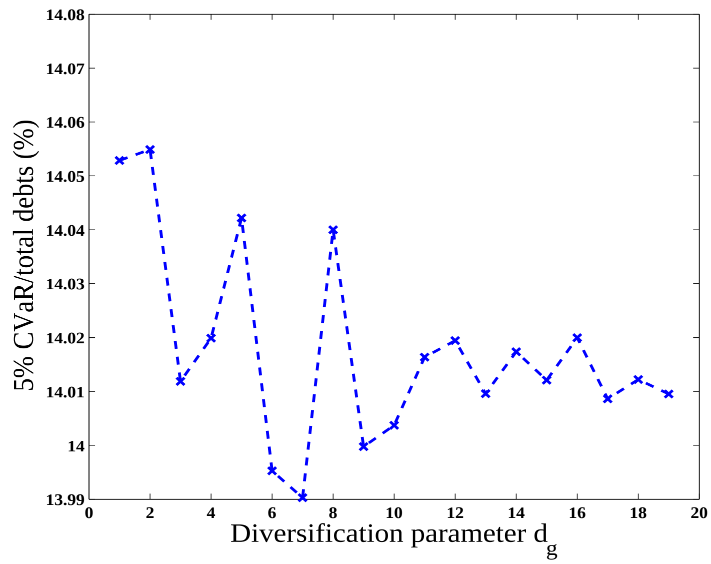


FIGURE 5.3: 5% CVaR vs. degree of diversification with $a_g = 0.2$

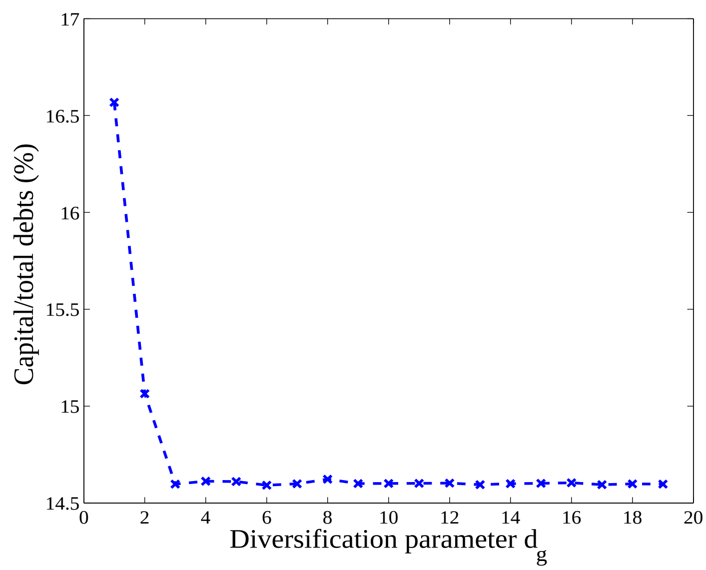


FIGURE 5.4: Capital vs. degree of diversification with $a_g = 0.2$

5.6.3 Single asset core-periphery model

Many recent empirical papers have found that interbank networks exhibit a core-periphery structure, see, e.g., Soramäki, Bech, Arnold, Glass, and Beyeler (2007) for the US interbank market, Bech and Atalay (2010) for the federal fund market, Craig and Von Peter (2014) for the German interbank network and in 't Veld and van Lelyveld (2014) for the Dutch banking system. Castiglionesi and Navarro (2008), Farboodi (2015) and Babus and Hu (2015) provide a network formation model that provides a theoretical explanation for these empirical results of core-periphery structure.

In a core-periphery network, there are a few highly connected banks in the core group and many sparsely connected banks in the periphery group. Moreover, a bank in the core holds a larger amount of assets than a bank at the periphery. Given this, I extend the random network examples in the previous section to ones exhibiting the core-periphery structure as in Elliott et al. (2014), Awiszus and Weber (2015) and Feinstein, Rudloff, and Weber (2015).

I follow a similar procedure to those found in the above papers in order to simulate core-periphery networks. In the network model of core-periphery structure, there are two groups: core and periphery, denoted by $T_1 = C$ and $T_2 = P$, respectively, in the economy. N_C and N_P are the number of core and periphery banks. Similar to the cross-holding matrix of Elliott et al. (2014), the liability matrix L can be described by a block matrix as follows:

$$L = \begin{pmatrix} CC & CP \\ PC & PP \end{pmatrix}. \tag{5.29}$$

where each block represents the liability from one group T_l to the other group T_k.

I simulate a core-periphery network as the following procedure:

1. Simulate IID. Bernoulli random variables G_{ij} with four different probabilities p^{CC}, p^{CP}, p^{PC} and p^{PP} for each block.

2. Fix the total liabilities $\sum_i \bar{p}_i$ in the economy. Allocate the aggregate interbank liabilities $a_g \sum_i \bar{p}_i$ in the fraction of x^{CC}, x^{CP}, x^{PC} and x^{PP} with x^{CC} +

$x^{CP} + x^{PC} + x^{PP} = 1$, e.g., the aggregate interbank liabilities for block CC is $a_g x^{CC} \sum_i \bar{p}_i$.

3. For each block of L, I put the equal amount of the interbank liabilities L_{ij} if $G_{ij} = 1$, e.g., for $(i,j) \in CC$, $Ł_{ij} = \frac{\bar{p}_i}{\left(\sum_{(l,k)\in CC} G_{lk}\right)}$ if $G_{ij} = 1$, otherwise $L_{ij} = 0$

4. Total external liability $(1 - a_g) \sum_i \bar{p}_i$ is allocated to each group in the fraction of $(x^{CC} + x^{CP}, x^{PC} + x^{PP})$ and equally distributed to the banks in the same group, e.g., if bank i is a core bank, $L_{iN+1} = \frac{1}{N_C}(1 - a_g)(x^{CC} + x_{cp}) \sum_i \bar{p}_i$.

Several points about the above simulation procedure are worth noting. Unlike the simulation of a homogenous network, I first simulate the adjacency matrix G for each block using different probabilities $(p^{CC}, p^{CP}, p^{PC}, p^{PP})$. Generally, in a core-periphery network, p^{CC} would be larger than p^{PP}, respectively.

To consider the asset size differences between the core and periphery banks, I fix the total amount of liabilities $\sum_i \bar{p}_i$ in the economy and then the allocation according to $(x^{CC}, x^{CP}, x^{PC}, x^{PP})$. x^{CC} is generally larger than x^{PP} in a core-periphery structure.

Finally, unlike Elliott, Golub, and Jackson (2014) and Awiszus and Weber (2015), I also consider the difference in external liabilities between core and periphery banks since the core banks need to borrow more in order to have larger assets than the periphery banks.

I use the same values for $(p^{CC}, p^{CP}, p^{PC}, p^{PP})$ and $(x^{CC}, x^{CP}, x^{PC}, x^{PP})$ as in Awiszus and Weber (2015) that are based on the German interbank data in Craig and von Peter (2014). In particular,

$$\begin{array}{llll} p^{CC} = 0.66 & p^{CP} = 0.15 & p^{PC} = 0.07 & p^{PP} = 0.001, \\ x^{CC} = 0.35 & x^{CP} = 0.16 & x^{PC} = 0.47 & x^{pp} = 0.02. \end{array}$$

In this section, I assume there is only one illiquid asset in the economy. Hence, the types of illiquid assets of core and periphery banks are the same. I set $N_C = 10$, $N_P = 90$ and $\sum_i \bar{p}_i = 1000$. The other parameters are the same as in Table 5.1.

Based on the simulation procedure and parameter above, Figure 5.5 illustrates the fraction of an individual bank's optimal capital to its total debt, i.e., $\frac{c_i^*}{\bar{p}_i}$. The first 10 banks are core banks and the remaining are periphery banks.

I find that the core and periphery groups affect the optimal capital. The average capital ratios $\frac{c_i^*}{\bar{p}_i}$ of core and periphery are 19.45% and 17.62%, respectively. This would support the idea of the currently updated capital regulations. It is designed to levy additional charges for capital on the systemically important financial institutions (SIFI), which correspond to the core banks in this example. However, as Figure 5.5 illustrated, the optimal capital of many periphery banks is in the range of optimal capital of the core banks. Hence, the core-periphery model with a single asset might not be enough to support the capital surcharge on SIFI.

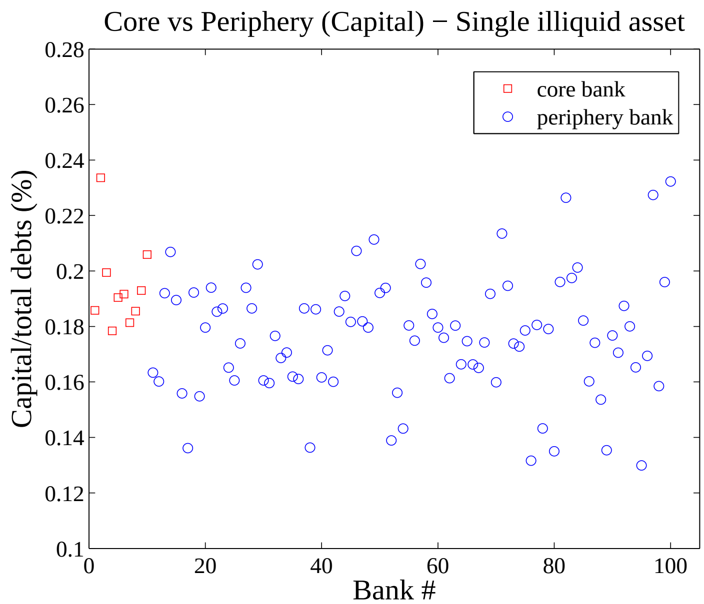


FIGURE 5.5: Optimal capital for individual banks in single asset model

5.6.4 Multi asset core-periphery model

I now consider the core-periphery model with multiple assets. For simplicity, I assume that there are two illiquid assets in the economy. The coefficients for the linear price impact function θ for each asset are set to 0.578×10^{-6} and 2.720×10^{-6}, respectively. The portfolio weights for illiquid assets of core and periphery banks are different. A core bank invests 80% and 20% of its asset into the more and the less illiquid assets respectively. A periphery bank invests in the opposite weight. In the real economy, the systemically important financial institutions are able to invest their assets in ways that the small and local financial institutions cannot. Hence, it is likely that they invest in different types of illiquid assets.

Figure 5.6 plots the fraction of the individual bank's optimal capital to its total debt, i.e., $\frac{c_i^*}{\bar{p}_i}$. As in Figure 5.5, the first 10 banks correspond to core banks and the remaining entities correspond to periphery banks. I see a significant difference in optimal capital between core and periphery banks. The average capital ratios $\frac{c_i^*}{\bar{p}_i}$

of core and periphery are 16.60% and 13.70%, respectively. This implies that the fire sales mechanism of different illiquid asset holdings by core and periphery banks is important in determining the optimal capital of each group. Since the difference between the optimal capitals of core and periphery groups increased, fire sales amplify the network effects on the optimal capital of banks.

