Time-Delay Control Using Conic Sectors

by

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A thesis submitted in partial fulfillment of the
requirements for the degree of Master of Science
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ABSTRACT

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Abstract

Time delays exist in many practical systems and ignoring them could degrade systems performance and even lead to instability. That's why it's of vital importance to study the stability and how to control systems with time delays. The main contribution of this thesis is to provide a novel way of establishing conic bounds for stable LTI systems with time-varying state delays. Two linear matrix inequality conditions are provided. Alone, they can be used as an analysis tool, determining conic bounds for LTI systems subject to bounded time-varying state delays. Combined with the Conic Sector Theorem, the conic bounds established by the LMIs can be used directly to design conic controllers, ensuring the closed-loop I-O stability.

Background knowledge about linear undelayed systems is reviewed, including the state-space equations, stability, controllability, and observability. The optimal state-feedback control problem is stated. Time-delays are then introduced into the system. The Lyapunov-Krasovskii based method and the input-output approach are discussed. The conic sector theorem is also stated as the basis of the main result.

This document is formulated in the following order. Chapter I gives the introduction, containing the motivation of this study, a review of notations, definitions, theorems, and some background knowledge. Chapter II provides a literature review. Five papers were chosen to be studied in detail and a comparative study was conducted. Numerical examples were given to show the results. Chapter III implements the main results of the author's original work.

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Notations

Here we give the notations used in this document.

$\mathbb{R}$	The set of real numbers
$\mathbb{R}^+$	The set of non-negative real numbers
$\mathbb{R}^n$	The space of real n-vectors
$\mathbb{R}^{n\times m}$	The space of real $n \times m$ matrices
$x(t)$	Vector valued function
$\dot{x}(t)$	The first time derivative of $x(t)$
P^T	Matrix transpose of P
$P > 0 (P \geq 0)$	Symmetric matrix P is positive (semi-) definite
I	Identity matrix with appropriate dimension
0	Zero matrix with appropriate dimension
$\lambda(P)$	Eigenvalue of matrix P
$\lambda_{min}(P), \lambda_{max}(P)$	The minimum and maximum eigenvalue of P
$\|\| \cdot \|\|_p$	The L_p norm
$L_2[0, \infty)$	The space of square-integrable functions on $[0, \infty)$
L_{2e}	The space of square-integrable functions on $[0, T]$, $T \in \mathbb{R}^+$
y_T	The truncation of y to [0,T]
$*$	The duplicated elements of a symmetric matrix

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Chapter 1

Introduction

In this chapter, we first state the motivation of studying time-delay systems and give some examples where time delays appear. Then we give the definitions and theorems used in this thesis.

1.1 Motivation

Time delays exist widely in the real world and affect many applications, including chemical processes, mechanical systems, and teleoperation systems [GCK03]. In the years 2020 and 2021, there is a very vivid example in the real world that shows the effect of time delays: the global pandemic of COVID-19.

The COVID-19 has taken millions of lives. The relatively long incubation period is one reason why it has a very strong spreading ability. If the incubation period is very short, that is, the symptoms are shown immediately when someone is infected, then we can give that person treatment and stop the spreading of the virus. But with a long incubation period, the symptoms are not shown immediately and people who get infected may not be aware of the infection and could spread the virus unconsciously. In this case, when modeling the spreading model of the virus, the incubation period is the time delay. A well known mathematical model for virus spreading is the SIR model [CMA20], where the S stands for the number of susceptible individuals, the I stands for the number of infectious individuals, and R stands for the the number of removed or deceased individuals. The simplest SIR

model can be expressed by the following ordinary differential equations:

$$\frac{dS}{dt} = -\frac{\beta IS}{N}$$
$$\frac{dI}{dt} = \frac{\beta IS}{N} - \gamma I$$
$$\frac{dR}{dt} = \gamma I$$

where β, γ are constants and depend on the data. The length of incubation period will definitely influence these constants.

Time delays can be found in many other examples:

- Communication delays in teleoperation systems [AS88].
- Drilling process on a drill [Fri14].
- The mean torque production model in internal combustion engine [KM93].
- Vehicular traffic flow models [Fri14].

It has been shown that in most cases, the system performance can be degraded with the appearance of time delays [SH12]. In some cases, time delays are introduced to the systems on purpose to stabilize systems with unknown dynamics as in [GHW01]. A great deal of attention has been paid to the systems with time delays. In many surveys such as [Ric03] and [GN03], the two main aspects of problems in this field are the stability analysis and controller synthesis.

Several approaches in the stability theory were developed. For linear time-invariant (LTI) systems, the frequency domain method is always used [BC63]. For a retarded type time-delay system, it is asymptotically stable if and only if all the roots of its characteristic equation have negative real parts. The Lyapunov-Krasovskii [Kra63] approach and the Lyapunov-Razumikhin [Raz60] approach are extensions of the Lyapunov stability theory, which are both widely used in developing stability conditions for time-delay systems. Compared with the classical Lyapunov function that depends only on the current state, a Lyapunov-Krasovskii functional depends on all values in the interval $[t - h(t), t]$. The

Lyapunov-Razumikhin theorem looses the restriction on the Lyapunov functional and has a larger applicable scope. The input-output (I-O) approach [LLLZ12] [LZL06], which can be applied to non-linear systems, is another widely used tool in the stability theory. In the aspect of the control design problem, many control approaches have been studied such as the sliding mode control (SMC) [FSF96] [JVM09].

In the input-output approach, one very useful analysis and synthesis tool is the Conic Sector Theorem. Originally proposed by Zames in [Zam66], the Conic Sector Theorem provides an I-O stability result that applies to both linear and nonlinear square systems. A lot of work has been done in this field. For instance, Gupta and Joshi proposed the Conic Sector Lemma [GJ94], providing linear matrix inequality (LMI) conditions to establish interior conic bounds to LTI undelayed systems. Similarly, Bridgeman and Forbes proposed the Exterior Conic Sector Lemma [BF15]. This work extended the application of the Conic Sector Theory. For systems with constant input, state, and output delays, Bridgeman and Forbes proposed stability conditions that consider all these three types of delays simultaneously in [Bri16]. In the comparative study [BF17], the author compared some of the existing I-O stability criteria. The Extended Conic Sector Theorem was proposed and served as a framework for comparison. A more detailed discussion on existing work will be given in Chapter II.

In this thesis, we focus our attention on square LTI systems with time-varying state delays. Some existing work focused on systems with constant delay [GC04], [UPP18]. Other work on time-varying delays focused mainly on the system gain [HEAH15], [EAHEH17]. We will apply the conic-sector based approach and characterize the system more tightly using general conic bounds.

1.2 Background

In this section, the background information which will play a very important role in the following chapters is introduced. We follow a natural order to illustrate it. First, the state-space model is stated, which will be the foundation of further study. Then, the definitions and some theorems on stability, controllability, and observability of linear systems are given. The Lyapunov stability is then demonstrated, providing another very widely used stability criterion. The H_2 optimal control problem is also stated. Then, the time-delays are taken into consideration. Two important tools, the Lyapunov–Krasovskii functional and the Conic Sector theorem are introduced.

1.2.1 State Equations

The state-space model is a important analysis tool in modern control theory. It can be used to represent a variety of systems, linear or non-linear, time-invariant or time-variant. In most cases, a nth-order differential equation can be represented as n 1st-order differential equations using the state equations. Since the state-space equations are in the vector and matrix form, it is very convenient to study systems with multiple input and output. There are many other advantages of using state-space equations to represent systems.

The general form of state representation for a LTI system is shown below:

$$\begin{aligned} \dot{x}(t) &= Ax(t) + Bu(t), \\ y(t) &= Cx(t) + Du(t), \end{aligned} \tag{1.1}$$

where $x(t) \in \mathbb{R}^n$ is the state vector, $u(t) \in \mathbb{R}^m$ is the input vector and $y(t) \in \mathbb{R}^p$ is the output vector. A, B, C, D are dynamics, input, output, and feedthrough matrices respectively. Define initial condition $x(t_0) = x_0$.

It should be noted that the state representation for a given system is not unique. The system matrices may change with different choice of the state vectors.

The solution of equation (1.1) is

$$x(t) = e^{At}x_0 + \int_0^t e^{A(t-\tau)}Bu(\tau)\,d\tau.$$

This can be proved directly by taking derivative to $x(t)$, thus the proof is omitted here. Take laplace transform to equation (1.1), we have

$$sX(s) = AX(s) + BU(s),$$
$$Y(s) = CX(s) + DU(s),$$

which implies

$$Y(s) = \big(C(sI - A)^{-1}B + D\big)U(s),$$

where $H(s) = C(sI - A)^{-1}B + D$ is the transfer function.

1.2.2 Stability, Controllability, and Observability

When studying control systems, it is important to study system properties such as stability, controllability, and observability. In the state-space model as stated above, the future state is determined by the current state and input. The current state and input also determine the current output. Intuitively, we want to study these system properties. For example, given an equilibrium, we want to know if it is stable or not, that is, if it can remain at the equilibrium after some suitable perturbation is applied to it. This is the stability problem. Also, we want to know if can we apply some control law to an unstable system and make it stable. The observability problem concerns whether or not the initial state can be uniquely determined given input and output.

Here we state some definitions and theorems without proof [WL$^+$07].

Definition 1.1 (Controllability, [WL$^+$07])

The linear system 1.1 is controllable if given any $x(t_0) = x_0$, there exists a continuous control law $u(t)$ for $t \in [t_0, t_f]$ such that $x(t_f) = 0$.

Theorem 1.1 (Controllable systems, [WL$^+$07])

The system 1.1 is controllable if and only if the controllability gramian

$$W(t_0, t_f) = \int_{t_0}^{t_f} e^{A(t_0-t)} BB^T e^{A^T(t_0-t)}\, dt$$

is invertible.

Definition 1.2 (Observability, [WL$^+$07])

The linear system 1.1 is observable over $[t_0, t_f]$ if given any $x(t_0) = x_0$ can be uniquely determined from $y(t)$ for $t \in [t_0, t_f]$.

Theorem 1.2 (Observable systems, [WL$^+$07])

The system 1.1 is observable if and only if the observablility gramian

$$M(t_0, t_f) = \int_{t_0}^{t_f} e^{A^T(t_0-t)} C^T C e^{A(t_0-t)}\, dt$$

is invertible.

Definition 1.3 (Uniform Stability, [WL$^+$07])

The equilibrium state $\tilde{x}$ is uniformly stable if $\exists \gamma > 0$ such that $||x(t)|| < \gamma ||x_0||$, $\forall x_0, t_0$.

1.2.3 Lyapunov Stability

The Lyapunov stability analysis is named after the Russian mathematician A. M. Lyapunov. Compared which the frequency domain method that applies to LTI systems, the Lyapunov stability can be applied to nonlinear systems, though the state-space realization is needed. It is well noted that the Lyapunov stability analysis provides a sufficient condition, that is, we cannot conclude that a system is unstable if one specific Lyapunov function violates the conditions stated below.

Here we give the Lyapunov stability theorem.

Theorem 1.3 (Lyapunov Stability, [WL+07])

Consider system $\dot{x} = f(x)$, let $x = 0$ be an equilibrium and let $D \subset \mathbb{R}^n$ be a domain containing $x = 0$. Let $V : D \to \mathbb{R}$ be a continuously differentiable function such that:

- $V(0) = 0$ and $V(x) > 0$ for $x \in D \setminus \{0\}$
- $\dot{V}(x) \leq 0$ for $x \in D$,

then $x = 0$ is stable. Morever, if

- $\dot{V}(x) < 0$ for $x \in D \setminus \{0\}$,

Then $x = 0$ is asymptotically stable.

1.2.4 H_2 Optimal Control

In section 1.2.2, we discussed the stability, controllability, and observability of systems. There are two questions concerning us:

- Will a stable system stay stable with some external disturbances?
- Can we design a control law to make an unstable plant stable?

These two questions are the motivation why we study feedback control. H_2 optimal control problem, also known as the Linear Quadratic Gaussian (LQG) problem, is one special case in the category of feedback control, the H_2 norm of the closed-loop transfer function is being minimized against the disturbances.

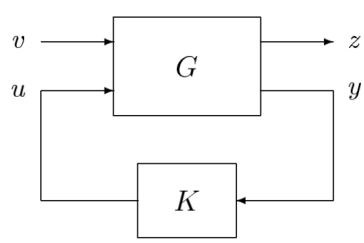


Figure 1.1: Feedback interconnection.

The standard H_2 optimal control problem is stated here [Toi98]. Consider the system shown in figure 1.1 with the partition of G, where v is the disturbance, u is the input, z is the controlled output and y is the measured output. Consider square system:

$$\begin{bmatrix} z \\ y \end{bmatrix} = \begin{bmatrix} G_{11} & G_{12} \\ G_{21} & G_{22} \end{bmatrix} \begin{bmatrix} v \\ u \end{bmatrix},$$

the closed-loop system $z = Fv$ has transfer function F given by:

$$F = G_{11} + G_{12}(I - KG_{22})^{-1}KG_{21}.$$

Consider cost function

$$J(K) = ||F||_2^2,$$

the H_2 optimal control problem is to find a control K which stabilizes the closed-loop system and minimizes the cost function $J(K)$.

Assume plant G has state-space realization

$$\begin{aligned} \dot{x}(t) &= Ax(t) + B_1 v(t) + B_2 u(t), \\ z(t) &= C_1 x(t) + D_{12} u(t), \\ y(t) &= C_2 x(t) + D_{21} v(t), \end{aligned}$$

and assume the following assumptions hold:

1. The pair $(A,\, B_2)$ is stabilizable. The pair $(C_1,\, A)$ has no unobservable modes on the imaginary axis.

2. $D_{12}^T D_{12}$ is invertible and $D_{12}^T C_1 = 0$.

3. The pair $(C_2,\, A)$ is detectable. The pair $(A,\, B_1)$ has no uncontrollable modes on the imaginary axis.

4. $D_{21} D_{21}^T$ is invertible and $D_{21} B_1^T = 0$.

In this document, we only focus on the optimal state-feedback control problem, and other problems are omitted. The optimal state-feedback control law is $K = -(D_{12}^T D_{12})^{-1} B_2^T S$, where S is the solution to the Algebraic Riccati Equation:

$$A^T S + SA - SB_2(D_{12}^T D_{12})^{-1} B_2^T S + C_1^T C_1 = 0.$$

1.2.5 Time-Delay Systems

Time-delay systems are usually described by the functional differential equations. They can be classified into two types, the retarded type and the neutral type.

The retarded type time-delay systems [Kha12] has the form

$$\dot{x}(t) = g(t, x(t), x(t-h)),$$

where $h > 0$ is the time delay.

