NONEMBEDDING AND NONEXTENSION RESULTS IN SPECIAL HOLONOMY

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Dedicated to Nigel Hitchin with great admiration, on the occasion of his 60th birthday

Abstract. Constructions of metrics with special holonomy by methods of exterior differential systems are reviewed and the interpretations of these construction as 'flows' on hypersurface geometries are considered.

It is shown that these hypersurface 'flows' are not generally well-posed for smooth initial data and counterexamples to existence are constructed.

Contents

1. Introduction

In the early analyses of metrics with special holonomy in dimensions 7 and 8, particularly in regards to their existence and generality, heavy use was made of the Cartan-Kähler theorem, essentially because the analyses were reduced to the

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study of overdetermined PDE systems whose natures were complicated by their diffeomorphism invariance. The Cartan-Kähler theory is well-suited for the study of such systems and the local properties of their solutions. However, the Cartan-Kähler theory is not particularly well-suited for studies of global problems for two reasons: First, it is an approach to PDE that relies entirely on the local solvability of initial value problems and, second, the Cartan-Kähler theory is only applicable in the real-analytic category.

Nevertheless, when there are no other adequate methods for analyzing the local generality of such systems, the Cartan-Kähler theory is a useful tool, and it has the effect of focussing attention on the initial value problem as an interesting problem in its own right. The point of this article is to clarify some of the existence issues involved in applying the initial value problem to the problem of constructing metrics with special holonomy. In particular, the role of the assumption of real-analyticity will be discussed, and examples will be constructed to show that one cannot generally avoid such assumptions in the initial value formulations of these problems.

The general approach can be outlined as follows: As is well-known (cf. [3]), the problem of understanding the local Riemannian metrics in dimension n whose holonomy is contained in a specified connected group $H \subset \mathrm{SO}(n)$ is essentially equivalent to the problem of understanding the H-structures in dimension n whose intrinsic torsion vanishes, or, equivalently, that are parallel with respect to the Levi-Civita connection of the Riemannian metric associated to the underlying $\mathrm{SO}(n)$-structure. In this article, an n-manifold M endowed with an H-structure $B \to M$ with vanishing intrinsic torsion will be said to be an *H-manifold*.[1] In particular, an H-manifold is a manifold M endowed with an H-structure B that is flat to first order.

It frequently happens (as it does for all of the cases to be considered here) that H acts transitively on S^{n-1} with stabilizer subgroup $K \subset H$ where

$$K = \big(\{1\} \times \mathrm{SO}(n-1)\big) \cap H.$$

In this case, an oriented hypersurface $N \subset M$ in an H-manifold M inherits, in a natural way, a K-structure $B' \to N$. Typically, this K-structure will not, itself, be torsion-free (unless N is a totally geodesic hypersurface in M), but will satisfy some weaker condition on its intrinsic torsion, essentially that its intrinsic torsion can be expressed in terms of the second fundamental form of N as a submanifold of M. The problem then becomes to determine whether these weaker conditions on a given K-structure $B' \to N^{n-1}$ are sufficient to imply that (N, B') can be induced by immersion into an H-manifold (M, B).

In the three cases to be considered in this article, in which H is one of $\mathrm{SU}(2) \subset \mathrm{SO}(4)$, $\mathrm{G}_2 \subset \mathrm{SO}(7)$, or $\mathrm{Spin}(7) \subset \mathrm{SO}(8)$, it will be shown that the weaker intrinsic torsion conditions on a hypersurface structure *are* sufficient to induce an embedding (in fact, essentially a unique one) *provided that the given K-structure is real-analytic with respect to some real-analytic structure on N.* It will also be shown, in each case, that there are K-structures $B' \to N^{n-1}$ that satisfy these weaker intrinsic torsion conditions that *cannot* be induced by an immersion into an H-manifold.

[1] While, strictly speaking, an H-manifold is a pair (M, B), it is common to refer to M as an H-manifold when the torsion-free H-structure B can be inferred from context.

