ON L-SPACE KNOTS OBTAINED FROM UNKNOTTING ARCS IN ALTERNATING DIAGRAMS

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Abstract. Let D be a diagram of an alternating knot with unknotting number one. The branched double cover of S^3 branched over D is an L-space obtained by half integral surgery on a knot K_D. We denote the set of all such knots K_D by $\mathcal{D}$. We characterize when $K_D \in \mathcal{D}$ is a torus knot, a satellite knot or a hyperbolic knot. In a different direction, we show that for a given $n > 0$, there are only finitely many L-space knots in $\mathcal{D}$ with genus less than n.

1. Introduction

A knot $K \subset S^3$ is an *L-space knot* if it admits a positive Dehn surgery to an L-space.[1] Examples include torus knots, and more broadly, Berge knots in S^3 [Ber]. In recent years, work by many researchers provided insight on the fiberedness [Ni07, Ghi08], positivity [Hed10], and various notions of simplicity of L-space knots [OS05b, Hed11, Krc14]. For more examples of L-space knots, see [Hom11, Hom16, HLV14, Mot16, MT14, Vaf15].

Since the double branched cover of an alternating knot is an L-space [OS05c], the Montesinos trick allows us to construct a knot with an L-space surgery for every alternating knot with unknotting number one [Mon73, OS05a]. The primary focus of this paper is to study the family of L-space knots arising in the branched double cover of alternating knots with unknotting number one.

Let (D, c) be an alternating knot diagram D with an unknotting crossing c. By cutting out the interior of a small ball containing c and taking the branched double cover we obtain the complement of a knot in S^3. We call this $K_{(D,c)}$, or K_D if we implicitly assume that an unknotting crossing has been chosen. We also call the arc connecting the two arcs of the unknotting crossing c of D the *unknotting arc*.

Let $\mathcal{D}$ denote the set of $K_{(D,c)}$ obtained by considering all reduced alternating diagrams with unknotting number one. Here, a diagram is *reduced* if it does not contain any nugatory crossings (see Figure 1).

Remark 1.1. It is worth noting that a diagram with unknotting number one is not enough to determine a knot in $\mathcal{D}$ on its own, one really does need to specify the unknotting crossing. For example, any alternating diagram for the knot 8_{13} possesses both a positive and a negative unknotting crossing. These give rise to different knots in $\mathcal{D}$, namely the torus knots $T_{2,7}$ and $T_{3,-5}$.

By the work of Thurston [Thu82], any knot in S^3 is precisely one of a torus knot, a satellite knot or a hyperbolic knot. In this paper, we characterize when each of these knot types arise in $\mathcal{D}$. When K_D is a satellite knot, its exterior $S^3 \setminus \nu(K_D)$ contains an incompressible, non-boundary parallel torus. Correspondingly, there will be a Conway sphere C in D.

Definition 1.2. Let (D, c) be an alternating diagram with an unknotting crossing c. Let C be a Conway sphere in D, disjoint from the unknotting arc specified by c. We will call the component of

[1] An *L-space* Y is a rational homology sphere with the simplest possible Heegaard Floer invariant, that is rk $\widehat{HF}(Y) = |H_1(Y;\mathbb{Z})|$.

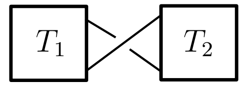

FIGURE 1. A nugatory crossing.

$S^3 \setminus C$ containing c the *interior* of C. We will call the other component the *exterior*. We say that C is *substantial* if it is visible[2], the interior of C contains more than one crossing and the exterior is not a rational tangle.

We may now state the main result of the paper.

Theorem 1.3. *Let (D, c) be an alternating diagram with an unknotting crossing c. The following are equivalent:*

(i) K_D is a satellite,
(ii) D contains a substantial Conway sphere.

If D is a 2-bridge knot, then we show that K_D is a torus knot. Conversely, it turns out that this is the only way that torus knots arise in $\mathcal{D}$.

Proposition 1.4. *The knot K_D is a torus knot if and only if D is a diagram of a 2-bridge knot.*

Combining Proposition 1.4 and Theorem 1.3 allows us to determine for which diagrams K_D is a hyperbolic knot.

Corollary 1.5. *Let (D, c) be an alternating diagram with an unknotting crossing c. Then K_D is a hyperbolic knot if and only if D is not a 2-bridge knot and D does not contain a substantial Conway sphere.*

Conjecturally, there are only finitely many L-space knots in S^3 with a given genus ([HW14, Conjecture 6.7] and [BM15, Conjecture 1.2]). As a final result, we verify this conjecture for knots in $\mathcal{D}$.

Proposition 1.6. *For a given $n > 0$ there are finitely many knots in $\mathcal{D}$ with genus less than n.*

Proof. Let $K_D \in \mathcal{D}$ correspond to a diagram (D, c). Using Montesinos trick [Mon73], we have $S^3_{\frac{d}{2}}(K_D) \cong \Sigma(D)$, for some d with $|d| = \det D$. Thus, by [McC14, Theorem 1.1], we have $\det D/2 \leq 4g(K_D)+3$. The result follows since there are finitely many alternating knots of a given determinant and each alternating knot has finitely many reduced alternating diagrams up to planar isotopy. □

The rest of the paper is organized as follows. In Section 2, we state Tsukamoto's theorem [Tsu09, Theorem 5] that gives a set of three moves (*flype, tongue*, and *twirl*) enabling us to go from a clasp diagram (Figure 3) to any diagram (D, c). We also determine the corresponding effect of these moves to the knots in $\mathcal{D}$. Theorem 1.3 and Proposition 1.4 are proved in Section 3.