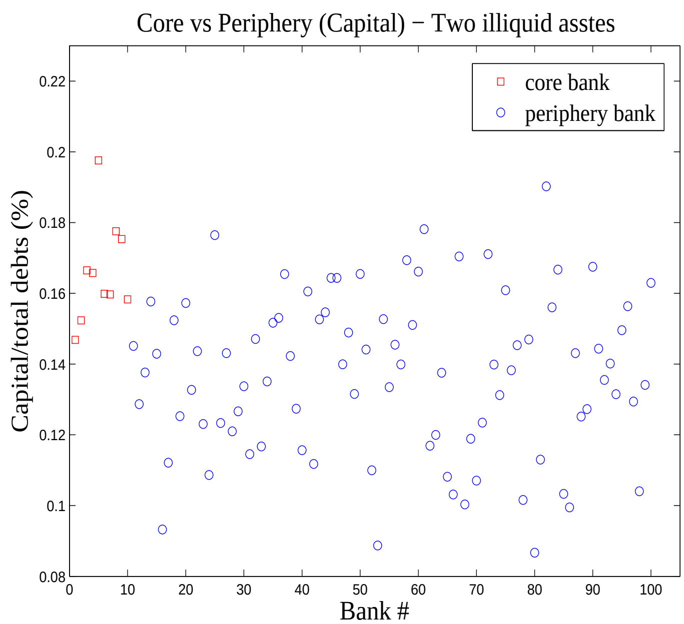


FIGURE 5.6: Optimal capital for individual banks in multiple asset model

One may wonder how significant core-periphery network structure is critical in determining the optimal capital. I simulate the network with different parameters of probability with $(p^{CC}, p^{CP}, p^{PC}, p^{PP}) = (0.1, 0.1, 0.1, 0.1)$ and compare the individual bank's optimal capital. Figure 5.7 plots the optimal capital ratio under the two different parameters of probability. The red squares and blue circles correspond to the capital ratios with $(p^{CC}, p^{CP}, p^{PC}, p^{PP}) = (0.66, 0.15, 0.07, 0.001)$ in Figure 5.6. The green triangles and magenta x's represent the capital ratio with $(p^{CC}, p^{CP}, p^{PC}, p^{PP}) = (0.1, 0.1, 0.1, 0.1)$. For a better understanding of the effect, I plot the histogram of the differences between the optimal capital ratio in Figure 5.8.

Even though the optimal capital of many banks has not significantly changed, the optimal capital of some banks has changed significantly. The largest difference in value is 8.37%. Hence, the network structure can have a substantial effect on the optimal capital of the economy.

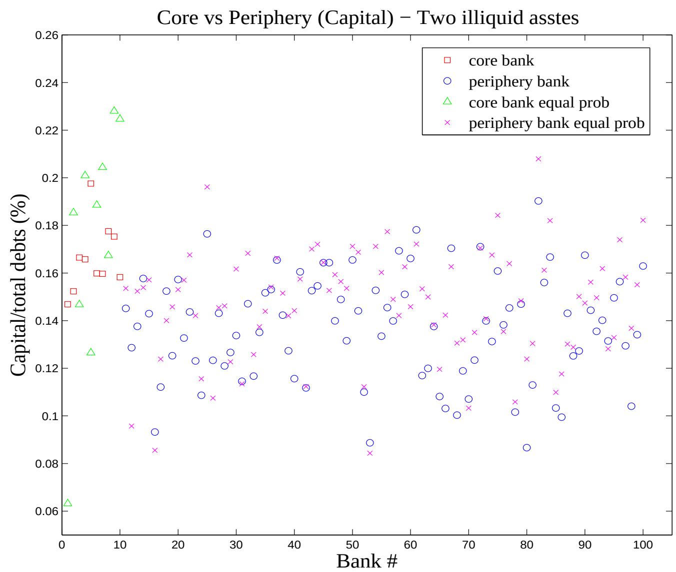


FIGURE 5.7: Optimal capitals with different connectivity probability in core-periphery structure

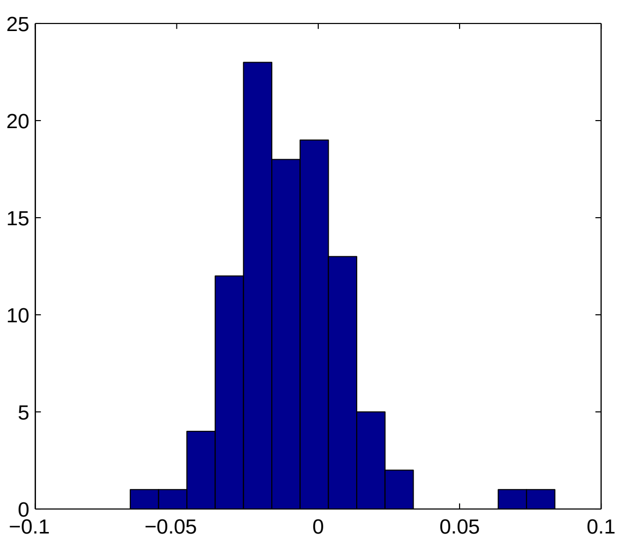


FIGURE 5.8: Histogram of the differences in optimal capital levels for individual banks in Figure 5.7

6

Conclusion and future directions

In this dissertation, I propose a framework to analyze the socially optimal capital requirements of interconnected financial institutions with fire sales externalities. I show that the optimal capital requirements problem without fire sales can be formulated as an LP and with the addition of default costs as a linear MIP. With fire sales, the optimization problems under socially optimal liquidation and prioritized liquidation rules can be formulated as a convex and convex mixed integer problem, respectively.

For models with default costs, due to the computational difficulty of solving a MIP problem to find the optimal capital requirements, I develop an iterative algorithm that produces an approximate solution and lower bounds. I show that the iterative algorithm improves the lower bound and eventually converges to an optimal solution. Moreover, I provide a method of finding an approximate solution and lower bounds based on information relaxations and an gradient penalty.

Numerical experiments show the effectiveness of the approach. The gap between the lower and upper bounds is small in the examples I study, implying that the approximate solution is nearly optimal. In some examples, the iterative algorithm finds the optimal solution only with one iteration.

I further investigate the combined effect of interconnected liabilities and fire sales in the problem of socially optimal capital requirements. The numerical examples show that the effects of interbank network and fire sales externalities can be different across banks and suggest that these phenomena should be considered in setting capital requirements.

There are a number of directions for future research. First, one may consider the optimal capital management of individual banks. Here, I assume that the individual bank holds capital that is equal to the capital requirements. However, individual banks set the capital based on their own private information. The decentralized behavior of banks would not necessarily be socially optimal. Hence, the individual bank's incentive problem should be considered.

Moreover, I can consider the optimal liquidation strategy of individual banks. Here I assume that the liquidation function is given but each bank's liquidation decision can be strategic. Feinstein (2015) studies such strategic liquidation rules where a bank maximizes the value of the total illiquid assets and shows the existence of the equilibrium payments and prices under the liquidation rule. However, it is not clear whether the equilibrium is unique and whether a bank would liquidate to maximize its equity or the value of total illiquid assets. Further investigation is needed in considering strategic liquidation rules.

The method of implementing the optimal capital requirements rule could also be explored. Since the optimal capital in our framework is the solution of an LP or MIP, it cannot be expressed as a simple formula. Hence, the framework may be more appropriate in stress testing. So it is natural to ask how to incentivize an individual bank to hold its capital close to the socially optimal level.

Appendix A

Proofs

A.1 Proposition 3.2.1

Following , $\mathrm{CVaR}_\alpha(Y)$ can be represented as follows:

$$\mathrm{CVaR}_\alpha(Y) \quad = \min_{\nu \in \mathbb{R}} \left\{ \nu + \tfrac{1}{\alpha}\mathbb{E}(-\nu - Y)^+ \right\}. \tag{A.1}$$

We then have

$$\begin{aligned} \mathrm{CVaR}_\alpha\left(-\sum_i l^*_{\omega i}(c)\right) &= \min_{\nu \in \mathbb{R}} \left\{ \nu + \tfrac{1}{\alpha}\mathbb{E}(-\nu + \sum_i l^*_{\omega i}(c))^+ \right\} \\ &= \min_{\nu \in \mathbb{R}} \left\{ \nu + \tfrac{1}{\alpha}\mathbb{E}(-\nu + \sum_i (\bar{p}_i - p^*_{\omega i}(c)))^+ \right\} \end{aligned} \tag{A.2}$$

where $l^*_{\omega i}(c) = (\bar{p}_i - a^*_{\omega i}(c))^+$.

The second equality follows from the fact that the debt payment of a defaulted bank is equal to its asset value, i.e., $(\bar{p}_i - a^*_{\omega i}(c))^+ = \bar{p}_i - p^*_{\omega i}(c)$, where $p^*_{\omega i}(c)$ is the equilibrium payment given c and $R_{\omega i}$. For ease of notation, from now on I will omit the notation c in $p^*_{\omega i}(c)$, i.e., $p^*_{\omega i} = p^*_{\omega i}(c)$.

Let $\nu^{*(EN)}$ denote the optimal solution of (A.2) with $p^*_{\omega i} = p^{*(EN)}_{\omega i}$. By the assumption of IID K samples of R_ω, I can express $\text{CVaR}\left(-\sum_i l_{\omega i}^{*(EN)}(c)\right)$ as follows:

$$\text{CVaR}\left(-\sum_i l_{\omega i}^{*(EN)}(c)\right) = \nu^{*(EN)} + \tfrac{1}{\alpha}\sum_\omega \tfrac{1}{K} u_\omega^{*(EN)} \tag{A.3}$$

where $u_\omega^{*(EN)} = \left(-\nu^{*(EN)} + \sum_i l_{\omega i}^{*(EN)}(c)\right)^+ = \left(-\nu^{*(EN)} + \sum_i(\bar{p}_i - p_{\omega i}^{*(EN)})\right)^+$.