Of course, such structures are not real-analytic with respect to any real-analytic structure on N.

The existence of the desired embedding in the analytic case is a consequence of the Cartan-Kähler theorem and, indeed, is implicit in the original analyses of G_2- and Spin(7)-structures to be found in my 1987 paper [3]. The examples constructed below, showing that existence can fail when one does not have real-analyticity, appear to be new.

There is another interpretation of the initial value problem that has been considered by a number of authors, in particular, Hitchin [8]: The idea of a 'flow' of K-structures that gives rise to a torsion-free H-structure. Let M be a connected n-manifold endowed with an H-structure and let g be the underlying metric. Let $N \subset M$ be an embedded, normally oriented hypersurface, and let $r : M \to [0, \infty)$ be the distance (in the metric g) from the hypersurface N. As is well-known, there is an open neighborhood $U \subset M$ of N on which there is a smooth function $t : U \to \mathbb{R}$ satisfying $|t| = r$ and $\mathrm{d}t \neq 0$ on U as well as the condition that its gradient along N is the specified oriented normal. There is then a well-defined smooth embedding $(t, f) : U \to \mathbb{R} \times N$, where, for $p \in U$, the function $f(p)$ is the closest point of N to p. In this way, U can be identified with an open neighborhood of $\{0\} \times N$ in $\mathbb{R} \times N$ and, in particular, each level set of t in U can be identified with an open subset of N. (When N is compact, there will be an $\epsilon > 0$ such that the level sets $t = c$ for $|c| < \epsilon$ will be diffeomorphic to N.) Thus, at least locally, one can think of the H-structure on U as a 1-parameter family of K-structures on N. When one imposes the condition that the H-structure on $U \subset \mathbb{R} \times N$ be torsion-free, this can be expressed as a first-order initial value problem with the given K-structure on $\{0\} \times N$ as the initial value. This first-order initial value problem is sometimes described as a 'flow', but this can be misleading, especially if it causes one to think in terms of parabolic or hyperbolic PDE.

Indeed, the character of this PDE problem is more like that of trying to use the Cauchy-Riemann equations $u_x = v_y$ and $v_x = -u_y$ to extend a complex-valued function defined on the imaginary axis $x = 0$ to a holomorphic function on a neighborhood of the imaginary axis. One knows that a necessary and sufficient condition for being able to do this is that the given function on the imaginary axis must be real-analytic. In the cases to be considered in this article, the requirements are not this strong since there will be cases in which the initial K-structure is not real-analytic and yet a solution of the initial value problem will exist. However, as will be seen, real-analyticity is a sufficient condition.

For background on the use of exterior differential systems in this article, the reader might consult [5].

I have included the case of SU(2)-structures on 4-manifolds because of its historical interest (it was the first case of special holonomy to be analyzed) and because the algebra is simpler. Also, because other approaches, based on the existence of local holomorphic coordinates, have been employed in this case, there is an instructive comparison to be made between those methods and the Cartan-Kähler approach. For this reason, I go into the SU(2)-case in some detail. I hope that the reader will find this as interesting as I have.

2. Beginnings

2.1. **Holonomy.** Let (M^n, g) be a connected Riemannian n-manifold.

2.1.1. *Parallel transport.* To g, one associates its Levi-Civita connection ∇, which defines, for a piecewise-C^1 curve $\gamma : [0,1] \to M$, a parallel transport

$$P^{\nabla}_{\gamma} : T_{\gamma(0)}M \to T_{\gamma(1)}M, \tag{2.1}$$

which is a linear g-isometry between the two tangent spaces.

2.1.2. *Group structure.* In 1918, J. Schouten [11] considered the set

$$H_x = \left\{ P^{\nabla}_{\gamma} \,\middle|\, \gamma(0) = \gamma(1) = x \right\} \subseteq \mathrm{O}(T_xM) \tag{2.2}$$

and called its dimension the number of *degrees of freedom* of g.

