Towards a Stability Condition on the Quintic Threefold

by

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Department of Mathematics
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David R. Morrison

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M. Ronen Plesser

Dissertation submitted in partial fulfillment of the requirements for the degree of
Doctor of Philosophy in the Department of Mathematics
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2010

Abstract

(Mathematics)

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Abstract

In this thesis we try to construct a stability condition on the quintic threefold. We have not succeeded in proving the existence of such a stability condition. However we have constructed a stability condition on a quotient category of a projective space that approximates the quintic. We conjecture the existence of a stability condition on the quintic generated by spherical objects and explore some consequences.

This thesis is dedicated to my mother, Banani Roy

Contents

List of Abbreviations and Symbols

Symbols

$\mathbb{P}^n$	Projective space of dimension n.
$\mathcal{H}^i$	Cohomology sheaf of a complex.
$\mathbb{R}$	The field of real numbers
$\mathbb{C}$	The field of complex numbers.
$D(X)$	The bounded derived category of an abelian category X.
θ_x	The skyscraper sheaf of a point x.
θ_X	The structure sheaf of a space X.

Acknowledgements

I should like to thank my adviser, Prof. Paul Aspinwall, for suggesting my dissertation problem, and guiding me through my research, and supporting my stay at Duke. He showed exemplary patience in dealing with me and without his encouragement I would never have discovered derived categories. I should also like to thank Prof.Ronen Plesser for teaching me all I know about quantum field theory and string theory.

I have an immense debt of gratitude towards Dr. Stefano Guerra. He collaborated with me in this project and without him much of the work would not have been possible. However, I am infinitely more grateful to him patiently discussing every wild idea I could think of and gently correcting the most basic mistakes in algebraic geometry!

I would like to thank my parents and my sister for their lifelong support, encouragement and indulgence; without them none of this would have been possible– no words can possibly describe how much I owe to them!

1

Introduction

Derived categories have long proved to be a powerful tool to analyse problems in algebraic geometry and topology. Recently they have found extensive applications in mathematical physics for studying D-branes in string theory. The theory of stability of D-branes was formalised by Bridgeland for any triangulated category and this definition is known as a stability condition on a triangulated category.

A stability condition roughly tells us how objects in a derived category are built up in terms of simple objects. In string theory this is equivalent to finding stable (physical) D-branes which combine to form other D-branes. The central charge in the physics context provides a good numerical invariant of a D-brane that governs this process of combination or decay. In a triangulated category T a stability condition is equivalent to giving a central charge map $Z : T \longrightarrow \mathbb{C}$. This map must be additive over triangles, which translates to charge conservation in physics. In terms of this function we can identify a set of objects/ D-branes, which we call stable which do not decay. All other objects/ D-branes, are built by successively adding these basic stable branes.

Bridgeland further proved that the set of all stability conditions, i.e all possible central charge maps and set of stable objects, can be thought of as a topological space with a well defined metric. This allows us to associate a complex manifold to every derived category and this is a very powerful geometric invariant. Unfortunately identifying the space of stability conditions is a very difficult task for most categories one encounters.

Compact Calabi-Yau manifolds are extremely interesting for both physics and mathematics. In string theory three dimensional Calabi-Yau manifolds are conjectured to be the true model for the compact part of spacetime. Mathematically Calabi-Yau manifolds are rich in structure and geometrically very interesting. One would of course like to construct a stability condition on a compact three dimensional Calabi-Yau. Unfortunately so far no one has succeeded in explicitly constructing stability conditions on them.

The quintic threefold is given by a fifth degree homogeneous equation and defines a hypersurface in $\mathbb{P}^4$. An example of a quintic would be, $x_0^5 + x_1^5 + x_2^5 + x_3^5 + x_4^5 = 0$, which is called the Fermat quintic. Quintics are Calabi-Yau manifolds. Our goal in this thesis is to construct an explicit stability condition on derived category of the quintic threefold. We use the embedding of the of the quintic in $\mathbb{P}^4$ and try to use the stability condition on $\mathbb{P}^4$ to induce a stability condition on the quintic. This proves to be too difficult. We conjecture the existence of a stability condition on the quintic which is generated by 4 spherical objects and explore some consequences.

From now on we refer to the bounded derived category of a space X as $D(X)$. As an approximation for the quintic we look at a quotient category of $D(\mathbb{P}^4)/T$

where T is the category generated by skyscraper sheaves of points that do not lie on the quintic. Basically T is the set of all objects that restrict to zero on the quintic. We construct an explicit stability condition on this quotient category using a stability condition defined on $D(\mathbb{P}^4)$ generated by the exceptional collection $\theta_{\mathbb{P}^4}, \Omega^1(1)[1], \Omega^2(2)[2], \Omega^3(3)[3], \Omega^4(4)[4]$. This is certainly not equivalent to a stability condition on the quintic because the space of homomorphisms between the objects are quite different, but this does preserve some of the features of the stability condition we seek to construct on the quintic.

The first chapter is a very quick review of the fundamentals of triangulated categories and derived categories. Most of the proofs are omitted as the material is very standard. Some results like a version of the nine lemma for triangulated categories, which are not commonly found in the literature, are proved in detail. The last few sections of the chapter introduce the idea of stability conditions. Even though it is motivated by physics we actually choose the classical theory of μ stability as our starting point and motivate Bridgeland's definition. We discuss t-structures in detail and with a slightly different emphasis from what is common in the literature.

In the last chapter we present our results. We start with a discussion of Beilinson spectral sequence which allows us to resolve any object in $D(\mathbb{P}^4)$ in terms of exceptional bundles. Derived categories are notoriously bad when it comes to "gluing" operations; more formally the derived categories do not form a sheaf and do not have nice fibre products. However t-structures on subcategories can be glued together with fairly restrictive conditions. This naturally leads to the question whether stability conditions, which are defined in terms of t-structures can be glued together. We prove that they can be, again with very restrictive conditions, namely the subcategories must form a semi-orthogonal decomposition of the category. This result was proved independently by Polishchuk and Collins; however our proof has a slightly different, more direct, constructive approach. As a direct corollary of this result we can prove that exceptional objects generate stability conditions on $D(\mathbb{P}^4)$.

In the next section we discuss in detail the structure of the derived category of the quintic. We try to see to what extent can the quintic be described in terms of it's embedding in $\mathbb{P}^4$. We present long exact sequences that facilitate the computation of homomorphism groups in terms of homomorphism groups on $D(\mathbb{P}^4)$ and derive some important conclusions as a result. Then we prove a very important generating theorem for $D(X)$, the derived category of the quintic. We show that it is split-generated by the pullback of $D(\mathbb{P}^4)$. This is not as strong as classical generation– basically where one can build up a category by "adding" and shifting a finite number of generating objects– but this is the best one can hope for.

Armed with these results we try to find the conditions we need to impose on a stability condition on $D(\mathbb{P}^4)$ such that it can be pulled back to the quintic. We derive the necessary conditions on the central charge and identify the generating objects for such a stability condition. We have not succeeded in proving the existence of a t-structure that includes the five spherical objects which are the pullback of the five exceptional objects. However we present a couple of results conjecturing the existence of such a t structure.

Then in the last section of the chapter we look at a quotient category of $D(\mathbb{P}^4)$ and prove that a stability condition on $D(\mathbb{P}^4)$ which obeys the same constraints as that required for inducing a stability condition on the quintic descends to the quotient and defines a consistent stability condition.

2

Triangulated and Derived Categories

We will collect in this chapter the most commonly used properties of triangulated categories and derived categories. Proofs will be omitted in most cases, and a sketch will be provided for less known results.

2.1 Triangulated Categories

Let A be an additive category and let $T : A \longrightarrow A$ be an additive autoequivalence of A called the shift functor. Let there be a set of distinguished triangles of the form,

$$P \longrightarrow Q \longrightarrow R \longrightarrow T(P)$$

where morphism between the distinguished triangles are defined by,

$$\begin{array}{ccccccc}
P & \longrightarrow & Q & \longrightarrow & R & \longrightarrow & T(P) \\
\downarrow{\scriptstyle f} & & \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle h} & & \downarrow{\scriptstyle T(f)} \\
P^{`} & \longrightarrow & Q^{`} & \longrightarrow & R^{`} & \longrightarrow & T(P)
\end{array}$$

and all squares are commutative

Before we proceed let us introduce a shorthand notation for T and distinguished triangles. We will denote T by [1]; in most applications T will be the shift functor for a complex. Also we will delete the last entry of triangle and write it succinctly as,

$$P \longrightarrow Q \longrightarrow R \cdot$$

The triangles follow the following axioms,

T1 Any triangle of the form,

$$P \xrightarrow{Id} P \longrightarrow 0$$

is a distinguished triangle.

Any morphism between two objects,

$$P \xrightarrow{f} Q$$

can be completed to a distinguished triangle,

$$P \xrightarrow{f} Q \longrightarrow Z$$

Z will be denoted as $C(f)$, or the cone of morphism f.

T2 A triangle of the form,

$$P \xrightarrow{f} Q \xrightarrow{g} R \xrightarrow{h} P[1]$$

is distinguished if and only if,

$$Q \xrightarrow{g} R \xrightarrow{h} P[1] \xrightarrow{-f[1]} Q[1]$$

is a distinguished triangle.

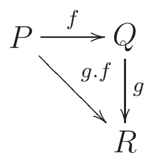
T3 The following diagram,

$$\begin{array}{ccccccc} P & \longrightarrow & Q & \longrightarrow & R & \longrightarrow & P[1] \\ {\scriptstyle f}\downarrow & & {\scriptstyle g}\downarrow & & {\scriptstyle h}\,\vdots & & {\scriptstyle f[1])}\downarrow \\ P' & \longrightarrow & Q' & \longrightarrow & R' & \longrightarrow & P'[1] \end{array}$$

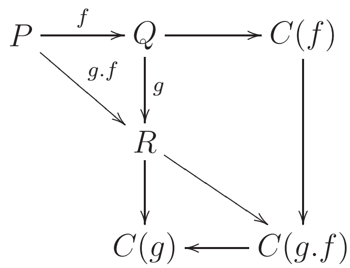
can be completed to a non canonical morphism of triangles. The map h is not necessarily unique.

T4- Octahedral Axiom This is, by far, the most difficult of all the axioms and often not necessary. Without this axiom, strictly speaking we have a pre-triangulated category. Unfortunately we will use this complicated axiom very frequently. The most general form of this axiom is difficult to state and it is an absolute nightmare to draw the necessary diagram. Instead we present a simplified version that would be most useful for our purposes.

Let us assume there is a composition of morphism,



From $T1$ we know that each morphism can be completed to a triangle. The octahedral axiom relates $C(f)$, $C(g)$ to $C(g.f)$ as follows,



where,

$$C(f) \longrightarrow C(g.f) \longrightarrow C(g)$$

is a distinguished triangle and all morphisms on the right hand squares commute. This property will be crucial in many applications relating to stability.

Exact Functors Let T, S be triangulated categories. A functor $F : T \longrightarrow S$ is called an exact functor if for all distinguished triangles,

$$A \longrightarrow B \longrightarrow C$$

in T,

$$F(A) \longrightarrow F(B) \longrightarrow F(C)$$

is a distinguished triangle in S. Exact functors are natural functors associated to triangulated categories.

Cohomological Functor A functor $H : T \longrightarrow A$, where T is triangulated and A abelian, is called cohomological if for all exact triangles,

$$A \longrightarrow B \longrightarrow C \longrightarrow A[1]$$

there is long exact sequence in A, as follows,

$$..H(C[n-1]) \longrightarrow H(A[n]) \longrightarrow H(B[n]) \longrightarrow H(C[n]) \longrightarrow H(A[n+1])..$$

2.1.1 Examples of triangulated categories

i)The commonest example and the one that will be mostly useful will be the homotopy category of the abelian category of chain complexes. The shift functor here is the usual homological shift functor of complexes and the cone of a morphism is the same as the cone construction of two complexes.

ii)Another triangulated category that we will encounter is the category of matrix factorizations.

iii)The derived category of any abelian category is triangulated. We will deal with this example in detail later.

2.2 Properties of Triangulated Categories

In this section we will list some of the most important properties of triangulated categories that will be most useful to us in this thesis.

2.2.1 Hom functor

We will consider only k-linear categories, i.e categories with Hom sets that are finite dimensional k-vector spaces. We write the usual long exact sequence that one gets on application of the Hom functor. We will denote $Hom(A, B[n])$ as $Hom^n(A, B)$. On the derived category Hom^n has the usual interpretation of $H^n RHom$ where $RHom$ is derived from the Hom functor on the abelian category. This also commonly called the Ext functor.

If,

$$P \longrightarrow Q \longrightarrow R$$

is an exact triangle, then on operating with $Hom(A, .)$ on the triangle we obtain,

$$...Hom^{n-1}(A, R) \longrightarrow Hom^n(A, P) \longrightarrow Hom^n(A, Q) \longrightarrow Hom^n(A, R)...$$

We will use the long exact sequence to prove a few properties of morphisms between exact triangles.

i)Let us have the induced morphism between the two triangles,

$$\begin{array}{ccccc}
P & \xrightarrow{\alpha} & Q & \xrightarrow{\beta} & R \\
{\scriptstyle h}\downarrow & & \downarrow{\scriptstyle m} & & \downarrow{\scriptstyle n} \\
P^{'} & \xrightarrow{\alpha^{'}} & Q^{'} & \xrightarrow{\beta^{'}} & R^{'}
\end{array}$$

The induced morphism $h^{`}$ is unique if and only if $Hom^{-1}(P, R^{`}) = 0$

Proof: We take $Hom(P, .)$ of the second triangle to obtain,

$$\cdots \longrightarrow Hom^{-1}(P, R^{'}) \longrightarrow Hom(P, P^{'}) \longrightarrow Hom(P, Q^{'}) \longrightarrow Hom(P, R^{'}) \longrightarrow \cdots$$

Now consider the first square and we see that $h \in Hom(P, P^{'})$ induces $\alpha^{'}.h \in Hom(P, Q^{'})$. We see that h is unique is equivalent to $Hom(P, P^{'}) \longrightarrow Hom(P, Q^{'})$, injective, which in turn implies that $Hom^{-1}(P, R^{`}) = 0$.

ii) Let us consider a diagram like the following,

$$\begin{array}{ccccc}
P & \xrightarrow{\alpha} & Q & \xrightarrow{\beta} & R \\
 & & \downarrow{\scriptstyle m} & & \\
P^{'} & \xrightarrow{\alpha^{'}} & Q^{'} & \xrightarrow{\beta^{'}} & R^{'}
\end{array}$$

One can complete this to a morphism of triangles, if $Hom(P, R^{'}) = 0$.

Proof: Let us apply $Hom(P, .)$ to the lower triangle to obtain,

$$\cdots \longrightarrow Hom(P, P^{'}) \longrightarrow Hom(P, Q^{'}) \longrightarrow Hom(P, R^{'}) \longrightarrow \cdots$$

If $Hom(P, R^{'}) = 0$, then $Hom(P, P^{'})$ surjects on $Hom(P, Q^{'})$. This means there exists a morphism $q : P \longrightarrow P^{'}$ such that $\alpha^{'}.q = m.\alpha$. Now we can use the proposition above to prove the claim.

iii) Let us consider the morphism of triangles,

$$\begin{array}{ccccc} P & \xrightarrow{\alpha} & Q & \xrightarrow{\beta} & R \\ \| h & & \| m & & \downarrow n \\ P^{'} & \xrightarrow{\alpha^{'}} & Q^{'} & \xrightarrow{\beta^{'}} & R^{'} \end{array}$$

Then $R \simeq R^{'}$

Proof: We take $Hom(X, .)$, where $X \in T$, where T is the triangulated category, to both triangles we obtain the following diagram of long exact sequences,

$$\begin{array}{ccccccccc} \cdots & \longrightarrow & Hom(X,P) & \longrightarrow & Hom(X,Q) & \longrightarrow & Hom(X,R) & \longrightarrow & \cdots \\ & & \| & & \| & & \downarrow n^{'} & & \\ \cdots & \longrightarrow & Hom(X,P^{'}) & \longrightarrow & Hom(X,Q^{'}) & \longrightarrow & Hom(X,R^{'}) & \longrightarrow & \cdots \end{array}$$

Properties of long exact sequences give us that $n^{'}$ is an isomorphism. Now we use Yoneda Lemma to claim that $R \simeq R^{'}$. This proves the claim.

iv)As a corollary of iii. we obtain, that if there is a triangle as follows,

$$P \xrightarrow{f} Q \longrightarrow R$$

then R is completely determined by f and can be denoted by $C(f)$

Proof: If possible let there be two triangles,

$$P \xrightarrow{f} Q \longrightarrow R_1$$

and,

$$P \xrightarrow{f} Q \longrightarrow R_2$$

This gives us,

$$\begin{array}{ccccc} P & \xrightarrow{f} & Q & \longrightarrow & R_1 \\ \| & & \| & & \downarrow h \\ P & \xrightarrow{f} & Q & \longrightarrow & R_2 \end{array}$$

By the earlier claim we know that, $R_1 \simeq R_2$. This allows us to use the notation $C(f)$ without ambiguity. (Note: the cone construction is not functorial, i.e the isomorphism is not canonical; so the preceding statement is slightly misleading if one were to consider only the abelian category.)

v)9 Lemma: This is a result that I have found to be often useful. Even though I am sure this is known, I have not seen the result in the literature. Hence I will give a complete proof.