The neutral type time-delay systems [Kha12] has the form

$$\frac{d}{dt}(x(t) - Dx(t-h)) = f(t, x_t),$$

where $h > 0$ is the time delay. In this thesis, we focus on the retarded type.

In the state-space representation, time delays can appear in the state vector, the input vector, and the output vector. They can also appear in the interconnections between plants as communication delays. Also, they can be time-invariant and time-varying. A square LTI system with the time-varying state, input, and output delays has the following form:

$$\begin{aligned} \dot{x}(t) &= Ax\big(t - h_s(t)\big) + Bu\big(t - h_i(t)\big), \\ y(t) &= Cx\big(t - h_s(t) - h_o(t)\big) + Du\big(t - h_i(t)\big), \end{aligned} \tag{1.2}$$

where $h_s(t)$, $h_i(t)$, and $h_o(t)$ are the time-varying state, input, and output delays respectively. In this thesis, we focus on systems with time-varying state delays. A more detailed definition of the system and delays will be shown in Chapter III.

1.2.6 Lyapunov–Krasovskii Method

With the Lyapunov stability analysis for undelayed systems in mind, it is very intuitive that researchers tried to apply it to systems with delays. The Lyapunov–Krasovskii functional based approach is the extension of Lyapunov stability to time-delay systems. Similar to the Lyapunov stability analysis, the Lyapunov–Krasovskii method also provides a sufficient condition. A common approach of applying this method is first, constructing a Lyapunov–Krasovskii functional and second, using different bounding techniques to establish the bound as stated in Theorem 1.2.6.

Here we follow the theorem in [SH12]

Theorem 1.4 (Lyapunov–Krasovskii Stability Theorem, [SH12])

The system 1.1 is asymptotically stable if there exists a quadratic Lyapunov–Krasovskii functional $V(\phi, \dot{\phi})$, such that for some $\varepsilon_i > 0$ $(i = 1, 2, 3)$ it satisfies

$$\varepsilon_1 ||\phi(0)||^2 \leq V(\phi, \dot{\phi}) \leq \varepsilon_2 ||\phi||_W^2,$$

and its derivative along the system trajectory

$$\dot{V}(\phi, \dot{\phi}) = \dot{V}(x_t, \dot{x}_t)|_{x_t=\phi},$$

satisfies

$$\dot{V}(\phi, \dot{\phi}) \leq \varepsilon_3 ||\phi(0)||^2.$$

1.2.7 Conic Systems

The Conic-Sector theorem was originally proposed by Zames in [Zam66]. During the past fifty years, researchers have made a lot of progress in this topic. As stated above, the Conic-Sector based approach is an input-output approach, focused not on the system's internal property, but the input and the output of the system. This feature can be seen clearly in the following definition.

For simplicity and readability, the definition of conic systems and the Conic Sector Theorem given here follow [Bri16]. Compared to the original definition, the definition here introduces a β term that depends only on the initial conditions.

Definition 1.4 (Conic Systems, [Bri16])

Consider a system $\mathcal{G}$ on L_{2e}, satisfying the following inequality:

$$\delta(-||\mathcal{G}u||_2^2 + (a+b)\langle \mathcal{G}u, u\rangle_T - ab||u||_{2T}^2) \geq \delta\beta, \tag{1.3}$$

$\forall u$ on L_{2e} and T on $\mathbb{R}^+$, where β depends only on the initial conditions and $a \leq b$. The system, $\mathcal{G}$, is:

- **interior conic** with conic bounds $a, b \in \mathbb{R}$ if $\delta = 1$, denoted as $\mathcal{G} \in$ cone $[a, b]$;
- **exterior conic** with conic bounds $a, b \in \mathbb{R}$ if $\delta = -1.$, denoted as $\mathcal{G} \in$ excone (a, b);
- **strictly interior conic** with conic bounds $a, b \in \mathbb{R}$ if $\delta = 1$, denoted as $\mathcal{G} \in$ cone (a, b) if $\mathcal{G} \in$ cone $[a+\epsilon, b-\epsilon]$ for some small $\epsilon > 0$;
- **strictly exterior conic** with conic bounds $a, b \in \mathbb{R}$ if $\delta = 1$, denoted as $\mathcal{G} \in$ cone $[a, b]$ if $\mathcal{G} \in$ cone $(a-\epsilon, b+\epsilon)$ for some small $\epsilon > 0$;

Theorem 1.5 (Conic Sector Theorem, [Bri16])

Let $r^T = [r_1^T, r_2^T] \in L_{2e} \times L_{2e}$ be the external signal and $\mathcal{G}_1, \mathcal{G}_2 : L_{2e} \to L_{2e}$ be a plant and controller in a negative feedback interconnection as described by

$$y = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} \mathcal{G}_1 u_1 \\ \mathcal{G}_2 u_2 \end{bmatrix} \quad u = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} r_1 - y_2 \\ r_2 + y_1 \end{bmatrix} \tag{1.4}$$

Suppose that $\mathcal{G}_1 \in$ cone $[a, b]$ for $a \in (-\infty, b), b \in (0, \infty)$. The close loop system, $\mathcal{G}$ defined as $y = \mathcal{G}u$ is I-O stable if:

1. $\mathcal{G}_2 \in$ excone $(-\frac{1}{a}, -\frac{1}{b})$, if $a \in [0, \infty)$, or
2. $\mathcal{G}_2 \in$ cone $(-\frac{1}{b}, -\frac{1}{a})$, if $a \in (\infty, 0]$.

Chapter 2

Literature Review

In this chapter, an overview of some recent research in the field of time-delay systems is given. As stated in Chapter I, two main aspects of interest are the stability theory and the control synthesis problem.

In studying the stability theory, one very common approach is using the Lyapunov–Kras ovskii functional based method, where a Lyapunov–Krasovskii functional is constructed at first, and different bounding techniques are used to find the upper and lower bound of the functional and the upper bound of its first order time derivative. These stability criteria are classified into two classes, delay-dependent and delay-independent, depending on whether the information of delays is included in the stability criteria. For example, in [SH12] and [Han02], different bounding techniques were used such as Jensen's inequality and Rayleigh's inequality to derive the delay-dependent stability criteria for systems with time-varying state delays. Another widely used approach is the input-output approach. In [ZGK12] and [HEAH15], the Scaled Small Gain theorem was applied to systems with time-varying state delays to derive the LMI conditions. In [XAG14], a passivation method that guarantees the I-O stability for systems with input and output delays was introduced.

In the control aspect, two important performance indices are the H_2 norm and the H_∞ norm. In [Xie08], the author started with the standard delay-independent H_2 performance analysis in [BEGFB94] and proposed less conservative results for systems with polytopic uncertainties. In [KKM98] a delay-independent stability criterion which assured the H_∞ performance was given for state-delayed systems with time-varying norm-bounded uncertainties. In [JVM10], the author showed how the H_2 norm of time-delay systems can be computed by solving the delay Lyapunov equation. In [KR91], the mixed H_2/H_∞ performance was considered. In [FSR04] and [FS02], the descriptor form was used to derive the

delay-dependent stabilizing state-feedback controllers for time-delay systems.

In this chapter, five representative papers are chosen to be discussed in detail about the approaches and bounding techniques used in these papers. Some of the techniques helped to derive the main results discussed in this paper, while others may not be used directly in this thesis but provided new perspectives to consider the problem faced by the author. At the end of this chapter, a numerical example is given, showing the differences and inner relation of these papers.

2.1 Less conservative delay-dependent stability criteria for linear systems with interval time-varying delays

In [SH12], the author provided a less conservative delay-dependent stability criteria for systems with time-varying state delays as stated below. Two numerical examples were given to show the advantages over other existing work.

$$\begin{aligned} \dot{x}(t) &= Ax(t) + A_d x\big(t - h(t)\big), t \geq 0 \\ x(\theta) &= \phi(\theta), \qquad\qquad \theta \in [-h_2, 0] \end{aligned} \tag{2.1}$$

and the time-varying delay satisfies the following condition:

$$\begin{aligned} 0 < h_m \leq h(t) \leq h_M, \\ \dot{h}(t) \leq \mu \end{aligned} \tag{2.2}$$

The derivation of this paper was based on the Lyapunov–Krasovskii stability theorem. The author constructed a new Lyapunov–Krasovskii functional and estimated a tighter upper bound of its derivative. Because fewer design matrix variables were introduced during the derivation, the results proposed in this paper were less conservative compared to some other work theoretically.

Theorem 2 in [SH12] provided LMI conditions where h_m, h_M and μ are known, while Corollary 1 applies to the same systems when μ is unknown. Theorem 4 is a special case of Theorem 2 where $h_m = 0$.

2.1.1 Useful Techniques

Here we summarize some bounding technique from [SH12].

Jensen's Inequality

For any constant matrix $M \in \mathbb{R}^{n\times n}$ which is positive definite, scalar $\lambda > 0$ and vector valued function $w : [0, \lambda] \to \mathbb{R}^n$ which integrations are well defined,

$$\Big(\int_0^\lambda w(s)\,ds\Big)^T M\Big(\int_0^\lambda w(s)\,ds\Big) \le \lambda \int_0^\lambda w^T(s)Mw(s)\,ds.$$

This can be proved by Schur Complement:

$$\begin{bmatrix} w^T(s)Mw(s) & w^T(s) \\ w(s) & M^{-1} \end{bmatrix} \ge 0,$$

integrating the above inequality from $[0, \lambda]$ gives us:

$$\begin{bmatrix} \int_0^\lambda w^T(s)Mw(s)\,ds & \int_0^\lambda w^T(s)\,ds \\ \int_0^\lambda w(s)\,ds & \int_0^\lambda M^{-1}\,ds \end{bmatrix} \ge 0,$$

and taking Schur Complement gives us the desired result.

Rayleigh's Inequality

For any constant matrix $M \in \mathbb{R}^{n\times n}$ which is positive definite and $x \in \mathbb{R}^n$, denote the minimum and maximum eigenvalue of M as $\lambda_{min}(M)$ and $\lambda_{max}(M)$ respectively, we have:

$$\lambda_{min}(M)||x||_2^2 \le x^T Mx \le \lambda_{max}(M)||x||_2^2.$$

Since $M > 0$, we have $M = Q^T \Lambda Q$ where Λ is diagonal matrix of eigenvalues of M and Q satisfies $Q^T = Q^{-1}$ and $Q^T Q = I$. We have:

$$\begin{aligned}
\lambda_{min}(M)||x||_2^2 &= \lambda_{min}(M)||Q^T x||_2^2 \\
&= \sum_k \lambda_{min}(M)[Q^T x]_2^2 \\
&\leq \sum_k \lambda_k(M)[Q^T x]_2^2 \\
&= x^T M x,
\end{aligned}$$

similarly, we have

$$\begin{aligned}
\lambda_{max}(M)||x||_2^2 &= \lambda_{max}(M)||Q^T x||_2^2 \\
&= \sum_k \lambda_{max}(M)[Q^T x]_2^2 \\
&\geq \sum_k \lambda_k(M)[Q^T x]_2^2 \\
&= x^T M x,
\end{aligned}$$

and thus prove the inequality.

Integration Trick

Consider a positive scalar h_1 and a well defined positive function $f : [t - h_1, t] \to \mathbb{R}$, we have the following inequality:

$$\begin{aligned}
\int_{-h_1}^{0} \int_{t+s}^{t} h_1 f(\alpha)\, d\alpha\, ds &\leq \int_{-h_1}^{0} \int_{t-h_1}^{t} h_1 f(\alpha)\, d\alpha\, ds \\
&= \int_{t-h_1}^{t} \int_{-h_1}^{0} h_1 f(\alpha)\, ds\, d\alpha \\
&= h_1^2 \int_{t-h_1}^{t} h(\alpha)\, d\alpha.
\end{aligned}$$

2.2 Stability analysis of linear systems with time varying delay: An input output approach

In [HEAH15], the author focused on the same systems as in [SH12], thus the state equations are omitted here. The author applied the Scaled Small Gain theorem (SSG) stated in [ZGK12]. Two numerical examples were given to show the less conservative results compared to [SH12].

To apply the SSG theorem, the original system was transformed into two subsystems. Instead of finding bounds for Lyapunov-Krasovskii functional, the author applied this input-output approach Similar to [SH12], in this paper, the author provided stability criteria when $h_m > 0$ and $h_m = 0$, and also when μ is known and unknown.

2.2.1 Useful Techniques

System Transformation

The original system was transformed into following two subsystems:

$$\begin{aligned} (S_1) &: z(t) = Gw(t), \\ (S_2) &: w(t) = \Delta z(t), \end{aligned} \tag{2.3}$$

where S_1 is known and S_2 is unknown with operator: $\Delta \in \mathcal{D} \triangleq \{\Delta : ||\Delta||_\infty \leq 1\}$. Specifically for system (2.3), let

$$w(t) = \frac{2}{h_{12}}\big[\big(x(t-h(t)\big) - \frac{1}{2}\big(x(t-h_1) + x(t-h_2)\big)\big],$$

and we get

$$S_1 = \begin{cases} \dot{x}(t) = Ax(t) + \frac{1}{2}A_d x(t-h_m) + \frac{1}{2}A_d x(t-h_M) + \frac{h_{12}}{2}A_d w(t), \\ z(t) = \dot{x}(t), \end{cases} \tag{2.4}$$

then, system S_1 is the known forward system.

Scaled Small Gain Theorem

Here we follow Lemma 1 in [ZGK12]: Consider (2.5), and assume S_1 is internally stable. The closed-loop system formed by S_1 and S_2 is robustly asymptotically stable for all Δ if there exist matrices $\{T_w, T_z\} \in T$ with

$$T \triangleq \left\{\{T_w, T_z\} \in \mathbb{R}^{w\times w} \times \mathbb{R}^{z\times z} : T_w, T_z \text{nonsingular}, ||T_w \circ \Delta \circ T_z^{-1}||_\infty \leq 1\right\}$$

such that the following SSG condition holds:

$$||T_w \circ G \circ T_z^{-1}||_\infty < 1.$$

2.3 Improved delay-independent H_2 performance analysis and memoryless state feedback for linear delay systems with polytopic uncertainties

In [Xie08], the author focused on the H_2 performance of the state-delayed systems with polytopic uncertainties. Started by the standard H_2 performance analysis given by [BEGFB94], the author introduced a parameter-dependent Lyapunov function and thus reduced the conservatism. A numerical example was given to show how the H_2 performance changes with respect to the design variable scalar.

The system of interests is shown in 2.5

$$\begin{aligned} \dot{x}(t) &= A(a)x(t) + A_d(a)x(t-h_1) + B(a)u(t), \\ y(t) &= C(a)x(t) + C_d(a)x(t-h_1), \end{aligned} \tag{2.5}$$

assume null initial condition, that is $\forall t \in [-h_1, 0]$, $x(t) = 0$.