It is easy to establish the identities

$$P^{\nabla}_{\bar\gamma} = \left(P^{\nabla}_{\gamma}\right)^{-1} \qquad \text{and} \qquad P^{\nabla}_{\gamma_2 * \gamma_1} = P^{\nabla}_{\gamma_2} \circ P^{\nabla}_{\gamma_1} \tag{2.3}$$

where $\bar\gamma$ is the reverse of γ and $\gamma_2 * \gamma_1$ is the concatenation of paths γ_1 and γ_2 satisfying $\gamma_1(1) = \gamma_2(0)$.

Consequently, $H_x \subset \mathrm{O}(T_xM)$ is a subgroup and

$$H_{\gamma(1)} = P^{\nabla}_{\gamma}\ H_{\gamma(0)}\ \left(P^{\nabla}_{\gamma}\right)^{-1}. \tag{2.4}$$

In particular, fixing a linear isometry $u : T_xM \to \mathbb{R}^n$, the conjugacy class of $H_u = uH_xu^{-1}$ in $\mathrm{O}(n)$ is well-defined, independent of the choice of $x \in M$ or the isometry $u : T_xM \to \mathbb{R}^n$. By abuse of terminology, we say that H is the *holonomy* of the metric g if $H \subset \mathrm{O}(n)$ is a group conjugate to some (and hence any) of the groups H_u.

For later reference, if $u : T_xM \to \mathbb{R}^n$ is fixed, we let

$$B_u = \left\{ u \circ P^{\nabla}_{\gamma} \,\middle|\, \gamma(1) = x \right\}. \tag{2.5}$$

This B_u is an H_u-subbundle of the orthonormal coframe bundle of g, i.e., it is an H_u-structure on M. By its very construction, it is invariant under ∇-parallel translation and, since ∇ is torsion-free, it follows that this H_u-structure admits a torsion-free compatible connection.

Conversely, let $H \subset \mathrm{O}(n)$ be a subgroup and let $B \to M$ be an H-structure on M. If $B \to M$ admits a compatible, torsion-free connection ∇, then B is said to be *torsion-free*. In this case, ∇ (necessarily unique) is the Levi-Civita connection of the underlying metric g on M and B is invariant under ∇-parallel translation, implying that $H_u \subset H$ for all $u \in B$. Thus, finding torsion-free H-structures on M provides a way to find metrics on M whose holonomy lies in H.

In this article, I will use the term *H-manifold* to denote an n-manifold M^n endowed with a torsion-free H-structure $B \to M$. (The intended embedding $H \subset \mathrm{O}(n)$ is to be understood from context.)

2.1.3. *Cartan's early results.* In 1925, É. Cartan [6] made the following statements:

(1) H_x is a Lie subgroup of $\mathrm{O}(T_xM)$, connected if M is simply-connected.
(2) If H_x acts reducibly on T_xM, then g is locally a product metric.

Cartan's first statement was eventually proved (to modern standards of rigor) by Borel and Lichnerowicz [2], and the second statement was globalized and proved by G. de Rham.

2.2. **A nontrivial case.** In dimensions 2 and 3, the above facts suffice to determine the possible holonomy groups of simply-connected manifolds.

Cartan [6][2] studied the first nontrivial case, namely, $n = 4$ and $H_x \simeq \mathrm{SU}(2)$, and observed that such metrics g

(1) have vanishing Ricci tensor,
(2) are what we now call 'self-dual', and
(3) locally (modulo diffeomorphism) depend on 2 functions of 3 variables.

While the first two observations are matters of calculation and/or definition, the third observation is nontrivial. However, Cartan gave no indication of his proof and, to my knowledge, never returned to this example again.