Let us consider a commuting square,

$$\begin{array}{ccc} A & \xrightarrow{a} & B \\ \downarrow{\scriptstyle c} & & \downarrow{\scriptstyle b} \\ C & \xrightarrow{d} & D \end{array}$$

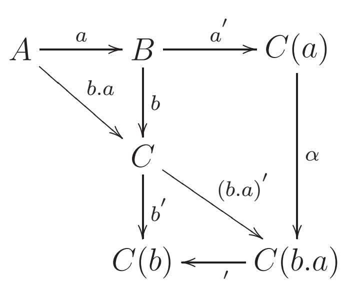
This can be completed to the following diagram,

$$\begin{array}{ccccc} A & \xrightarrow{a} & B & \xrightarrow{a'} & C(a) \\ \downarrow{\scriptstyle c} & & \downarrow{\scriptstyle b} & & \downarrow{\scriptstyle m} \\ C & \xrightarrow{d} & D & \xrightarrow{d'} & C(d) \\ \downarrow{\scriptstyle c'} & & \downarrow{\scriptstyle b'} & & \\ C(c) & \xrightarrow{n} & C(b) & & \end{array}$$

We will show that $C(m) \cong C(n)$ and that the the diagram can be completed to,

$$\begin{array}{ccccc} A & \xrightarrow{a} & B & \xrightarrow{a'} & C(a) \\ \downarrow{\scriptstyle c} & & \downarrow{\scriptstyle b} & & \downarrow{\scriptstyle m} \\ C & \xrightarrow{d} & D & \xrightarrow{d'} & C(d) \\ \downarrow{\scriptstyle c'} & & \downarrow{\scriptstyle b'} & & \downarrow{\scriptstyle m'} \\ C(c) & \xrightarrow{n} & C(b) & \xrightarrow{n'} & K \end{array}$$

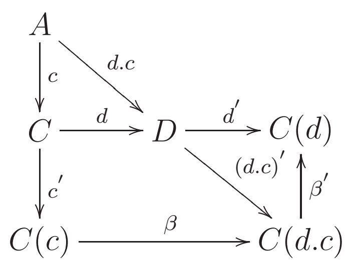
Proof: Let us first consider the octahedral axiom for $b.a$,



This gives us the triangle,

$$C(a) \xrightarrow{\alpha} C(b.a) \xrightarrow{\alpha'} C(b)$$

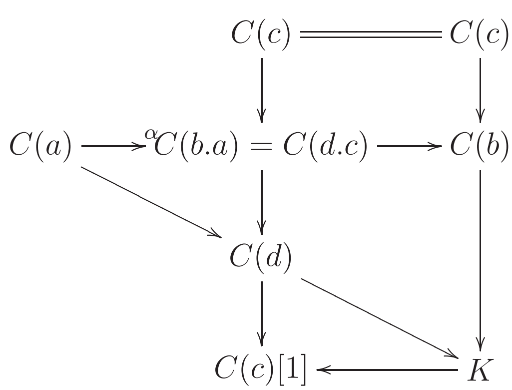
Similarly for $d.c$, we obtain,



and the triangle,

$$C(c) \xrightarrow{\beta} C(d.c) \xrightarrow{\beta'} C(d)$$

Now consider the two triangles involving cones of the morphisms and apply octahedral axiom on them to obtain,



From this we obtain the bottom and right edge of our 3X3 diagram by reading off the two new triangles we have obtained,

$$
\begin{array}{ccccc}
 & & & & C(a) \\
 & & & & \downarrow \\
 & & & & C(d) \\
 & & & & \downarrow \\
C(c) & \longrightarrow & C(b) & \longrightarrow & K
\end{array}
$$

where both edges are distinguished triangles. The proof for the commutation if the squares is a long and tedious diagram chase.

2.3 Admissible Subcategories and Exceptional Collections

2.3.1 Admissible Subcategories

In this section we discuss the notions of admissible subcategories and exceptional collections, which are special cases of admissible subcategories. These constructions have wide applications for projective spaces and toric varieties and are prime examples of stability conditions to be discussed later.

Definition Let T be a triangulated category and S be a full triangulated subcategory. If the inclusion functor, $i : S \longrightarrow T$ has a right adjoint, $\pi : T \longrightarrow S$, then S is said to be an admissible subcategory. Note that depending on which adjoint i has, S is either left admissible or right admissible– for our purposes we will deal with left admissible subcategories only. The treatment for right admissible subcategories is exactly the same.

Definition Given a triangulated subcategory S of T, we can define a triangulated subcategory, $S^{\perp}$, defined as $S^{\perp} = (X \in T : Hom(Y, X) = 0, \forall Y \in S)$

The main result we need is the following;

Proposition A full triangulated subcategory S of T is admissible, if and only if, for all $A \in T$, there exists,

$$A_S \longrightarrow A \longrightarrow A_{S^\perp}$$

where, $A_S \in S$ and $A_{S^\perp} \in S^\perp$

2.3.2 Exceptional Collection

Definition An object $A \in T$ in a k-linear triangulated category T is called exceptional if,

$Hom(A, A[i]) = 0$, if $i \neq 0$

and, $Hom(A, A) = k$.

Definition An exceptional sequence A_1, A_2,..A_n is called an exceptional sequence if,

$Hom(A_m, A_n[i]) = 0$, if $\forall m > n$ or $i \neq 0$ if $m = n$

and, $Hom(A_m, A_n) = k$ if, $m = n$.

An exceptional collection is called full if the triangulated subcategory generated by A_1, A_2,..A_n equals T.

Definition A sequence of admissible subcategories, S_1, S_2,..$S_n \subset T$ is called semiorthogonal, if $\forall i > j$, $S_j \subset S_i^\perp$. Such a sequence defines a semiorthogonal decomposition of T, if S_i generate T.

2.4 Quotient of Triangulated Categories

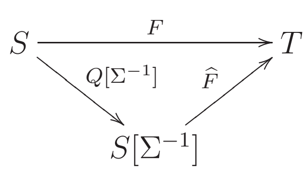
In this section we will briefly discuss the quotient of a triangulated category. The quotient process is similar to localization in commutative algebra and indeed will give us a calculus of fractions, in terms of diagrams.

Definition Let $F : S \longrightarrow T$ be a functor. F is said to make a morphism $\sigma \in Mor(S)$ invertible, if $F\sigma$ is invertible in T. The set of morphisms that are invertible under F will be denoted by $\Sigma(F)$.

Definition Given a category S and a set of morphisms Σ we formally define the quotient category, $S[\Sigma^{-1}]$ and a canonical quotient functor $Q[\Sigma^{-1}] : S \longrightarrow S[\Sigma^{-1}]$ by the following properties,

i)$Q[\Sigma^{-1}](\sigma)$ is invertible $\forall \sigma \in \Sigma$.

ii)If a functor $F : S \longrightarrow T$ makes the morphisms in Σ invertible, then F factors through as in the following diagram.



We give a quick construction of the quotient category without proofs. First we define the notion of a multiplicative system of morphisms.

Definition Let C be a category. A collection S of morphisms in C is called a multiplicative system if the following axioms hold:

i) If $f, g \in S$, then fg exists and $fg \in S$. Also $\forall X \in C$, $Id_X \in S$.

ii) Any diagram

$$\begin{array}{ccc} & & Z \\ & & \downarrow{\scriptstyle s} \\ X & \xrightarrow{u} & Y \end{array}$$

with $s \in S$ can be completed to a commutative diagram,

$$\begin{array}{ccc} W & \xrightarrow{v} & Z \\ \downarrow{\scriptstyle t} & & \downarrow{\scriptstyle s} \\ X & \xrightarrow{u} & Y \end{array}$$

with $t \in S$. The same statement, with arrows reversed, holds true as well.

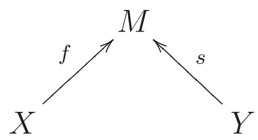
iii) If $f, g : X \longrightarrow Y$ are morphisms in C, then the following condition holds,

There exists an $s : Y \longrightarrow Y^{'}$, $s \in S$: $sf = sg$, if and only if, there exists a $t : X^{'} \longrightarrow X$, $t \in S$: $ft = gt$.

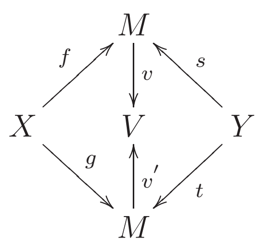
We will take quotients only with respect to multiplicative systems Σ. Now we sketch briefly the construction of the category, $S[\Sigma^{-1}]$.

i) The objects of $S[\Sigma^{-1}]$ are same as the objects of S.

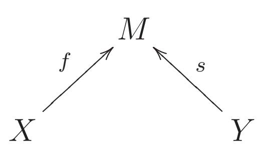
ii) The morphisms in $S[\Sigma^{-1}]$ are "roof diagrams" which are diagrams in S of the following form,



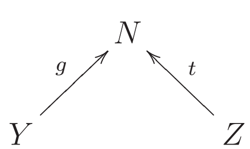
where $s \in \Sigma$. This defines a morphism in $S[\Sigma^{-1}]$, $\phi : Q(X) \longrightarrow Q(Y)$ as $Q(s)$ is an isomorphism. One can easily see that two different pairs (f, s) and (g, t) define the same morphism if the following hold,



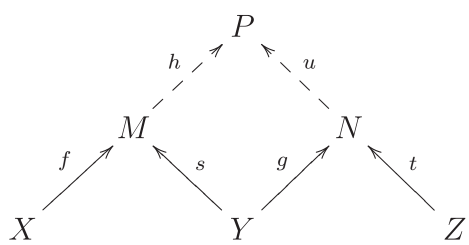
iii) Composition of two morphisms,



and,



is defined uniquely because of the multiplicative system hypotheses. Roughly the construction goes as follows,



where the existence of h and $u \in \Sigma$ and the commutativity of the top square are guaranteed by the axioms of multiplicative systems.

2.4.1 Multiplicative systems and thick subcategories

Even though the definition of a quotient in terms of multiplicative systems satisfactorily parallels the theory of localization in rings, in categories a more concrete description in terms of objects is more helpful at times. It turns out that if one considers the cones of all morphisms in a multiplicative system then they form a triangulated subcategory and it is precisely this subcategory that goes to zero on localization. More precisely, let Σ be a multiplicative system of morphisms. Then $C(\Sigma) = X : X = Cone(\sigma), \sigma \in \Sigma$ is a triangulated subcategory. Furthermore it is a thick triangulated subcategory in the sense of Verdier. We will use the simpler definition due to Ricard – equivalent to Verdier's formulation– and define a thick subcategory as a triangulated subcategory T, such that if, $A \oplus B \in T$ if and only if $A, B \in T$. What is more interesting is that if we take a thick subcategory T then it gives rise to a multiplicative system $\Sigma(T) = \sigma : C(\sigma) \in T$. To prove that they obey the axioms of multiplicative system requires a fair bit of triangle chasing and will be omitted. The most common example of a thick triangulated category which are useful in practise arise as kernels of exact functors between triangulated categories or cohomological functors from a triangulated category to an abelian category.

Let S be a thick triangulated subcategory of T. Let $\Sigma(S)$ be the multiplicative set of morphisms with cones in S. Then we can take the localization $Q : T \longrightarrow T[\Sigma(S)^{-1}]$ which we will compactly write as, T/S or $\frac{T}{S}$. Let us consider the action of Q on a triangle in T as follows,

$$A \xrightarrow{\sigma} B \longrightarrow C$$

where, $\sigma \in \Sigma(S)$ and therefore $C \in S$. Since by definition $Q(\sigma)$ is invertible and Q is an exact functor, we obtain the following triangle,

$$Q(A) \overset{Q(\sigma)}{=\!=\!=} Q(B) \longrightarrow Q(C)$$

which means $Q(C) = 0$. Therefore the subcategory S goes to zero under Q justifying the quotient notation.

It is probably worth noting here that computing Hom groups in the quotient is in general a very difficult task and there exists, to the best of my knowledge, no general way of tackling that issue. To appreciate why this might be the case, we remind ourselves that a morphism in $\frac{T}{S}$, $\phi : A \longrightarrow B$ is equivalent a class of roof diagrams in T of the following form;

$$\begin{array}{ccccc} & & M & & \\ & \overset{f}{\nearrow} & & \overset{s}{\nwarrow} & \\ A & & & & B \end{array}$$

where $s \in \Sigma(S)$, and the usual equivalence relation of roof diagrams apply. In general categorizing this group without any other information is very difficult.

2.5 Derived Categories

Finally we are ready with all the ingredients to define the central ingredient of what is to follow: the derived category of an abelian category. Let us first sketch the construction before looking at some of the properties that make derived categories so useful for geometric applications.

2.5.1 Homotopy Category

First off we start with an abelian category A. This category of course is not triangulated. Then we consider the category $C(A)$ which is the category of complexes with entries in A and d is the differential. $(C(A), d.)$ is abelian is not triangulated either. We take a quotient of $C(A)$ with respect to the equivalence relation of homotopy: basically we are taking a localization with respect to the multiplicative system of homotopic morphisms. We will call the resultant category $C(A)/Ht = K(A)$. The homotopy category $K(A)$ turns out to be triangulated.

We define the translation functor as the usual homological shift of a complex $[n]$ defined as $F.[n]_m = F_{n+m}$ and $d[n]_m = d_{n+m}$. The distinguished triangles are defined by the cone of a chain map f,

$$\begin{array}{ccccccc} \cdots & \longrightarrow & A_{i-1} & \xrightarrow{\delta_{i-1}} & A_i & \xrightarrow{\delta_i} & \cdots \\ & & \downarrow f_{i-1} & & \downarrow f_i & & \\ \cdots & \longrightarrow & B_{i-1} & \xrightarrow{d_{i-1}} & A_i & \xrightarrow{di} & \cdots \end{array}$$

as follows, $Cone(f)_i = A_i \oplus B_{i+1}$

and the differential $(d_{Cone(f)})_i$ is given by the following matrix.

$$\begin{pmatrix} \delta_i & f_i \\ 0 & d_i \end{pmatrix}$$

Note: The discerning reader will notice a complete absence of the ubiquitous $(-1)^n$ signs that plague these definitions. Those signs are useful only in the book keeping of the diagram chases and have been suppressed, though fully understood to be there so as to make the differentials nilpotent.

Using homotopy equivalence one can show that,

$$A_. \xrightarrow{f_.} B_. \longrightarrow Cone(f)$$

forms a distinguished triangle thereby giving $K(A)$ the structure of a triangulated category.

Now consider the cohomology functor– the differentials still square to zero– defined from $H : K(A) \longrightarrow Ab$. The kernel of this functor are all complexes with zero cohomology– or complexes quasi-isomorphic to zero which are called acyclic objects. By our earlier discussion this is thick, or in other words the set of quasi-isomorphisms form a multiplicative system, and we can take the localization of $K(A)$ with respect to quasi-isomorphisms. The resultant category, which is triangulated, is called the derived category, $D(A)$ of the abelian category A.

To understand what happens when we pass to the derived category, it is best to think in terms of resolutions in $C(A)$. Let an object $F \in C(A)$ have a right or left resolution of the form, $F \xrightarrow{f} A_0 \xrightarrow{d_0} A_1 \xrightarrow{d_1}$

or,

$$\longrightarrow A_{-2} \longrightarrow A_{-1} \longrightarrow A_0 \xrightarrow{f} F$$

Let us consider the latter resolution for simplicity. We can rewrite this resolution as follows,

$$\begin{array}{ccccccc} \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & F \\ & \downarrow{\scriptstyle 0} & & \downarrow{\scriptstyle 0} & & \downarrow{\scriptstyle f} \\ \longrightarrow & A_{-2} & \longrightarrow & A_{-1} & \longrightarrow & A_0 \end{array}$$

The complex

$$\longrightarrow 0 \longrightarrow 0 \longrightarrow F$$

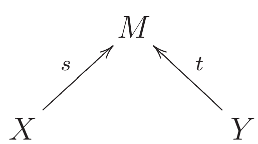
and the complex,

$$\longrightarrow A_{-2} \longrightarrow A_{-1} \longrightarrow A_0$$

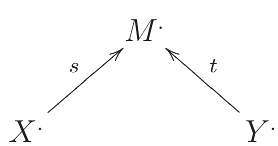
have the same cohomologies, as $H^0(A_.) = F$. So these two complexes are quasi-isomorphic and map to the same object in the derived category. Thus in the derived category objects are represented by their resolutions.