First, a new LMI condition for H_2 performance analysis tool for the nominal systems with state delays was given. Then, the polytopic uncertainties were taken into consideration. A memoryless state-feedback controller which guarantees the H_2 performance was then designed for the uncertain systems.

2.3.1 Useful Techniques

Polytopic-type Uncertainties

In systems 2.5, the polytopic uncertainty domain is defined as follows:

$$\phi := \big(A(a), A_d(a), B(a), C(a), C_d(a)\big) = \sum_{i=1}^{N} a_i(A_i, A_{d,i}, B_i, C_i, C_{d,i}),$$
$$a_i \geq 0, i = 1, ..., N, \quad \sum_{i=1}^{N} a_i = 1,$$

where $A_i, A_{d,i}, B_i, C_i, C_{d,i}, i = 1, ..., N$ are constant matrices. Assume that the system matrices $\big(A(a), A_d(a), B(a), C(a), C_d(a)\big)$ in 2.5 belongs to ϕ, then, it can be expressed as a linear combination of $(A_i, A_{d,i}, B_i, C_i, C_{d,i})$.

The LMI conditions at the vertices $(A_i, A_{d,i}, B_i, C_i, C_{d,i})$ can be viewed as the 'worst case.' If the conditions hold at every vertex, then it must hold everywhere inside the polytope constructed by the vertices.

Schur Complement

Schur complement is a very common technique in deriving LMI conditions. Here we give the statement and omit the proof.

Define $M = \begin{bmatrix} A & B \\ C & D \end{bmatrix}$, the schur complement of D is:

$$M/D := A - BD^{-1}C,$$

further, M is positive definite if and only if D and its schur complementM/D are positive definite.

2.4 Delay-dependent robust stabilization of uncertain state-delayed systems

In [MPKL01], the author focused on uncertain systems with constant state delay and also applied the Lyapunov–Krasovskii approach. Two numerical examples were given to show the advantages of the proposed approach over some existing results.

The system of interests is shown in 2.6

$$\begin{aligned} \dot{x}(t) &= \big(A + DF(t)E\big)x(t) + \big(A_d + D_d F_d(t) E_d\big)x(t-h) + \big(B + DF(t)E_b\big)u(t), \\ x(t) &= \phi(t) \qquad t \in [-h, 0], \end{aligned} \tag{2.6}$$

where $A, A_d, B, D, D_d, E, E_d, E_b$ are constant matrices with appropriate dimension and $F(t)$ and $F_d(t)$ are time-varying uncertainties satisfy $||F(t)|| \leq 1$, $||F_d(t)|| \leq 1$.

The author started with the nominal unforced state-delayed systems, then extended the conditions to design a stabilizing controller. In the case where the size of delay was unknown, a memoryless controller was designed while when the size was known, a delayed state feedback controller was designed. For the uncertain systems, the author also started with the analysis problem and then extended it to the controller synthesis problem. S-procedure was used to deal with the norm bounded uncertainties.

Note that in the controller synthesis problem, a non-linear term appeared and thus made the conditions unsolvable using convex optimization methods. An equivalent non-linear minimization problem was stated and an algorithm to solve it was proposed.

2.4.1 Useful Techniques

Upper Bound of Inner Product

For any $a \in \mathbb{R}^n$, $b \in \mathbb{R}^{2n}$, $X \in \mathbb{R}^{n\times n}$, $Y \in \mathbb{R}^{n\times 2n}$, $Z \in \mathbb{R}^{2n\times 2n}$, $N \in \mathbb{R}^{2n\times n}$ and

$$\begin{bmatrix} X & Y \\ Y^T & Z \end{bmatrix} \geq 0,$$

we have:

$$-2b^T Na \leq \begin{bmatrix} a \\ b \end{bmatrix}^T \begin{bmatrix} X & Y - N^T \\ Y^T - N & Z \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix}.$$

The proof is shown here:

$$\begin{aligned}
&\begin{bmatrix} a \\ b \end{bmatrix}^T \begin{bmatrix} X & Y - N^T \\ Y^T - N & Z \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} \\
=& \begin{bmatrix} a \\ b \end{bmatrix}^T \begin{bmatrix} X & Y \\ Y^T & Z \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} - \begin{bmatrix} a \\ b \end{bmatrix}^T \begin{bmatrix} 0 & N^T \\ N & 0 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} \\
=& \begin{bmatrix} a \\ b \end{bmatrix}^T \begin{bmatrix} X & Y \\ Y^T & Z \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} - 2b^T Na \\
\geq& -2b^T Na.
\end{aligned}$$

Pre-and-Post Multiplication

This is a common trick usually used in the derivation of closed-loop systems with state feedback when the LMI conditions for unforced systems were already established. A common term in LMI conditions for unforced systems is PA and state feedback control is usually modeled as $u(t) = Kx(t)$ with a feedback gain. Then, the closed-loop systems have state-space realization

$$\dot{x}(t) = (A + BK)x(t),$$

replacing A with $A + BK$ introduces the nonlinear term PBK and thus cannot be solved with convex optimization software. In this case, to eliminate this nonlinear term, we pre-and-post multiply PBK with P^{-1} and let KP^{-1} be the new design variable.

The theoretical basis is shown here, $\forall M \in \mathbb{R}^{n\times n}$ and $T \in \mathbb{R}^{n\times m}$, if M is positive definite and T is full rank, then $T^T MT$ is also positive definite.

S-procedure

As described in [BEGFB94], in the derivation of LMIs, there exists such constraints that, if some quadratic functions are all negative, than some other quadratic functions are negative.

Consider $\zeta \in \mathbb{R}^n$, $u_i \in \mathbb{R}^n$, $v_i \in \mathbb{R}$, $T_i = T_i^T \in \mathbb{R}^{n\times n}$ and quadratic function

$$\begin{aligned} F_i(\zeta) &= \zeta^T T_i \zeta + 2u_i^T \zeta + v_i \\ &= \begin{bmatrix} \zeta \\ 1 \end{bmatrix}^T \begin{bmatrix} T_i & u_i \\ u_i^T & v_i \end{bmatrix} \begin{bmatrix} \zeta \\ 1 \end{bmatrix}. \end{aligned}$$

Consider following condition:

$$\forall \zeta, F_0(\zeta) \geq 0 \text{ such that } F_i(\zeta) \geq 0, \quad i = 1, 2, ..., p. \tag{2.7}$$

It can be seen that 2.7 holds if the following condition holds:

$$\exists \tau_i \geq 0, \quad i = 1, 2, ..., p, \text{ such that } \forall \zeta, F_0(\zeta) - \sum_{1=1}^{p} \tau_i F_i(\zeta) \geq 0, \tag{2.8}$$

which can be written as

$$\begin{bmatrix} T_0 & u_0 \\ u_0^T & v_0 \end{bmatrix} - \sum_{1=1}^{p} \tau_i \begin{bmatrix} T_i & u_i \\ u_i^T & v_i \end{bmatrix} \geq 0, \tag{2.9}$$

and 2.9 is a LMI problem.

2.5 Robust sampled-data stabilization of linear systems: an input delay approach

[FSR04] focused on the continuous-time systems with piecewise-continuous input as stated in (2.6). Based on the descriptor representation, a new approach of robust sampled-data stabilization is given. The LMI conditions guarantee stability for any sampling interval less or equal to a maximum h. The stability condition with a saturated sampled-data controller was also derived.

In the derivation process, the LMI conditions first proposed were bilinear in two independent matrix variables. Applying the Schur complement and pre-post multiplication in above mentioned sections, the LMI conditions were obtained.

$$\begin{aligned}
\dot{x}(t) &= Ax(t) + Bu(t), \\
u(t) &= Kx(t - \tau(t)), \\
\tau(t) &= t - t_k, \quad t_k \leq t < t_{k+1},
\end{aligned} \tag{2.10}$$

assume $t_{k+1} - t_k \leq h$, where h is the maximum sampling interval.

2.5.1 Useful Techniques

Descriptor Representation

As stated in [Fri14], a descriptor system may have impulsive solutions. Consider the following undelayd system:

$$E\dot{x}(t) = Ax(t),$$

where $x(t) \in \mathbb{R}^n$, E and A are all $n - by - n$ matrices with $rank(E) = n_1 \leq n$.

The following descriptor representation introduced no impulsive solutions since $y(t)$ is multiplied by nonsingular matrix I.

$$\dot{x}(t) = y(t), \quad 0 = -y(t) + Ax(t) + BKx(t - h(t)).$$

Leibniz's Rule

The Leibniz's rule is widely used in existing work, we put it here because we encountered it in this paper for the first time.

$$\frac{d}{dx}\left(\int_{a(x)}^{b(x)} f(x,t)\,dt\right) = f\big(x, b(x)\big) \cdot \frac{d}{dx}b(x) - f\big(x, a(x)\big) \cdot \frac{d}{dx}a(x) + \int_{a(x)}^{b(x)} \frac{\partial}{\partial x} f(x,t)\,dt.$$

Specifically in this paper, the author applied Leibniz's rule to the following term twice and get:

$$\begin{aligned}
&\frac{d}{dt}\left(\int_{-h}^{0}\int_{t+\theta}^{t} y^T(s)Ry(s)\,ds\,d\theta\right)\\
=&\int_{-h}^{0}\frac{\partial}{\partial t}\int_{t+\theta}^{t} y^T(s)Ry(s)\,ds\,d\theta\\
=&hy^T(t)Sy(t)-\int_{-h}^{0} y^T(t+\theta)Sy(t+\theta)\,d\theta.
\end{aligned}$$

2.6 Numerical Examples

Two numerical examples are given here. In this section, we implemented Theorem 2 in [SH12], Theorem 3 and Lemma 1 in [Xie08], Theorem 1 in [HEAH15], Theorem 4 in [MPKL01] by letting $L = R$, and Lemma 2.3 in [FSR04]. It should be noted that in the implementation of [HEAH15], for the example used in this paper, our results are slightly different as the results stated in the paper. Since the author did not state what solver they used, we assume the difference in results is because of the different precision between different solvers.

2.6.1 Problem Statement

The systems used in this section is given below:

$$\begin{aligned}
\dot{x}(t) &= Ax(t) + A_d x(t-h_1) + Bu(t-h_u),\\
y(t) &= Cx(t) + C_d x(t-h_1),
\end{aligned} \tag{2.11}$$

where x, u, y are the state, input, and output vectors respectively. h_1 and h_u are state and input delay respectively which satisfy:

$$\begin{aligned} &h_1 = h_1(t), \\ &0 < h_m \leq h_1(t) \leq h_M, \\ &\dot{h}_1(t) < \mu, \end{aligned} \tag{2.12}$$

for system $\mathcal{G}_1$, the matrices A, A_d, B, C, C_d are given by:

$$A = \begin{bmatrix} 0 & 1 \\ -1+g_1 & -2 \end{bmatrix}, \quad A_d = \begin{bmatrix} 0 & 0 \\ -1 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} g_2 \\ 1 \end{bmatrix},$$
$$C = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \quad C_d = 0,$$

and for system $\mathcal{G}_2$, the matrices A, A_d, B, C, C_d are given by:

$$A = \begin{bmatrix} 0 & 1 \\ g_1 & -2 \end{bmatrix}, \quad A_d = \begin{bmatrix} 0 & 0 \\ -1 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} g_2 \\ 1 \end{bmatrix},$$
$$C = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \quad C_d = 0,$$

g_1, $g_2 \in [-0.1, 0.1]$, assume null initial condition.

We first use the method from [SH12] and [HEAH15], to determine the upper bound for the nominal unforced system. Then, we compare three different control methods given by [MPKL01], [FSR04] and [Xie08]. Both nominal system and uncertain system were considered. Since in different papers, the systems studied has different state equation, a specific definition of systems is given in table 2.1 and table 2.2.

Table 2.1: Specific matrix parameters used in the numerical example for system $\mathcal{G}_1$ with nominal states.

	A	A_d	B	C	C_d
[SH12]	$\begin{bmatrix} 0 & 1 \\ -1 & -2 \end{bmatrix}$	$\begin{bmatrix} 0 & 0 \\ -1 & 1 \end{bmatrix}$	0	0	0
[HEAH15]	$\begin{bmatrix} 0 & 1 \\ -1 & -2 \end{bmatrix}$	$\begin{bmatrix} 0 & 0 \\ -1 & 1 \end{bmatrix}$	0	0	0
[Xie08]	$\begin{bmatrix} 0 & 1 \\ -1 & -2 \end{bmatrix}$	$\begin{bmatrix} 0 & 0 \\ -1 & 1 \end{bmatrix}$	$\begin{bmatrix} 0 \\ 1 \end{bmatrix}$	$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$	0
[FSR04]	$\begin{bmatrix} 0 & 1 \\ -1 & -2 \end{bmatrix}$	0	$\begin{bmatrix} 0 \\ 1 \end{bmatrix}$	$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$	0
[MPKL01]	$\begin{bmatrix} 0 & 1 \\ -1 & -2 \end{bmatrix}$	$\begin{bmatrix} 0 & 0 \\ -1 & 1 \end{bmatrix}$	$\begin{bmatrix} 0 \\ 1 \end{bmatrix}$	$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$	0

Table 2.2: Specific matrix parameters used in the numerical example for system $\mathcal{G}_2$ with nominal states.

	A	A_d	B	C	C_d
[SH12]	$\begin{bmatrix} 0 & 1 \\ 0 & -2 \end{bmatrix}$	$\begin{bmatrix} 0 & 0 \\ -1 & 1 \end{bmatrix}$	0	0	0
[HEAH15]	$\begin{bmatrix} 0 & 1 \\ 0 & -2 \end{bmatrix}$	$\begin{bmatrix} 0 & 0 \\ -1 & 1 \end{bmatrix}$	0	0	0
[Xie08]	$\begin{bmatrix} 0 & 1 \\ 0 & -2 \end{bmatrix}$	$\begin{bmatrix} 0 & 0 \\ -1 & 1 \end{bmatrix}$	$\begin{bmatrix} 0 \\ 1 \end{bmatrix}$	$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$	0
[FSR04]	$\begin{bmatrix} 0 & 1 \\ 0 & -2 \end{bmatrix}$	0	$\begin{bmatrix} 0 \\ 1 \end{bmatrix}$	$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$	0
[MPKL01]	$\begin{bmatrix} 0 & 1 \\ 0 & -2 \end{bmatrix}$	$\begin{bmatrix} 0 & 0 \\ -1 & 1 \end{bmatrix}$	$\begin{bmatrix} 0 \\ 1 \end{bmatrix}$	$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$	0

2.6.2 Maximum Allowed Time-Varying Delays

Using theorem 2 in [SH12], the admissible upper bounds h_M with various h_m and $\mu = 0.3$ for the nominal states of $\mathcal{G}_1$ and $\mathcal{G}_2$ are shown in table 2.3. Note that "NaN" means such an upper bound does not exist. For both systems, we test constant delays and sinusoidal delays. We use the phase portrait to show the simulation results.