While I cannot be sure, I believe that it is likely that Cartan had a proof in mind along the following lines.[3]

The associated SU(2)-structure $B \to M$ of such a metric g satisfies structure equations of the form

$$\begin{aligned} \mathrm{d}\omega &= -\,\theta\wedge\omega,\\ \mathrm{d}\theta &= -\,\theta\wedge\theta + R\big(\omega\wedge\omega\big),\\ \mathrm{d}R &= -\,\theta.R + R'(\omega). \end{aligned} \tag{2.6}$$

where

(1) the tautological 1-form ω takes values in $\mathbb{R}^4$,
(2) the connection 1-form θ takes values in $\mathfrak{su}(2) \subset \mathfrak{so}(4)$,
(3) the curvature function R takes values in W_4, the 5-dimensional (real) irreducible representation of SU(2) that lies in $\mathrm{Hom}\big(\Lambda^2(\mathbb{R}^4), \mathfrak{su}(2)\big)$, and
(4) the derived curvature function R' takes values in V_5, the 6-dimensional complex irreducible representation of SU(2) that lies in $\mathrm{Hom}(\mathbb{R}^4, W_4)$.

Calculation shows that the subspace V_5 is an involutive tableau in $\mathrm{Hom}(\mathbb{R}^4, W_4)$, with character sequence $(s_1, s_2, s_3, s_4) = (5, 5, 2, 0)$. Since the last nonzero character of this tableau is $s_3 = 2$, Cartan's generalization of the third fundamental theorem of Lie applies to the structure equation (2.6) to yield his third observation above.

2.3. **The hyperKähler viewpoint.** Riemannian manifolds (M^4, g) with

$$H_x \simeq \mathrm{SU}(2) \subset \mathrm{SO}(4)$$

are nowadays said to be *hyperKähler*, and we understand them as special cases of Kähler manifolds. In fact, using our understanding of complex and Kähler geometry, we now arrive at Cartan's result by a somewhat different route:

Because the subgroup $\mathrm{SU}(2) \subset \mathrm{SO}(4)$ acts trivially on the space of self-dual 2-forms on $\mathbb{R}^4$, when (M^4, g) has $H_x \simeq \mathrm{SU}(2)$, then there exist three g-parallel self-dual 2-forms on M, say Υ_1, Υ_2, and Υ_3, such that

$$\Upsilon_i \wedge \Upsilon_j = 2\delta_{ij}\,\mathrm{d}V_g\,. \tag{2.7}$$

Conversely, a triple $\Upsilon = (\Upsilon_1, \Upsilon_2, \Upsilon_3)$ of closed 2-forms on M that satisfies

$$ {\Upsilon_1}^2 = {\Upsilon_2}^2 = {\Upsilon_3}^2 \neq 0 \qquad \text{while} \qquad \Upsilon_2\wedge\Upsilon_3 = \Upsilon_3\wedge\Upsilon_1 = \Upsilon_1\wedge\Upsilon_2 = 0 \tag{2.8}$$

[2]Especially note Chapitre VII, Section II.

[3]He had developed all of the tools necessary for this proof in his famous series of papers on pseudo-groups and the equivalence problem and, using those results, it would have been a simple observation for him.

is easily shown to be g-parallel and self-dual with respect a unique metric on M for which (2.7) holds. Moreover, the holonomy of g will then preserve the SU(2)-structure B_u.

Now, given a triple $(\Upsilon_1, \Upsilon_2, \Upsilon_3)$ of closed 2-forms on M satisfying (2.8), one can prove (using the Newlander-Nirenberg theorem) that each point of M lies in a local coordinate chart $z = (z^1, z^2) : U \to \mathbb{C}^2$ for which there exists a real-analytic function $\phi : z(U) \to \mathbb{R}$ so that

$$\Upsilon_2 + \mathrm{i}\,\Upsilon_3 = \mathrm{d}z^1 \wedge \mathrm{d}z^2 \qquad \text{and} \qquad \Upsilon_1 = \tfrac{1}{2}\mathrm{i}\,\partial\bar{\partial}\phi, \tag{2.9}$$

where ϕ satisfies the elliptic Monge-Ampère equation

$$\det\left(\frac{\partial^2\phi}{\partial z^i \partial \bar{z}^j}\right) = 1 \tag{2.10}$$

and the strict pseudo-convexity condition with respect to z given by

$$\left(\frac{\partial^2\phi}{\partial z^i \partial \bar{z}^j}\right) > 0. \tag{2.11}$$