One might wonder why construct the derived category at all, which admittedly has a rather convoluted construction. We shall briefly summarize a very lucid argument provided Richard Thomas– "Complexes good, homology bad." Let us see in some detail why this may be the case and how derived categories ride to our rescue.

It is a well known theorem in topology, due to Massey, that two spaces X and Y are homotopy equivalent if and only if there exist the following diagram,



where M is a topological space, such that the corresponding chain maps of complexes in the diagram,



induce isomorphisms on cohomology, i.e $H^q(X^\cdot) \cong H^q(M^\cdot) \cong H^q(Y^\cdot)$. Here $X^\cdot$ refers to the complex corresponding to X due to simplicial decomposition.

The complex $X^\cdot$ then contains all the information about X and it would be nice to have it as an invariant of topology. One can already see the reason for the derived category– roof diagrams with quasi-isomorphisms of course arise naturally because of the quotient.

Let us consider another example where not only are complexes "good", but homology would actually be "bad". Let us consider construction of coholomogy in topology. We recall that one does not take simply the dual of the homology groups, $H^* \neq Hom(H_*, \mathbb{Z})$. There is a good reason for that– the naive dual would lose all information about the torsion of the complex. Instead we actually take the entire complex and dualize it first and then take the homology of the complex. If we consider,

$$\mathbb{Z} \xrightarrow{X2} \mathbb{Z}$$

we obtain the homology groups are $H_1 = 0$ and $H_0 = \mathbb{Z}/2$. Note if we take dual of the homology groups we get zero in both degrees. However dualizing the complex first and then taking homology does give us the expected cohomology groups, $H^1 = \mathbb{Z}/2$ and $H^0 = 0$, which does preserve the torsion data.

A more relevant example would be the case of torsion sheaves in the abelian category of coherent sheaves of a space. Torsion sheaves are not easy to deal with. On the other hand locally free sheaves are very easy to manipulate. Any torsion sheaf can be resolved by locally free sheaves. For example the skyscraper sheaf of points, θ_x on $\mathbb{P}^n$ can be written as a Koszul resolution of the form,

$$0 \longrightarrow \Lambda^n\omega \longrightarrow \Lambda^{n-1}\omega \longrightarrow \cdots\cdots \longrightarrow \Lambda^1\omega \longrightarrow \theta_{\mathbb{P}^n} \longrightarrow \theta_x$$

Now working with locally free sheaves like $\Lambda^r\omega$ is pretty easy, whereas dealing with θ_x is not so. However by our earlier conclusion that in the derived category objects are replaced by their resolutions, we see how the skyscraper sheaf of a point becomes equivalent to a finite length complex of locally free sheaves in $D^b(\mathbb{P}^4)$.

2.5.2 Derived Functors

Once we have defined the derived categories the natural question would be to ask what are the functors between these categories. Of course that class is rather large. We will instead focus on the class of functors induced by the underlying abelian category. Let us consider the derived category $D(A)$ and $D(B)$ of abelian categories A and B. Let $F : A \longrightarrow B$. We would like to formulate a notion of a "derived functor" $DF : D(A) \longrightarrow D(B)$.

First off we note that if $F : K(A) \longrightarrow K(B)$ is an exact functor of triangulated categories then, F will induce a commutative diagram,

$$\begin{array}{ccc} K(A) & \xrightarrow{F} & K(B) \\ \downarrow{\scriptstyle Q_A} & & \downarrow{\scriptstyle QB} \\ D(A) & \xrightarrow{\overline{F}} & D(B) \end{array}$$

if the following conditions are true,

i) $F(\Sigma_A) \longrightarrow \Sigma_B$, where Σ are quasi-isomorphisms.

ii)If $H^n(C) = 0$, $\forall n$, then $H^n(F(C)) = 0$, $\forall n$, that is acyclic complexes are mapped to acyclic complexes via the functor F.

It is not difficult to see that the universal property of quotient by quasi-isomorphisms immediately implies existence of $\overline{F}$. However most interesting functors that arise in geometry do not obey the two required conditions. We need a slightly more complicated construction for those cases.

To get to the construction we briefly remind ourselves the important properties of injective and projective resolutions.

Definition An abelian category A is said to have "enough injectives" (enough projectives) if $\forall X \in A$, there exists an injective morphism,(surjective morphism p) $i : A \longrightarrow I$, $I \in A$ is injective.($p : P \longrightarrow A$, $P \in A$ is projective.)

Definition An injective resolution of an object $X \in A$ is defined to be an exact sequence,

$$0 \longrightarrow X \longrightarrow I^0 \longrightarrow I^1 \longrightarrow I^2 \longrightarrow \cdots$$

where I^j are all injective objects.

Definition A projective resolution of an object $X \in A$ is defined to be an exact sequence,

$$0 \longleftrightarrow X \longleftarrow P^0 \longleftarrow P^{-1} \longleftarrow P^{-2} \longleftarrow \cdots$$

where P^j are all projective objects.

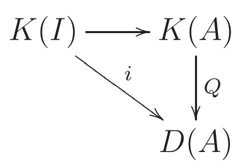
We have already explained how in the derived category objects are replaced by their resolutions. Let us formalize the result in a series of propositions and lemma that are easily proved.

Proposition: Let A be an abelian category. Let $K(A)$ be the corresponding homotopy category. If A has enough injectives(projectives) then $\forall X^{\cdot} \in K(A)$ there exists a complex $I^{\cdot}$ of injectives ($P^{\cdot}$ of projectives) such that there is a quasi-isomorphism $X^{\cdot} \longrightarrow I^{\cdot}$ ($P^{\cdot} \longrightarrow X^{\cdot}$)

Proof: The proof is easy if one recalls that projective resolutions are unique upto homotopy.

The crucial result is the following proposition.

Proposition: If A is an abelian category with enough injectives and let I be the additive subcategory of all injectives, $I \hookrightarrow A$. Passing to the homotopy category we obtain, $K(I) \longrightarrow K(A)$. By earlier proposition any X is quasi-isomorphisc to $J^{\cdot} \in I$. Therefore we have the following diagram,



The induced functor i is an equivalence.

Proposition:(Construction of Right Derived Functor)Let A, B be abelian categories and $F : A \longrightarrow B$ be an left exact functor. We note that F is exact on the subcategory of injective objects. Then the right derived functor RF is constructed from the following diagram,

$$\begin{array}{ccccc} K(I) & \longrightarrow & K(A) & \xrightarrow{KF} & K(B) \\ & {\scriptstyle i}\searrow \nwarrow{\scriptstyle i^{-1}} & \downarrow{\scriptstyle Q_A} & & \downarrow{\scriptstyle Q_B} \\ & & D(A) & \overset{RF}{\dashrightarrow} & D(B) \end{array}$$

Using this diagram we can define $RF = KF \bullet i^{-1}$. This is well defined as injective resolutions are defined upto quasi-isomorphisms.

A left derived functor for a right exact functor is constructed exactly similarly only that one has to replace the injective resolution with either a projective or flat resolution. In a geometric setting one almost always can use a locally free resolution for all cases. There are quite a few subtleties that have been suppressed in this quick sketch– existence of enough injectives, the proper resolutions demand careful attention to whether the category is bounded or only semi-unbounded. We have ignored these details as the essence of the construction is the same in all cases.

A couple of very common derived functors are induced from $Hom(A, -)$ which is left exact and $\otimes$ which is a right exact functor. The corresponding derived functors are represented as $RHom$ and $\otimes^L$.

We will denote the homologies of the derived functor as, $H^n RF \equiv R^n F$. We list some important properties of the derived functor in the following proposition

Proposition i)$H^0 F = F$.

ii) If,

$$0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$$

is a short exact sequence and F be a left exact functor. Let us assume that the right derived functor RF exists. Then we have the following long exact sequence,

$$0 \longrightarrow F(A) \longrightarrow F(B) \longrightarrow F(C) \longrightarrow$$

$$R^1F(A) \longrightarrow R^1F(B) \longrightarrow R^1F(C) \longrightarrow$$

$$\cdots \longrightarrow R^iF(B) \longrightarrow R^iF(B) \longrightarrow R^iF(B) \longrightarrow \cdots$$

Proposition: $R^iHom(A, B) = Hom_{D(X)}(A, B[i]) = Ext^i(A, B)$ where the last equality holds if the Yoneda extension is defined.

2.6 Derived Category of Coherent Sheaves and Derived Functors in Geometry

In this section we briefly sketch the construction of the derived category of coherent sheaves, the associated derived functors and those geometric constructions that will be used later. Let X be a reduced scheme– for our purposes a variety would do just as well– and $QCoh(X)$ be the abelian category of quasi-coherent sheaves on X and $Coh(X)$ be the abelian category of coherent sheaves on X. The derived category of coherent sheaves on X, $D(X)$ is defined to be $D_{Coh(X}(QCoh(X))$, i.e the derived category of complexes of quasi-coherent sheaves with coherent cohomology sheaves. In general $D(X)$ is extremely difficult to describe in concrete details. In this section we collect the most important properties of $D(X)$ and the derived functors that will be relevant for our future work.

2.6.1 Derived Functors and some properties

In classical algebraic geometry given a morphism f between two schemes X and Y (we can assume the schemes to be projective) we have four canonically defined functors associated to the abelian category of coherent sheaves on these spaces. The four functors are $f^* : Coh(Y) \longrightarrow Coh(X)$, $f_* : Coh(X) \longrightarrow Coh(Y)$, $f^! : Coh(Y) \longrightarrow Coh(X)$ and $f_! : Coh(X) \longrightarrow Coh(Y)$. When we pass to the derived categories we have the corresponding derived functors $Lf^* : D(Y) \longrightarrow D(X)$, $Rf_* : D(X) \longrightarrow D(Y)$, $Lf^! : D(Y) \longrightarrow D(X)$, $Rf_! : D(X) \longrightarrow D(Y)$. Now these functors form adjoint pairs, $f_! \dashv f^* \dashv f_* \dashv f^!$. It is a general result if $F \dashv G$ then $LF \dashv RG$. Therefore we have the same adjoint relationships for the derived functors. From now on we will drop the L or R before the derived functor sign and the derived nature of the functor will be understood from context.

We state without proof a couple of results that will be used in the future. The proofs are not difficult but require spectral sequences and the idea of a spanning set and are not really relevant to our main focus and applications.

Proposition If $f : T \hookrightarrow S$ be a closed embedding, then for any $F \in D(S)$ one has, $Supp(F) \cap T = Supp(f^*F)$

Projection Formula If $f : X \longrightarrow Y$ is a proper morphism of projective schemes then for any $F \in D(X)$ and $G \in D(Y)$ there exists a natural isomorphism,

$$f_* F \otimes G = f_*(F \otimes f^* G).$$

Proposition If $f : X \longrightarrow Y$ is a projective morphism and $F, G \in D(Y)$ then,

$$f^*(F \otimes G) = f^* F \otimes f^* G$$

The proofs are done by taking resolutions in terms of locally free sheaves and using the fact these functors, which are really derived functors, are exact on locally frees and reduce to the underived functors on coherent sheaves where we have the same relations.

There is a lot more to be said about derived categories of coherent sheaves. We have discussed only those topics that we will use in this thesis.

2.7 Stability on an Abelian Category

To motivate the definition of μ stability we start with a simple example– the abelian category of coherent sheaves on an algebraic curve C over $\mathbb{C}$. Let us look at the additive (in the sense of short exact sequences) numerical invariants of the objects in this category: two candidates would be the rank of a bundle and degree of a bundle. Let us associate to each coherent sheaf $\mathcal{E}$ a function called slope, $\mu : Coh(X) \longrightarrow \mathbb{R}$, such that $\mu(\mathcal{E}) = \frac{deg(\mathcal{E})}{rank(\mathcal{E})}$. For the time being we assume our sheaf is torsion free and has non-zero rank. One of the interesting consequences of this function is the following proposition.

Proposition The function μ satisfies the see-saw property which means that if there is a short exact sequence,

$$0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$$

then,

$$\mu(A) < \mu(B) \Longleftrightarrow \mu(B) < \mu(C)$$

and,

$$\mu(A) > \mu(B) \Longleftrightarrow \mu(B) > \mu(C)$$

Proof Let us prove the first iff condition– the second one is exactly similar. Let us write $r(A)$ for rank of A and $d(A)$ for degree of A. Then,

$$\mu(B) = \frac{d(A)+d(C)}{r(A)+r(C)} > \mu(A) = \frac{d(A)}{r(A)}$$

$$\Rightarrow r(A)d(C) - d(A)r(C) > 0$$

$$\Rightarrow \mu(C) - \mu(B) = \frac{d(C)r(A)-d(A)r(C)}{r(C)(r(A)+r(C))} > 0$$

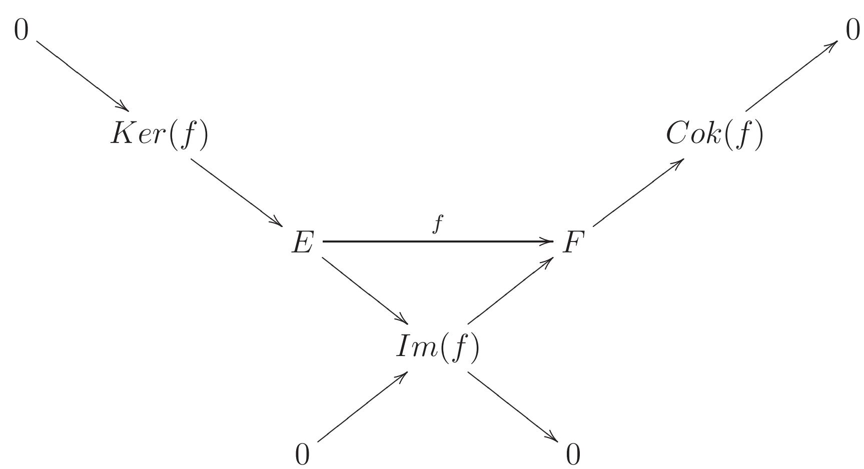
Essentially this says that $\mu(B)$ lies between $\mu(A)$ and $\mu(C)$.

An easier way of doing this proof would have been a simple geometric construction– define a function $Z : Coh(X) \longrightarrow \mathbb{C} : \ Z(A) = r(A) + id(A)$. Then $arg(Z) = \mu$ and of course Z is additive over short exact sequence. The see-saw property follows easily from the parallelogram formed by adding $Z(A) + Z(C) = Z(B)$. By convention we take $Z(T) = \infty$ when T is a torsion sheaf.

Definition of μ stability A torsion free coherent sheaf E is said to be μ semi-stable (sometimes referred to as slope stable) if for all subsheaves $F \hookrightarrow E$, $\mu(F) \leq \mu(E)$ and for all quotient sheaves $E \twoheadrightarrow G$, $\mu(E) \leq \mu(G)$.

Lemma If E and F are semi-stable and $\mu(E) > \mu(F)$ then $Hom(E, F) = 0$.

Proof Let us say there exists a nonzero $f : E \longrightarrow F$. Consider the kernel-image series,



where all the diagonal lines are short exact sequences. Now using the see-saw property of μ over short exact sequence and the definition of stability we obtain the following inequalities.

$\mu(E) < \mu(Im(f)) < \mu(F)$ and this is obviously a contradiction.

Theorem(Harder-Narasimhan filtration) Given a μ stability condition on $Coh(X)$ for any coherent sheaf F we have a unique filtration called the Harder-Narasimhan filtration (abbreviated as H-N filtration),

$$E_0 \hookrightarrow E_1 \hookrightarrow E_2 \hookrightarrow \cdots \quad \hookrightarrow E_n = F$$

with the following properties,

i)$E_i/E_{i-1} = F_i$ is semi-stable.

ii)$\mu(F_i) > \mu(F_{i+1}) \forall i$

Proof The details of the proof are somewhat involved and we shall omit the proof.

Now we try to formalize the notion of slope stability. First off note that the function Z we defined is really defined from the Grothendieck group $K(Coh(X)) \longrightarrow \mathbb{C}$ because it is additive over short exact sequence. We will call such a function a central charge. We formally define the requirements of a central charge function.

Definition A centre slope function or central charge on an abelian category A is defined as a function $Z : A \longrightarrow \mathbb{C}$ such that $Z(E) \in H$ for all $E \in A$ where $H = (r \exp(i\pi\phi), r \geq 0, 0 < \phi \leq 1)$ is the upper half plane.