On the other hand, let ν^{LP}, p^{LP} and u^{LP} denote the optimal solution of the linear programming problem (3.6) in Proposition 3.2.1. Then we need to show

$$\nu^{*(EN)} + \frac{1}{\alpha K}\sum_\omega u_\omega^{*(EN)} = \nu^{LP} + \frac{1}{\alpha K}\sum_\omega u_\omega^{LP} \tag{A.4}$$

i) First, I claim

$$\nu^{*(EN)} + \frac{1}{\alpha K}\sum_\omega u_\omega^{*(EN)} \geq \nu^{LP} + \frac{1}{\alpha K}\sum_\omega u_\omega^{LP} \tag{A.5}$$

Clearly, $\nu^{*(EN)}, p^{*(EN)}$ and $u^{*(EN)}$ are feasible in (3.6). This leads to the above inequality (A.15) since ν^{LP}, p^{LP} and u^{LP} is an optimal solution of (3.6).

ii) Next, I show the opposite inequality of (A.15). That is

$$\nu^{*(EN)} + \frac{1}{\alpha K}\sum_\omega u_\omega^{*(EN)} \leq \nu^{LP} + \frac{1}{\alpha K}\sum_\omega u_\omega^{LP} \tag{A.6}$$

Notice that p_ω^{LP} satisfies the limited liability constraints, i.e.,

$$\begin{aligned} p_{\omega i}^{LP} &\leq \bar{p}_i \\ p_{\omega i}^{LP} &\leq R_{\omega i}(c_i + \bar{p}_i - \sum_k \bar{p}_{ki}) + \sum_k \pi_{ki}\, p_{\omega k}^{LP} \end{aligned} \tag{A.7}$$

Hence,

$$-\sum_i(\bar{p}_i - p_{\omega i}^{*(EN)}) \geq -\sum_i(\bar{p}_i - p_{\omega i}^{LP}) \quad \text{for all } \omega, \tag{A.8}$$

since $p^{*(EN)}$ is an optimal solution of the EN linear program (3.2) with the objective function $f(p) = \sum_i p_i$. We then have

$$-l_{\omega i}^{*(EN)} \geq -l_{\omega i}^{LP} \quad \text{for all } \omega, \tag{A.9}$$

where $l_{\omega i}^{LP} = \bar{p}_i - p_{\omega i}^{LP}$.

It then follows

$$\mathrm{CVaR}_\alpha \left(-\sum_i l_{\omega i}^{*(EN)} \right) \leq \mathrm{CVaR}_\alpha \left(-\sum_i l_{\omega i}^{LP} \right) \tag{A.10}$$

by the monotonicity of the coherent risk measure.

On the other hand, we have

$$\begin{aligned} \mathrm{CVaR}_\alpha \left(-\sum_i l_{\omega i}^{LP} \right) &= \min_{\nu \in \mathbb{R}} \left\{ \nu + \tfrac{1}{\alpha K} \sum_\omega \left(-\nu + \sum_i (\bar{p}_i - p_{\omega i}^{LP}) \right)^+ \right\} \\ &\leq \nu^{LP} + \tfrac{1}{\alpha K} \sum_\omega u_\omega^{LP} \end{aligned} \tag{A.11}$$

The inequality follows from the fact that

$$u_\omega^{LP} = \left(-\nu^{LP} + \sum_i \bar{p}_i - p_{\omega i}^{LP}(c) \right)^+ \tag{A.12}$$

since the linear program (3.6) minimizes $\nu + \frac{1}{\alpha K} \sum_\omega u_\omega$ and u_ω^{LP} satisfies the constraints

$$\begin{aligned} & u_\omega^{LP} \geq -\nu + \sum_i (\bar{p}_i - p_{\omega i}^{LP}) \\ & u_\omega^{LP} \geq 0. \end{aligned} \tag{A.13}$$

We then have

$$\mathrm{CVaR}_\alpha \left(-\sum_i l_{\omega i}^{*(EN)} \right) \leq \nu^{LP} + \frac{1}{\alpha K} \sum_\omega u_\omega^{LP}. \tag{A.14}$$

Thus, the inequality (A.6) holds.

Since both (A.15) and (A.6) holds, we then have

$$\nu^{*(EN)} + \frac{1}{\alpha K}\sum_{\omega} u_{\omega}^{*(EN)} = \nu^{LP} + \frac{1}{\alpha K}\sum_{\omega} u_{\omega}^{LP} \tag{A.15}$$

A.2 Proposition 3.3.1

i) First, I prove

$$\sum_i \hat{c}_i^* \geq \sum_i \hat{c}_i^{LP} \tag{A.16}$$

where $\hat{c}^*$ and $\hat{c}^{LP}$ denote the optimal capital in the problem (3.8) and (3.9), respectively. From Proposition 3.2.1, I know that there exists $\hat{\nu}^*$ and $\hat{l}^*$ such that

$$\mathrm{CVaR}_{\alpha}\left(\sum_i (\bar{p}_i - p_{\omega i}^*(\hat{c}^*))^+\right) = \hat{\nu}^* + \frac{1}{\alpha K}\sum_{\omega} \hat{l}_{\omega}^*. \tag{A.17}$$

$$\hat{l}_{\omega}^* = \left(-\hat{\nu}^* + \sum_i (\bar{p}_i - p_{\omega i}^*(\hat{c}))^+\right)^+ \tag{A.18}$$

Moreover, $p_{\omega}^*(\hat{c}^*)$ satisfies $p_{\omega}^*(\hat{c}^*) = \min(\bar{p}_i, R_{\omega i}(\hat{c}_i^* + \bar{p}_i - \sum_k \bar{p}_{ki}) + \sum_k \pi_{ki} p_{\omega k}^*(\hat{c}^*))$

It then follows that $\hat{c}^*$, $\hat{\nu}^*$, $\hat{l}^*$ and $p_{\omega}^*(\hat{c}^*)$ are feasible in (3.9).

ii) Now I prove the opposite of inequality (A.16). Let $\hat{c}^{LP}$, $\hat{\nu}^{LP}$, $\hat{l}^{LP}$ and $\hat{p}^{LP}$ denote the optimal solution of (3.9). From Proposition 3.2.1,

$$\mathrm{CVaR}_{\alpha}\left(\sum_i (\bar{p}_i - p_{\omega i}^*(\hat{c}^{LP}))^+\right) \leq \hat{\nu}^{LP} + \frac{1}{\alpha K}\sum_{\omega} \hat{l}_{\omega}^{LP} \tag{A.19}$$

since $\mathrm{CVaR}_{\alpha}\left(\sum_i (\bar{p}_i - p_{\omega i}^*(\hat{c}^{LP}))^+\right)$ is equal to the minimum value of $\nu + \frac{1}{\alpha K}\sum_{\omega} l_{\omega}$ where ν, l and p satisfy the last three constraints in (3.9). This leads to

$$\mathrm{CVaR}_{\alpha}\left(\sum_i (\bar{p}_i - p_{\omega i}^*(\hat{c}^{LP}))^+\right) \leq \gamma \tag{A.20}$$

since $\hat{\nu}^{LP} + \frac{1}{\alpha K}\sum_{\omega} \hat{l}_{\omega}^{LP} \leq \gamma$. It then follows that $\hat{c}^{LP}$ is feasible in (3.8). Hence,

$$\sum_i \hat{c}_i^* \leq \sum_i \hat{c}_i^{LP} \tag{A.21}$$

Together, inequalities (A.16) and (A.21) prove Proposition 3.3.1.

A.3 Proposition 3.3.2

Let $\tilde{c}^{LP}$, $\tilde{\nu}^{LP}$, $\tilde{u}^{LP}$ and $\tilde{p}^{LP}$ denote the optimal solution of linear program (3.11). Moreover, c^* and $p^{*(EN)}(c^*)$ denote the optimal capital in the problem (3.10) and the EN equilibrium payment given c^*, respectively.

i) First, I prove

$$\sum_i c_i^* + \beta\,\mathrm{CVaR}_\alpha\left(\sum_i(\bar{p}_i - p_{\omega i}^{*(EN)}(c^*))^+\right) \leq \sum_i \tilde{c}_i^{LP} + \beta\,\mathrm{CVaR}_\alpha\left(\sum_i(\bar{p}_i - \tilde{p}_{\omega i}^{LP})^+\right) \tag{A.22}$$

Obviously,

$$\sum_i c_i^* + \beta\,\mathrm{CVaR}_\alpha\left(\sum_i(\bar{p}_i - p_{\omega i}^{*(EN)}(c^*))^+\right) \leq \sum_i \tilde{c}_i^{LP} + \beta\,\mathrm{CVaR}_\alpha\left(\sum_i(\bar{p}_i - p_{\omega i}^{*(EN)}(\tilde{c}^{LP}))^+\right) \tag{A.23}$$

where $p_{\omega i}^{*(EN)}(\tilde{c}^{LP})$ denotes the EN equilibrium payment given $\tilde{c}^{LP}$. From Proposition 3.2.1, I know

$$\tilde{\nu}^{LP} + \frac{1}{\alpha K}\sum_{\omega} \tilde{u}_{\omega}^{LP} = \mathrm{CVaR}_\alpha\left(\sum_i(\bar{p}_i - p_{\omega i}^{*(EN)}(\tilde{c}^{LP}))^+\right). \tag{A.24}$$

since $\nu + \frac{1}{\alpha K} \sum_{\omega} l_\omega$ is in the objective function of (3.11) and it is the minimum value given the optimal value $\tilde{c}^{LP}$. Hence, inequality (A.22) follows.

ii) I now prove the opposite of inequality (A.22).

Let $\nu^{*(EN)}$ and $l^{*(EN)}$ achieve the equivalent to CVaR, i.e.,

$$\begin{aligned} \mathrm{CVaR}_\alpha\left(\sum_i (\bar{p}_i - p^{*(EN)}_{\omega i}(c^*))^+\right) &= \nu^{*(EN)} + \frac{1}{\alpha K}\sum_\omega l^{*(EN)}_\omega \\ l^{*(EN)}_\omega &= \left(-\nu^{*(EN)} + \bar{p}_i - p^{*(EN)}_{\omega i}(c^*)\right)^+ \end{aligned} \tag{A.25}$$

Hence, $c^{*(EN)}$, $p^{*(EN)}(c^*)$, $\nu^{*(EN)}$ and $l^{*(EN)}$ are feasible in the linear program (3.11). This leads to

$$\sum_i c^{*(EN)}_i + \beta(\nu + \frac{1}{\alpha K}\sum_\omega l^{*(EN)}_\omega) \geq \sum_i \tilde{c}^{LP}_i + \beta(\nu + \frac{1}{\alpha K}\sum_\omega \tilde{u}^{LP}_\omega), \tag{A.26}$$

which is equivalent to

$$\begin{aligned} &\sum_i c^*_i + \beta\,\mathrm{CVaR}_\alpha\left(\sum_i (\bar{p}_i - p^{*(EN)}_{\omega i}(c^*))^+\right) \\ &\qquad \geq \sum_i \tilde{c}^{LP}_i + \beta\,\mathrm{CVaR}_\alpha\left(\sum_i (\bar{p}_i - \tilde{p}^{LP}_{\omega i})^+\right) \end{aligned} \tag{A.27}$$

From the inequalities (A.22) and (A.27), I have

$$\begin{aligned} &\sum_i c^*_i + \beta\,\mathrm{CVaR}_\alpha\left(\sum_i (\bar{p}_i - p^{*(EN)}_{\omega i}(c^*))^+\right) \\ &\qquad = \sum_i \tilde{c}^{LP}_i + \beta\,\mathrm{CVaR}_\alpha\left(\sum_i (\bar{p}_i - \tilde{p}^{LP}_{\omega i})^+\right) \end{aligned} \tag{A.28}$$

A.4 Proposition 4.1.1

i) I claim an optimal solution to the following optimization problem with logical constraints is an RV equilibrium payment:

$$\begin{array}{lll} \underset{p}{\text{maximize}} & \sum_i p_i & \\ & p_i \leq \bar{p}_i & \text{if } y_i + \sum_k \pi_{ki}\, p_k \geq \bar{p}_i \\ & p_i \leq (1-\beta)\left(y_i + \sum_k \pi_{ki}\, p_k\right) & \text{otherwise} \end{array} \tag{A.29}$$

I prove by contradiction. Let p^{LC} be an optimal solution to (A.29). Suppose p^{LC} is not a fixed point of Φ^{RV}. This implies there exists i such that $p_i^{LC} < \bar{p}_i$ or $p_i^{LC} < (1-\beta)\left(y_i + \sum_k \pi_{ki}\, p_k^{LC}\right)$. If p_i^{LC} is increased by ϵ, then the asset value of bank j, i.e., $y_j + \sum_k \pi_{kj}\, p_k$ is increased at least by $\pi_{ij}\epsilon$. This means that p_j should be the same or increased with $p_i^{LC} + \epsilon$. This results that $p_i^{LC} + \epsilon$ is feasible as well, which contradicts the fact that p^{LC} is an optimal solution of (A.29). Thus, p^{LC} is a fixed point of Φ^{RV}. Finally, p^{LC} is the largest fixed point of Φ^{RV} since it maximizes $\sum_i p_i$.