In fact, if one fixes an open set $U \subset \mathbb{C}^2$ and chooses a smooth, strictly pseudo-convex function $\phi : U \to \mathbb{R}$ satisfying (2.10), then the formulae (2.9) define a triple $(\Upsilon_1, \Upsilon_2, \Upsilon_3)$ on U consisting of g_ϕ-parallel, self-dual 2-forms where

$$g_\phi = \frac{\partial^2\phi}{\partial z^i \partial \bar{z}^j}\,\mathrm{d}z^i \circ \mathrm{d}\bar{z}^j. \tag{2.12}$$

Thus, the holonomy of g_ϕ is (conjugate to) a subgroup of SU(2) and it is not difficult to show that, for a generic strictly pseudoconvex ϕ satisfying (2.10), the holonomy of g_ϕ is (conjugate to) the full SU(2).

Note that, because (2.10) is an analytic elliptic PDE at a strictly pseudoconvex solution ϕ, all of its strictly pseudoconvex solutions are real-analytic. In particular, a Riemannian metric g on M^4 with holonomy SU(2) is real-analytic in harmonic coordinates (as would have followed anyway from its Ricci-flatness and a result of Deturck and Kazdan [7]).

Now, one does not normally think of solving an elliptic PDE by an initial value problem, but, of course, in the analytic category, there is nothing wrong with such a procedure. Indeed, the Cauchy-Kovalewskaya theorem implies that, in this particular case, one can specify ϕ and its normal derivative along a hypersurface, say, $\mathrm{Im}(z^2) = 0$, as essentially arbitrary real-analytic functions (subject only to an open condition that guarantees the strict pseudoconvexity of the resulting solution) and thereby determine a unique strictly pseudoconvex solution of (2.10). Thus, in this sense, one sees, again, that the 'general' metric with holonomy in SU(2) modulo diffeomorphism depends on two functions of three variables, in agreement with Cartan's claim.

3. HyperKähler 4-manifolds

3.1. **An exterior differential systems proof.** One can use the characterization of SU(2) as the stabilizer of three 2-forms in dimension 4 as the basis of another analysis of the existence problem via an EDS (= 'exterior differential system').

Let M^4 be an analytic manifold and let $\boldsymbol{\Upsilon}$ be the tautological 2-form on $\Lambda^2(T^*M)$. Let

$$X^{17} \subset \left(\Lambda^2(T^*M)\right)^3 \tag{3.1}$$

be the submanifold consisting of triples $(\beta_1, \beta_2, \beta_3) \in \Lambda^2(T_x^*M)$ such that

$$(3.2) \qquad {\beta_1}^2 = {\beta_2}^2 = {\beta_3}^2 \neq 0, \quad \text{and} \quad \beta_1 \wedge \beta_2 = \beta_3 \wedge \beta_1 = \beta_2 \wedge \beta_3 = 0.$$

Let $\pi_i : X \to \Lambda^2(T^*M)$ for $1 \leq i \leq 3$ denote the projections onto the three factors. The pullbacks $\boldsymbol{\Upsilon}_i = \pi_i^*(\boldsymbol{\Upsilon})$ define an EDS on X

$$(3.3) \qquad \mathcal{I} = \{\mathrm{d}\boldsymbol{\Upsilon}_1, \mathrm{d}\boldsymbol{\Upsilon}_2, \mathrm{d}\boldsymbol{\Upsilon}_3\}.$$

The basepoint projection $\pi : X \to M$ makes X into a bundle over M whose fibers are diffeomorphic to the 13-dimensional homogeneous space $\mathrm{GL}(4, \mathbb{R})/\,\mathrm{SU}(2)$ and whose sections $\sigma : M \to X$ correspond to the SU(2)-structures on M.

An $\mathcal{I}$-integral manifold $Y^4 \subset X$ transverse to $\pi : X \to M$ then represents a choice of three closed 2-forms Υ_i on an open subset $U \subset M$ that satisfy the algebra conditions needed to define an SU(2)-structure on U.