Definition Let A be an abelian category and let $Z : K(A) \longrightarrow \mathbb{C}$ be a central charge as defined above. Then an object $a \in A$ is said to be semi-stable if for all non-zero subobjects $b \hookrightarrow a$, $\phi(b) \leq \phi(a)$, or equivalently for all non-zero quotients $a \twoheadrightarrow c$, $\phi(a) \leq \phi(c)$.

In summary to define a stability condition on an abelian category A we need two ingredients; a central charge $Z : K(A) \longrightarrow \mathbb{C}$ and for each object a unique harder-Narasimhan filtration where the graded quotients are semistable objects in the sense defined above. A natural question to ask would be, given a map $Z : K(A) \longrightarrow \mathbb{C}$ does it define a stability condition. Bridgeland gives a very satisfactory theorem that answers this question.

Proposition(Bridgeland) Let $Z : K(A) \longrightarrow \mathbb{C}$ be a central charge function. Then Z has the Harder-Narasimhan property, i.e every object has a unique Harder-Narasimhan filtration, if the two following conditions hold,

i)There is no infinite sequence of subobjects in A,

$... \subseteq E_{n-2} \subseteq E_{n-1} \subseteq E_n$

where $\phi(E_{n-k}) < \phi(E_{n-k+1})$ for all k.

ii) There is no infinite sequence of quotient objects in A,

$E_0 \twoheadrightarrow E_1 \twoheadrightarrow E_2 \twoheadrightarrow ....$

where $\phi(E_i) > \phi(E_{i+1})$ for all i.

2.8 t-Structures

Before we generalize the notion of stability to a triangulated category we need to define and develop some basic properties of t-structures on a triangulated category.

The results are standard and the proofs are long, but fairly easy, exercise in manipulating triangles in a triangulated category. We will omit most of the proofs and simply state the results.

Definition Let T be a triangulated category. Let $T^{\leq 0}$ and $T^{\geq 0}$ be full subcategories. We say the pair $T^{\leq 0}$ and $T^{\geq 0}$ defines a t-structure on T if the following conditions are satisfied. We use the notation $T^{\leq n} = T^{\leq 0}[-n]$ and $T^{\geq n} = T^{\geq 0}[-n]$.

i)$T^{\leq 1} \subset T^{\leq 0}$ and $T^{\geq 1} \subset T^{\geq 0}$.

ii) $Hom_T(X, Y) = 0$ for all $X \in T^{\leq 0}$ and $Y \in T^{\geq 1}$

iii) For any $F \in T$ there exists a unique triangle,

$$F_0 \longrightarrow F \longrightarrow F_1$$

such that $F_0 \in T^{\leq 0}$ and $F_1 \in T^{\geq 1}$.

Definition The full subcategory $T^{\leq 0} \cap T^{\geq 0} \equiv A$ is called the heart of a t-structure.

This definition of a t-structure is the traditional one. However this is not very transparent as to what it exactly means. We first state a couple of propositions that follow from the definition and then the intuitive meaning of a t-structure ought to be very clear.

Proposition The heart of a t-structure A is an abelian category.

Proposition $A \subset T$ is the heart of a t-structure if and only if the following two conditions hold,

i)For all $X, Y \in A$, $Hom(X, Y[-k]) = 0$ for all $k > 0$.

ii)For any object $Z \in T$ there exists a unique filtration of the following form,

$$\xymatrix{0 \ar[r] & E_0 \ar[r] \ar[d] & E_1 \ar[r] \ar[d] & \cdots\cdots \ar[r] & E_{n-1} \ar[r] \ar[d] & Z \ar[d] \\ & F_0 \ar[ul] & F_1 \ar[ul] & F_2 \ar[ul] & F_{n-1} \ar[ul] & F_n \ar[ul]}$$

such that $F_i \neq 0$ and $F_i[i] \in A$.

These last two propositions are particularly helpful in both defining and using t-structures in practice. The first example, and the one which should give us an intuitive understanding of t-structure, is the case of cohomology sheaves.

Example– Canonical t-Structure Let $D(A)$ be the derived category of coherent sheaves. The canonical t-structure is defined as follows,

i)$T^{\leq 0} = (F^\bullet : \mathcal{H}^i(F) = 0, \forall i > 0)$ and $T^{\geq 0} = (F^\bullet : \mathcal{H}^i(F) = 0, \forall i < 0)$

ii)We see that A the heart of the t-structure is nothing but the abelian category of coherent sheaves. Now for a complex $F^\bullet$ the cohomology sheaves form a long exact sequence. More precisely if we have a triangle of complexes,

$$F^\bullet \longrightarrow G^\bullet \longrightarrow H^\bullet$$

then we have a long exact sequence of cohomology sheaves,

$$\cdots \longrightarrow \mathcal{H}^{n-1}(H^\bullet) \longrightarrow \mathcal{H}^n(F^\bullet) \longrightarrow \mathcal{H}^n(G^\bullet) \longrightarrow \mathcal{H}^n(H^\bullet) \longrightarrow \mathcal{H}^{n+1}(F^\bullet) \longrightarrow \cdots$$

This leads us to ask whether in general the graded quotients in a filtration given by a t-structure form a long exact sequence. The answer turns out to be positive and we summarize this in the next proposition. Before that we need a couple of definitions.

Definition-Truncation Functor We define a functor $\tau^{\leq 0} : T \longrightarrow T^{\leq 0}$ which acts on each object as follows. Given an object in T by definition we have

$$F_0 \longrightarrow F \longrightarrow F_1$$

such that $F_0 \in T^{\leq 0}$ and $F_1 \in T^{\geq 1}$

$\tau^{\leq 0}(F) = F_0$. We similarly define $\tau^{\geq 0}$, $\tau^{\leq n}$ and $\tau^{\geq n}$

The key to these definitions is the fact that by definition for any object F we have a unique decomposition for all n into,

$$F_{\leq n} \longrightarrow F \longrightarrow F_{>n}$$

such that $F_{\leq n} \in T^{\leq n}$ and $F_{>n} \in T^{>n}$

and also,

$$F_{<n} \longrightarrow F \longrightarrow F_{\geq n}$$

such that $F_{<n} \in T^{<n}$ and $F_{\geq n} \in T^{\geq n}$

Basically the τ functors project onto the right subcategory using the unique decomposition of an object.

Definition- Cohomological Functor for t-Structure We define functors $H^n_t = \tau^{\geq n}\tau^{\leq n}$. If we keep the filtration in mind we see the functor H^n_t projects an object to it's n'th graded quotient.

Proposition H^n_t is a cohomological functor, i.e given a t-structure in a triangulated category T and an exact triangle,

$$A \longrightarrow B \longrightarrow C$$

we have a long exact sequence,

$$\cdots \longrightarrow H^{n-1}_t(C) \longrightarrow H^n_t(A) \longrightarrow H^n_t(B) \longrightarrow H^n_t(C) \longrightarrow H^{n+1}_t(A) \longrightarrow \cdots$$

Why t-Structures at all? Finally we may ask why does one need such a complicated definition for a t-structure. One answer is that a t structure gives a filtration for any object and if the t-structure is bounded, i.e the filtrations are finite, then a t-structure gives valuable information about how an object is built up through extensions of "simpler" objects which belong to an abelian category.

The better answer is actually quite profound. Let us concentrate on bounded derived categories to keep things simple. Let us have two non-equivalent abelian categories A and B. It is possible that $D(A) \simeq D(B)$ once we pass to the derived category. We can invert this question and ask given a derived category $D(A_0)$ are there other abelian categories A_i, all non-equivalent, such that $D(A_i) \simeq D(A)$. One way to hunt for these categories, or prove equivalence of two derived categories would be to construct t-structures which have these abelian categories as their hearts. More precisely if two bounded t-structures with hearts A_1 and A_2 can be defined on T then $D(A_1) \simeq D(A_2)$. Bridgeland used this technique to prove equivalence of two derived categories for a flop.

2.9 Bridgeland Stability Condition for Triangulated Categories

Bridgeland,(1), generalised the definition of stability on an abelian category to triangulated categories. Let us limit our discussion to derived categories though it works just as well for any triangulated category.

K group of Derived Categories Given $D(A)$ there is a canonical embedding of $A \longrightarrow D(A)$. This induces a map on K-theories, $K(A) \longrightarrow K(D)$. This is actually an equivalence. The inverse functor is constructed by mapping a complex $F^\bullet \longrightarrow \oplus_i (-1)^i H^i(F^\bullet)$.

Definition of Bridgeland Stability A stability condition on a triangulated category T is a pair (Z, P) where $Z : K(T) \longrightarrow \mathbb{C}$, which is called a central charge, and full additive subcategories $P(\phi) \subseteq T$ for each $\phi \in \mathbb{R}$ which satisfy the following axioms,

i)If $X \in P(\phi)$ then $Z(X) = m(X) \exp(i\pi\phi)$ where $m(X) \in \mathbb{R}_{\geq 0}$

ii) For all $\phi \in \mathbb{R}$, $P(\phi + 1) = P(\phi)[1]$.

iii) If $X_i \in P(\phi_i)$ then $i > j$, then $Hom_T(X_i, X_j) = 0$.

iv) For any object $Z \in T$ there exists a unique filtration of the following form,

$$\begin{array}{ccccccccccc} 0 & \longrightarrow & E_0 & \longrightarrow & E_1 & \longrightarrow & \cdots\cdots & \longrightarrow & E_{n-1} & \longrightarrow & Z \\ & \nwarrow & \downarrow & \nwarrow & \downarrow & \nwarrow & & \nwarrow & \downarrow & \nwarrow & \downarrow \\ & & F_0 & & F_1 & & F_2 & & F_{n-1} & & F_n \end{array}$$

such that $F_i \neq 0$ and $F_i \in P(\phi_i)$ and $\phi_i \geq \phi_{i+1}$

The last condition is reminiscent of t-structures and their filtrations and we see that a t-structure provides a "coarse" filtration of the last kind where $\phi_i = i \in \mathbb{Z}$. This leads us to the next, and perhaps the all important theorem due to Bridgeland, which describes a stability condition in terms of t-structures and their hearts.

Proposition- Bridgeland To give a stability condition on a triangulated category T in the sense defined above, is equivalent to giving a bounded t-structure on T and giving a centre slope function Z on the heart A of the t-structure, which has the Harder-Narasimhan property.

Proof This is such an important theorem and so central to our future work that we give a sketch of the proof.

Suppose we are given a stability condition (Z, P) on a triangulated category T then we can define a t-structure as follows. Given an object F we have a minimum and maximum phase in terms of it's filtration. Let us denote them as ϕ_{min} and ϕ_{max} We define $T^{\leq 0} = (X : \phi_{max}(X) \leq 0)$ and $T^{\geq 0} = (X : \phi_{min}(X) \geq 0)$. One can show this defines a consistent t-structure. The heart of the t-structure is given by the category $P(0,1] = (X : 0 \leq \phi_{min}(X) \leq .... \leq \phi_{max}(X) \leq 1)$.

The other direction is pretty obvious. If we have a centre-slope function on the heart A then every object in the heart, F_n has a filtration such that,

$$\begin{array}{ccccccccccc} 0 & \longrightarrow & E_0 & \longrightarrow & E_1 & \longrightarrow \cdots\cdots \longrightarrow & & E_{n-1} & \longrightarrow & F_i \\ & \nwarrow & \downarrow & \nwarrow & \downarrow & \nwarrow & \nwarrow & \downarrow & \nwarrow & \downarrow \\ & & h_0^i & & h_1^i & h_2^i & & h_{n-1}^i & & h_n^i \end{array}$$

where each $\phi(h_m^i) \geq \phi(h_{m+1}^i)$

Now each object in T has a filtration due to the t-structure of the following form,

$$\begin{array}{ccccccccccc} 0 & \longrightarrow & E_0 & \longrightarrow & E_1 & \longrightarrow \cdots\cdots \longrightarrow & & E_{n-1} & \longrightarrow & Z \\ & \nwarrow & \downarrow & \nwarrow & \downarrow & \nwarrow & \nwarrow & \downarrow & \nwarrow & \downarrow \\ & & F_0 & & F_1 & F_2 & & F_{n-1} & & F_n \end{array}$$

where $F_i \in A[-i]$. Now we can "glue" these filtrations together to give an overall filtration where the phase inequality condition is maintained. The proof that such a gluing is possible is given in the second chapter.

What the Bridgeland definition does is handle the question of stability in two steps: in the first one uses a t-structure to provide a coarse filtration with quotients in some abelian category and then the centre slope function provides a finer filtration of each graded quotient inside the abelian category.

For completeness sake one should mention that Bridgeland proved a far stronger theorem where he gave a topology to the space of all stability conditions on a triangulated category and proved that the space has is a complex manifold. However we do not investigate the space of stability conditions in this thesis and hence we will refrain from a discussion of this very important topic. There are only very few explicit examples of stability conditions. Some explicit example relevant to our work can be found in, (2), (3), (4), (5) etc.

3

Construction of Stability Conditions

3.1 Introduction

In this chapter we try to construct a stability condition on the quintic threefold X. We have a conjecture as to what should be part of the stability condition, but we have not succeeded in proving a key existence theorem which would rigorously construct a stability condition. The key to our approach is to induce a stability condition on the quintic through it's embedding $X \hookrightarrow \mathbb{P}^4$. $D(\mathbb{P}^4)$ has a well known set of stability conditions constructed by the exceptional objects that generate the derived category. To this end we extensively study the stability condition in $D(\mathbb{P}^4)$. However in the end we construct a stability condition on the triangulated category $D(\mathbb{P}^4)/Ker(f^*)$ where $Ker(f^*)$ is the thick triangulated category of objects that pull-back to zero on the quintic.

We give an outline of how the chapter is organized. In section 2 we construct a set of stability condition on $D(\mathbb{P}^4)$. We state the all important Beilinson spectral sequence that allows the construction of the stability condition. We prove a couple of useful, if rather technical, results that allow us to convert resolutions into filtrations. This is a very useful trick to doing explicit tests of stability.

In section 3 we give a more direct proof of a theorem proved by Polishchuk and Collins, (6), which states that any triangulated category T which has a semi-orthogonal decomposition in terms of triangulated subcategories $A_0, A_1, ...A_n$ can be given a stability condition induced from the stability condition on each subcategory. We give a list of cases where this technique can be applied with some hope of success. As a direct corollary of this theorem we construct a stability condition on $D(\mathbb{P}^4)$ induced by a collection of exceptional objects in it.

Section 4 is rather technical in nature. However the results are absolutely central to understanding the structure of $D(X)$ and it's embedding in $D(\mathbb{P}^4)$. We show the embedding is neither full or faithful. We also construct a long exact sequence to compute $Hom_X(f^*a, f^*b)$, which is difficult to compute, in terms of $Hom^i_{\mathbb{P}^4}(a, b)$ which are easier to compute. We prove a very important structure theorem about $D(X)$ by showing $f^*D(\mathbb{P}^4)$ split generates $D(X)$ in the the sense for any object $F \in D(X)$, $F \oplus F \otimes f^*\omega^3[5] \in f^*D(\mathbb{P}^4)$. Split generation is not as strong a condition as classical generation by cones and shifts alone, but that is the closest one can get to a generation theorem about a Calabi-Yau hypersurface. We also prove that we need only four spherical objects to split generate $D(X)$ whereas one needs five exceptional objects to generate $D(\mathbb{P}^4)$. Finally we prove that $Ker(f^*)$ which is the thick triangulated category of objects that pull-back to zero on the quintic is composed of objects whose cohomology sheaves are skyscraper sheaves of points that do not lie on the quintic.

In section 5 we get to the heart of the problem and try to see what conditions need to be imposed to pull back a stability condition from $D(\mathbb{P}^4)$. We find that we need to impose the condition that the central charge of a skyscraper sheaf of a point on $\mathbb{P}^4$ needs to be zero. However we run into problems as soon as we pullback on $D(X)$ because there is no obvious way to finitely filter a complex in $D(X)$. $f^*D(\mathbb{P}^4)$ is not Karoubi complete– in the sense an idempotent morphism will not split inside it– and can not even support a bounded t-structure. The Karoubi completion of $f^*D(\mathbb{P}^4)$ is the full $D(X)$. The problem is to identify objects that lie outside of $f^*D(\mathbb{P}^4)$ that will provide a t-structure for $D(X)$. We have not been able to prove the existence of such a t-structure.