ACKNOWLEDGEMENTS

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[2]A Conway sphere is *visible* if it intersects the plane of the diagram in a single simple closed curve. (cf. [Thi91, Section 3]).

2. Almost alternating diagrams of the unknot

Recall that a diagram D is said to be *almost alternating* if it is non-trivial, non-alternating and can be changed into an alternating diagram by changing a single crossing. Given an almost alternating diagram we call the crossing which can be changed to obtain an alternating diagram the *dealternator*. We will refer to the crossing arc of the dealternator as the *dealternating arc*.

As the following remark shows, for every diagram (D, c) there is a reformulation of $K_{(D,c)}$ in terms of dealternating arcs. We will sometimes find it convenient to use this alternative approach.

Remark 2.1. Given an alternating diagram with an unknotting crossing (D, c), let $\widetilde{D}$ be the almost alternating diagram of the unknot obtained by changing c. Since $\widetilde{D}$ is unknotted, taking the double branched cover lifts its dealternating arc to a knot in S^3, which can easily seen to be $K_{(D,c)}$.

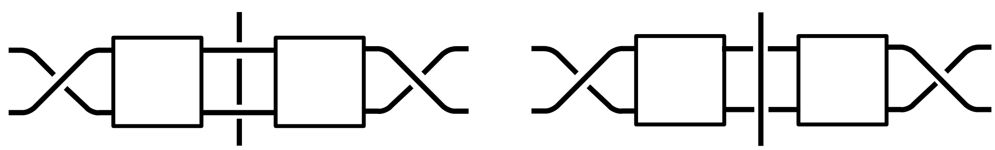

Figure 2. Flyped tongues. In both pictures the dealternator is the top central crossing.

Tsukamoto has studied the structure of almost alternating diagrams of the unknot. The key result we require shows that most almost alternating diagrams of the unknot contain a flyped tongue, where a *flyped tongue* is a region appearing in as in Figure 2 [Tsu09, Theorem 4].

Theorem 2.2 (Tsukamoto)**.** *Any reduced almost alternating diagram of the unknot which is not an unknotted clasp diagram (see Figure 3) contains a flyped tongue.*

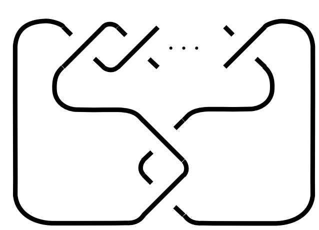

Figure 3. An unknotted clasp diagram

A reduced almost alternating diagram which contains a flyped tongue can be isotoped to an almost alternating diagram with fewer crossings by twisting the two tangles so that the outer most two crossings are removed from the diagram. If the result of this isotopy is not reduced, then we can further perform a Reidemeister I move to obtain a reduced diagram. Given an almost alternating diagram of the unknot, Theorem 2.2 then allows us to carry out a sequence of such isotopies. Carrying out this argument more carefully, allows one to show that all reduced alternating diagrams with an unknotting crossing can be built up by a sequence of simple local moves. The following is immediate from [Tsu09, Theorem 5].

Theorem 2.3 (Tsukamoto)**.** *Let (D, c) be a reduced alternating diagram with an unknotting crossing. Then there are a sequence of reduced alternating diagrams D_i with unknotting crossings c_i,*

$$(D_1, c_1) \to \cdots \to (D_p, c_p) = (D, c),$$

such that (D_1, c_1) is a clasp diagram (see Figure 3), and for each i, (D_i, c_i) is obtained from (D_{i-1}, c_{i-1}) by either a flype fixing the c_{i-1}, a twirl move, or a tongue move (see Figure 5)

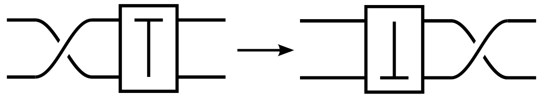

FIGURE 4. A flype move

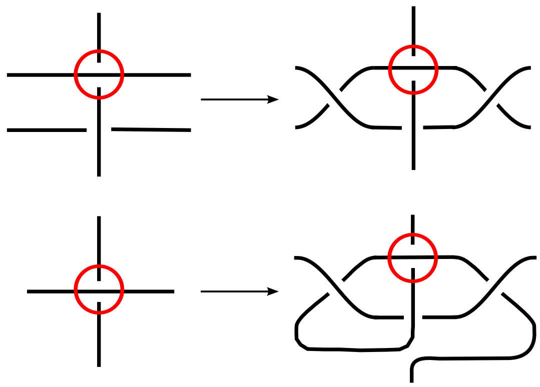

FIGURE 5. Up to reflection, the tongue move (top) and a twirl move (bottom). In each case the new unknotting crossing is marked by the red circle.