Moreover, since $y_i + \sum_k \pi_{ki}\, p_k < \bar{p}_i$ implies $p_i \leq (1-\beta)\left(y_i + \sum_k \pi_{ki}\, p_k\right) \leq \bar{p}_i$, $p_i \leq \bar{p}_i$ holds at any p_i. Hence I can remove the logical constraint $y_i + \sum_k \pi_{ki}\, p_k \geq \bar{p}_i$.

ii) Next I use a mixed integer programming formulation for the logical constraints of whether a bank defaults or not. Let x_i be an integer variable which equals 1 if $y_i + \sum_k \pi_{ki}\, p_k < \bar{p}_i$ and equals 0 otherwise. This can be expressed as

$$-M_2 x_i \leq y_i + \sum_k \pi_{ki}\, p_{\omega i} - \bar{p}_i \leq M_2(1 - x_i) - \epsilon, \tag{A.30}$$

where M_2 is a large enough number such that the largest and smallest value of $y_i + \sum_k \pi_{ki}\, p_{\omega i} - \bar{p}_i$ is bounded by M_2 and $-M_2$ and ϵ is a small enough number such that $y_i + \sum_k \pi_{ki}\, p_{\omega i} - \bar{p}_i \leq \epsilon$ when $y_i + \sum_k \pi_{ki}\, p_{\omega i} - \bar{p}_i$ is negative. In addition, $x_i = 0$ when $y_i + \sum_k \pi_{ki}\, p_k - \bar{p}_i = 0$.

Hence, I can replace the logical constraint in (A.29) with the following constraints:

$$p_i \;\leq\; (1-\beta)\left(y_i + \sum_k \pi_{ki}\, p_k\right) + M_1)(1 - x_i). \tag{A.31}$$

By combining (A.74) and (A.31), I get

$$\begin{aligned}
\underset{p}{\text{maximize}} \quad & \sum_i p_i \\
& p_i \leq \bar{p}_i \\
& p_i \leq (1-\beta)\left(y_i + \sum_k \pi_{ki}\, p_k\right) + M_1(1-x_i) \\
& -M_2 x_i \leq y_i + \sum_k \pi_{ki}\, p_k - \bar{p}_i \\
& M_2(1-x_i) - \epsilon \geq y_i + \sum_k \pi_{ki}\, p_k - \bar{p}_i \\
& x_i \in \{0, 1\}\,.
\end{aligned}$$

Finally, I can remove $-\epsilon$ in the constraint $M_2(1-x_i) - \epsilon \geq y_i + \sum_k \pi_{ki}\, p_k - \bar{p}_i$: since $\sum_i p_i$ is maximized at an optimal solution, x_i always equal 0 when $y_i + \sum_k \pi_{ki}\, p_i - \bar{p}_k = 0$ even when $M_2(1-x_i) \geq y_i + \sum_k \pi_{ki}\, p_k - \bar{p}_i$.

A.5 Proposition 4.2.1

Basically, it follows by the same reasoning as the proof of Proposition 3.3.2. Let $\tilde{c}^{RV}$, $\tilde{p}^{RV}$, $\tilde{x}^{RV}$, $\tilde{u}^{RV}$ and $\tilde{\nu}^{RV}$ denote the optimal solution of a mixed integer programming problem (4.4). Moreover, c^* and $p^{*(RV)}(c^*)$ denote the optimal capital in the problem (3.10) and the RV equilibrium payment given c^*, respectively.

i) First, I prove

$$\sum_i c_i^* + \lambda\,\mathrm{CVaR}_\alpha\left(\sum_i (\bar{p}_i - p_{\omega i}^{*(RV)}(c^*))^+\right)$$

$$\leq \sum_i \tilde{c}_i^{RV} + \lambda\,\mathrm{CVaR}_\alpha\left(\sum_i (\bar{p}_i - \tilde{p}_{\omega i}^{RV})^+\right) \quad \text{(A.32)}$$

Following the same reasoning of Proposition 3.2.1, I know

$$\tilde{\nu}^{RV} + \frac{1}{\alpha K}\sum_\omega \tilde{u}_\omega = \mathrm{CVaR}_\alpha\left(\sum_i (\bar{p}_i - \tilde{p}_{\omega i}^{RV})^+\right) \quad \text{(A.33)}$$

since the term $\nu + \frac{1}{\alpha K}\sum\limits_\omega l_\omega$ is in the objective function of (4.4) and the equilibrium payments are an optimal solution of MIP (4.3). Hence, the inequality (A.32) follows.

ii) I now prove the opposite of the inequality (A.32).

Let $\nu^{*(RV)}$ and $u^{*(RV)}$ are the solutions of (A.1) for $p^{*(RV)}$, i.e.,

$$\begin{aligned} &\mathrm{CVaR}_\alpha\left(\sum_i (\bar{p}_i - p_{\omega i}^{*(RV)}(c^*))^+\right) = \nu^{*(RV)} + \tfrac{1}{\alpha K}\sum_\omega u_\omega^{*(RV)} \\ &u_\omega^{*(RV)} = \left(-\nu^{*(RV)} + \bar{p}_i - p_{\omega i}^{*(RV)}(c_i)^*\right)^+ \end{aligned} \quad \text{(A.34)}$$

Hence, $c^{*(RV)}$, $p^{*(RV)}$, $\nu^{*(RV)}$ and $u^{*(RV)}$ are feasible in the MIP (4.4). This leads to

$$\sum_i c_i^{*(RV)} + \lambda(\nu + \frac{1}{\alpha K}\sum_\omega u_\omega^{*(RV)}) \geq \sum_i \tilde{c}_i^{RV} + \lambda(\nu + \frac{1}{\alpha K}\sum_\omega \tilde{u}_\omega^{RV}), \quad \text{(A.35)}$$

which is equivalent to

$$\sum_i c_i^* + \lambda\,\mathrm{CVaR}_\alpha\left(\sum_i (\bar{p}_i - p_{\omega i}^{*(RV)}(c^*))^+\right)$$

$$\geq \sum_i \tilde{c}_i^{RV} + \lambda\,\mathrm{CVaR}_\alpha\left(\sum_i (\bar{p}_i - \tilde{p}_{\omega i}^{RV})^+\right) \quad \text{(A.36)}$$

From the inequalities (A.32) and (A.36), I have

$$\sum_i c_i^* + \lambda \,\mathrm{CVaR}_\alpha \left(\sum_i (\bar{p}_i - p_{\omega i}^{*(RV)}(c^*))^+ \right)$$
$$= \sum_i \tilde{c}_i^{RV} + \lambda \,\mathrm{CVaR}_\alpha \left(\sum_i (\bar{p}_i - \tilde{p}_{\omega i}^{RV})^+ \right). \quad (A.37)$$

A.6 Lemma 4.3.1

By (4.5), for any $\mathbb{Q} \in \mathscr{Q}_\alpha$, $\mathrm{CVaR}(-l_\omega^{*(RV)}(c)) \geq \mathbb{E}_\mathbb{Q}[l_\omega^{*(RV)}(c)]$. This leads to the inequality (4.9).

Now I show that if $V = \widetilde{V}_\mathbb{Q}$ then $\mathrm{CVaR}_\alpha \left(-l_\omega^{*(RV)}(c^{RV}) \right) = \mathbb{E}_\mathbb{Q}[l_\omega^{*(RV)}(c^{RV})]$.

By the definition of $\mathbb{Q}^*$,

$$\begin{aligned} V &= \sum_i c_i^{RV} + \lambda \,\mathrm{CVaR}_\alpha \left(-l^{*(RV)}(c^{RV}) \right) \\ &= \sum_i c_i^{RV} + \lambda \,\mathbb{E}_{\mathbb{Q}^*}[l_\omega^{*(RV)}(c^{RV})] \end{aligned} \quad (A.38)$$

Let $\tilde{c}$ denote the optimal solution for $\widetilde{V}_\mathbb{Q}$, i.e.,

$$\widetilde{V}_\mathbb{Q} = \sum_i \tilde{c}_i + \lambda \,\mathbb{E}_\mathbb{Q}[l_\omega^{*(RV)}(\tilde{c})]. \quad (A.39)$$

Since $\tilde{c}$ is optimal to (4.8), I have

$$\sum_i c_i^{RV} + \lambda \,\mathbb{E}_\mathbb{Q}[l_\omega^{*(RV)}(c^{RV})] \geq \sum_i \tilde{c}_i + \lambda \,\mathbb{E}_\mathbb{Q}[l_\omega^{*(RV)}(\tilde{c})] = \widetilde{V}_\mathbb{Q}. \quad (A.40)$$

On the other hand, $\mathbb{Q}^*$ achieve the largest $\mathbb{E}_\mathbb{Q}[l_\omega^{*(RV)}(c^{RV})]$ among all $\mathbb{Q}$. This leads to

$$V = \sum_i c_i^{RV} + \lambda \,\mathbb{E}_{\mathbb{Q}^*}[l_\omega^{*(RV)}(c^{RV})] \geq \sum_i c_i^{RV} + \lambda \,\mathbb{E}_\mathbb{Q}[l_\omega^{*(RV)}(c^{RV})]. \quad (A.41)$$

$V = \widetilde{V}_{\mathbb{Q}}$ implies that the right and the left hand side of (A.41) are equal. Hence, I have

$$\sum_i c_i^{RV} + \lambda\, \mathbb{E}_{\mathbb{Q}^*}[l_\omega^{*(RV)}(c^{RV})] \;\; = \sum_i c_i^{RV} + \lambda\, \mathbb{E}_{\mathbb{Q}}[l_\omega^{*(RV)}(c^{RV})]. \tag{A.42}$$

This leads to $\mathbb{E}_{\mathbb{Q}^*}[l_\omega^{*(RV)}(c^{RV})] = \mathbb{E}_{\mathbb{Q}}[l_\omega^{*(RV)}(c^{RV})]$.