As an approximation to $D(X)$ we can try to look at it's embedded image in $D(\mathbb{P}^4)$ in the sense we look at $f_*D(X)$. The problem of inducing a stability condition on this subcategory is that it is not an admissible category of $D(\mathbb{P}^4)$. The next candidate, and in some sense the best, should be the derived category of coherent sheaves on $\mathbb{P}^4$ which are supported only on the quintic X. We call this category $D_X(\mathbb{P}^4)$. This has an embedding in $D_{\mathbb{P}^4}(\mathbb{P}^4)$; unfortunately this too is not an admissible subcategory. We remind the reader that A is an admissible subcategory of T, when the inclusion functor, $i : A \hookrightarrow T$ has a right adjoint π. This is the closest approximation of a splitting one can have in a triangulated category. We consider the category $D(\mathbb{P}^4)/Ker(f^*)$ where $Ker(f^*)$ is the thick triangulated category of objects that pull-back to zero on the quintic. This category is not as geometric as the other possible choices, but we succeed in inducing a stability condition on this category. By construction this category has exactly the same constraint condition, the central charge of a skyscraper sheaf of a point on $\mathbb{P}^4$ is zero, and in some sense gives an approximation to the stability condition on the quintic. We show that both $f_*D(X)$ and $D_X(\mathbb{P}^4)$ have natural functors into this category and we show that this functor is faithful though crucially not essentially surjective. We would have hoped that this category would be equivalent to $D_X(\mathbb{P}^4)$. However that is not true– it turns out that equivalence of the categories is equivalent to existence of a right adjoint to the quotient map which does not exist.

Finally at the end of the section 5 we give a proof for the stability condition on the quotient category. Basically the proof shows that for locally free sheaves, or their shifts, the Hom groups remain unchanged under the quotient process. We use this fact to construct a stability condition in this category.

3.2 Stability Condition on $\mathbb{P}^4$

3.2.1 Beilinson Spectral Sequence

Proposition:(Beilinson) For any coherent sheaf F on $\mathbb{P}^n$ there exist two natural spectral sequences:

$E_1^{r,s} = H^s(\mathbb{P}^n, F(r)) \otimes \Omega^{-r}(-r)$ converges to $E^{r+s} = F$ if $r+s=0$ and $E^{r+s}=0$ if $r+s \neq 0$

and,

$E_1^{r,s} = H^s(\mathbb{P}^n, F(r) \otimes \Omega^{-r}(-r)) \otimes \theta_{\mathbb{P}^n}(r)$ converges to $E^{r+s} = F$ if $r+s=0$ and $E^{r+s}=0$ if $r+s \neq 0$

Proposition:(Existence of Exceptional Collection) The Beilinson spectral sequence proves that there exist two exceptional collections on $D^b(\mathbb{P}^n)$. Any collection of the form $\theta_{\mathbb{P}^n}(a), \theta_{\mathbb{P}^n}(a+1), ..., \theta_{\mathbb{P}^n}(a+n)$ form a full exceptional collection on $\mathbb{P}^n$. Also $\theta_{\mathbb{P}^n}, \Omega^1(1), ..., \Omega^n(n)$ forms a full exceptional collection on $\mathbb{P}^n$.

We remind the reader of the definition of an exceptional collection,

Definition A collection of objects A_1, A_2,..A_n is called an exceptional sequence if,

$Hom(A_m, A_n[i]) = 0$, if $\forall m > n$ or $i \neq 0$ if $m = n$

and, $Hom(A_m, A_n) = k$ if, $m = n$.

An exceptional collection is called full if the triangulated subcategory generated by A_1, A_2,..A_n equals T.

Definition An exceptional collection is called strong if $Hom(A_i, A_j[l]) = 0$, $\forall i, j$ and $l \neq 0$.

Proposition The two exceptional collections constructed above are strong.

We illustrate how Beilinson Spectral sequence is used through a couple of explicit examples. Let us consider the skyscraper sheaf of a point θ_x. If we choose the strong exceptional sequence $\theta_{\mathbb{P}^4}(-4), \theta_{\mathbb{P}^4}(-3), \theta_{\mathbb{P}^4}(-2), \theta_{\mathbb{P}^4}(-1), \theta_{\mathbb{P}^4}$. The beilinson spectral sequence, once unravelled, gives us the following resolution,

$$0 \longrightarrow \theta_{\mathbb{P}^4}(-4) \longrightarrow \theta_{\mathbb{P}^4}(-3)^4 \longrightarrow \theta_{\mathbb{P}^4}(-2)^6 \longrightarrow \theta_{\mathbb{P}^4}(-1)^4 \longrightarrow \theta_{\mathbb{P}^4} \longrightarrow \theta_x$$

We note this is nothing but the Koszul resolution of the point. We can also write θ_x in terms of the other sequence, $\theta_{\mathbb{P}^4}, \Omega^1(1), \Omega^2(2), \Omega^3(3), \Omega^4(4)$. The resolution in terms of these objects is as follows,

$$0 \longrightarrow \Omega^4(4) \longrightarrow \Omega^3(3) \longrightarrow \Omega^2(2) \longrightarrow \Omega^1(1) \longrightarrow \theta_{\mathbb{P}^4} \longrightarrow \theta_x$$

Another example would be that of a curve θ_C in $\mathbb{P}^4$. In terms of $\theta_{\mathbb{P}^4}(i)$ the resolution is,

$$0 \longrightarrow \theta_{\mathbb{P}^4}(-3) \longrightarrow \theta_{\mathbb{P}^4}(-2)^3 \longrightarrow \theta_{\mathbb{P}^4}(-1)^3 \longrightarrow \theta_{\mathbb{P}^4} \longrightarrow \theta_C$$

We can try to rewrite this in terms of the other set but that is a cumbersome calculation. We can bypass using the spectral sequence explicitly if we write down the "change of base" resolutions, i.e, rewrite the objects in one exceptional collection in terms of the objects in the other basis.

3.2.2 A Stability Condition on $\mathbb{P}^4$

In this section we sketch a proof for the construction of a stability condition on $\mathbb{P}^4$. We consider the skyscraper sheaf of a point as an example. The skyscraper sheaf has a resolution,

$$\Omega^4(4) \longrightarrow \Omega^3(3) \longrightarrow \Omega^2(2) \longrightarrow \Omega^1(1) \longrightarrow \theta_{\mathbb{P}^4} \longrightarrow \theta_x$$

We need a couple of preliminary results to turn this into filtration.

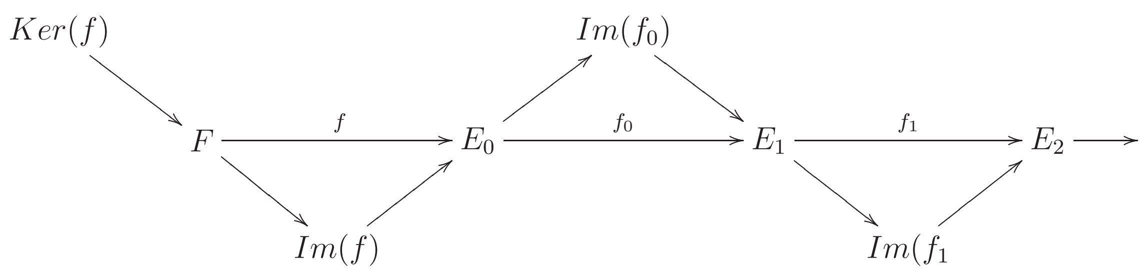
Turning resolutions to filtrations Our goal is to turn a resolution in an abelian category into a filtration by exact triangles in the derived category. To this end we need the two following results.

Proposition If A is an abelian category and $F \in A$ has an exact resolution,

$$0 \longrightarrow F \xrightarrow{f} E_0 \longrightarrow E_1 \longrightarrow E_2 \longrightarrow \cdots \longrightarrow E_n \longrightarrow 0$$

then we can construct a canonical filtration for F in $D(A)$ from this resolution. We will call this resolution an "injective" type resolution.

Proof: Let us break up the resolution in a *ker coker* series,



Since the resolution is exact we have $Im(f_i) = Ker(f_{i+1})$. This gives us a sequence of triangles,

$$F = Ker(f_0) \longrightarrow E_0 \longrightarrow Im(f_0)$$

$$Im(f_0) = Ker(f_1) \longrightarrow E_1 \longrightarrow Im(f_2)$$

$\vdots$

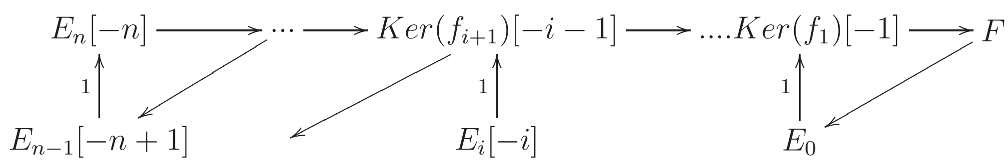
$\vdots$

$$Im(f_{i-1}) = Ker(f_i) \longrightarrow E_i \longrightarrow Im(f_i)$$

We rotate the triangle to write every triangle in the form,

$$Ker(f_{i+1})[-1] \longrightarrow Ker(f_i) \longrightarrow E_i$$

and then put the triangle together to obtain a filtration,



This gives us the required filtration.

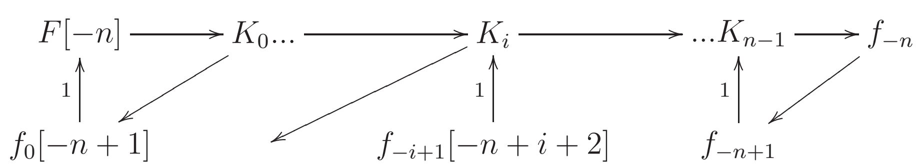
The next proposition constructs a filtration when we have a "projective" type resolution.

Proposition Let A be an abelian category and $F \in A$ have a resolution of the form,

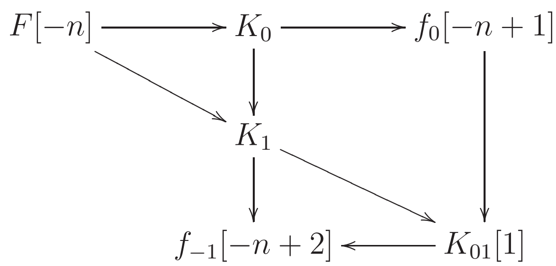
$$0 \longrightarrow f_{-n} \longrightarrow f_{-n+1} \longrightarrow \cdots\cdot \qquad f_{-1} \longrightarrow f_0 \longrightarrow F \longrightarrow 0$$

then F has a canonical filtration in $D(A)$.

Proof This can be thought of as an "injective" type resolution for f_{-n}. Using the last proposition we write a filtration as,



Now we will put triangles together using the octahedral axiom as follows,

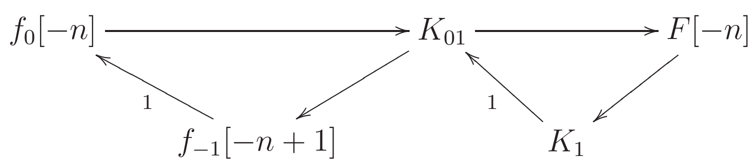


So we have two triangles now,

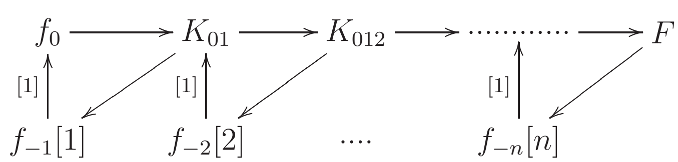
$$K_{01} \longrightarrow F[-n] \longrightarrow K_1$$

$$f_0[-n] \longrightarrow K_{01} \longrightarrow f_{-1}[-n+1]$$

Putting these two triangles together we have,



Carrying on in the same fashion and successively applying octahedral axiom and keeping careful track of the homological shifts, we obtain the filtration,



As an illustration of these propositions we prove the following, *Proposition* The skyscraper sheaf of a point θ_x has a filtration,

$$\begin{array}{ccccccccc} \theta & \longrightarrow & K_1 & \longrightarrow & K_2 & \longrightarrow & K_3 & \longrightarrow & \theta_x \\ {\scriptstyle [1]}\uparrow & \swarrow & {\scriptstyle [1]}\uparrow & \swarrow & {\scriptstyle [1]}\uparrow & \swarrow & {\scriptstyle [1]}\uparrow & \swarrow & \\ \Omega^1[1] & & \Omega^2[2] & & \Omega^3[3] & & \Omega^4[4] & & \end{array}$$

Proof We take the resolution,

$$\Omega^4(4) \longrightarrow \Omega^3(3) \longrightarrow \Omega^2(2) \longrightarrow \Omega^1(1) \longrightarrow \theta_{\mathbb{P}^4} \longrightarrow \theta_x$$

and then apply the proposition to derive the claimed filtration.

Though we have not proved it yet, there exists a stability condition on $D(\mathbb{P}^4)$ generated by $< \Omega^4(4)[4], \Omega^3(3)[3], \Omega^2(2)[2], \Omega^1(1)[1], \theta_{\mathbb{P}^4} >$ where we can assign them central charges $Z(\Omega^i(i)[i] = \exp^{\frac{2\pi i}{5}}$. Given this definition we can clearly see that the point θ_x is in the heart of the stability condition. However should we choose say, $< \Omega^4(4)[6], \Omega^3(3)[3], \Omega^2(2)[2], \Omega^1(1)[1], \theta_{\mathbb{P}^4} >$ as the new objects generating the heart– note the central charge stays the same– then we clearly have that the sky-scraper sheaf of the point is unstable. The way we can see that is to take the long exact sequence in cohomological functor for the particular t-structure. We see that the last triangle in the filtration for the point,

$$K_3 \longrightarrow \theta_x \longrightarrow \Omega^4[6][-2]$$

where we have deliberately written the last term in this particular form to emphasize it is in the $[-2]$ translate of the heart. Now the long exact sequence gives us,

$$\cdots \longrightarrow H_s^i(K_3) \longrightarrow H_s^i(\theta_x) \longrightarrow H_s^i(\Omega^4[6][-2]) \longrightarrow \cdots$$

where H_s is the cohomological functor for this t-structure. Since the last term is non zero only for $i = 2$ we have that θ_x has two cohomologies– in degree 0 and 2– and hence can not belong to the heart and can not be stable.

We generalize this idea in the next section and prove a more general result for semi-orthogonal decompositions which is a more general case of an exceptional collection. That will imply a stability condition on $D^b(\mathbb{P}^4)$. Macri et.al, (2) first constructed stability conditions given by exceptional objects.

3.3 Gluing stability condition

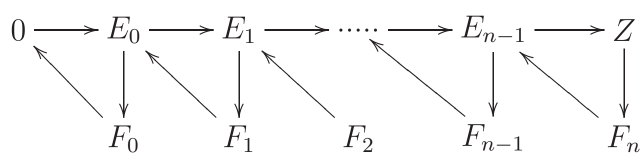
Let us start with a proposition that shows how to glue filtrations.

Proposition Let

$$X \longrightarrow Y \longrightarrow Z$$

be an exact triangle. Let X and Z have filtrations with graded quotients (H_i) and (F_i). Then the filtrations can be glued together to provide a filtration for Y with graded quotients (H_i, F_j) .

Proof This is a simple result due to the octahedral axiom for triangulated categories. Let,



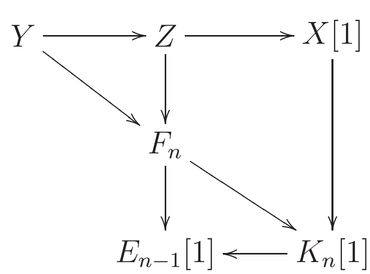
be the HN filtration for Z. Let us look at the triangles

$$E_{n-1} \longrightarrow Z \longrightarrow F_n$$

and

$$X \longrightarrow Y \longrightarrow Z$$

We use octahedral axiom on these two triangles to obtain,

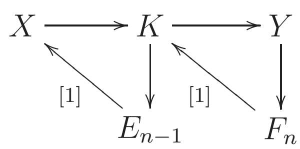


This gives two triangles,

$$K_n \longrightarrow Y \longrightarrow F_n$$

$$X \longrightarrow K_n \longrightarrow E_{n-1}$$

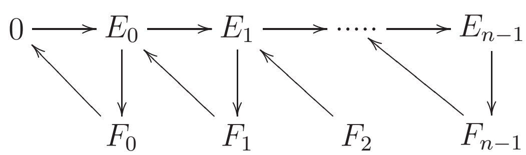
which gives us the following filtration of Y



We note that we have introduced F_n as a graded quotient of Y. Also the last triangle is,

$$X \longrightarrow K_n \longrightarrow E_{n-1}$$

and now we repeat the same process for E_{n-1} which has a filtration,



We repeat this process and create a filtration for Y with F_i as the graded quotients. Since X is at the left hand edge of the triangle, any filtration for X is automatically added to the filtration created.

Using this result we can prove the following simple result,

Proposition Let A be an admissible subcategory of a triangulated category T. Let A have a stability condition with heart $\mathcal{H}_A$ and $A^{\perp}$ have a stability condition with heart $\mathcal{H}_{A^{\perp}}$. Let us assume the following conditions hold,

i)$Hom^i(h_{A^{\perp}}, h_A) = 0 \ \forall i \leq 0$.

ii)Let Z_A and $Z_{A_{\perp}}$ be the central charges on A and $A^{\perp}$. Let us assume that

$\phi(A) \geq \phi(A^{\perp})$, i.e, all the phases of objects in the heart $\mathcal{H}_A$ are greater than the phases of all objects in the heart $\mathcal{H}_{A^{\perp}}$.