Table 2.3: Admissible upper bounds h_M with various h_m and $\mu = 0.3$ for the nominal states of $\mathcal{G}_1$ and $\mathcal{G}_2$.

h_m	0	1	2	3
Admissible upper bounds h_M for $\mathcal{G}_1$	2.35	2.35	2.58	3.47
Admissible upper bounds h_M for $\mathcal{G}_2$	1.16	1.29	NaN	NaN

We first tested the constant delays. For a constant delay, we need to find an interval in which the upper bound h_M does not exist. As far as we tested, system $\mathcal{G}_1$ is always asymptotically stable with a constant delay $\tau \in [0, 4.67]$. For system $\mathcal{G}_2$, we find that for $h_m = 1.41$ we cannot find an admissible upper bound. This means that according to [SH12], we can conclude that $\mathcal{G}_2$ with any constant delay $\tau \in [0, 1.41]$ is asymptotically stable. Because theorem 2 in [SH12] provides a sufficient condition, we cannot conclude that the system is unstable with a delay greater than 1.41. Figure 2.1 shows the simulation result. We can see that at $\tau = 1.4$ the system is stable while at $\tau = 1.7$ the system is unstable.

We chose a sinusoidal type function $\tau(t) = a_1 + a_2 sin(\omega t)$ as the time-varying delay. We tested different sets of parameters as shown in table 2.4 for system $\mathcal{G}_q$. The theoretical results are concluded from table 2.3. Figure 2.2 shows the simulation results. We can see that the simulation results match the theoretical results for all these different sets of time-varying delay.

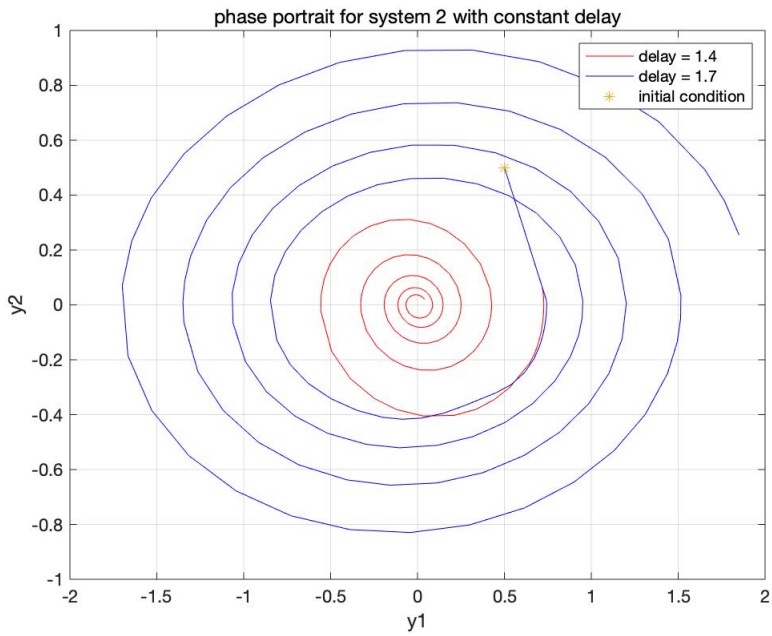


Figure 2.1: Phase portrait for system $\mathcal{G}_2$ starting from $(0.5, 0.5)$ with different constant delays. The system is stable with $\tau = 1.4$ and unstable with $\tau = 1.7$.

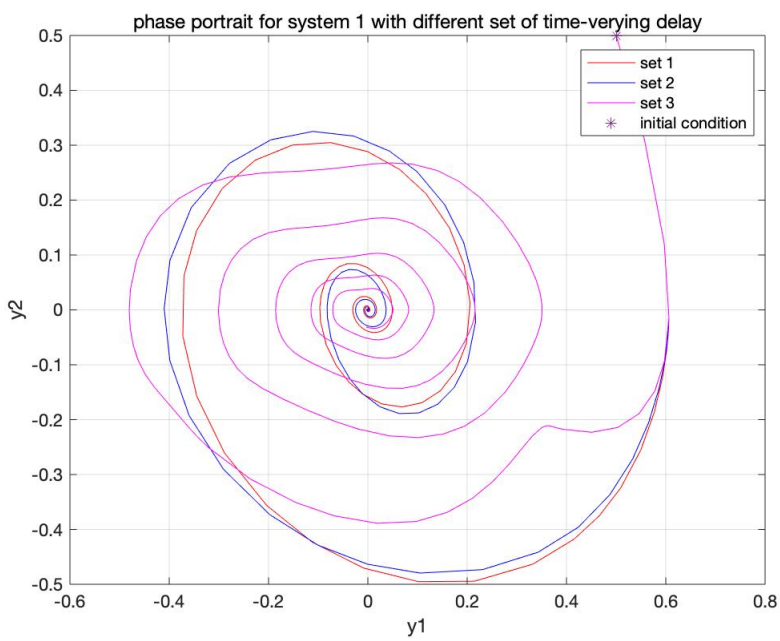


Figure 2.2: Phase portrait for system $\mathcal{G}_1$ starting from $(0.5, 0.5)$ with different time-varying delays. Systems with three different sets of delay are all stable.

Table 2.4: Theoretical and simulation result from different parameter sets for system $\mathcal{G}_1$. The simulation results match the theoretical results for all these different sets of time-varying delays.

Parameterset	a_1	a_2	ω	theoretical	simulation
1	0.5	0.2	0.5	asymptotically stable	asymptotically stable
2	0.5	0.4	0.5	asymptotically stable	asymptotically stable
3	2	0.25	2	asymptotically stable	asymptotically stable

For system $\mathcal{G}_2$, the simulation results are also expected. Table 2.5 shows the different set of parameters and expected results and figure 2.3 shows the simulation results. It can be seen that for set 3, the simulation result shows the system is unstable.

Table 2.5: Theoretical and simulation result from different parameter sets for system $\mathcal{G}_2$. The simulation results match the theoretical results for all these different sets of time-varying delays.

Parameterset	a_1	a_2	ω	theoretical	simulation
1	0.5	0.2	0.5	asymptotically stable	asymptotically stable
2	0.5	0.4	0.5	asymptotically stable	asymptotically stable
3	2	0.25	2	cannot determine	unstable

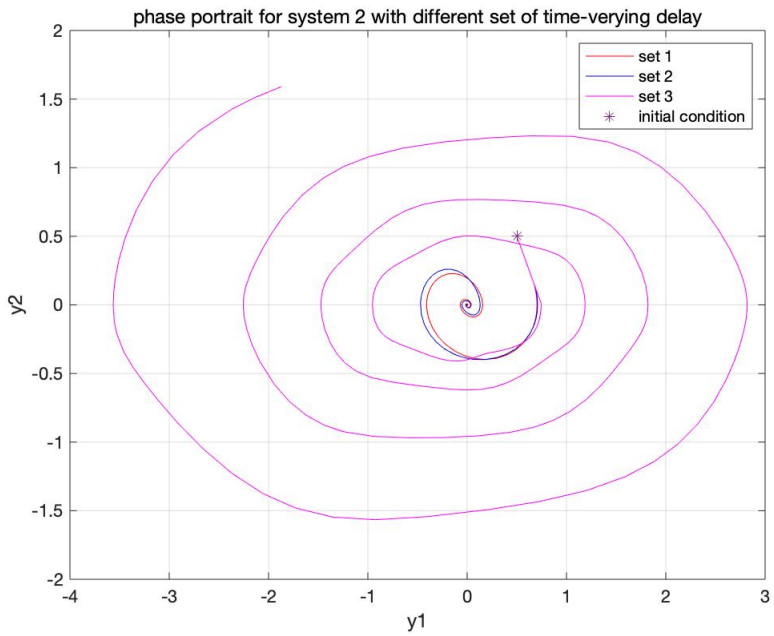


Figure 2.3: Phase portrait for system $\mathcal{G}_2$ starting from $(0.5, 0.5)$ with different time-varying delays. Systems with delay set 1 and set 2 are stable, while with set 3 of delay is unstable.

2.6.3 Study on Different Control Methods

In this section, we first use pre-developed code to find the control using each method. For [Xie08] and [FSR04], we find solution for both nominal states and state with polytopic uncertainties. We also show the control law using Lemma 1 in [Xie08]. The control laws for system $\mathcal{G}_1$ and $\mathcal{G}_2$ are shown in table 2.6 and table 2.7 respectively. It is notable that in [Xie08], the author provided a delay-independent condition, thus there is no maximum delay for 'Lemma 1,' 'Xie Nominal' and 'Xie Polytopic.'

We apply these controls to the time-delay system and use MATLAB 'dde23' function to find the simulation results. In this section, we set the delay to be 0.4 and a null initial condition. Theoretically, since the delay we chose is less than the maximum delay for delay-dependent conditions, both systems with different control laws should be asymptotically stable. Figure 2.4 and figure 2.5 show the simulation results for both systems respectively

and prove the theoretical results.

Table 2.6: Control laws and maximum allowed delays for system $\mathcal{G}_1$. The maximum allowed delay is ∞ for the delay-independent conditions.

Different controller's name	controller	maximum allowed delay
Lemma 1	$K_1 = [-1.0072, -0.7434]$	∞
Xie Nominal	$K_2 = [-0.7922, 0.0157]$	∞
Xie Polytopic	$K_3 = [-0.6258, 0.0788]$	∞
Fridman Nominal	$K_4 = [0.3574, 0.5227]$	1.4500
Fridman Polytopic	$K_5 = [0.4207, 0.5200]$	1.2700
Moon Nominal	$K_6 = [0.5877, -0.3351]$	0.7300
Moon Polytopic	$K_7 = [-1.1763, -2.6670]$	0.3100

Table 2.7: Control laws and maximum allowed delay for system $\mathcal{G}_2$. The maximum allowed delay is ∞ for the delay-independent conditions.

Different controller's name	controller	maximum allowed delay
Lemma 1	$K_1 = [-2.4554, -1.4578]$	∞
Xie Nominal	$K_2 = [-1.7400, -0.4048]$	∞
Xie Polytopic	$K_3 = [-1.8379, -0.3806]$	∞
Fridman Nominal	$K_4 = [-0.1188, 0.7441]$	0.9000
Fridman Polytopic	$K_5 = [-0.2741, 0.5233]$	0.8700
Moon Nominal	$K_6 = [-0.4122, -0.3351]$	0.7300
Moon Polytopic	$K_7 = [-2.1763, -2.6670]$	0.3100

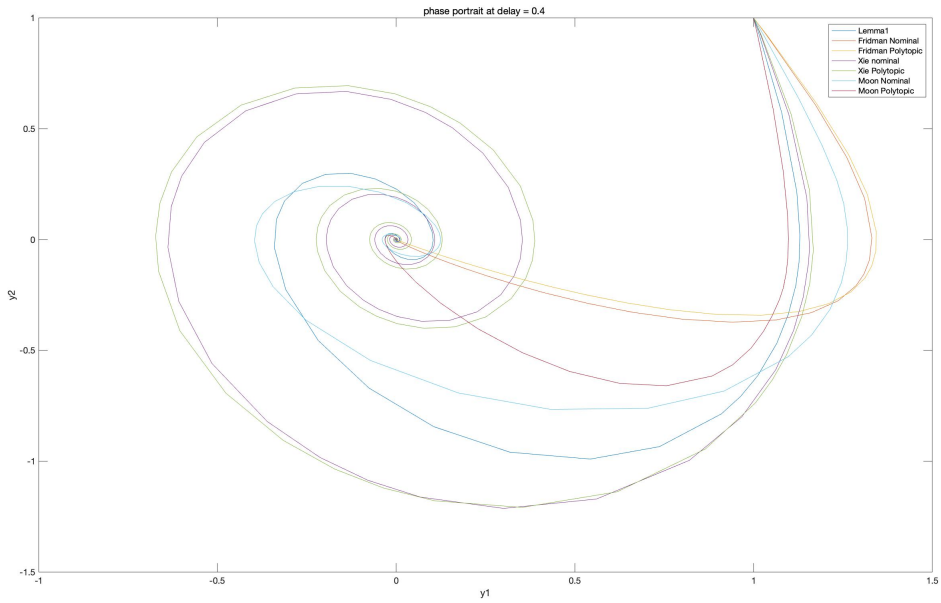


Figure 2.4: Phase portrait for system $\mathcal{G}_1$ starting from $(0.5, 0.5)$ with different control at $\tau = 0.4$. All control leads to stability but follow different trajectories.

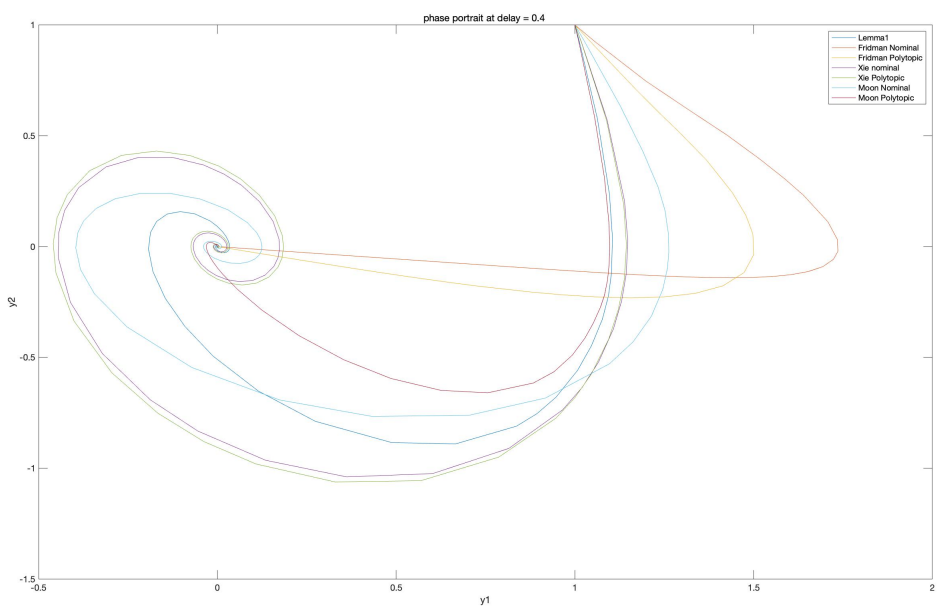


Figure 2.5: Phase portrait for system $\mathcal{G}_2$ starting from $(0.5, 0.5)$ with different control at $\tau = 0.4$. All control leads to stability but follow different trajectories.