Calculation shows that $\mathcal{I}$ is involutive. In particular, a 3-dimensional real-analytic $\mathcal{I}$-integral manifold $P \subset X$ that is transverse to the fibers of π can be 'thickened' to a 4-dimensional integral manifold that is transverse to the fibers of π. This 'thickening' will not be unique, however, because of the invariance of the ideal $\mathcal{I}$ under the obvious action induced by the diffeomorphisms of M.

3.2. **A sharper result.** Suppose that (M^4, g) has holonomy SU(2) and let Υ_i be three g-parallel 2-forms on M satisfying

$$(3.4) \qquad \Upsilon_i \wedge \Upsilon_j = 2\delta_{ij}\, \mathrm{d}V_g\,.$$

If $N^3 \subset M$ is an oriented hypersurface, with oriented normal $\mathbf{n}$, then there is a coframing η of N defined by

$$(3.5) \qquad \eta = \begin{pmatrix} \eta_1 \\ \eta_2 \\ \eta_3 \end{pmatrix} = \begin{pmatrix} \mathbf{n} \lrcorner \Upsilon_1 \\ \mathbf{n} \lrcorner \Upsilon_2 \\ \mathbf{n} \lrcorner \Upsilon_3 \end{pmatrix}$$

and it satisfies

$$(3.6) \qquad N^* \begin{pmatrix} \Upsilon_1 \\ \Upsilon_2 \\ \Upsilon_3 \end{pmatrix} = \begin{pmatrix} \eta_2 \wedge \eta_3 \\ \eta_3 \wedge \eta_1 \\ \eta_1 \wedge \eta_2 \end{pmatrix} = *_\eta \eta.$$

where $*_\eta$ is the Hodge star associated to the metric $g_\eta = {\eta_1}^2 + {\eta_2}^2 + {\eta_3}^2$ and orientation $\eta_1 \wedge \eta_2 \wedge \eta_3 > 0$.

In particular, note that the coframing η is not arbitrary, but satisfies the system of three first-order PDE

$$(3.7) \qquad \mathrm{d}(*_\eta \eta) = N^* \mathrm{d}\Upsilon = 0.$$

An alternative expression of the involutivity of the system $\mathcal{I}$ that is better adapted to the initial value problem then becomes the following existence and uniqueness result:

Theorem 1. *Let η be a real-analytic coframing of N such that $\mathrm{d}(*_\eta \eta) = 0$. There exists an essentially unique embedding of N into a $\mathrm{SU}(2)$-holonomy manifold (M^4, g) that induces the given coframing η in the above manner.*

Remark 1 (Essential Uniqueness). The meaning of this term is as follows: If N can be embedded into two different SU(2)-holonomy manifolds (M_1, g_1) and (M_2, g_2) in such a way that both embeddings induce the same coframing η on N by the above pullback formula, then there are open neighborhoods $U_i \subset M_i$ of the images of N

and a diffeomorphism $f : U_1 \to U_2$ that is the identity on the image of N that pulls the metric g_2 back to the metric g_1.

Proof. Write $\mathrm{d}\eta = -\theta{\wedge}\eta$ where $\theta = -{}^t\theta$. (This θ exists and is unique by the Fundamental Lemma of Riemannian geometry.)

On $N \times \mathrm{GL}(3,\mathbb{R})$ define[4]

$$\boldsymbol{\omega} = g^{-1}\,\eta \qquad \text{and} \qquad \gamma = g^{-1}\mathrm{d}g + g^{-1}\theta g, \tag{3.8}$$

so that $\mathrm{d}\boldsymbol{\omega} = -\gamma{\wedge}\boldsymbol{\omega}$. On $X = \mathbb{R} \times N \times \mathrm{GL}(3,\mathbb{R})$ define the three 2-forms