Then we can glue the stability conditions to produce a stability condition on T with heart $< \mathcal{H}_A, \mathcal{H}_{A^{\perp}} >$.

Proof Let us first look at the conditions and see what they mean. First note that by the semi-orthogonality condition, $Hom(A, A^{\perp}) = 0$ for all objects in A and $A^{\perp}$. So the only homomorphisms can be from $A^{\perp}$ to A. By the first condition we require that the only nonzero Hom are in degree 1 and above, i.e Ext^1 and above. An example of such a situation would be the collection $\theta_{\mathbb{P}^4}, \Omega^1(1)[1], \Omega^2(2)[2], \Omega^3(3)[3], \Omega^4(4)[4]$ in $D(\mathbb{P}^4)$. The second condition is imposed to ensure a proper phase ordering in a Harder Narasimhan filtration. It is probably worth mentioning that if we impose a more restrictive condition on the Homs and make them only Ext^2 and higher then we do not even need the second condition! However the most interesting stability conditions we will encounter will always be Ext^1 collections.

To construct a stability condition we need to first construct a t-structure and then give a central charge on the heart of the t-structure that obeys the Harder-Narasimhan property. First we construct a glued t-structure.

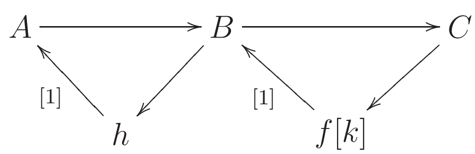
From the semi-orthogonality property we have that for any object $X \in T$ we have a decomposition,

$$X_A \longrightarrow X \longrightarrow X_{A^{\perp}}$$

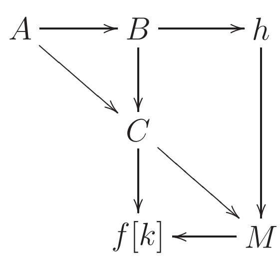
where $X_A \in A$ and $X_{A^{\perp}} \in A^{\perp}$. Since we have stability conditions on A and $A^{\perp}$

we have ready made filtrations for X_A and $X_{A^\perp}$. Using the earlier proposition we can glue the filtrations to produce a filtration for X. Also note that the graded quotients are exactly those of X_A and $X_{A^\perp}$. Let us assume that the last quotient of $X_{A^\perp}$ be in the $[m]$'th shift of the heart $\mathcal{H}_{A^\perp}$, whereas the first term of the filtration of X_A be in the $[n]$'th shift of $\mathcal{H}_A$. If $m > n$ then it violates the condition of t-structure. We need a way to shift the quotients around so that we have the right definition for a t-structure.

To solve this last problem we will need the Ext^1 condition. Let us consider a typical sequence of triangles to show how we can exchange the order.



where $k \geq 1$. As usual we will use the octahedral axiom to glue triangles together,



This gives us two triangles,

$$A \longrightarrow C \longrightarrow M$$

and,

$$h \longrightarrow M \longrightarrow f[k]$$

We have not really done anything new here– putting the triangles back together and using the filtration-gluing proposition we have exactly the same filtration we started with. The second triangle is more interesting. This triangle is produced by a morphism $Hom(f[k], h[1])$ i.e, $Hom^{1-k}(f, h)$ By our earlier assumption $Hom^i(f, h) = 0$ $\forall i \leq 0$. However as $k \geq 1$ we have $i - k \leq 0$ which precisely the condition we imposed. Therefore $Hom^{1-k}(f, h) = 0$ which in turn implies $M = h \oplus f[k]$. Now we can easily flip the order of h and $f[k]$. We can continue doing this until all the homological shifts are in the right order and then we can put the triangles back together to produce the Harder-Narasimhan filtration required for a t-structure.

Now we see the need of the phase ordering condition. Within each shift of the glued heart we always have objects from $\mathcal{H}_A$ to the left of objects of $\mathcal{H}_{A^\perp}$ and the phase order is precisely the condition we need to construct a stability condition.

3.3.1 Applications and examples

We list a few examples where the proposition above can be used.

$D^b(\mathbb{P}^n)$ *and exceptional sequences* Beilinson proved the existence of full exceptional sequences in $\mathbb{P}^n$ which provide a semi-orthogonal decomposition. In general any triangulated category with a full exceptional collection can be given a stability condition by the proposition proved above. Examples would include toric varieties with exceptional collections.

Inducing stability via morphisms In general it is difficult to functorially induce t-structures or stability via morphisms between categories. However there is one example where one can produce a semi-orthogonal decomposition and one can indeed use the technique listed above.

Let,

$$X \xrightarrow{f} Y$$

which induces the following functors between the derived categories,

$$D^b(X) \underset{f^*}{\overset{f_*}{\rightleftarrows}} D^b(Y)$$

Let us assume the following condition holds,

$$f_*\theta_X = \theta_Y$$

Then $f^*(Y)$ and $Kerf_*$ gives a semi-orthogonal decomposition of $D^b(X)$ and it can be given a stability condition as long as the conditions of proposition 1 are satisfied.

To prove the claim we note the chain of equalities,

$$f_*\theta_X = \theta_Y$$

$$\Rightarrow f_*\theta_X \otimes F = F, \forall F \in Y$$

$$\Rightarrow f_*f^*\theta_Y \otimes F = F$$

Use of projection formula gives,

$$\Rightarrow f_*(\theta_Y \otimes f^*F) = F$$

$\Rightarrow f_*f^*F = F$

This gives us that $Hom_X(f^*A, f^*B) = Hom_Y(A, f_*f^*B) = Hom_Y(A, B), \forall A, B \in Y$

Now we prove the subcategory f^*Y is triangulated. It is obvious that this closed under homological shifts. Now consider

$$f^*A \xrightarrow{a} f^*B \longrightarrow C$$

Using the Hom result we know that $a = f^*b$, for some $b \in End(Y)$. Taking f_* of the triangle we obtain the following morphism of triangles,

$$\begin{array}{ccccc} A & \xrightarrow{b} & B & \longrightarrow & f_*C \\ \| & & \| & & \downarrow \\ A & \xrightarrow{b} & B & \longrightarrow & Cone(b) \end{array}$$

If we take f^* of the lower triangle we obtain that $C = f^*(Cone(b)$. This proves f^*Y is triangulated.

We use the canonical morphism $f^*f_*A \longrightarrow A$ to obtain the triangle,

$$f^*f_*A \longrightarrow A \longrightarrow C$$

Taking f_* of this triangle gives us $f_*C = 0$

This proves that f^*Y and $Kerf_*$ form a semiorthogonal decomposition of X. Note, that $Kerf_* = \Phi$ would imply equivalence of the categories and of course the stability conditions are automatically induced.

This method can be generalized for any pair of adjoint functors F, G between two categories S and T, such that either $FG = Id$ or $GF = Id$.

Bridgeland, (7), uses a similar technique to produce t-structures for the flop.

3.4 Closed Embedding of the Quintic in $\mathbb{P}^4$

In this section we consider the closed embedding of a 3-dimensional Calabi-Yau quintic hypersurface $X \hookrightarrow \mathbb{P}^4$. One of the commonest examples is the Fermat quintic given by $x_0^5 + x_1^5 + x_2^5 + x_3^5 + x_4^5 = 0$. All the results in the section hold for any Calabi-Yau quintic threefold, not just the Fermat quintic.
At the level of the derived categories we have the diagram:

$$D^b(X) \underset{f^*}{\overset{f_*}{\rightleftarrows}} D^b(\mathbb{P}^4)$$

Our goal is to compute various Hom groups in $D(X)$, which are difficult to describe, in terms of Hom groups in $D(\mathbb{P}^4)$ which have more combinatorial description. Though rather technical, this section provides the backbone of the results that we derive later.

We list some of the important results that we use in the later sections.

Proposition : For any object $F \in D(\mathbb{P}^4)$ we have an exact triangle of the form,

$$F \otimes \omega \longrightarrow F \longrightarrow f_* f^* F$$

Proof First let us consider the exact sequence that defines the embedding of the quintic X which is a fifth degree equation,

$$0 \longrightarrow \theta_{\mathbb{P}^4}(-5) \longrightarrow \theta_{\mathbb{P}^4} \longrightarrow f_* \theta_X \longrightarrow 0$$

where θ_X is the structure sheaf of the quintic X. Now $\theta_{\mathbb{P}^4}(-5)$ happens to be the canonical bundle of $\mathbb{P}^4$ which we denote by $\omega_{\mathbb{P}^4}$. Therefore in the derived category we have the exact triangle,

$$\omega_{\mathbb{P}^4} \longrightarrow \theta_{\mathbb{P}^4} \longrightarrow f_* \theta_X$$

We tensor with F to obtain,

$$F \otimes \omega_{\mathbb{P}^4} \longrightarrow F \longrightarrow F \otimes f_* \theta_X$$

However $\theta_X = f^* \theta_{\mathbb{P}^4}$, which gives us the following once we use the projection formula,

$$f_* f^* \theta_{\mathbb{P}^4} \otimes F = f_*(f^* \theta_{\mathbb{P}^4} \otimes f^* F) = f_* f^* F$$

This proves the required result.

This result can be used to prove a very useful result for Hom groups.

Proposition For any $F, G \in D(\mathbb{P}^4)$ we have the following long exact sequence,

$$\cdots \longrightarrow Hom^n_{\mathbb{P}^4}(G, \omega \otimes F) \longrightarrow Hom^n_{\mathbb{P}^4}(G, F) \longrightarrow Hom^n_X(f^*G, f^*F) \longrightarrow \cdots$$

which can be written equivalently as,

$$\cdots \longrightarrow Hom^n_{\mathbb{P}^4}(G, F) \longrightarrow Hom^n_X(f^*G, f^*F) \longrightarrow (Hom^{3-n}_{\mathbb{P}^4}(F, G))^* \longrightarrow \cdots$$

Proof The first long exact sequence is obtained easily by taking $Hom(G, \bullet)$ of the exact triangle in the last proposition and using the right adjoint property of f_*, i.e, note that $Hom_{\mathbb{P}^4}(G, f_*f^*F) = Hom_X(f^*G, f^*F)$.

To obtain the second identity we have to use the definition of the Serre functor on $\mathbb{P}^4$. For any smooth projective variety X the Serre functor is $S = \bullet \otimes \omega[dimX]$. For us $dimX = 4$, Using this we can rewrite $Hom^{n+1}(G, F \otimes \omega)$ as follows,

$$Hom^{n+1}(G, F \otimes \omega) = Hom(G, F \otimes \omega[4[n-3]) = Hom(G, S(F[n-3])) =$$

$$(Hom(F[n-3], G))^* = (Hom(F, G[3-n]))^* = (Hom^{3-n}(F, G))^*$$

Now the first long exact sequence is,

$$\cdots \longrightarrow Hom^n_{\mathbb{P}^4}(G, F) \longrightarrow Hom^n_X(f^*G, f^*F) \longrightarrow Hom^{n+1}_{\mathbb{P}^4}(G, \omega \otimes F) \longrightarrow \cdots$$

Using the expression for $Hom^{n+1}(G, \omega \otimes F)$ we derived, we obtain the second long exact sequence.

This long exact sequence makes it possible to calculate Hom groups of objects in $f^*D(\mathbb{P}^4) \subset D(X)$ entirely in terms of Hom groups in $D(\mathbb{P}^4)$ up to the usual extension issues. In fact the term $(Hom^{3-n}(F,G))^*$ explicitly shows how the Serre dual Hom groups arise on the Calabi-Yau which has Serre functor [3] and consequently the Serre dual relation on Hom groups is as follows,

$$Hom^n_X(F,G) = (Hom^{3-n}_X(G,F))^*.$$

The following proposition shows an explicit example of this phenomenon.

Proposition If E is an exceptional object in $D(\mathbb{P}^4)$, i.e $Hom^n_{\mathbb{P}^4}(E,E) = k$ if $n = 0$ and zero otherwise, then $f^*E \in D(X)$ is spherical, i.e $Hom^n_X(f^*E, f^*E) = k$ if $n = 0, 3$ and zero otherwise

Proof Let us look at the second long exact sequence for $n = 0$ and $n = 3$. For $n = 0$ we have,

$$\begin{array}{ccccccc} (Hom^4_{\mathbb{P}^4}(E,E))^* & \longrightarrow & Hom_{\mathbb{P}^4}(E,E) & \longrightarrow & Hom_X(f^*E,f^*E) & \longrightarrow & (Hom^3_{\mathbb{P}^4}(E,E))^* \\ \| & & \| & & \| & & \| \\ 0 & \longrightarrow & k & = & k & \longrightarrow & 0 \end{array}$$

Similarly for $n = 3$ we have,

$$\begin{array}{ccccccc} Hom^3_{\mathbb{P}^4}(E,E) & \longrightarrow & Hom^3_X(f^*E,f^*E) & \longrightarrow & (Hom_{\mathbb{P}^4}(E,E))^* & \longrightarrow & Hom^4_{\mathbb{P}^4}(E,E) \\ \| & & \| & & \| & & \| \\ 0 & \longrightarrow & k & = & k & \longrightarrow & 0 \end{array}$$

This shows how even though $Hom^3_{\mathbb{P}^4}(E,E) = 0$, one grows a non zero $Hom^3_X(f^*E, f^*E)$.

In general if $Hom_{\mathbb{P}^4}(A,B) = 0$ then $Hom_X(f^*A, f^*B)$ is generally nonzero as can be seen from the long exact sequence above. However for a special class of objects– a strong exceptional sequence– one does indeed have the zero Homs pulling back to zero Homs.

Proposition If one has a strong exceptional sequence, i.e an exceptional sequence A_1, A_2,..A_n such that ,

$Hom(A_m, A_n[i]) = 0$, if $\forall m > n$ or $i \neq 0$ if $m = n$

and, $Hom(A_m, A_n) = k$ if, $m = n$.

then $Hom_{\mathbb{P}^4}(A_m, A_n) = 0$ implies $Hom_X(f^*A_m, f^*A_n) = 0$.

Proof We look at the $n = 0$ part of the long exact sequence we see,

$$\cdots \longrightarrow Hom_{\mathbb{P}^4}(A_m, A_n) \longrightarrow Hom_X(f^*A_m, f^*A_n) \longrightarrow (Hom^3_{\mathbb{P}^4}(A_n, A_m))^* \longrightarrow \cdots$$

Because the sequence is strong all higher Homs are zero and thus the last term, $(Hom^3_{\mathbb{P}^4}(A_n, A_m))^*$, vanishes as well the first term $Hom_{\mathbb{P}^4}(A_m, A_n)$, thereby giving the result.

Proposition For any object F, on $D(X)$ there exists an exact triangle of the form,

$$F \otimes f^*\omega[1] \longrightarrow f^*f_*F \longrightarrow F$$

where $\omega = \theta_{\mathbb{P}^4}(-5)$ is the canonical bundle on $\mathbb{P}^4$.

Proof The proof is a rather tricky use of Fourier-Mukai transforms and can be found in Bondal and Orlov.

As an example of how this triangle can be used we prove the following lemma,

Lemma i) If $Hom^2_X(F, F \otimes f^*\omega) = 0$ then $f^*f_*F = F \oplus F[1]$.

ii) As an immediate consequence we have , $f^*f_*\theta_X = \theta_X \oplus \theta_X[1]$ where θ_X is the structure sheaf of the quintic.

iii)Also we have $f^*f_*\theta_x = \theta_x \oplus \theta_x[1]$ where θ_x is the skyscraper sheaf of a point.

Proof i) Let us take the exact triangle in the proposition above. We have,

$$F \otimes f^*\omega[1] \longrightarrow f^*f_*F \longrightarrow F \xrightarrow{0} F \otimes f^*\omega[2]$$

which immediately gives us that the triangle "splits" and that the middle-term is the direct sum of the first and the last.

ii)We have to compute $Hom^2_X(\theta_X, \theta_X \otimes f^*\omega_{\mathbb{P}^4})$. However $\theta_X = f^*\theta_{\mathbb{P}^4}$. Therefore we have to compute $Hom^2_X(f^*\theta_{\mathbb{P}^4}, \theta_{\mathbb{P}^4} \otimes f^*\omega^4_{\mathbb{P}}) = Hom^2_X(f^*\theta_{\mathbb{P}^4}, f^*\theta_{\mathbb{P}^4}(-5))$

We use the long exact sequence we constructed earlier for such computations and obtain,

$$...Hom^2_{\mathbb{P}^4}(\theta_{\mathbb{P}^4}, \theta_{\mathbb{P}^4}(-5)) \longrightarrow Hom^2_X(f^*\theta_{\mathbb{P}^4}, f^*\theta_{\mathbb{P}^4}(-5)) \longrightarrow (Hom^1_{\mathbb{P}^4}(\theta_{\mathbb{P}^4}(-5), \theta_{\mathbb{P}^4}))^*$$

However all higher Hom s are zero giving us $Hom^2_X(f^*\theta_{\mathbb{P}^4}, f^*\theta_{\mathbb{P}^4}(-5)) = 0$ and hence the result.

iii) We observe that $\theta_x \otimes \theta(-5) \simeq \theta_x$ and of course $Hom^2(\theta_x, \theta_x) = 0$, which immediately gives the result. $f_*\theta_x$ is a skyscraper sheaf of a point on $\mathbb{P}^4$ and this result shows that the K-class of the pullback of a point is zero, because $[\theta_x[1]] = -[\theta_x]$. This seemingly innocuous result is actually quite significant as we will see later.