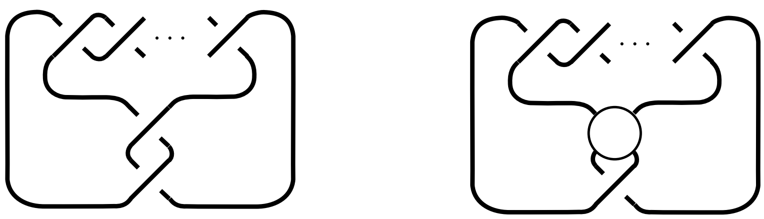

FIGURE 6. A clasp diagram (left) and the rational tangle obtained by excising a ball containing an unknotting crossing (right).

2.1. **The corresponding operations in $\mathcal{D}$.** In this section, we turn our focus to the moves, used in Theorem 2.3, that allow us to build up any alternating diagram with unknotting number one. In principle, by studying the effect of these moves on the knots in $\mathcal{D}$, one should be able to obtain an understanding of the knots arising in $\mathcal{D}$. In practice, however, this approach is not particularly straightforward to carry out. As we will see, the effects of a twirl move can be understood easily enough, but the effects of a tongue move are much more subtle.

The diagrams from which Theorem 2.3 allows us to build all other alternating knots with unknotting number one are the clasp diagrams. According to the following proposition, the clasp diagrams are precisely those which give rise to the unknot in $\mathcal{D}$.

Proposition 2.4. *K_D is the unknot if and only if D is a clasp diagram.*

Proof. For any odd integer d, the manifold $S^3_{d/2}(U)$ is homeomorphic to the lens space $L(d, 2)$. Thus if $K_D = U$, then D must be a clasp diagram. Conversely, as shown in Figure 6, when a ball containing an unknotting crossing is cut out, we obtain a rational tangle. Thus if D is a clasp diagram then the complement of K_D is a solid torus and hence K_D is the unknot. □

Remark 2.5. Note that as the flypes in Theorem 2.3 can be chosen to fix the unknotting crossing, they will clearly leave the knot K_D unchanged.

A twirl move corresponds to a cabling operation. It is a special case of a more general operation for producing satellite knots in $\mathcal{D}$ studied in the following section.

Proposition 2.6. *If (D', c') is obtained from (D, c) by a twirl move, then $K_{D'}$ is obtained from K_D by taking a cable with winding number two.*

Proof. This is immediate from Figure 7 which shows how the dealternating arc changes under introducing a twirl. □

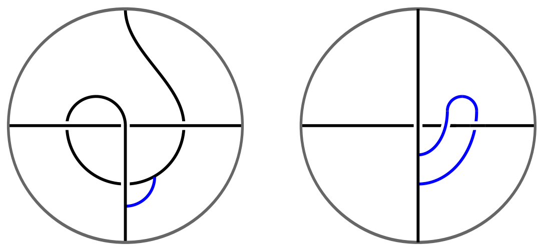

FIGURE 7. The diagram on the left is obtained from a twirl move on the right one. The blue arc shows how the unknotting arc in D' appears in D after reversing the twirl move.

Finally we turn our attention to the tongue move. As we will see this corresponds to a band sum operation in $\mathcal{D}$. In certain cases, it can be described precisely what the band sum is, however, in general, different tongue moves on the same diagram will result in different knots in $\mathcal{D}$. For example, if D is an alternating diagram of 8_{14} there is an unknotting crossing for which $K_D = T_{3,5}$. There is a tongue move to a ten-crossing diagram D' with $K_{D'} = T_{4,7}$ and a tongue move to a different ten-crossing diagram D'' for which $K_{D''}$ is a positive braid with braid index equal to six.

Proposition 2.7. *If (D', c') is obtained from (D, c) by a tongue move, then $K_{D'}$ is obtained by taking a band sum of K_D with another knot.*

Proof. This follows from Figure 8. It is clear that the lift of the new dealternating arc is obtained by taking a band sum of knots obtained by lifting the blue and red arcs to the double branched cover. Note that the lift of the blue arc is K_D. □

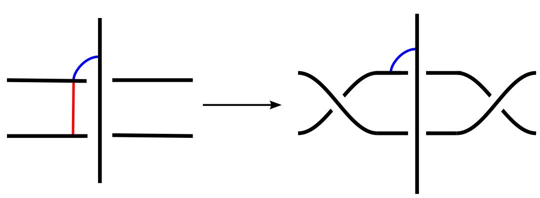

FIGURE 8. A tongue move corresponds to a band sum operation in $\mathcal{D}$.

3. THE GEOMETRY OF KNOTS IN $\mathcal{D}$

In this section we study which types of L-space knots arise in $\mathcal{D}$. Precisely, we show when K_D is a torus knot, a satellite knot or a hyperbolic knot.

3.1. **Torus knots in $\mathcal{D}$.** We have already seen from Proposition 2.4 when the unknot arises in $\mathcal{D}$. We now generalize this to show that torus knots correspond to 2-bridge knots with unknotting number one.