Thus, I have $\mathrm{CVaR}_\alpha\left(-l^{*(RV)}(c^{RV})\right) = \mathbb{E}_{\mathbb{Q}}[l_\omega^{*(RV)}(c^{RV})]$.

A.7 Proposition 4.3.1

Since Ω_k strictly increases and is a subset of Ω, the iteration is certain to terminate. The question is if $\Omega^{(k)} \subset \Omega_k$, whether it obtains the optimal solution or not.

From the same reasoning of Proposition 4.2.1, the optimal solution p^{Ω_k} of the restricted problem (4.13) is the equilibrium payment given R_ω and c^{Ω_k}. Hence, the aggregate loss $l_\omega^{*(RV)}(c^{\Omega_k}) = \sum_i (\bar{p}_i - p_{\omega i}^{\Omega_k})^+$ is the loss of the equilibrium payments given c^{Ω_k}.

$\Omega^{(k)}$ takes the αK worst scenarios given c^{Ω_k} from Ω. Hence, if αK scenarios of the $\Omega^{(k)}$ belongs to the restricted set Ω_k, these are the worst αK scenarios of the restricted set Ω_k as well.

This implies that the CVaR of the upper and the lower bound, i.e., $\mathcal{V}(\Omega_k)$ and $V(c^{\Omega_k})$, respectively, are equal since the αK worst scenarios of lower and upper bounds are the same.

Therefore, if $\Omega^{(k)} \subset \Omega_k$, the upper and lower bounds are equal.

A.8 Lemma 4.4.1

Similar to Rogers and Veraart (2013), I first show that the mapping ϕ^{RV2} has the following two properties:

i) First, ϕ^{RV2} is bounded above by $\bar{p}$: for any p we have $\phi^{RV2}(p) \leq \bar{p}$.

This property follows directly from the definition of ϕ^{RV2}.

ii) Second ϕ^{RV2} is monotone: if $\tilde{p} \leq p$, then $\phi^{RV2}(\tilde{p}) \leq \phi^{RV2}(p)$.

To prove monotonicity, we let x_i and $\tilde{x_i}$ denote binary variables with p and $\tilde{p}$, respectively, which satisfies Definition 4.4.1. Notice that x_i decreases in in any p_k. And x_i increases in x_j for $j \neq i$. Hence, $\tilde{x_i} \geq x_i$, i.e., $\tilde{x} \geq x$.

Now we let $\mathfrak{D} = \{i : \bar{p}_i > (1 - \sum_j \beta_{ij} x_j)(y_i + \sum_k \pi_{ki}\, p_k)\}$ and $\tilde{\mathfrak{D}} = \{i : \bar{p}_i > (1 - \sum_j \beta_{ij} \tilde{x_j})(y_i + \sum_k \pi_{ki}\, \tilde{p_k})\}$. Clearly, $\mathfrak{D} \subset \tilde{\mathfrak{D}}$ since $\tilde{p} \leq p$ and $\tilde{x} \geq x$. Hence for $i \in \mathfrak{D}$, $\phi^{RV2}(\tilde{p})_i = (1 - \sum_j \beta_{ij} \tilde{x_j})(y_i + \sum_k \pi_{ki}\, \tilde{p_k}) \leq (1 - \sum_j \beta_{ij} x_j)(y_i + \sum_k \pi_{ki}\, p_k) = \phi^{RV2}(p)_i$. And for $i \in \tilde{\mathfrak{D}} \backslash \mathfrak{D}$, $\phi^{RV2}(\tilde{p})_i = (1 - \sum_j \beta_{ij} \tilde{x_j})(y_i + \sum_k \pi_{ki}\, \tilde{p_k}) < \bar{p}_i = \phi^{RV2}(p)_i$.

Hence with above properties, we follow the same procedure as in the proof of Theorem 3.1 of Rogers and Veraart (2013). That is, we consider a sequence of vectors $p^{(n)}$, $n = 0, 1, \ldots$, defined recursively by $p^{(0)} = \bar{p}$,

$$p^{(n+1)} = \phi^{RV2}(p^{(n)}) \tag{A.43}$$

From i), we know $p^{(1)} \leq p^{(0)} = \bar{p}$. And

$$p^{(n+1)} \leq p^{(n)} \tag{A.44}$$

by induction. Since $p^{(n)}$ are nonnegative, there is a monotone limit $p^* :=\downarrow lim_{n\to\infty} p^{(n)}$. Also the set $D_n = \{i : p_i^{(n)} < \bar{p}_i\}$ is nondecreasing in n and will converges to the set $D^* = \{i : p_i^* < \bar{p}_i\}$. Moreover, ϕ^{RV2} is continuous from above since $x_i = 0$ if $(1 - \sum_{j \neq i} \beta_{ij} x_j)(e_i + \sum_k \pi_{ki}\, p_k) >= \bar{p}_i$ and $x_i = 0$ otherwise. Hence, p^* satisfies

$$p^* = \phi^{RV2}(p^*), \tag{A.45}$$

that is, p^* is a payment equilibrium.

p^* is the largest payment equilibrium since p^* is the convergence of monotone decreasing vector $p^{(n)}$ with $p^{(0)} = \bar{p}$ and any clearing vector is bounded above by $\bar{p}$.

We can also construct the least payment equilibrium p_* with $p^{(0)} = 0$ following the proof of Theorem 3.1 of Rogers and Veraart (2013).

A.9 Proposition 4.4.1

First, following proof of Proposition 4.1.1, I can derive the following Big-M formulation:

$$\begin{aligned} \underset{p}{\text{maximize}} \quad & \sum_i p_i \\ \text{subject to} \quad & p_i \leq \bar{p}_i \\ & p_i \leq \left(1 - \sum_j \beta_{ij} x_j\right)\left(y_i + \sum_k \pi_{ki}\, p_k\right) \\ & -M_2 x_i \leq \left(1 - \sum_{j \neq i} \beta_{ij} x_j\right)\left(y_i + \sum_k \pi_{ki}\, p_k\right) - \bar{p}_i \\ & M_2(1 - x_i) \geq \left(1 - \sum_{j \neq i} \beta_{ij} x_j\right)\left(y_i + \sum_k \pi_{ki}\, p_k\right) - \bar{p}_i \\ & x_j \in \{0, 1\}\,. \end{aligned} \tag{A.46}$$

And I need to replace the product terms of integer variable x_i and the continuous variable, $y_i + \sum_k \pi_{ki}\, p_k$. I use the basic technique of reformulating a product term of integer variables and continuous variables. w_{ij} can be represented by the constraints:

$$\begin{aligned} & w_{ij} \leq \bar{a}_i x_j \\ & w_{ij} \leq y_i + \sum_k \pi_{ki}\, p_k \\ & w_{ij} \geq \bar{a}_i (x_j - 1) + y_i + \sum_k \pi_{ki}\, p_k \\ & w_{ij} \geq 0 \end{aligned} \tag{A.47}$$

where $\bar{a}_i \geq y_i + \sum_k \pi_{ki}\, p_k$.

According to (A.47), w_{ij} equals $y_i + \sum_k \pi_{ki} p_k$ if $x_j = 1$ and zero otherwise. Using this technique, I get the following MIP formulation in Proposition 4.4.1.

A.10 Lemma 4.5.1

Note that given the same liability matrix L and the realized asset y, $p^{*(RV)} \leq p^{*(EN)}$ since $\Phi^{RV}(p) \leq \Phi^{EN}(p)$ and $p^{*(RV)}$ and $p^{*(EN)}$ are fixed points of $\Phi^{RV}(p)$ and $\Phi^{EN}(p)$, respectively.

Hence,

$$\mathbb{E}V_\omega(c^*) \geq \mathbb{E}\hat{V}_\omega(c^*) \geq \mathbb{E}\hat{V}_\omega(c^{EN}). \tag{A.48}$$

A.11 Lemma 4.5.2

This directly follows from Lemma 2.1 in Brown and Smith (2014).

$$\mathbb{E}V_\omega(c^*) \geq \mathbb{E}[V_\omega(c^*) + z_\omega(c^*)] \geq \mathbb{E}[\min_{c \geq 0} V_\omega(c) + z_\omega(c)]. \tag{A.49}$$

The first inequality holds since $\mathbb{E}[z_\omega(c)] \leq 0$ for any c and the second inequality follows from the fact that the information constraints are relaxed in $\mathbb{E}[\min_{c \geq 0} V_\omega(c) + z_\omega(c)]$.

A.12 Lemma 4.5.3

The right hand side problem in (A.50) can be written as $\mathbb{E}[\min_{c \geq 0} V_\omega(c) + \mathbb{E}\hat{V}_\omega(c) - \hat{V}_\omega(c)]$. It then follows

$$\mathbb{E}[\min_{c \geq 0} V_\omega(c) + \mathbb{E}\hat{V}_\omega(c) - \hat{V}_\omega(c)] \geq \mathbb{E}\hat{V}_\omega(c^{EN}) \tag{A.50}$$

since $V_\omega(c) - \hat{V}_\omega(c) \geq 0$ and $\min_{c \geq 0} \mathbb{E}\hat{V}_\omega(c) = \mathbb{E}\hat{V}_\omega(c^{EN})$.

A.13 Lemma 4.5.4

Note that $\mathbb{E}[z_{\nabla\omega}(c^{EN})] = 0$ since c^{EN} is an optimal solution of $\min_{c\geq 0} \mathbb{E}\hat{V}_\omega(c)$. Hence, $\mathbb{E}[\min_{c\geq 0} V_\omega(c) + z_{\nabla\omega}(c)]$ is a lower bound to $\mathbb{E}V_\omega(c^*)$.