To compare the performance of these control laws, we focus on the step response. We choose five indices: steady-state value, rise time, peak time, settling time, and overshoot. Also, we want to see how the value of delay influences these indices. For both systems, the delay range we choose is from 0 to 2, because it contains the value of maximum delay for the delay-dependent control. Theoretically, we expect the systems to be asymptotically stable for delay less than the maximum delay if any, and for the delay greater than that maximum value, we cannot conclude anything. Figure 2.6 and 2.7 show the results for system $\mathcal{G}_1$ and system $\mathcal{G}_1$ with nominal states respectively.

For system $\mathcal{G}_1$, we can see that the steady-state value for each control law does not change with delay. For 'Fridman Nominal' and 'Fridman Polytopic,' the output state does not have oscillation and the overshoot for these two controllers are zero. Also, there is an increasing trend for both peak time, rise time, and settling time. For the other four controllers, the rise time and peak time do not change, while the overshoot and settling time both increase with the increase of delay.

For 'Moon Nominal,' we can see both overshoot and settling time start at the lowest value and increase rapidly. This result shows that the delay has a greater influence on Moon's method than other controllers. For 'Moon Polytopic,' the overshoot and settling time are both lower than 'Moon Nominal.'

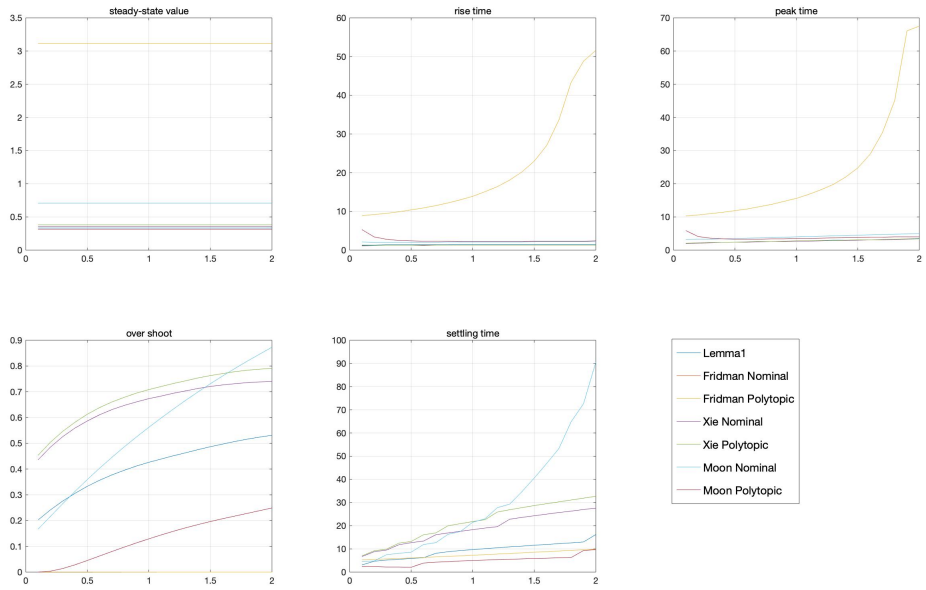


Figure 2.6: Steady-state value, rise time, peak time, overshoot and settling time for system $\mathcal{G}_1$ with nominal states.

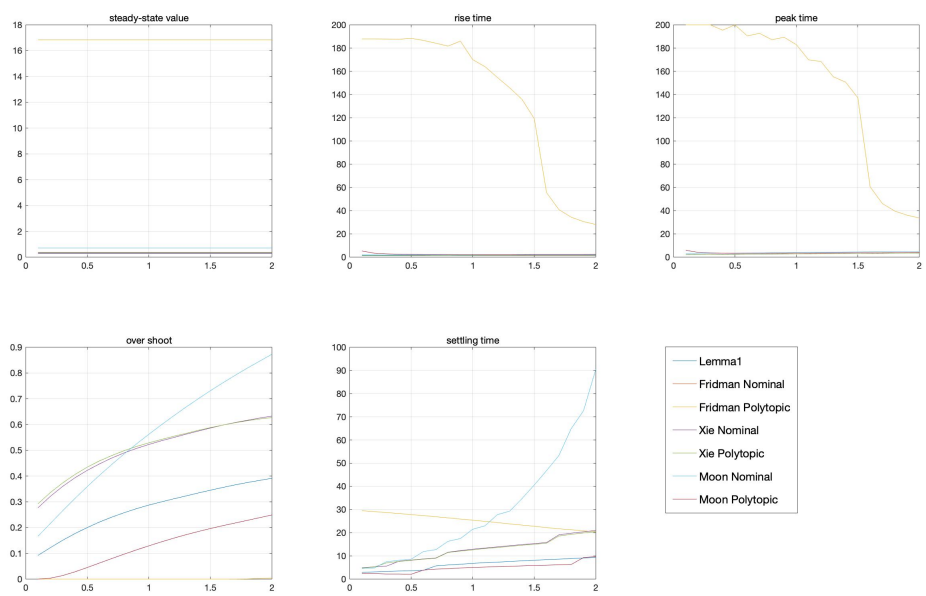


Figure 2.7: Steady-state value, rise time, peak time, overshoot and settling time for system $\mathcal{G}_2$ with nominal states.

For system $\mathcal{G}_1$, it can be seen that the 'Fridman Nominal' and 'Fridman Polytopic' show a different trend in the rise time, peak time, and settling time, while the other controllers show the same trend as system $\mathcal{G}_1$. Figure 2.8 shows the output state for 'Fridman Nominal' and 'Fridman Polytopic' with different delays for system $\mathcal{G}_2$. We can see that for system $\mathcal{G}_2$, the output state does not have oscillation when the delay is relatively small. With the increase of delay, the output state shows the oscillation, and the rise time, peak time, and settling time all decrease. Also, from figure 2.7, the overshoot of Fridman's methods shows an increasing trend.

The above results are all for the nominal states. From figure 2.6 and figure 2.7, we see that for Xie's paper, the polytope controller has worse performance than the nominal controller. We can also see that for Fridman's method, the performance of the polytope controller and nominal controller show no difference.

Figure 2.9 and figure 2.10 show the step response for system 1 and system 2 with polytopic uncertainties. To conclude the results, we give the uncertain parameters a specific value for each iteration and use the mean value of the different indices as the final value.

We can see that Fridman's method has a better performance in the rise time and peak time for system 1. 'Moon Nominal' shows a better performance in overshoot and settling time for both systems. And same as the system with nominal states, "Moon Polytopic" shows a better performance in all indices than 'Moon Nominal.' The reason for the decrease of 'Moon Polytopic' in the rise time for system $\mathcal{G}_1$ is the same as we discussed above for Fridman's method, the step response shows oscillation when the delay is small and no oscillation when delay increase.

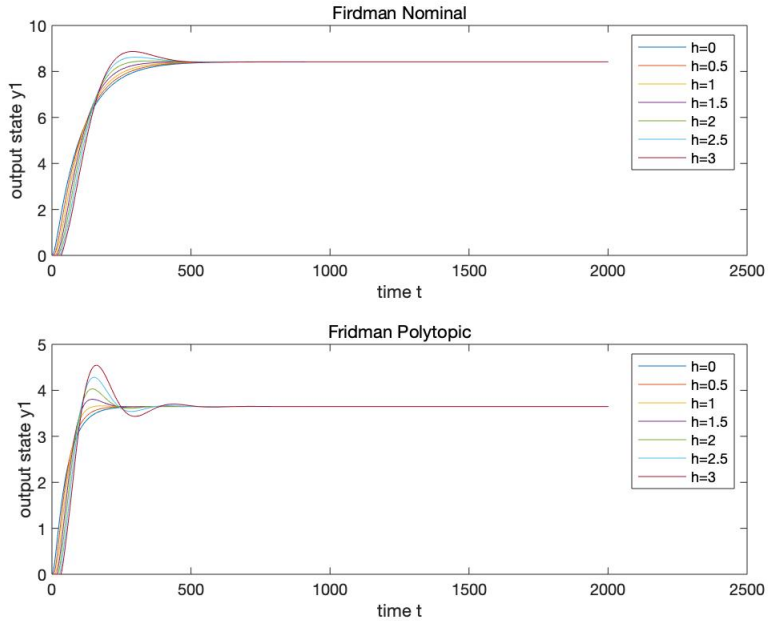


Figure 2.8: Output state for system $\mathcal{G}_2$ with 'Fridman Nominal' and 'Fridman Polytope' with different delays. The oscillation appears with the increase of delay.

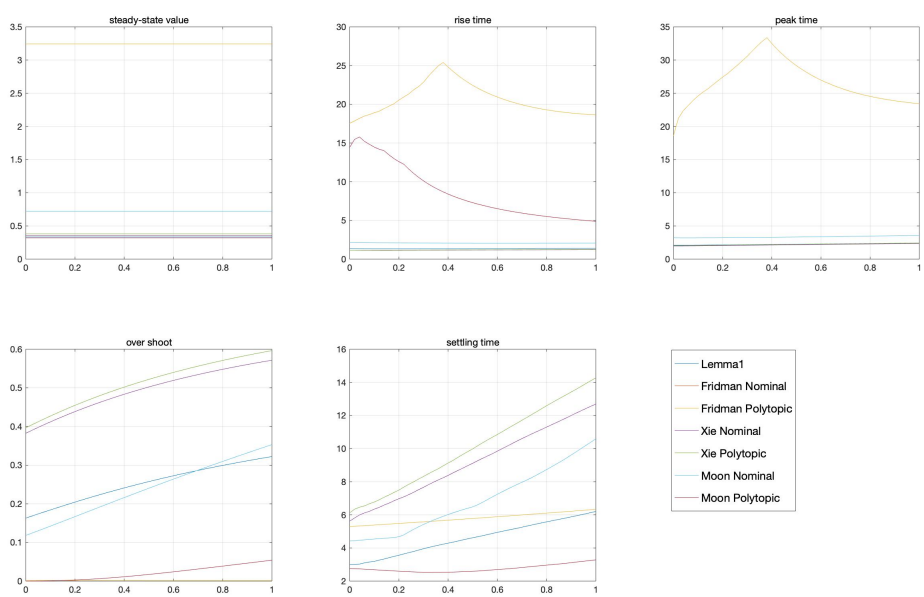


Figure 2.9: Steady-state value, rise time, peak time, overshoot and settling time for system $\mathcal{G}_1$ with polytopic uncertainties.

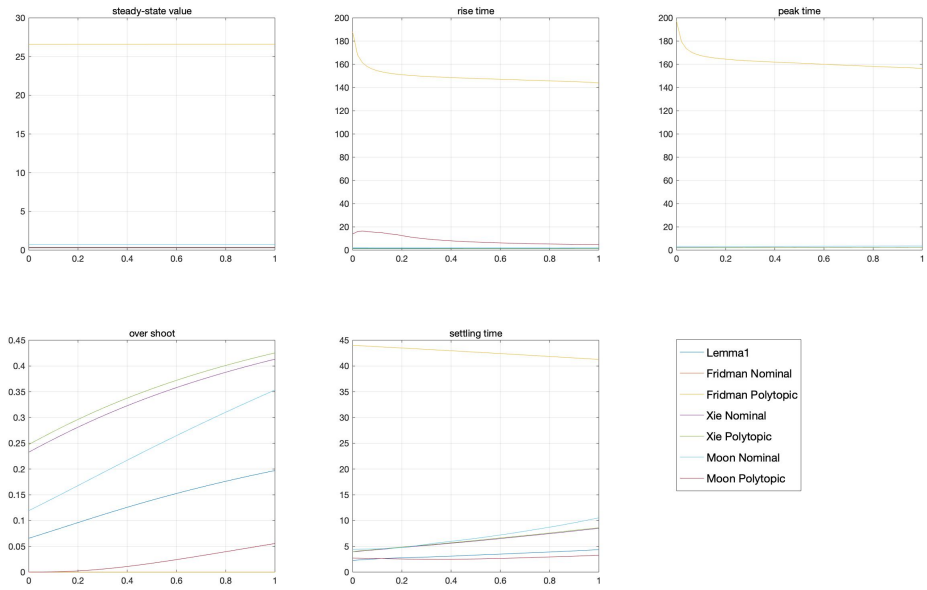


Figure 2.10: Steady-state value, rise time, peak time, overshoot and settling time for system $\mathcal{G}_2$ with polytopic uncertainties.

2.6.4 Conclusion

In this comparative study, stability criteria and step response are tested. For each different control law, we cannot conclude that one is better than the others, but we do show the difference. From table 6 and table 7, we can see for the delay-dependent controls, 'Fridman Nominal' provides a less conservative result and from the step response, we find that Fridman's method will lead to a non-oscillation step response. Also, the control laws by Fridman and Moon ensure asymptotical stability for delays much larger than the maximum delay provided in their paper.

Chapter 3

Main Results

In this chapter, we introduce a novel approach for establishing robust stability criteria for systems with time-varying state delays based on the conic sector system as stated in Chapter I. This paper is an extension to the work of [Bri16]. LMI conditions are proposed to achieve the goal. Compared to other work on stability criteria such as [SH12], this approach may not be the least conservative one. However, information about the conic bounds is implied. Using the Conic Sector Theory and combining with existing conic controller synthesis methods such as [BCF14] and [BF14], a sub-optimal H_2 controller can be designed such that the closed-loop is guaranteed.

3.1 Preliminaries

3.1.1 Conic Systems

For clarity and better reference, we restate the definition of conic systems and the Conic-Sector theorem.

Definition 3.1 (Conic Systems, [Bri16])

Consider a system $\mathcal{G}$ on L_{2e}, satisfying the following inequality:

$$\delta(-||\mathcal{G}u||_2^2 + (a+b)\langle \mathcal{G}u, u\rangle_T - ab||u||_{2T}^2) \geq \delta\beta \tag{3.1}$$

forall u on L_{2e} and T on $\mathbb{R}^+$, where β depends only on the initial conditions and $a \leq b$. The system, $\mathcal{G}$, is:

- **interior conic** with conic bounds $a, b \in \mathbb{R}$ if $\delta = 1$, denoted as $\mathcal{G} \in$ cone $[a, b]$;
- **exterior conic** with conic bounds $a, b \in \mathbb{R}$ if $\delta = -1.$, denoted as $\mathcal{G} \in$ excone (a, b);
- **strictly interior conic** with conic bounds $a, b \in \mathbb{R}$ if $\delta = 1$, denoted as $\mathcal{G} \in$ cone (a, b) if $\mathcal{G} \in$ cone $[a + \epsilon, b - \epsilon]$ for some small $\epsilon > 0$;
- **strictly exterior conic** with conic bounds $a, b \in \mathbb{R}$ if $\delta = 1$, denoted as $\mathcal{G} \in$ cone $[a, b]$ if $\mathcal{G} \in$ cone $(a - \epsilon, b + \epsilon)$ for some small $\epsilon > 0$;

Theorem 3.1 (Conic Sector Theorem, [Bri16])

Let $r^T = [r_1^T, r_2^T] \in L_{2e} \times L_{2e}$ be the external signal and $\mathcal{G}_1, \mathcal{G}_2 : L_{2e} \to L_{2e}$ be a plant and controller in a negative feedback interconnection as described by

$$y = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} \mathcal{G}_1 u_1 \\ \mathcal{G}_2 u_2 \end{bmatrix} \quad u = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} r_1 - y_2 \\ r_2 + y_1 \end{bmatrix} \tag{3.2}$$

Suppose that $\mathcal{G}_1 \in$ cone $[a, b]$ for $a \in (-\infty, b), b \in (0, \infty)$. The close loop system, $\mathcal{G}$ defined as $y = \mathcal{G}u$ is I-O stable if:

1. $\mathcal{G}_2 \in$ excone $(-\frac{1}{a}, -\frac{1}{b})$, if $a \in [0, \infty)$, or
2. $\mathcal{G}_2 \in$ cone $(-\frac{1}{b}, -\frac{1}{a})$, if $a \in (\infty, 0]$.