Proposition 1.4. *The knot K_D is a torus knot if and only if D is a diagram of a 2-bridge knot.*

Proof. If $S^3_{d/2}(T_{r,s}) \cong \Sigma(L)$ is the double branched cover of an alternating knot L, then $d = 2rs \pm 1$ [McC14]. In particular, $\Sigma(L)$ must be the lens space $L(2rs \pm 1, 2r^2)$ [Mos71]. Since the only knots in S^3 with lens space branched double covers are the 2-bridge knots, this shows that if K_D is a torus knot, then D is a diagram of a 2-bridge knot.

Conversely, if D is an alternating diagram of a 2-bridge knot, then there is a Conway sphere C passing through the unknotting crossing, such that the both components of D in $S^3 \setminus C$ are rational tangles (cf. Figure 9). Consider the diagram of the unknot obtained by changing the unknotting crossing in C. Since both sides of C contain a rational tangle, we see that C lifts to an unknotted torus in S^3 (it bounds a solid torus on both sides) upon taking the double branched cover. Since the unknotting arc in D can be isotoped to lie on C, K_D must lie on an unknotted torus in S^3. In particular, it is a torus knot, as required. □

Remark 3.1. One can also appeal to the Cyclic Surgery Theorem [CGLS87] to show that a 2-bridge diagram yields a torus knot.

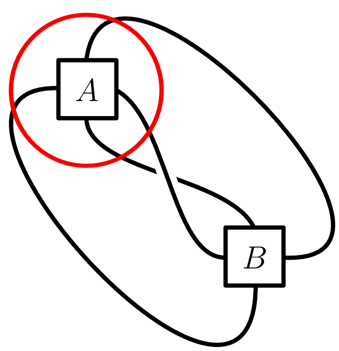

FIGURE 9. An unknotting number one 2-bridge diagram with a Conway sphere (marked in red) onto which the unknotting arc can be isotoped. Both A and B are rational.

Remark 3.2. By exhibiting a 2-bridge diagram along with an unknotting crossing for each torus knot, it can be proved that all torus knots are in $\mathcal{D}$.

3.2. **Constructing satellite knots in $\mathcal{D}$.** Now we turn our attention to the satellite knots in $\mathcal{D}$. In this section we give a general construction for producing alternating diagrams with an unknotting crossing for the resulting knot in $\mathcal{D}$ is a satellite knot. For this construction we need the following definition.

Definition 3.3. An *alternating unknotting tangle* is an alternating tangle T containing a distinguished crossing c_T, with the property that after changing c_T we obtain a tangle that is isotopic, relative to the boundary, to a single crossing. Moreover, we require that the crossings connected by an arc to the boundary appear as in Figure 10.

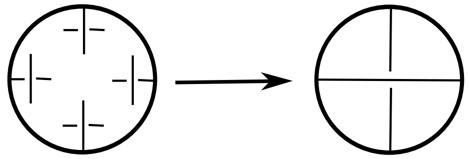

FIGURE 10. A diagram showing the crossing on the outside of an alternating unknotting tangle and the tangle obtained by changing the distinguished crossing.

Remark 3.4. An alternating unknotting tangle T is defined in such a way that if we replace the unknotting crossing of (D, c) with T to obtain a new alternating diagram, then this alternating diagram will have unknotting number one with the distinguished crossing c_T as an unknotting crossing.

Construction 3.5. *Start with D, an alternating diagram with an unknotting crossing c. Let (T, c_T) be an alternating unknotting tangle with more than one crossing. We replace c with T to obtain an alternating diagram (D', c_T) with unknotting number one.*

Lemma 3.6. *If D is not a clasp diagram, then $K_{D'} \in \mathcal{D}$ is a satellite knot with companion given by K_D.*

Proof. Starting with D' and changing the distinguished crossing c_T of T, we obtain a diagram that is isotopic to the diagram obtained by changing c in D, i.e. an almost alternating diagram of the unknot. Using Proposition 2.4, if D is not a clasp diagram then K_D is not the unknot. The lift of the boundary of T is a torus R in the knot complement $S^3 \setminus \nu(K_{D'})$. Note that on one side of R, we have the knot complement $S^3 \setminus \nu(K_D)$. To complete the proof we use the following claim.

Claim. *As T consists of more than one crossing, we may replace c_T with a rational tangle in order to obtain a non-rational tangle.*

Proof of Claim. Let (T', c') be an alternating unknotting tangle for which every replacement of c' by a rational tangle results in a rational tangle. We will show that T' must consist of a single crossing. Let $\widetilde{D}$ be the alternating diagram with unknotting number one obtained by inserting T' in place of the unknotting crossing in a clasp diagram. Observe that whenever we replace c' in $\widetilde{D}$ with a rational tangle we obtain a diagram of a 2-bridge knot or link. This shows that $K_{\widetilde{D}}$ is knot for which every Dehn surgery is a lens space. It follows that $K_{\widetilde{D}}$ is the unknot and thus, by Proposition 2.4, that $\widetilde{D}$ is a clasp diagram. Therefore, we see that T' must have consisted of a single crossing only. □

The claim shows that we may replace the distinguished crossing c_T with a rational tangle to obtain a non-rational tangle T'. This shows that $K_{D'}$ admits surgeries under which R survives as an incompressible torus. In particular, R is not boundary parallel, showing that $K_{D'}$ is a non-trivial satellite knot with companion K_D. □

An example of Construction 3.5 is given in Figure 11, in $\mathcal{D}$ this produces satellite knots for which the companion is a trefoil. Another example of this construction is given by the twirl move, in which case the effect in $\mathcal{D}$ is a cabling operation. (See Proposition 2.6.)