The inner problem can be written as:

$$\mathbb{E}[\min_{c\geq 0} V_\omega(c) + [\nabla_c \mathbb{E}\hat{V}_\omega(c^{EN}) - \nabla_c \hat{V}_\omega(c^{EN})]^\top (c - c^{EN}) + \mathbb{E}\hat{V}_\omega(c^{EN}) - \hat{V}_\omega(c^{EN})]. \tag{A.51}$$

I can remove $\mathbb{E}\hat{V}_\omega(c^{EN}) - \hat{V}_\omega(c^{EN})$ since $\mathbb{E}[\mathbb{E}\hat{V}_\omega(c^{EN}) - \hat{V}_\omega(c^{EN})] = 0$

Then,

$$\begin{aligned}
&\mathbb{E}[\min_{c\geq 0} V_\omega(c) + \nabla_c \mathbb{E}\hat{V}_\omega(c^{EN})^\top (c - c^{EN}) - \nabla_c \hat{V}_\omega(c^{EN})^\top (c - c^{EN})] \\
&\geq \mathbb{E}[\min_{c\geq 0} V_\omega(c) - \nabla_c \hat{V}_\omega(c^{EN})^\top (c - c^{EN})] \\
&\geq \mathbb{E}[\min_{c\geq 0} \hat{V}_\omega(c) - \nabla_c \hat{V}_\omega(c^{EN})^\top (c - c^{EN})] \\
&= \mathbb{E}\hat{V}_\omega(c^{EN})
\end{aligned} \tag{A.52}$$

The first inequality follow from the fact that $\nabla_c \mathbb{E}\hat{V}_\omega(c^{EN})^\top (c - c^{EN}) = 0$ since c^{EN} is an optimal solution of $\min_{c\geq 0} \mathbb{E}\hat{V}_\omega(c)$.

The second inequality holds since $V_\omega(c) \geq \hat{V}_\omega(c)$.

The last inequality follows from the fact that $\min_{c\geq 0} \hat{V}_\omega(c) - \nabla_c \hat{V}_\omega(c^{EN})^\top (c - c^{EN})$ is a convex optimization problem and c^{EN} is an optimal solution of $\min_{c\geq 0} \hat{V}_\omega(c) - \nabla_c \hat{V}_\omega(c^{EN})^\top (c - c^{EN})$ as well since the gradient of $\hat{V}_\omega(c) - \nabla_c \hat{V}_\omega(c^{EN})^\top (c - c^{EN})$ $(= \nabla_c \hat{V}_\omega(c) - \nabla_c \hat{V}_\omega(c^{EN}))$ is zero at $c = c^{EN}$.

A.14 Lemma 5.1.1

First, notice that $q^{FS} = f\left(\sum_i \min\left(\frac{(\bar{p}_i - c_i - \sum_k \pi_{ki}\, p_k^{FS})^+}{q^{FS}}, y_i\right)\right)$ since

$f\left(\sum_i \min\left(\frac{(\bar{p}_i - c_i - \sum_k \pi_{ki}\, p_k)^+}{q}, y_i\right)\right)$ and the objective function $qy_i + c_i + \sum_k \pi_{ki}\, p_k$ are increasing in q.

Second, suppose there exists an s such that $p_s^{FS} < \bar{p}_s$ and $p_s^{FS} < q^{FS} y_s + \gamma_s + \sum_k \pi_{ks}\, p_k^{FS}$. I can increase p_s^{FS} to $p_s^{FS} + \epsilon$ such that the third inequality constraints for the other $j \neq s$ still hold, i.e., $p_j^{FS} \leq q^{FS} y_j + \gamma_j + \sum_{k \neq s} \pi_{kj}\, p_k^{FS} + \pi_{sj}\,(p_s^{FS} + \epsilon)$.

Moreover, with $p_s^{FS} + \epsilon$ and p_k^{FS}, $k \neq s$, the first inequality constraint, i.e., $q^{FS} \leq f\left(\sum_i \min\left(\frac{(\bar{p}_i - c_i - \sum_k \pi_{ki}\, p_k^{FS})^+}{q^{FS}}, y_i\right)\right)$, still holds since the right hand side of the inequality increases as p_k^{FS} gets larger. Hence $p_s^{FS} + \epsilon$, p_k^{FS} and q^{FS} are feasible in (5.2) and its objective value is increased by ϵ.

This contradicts the claims that p^{FS} and q^{FS} are optimal. This implies that $p_i^{FS} = \bar{p}_i$ or $p_i^{FS} = q^{FS} y_i + c_i + \sum_k \pi_{ki}\, p_k^{FS}$.

It then follows that $p_i^{FS} = \min\left(q^{FS} y_i + c_i + \sum_k \pi_{ki}\, p_k^{FS}, \bar{p}_i\right)$.

In sum, $q^{FS} = f\left(\sum_i \min\left(\frac{(\bar{p}_i - c_i - \sum_k \pi_{ki}\, p_k^{FS})^+}{q^{FS}}, y_i\right)\right)$ and

$p_i^{FS} = \min\left(q^{FS} y_i + c_i + \sum_k \pi_{ki}\, p_k^{FS}, \bar{p}_i\right)$.

A.15 Proposition 5.1.1

i) If bank i does not default at the equilibrium, then

$$\bar{p}_i \leq q^{*(FS)} y_i + c_i + \sum_k \pi_{ki}\, p_k^{*(FS)} \qquad \text{(A.53)}$$

since the asset value of non-default bank, $q^{*(FS)}y_i + c_i + \sum_k \pi_{ki}\, p_k^{*(FS)}$, is larger than the total debt $\bar{p}_i$. $y_i \geq 0$. Consequently,

$$\frac{(\bar{p}_i - c_i - \sum_k \pi_{ki}\, p_k^{*(FS)})^+}{q^{*(FS)}} \leq y_i. \tag{A.54}$$

Hence, (5.3) holds for a non-default bank i since a non-defaulted bank pays debts in full, i.e., $p_i^{*(FS)} = \bar{p}_i$.

ii) If bank i defaults at the equilibrium, the opposite inequality (A.54) holds. That is,

$$\frac{(\bar{p}_i - c_i - \sum_k \pi_{ki}\, p_k^{*(FS)})^+}{q^{*(FS)}} > y_i. \tag{A.55}$$

Consequently,

$$\min\left(\frac{(\bar{p}_i - c_i - \sum_k \pi_{ki}\, p_k^{*(FS)})^+}{q^{*(FS)}}, y_i\right) = y_i. \tag{A.56}$$

The equilibrium payment is

$$p_i^{*(FS)} = q^{*(FS)}y_i + c_i + \sum_k \pi_{ki}\, p_k^{*(FS)} \tag{A.57}$$

This yields

$$\frac{p_i^{*(FS)} - c_i - \sum_k \pi_{ki}\, p_k^{*(FS)}}{q^{*(FS)}} = y_i. \tag{A.58}$$

$y_i \geq 0$. Hence, (5.3) holds for a defaulted bank i as well.

A.16 Proposition 5.1.2

I show that (5.2) and (5.6) are equivalent. Let $z_i = (p_i - c_i - \sum_k \pi_{ki} p_k)^+$. $z_i = (p_i - c_i - \sum_k \pi_{ki} p_k)^+$ is equivalent to $z_i \geq p_i - c_i - \sum_k \pi_{ki} p_k$ and $z_i \geq 0$ since the objective function of (5.2) is increasing in q and $q \leq f\left(\frac{\sum_i z_i}{q}\right)$. Recall that $f(x)$ is a decreasing function. Hence, I can replace $z_i = (p_i - c_i - \sum_k \pi_{ki} p_k)^+$ with $p_i \leq z_i + c_i + \sum_k \pi_{ki} p_k$ and $z_i \geq 0$ in (5.2).

Since $z_i \leq y_i q$ from (5.4), $p_i \leq q y_i + c_i + \sum_k \pi_{ki} p_k$ can be replaced with $p_i \leq z_i + c_i + \sum_k \pi_{ki} p_k$ and $z_i \leq y_i q$.

Finally, from (5.3) and (5.5), I can replace $q \leq f\left(\sum_i \min\left(\frac{(\bar{p}_i - c_i - \sum_k \pi_{ki} p_k)^+}{q}, y_i\right)\right)$ with $q \leq f\left(\frac{\sum_i z_i}{q}\right)$ in (5.2). It then follows $\sum_i z_i \leq \; q f^{-1}(q)$ since $f(x)$ is non-decreasing.

A.17 Proposition 5.1.3

With this equality, I show the following property of the equilibrium price $q^{*(FS)}$ and the nominal asset sales z^*.

Lemma A.17.1. *The equilibrium price $q^{*(FS)}$ and nominal asset sales z^* satisfies the following equation:*

$$q^{*(FS)} = \frac{q_0 + \sqrt{q_0^2 - 4\theta \sum_i z_i^*}}{2} \tag{A.59}$$

Proof. Note that since $q^{*(FS)}$ and $p^{*(FS)}$ is a fixed point of the operator ϕ^{FS},

$$\begin{aligned} q^{*(FS)} &= f\left(\sum_i \min\left(\frac{(\bar{p}_i - c_i - \sum_k \pi_{ki} p_k^{*(FS)})^+}{q^{*(FS)}}, y_i\right)\right) \\ &= f\left(\sum_i \frac{(p_i^{*(FS)} - c_i - \sum_k \pi_{ki} p_k^{*(FS)})^+}{q^{*(FS)}}\right) \\ &= f\left(\sum_i \frac{z_i^*}{q^{*(FS)}}\right) \end{aligned} \tag{A.60}$$

The second equality follows from Proposition 5.1.1.

Since I assume $f(x) = q_0 - \theta x$, I have

$$q^{*(FS)} = q_0 - \theta \sum_i \frac{z_i^*}{q^{*(FS)}} \tag{A.61}$$

Hence, $q^{*(FS)}$ is a solution of the following quadratic equation.

$$q^2 - q_0 q^2 - \theta \sum_i z_i^* = 0 \tag{A.62}$$

The roots of the quadratic equation (A.62) are

$$q = \frac{q_0 \pm \sqrt{q_0^2 - 4\theta \sum_i z_i^*}}{2} \tag{A.63}$$

Note that the smaller root of the quadratic equation (A.62) cannot be the equilibrium price $q^{*(FS)}$. To see this, I claim that

$$q^{*(FS)} > \frac{q_0}{2}, \tag{A.64}$$

i.e., the equilibrium price $q^{*(FS)}$ is greater than $\frac{q_0}{2}$. Let y_{tot} denote the total number of shares of illiquid assets in the economy, which represents the maximum number of shares of illiquid assets that can be liquidated in the economy. Generally, $y_{tot} = \sum_i y_i$. Hence the domain of the inverse demand function $f(x)$ is $[0, y_{tot}]$. By the assumption (iii) of the inverse demand function $f(x)$ in AFM (2015), $xf(x) = x(q_0 - \theta x)$ is increasing on $[0, y_{tot}]$. It then follows that $q_0 - 2\theta x > 0$ and therefore $\theta < \frac{q_0}{2y_{tot}}$. Using this inequality, I have

$$f(y_{tot}) = q_0 - 2\theta y_{tot} > \frac{q_0}{2}, \tag{A.65}$$

i.e., the worst case price where all assets are liquidated is greater than $\frac{q_0}{2}$. Thus, the equilibrium price $q^{*(FS)}$ is greater than $\frac{q_0}{2}$ as well. This leads to (A.59). □