$$\begin{pmatrix}\boldsymbol{\Upsilon}_1\\ \boldsymbol{\Upsilon}_2\\ \boldsymbol{\Upsilon}_3\end{pmatrix} = \begin{pmatrix}\mathrm{d}t{\wedge}\boldsymbol{\omega}_1 + \boldsymbol{\omega}_2{\wedge}\boldsymbol{\omega}_3\\ \mathrm{d}t{\wedge}\boldsymbol{\omega}_2 + \boldsymbol{\omega}_3{\wedge}\boldsymbol{\omega}_1\\ \mathrm{d}t{\wedge}\boldsymbol{\omega}_3 + \boldsymbol{\omega}_1{\wedge}\boldsymbol{\omega}_2\end{pmatrix} = \mathrm{d}t \wedge \boldsymbol{\omega} + *_\omega \boldsymbol{\omega}. \tag{3.9}$$

Let $\mathcal{I}$ be the ideal on X generated by $\{\mathrm{d}\boldsymbol{\Upsilon}_1, \mathrm{d}\boldsymbol{\Upsilon}_2, \mathrm{d}\boldsymbol{\Upsilon}_3\}$. One calculates

$$\mathrm{d}\boldsymbol{\Upsilon} = \bigl({}^t\gamma - (\mathrm{tr}\,\gamma)I_3\bigr) \wedge *_\omega\boldsymbol{\omega} + \gamma \wedge \boldsymbol{\omega} \wedge \mathrm{d}t. \tag{3.10}$$

Consequently, $\mathcal{I}$ is involutive, with characters $(s_1, s_2, s_3, s_4) = (0, 3, 6, 0)$.

Since $\mathrm{d}(*_\eta\eta) = 0$, the locus $L = \{0\} \times N \times \{I_3\} \subset X$ is a regular, real-analytic integral manifold of the real-analytic ideal $\mathcal{I}$. (Note that L is just a copy of N.)

By the Cartan-Kähler Theorem, L lies in a unique 4-dimensional $\mathcal{I}$-integral manifold $M \subset X$. The $\boldsymbol{\Upsilon}_i$ thus pull back to M to be closed and to define the desired SU(2)-structure forms Υ_i on M inducing η on $L = N$. □

It is natural to ask whether it is necessary to assume that η be real-analytic for the conclusion of Theorem 1. The following result shows that one cannot weaken this assumption to 'smooth' and still get the same conclusion:

Theorem 2. *If η is a coframing on N^3 that is not real-analytic in any local coordinate system and*

$$\mathrm{d}(*_\eta\eta) = 0 \qquad \textit{and} \qquad *_\eta({}^t\eta \wedge \mathrm{d}\eta) = 2C \tag{3.11}$$

for some constant C, then η cannot be induced by immersing N into an SU(2)*-manifold.*

Smooth-but-not-real-analytic coframings η satisfying (3.11) *do exist locally.*

Proof. Suppose that Υ_i $(1 \le i \le 3)$ are the parallel 2-forms on an (M^4, g) with holonomy SU(2) and let $N^3 \subset M$ be an oriented hypersurface.

Calculation yields that the induced co-closed coframing η on N satisfies

$$*_\eta({}^t\eta \wedge \mathrm{d}\eta) = 2H \tag{3.12}$$

where H is the mean curvature of N in M.

Now, since g is Ricci-flat, it is real-analytic in g-harmonic coordinates. In particular, such coordinate systems can be used to define a real-analytic structure on M, which is the one that we will mean henceforth. In particular, since the forms Υ_i are g-parallel, they, too, are real-analytic with respect to this structure. If H is constant, then elliptic regularity implies that N must be a real-analytic hypersurface in M and hence η must also be real-analytic.

[4]In the following formulae, I regard $g : N \times \mathrm{GL}(3,\mathbb{R}) \to \mathrm{GL}(3,\mathbb{R})$ as the projection onto the second factor. Also, to save writing, I will write η for $\pi_1^*\eta$ where $\pi : N \times \mathrm{GL}(3,\mathbb{R}) \to N$ is the projection onto the first factor.