Split Generation of the Quintic As mentioned earlier the derived category of the Calabi-Yau is extremely difficult to describe explicitly. One way triangulated categories can be described is to find a finite generating set, in the sense a family of objects such that the entire category is generated by cones and translations. There is no obvious way to construct such a set for the quintic threefold. However the Calabi-Yau quintic threefold has a nice codimension one embedding in $\mathbb{P}^4$ which has a nice combinatorial structure as we have seen in the last section. Can we describe the quintic better in terms of the embedding? The answer turns out to be nearly, but not quite! We explore this in the following proposition.

Definition Let $S \hookrightarrow T$ be a triangulated subcategory of the triangulated category T. S is called a split generator of T if any object in T can be written in a finite number of cones, translates and summands of objects in S.

Proposition $f^*D(\mathbb{P}^4)$ is a split exact generator of $D(X)$ where X is the quintic threefold.

Proof We turn to the exact triangle,

$$f^*f_*F \longrightarrow F \longrightarrow F \otimes f^*\omega[2]$$

We write the same triangle for $F \otimes f^*\omega[2]$ and obtain,

$$f^*f_*(F \otimes f^*\omega[2]) \longrightarrow F \otimes f^*\omega[2] \longrightarrow F \otimes f^*\omega^2[4]$$

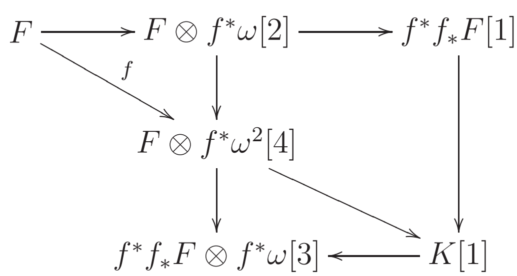
Using the projection formula we can simplify the first term as,

$$f^*f_*(F \otimes f^*\omega[2]) = f^*(f_*F \otimes \omega[2]) = f^*f_*F \otimes f^*\omega[2]$$

Then we use the octahedral axiom on the composition of the morphisms,

$$F \longrightarrow F \otimes f^*\omega[2] \longrightarrow F \otimes f^*\omega^2[4]$$

as follows,



Thus we have two exact triangles,

i) $F \xrightarrow{f} F \otimes f^*\omega^2[4] \longrightarrow K[1]$

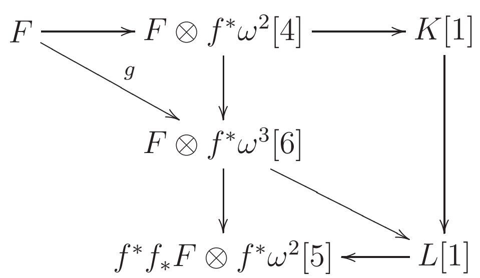
ii) $f^*f_*F[1] \longrightarrow K[1] \longrightarrow f^*f_*F \otimes f^*\omega[3]$

or equivalently,

$$f^*f_*F \otimes f^*\omega[1] \longrightarrow f^*f_*F \longrightarrow K$$

where K is defined through the last triangle.

Now we repeat the process all over and use octahedral axiom for composition again to obtain,



This gives us two triangles,

$$F \xrightarrow{g} F \otimes f^*\omega^3[6] \longrightarrow L[1]$$

$$K[1] \longrightarrow L[1] \longrightarrow f^*f_*F \otimes f^*\omega^2[5]$$

The important part of the proof is the show $g = 0$. This will imply that $L = F \oplus F \otimes f^*\omega^3[5]$. However L is given as a cone of K and another object, both of which are in the triangulated extension of $f^*D(\mathbb{P}^4)$. This proves $f^*D(\mathbb{P}^4)$ split generates $D(X)$. The rest of the proof is to show that $g = 0$ and we prove it using induction.

Any complex F has a filtration in terms of it's cohomology sheaves which are locally free. Since we are in the bounded derived category we have a finite filtration of the following form,

$$\begin{array}{ccccccccccc}
\mathcal{H}_0 = F_0 & \longrightarrow & F_1 & \longrightarrow & F_2 & \longrightarrow & \cdots & \longrightarrow & F_{n-1} & \longrightarrow & F_n = F \\
& {\scriptstyle [1]}\nwarrow & \downarrow & {\scriptstyle [1]}\nwarrow & \downarrow & & & {\scriptstyle [1]}\nwarrow & \downarrow & {\scriptstyle [1]}\nwarrow & \downarrow \\
& & \mathcal{H}_1 & & \mathcal{H}_2 & & & & \mathcal{H}_{n-1} & & \mathcal{H}_n
\end{array}$$

Since $f^*\omega$ is locally free (hence flat) the filtration for $F \otimes f^*\omega^2$ is given in the p'th triangle,

$$F_{p-1} \otimes f^*\omega^2 \longrightarrow F_p \otimes f^*\omega^2 \longrightarrow \mathcal{H}_p \otimes f^*\omega^2$$

Since $dim(X) = 3$ by Serre's theorem $Hom^n(\mathcal{H}_p, \mathcal{H}_q \otimes f^*\omega^2) = 0$ for all $n \geq 4$. Now we use the induction step. Let's consider the first triangle in the filtration for each complex,

$$F_{n-1} \longrightarrow F \longrightarrow \mathcal{H}_n$$

and,

$$G_{n-1} \longrightarrow G \longrightarrow \mathcal{L}_n$$

We have used G to denote $F \otimes f^*\omega^3$ and $\mathcal{L}_n$ as it's cohomology sheaves. Now we apply $Hom(\mathcal{H}_n, \bullet)$ on the second triangle to obtain,

$$..Hom^4(\mathcal{H}_n, \mathcal{L}_n) \longrightarrow Hom^5(\mathcal{H}_n, G_{n-1}) \longrightarrow Hom^5(\mathcal{H}_n, G_n) \longrightarrow Hom^5(\mathcal{H}_n, \mathcal{L}_n)..$$

Because of dimensional reasons $Hom^4(\mathcal{H}_n, \mathcal{L}_n) = 0$ and $Hom^5(\mathcal{H}_n, \mathcal{L}_n) = 0$. So we have, $Hom^5(\mathcal{H}_n, G_n) \simeq Hom^5(\mathcal{H}_n, G_{n-1})$. We can repeat this and in a finite number of steps, n to be exact, we have $Hom^5(\mathcal{H}_n, G_n) \simeq Hom^5(\mathcal{H}_n, G_0) = Hom^5(\mathcal{H}_n, \mathcal{L}_0) = 0$. Recall $\mathcal{L}_0$ is a locally free sheaf as well and hence by dimensional reasons we have the 0. So in the end we have proved, $Hom^5(\mathcal{H}_n, G_n) = 0$. We can prove $Hom^n(\mathcal{H}_n, G_n) = 0 \ \forall n \geq 5$ using exactly the same argument.

Now apply $Hom(\bullet, G_n)$ to the first triangle to obtain,

$$Hom^6(\mathcal{H}_n, G_n) \longrightarrow Hom^6(F_n, G_n) \longrightarrow Hom^6(F_{n-1}, G_n) \longrightarrow Hom^7(\mathcal{H}_n, G_n)$$

The first and the last term are zero because $Hom^n(\mathcal{H}_n, G_n) = 0 \ \forall n \geq 5$ which gives us the isomorphism, $Hom^6(F_n, G_n) \simeq Hom^6(F_{n-1}, G_n)$. Again n ietration of this step will give us, $Hom^6(F_n, G_n) \simeq Hom^6(F_0, G_n)$. But F_0 is locally free and applying $Hom(F_0, \bullet)$ on the second triangle gives us exactly what we obtained for $\mathcal{H}_n$, namely, $Hom^n(F_0, G_n) = 0 \ \forall n \geq 5$. This proves that $Hom^6(F_n, G_n) = 0$. G_n is just the shorthand for $F \otimes f^*\omega^3$ and we have proved that $Hom^6(F, F \otimes f^*\omega^3) = 0$. Hence we have proved $g = 0$.

As a direct consequence of the proof for split generation we have the following result,

Proposition If $< E_0, E_1, ..., E_4 >$ is any exceptional collection that generates $D(\mathbb{P}^4)$, then four of the f^*E_is, which are all spherical objects by our earlier result, split generate $D(X)$. In particular if we choose the exceptional collection as $\theta_{\mathbb{P}^4}, \Omega^1(1), ..., \Omega^4(4)$, or their translates, then $f^*\theta_{\mathbb{P}^4} = \theta_X, f^*\Omega^1(1), f^*\Omega^2(2), f^*\Omega^3(3)$ split-generate $D(X)$.

Proof The first important point is to note that whereas $D(\mathbb{P}^4)$ is generated by five objects, $D(X)$ is split-generated by only four of them. The fact all five of them split generate $D(X)$ follows from the proposition above. We have also proved earlier that f^*E_i is spherical if E_i is exceptional. There are points on $\mathbb{P}^4$ which do not lie on the quintic and hence pull-back to zero (The fact that the K-class of the pull-back of any point is zero is significant– this allows a certain consistency for our future work). The skyscraper sheaf of the point has a resolution in terms of E_is by Beilinson spectral sequence. This immediately imposes a zero condition on f^*E_is eliminating one of them.

To see this explicitly let us consider the Beilinson resolution of a point in terms of $\Omega^i(i)$. The resolution is,

$$0 \longrightarrow \Omega^4(4) \longrightarrow \Omega^3(3) \longrightarrow \Omega^2(2) \longrightarrow \Omega^1(1) \longrightarrow \theta_{\mathbb{P}^4} \longrightarrow \theta_x$$

Let us assume $x \nsubseteq X$. Then $f^*\theta_x = 0$. f^* is exact on locally frees and hence we have,

$$0 \longrightarrow f^*\Omega^4(4) \longrightarrow f^*\Omega^3(3) \longrightarrow f^*\Omega^2(2) \longrightarrow f^*\Omega^1(1) \longrightarrow \theta_X \longrightarrow 0$$

This means we can write $f^*\Omega^4(4)$ in terms of the rest. This proves the claim.

Finally we look at the functors f^* and f_* and find an explicit description of their kernels.

Proposition $f_*F = 0$ if and only if $F = 0$

Proof We use the exact triangle,

$$f^*f_*F \longrightarrow F \longrightarrow F \otimes f^*\omega[2]$$

to note that if, $f_*F = 0$, then from the triangle above we can conclude that, $F^\cdot \longrightarrow F \otimes f^*\omega[2]$ is an isomorphism. Now we argue in the same spirit as the proof for split generation– we iterate this process to obtain,

$$F \simeq F \otimes f^*\omega[2] \simeq F \otimes f^*\omega^2[4] \simeq F \otimes f^*\omega^3[6]$$

But we have already proved that $Hom^6(F, F \otimes f^*\omega^3) = 0$. This is a contradiction unless $F = 0$. This proves the claim.

Proposition If $F \in D(\mathbb{P}^4)$ and $f^*F = 0$. Then $\mathcal{H}(F)$, the cohomology sheaves of F, are skyscraper sheaves of points not lying on the quintic X.

Proof Let $f^*F = 0$. We use the fact that if X is a closed subscheme of Y, $X \hookrightarrow Y$, then $Supp(f^*G) = Supp(G) \cap X$ where G is a sheaf. If $f^*F = 0$, then $\mathcal{H}^q(f^*F) = f^*(\mathcal{H}^qF) = 0$. Therefore $Supp(H^qF)$ lies in the complement of X. However the ideal sheaf of the quintic is very ample and any subscheme of dimension greater than 0 intersects quintic at some point. Therefore $Supp(H^qF)$ must have dimension 0 and hence a skyscraper sheaf of a point not lying on the quintic. This proves the claim.

So far we have tried to deduce as much as we could about the structure of $D(X)$ from the embedding. We saw that the embedding is not faithful, the long exact sequence in Homs prove that. We did find a somewhat satisfactory generating property of

$D(X)$ in terms of spherical objects. We have also explicitly described the kernels of the functors f^* and f_*.

3.5 An Attempt Towards a Stability Condition on the Quintic

If our goal is to induce a stability condition on the quintic via a stability condition on $D(\mathbb{P}^4)$ then the first question we must ask is how to define a central charge on $D(X)$. Obviously we would like to use the embedding,

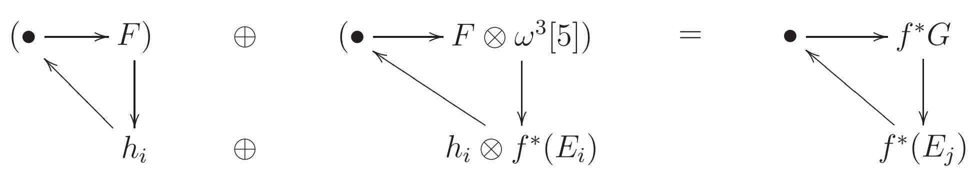
$$D(X) \underset{f^*}{\overset{f_*}{\rightleftarrows}} D(\mathbb{P}^4)$$

We define the central charge of $Z(f^*F) = Z(F)$. However we know that $f^*\theta_x = 0$ if $x \nsubseteq X$ and θ_x is the skyscraper sheaf of a point. Therefore, for consistency $Z(f^*\theta_x)$ had better be zero. However, $\theta_x = f_*\theta_p$ for some point p on the quintic if the point x lies in the image of the embedded quintic. We proved that $f^*\theta_x = \theta_p \oplus \theta_p[1]$ and hence the K-class $[f^*\theta_x] = 0$. Thus we have a consistent condition that the K-class of the pullback of any point on $\mathbb{P}^4$ is zero. This implies that we must choose a central charge on $D(\mathbb{P}^4)$ such that $Z(f^*\theta_x) = Z(\theta_x) = 0$. This is the first condition that we must impose on any central charge if we hope to get a stability condition on $D(X)$ using that map.

We have to extend this map. We know that not every object in $D(X)$ is in $f^*D(\mathbb{P}^4)$. Aspinwall proved that the rational K-theory of $D(X)$ is generated by $f^*D(\mathbb{P}^4)$. We proved a very similar statement when we showed that the quintic is split generated by $f^*D(\mathbb{P}^4)$. So by giving a central charge on $f^*D(\mathbb{P}^4)$ we have given a central charge on the entire $D(X)$ in a consistent manner.

Now the second question is can we find a t-structure such that the central charge we have chosen has the Harder-Narasimhan property on the heart of that t-structure. If we inspect the stability conditions we have constructed on $D(\mathbb{P}^4)$, the hearts were generated by exceptional objects E_i. The pullback of exceptional objects, f^*E_i are spherical objects and these spherical objects must be part of the generating set of the heart of any t-structure in $D(X)$. Our first guess might be that the abelian category generated by $< f^*E_i >$ should be enough to generate a t-structure. However this must be false because it would imply that $D(X)$ is classically generated by $< f^*E_i >$ which we know is not true.

We know that $D(X)$ is split-generated by $f^*D(\mathbb{P}^4)$ and for any $F \in D(X)$, $F \oplus F \otimes f^*\omega^3[5] \in f^*D(\mathbb{P}^4)$. We know for certain that there must be objects not in $f^*D(\mathbb{P}^4)$ which are part of the heart of a stability condition. How do we detect these objects? One hope might be to use the split generation condition. Let us say that h_i are the semi-stable objects in the filtration of an object F. The filtration for $f^*\omega^3$ can be computed in terms of it's filtration in $D(\mathbb{P}^4)$. Schematically we have,



We immediately run into a big problem: tensors behave badly with respect to t-structures. Once we have tensored two semi-stable objects the product need not be semi-stable, in fact mostly they are not. Far worse is that we do not have any nice way of finding the filtration, i.e semi-stable quotients, of the tensor product of two semi-stable objects. This approach does not seem to have an obvious solution either. The central problem is there is no obvious way of constructing a t-structure with

f^*E_i as part of the heart. One needs a geometric way to filter an arbitrary complex in $D(X)$ and we have not succeeded not doing so.