3.1.2 Systems Formulation

This paper considers state delay system, $y = \mathcal{G}u$ defined as

$$\begin{aligned} \dot{x}(t) &= Ax_d(t) + Bu(t), \\ y(t) &= Cx_d(t) + Du(t) \end{aligned} \tag{3.3}$$

where A, B, C, D are system dynamics, input, output, and feedthrough matrices respectively. 'd' denotes a function's delayed counterpart, defined as

$$u_d(t) = \begin{cases} u\big(t - h(t)\big) & t > h(t) \\ u_0\big(t - h(t)\big) & t \leq h(t), \end{cases} \tag{3.4}$$

where the time-varying delay satisfies

$$0 < h_m \leq h(t) \leq h_M,$$
$$\dot{h}_m \leq \dot{h}(t) \leq \dot{h}_M < 1, \tag{3.5}$$

and define initial states as

$$\dot{x}_0(t) = Ax_0\big(t - h(t)\big) + Bu(t) \quad \forall t \leq 0. \tag{3.6}$$

3.1.3 Useful Inequalities

In this part, three inequalities used in the proof of theorem 3.1 and theorem 3.2 are given. Concise proof for inequality 3.2 and 3.3 are provided.

Inequality 3.1

$\forall u, v \in \mathbb{R}^n$ and $R > 0$,

$$2v^T u \geq -v^T R v - u^T R^{-1} u. \tag{3.7}$$

Inequality 2.1 is a special case for the 'Upper Bound of Inner Product' in section 2.4.1, with $Y = N = I$, $X = Z^{-1}$.

Inequality 3.2

$\forall v \in \mathbb{R}^n$, $S \geq 0$, $h > 0$ and $\phi_1(t) \leq 0$,

$$-\int_0^T \int_{-h}^0 v^T\big(t + \phi_1(t_1)\big) S v\big(t + \phi_1(t_1)\big)\, dt_1\, dt \geq$$
$$-\int_0^T h v^T(t) S v(t)\, dt - \int_{-h}^0 \int_{\phi_1(t_1)}^0 v^T(t_2) S v(t_2)\, dt_2\, dt_1. \tag{3.8}$$

Proof of inequality 3.2

Leibniz's Rule helps to prove this inequality.

$$\begin{aligned}
&\int_{-h}^{0}\Big(\int_{T+\phi_1(t_1)}^{T} v^T(t_2)Sv(t_2)\,dt_2 - \int_{\phi_1(t_1)}^{0} v^T(t_2)Sv(t_2)\,dt_2\,\Big)dt_1\\
=&\int_{-h}^{0}\int_{0}^{T}\frac{d}{dt}\Big(\int_{t+\phi_1(t_1)}^{t} v^T(t_2)Sv(t_2)\,dt_2\Big)\,dt\,dt_1\\
=&\int_{-h}^{0}\int_{0}^{T}\left[v^T(t)Sv(t) - v^T(t+\phi_1(t_1))Sv(t+\phi_1(t_1))\right]dt\,dt_1,
\end{aligned}$$

here, we apply the Leibniz's Rule and get

$$\begin{aligned}
&\int_{-h}^{0}\int_{0}^{T}\left[v^T(t)Sv(t) - v^T(t+\phi_1(t_1))Sv(t+\phi_1(t_1))\right]dt\,dt_1,\\
=&\int_{0}^{T}\int_{-h}^{0} v^T(t)Sv(t)\,dt_1\,dt - \int_{0}^{T}\int_{-h}^{0} v^T(t+\phi_1(t_1))Sv(t+\phi_1(t_1))\,dt_1\,dt\\
=&\int_{0}^{T} hv^T(t)Sv(t)\,dt - \int_{0}^{T}\int_{-h}^{0} v^T(t+\phi_1(t_1))Sv(t+\phi_1(t_1))\,dt_1\,dt,
\end{aligned}$$

since $S \geq 0$, $\phi_1 \leq 0$ and $h > 0$, we can conclude that $\int_{-h}^{0}\int_{T+\phi_1(t_1)}^{T} v^T(t_2)Sv(t_2)\,dt_2 \geq 0$ and this completes the proof.

Inequality 3.3

Consider a non-negative continuous integrable function $f(t) \geq 0$, we can always have

$$\int_{0}^{T} f(t)\,dt \geq \int_{0}^{T}\big(1-\dot{h}(t)\big)f\big(t-h(t)\big)\,dt + \int_{0}^{-h(0)} f(t)\,dt. \tag{3.9}$$

Proof of inequality 3.3

Let $F(t) = \int f(t)\,dt$.

$$\int_{0}^{T} f(t)\,dt = F(T) - F(0)$$

$$
\begin{aligned}
&\geq F\big(T - h(T)\big) - F(0) \\
&= F\big(T - h(T)\big) - F\big(-h(0)\big) + F\big(-h(0)\big) - F(0) \\
&= F\big(T - h(T)\big) - F\big(-h(0)\big) + \int_0^{-h(0)} f(t)\,dt \\
&= \int_0^T f\big(t - h(t)\big)\big(1 - \dot{h}(t)\big)\,dt + \int_0^{-h(0)} f(t)\,dt.
\end{aligned}
$$

3.2 Stability Criteria

Here we propose our main results. Conic bounds are established for LTI systems with time-varying state delays. Two LMI conditions are given along with the proof.

Theorem 3.2

Consider a square, LTI system $\mathcal{G}$ with a time-varying state delay $h(t)$ as defined in 3.3 and 3.5. Then,

$$
\begin{aligned}
&\mathcal{G} \in cone[a, b] \text{ if } \delta = 1 \text{ and} \\
&\mathcal{G} \in excone(a, b) \text{ if } \delta = -1
\end{aligned}
$$

if there exists $P, R_1, R_2, R_3, R_4 > 0$ such that $a < b$, $M_1(h(t)) \leq 0$ and $M_2(h(t), \dot{h}(t)) \leq 0$, where M_1 and M_2 are defined following:

$$
\begin{aligned}
&M_1(h(t)) = \\
&\qquad \begin{bmatrix} \hat{M}_{11} & \sqrt{h(t)}PA & \sqrt{h(t)}PA \\ \sqrt{h(t)}A^T P & -R_1 & 0 \\ \sqrt{h(t)}A^T P & 0 & -R_2 \end{bmatrix} \\
&\hat{M}_{11} = PA + A^T P + h_M A^T (R_1 + R_3) A,
\end{aligned}
$$

$$M_2(h(t), \dot{h}(t)) =$$

$$\begin{bmatrix} M_{11} & M_{12} & M_{13} & M_{13} & 0 & 0 \\ * & M_{22} & 0 & 0 & M_{25} & M_{25} \\ * & * & -R_1 & 0 & 0 & 0 \\ * & * & * & -R_2 & 0 & 0 \\ * & * & * & * & -R_3 & 0 \\ * & * & * & * & * & -R_4 \end{bmatrix}$$

$$M_{11}\big(\dot{h}(t)\big) = \big(1 - \dot{h}(t)\big)\hat{M}_{11} + \delta C^T C,$$

$$M_{13}\big(h(t), \dot{h}(t)\big) = -\sqrt{h(t)\big(1 - \dot{h}(t)\big)}PA,$$

$$M_{22} = h_M B^T (R_2 + R_4)B - \delta\big((r^2 - c^2) - D^T D + D^T c + cD\big),$$

$$M_{25}\big(h(t)\big) = -\sqrt{h(t)}B^T P.$$

Proof of Theorem 3.2

As defined in 3.1, β depends only on the initial conditions. To establish conic bounds, it is important to determine which term contributes to β. In this state-delay case, we can see that $\forall t \leq 0$, $x(t)$ depends only on initial conditions.

Consider a Lyapunov function. Suppose $\exists P = P^T > 0$ and $V(t) = x^T(t)Px(t)$.

$$\begin{aligned} & x^T(T)Px(T) - x^T(0)Px(0) \\ &= \int_0^T \frac{d}{dt}\big(x^T(t)Px(t)\big)\,dt \\ &= \int_0^T 2x^T(t)P\dot{x}(t)\,dt \\ &= \int_0^T 2x^T(t)P\big(Ax_d(t) + Bu(t)\big)\,dt \\ &= \int_0^T 2x^T(t)PAx_d(t)\,dt + \int_0^T 2x^T(t)PBu(t)\,dt, \end{aligned}$$

thus, we have:

$$
\begin{aligned}
0 &= x^T(T)Px(T) - x^T(0)Px(0) \\
&- \int_0^T 2x^T(t)PAx_d(t)\,dt - \int_0^T 2x^T(t)PBu(t)\,dt \\
&\geq -x^T(0)Px(0) - \int_0^T 2x^T(t)PAx_d(t)\,dt - \int_0^T 2x^T(t)PBu(t)\,dt \\
&= \alpha_1 + \alpha_2 + \alpha_3 - \int_0^T 2x^T(t)PAx(t)\,dt - \int_0^T 2x_d(t)^T PBu(t)\,dt, \qquad (3.10)
\end{aligned}
$$

where:

$$
\begin{aligned}
\alpha_1 &= -x^T(0)Px(0), \\
\alpha_2 &= 2\int_0^T x^T(t)PA\big(x(t) - x_d(t)\big)\,dt, \\
\alpha_3 &= 2\int_0^T \big(x_d(t) - x(t)\big)^T PBu(t)\,dt.
\end{aligned}
$$

Here, α_1 is a constant depending only on the initial conditions, while α_2 and α_3 are not. Next, α_2 and α_3 will be expressed in terms of constants depending only on the initial conditions and matrix inequality terms. Use the first theorem of calculus and 3.3, we have

$$
\begin{aligned}
\alpha_2 &= 2\int_0^T x^T(t)PA\left(x(t) - x_d(t)\right)\,dt \\
&= 2\int_0^T \int_{t-h(t)}^t x^T(t)PA\dot{x}^T(t_1)\,dt_1\,dt \\
&= 2\int_0^T \int_{t-h(t)}^t x^T(t)PA\left(Ax_d(t_1) + Bu(t_1)\right)\,dt_1\,dt, \qquad (3.11)
\end{aligned}
$$

applying inequality 3.7 to α_2 twice, with $v_1^T = x^T(t)PA, u_1 = Ax_d(t_1), v_2^T = x^T(t)PA, u_2 = Bu(t_1)$ and rearranging, we have:

$$
\alpha_2 \geq \alpha_4 + \alpha_5 - \int_0^T h(t)\big(x(t)^T PA(R_1^{-1} + R_2^{-1})A^T Px(t)\big)\,dt, \qquad (3.12)
$$

where α_4 and α_5 are as follows:

$$\begin{aligned}
\alpha_4 &= -\int_0^T \int_{t-h(t)}^t x_d^T(t_1) A^T R_1 A x_d(t_1)\, dt_1\, dt \\
&= -\int_0^T \int_{-h(t)}^0 x^T\big(t+t_1-h(t_1)\big) A^T R_1 A x\big(t+t_1-h(t_1)\big)\, dt_1\, dt \\
&\geq -\int_0^T \int_{-h_M}^0 x^T\big(t+t_1-h(t_1)\big) A^T R_1 A x\big(t+t_1-h(t_1)\big)\, dt_1\, dt, \\
\alpha_5 &= -\int_0^T \int_{t-h(t)}^t u^T(t_1) B^T R_2 B u(t_1)\, dt_1\, dt \\
&= -\int_0^T \int_{-h(t)}^0 u^T(t+t_1) B^T R_2 B u(t+t_1)\, dt_1\, dt \\
&\geq -\int_0^T \int_{-h_M}^0 u^T(t+t_1) B^T R_2 B u(t+t_1)\, dt_1\, dt.
\end{aligned}$$

Similar to α_2, for α_3 we have:

$$\begin{aligned}
\alpha_3 &= 2\int_0^T \big(x_d(t)-x(t)\big)^T PBu(t)\, dt \\
&= -2\int_0^T \int_{t-h(t)}^t u^T(t) B^T P \dot{x}(t_1)\, dt_1\, dt \\
&= -2\int_0^T \int_{t-h(t)}^t u^T(t) B^T P\big(Ax_d(t_1)+Bu(t_1)\big)\, dt_1\, dt,
\end{aligned}$$

applying inequality 3.7 to α_3, with $v_3^T = -u^T(t)B^TP, u_3 = Ax_d(t_1), v_4^T = -u^T(t)B^TP, u_4 = Bu(t_1)$ and rearranging, we have:

$$\alpha_3 \geq \alpha_6 + \alpha_7 - \int_0^T h(t)\big(u^T(t)B^TP(R_3^{-1}+R_4^{-1})PBu(t)\big)\, dt, \tag{3.13}$$

where

$$\begin{aligned}
\alpha_6 &= -\int_0^T \int_{t-h(t)}^t x_d^T(t_1) A^T R_3 A x_d(t_1)\, dt_1\, dt \\
&= -\int_0^T \int_{-h(t)}^0 x^T\big(t+t_1-h(t_1)\big) A^T R_3 A x\big(t+t_1-h(t_1)\big)\, dt_1\, dt
\end{aligned}$$

$$
\begin{aligned}
&\geq -\int_0^T \int_{-h_M}^0 x^T\big(t+t_1-h(t_1)\big)A^T R_3 A x\big(t+t_1-h(t_1)\big)\,dt_1\,dt,\\
\alpha_7 &= -\int_0^T \int_{t-h(t)}^t u^T(t_1)B^T R_4 B u(t_1)\,dt_1\,dt\\
&= -\int_0^T \int_{-h(t)}^0 u^T(t+t_1)B^T R_4 B u(t+t_1)\,dt_1\,dt\\
&\geq -\int_0^T \int_{-h_M}^0 u^T(t+t_1)B^T R_4 B u(t+t_1)\,dt_1\,dt.
\end{aligned}
$$