Now from (5.3) and (A.59), I can express the payment equilibrium $p^{*(FS)}$ with $f(x) = q_0 - \theta x$ as a fixed point of the following operator

$$\widetilde{\phi}^{FS}(p) = \min\left(y_i \left(\frac{q_0 + \sqrt{q_0^2 - 4\theta \sum_i z_i(p)}}{2} \right) + c_i + \sum_k \pi_{ki}\, p_k, \bar{p}_i \right) \tag{A.66}$$

where $z_i(p) = (p_i - c_i - \sum_k \pi_{ki}\, p_k)^+$. Following the same reasoning as in Lemma 5.1.1, the optimal solution of the following optimization problem is the equilibrium payment:

$$\begin{aligned}
\underset{p,z}{\text{maximize}} \quad & \sum_i \left(y_i \left(\frac{q_0 + \sqrt{q_0^2 - 4\theta \sum_i z_i}}{2} \right) + c_i + \textstyle\sum_k \pi_{ki}\, p_k \right) \\
\text{subject to} \quad & p_i \le \bar{p}_i \\
& p_i \le y_i \left(\frac{q_0 + \sqrt{q_0^2 - 4\theta \sum_i z_i}}{2} \right) + c_i + \sum_k \pi_{ki}\, p_k \\
& z_i = (p_i - c_i - \textstyle\sum_k \pi_{ki}\, p_k)^+ \\
& p \ge 0.
\end{aligned} \tag{A.67}$$

Since the objective function is decreasing in z, the equality constraint, $z_i = (p_i - c_i - \sum_k \pi_{ki}\, p_k)^+$, is equivalent to $z_i \ge p_i - c_i - \sum_k \pi_{ki}\, p_k$ and $z_i \ge 0$. Moreover, since $p_i - c_i - \sum_k \pi_{ki}\, p_k \le y_i \left(\frac{q_0 + \sqrt{q_0^2 - 4\theta \sum_i z_i}}{2} \right)$ I can express the second inequality constraint of (A.67) as

$$z_i \le \; y_i \left(\frac{q_0 + \sqrt{q_0^2 - 4\theta \sum_i z_i}}{2} \right). \tag{A.68}$$

Therefore, the optimization problem (A.67) is equivalent to the optimization problem (5.9) in Proposition 5.1.3.

Moreover, $\frac{q_0 + \sqrt{q_0^2 - 4\theta \sum_i z_i}}{2}$ is concave in z. It then follows that the objective function of (5.9) is concave in p and z and the constraints define a convex set in p and z. Thus, (5.9) is a convex optimization problem.

A.18 Proposition 5.2.2

Following the same reasoning of Lemma 5.1.1, (p^*, q^*) are optimal to the following optimization problem:

$$\begin{aligned} \underset{p,q,z}{\text{maximize}} \quad & \sum_i \left(\sum_m y_{im} q_m + c_i + \sum_k \pi_{ki}\, p_k \right) \\ \text{subject to} \quad & p_i \le \bar{p}_i \\ & p_i \le \sum_m y_{im} q_m + c_i + \sum_k \pi_{ki}\, p_k \\ & q_m \le f_m \left(\sum_i \min \left(\frac{s_{im}(p,q)}{q_m}, y_{im} \right) \right) \\ & p, q, z \ge 0. \end{aligned} \tag{A.69}$$

Moreover, following the same reasoning of Proposition 5.1.1,

$$(p_i^* - c_i - \sum\nolimits_k \pi_{ki}\, p_k^*)^+ = \min \left(\sum_m q_m y_{im}, (\bar{p}_i - c_i - \sum\nolimits_k \pi_{ki}\, p_k^*)^+ \right). \tag{A.70}$$

Let $z_{im} = \min\left(q_m y_{im}, s_{im}(p,q)\right)$. By (A.70) and the minimal liquidation condition (5.11), at the optimal p^*, q^* and z^*, I have

$$\sum_m z_{im}^* = (p_i^* - c_i - \sum_k \pi_{ki}\, p_k^*)^+. \tag{A.71}$$

As in the proof of Proposition 5.1.2, (A.71) is equivalent to $\sum\limits_m z_{im}^* \ge p_i^* - c_i - \sum\limits_k \pi_{ki}\, p_k^*$ and $\sum\limits_m z_{im}^* \ge 0$ since the objective function (A.69) is increasing in q_m and $q_m \le f_m\left(\frac{\sum_m z_{im}}{q_m}\right)$, where $f_m(x)$ is a decreasing function. Since the liquidation amount is nonnegative, $z_{im}^* \ge 0$ for all i and m.

Hence, $q_m \le f_m \left(\sum\limits_i \min \left(\frac{s_{im}(p,q)}{q_m}, y_{im} \right) \right)$ can be replaced with

$$\begin{aligned} & q_m \le f_m \left(\left(\frac{\sum_i z_{im}}{q_m} \right) \right) \\ & \sum_m z_{im} \ge p_i - c_i - \sum_k \pi_{ki}\, p_k \\ & z_{im} \ge 0. \end{aligned} \tag{A.72}$$

Moreover, $z_{im} \leq y_{im} q_m$ and $z_{im} \leq \; s_{im}(p,q)$ since $z_{im} = \min\left(q_m y_{im}, s_{im}(p,q)\right)$. $p_i \leq \sum_m y_{im} q_m + c_i + \sum_k \pi_{ki}\, p_k$ is redundant since $\sum_m z_{im} \geq p_i - c_i - \sum_k \pi_{ki}\, p_k$ and $z_{im} \leq y_{im} q_m$ for all i and m.

Finally, as in the Proposition 5.1.3, $q_m \leq f_m\left(\left(\frac{\sum_i z_{im}}{q_m}\right)\right)$ can be replace with $q_m = Q_m(z_m)$, where $Q_m(z_m) = \dfrac{q_{m0} + \sqrt{q_{m0}^2 - 4\theta_m \sum_i z_{im}}}{2}$ and q_{m0} is the price without any liquidation and it is equal to 1 in this dissertation.

$q_m = Q_m(z_m)$ can be replace with $q_m \leq Q_m(z_m)$ since the objective function of (A.69) is increasing in q_m. Therefore, (A.69) is equivalent to (5.14).

A.19 Lemma 5.2.1

In proving Proposition 5.2.2, I simply used the fact that the objective function is nondecreasing in p and q. Hence Lemma 5.2.1 follows from the proof of Proposition 5.2.2.

A.20 Proposition 5.2.3

Assume that $p^\sharp$ and $z^\sharp$ are optimal to (5.15), and let $q_m^\sharp = Q_m(z_m^\sharp)$. I define the liquidation function for given $p^\sharp$ and $q^\sharp$ by $s_{im}(p^\sharp, q^\sharp) = z_{im}^\sharp$.

If $s_{im}(p^\sharp, q^\sharp)$ satisfies the minimal liquidation condition (5.11), Proposition 5.2.3 directly follows from Proposition 5.2.2.

First note that $p_i^\sharp = \sum_m z_{im}^\sharp + c_i + \sum_k \pi_{ki}\, p_k^\sharp$. If bank i defaults, $p_i^\sharp = \sum_m z_{im}^\sharp + c_i + \sum_k \pi_{ki}\, p_k^\sharp = \sum_m y_{im} q^\sharp + c_i + \sum_k \pi_{ki}\, p_k^\sharp$. Even if bank i does not default, $p_i^\sharp = \overline{p}_i = \sum_m z_{im}^\sharp + c_i + \sum_k \pi_{ki}\, p_k^\sharp$ since $Q_m(z)$ is decreasing in z.

If $s_{im}(p^\sharp, q^\sharp) = y_{im} q_m^\sharp$ for all m, $\sum_m \min\left(q_{im}^\sharp y_{im}, s_{im}(p^\sharp, q^\sharp)\right)$

$= \min\left(\sum_m q^{\sharp}_{im} y_{im}, (\bar{p}_i - c_i - \sum_k \pi_{ki}\, p^{\sharp}_k)^+\right) = \sum_m q^{\sharp}_{im} y_{im}$. If there exists an m such that $s_{im}(p^{\sharp}, q^{\sharp}) < y_{im} q^{\sharp}_m$, then $\sum_m \min\left(q^{\sharp}_{im} y_{im}, s_{im}(p^{\sharp}, q^{\sharp})\right)$

$= \min\left(\sum_m q^{\sharp}_{im} y_{im}, (\bar{p}_i - c_i - \sum_k \pi_{ki}\, p^{\sharp}_k)^+\right) = \left(\bar{p}_i - c_i - \sum_k \pi_{ki}\, p^{\sharp}_k\right)^+$. Hence, the minimal liquidation condition is satisfied.

A.21 Proposition 5.2.4

From Proposition 5.2.2 and (5.16), the equivalent optimization formulation under the prioritized rule is formulated as (5.14) with $\widehat{s}_{im}(p,q) = \left(\bar{p}_i - c_i - \sum_k \pi_{ki} p_k - \sum_{l=1}^{m-1} q_l y_{il}\right)^+$. And $\widehat{s}_{im}(p,q) = \bar{p}_i - c_i - \sum_k \pi_{ki} p_k - \sum_{l=1}^{m-1} q_l y_{il}$ if $\bar{p}_i - c_i - \sum_k \pi_{ki} p_k - \sum_{l=1}^{m-1} q_l y_{il}$ is positive and $\widehat{s}_{im}(p,q) = 0$ otherwise.

Hence (5.14) can be written as

$$
\begin{aligned}
\underset{p,q,z}{\text{maximize}} \quad & \sum_i \left(\sum_m y_{im} q_m + c_i + \sum_k \pi_{ki}\, p_k\right) \\
\text{subject to} \quad & p_i \le \bar{p}_i \\
& p_i \le \sum_m z_{im} + c_i + \sum_k \pi_{ki}\, p_k \\
& z_{im} \le \; y_{im} q_m \\
& z_{im} \le \; \left(\bar{p}_i - c_i - \sum_k \pi_{ki} p_k - \sum_{l=1}^{m-1} q_l y_{il}\right)^+ \\
& q_m \le Q_m(z_m) \\
& p, q, z \ge 0.
\end{aligned}
\tag{A.73}
$$

As in the proof of Proposition 4.1.1I use a mixed integer programming formulation for the logical constraints of whether $\bar{p}_i - c_i - \sum_k \pi_{ki} p_k - \sum_{l=1}^{m-1} q_l y_{il}$ is positive or not. Let x_{im} be an integer variable which equals 1 if $\bar{p}_i - c_i - \sum_k \pi_{ki} p_k - \sum_{l=1}^{m-1} q_l y_{il} \ge 0$ and equals 0 otherwise. This can be represented by the "big-M" constraint:

$$-\mathcal{M}_{im}(1-x_{im}) \leq \overline{p}_i - c_i - \sum_k \pi_{ki} p_k - \sum_{l=1}^{m-1} q_l y_{il} \leq \mathcal{M}_{im} x_{im} - \epsilon, \tag{A.74}$$

where $\mathcal{M}_{im}$ is a large enough number such that the largest and smallest value of $\overline{p}_i - c_i - \sum_k \pi_{ki} p_k - \sum_{l=1}^{m-1} q_l y_{il}$ is bounded by $\mathcal{M}_{im}$ and $-\mathcal{M}_{im}$ and ϵ is a small enough number such that $\overline{p}_i - c_i - \sum_k \pi_{ki} p_k - \sum_{l=1}^{m-1} q_l y_{il} \leq \epsilon$ when $\overline{p}_i - c_i - \sum_k \pi_{ki} p_k - \sum_{l=1}^{m-1} q_l y_{il}$ is negative. In addition, $x_i = 0$ when $\overline{p}_i - c_i - \sum_k \pi_{ki} p_k - \sum_{l=1}^{m-1} q_l y_{il} = 0$.