The following lemma will be useful for finding alternating unknotting tangles inside an alternating diagram with unknotting number one.

Lemma 3.7. *Let D be a reduced alternating diagram with unknotting crossing c in the interior of a visible Conway sphere C. If the exterior of C contains at least one crossing and C is maximal, in*

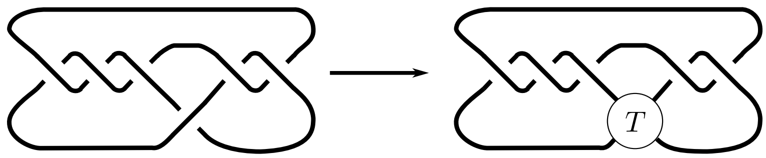

FIGURE 11. Inserting an alternating unknotting tangle to obtain a new alternating diagram with unknotting number one.

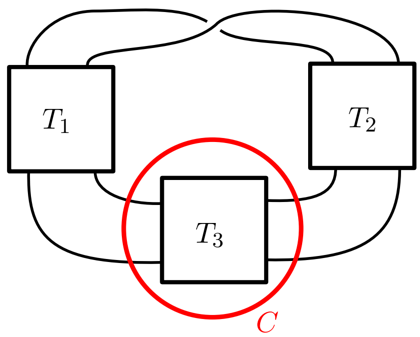

FIGURE 12. A diagram with a Conway sphere which doesn't satisfy the maximality condition of Lemma 3.7. By twisting either T_1 or T_2, we can flype to increase the number of crossings in the interior of C.

the sense that no flype can increase the number of crossings in the interior of C, then the interior of C is an alternating unknotting tangle with the distinguished crossing given by c. In particular D can be obtained from a smaller diagram by an application of Construction 3.5.

Proof. We prove the lemma by induction on the number of crossings in the interior of C. When the interior of C contains only a single crossing there is nothing to prove. Thus, we will assume from now on that the interior of C contains more than just the unknotting crossing. Note also that the conditions on C guarantee that D is not a clasp diagram; whenever we have a visible Conway sphere in a clasp diagram with at least one crossing in its exterior, we can always find a flype which carries one of the crossings into interior.

Thus, if we let D' be the almost alternating diagram obtained by changing c, we can apply Theorem 2.2 to see that D' contains a flyped tongue. By considering carefully how C can intersect a flyped tongue, we see that the maximality condition we have imposed implies that the interior of C in D' is either just the dealternator or it contains the entire flyped tongue. As we have assumed that the interior of C contains more than one crossing, it follows that the interior of C contains a flyped tongue. Therefore, if we let T' be the tangle contained in the interior of C in D', we see

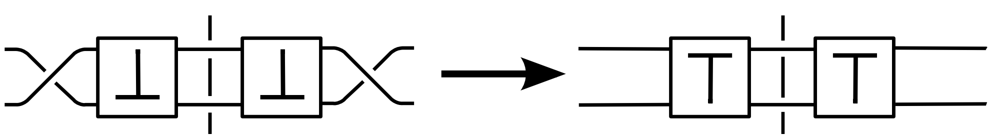

FIGURE 13. Reducing T' to T'' by twisting the flyped tongue.

that T' contains a flyped tongue. We may perform an isotopy as in Figure 13 on this flyped tongue to obtain an almost alternating diagram D'' with fewer crossings in the interior of C. If D'' is not reduced then we may perform Reidemeister I moves to remove any nugatory crossings. Note that none of these Reidemeister I moves can remove the dealternating crossing, since if they did, we would obtain a reduced alternating diagram of the unknot. Thus we can assume further that D'' is reduced. Let T'' denote the interior of C in D''. As the exterior of C is the same in both D' and D'', we see that D'' also satisfies the hypotheses that no flype can increase the number of crossings in the interior of C. Thus by the inductive hypothesis we can assume that T'' was obtained by changing the unknotting crossing in an alternating unknotting tangle. Since D' can be obtained from D'' by an isotopy fixing the exterior of C, it follows that the interior of C in D is also an alternating unknotting tangle, as required. □

Given a reduced alternating unknotting tangle T, we can insert it into an alternating diagram as in Construction 3.5. If the diagram we inserted T into is chosen as in Figure 11, for example, then we notice that the Conway sphere containing T satisfies the hypotheses of Lemma 3.7. Thus the proof of Lemma 3.7 shows that after changing the distinguished crossing in T, we obtain a tangle containing a flyped tongue. Thus in the same way that Theorem 2.3 follows from Theorem 2.2, we can obtain an analogue to Theorem 2.3 for alternating unknotting tangles.