Table 3.1: Parameters used for deriving inequalities 3.14

	$\phi_1(t_1)$	v	S
α_4	$t_1 - h(t_1)$	x	$A^T R_1 A$
α_5	t_1	u	$B^T R_2 B$
α_6	$t_1 - h(t_1)$	x	$A^T R_3 A$
α_7	t_1	u	$B^T R_4 B$

Selecting parameters according to table 3.1 and applying inequality 3.8 to α_4, α_5, α_6 and α_7, we get:

$$
\begin{aligned}
\alpha_4 &\geq -\int_0^T h_M x^T(t)A^T R_1 A x(t)\,dt - \alpha_8,\\
\alpha_5 &\geq -\int_0^T h_M u^T(t)B^T R_2 B u(t)\,dt - \alpha_9,\\
\alpha_6 &\geq -\int_0^T h_M x^T(t)A^T R_3 A x(t)\,dt - \alpha_{10},\\
\alpha_7 &\geq -\int_0^T h_M u^T(t)B^T R_4 B u(t)\,dt - \alpha_{11},
\end{aligned}
\tag{3.14}
$$

where

$$
\begin{aligned}
\alpha_8 &= -\int_{-h}^0 \int_{t_1-h(t_1)}^0 x^T(t_2)A^T R_1 A x(t_2)\,dt_2\,dt_1,\\
\alpha_9 &= -\int_{-h}^0 \int_{t_1}^0 u^T(t_2)B^T R_2 B u(t_2)\,dt_2\,dt_1,
\end{aligned}
$$

$$\alpha_{10} = -\int_{-h}^{0}\int_{t_1-h(t_1)}^{0} x^T(t_2)A^T R_3 A x(t_2)\,dt_2\,dt_1,$$
$$\alpha_{11} = -\int_{-h}^{0}\int_{t_1}^{0} u^T(t_2)B^T R_4 B u(t_2)\,dt_2\,dt_1.$$

Recall inequality 3.8 with $\phi_1(t_1) \leq 0$ and 3.6, we see that α_8, α_9, α_{10}, α_{11} are all constants of only initial condition. Combining 3.1 and 3.14, we have

$$0 \geq \alpha_1 + \alpha_8 + \alpha_9 + \alpha_{10} + \alpha_{11} + \int_0^T v(t)^T S v(t)\,dt, \tag{3.15}$$

where;

$$S = \begin{bmatrix} -S_1 & 0 & 0 \\ 0 & 0 & -PB \\ 0 & -B^T P & -S_2 \end{bmatrix} \quad v(t) = \begin{bmatrix} x(t) \\ x_d(t) \\ u(t) \end{bmatrix}$$
$$S_1 = PA + A^T P + h_M A^T(R_1 + R_3)A + h(t)PA(R_1^{-1} + R_2^{-1})A^T P,$$
$$S_2 = h_M B^T(R_2 + R_4)B + h(t)B^T P(R_3^{-1} + R_4^{-1})PB.$$

S_2 is always positive definite and S_1 can be made negative definite. Taking Schur complement of $M_1(h(t)) \leq 0$ we get $S_1 \leq 0$. Applying inequality 3.9 we get

$$\begin{aligned} &\int_0^T x(t)^T(-S_1)x(t)\,dt \\ &\geq \int_0^T \big(1 - \dot{h}(t)\big)x_d(t)^T(-S_1)x_d(t)\,dt + \int_0^{-h(0)} x(t)^T(-S_1)x(t)\,dt \\ &= \int_0^T \big(1 - \dot{h}(t)\big)x_d(t)^T(-S_1)x_d(t)\,dt + \alpha_{12}, \end{aligned}$$

where

$$\alpha_{12} = \int_0^{-h(0)} x(t)^T(-S_1)x(t)\,dt,$$

depends only on the initial conditions. Then, we arrive

$$0 \geq \alpha_1 + \alpha_8 + \alpha_9 + \alpha_{10} + \alpha_{11} + \alpha_{12} + \int_0^T \hat{v}(t)^T \hat{S} \hat{v}(t)\,dt, \tag{3.16}$$

where;

$$\hat{S} = \begin{bmatrix} -(1 - \dot{h}(t))S_1 & -PB \\ -B^T P & -S_2 \end{bmatrix}, \quad \hat{v}(t) = \begin{bmatrix} x_d(t) \\ u(t) \end{bmatrix}.$$

Substituting 3.3 and 3.14 into 3.1, the conic bound is established. Note that

$$
\begin{aligned}
&\delta(r^2||u||_{2T}^2 - ||G_d u - cu||_{2T}^2) \\
=&\delta \int_0^T r^2 u^T u - (Cx_d + Du - cu)^T (Cx_d + Du - cu)\, dt \\
=&\delta \int_0^T \hat{v}^T(t) \begin{bmatrix} -C^T C & -C^T D + C^T c \\ -D^T C + cC & (r^2 - c^2) - D^T D + D^T c + cD \end{bmatrix} \hat{v}(t)\, dt \\
\geq&\alpha_1 + \alpha_8 + \alpha_9 + \alpha_{10} + \alpha_{11} + \alpha_{12} + \int_0^T \hat{v}(t)^T \hat{M} \hat{v}(t)\, dt,
\end{aligned}
$$

where:

$$
\begin{aligned}
\hat{M} &= \begin{bmatrix} \hat{M}_{11} & \hat{M}_{12} \\ \hat{M}_{21} & \hat{M}_{22} \end{bmatrix} \\
\hat{M}_{11} &= -(1 - \dot{h}(t))S_1 - \delta C^T C, \\
\hat{M}_{12} &= -PB - \delta(C^T D - C^T c), \\
\hat{M}_{21} &= -B^T P - \delta(D^T C - cC), \\
\hat{M}_{22} &= -S_2 + \delta\big((r^2 - c^2) - D^T D + D^T c + cD\big).
\end{aligned}
$$

$\hat{M}$ has to be made positive semi-definite to make sure the above showed integral is bounded below, that is $\hat{M} \geq 0$. Take Schur complement to $\hat{M}$ four times and we arrive $M_2(h(t), \dot{h}(t)) \leq 0$ and this proves theorem 3.2.

The above showed result can be made less conservative by introducing a new parameter λ, as stated below.

Theorem 3.3

Consider a square, LTI system $\mathcal{G}$ with a time-varying state delay $h(t)$ as defined in 3.3 and 3.5. Then,

$$\mathcal{G} \in cone[a, b] \text{ if } \delta = 1 \text{ and}$$

$$\mathcal{G} \in excone(a, b) \text{ if } \delta = -1$$

if there exists $P, R_1, R_2, R_3, R_4, R_5, R_6 > 0$ and $\lambda \in (0, 1]$ such that $a < b$, $Q_1(h(t)) \leq 0$ and $Q_2(h(t), \dot{h}(t)) \leq 0$, where Q_1 and Q_2 are defined following:

$$Q_1\big(h(t)\big) =$$

$$\begin{bmatrix} \hat{Q}_{11} & \sqrt{\lambda h(t)}PA & \sqrt{\lambda h(t)}PA \\ \sqrt{\lambda h(t)}A^T P & -R_1 & 0 \\ \sqrt{\lambda h(t)}A^T P & 0 & -R_2 \end{bmatrix}$$

$$\hat{Q}_{11} = \lambda(PA + A^T P) + h_M A^T\big(\lambda(R_1 + R_3) + (1 - \lambda)(R_5 + R_6)\big)A,$$

$$Q_2\big(h(t), \dot{h}(t)\big) =$$

$$\begin{bmatrix} Q_{11} & Q_{12} & Q_{13} & Q_{13} & Q_{15} & Q_{15} & 0 & 0 \\ * & Q_{22} & 0 & 0 & 0 & 0 & Q_{27} & Q_{27} \\ * & * & -R_1 & 0 & 0 & 0 & 0 & 0 \\ * & * & * & -R_2 & 0 & 0 & 0 & 0 \\ * & * & * & * & -R_5 & 0 & 0 & 0 \\ * & * & * & * & * & -R_6 & 0 & 0 \\ * & * & * & * & * & * & -R_3 & 0 \\ * & * & * & * & * & * & * & -R_4 \end{bmatrix}$$

$$Q_{11}\big(h(t)\big) = (1 - \dot{h}(t))\hat{Q}_{11} + (1 - \lambda)(PA + A^T P) + \delta C^T C,$$

$$Q_{12} = PB + \delta(C^T D - C^T c),$$

$$Q_{13}\big(h(t), \dot{h}(t)\big) = -\sqrt{\lambda h(t)\big(1 - \dot{h}(t)\big)}PA,$$

$$Q_{15} = -\sqrt{\lambda h(t)}PA,$$

$$Q_{22} = h_M B^T(R_2 + R_4)B - \delta\big((r^2 - c^2) - D^T D \quad + D^T c + cD\big),$$

$$Q_{27}\big(h(t)\big) = -\sqrt{h(t)}PB.$$

Proof of Theorem 3.3

Use the same Lyapunov function as in the proof of theorem 3.2. Let $\theta = -\int_0^T 2x^T(t)PAx_d(t)\,dt$. It can be seen that $\forall\lambda \in (0,1]$

$$\begin{aligned}\theta =& -\int_0^T 2x^T(t)PAx_d(t)\,dt\\ =& -\lambda\int_0^T 2x^T(t)PAx_d(t)\,dt - (1-\lambda)\int_0^T 2x^T(t)PAx_d(t)\,dt\\ =&\lambda\theta_1 + (1-\lambda)\theta_2,\end{aligned}$$

where

$$\begin{aligned}\theta_1 &= \alpha_2 - \int_0^T 2x^T(t)PAx(t)\,dt,\\ \theta_2 &= \alpha_{13} - \int_0^T 2x_d^T(t)PAx_d(t)\,dt,\\ \alpha_{13} &= 2\int_0^T -\left(x^T(t) - x_d^T(t)\right)PAx_d(t)\,dt.\end{aligned}$$

In the proof of theorem 3.1, we already have

$$\alpha_2 \geq \alpha_4 + \alpha_5 - \int_0^T h(t)\big(x(t)^T PA(R_1^{-1} + R_2^{-1})A^T Px(t)\big)\,dt.$$

The derivation process of α_2 and α_3 are omitted here since they are the same process as in the proof of theorem 3.1. We now process α_{13} following a similar manner of α_2

$$\begin{aligned}\alpha_{13} &= 2\int_0^T -\big(x^T(t) - x_d^T(t_1)\big)PAx_d(t_1)\,dt\\ &= 2\int_0^T\int_{t-h(t)}^t x_d^T(t)PA\big(Ax_d(t_1) + Bu(t_1)\big)\,dt_1\,dt,\end{aligned} \tag{3.17}$$

applying inequality 3.7 to α_{13} with $v_5^T = x(t)_d^T PA, u_5 = Ax_d(t_1), v_6^T = x(t)_d^T PA, u_6 = Bu(t_1)$ and rearranging, we have:

$$\alpha_{13} \geq \alpha_4 + \alpha_5 - \int_0^T h(t)\left[x(t)_d^T PA(R_5^{-1} + R_6^{-1})A_d^T Px(t)\right]\,dt, \tag{3.18}$$

where α_4 and α_5 are defined in (9). Thus,

$$\begin{aligned}\theta &= \lambda\theta_1 + (1-\lambda)\theta_2 \\ &\geq \alpha_4 + \alpha_5 - \lambda \int_0^T 2x^T(t)PAx(t)\,dt \\ &+ (1-\lambda)\int_0^T 2x^T(t-h(t))PAx(t-h(t))\,dt. \end{aligned} \tag{3.19}$$

For now, we have

$$0 \geq \alpha_1 + \alpha_8 + \alpha_9 + \alpha_{10} + \alpha_{11} + \int_0^T v(t)^T N v(t)\,dt, \tag{3.20}$$

where

$$N = \begin{bmatrix} -N_1 & 0 & 0 \\ 0 & -N_2 & -PB \\ 0 & -B^TP & -S_2 \end{bmatrix}, \quad v(t) = \begin{bmatrix} x(t) \\ x_d(t) \\ u(t) \end{bmatrix},$$

$$N_1 = \lambda S_1 + (1-\lambda)\left(h_M(R_5+R_6)\right),$$

$$N_2 = (1-\lambda)\left(h(t)PA(R_5^{-1}+R_6^{-1})A^TP + A^TP + PA\right).$$

Again, N_2 is always positive definite, while N_1 can be made negative definite. Taking Schur complement of $Q_1(h(t)) \leq 0$ we get $N_1 \leq 0$. Applying inequality 3.9 we get

$$\begin{aligned} &\int_0^T x(t)^T(-N_1)x(t)\,dt \\ \geq &\int_0^T (1-\dot{h}(t))x_d(t)^T(-N_1)x_d(t)\,dt + \int_0^{-h(0)} x(t)^T(-N_1)x(t)\,dt \\ = &\int_0^T (1-\dot{h}(t))x_d(t)^T(-N_1)x_d(t)\,dt + \alpha_{14}, \end{aligned}$$

where

$$\alpha_{14} = \int_0^{-h(0)} x(t)^T(-N_1)x(t)\,dt$$

depends only on the initial conditions. Then, we arrive

$$0 \geq \alpha_1 + \alpha_8 + \alpha_9 + \alpha_{10} + \alpha_{11} + \alpha_{14} + \int_0^T \hat{v}(t)^T \hat{N} \hat{v}(t)\,dt, \tag{3.21}$$

where

$$\hat{N} = \begin{bmatrix} \hat{N}_1 & -PB \\ -B^T P & -S_2 \end{bmatrix} \quad \hat{v}(t) = \begin{bmatrix} x_d(t) \\ u(t) \end{bmatrix}$$

$$\hat{N}_1 = -(1 - \dot{h}(t))N_1 - N_2$$

Substituting 3.1 and 3.21 into 3.3, we have

$$\begin{aligned} &\delta(r^2||u||_{2T}^2 - ||G_d u - cu||_{2T}^2) \\ \geq &\alpha_1 + \alpha_8 + \alpha_9 + \alpha_{10} + \alpha_{11} + \alpha_{14} + \int_0^T \hat{v}(t)^T \hat{U} \hat{v}(t)\, dt, \end{aligned}$$

where:

$$\hat{U} = \begin{bmatrix} \hat{N}_1 - \delta C^T C & -PB - \delta(C^T D - C^T c) \\ -B^T P - \delta(D^T C - cC) & -S_2 + \delta((r^2 - c^2) - D^T D + D^T c + cD) \end{bmatrix}.$$

Take Schur complement to $\hat{U}$ six times and we arrive $Q_2(h(t), \dot{h}(t)) \leq 0$.