Let us now look at subcategories of $D(\mathbb{P}^4)$ which may be good approximations for $D(X)$. Right away we must make it very clear that any subcategory of $D(\mathbb{P}^4)$ will differ crucially from $D(X)$ in their internal $Homs$– the Serre functors are completely different on the two categories. With this caveat let us try to look at a few possible choices.

Stability Condition on a Quotient Category Our first attempt is to look at the kernel of the functor f^*. We have already proved that this is a thick subcategory of objects F such that their cohomology sheaves $\mathcal{H}_i$ are skyscraper sheaves of points θ_x such that x is not in the quintic. We know that on pulling back to the quintic these objects pull back to zero. So a quotient of $D(\mathbb{P}^4)$ by these objects approximate the quintic in some sense. From now on we specifically focus on the stability condition generated by the collection $< \Omega^i(i)[i] >$. We need to impose the condition that $Z(\theta_x) = 0$. We recall that θ_x has a resolution as follows,

$$\Omega^4(4) \longrightarrow \Omega^3(3) \longrightarrow \Omega^2(2) \longrightarrow \Omega^1(1) \longrightarrow \theta_{\mathbb{P}^4} \longrightarrow \theta_x$$

This shows that, $Z(\theta_x) = Z(\theta_{\mathbb{P}^4}) - Z(\Omega^1(1)) + Z(\Omega^2(2)) - Z(\Omega^3(3)) + Z(\Omega^4(4))$ or equivalently, $Z(\theta_x) = Z(\theta_{\mathbb{P}^4}) + Z(\Omega^1(1)[1]) + Z(\Omega^2(2)[2]) + Z(\Omega^3(3)[3]) + Z(\Omega^4(4)[4])$

It is rather nice that the right collection is an Ext^1 collection and we can glue them together to produce a stability condition. The only condition we require is that, $Z(\theta_{\mathbb{P}^4}) + Z(\Omega^1(1)[1]) + Z(\Omega^2(2)[2]) - Z(\Omega^3(3)[3]) + Z(\Omega^4(4)[4]) = 0$. So far the order of the phases does not matter as there are no $Homs$ involved between the objects. However we choose to order the phases such that $\phi(\Omega^3(3)[3]) > \phi(\Omega^2(2)[2]) > \phi(\Omega^1(1)[1]) > \phi(\theta_{\mathbb{P}^4})$. We deliberately exclude $Z(\Omega^4(4)[4])$ because $Z(f^*\Omega^4(4)[4])$ must be a linear combination of four objects we included in the list. We note that the phases of all semi-stable objects must be in the upper half plane. Given the constraint equation on the central charges if we ensure $Z(\theta_{\mathbb{P}^4}), Z(\Omega^1(1)[1]), Z(\Omega^2(2)[2]), Z(\Omega^3(3)[3])$ lie in the upper half plane then $Z(\Omega^4(4)[4])$ must necessarily lie in the lower half plane. To fix this we choose $\Omega^4(4)[5]$ as the object that belongs to the heart. Now the central charge lies in the upper half plane. By our earlier proposition as long as we have Ext^1 or higher we can glue together to form a stability condition and this collection will still produce a valid stability condition.

Our choices of central charge have not been quite as arbitrary as it might have seemed. First we imposed the constraint condition to ensure that central charge behaved well when pulled back to the quintic. Also the reason we choose to order the phases has to do with the extra Serre dual morphisms on the quintic. On $D(\mathbb{P}^4)$ there is a non zero $Ext^3(\Omega^3(3)[3], \theta)$. On the quintic this will give a nonzero $Hom(f^*\theta, f^*\Omega^3(3)[3])$. This implies that $\phi(\theta) < \phi(\Omega^3(3)[3])$ if we are to define a consistent stability condition. The other two objects, $\Omega^1(1)[1]$ and $\Omega^2(2)[2]$ do not have any new Homs. This leaves us with $\Omega^4(4)[5]$; this will most certainly have new Homs due to Serre duality with $\Omega^2(2)[2]$. Worse yet, it will develop negative Hom s with $\Omega^1(1)[1]$ and θ which is strictly not allowed for any object in the heart of a stability condition. However this is not a problem because of the following,

Proposition If there is a stability condition on $D(X)$ such that, $\theta_X, f^*\Omega^1(1)[1], f^*\Omega^2(2)[2], f^*\Omega^3(3)[3]$ are in the heart of the stability condition, then $f^*\Omega^4(4)$ is not in the heart of the stability condition. If H is the heart of the stability condition then $f^*\Omega^4(4) \in H[-3]$.

Proof Since $f^*(\theta_x) = 0$ in $D(X)$ we have the following exact sequence,

$$0 \longrightarrow \Omega^4(4) \longrightarrow \Omega^3(3) \longrightarrow \Omega^2(2) \longrightarrow \Omega^1(1) \longrightarrow \theta_{\mathbb{P}^4} \longrightarrow 0$$

This is an injective type resolution for $\Omega^4(4)$ and using our earlier proposition we can write a filtration for it as follows,

$$\begin{array}{ccccccc}
\theta_X[-3] & \longrightarrow & \bullet & \longrightarrow & \bullet & \longrightarrow & \Omega^4(4) \\
 & {\scriptstyle [1]}\nwarrow & \downarrow & {\scriptstyle [1]}\nwarrow & \downarrow & {\scriptstyle [1]}\nwarrow & \downarrow \\
 & & f^*\Omega^1(1)[-2] & & f^*\Omega^2(2)[-1] & & f^*\Omega^3(3)
\end{array}$$

or in terms of our chosen set of generators,

$$\begin{array}{ccccccc}
\theta_X[-3] & \longrightarrow & \bullet & \longrightarrow & \bullet & \longrightarrow & \Omega^4(4) \\
 & {\scriptstyle [1]}\nwarrow & \downarrow & {\scriptstyle [1]}\nwarrow & \downarrow & {\scriptstyle [1]}\nwarrow & \downarrow \\
 & & f^*\Omega^1(1)[1][-3] & & f^*\Omega^2(2)[2][-3] & & f^*\Omega^3(3)[3][-3]
\end{array}$$

This proves the claim.

This is consistent with our earlier proof that $D(X)$ is split-generated by only four objects. Since we have proved that $f^*\Omega^4(4) \in H[-3]$ it is obvious that $f^*\Omega^4(4)[5] \in H[2]$ which explains the Hom^{-1} and Hom^{-2} that it develops with objects in the heart.

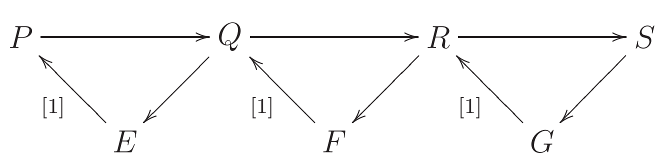
With all these preliminaries let us now concentrate on the quotient category. Note because we are quotienting by θ_x the quotient category obeys the same constraint condition as the quintic. The only thing left to prove is that there are no new Homs that violate the phase ordering.

In general it is almost impossible to explicitly compute $Hom_{A/B}(\bullet, \bullet)$ even if all the $Hom_A(\bullet, \bullet)$ are known. However because we are quotienting by the skyscraper sheaves of points we can actually explicitly compute the $Homs$ in this case. The detailed proof is in the next section, but we mention the result here and explore some consequences.

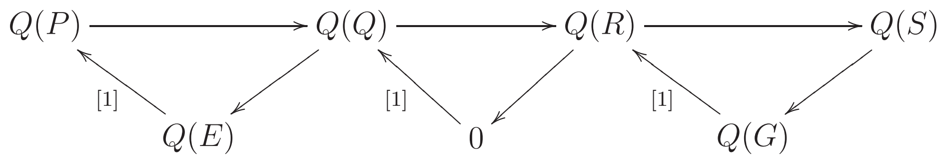
Proposition Let F and G be locally free sheaves or translates of them
If $Hom^n_X(F, G) = 0$ for $\forall n \leq 0$ then, $Hom^n_{\frac{X}{T}}(F, G) = 0$ for $\forall n \leq 0$. T here is a thick subcategory generated by complexes with support on skyscraper sheaves of points.

The objects under consideration obey the conditions of this proposition and this proves that once we pass to $D(\mathbb{P}^4)/Ker(f^*)$ then we the generating set of stable objects do not develop any new Hom^i where $i \leq 0$, that is we still have an Ext^1 or higher set of generators. The t-structure filtration required for a stability condition is inherited from $D(\mathbb{P}^4)$. The only problem that can arise is if one of the graded quotients go to zero under the quotient map $Q : D(\mathbb{P}^4) \longrightarrow D(\mathbb{P}^4)/Ker(f^*)$. We can show that this does not pose a problem as we can simply "glue" together the filtration.

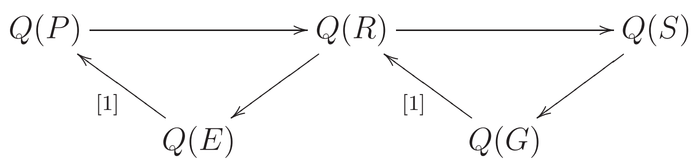
Let us assume that a part of the filtration is,



Suppose under the quotient functor Q, $Q(F) = 0$. Then in the quotient category we have,



But that immediately implies an isomorphism $Q(Q) \simeq Q(R)$ which in turn means that we can write the filtration as,



In conclusion let us point out that if consider the category $D_X(\mathbb{P}^4)$, i.e the derived category of coherent sheaves on $\mathbb{P}^4$ with support on the quintic X, then this category has a natural functor $P : D_X(\mathbb{P}^4) \longrightarrow D(\mathbb{P}^4)/Ker(f^*)$. We describe this functor in the next proposition. The proof is very close in spirit to the proof of Hom in quotient category and is relegated to the next subsection.

Proposition $\quad Hom_{\mathbb{P}^4)/Ker(f^*}(P(A), P(B)) = Hom_{D_X(\mathbb{P}^4)}(A, B)$

We would expect that $D_X(\mathbb{P}^4)$ would be equivalent to $D(\mathbb{P}^4)/Ker(f^*)$. However this is difficult to prove. The main difficulty is in constructing the inverse functor of P. One can see the existence of such a functor would imply the existence of an adjoint for the quotient functor Q which would be equivalent to requiring a semi-orthogonal decomposition of $D(\mathbb{P}^4)$ with $D_X(\mathbb{P}^4)$ as an admissible subcategory. However there is no obvious way of defining a projection functor on to $D_X(\mathbb{P}^4)$.

Proofs

Proposition Let F and G be locally free sheaves or translates of them. If $Hom^n_X(F,G) = 0$ for $\forall n \leq 0$ then, $Hom^n_{\frac{X}{T}}(F,G) = 0$ for $\forall n \leq 0$. T is a thick subcategory generated by complexes with support on skyscraper sheaves of points.

Proof $Hom(F,G)$ in $\frac{X}{T}$ is characterized by this roof diagram in X

$$\begin{array}{ccccc} & & F & & \\ & & \downarrow & & \\ G & \longrightarrow & K & \longrightarrow & T \end{array}$$

where, T lies in the quotient subcategory and is killed by the quotient.

First we consider the triangle

$$G \longrightarrow K \longrightarrow T$$

Proposition i) $\mathcal{H}^n(K) = \mathcal{H}^n(T)$, $\forall n \neq 0$

ii) $\mathcal{H}^0(K) = G \oplus \mathcal{H}^0(T)$

Proof The long exact sequence in cohomology gives us,

$$\cdots \longrightarrow \mathcal{H}^n(G) \longrightarrow \mathcal{H}^n(K) \longrightarrow \mathcal{H}^n(T) \longrightarrow \cdots$$

Since G is assumed to be a sheaf, $\mathcal{H}^n(G) = 0$, $\forall n \neq 0$, which implies that $\mathcal{H}^n(K) = \mathcal{H}^n(T)$, $\forall n \neq -1, 0$. For $n = -1, 0$ we consider the exact sequence,

$$0 \longrightarrow \mathcal{H}^{-1}(K) \longrightarrow \mathcal{H}^{-1}(T) \longrightarrow G \longrightarrow \mathcal{H}^{0}(K) \longrightarrow \mathcal{H}^{0}(T) \longrightarrow 0$$

Recall that T has cohomology sheaves that are skyscraper sheaves. We need some basic facts about Hom groups of skyscraper sheaves to prove the proposition. If θ_x is a skyscraper sheaf and G is a locally free sheaf then,

$$Hom^n(\theta_x, G) = Hom(\theta_x, G[n]) = (Hom(G[n], \theta_x \otimes \omega[d])^*,$$

where ω is the canonical bundle and d is dimension of X and the dualizing process was just the Serre dual. Since G is assumed to be locally free we have, $(Hom(G[n], \theta_x \otimes \omega[d])^* = (Hom(\theta_X, G^\vee \otimes \theta_x \otimes \omega[d-n])^*$

Therefore the only nonzero Hom is for $n = d$. Therefore the connecting homomorphism between $\mathcal{H}^{-1}(T) \longrightarrow G$ must be zero giving us,

$$0 \longrightarrow G \longrightarrow \mathcal{H}^{0}(K) \longrightarrow \mathcal{H}^{0}(T) \longrightarrow 0$$

We have also proved that $\mathcal{H}^{-1}(K) = \mathcal{H}^{-1}(T)$. Since $Hom^1(\mathcal{H}^0(T), G) = 0$ as well, the short exact sequence splits and this gives us the desired result.

Now we look at $Hom(F, K)$. We use the standard spectral sequence for filtered complex, which gives us,

$E^2_{p,q} = Hom^p(F, \mathcal{H}^q(K))$ converges to $Hom^{p+q}(F, K)$. We use the result we proved about $\mathcal{H}^r(K))$ to obtain that

i) If $q \neq 0$, then, $Hom^p(F, \mathcal{H}^q(K)) = Hom^p(F, \mathcal{H}^q(T))$

ii) If $q = 0$, then, $Hom^p(F, \mathcal{H}^0(K)) = Hom^p(F, G \oplus \mathcal{H}^0(T))$

This shows that $\forall p \leq 0$ any new morphism factors through T and dies in the quotient. This proves the proposition.

Proposition There is a functor $P : D_X(\mathbb{P}^4) \longrightarrow D(\mathbb{P}^4)/Ker(f^*)$ which is given by the composition of the inclusion functor $i : D_X(\mathbb{P}^4) \longrightarrow D(\mathbb{P}^4)$ and $Q : D(\mathbb{P}^4) \longrightarrow D(\mathbb{P}^4)/Ker(f^*)$, i.e :$P = Q \bullet i$. Under this functor the following holds,

$Hom_{D(\mathbb{P}^4)/Ker(f^*)}(P(A), P(B)) = Hom_{D_X(\mathbb{P}^4)}(A, B)$.

Proof The proof is very similar to the preceding one. Let A, B be two objects in $D_X(\mathbb{P}^4)$. Then, $i(A), i(B)$ are supported only on X. Let us work out $Hom^k(i(A), F)$ and $Hom^k(F, i(A))$ where $F \in Ker(f^*)$. Use of the spectral sequence used above gives us, $E^2_{pq} = Hom^p(i(A), \mathcal{H}^q(F)) = Hom^p(i(A), \theta_x)$. To compute an arbitrary $Hom^p(i(A), \theta_x)$ we use the spectral sequence again to obtain, $E^2_{pq} = Hom^p(\mathcal{H}^q(i(A)), \theta_x) = Hom^p(, \theta_{\mathbb{P}^4}, \mathcal{H}^q(i(A))^* \otimes \theta_x) = H^0(i(A)^*_x) = 0$,since by definition $i(A)$ has no support at x.

To compute $Hom^k(F, i(A))$ we use the exact same spectral sequences to reduce it to a computation of $Hom^p(\theta_x = \mathcal{H}(F), \mathcal{H}^q(i(A)))$. Now we use Serre-duality to turn this into, $Hom(\mathcal{H}(F) \otimes \omega^{-1}[p-4], \theta_x)^* = H^{4-p}(\mathcal{H}(F)_x \otimes \omega_x^{-1}) = 0$, because $i(A)$ has no support at x.

Let us now consider a roof diagram,

$$
\begin{array}{ccccc}
 & & i(B) & & \\
 & & \downarrow & & \\
i(A) & \longrightarrow & K & \longrightarrow & F
\end{array}
$$

where as usual $F \in Ker(f^*)$ and is killed by the quotient. By the discussion above we know, $Hom^1(F, i(A)) = 0$ which implies the triangle necessarily splits, giving us $K = i(A) \oplus F$. This implies that we have $Hom(K, i(B)) = Hom(i(A), i(B)) \oplus Hom(F, i(B))$. The last term is zero showing there are no new morphisms after quotienting. This gives the desired result.

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