Corollary 3.8. *Any reduced alternating unknotting tangle can be built up from a single crossing by a sequence of flypes, tongue moves and twirl moves.*

3.3. **Substantial Conway spheres.** A satellite knot K_D in $\mathcal{D}$ generated by Construction 3.5 corresponds to an alternating knot (D, c) containing a Conway sphere. The goal of this subsection is to detect when the converse is true. That is, for what types of Conway spheres in (D, c) the knot K_D is a satellite knot. Recall from Section 1 that a Conway sphere C in (D, c) is called substantial if it is visible, the interior of C (i.e. the component of $S^3 \setminus C$ containing c) contains more than one crossing and the exterior of C is not a rational tangle.

Remark 3.9. Note that a substantial Conway sphere need not be essential, since the interior may be a rational tangle.

Proposition 3.10. *Let (D, c) be an alternating diagram with an unknotting crossing c containing a substantial Conway sphere. Then the corresponding knot K_D in $\mathcal{D}$ is a satellite knot with companion also in $\mathcal{D}$.*

Proof. Let C be the substantial Conway sphere in D. We may perform a sequence of flype moves on D to obtain a diagram D' that maximizes the number of crossings in the interior of C. Since the exterior of C in D is not rational, the exterior of C in D' will also not be non-trivial. In particular, this D' satisfies the hypotheses of Lemma 3.7. It follows that D' arises by Construction 3.5. Consequently, $K_D = K_{D'}$ is a satellite in $\mathcal{D}$. □

Lemma 3.11. *Let $K_D \in \mathcal{D}$ arise from the diagram (D, c). If K_D is a satellite knot, then D contains a Conway sphere C, satisfying either*

(1) C is substantial or
(2) C is hidden and, moreover, the tangle in the exterior of C is not rational.

Remark 3.12. Note that, by construction, K_D is strongly invertible, that is, there is an involution on the knot complement $S^3 \setminus \nu(K_D)$ that fixes a pair of arcs meeting the boundary torus transversally in four points.

Lemma 3.11 is proven by first finding an incompressible, non-boundary parallel torus in the exterior of K_D (not necessarily the one we get from the assumption that K_D is a satellite knot). Given the strong inversion of $S^3 \setminus \nu^\circ(K_D)$, we quotient the torus to get a Conway sphere C. By carefully exploring C we see that the only possibilities are the ones stated in the lemma. For the first part of the argument (i.e. finding the torus), we appeal to the following theorem that we state without proof. The theorem follows directly from [Hol91, Corollary 4.6]

Theorem 3.13 (Holzmann). *Let M be an irreducible, P^2-irreducible three-manifold with an involution ι. Let also M contain an incompressible torus. Suppose M is not an orientable Seifert fiber space over the 2-sphere with four exceptional fibers. Then there is a two-sided incompressible torus $R \subset \mathring{M}$, transverse to the fix point set of ι, with either $\iota(R) \cap R = \emptyset$ or $\iota(R) = R$.*

Proof of Lemma 3.11. Let $\widehat{D} \subset S^3 \setminus (\mathring{B}^3)$ be the diagram obtained by cutting out a small ball containing the unknotting crossing c. The double branched cover over $\widehat{D}$ is the knot exterior $S^3 \setminus \nu(K_D)$. Since K_D is a satellite knot, $S^3 \setminus \nu(K_D)$ contains an incompressible torus R'. If ι is the covering involution on $S^3 \setminus \nu(K_D)$ (c.f. Remark 3.12), then, using Theorem 3.13, we get that $S^3 \setminus \nu(K_D)$ contains an incompressible torus R, transverse to the fixed set of ι, such that either $\iota(R) = R$ or $\iota(R) \cap R = \emptyset$. Moreover, since we can also apply Theorem 3.13 to any Dehn filling of $S^3 \setminus \nu(K_D)$ and, in particular, all those in which R' remains incompressible, we can assume that R is not boundary parallel. An Euler characteristic argument shows that the quotient of R by the involution is either a torus disjoint from $\widehat{D}$ or a Conway sphere (corresponding to either $\iota(R) \cap R = \emptyset$ or $\iota(R) = R$, respectively). We remind the reader that prime alternating knots are not satellites knots [Men84], and in particular, the exterior of D cannot contain an incompressible non-boundary parallel torus. Thus the possibility that the quotient of R be a torus does not occur. Therefore, we get a Conway sphere C in (D, c) which does not intersect the unknotting arc specified by c. Since R is incompressible in $S^3 \setminus \nu(K_D)$, we see that the exterior of C cannot be a rational tangle. Thus if C is hidden, we see that (2) of the lemma holds. If C is visible, then the interior of C must contain more than one crossing, as otherwise R would have been boundary parallel in $S^3 \setminus \nu(K_D)$ contradicting the assumption that K_D is a satellite knot. This shows that if C is visible, it must be substantial. □

Lemma 3.14. *Let (D, c) contain a hidden Conway sphere disjoint from the unknotting arc specified by c. Let also the exterior of the Conway sphere be non-rational. Then either D contains a substantial Conway sphere or D is a diagram of a 2-bridge knot.*

Proof. Menasco, in [Men84], shows that if an alternating diagram D contains a hidden Conway sphere, D must appear as in Figure 14(a). See [Thi91, Figure 3(i)]. We may assume that the unknotting crossing c is contained in the tangle X. Observe that if any of Y, Z or W is not rational, then the visible Conway sphere containing it is a substantial one. Therefore, we may assume that the tangles Y, Z and W are all rational.