Note that in theorem 3.2, λ is assume to be from $(0, 1]$. This is because when $\lambda = 0$, $N_1 = h_M(R_5 + R_6)$ is always positive definite and we can not apply inequality 3.9 to $-N_1$ in this case. Also note that when $\lambda = 1$, theorem 3.2 degrade to theorem 3.2. We will show this equivalence through a numerical example.

The LMIs in the above proposed theorems are time-varying, they are not solvable for Yalmip. It is vital to find a polytope to numerically solve the LMIs. In this case, eight vortices are required and listed below. A different choice of vortices would affect the conservativeness of the results.

$$\begin{aligned} v_1 &= (h_m, \dot{h}_m, 0), \\ v_2 &= (h_M, \dot{h}_m, 0), \\ v_3 &= (h_m, \dot{h}_M, 0), \\ v_4 &= (h_M, \dot{h}_M, 0), \end{aligned}$$

$$
\begin{aligned}
v_5 &= (h_m, \dot{h}_m, \frac{1}{2}\sqrt{h_M(1-\dot{h}_m)} + \frac{1}{2}\sqrt{h_m(1-\dot{h}_m)}),\\
v_6 &= (h_M, \dot{h}_m, \frac{1}{2}\sqrt{h_M(1-\dot{h}_m)} + \frac{1}{2}\sqrt{h_m(1-\dot{h}_m)}),\\
v_7 &= (h_m, \dot{h}_M, \sqrt{h_M(1-\dot{h}_m)}),\\
v_8 &= (h_M, \dot{h}_M, \frac{1}{2}\sqrt{h_M(1-\dot{h}_m)} + \frac{1}{2}\sqrt{h_m(1-\dot{h}_m)}).
\end{aligned}
$$

3.3 Numerical Examples

In this section, a numerical example is given to illustrate the strength of theorem 3.2. In part A, theorem 3.2 is used as a stability criterion to determine the maximum allowed state delay and compared with [SH12]. It is reasonable that the conic bound found in part A is big. In part B, the maximum delay is assumed to be 2.4, and theorem 3.2 is used to find the minimum conic radius under this condition. Two controllers are designed in this part, one is a $\mathcal{H}_2$ optimal controller and another is a conic controller, to show the closed-loop stability.

Here, we use the heat exchanger example given by [Bri16]. The parameters can be found in the cited paper and the standard state-space realization following 3.3 is given by

$$
\begin{aligned}
A &= \begin{bmatrix} -0.1917 & 0.0775 \\ 0.0192 & -0.1042 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 0.085 \end{bmatrix},\\
C &= \begin{bmatrix} 1 & 0 \end{bmatrix}, \qquad D = 0.
\end{aligned}
$$

Since we are dealing with time-varying delays, we further assume that $\dot{h}_M = 0.5$ and $\dot{h}_m = -0.5$.

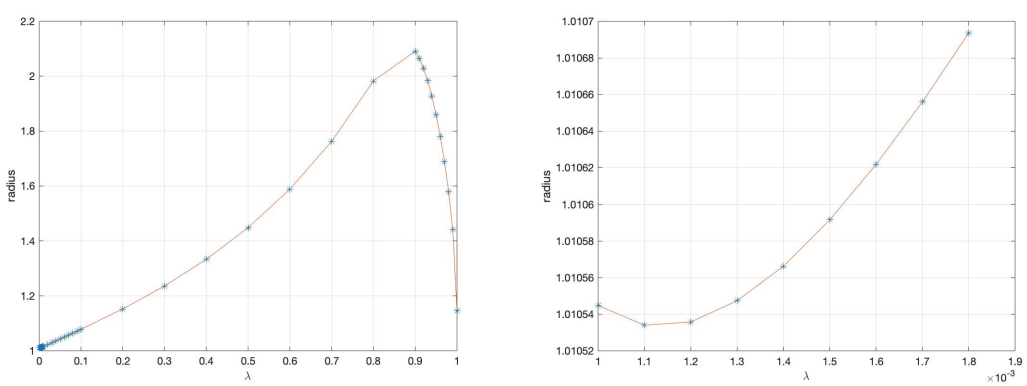
MATLAB [MAT18], YALMIP [Löf04] and MOSEK [ApS19] are used to implement the LMIs and solve the problem.

3.3.1 Stability Criteria

For the given state-space realization, we find the maximum allowed time-varying delay is $h_M = 4.4$ at $\lambda = 1$, with $a = -9.2687$ and $b = 5.0478$. In [SH12], let $A = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ and $A_1 = \begin{bmatrix} -0.1917 & 0.0775 \\ 0.0192 & -0.1042 \end{bmatrix}$, we find the maximum allowed delay is $h_M = 6.67$. Though [SH12] gives a larger allowed delay, it is a stability criteria applicable to linear systems. However, theorem 3.2 can be served not only as a stability criteria, it can be used directly to design controllers.

3.3.2 Closed-loop Stability

In this part, we assume the maximum allowed delay is $h_M = 2.4$ and establish new conic bound using theorem 3.2. Figure 3.1(a) and figure 3.1(b) show the process of how we find the minimum conic radius. The lower bound is $a = -0.9651$ and the upper bound is $b = 1.0560$. This is achieved at $\lambda = 0.0011$. Note that when $\lambda \in (0, 0.9]$, the conic radius decrease at the beginning and then increase. At $\lambda = 1$, the conic radius is not necessarily the largest.



(a) conic radius for λ from 0 to 1.

(b) conic radius for λ from 0.001 to 0.0018.

Figure 3.1: Iterating over λ to search for the minimum conic radius.

To design a $\mathcal{H}_2$ optimal controller, we assume the following standard state-space realization:

$$\begin{aligned}
\dot{x} &= Ax + B_1 w + B_2 u, \\
z &= C_1 x + D_{12} u, \\
y &= C_2 x + D_{21} w,
\end{aligned} \tag{3.22}$$

with

$$\begin{aligned}
A &= \begin{bmatrix} -0.1917 & 0.0775 \\ 0.0192 & -0.1042 \end{bmatrix}, \quad B_2 = \begin{bmatrix} 0 \\ 0.085 \end{bmatrix}, \\
B_1 &= \begin{bmatrix} \sqrt{0.1} & 0 & 0 \\ 0 & \sqrt{30} & 0 \end{bmatrix}, \\
C_2 &= \begin{bmatrix} 1 & 0 \end{bmatrix}, \quad D_{21} = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix}, \\
C_1 &= \begin{bmatrix} \sqrt{30} & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, \quad D_{12} = \begin{bmatrix} 0 \\ 0 \\ \sqrt{0.1} \end{bmatrix}.
\end{aligned}$$

Also, assume that the following assumptions hold:

1. The pair $(A,\, B_2)$ is stabilizable. The pair $(C_1,\, A)$ has no unobservable modes on the imaginary axis
2. $D_{12}^T D_{12}$ is invertible and $D_{12}^T C_1 = 0$
3. The pair $(C_2,\, A)$ is detectable. The pair $(A,\, B_1)$ has no uncontrollable modes on the imaginary axis.
4. $D_{21} D_{21}^T$ is invertible and $D_{21} B_1^T = 0$.

The $\mathcal{H}_2$ optimal controller has the following state-space realization

$$\{\begin{bmatrix} -0.8973 & 0.0775 \\ -4.9022 & -0.4214 \end{bmatrix}, \begin{bmatrix} 0.7056 \\ 4.3117 \end{bmatrix}, \begin{bmatrix} -7.1728 & -3.7322 \end{bmatrix}, 0\}.$$

Follow [BF14], the corresponding conic controller has the following state-space realization

$$\{\begin{bmatrix} -0.8973 & 0.0775 \\ -4.9022 & -0.4214 \end{bmatrix}, \begin{bmatrix} 0.7056 \\ 4.3117 \end{bmatrix}, \begin{bmatrix} -0.3705 & -0.2425 \end{bmatrix}, 0\}.$$

Figure 3.2 and 3.3 show the closed-loop response with a $\mathcal{H}_2$ optimal controller and a conic controller respectively.We see that the $\mathcal{H}_2$ controller stabilizes the system when h_M is relatively small and destabilizes the system when h_M grows, while a conic controller always stabilizes the system.

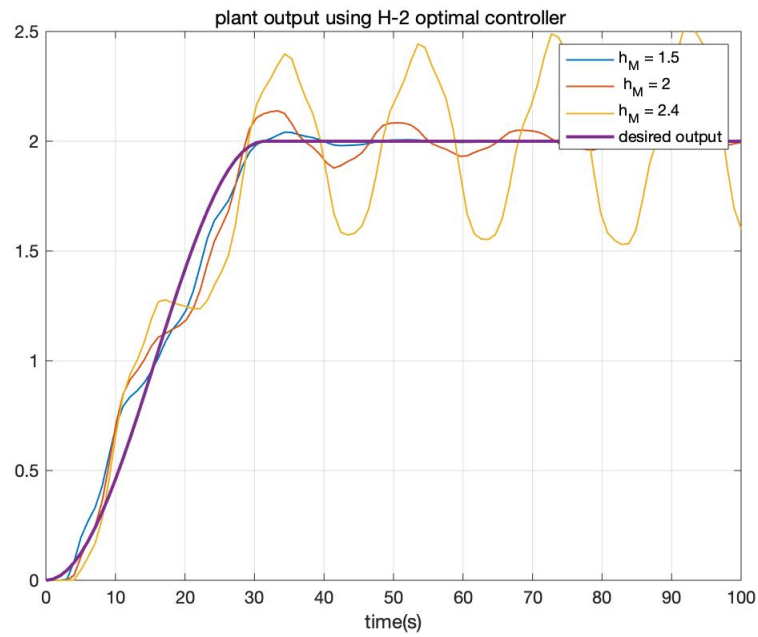


Figure 3.2: Closed-loop response using $\mathcal{H}_2$ optimal controller with different state delays. The closed-loop system is stable with relatively small delays and unstable with larger delays

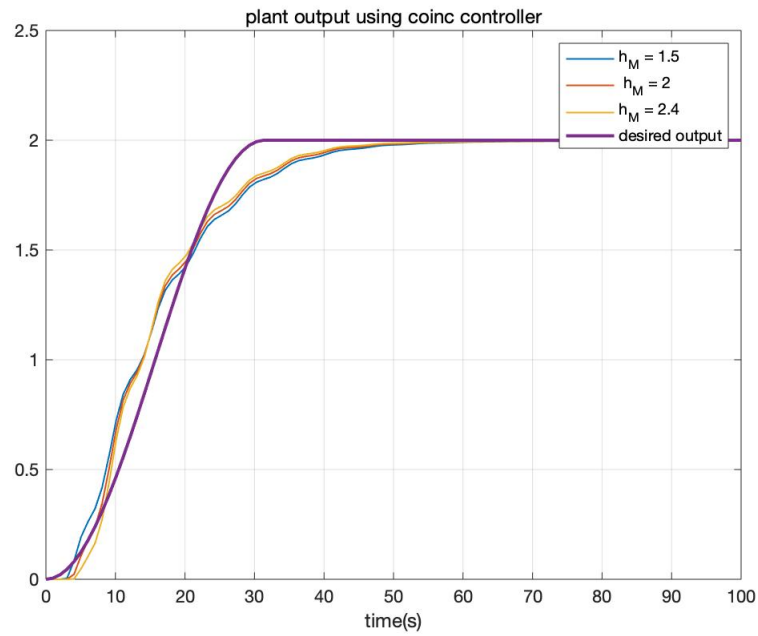


Figure 3.3: Closed-loop response using conic controller with different state delays. The closed-loop stability is always guaranteed.

3.4 Conslusion

In this chapter, a novel way of establishing conic bounds for stable LTI systems with time-varying state delays is proposed. Two LMI conditions are provided together with a numerical example showing that they can serve both as an analysis tool and controller synthesis tool. Detailed mathematical proof is given.

During the derivation, we encountered many difficulties. Different techniques were used to overcome those barriers. One of the main difficulties encountered is in equation 3.10. The term $x^T(t)PAx_d(t)$ is the problematic term. Ideally, we want turn it in to two terms:

$$\begin{aligned} x^T(t)PAx_d(t) &= x_d^T(t)PAx_d(t) + \big(x^T(t) - x_d^T(t)\big)PAx_d(t) \\ &= x_d^T(t)PAx_d(t) + \int_{t-h(t)}^{t} \dot{x}^T(t_1)PAx_d(t)\,dt_1. \end{aligned}$$

However, following this direction, we ended up in a term that is strictly negative while we want it to be positive definite to derive the LMI conditions. Since we cannot foresee this situation, we had to replicate the derivation several times to finally figure out which is the correct direction.

Chapter 4

Conclusion

4.1 Contributions

The main contribution of this thesis is using the conic-sector based approach to extended existing work [Bri16] to the case with time-varying state delays. Linear matrix inequality conditions are provided and validated using a numerical example.

The literature review of some existing works is another contribution of this thesis. In Chapter II, a summary of some existing work is given, together with the bounding techniques used in those papers. This part can serve as an introduction to starters who are not very familiar with this field. When the author first started to study time-delay systems, the massive existing work was confusing. There were many papers: some focused on time-varying delays while others focused on the constant delay; some applied the Lyapunov-Krasovskii based approach, while others used the input-output approach; some provided only stability conditions, while others also give controller synthesis methods. The survey papers [Ric03] and [GN03] helped a lot in finding ways to categorize these papers. Another difficulty is the bounding techniques. Most of the existing papers assume the readers are familiar with this field, thus the proofs are sometimes very brief, which make it difficult for the starters to understand. The literature presented here can serve as a guide or reference to help new researchers understand these techniques.

4.2 Future Work

This section gives three directions for future work.

4.2.1 Apply the Proposed Theorem to Practical Systems

The theories are developed to solve practical problems and serve society. It is of vital importance that the proposed theories can be applied to address real-world problems. Up to now, the stability criteria proposed in Chapter III have only been validated by numerical examples. Finding practical systems which have slowly varying delays that satisfy the constraints we made and validate our theorem using those systems can be the very next steps.

4.2.2 Apply Different Bounding Techniques to Reach Less Conservative Results

In the derivation process, besides the inequalities stated in section 3.1.3, we mainly use Leibniz's Rule and The Second Fundamental Theorem of Calculus. The approach we applied is very intuitive. There can be other bounding techniques to help derive the conic bounds, and using them could probably reduce conservatism.

4.2.3 Extend to Systems with Time-varying Input, Output, and State delays

As stated in Chapter I, delays can appear almost everywhere and in this thesis, we focused only on systems with time-varying state delays. To expand the applicable scope of this approach, we want to study systems with time-varying input, output, and state delays, which will be the much more general case.

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