With this, I can replace $z_{im} \leq \left(\overline{p}_i - c_i - \sum_k \pi_{ki} p_k - \sum_{l=1}^{m-1} q_l y_{il}\right)^+$ with

$$\begin{aligned} z_{im} \leq &\ \overline{p}_i - c_i - \sum_k \pi_{ki} p_k - \sum_{l=1}^{m-1} q_l y_{il} + \mathcal{M}_{im}(1-x_{im}) \\ z_{im} \leq &\ \mathcal{M}_{im} x_{im}. \end{aligned} \tag{A.75}$$

A.22 Proposition 5.3.2

The formulation follows from the same logic as the proof of Proposition 5.2.2 with the total liquidation amount of $\max\left((\tau+1)\overline{p}_i - \tau a_i(p,q), \bar{p}_i^s\right)$. And new logical constraints representing the seniority between p_i^s and p_i^l, i.e., $p_i^l = 0$ if $p_i^s < \bar{p}_i^s$ and $p_i^l \geq 0$ if $p_i^s = \bar{p}_i^s$ are expressed with the last constraints $\frac{p_i^l}{\bar{p}_i^l} \leq w_i \leq \frac{p_i^s}{\bar{p}_i^s}$ with the integer variable $w_i \in \{0,1\}$.

Bibliography

Acemoglu, D., Ozdaglar, A., and Tahbaz-Salehi, A. (2015), "Systemic risk and stability in financial networks," *American Economic Review*, 105, 564–608.

Acharya, V. V., Pedersen, L. H., Philippon, T., and Richardson, M. P. (2010), "Measuring systemic risk," *Working paper, NYU.*

Adrian, T. and Brunnermeier, M. K. (2016), "CoVaR," *American Economic Review*, Forthcoming.

Adrian, T., Covitz, D. M., and Liang, N. (2013), "Financial stability monitoring," *Federal Reserve Bank of New York Staff Reports.*

Allen, F., Babus, A., and Carletti, E. (2012), "Asset commonality, debt maturity and systemic risk," *Journal of Financial Economics*, 104, 519–534.

Amini, H., Filipović, D., and Minca, A. (2015), "To fully net or not to net: adverse effects of partial multilateral netting," *Operations Research*, Forthcoming.

Amini, H., Filipović, D., and Minca, A. (2016), "Uniqueness of equilibrium in a payment system with liquidation costs," *Operations Research Letters*, 44, 1 – 5.

Artzner, P., Delbaen, F., Eber, J.-M., and Heath, D. (1999), "Coherent measures of risk," *Mathematical finance*, 9, 203–228.

Awiszus, K. and Weber, S. (2015), "The Joint impact of bankruptcy costs, fire sales and cross-holdings on systemic risk in financial networks," *Working paper.*

Babus, A. and Hu, T.-W. (2015), "Endogenous intermediation in over-the-counter markets," *Working paper.*

Bech, M. L. and Atalay, E. (2010), "The topology of the federal funds market," *Physica A: Statistical Mechanics and its Applications*, 389, 5223–5246.

Bernanke, B. S. (2013), "Stress Testing Banks: What Have We Learned?" in *Maintaining Financial Stability: Holding a Tiger by the Tail (conference volume).*

Bernanke, B. S. (2014), "The federal reserve: looking back, looking forward," in *speech delivered at the Annual Meeting of the American Economic Association, Philadelphia, January*, vol. 3.

Bertsimas, D. and Brown, D. B. (2009), "Constructing uncertainty sets for robust linear optimization," *Operations research*, 57, 1483–1495.

Bisias, D., Flood, M. D., Lo, A. W., and Valavanis, S. (2012), "A survey of systemic risk analytics," *US Department of Treasury, Office of Financial Research.*

Bolton, P. and Freixas, X. (2006), "Corporate finance and the monetary transmission mechanism," *Review of financial Studies*, 19, 829–870.

Bookstaber, R., Cetina, J., Feldberg, G., Flood, M., and Glasserman, P. (2014), "Stress tests to promote financial stability: Assessing progress and looking to the future," *Journal of Risk Management in Financial Institutions*, 7, 16–25.

Brown, D. B. and Smith, J. E. (2014), "Information relaxations, duality, and convex stochastic dynamic programs," *Operations Research*, 62, 1394–1415.

Caballero, R. J. and Simsek, A. (2013), "Fire sales in a model of complexity," *The Journal of Finance*, 68, 2549–2587.

Capponi, A. and Larsson, M. (2015), "Price contagion through balance sheet linkages," *Review of Asset Pricing Studies*, 5, 227–253.

Caruana, J. (2015), "Global financial interconnectedness: Why does it matter and how to analyse it?" *Third BIS Research Network meeting: Global Financial Interconnectedness.*

Castiglionesi, F. and Navarro, N. (2008), "Optimal fragile financial networks," *SSRN working paper.*

Chen, C., Iyengar, G., and Moallemi, C. C. (2013), "An axiomatic approach to systemic risk," *Management Science*, 59, 1373–1388.

Chen, C., Iyengar, G., and Moallemi, C. C. (2014), "Asset-based contagion models for systemic risk," *Working paper.*

Cifuentes, R., Ferrucci, G., and Shin, H. S. (2005), "Liquidity risk and contagion," *Journal of the European Economic Association*, 3, 556–566.

Craig, B. and Von Peter, G. (2014), "Interbank tiering and money center banks," *Journal of Financial Intermediation*, 23, 322–347.

Davydenko, S. A., Strebulaev, I. A., and Zhao, X. (2012), "A market-based study of the cost of default," *Review of Financial Studies*, 25, 2959–2999.

Duarte, F. and Eisenbach, T. M. (2015), "Fire-sale spillovers and systemic risk," *FRB of New York Staff Report*.

Eisenberg, L. and Noe, T. H. (2001), "Systemic risk in financial systems," *Management Science*, 47, 236–249.

Elliott, M., Golub, B., and Jackson, M. O. (2014), "Financial networks and contagion," *American Economic Review*, 104, 3115–3153.

Elsinger, H., Lehar, A., and Summer, M. (2006), "Risk assessment for banking systems," *Management science*, 52, 1301–1314.

Erdős, P. and Rényi, A. (1959), "On random graphs," *Publicationes Mathematicae Debrecen*, 6, 290–297.

Farboodi, M. (2015), "Intermediation and voluntary exposure to counterparty risk," *Working Paper, Princeton University*.

Feinstein, Z. (2015), "Financial contagion and asset liquidation strategies," .

Feinstein, Z. and El-Masri, F. (2015), "The Effects of leverage requirements and fire sales on financial contagion via asset liquidation strategies in financial networks," *Submitted for publication*.

Feinstein, Z., Rudloff, B., and Weber, S. (2015), "Measures of systemic risk," *Submitted for publication*.

Föllmer, H. and Schied, A. (2002), "Convex measures of risk and trading constraints," *Finance and stochastics*, 6, 429–447.

Giesecke, K. and Kim, B. (2011), "Systemic risk: What defaults are telling us," *Management Science*, 57, 1387–1405.

Glasserman, P. and Tangirala, G. (2015), "Are the Federal Reserves stress test results predictable?" *Working Paper 15-02, Office of Financial Research*.

Glasserman, P. and Young, H. P. (2015a), "How likely is contagion in financial networks?" *Journal of Banking & Finance*, 50, 383–399.

Glasserman, P. and Young, P. (2015b), "Contagion in Financial Networks," *Working Paper 15-21, Office of Financial Research*.

Greenwood, R., Landier, A., and Thesmar, D. (2015), "Vulnerable banks," *Journal of Financial Economics*, 115, 471–485.

Haldane, A. G. (2009), "Rethinking the financial network," *Speech presented at the Financial Student Association, Amsterdam*.

in 't Veld, D. and van Lelyveld, I. (2014), "Finding the core: Network structure in interbank markets," *Journal of Banking & Finance*, 49, 27–40.

James, C. (1991), "The losses realized in bank failures," *The Journal of Finance*, 46, 1223–1242.

Pflug, G. C. (2000), "Some remarks on the value-at-risk and the conditional value-at-risk," in *Probabilistic constrained optimization*, Springer.

Pritsker, M. (2012), "Enhanced stress testing and financial stability," *Working paper, Federal Reserve Bank of Boston.*

Rockafellar, R. T. and Uryasev, S. (2000), "Optimization of conditional value-at-risk," *Journal of risk*, 2, 21–42.

Rogers, L. C. and Veraart, L. A. (2013), "Failure and rescue in an interbank network," *Management Science*, 59, 882–898.

Shin, H. S. (2008), "Risk and liquidity in a system context," *Journal of Financial Intermediation*, 17, 315–329.

Soramäki, K., Bech, M. L., Arnold, J., Glass, R. J., and Beyeler, W. E. (2007), "The topology of interbank payment flows," *Physica A: Statistical Mechanics and its Applications*, 379, 317–333.

The Board of Governors of the Federal Reserve System (2016), "Comprehensive capital analysis and review 2016 summary instructions," `http://www.federalreserve.gov/bankinforeg/ccar.htm`.

Webber, L. and Willison, M. (2011), "Systemic capital requirements," *Bank of England Working Paper.*

Yellen, J. (2013), "Interconnectedness and systemic risk: Lessons from the financial crisis and policy implications," *Board of Governors of the Federal Reserve System, Washington, DC.*

Biography

Jong Soo Hong is a Ph.D. candidate in Decision Sciences at the Fuqua School of Business, Duke University. He was born in Seoul, South Korea on July 18, 1978. Prior to joining Duke, he earned his B.S. in Civil, Urban and Geosystem Engineering and B.A. in Economics from Seoul National University in February 2005. After that, he came to the U.S. and received his M.S. in Financial Mathematics from the University of Chicago in May 2006 and subsequently his M.S in Operations Research from Columbia University in May 2008. He expects to graduate with a Doctor of Philosophy in Business Administration from Duke University's Fuqua School of Business in June 2016