Now consider Figure 14(b). Since the tangle contained in the exterior of the Conway sphere is not rational, we see that both Z and W are non-trivial. Thus if we consider again Figure 14(a), we see that the exterior of the visible Conway sphere containing X is not rational. Thus if X contains more than one crossing, the visible Conway sphere surrounding X is substantial. It remains to consider the case where X consists of a single crossing. We will show that in this case D is a diagram of a 2-bridge knot.

If X only consists of c, then the diagram appears as in Figure 14(c). After changing the unknotting crossing there is an isotopy to the non-prime diagram shown in Figure 14(d). Since this is the unknot we see that both summands must be unknotted. However, note that the summand containing Y is alternating. This implies that Y must be a trivial tangle. However, as we are assuming that Z and W are rational, we see that the diagram we started with is isotopic to a diagram of a 2-bridge knot. □

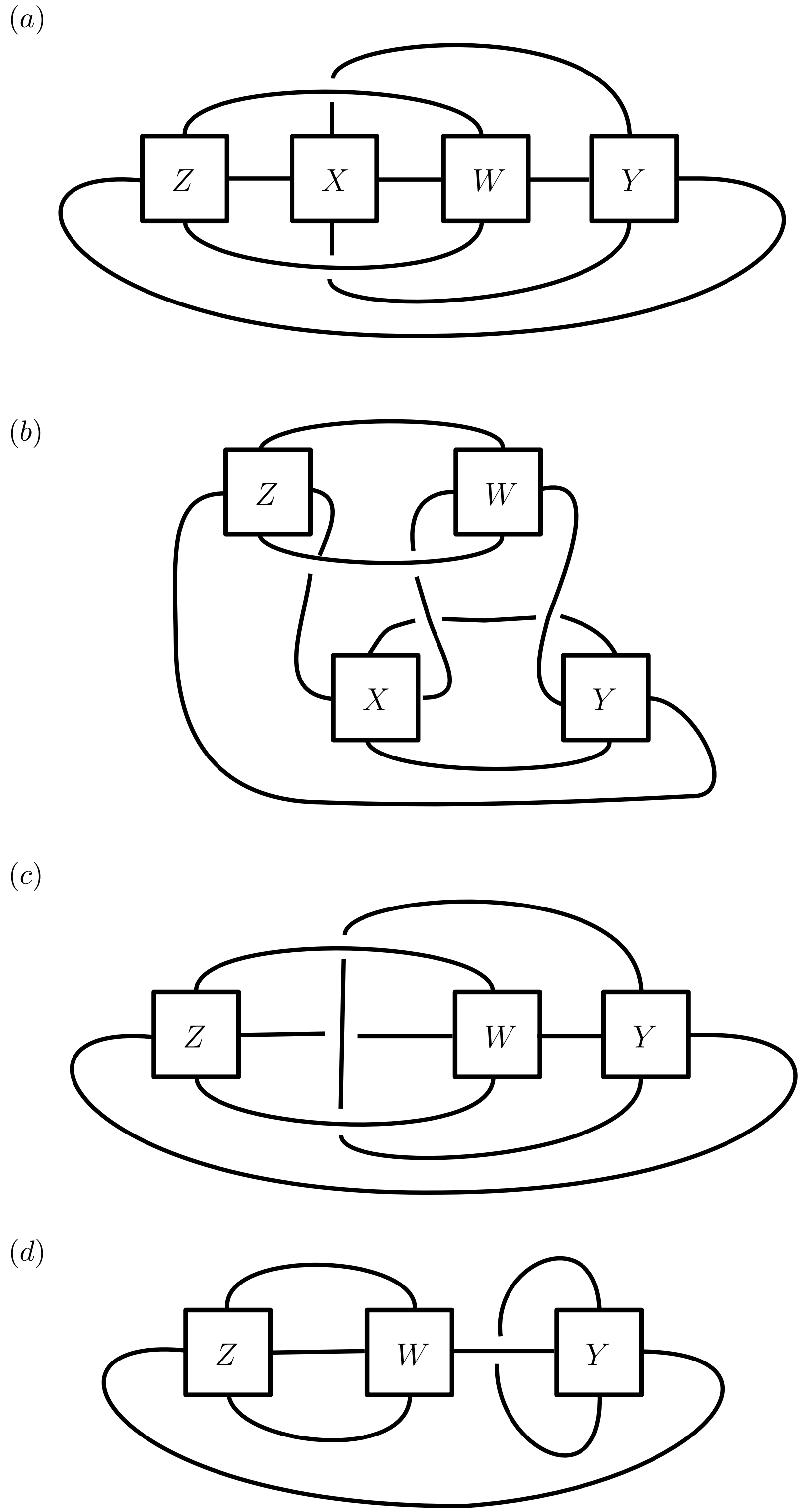

FIGURE 14. (a) Shows the standard form of a diagram containing a hidden Conway sphere. (b) Shows a non-alternating diagram in which the Conway sphere is clearly present. (c) Shows the diagram when X contains only the unknotting crossing. (d) Shows the non-prime diagram which can be obtained after changing the unknotting crossing in (c).

This provides the final step required for our main result on satellite knots in $\mathcal{D}$.

Proof of Theorem 1.3. $(ii) \Rightarrow (i)$: If D contains a substantial Conway sphere, then Proposition 3.10 shows that K_D is a satellite knot.

$(i) \Rightarrow (ii)$: If K_D is a satellite, then Lemma 3.11 together with Lemma 3.14 imply that either K_D contains a substantial Conway sphere or D is a diagram of a 2-bridge knot. However, Proposition 1.4 shows that D is not a diagram of a 2-bridge knot, so D must contain a substantial Conway sphere. □

References

[BM15] Kenneth Baker and Kimihiko Motegi, *Twist families of L-space knots, their genera, and Seifert surgeries*, preprint (2015), arXiv:1506.04455.

[Ber] John Berge, *Some knots with surgeries yielding lens spaces*, Unpublished manuscript.

[CGLS87] Marc Culler, Cameron Gordon, John Luecke, and Peter Shalen, *Dehn surgery on knots*, Ann. of Math. **125** (1987), no. 2, 237–300.

[Ghi08] Paolo Ghiggini, *Knot Floer homology detects genus-one fibred knots*, Amer. J. Math. **130** (2008), no. 5, 1151–1169.

[Hed10] Matthew Hedden, *Notions of positivity and the Ozsváth-Szabó concordance invariant*, J. Knot Theory Ramifications **19** (2010), no. 5, 617–629.

[Hed11] ______, *On Floer homology and the Berge conjecture on knots admitting lens space surgeries*, Trans. Amer. Math. Soc. **363** (2011), no. 2, 949–968.

[HW14] Matthew Hedden and Liam Watson, *On the geography and botany of knot Floer homology*, preprint (2014), arXiv:1404.6913.

[Hol91] W. H. Holzmann, *An equivariant torus theorem for involutions*, Trans. Amer. Math. Soc. **326** (1991), no. 2, 887–906.

[Hom11] Jennifer Hom, *A note on cabling and L-space surgeries*, Algebr. Geom. Topol. **11** (2011), no. 1, 219–223.

[Hom16] ______, *Satellite knots and L-space surgeries*, Bull. London Math. Soc. **48** (2016), no. 5, 771–778.

[HLV14] Jennifer Hom, Tye Lidman, and Faramarz Vafaee, *Berge-Gabai knots and L-space satellite operations*, Algebr. Geom. Topol. **14** (2014), no. 6, 3745–3763.

[Krc14] David Krcatovich, *A restriction on the Alexander polynomials of L-space knots*, preprint (2014), arXiv:1408.3886.

[McC14] Duncan McCoy, *Bounds on surgeries branching over alternating knots*, arXiv:1412.0906 (2014).

[Men84] William Menasco, *Closed incompressible surfaces in alternating knot and link complements*, Topology **23** (1984), no. 1, 37–44.

[Mon73] Jose Montesinos, *Seifert manifolds that are ramified two-sheeted cyclic coverings*, Bol. Soc. Mat. Mexicana (2) **18** (1973), 1–32.

[Mos71] Louise Moser, *Elementary surgery along a torus knot*, Pacific J. Math. **38** (1971), 737–745.

[Mot16] Kimihiko Motegi, *L-space surgery and twisting operation*, Algebr. Geom. Topol. **16** (2016), no. 3, 1727–1772.

[MT14] Kimihiko Motegi and Kazushige Tohki, *Hyperbolic L-space knots and exceptional Dehn surgeries*, J. Knot Theory Ramifications **23** (2014), no. 14.

[Ni07] Yi Ni, *Knot Floer homology detects fibred knots*, Invent. Math. **170** (2007), no. 3, 577–608.

[OS05a] Peter Ozsváth and Zoltán Szabó, *Knots with unknotting number one and Heegaard Floer homology*, Topology **44** (2005), no. 4, 705–745.

[OS05b] ______, *On knot Floer homology and lens space surgeries*, Topology **44** (2005), no. 6, 1281–1300.

[OS05c] ______, *On the Heegaard Floer homology of branched double-covers*, Adv. Math. **194** (2005), no. 1, 1–33.

[Thi91] Morwen Thistlethwaite, *On the algebraic part of an alternating link*, Pacific J. Math. **151** (1991), no. 2, 317–333.

[Thu82] William Thurston, *Three dimensional manifolds, Kleinian groups and hyperbolic geometry*, Bulletin of the A.M.S. **6** (1982), no. 3, 357–382.

[Tsu09] Tatsuya Tsukamoto, *The almost alternating diagrams of the trivial knot*, J. Topol. **2** (2009), no. 1, 77–104.

[Vaf15] Faramarz Vafaee, *On the knot Floer homology of twisted torus knots*, Int. Math. Res. Not. IMRN (2015), no. 15, 6516–6537.

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