Thus, if η is a non-real-analytic coframing on N^3 that satisfies (3.11) for some constant C, then η cannot be induced on N by an embedding into an SU(2)-manifold.

To finish the proof, I will now show how to construct a coframing η on an open subset of $\mathbb{R}^3$ that is not real-analytic in any coordinate system and yet satisfies (3.11).

To begin, note that, if a coframing η on N^3 is real-analytic in any coordinate system at all, it will be real-analytic in η-*harmonic* coordinates, i.e., local coordinates $x : U \to \mathbb{R}^3$ satisfying

$$\mathrm{d}\big(*_\eta \mathrm{d}x\big) = 0. \tag{3.13}$$

Now, fix a constant C and consider a coframing $\eta = g(x)^{-1}\,\mathrm{d}x$ on $U \subset \mathbb{R}^3$ where $g : U \to \mathrm{GL}(3,\mathbb{R})$ is a mapping satisfying the first-order, quasi-linear system

$$\mathrm{d}(*_\eta \eta) = 0, \qquad *_\eta({}^t\eta \wedge \mathrm{d}\eta) = 2C, \qquad \mathrm{d}\big(*_\eta \mathrm{d}x\big) = 0. \tag{3.14}$$

The system (3.14) consists of 7 equations for the 9 unknown entries of g.

Calculation shows this first-order system to be underdetermined elliptic (i.e., its symbol is surjective at every real covector ξ). By standard theory, it has smooth local solutions that are not real-analytic.

Taking a non-real-analytic solution g, the resulting η will not be real-analytic in the x-coordinates, which, by construction, are η-harmonic. Thus, such an η is not real-analytic in any local coordinate system. □

Remark 2 (The 'flow' interpretation). The condition $\mathrm{d}(\mathrm{d}t{\wedge}\omega + *\omega) = 0$ has sometimes been described as an 'SU(2)-flow' on coframings of N. In fact, this closure condition can be written in the 'evolutionary' form

$$\frac{\mathrm{d}}{\mathrm{d}t}\,\omega \;= *_\omega(\mathrm{d}\omega) - \tfrac{1}{2} *_\omega({}^t\omega \wedge \mathrm{d}\omega)\;\omega. \tag{3.15}$$

By Theorem 1, if η on N^3 is real-analytic and satisfies $\mathrm{d}(*_\eta\eta) = 0$, then (3.15) has a solution in a neighborhood of $t = 0$ in $\mathbb{R} \times N$ that satisfies the initial condition

$$\omega_{|t=0} = \eta. \tag{3.16}$$

One does not normally think of evolution equations as having to have real-analytic initial data. However, Theorem 2 shows that some such regularity assumption must be made.

Remark 3 (Coclosed coframings with specified metric). Given a Riemannian 3-manifold (N, g), it is an interesting question as to when there exist (either locally or globally) a g-orthonormal coframing $\eta = (\eta_1, \eta_2, \eta_3)$ that is coclosed.[5]

One can formulate this as an EDS for the section $\eta : N \to B$ of the g-orthonormal coframe bundle $B \to N$. The natural EDS for this is not involutive, but a slight extension is and is worth describing here.

Let ω_i and $\omega_{ij} = -\omega_{ji}$ (where $1 \le i, j \le 3$) be the tautological and Levi-Civita connection forms on B. In particular, g pulls back to B to be ${\omega_1}^2{+}{\omega_2}^2{+}{\omega_3}^2$ and these forms satisfy the structure equations

$$\mathrm{d}\omega_i = -\omega_{ij} \wedge \omega_j \qquad \mathrm{d}\omega_{ij} = -\omega_{ik} \wedge \omega_{kj} + \tfrac{1}{2} R_{ijkl}\,\omega_k \wedge \omega_l\,. \tag{3.17}$$

Because of their tautological reproducing property, one has $\eta^*(\omega_i) = \eta_i$. Consequently, the g-orthonormal coframing η is coclosed if and only if, when regarded as

[5] Of course, *closed* g-orthonormal coframings exist locally if and only if g is flat.

a section $\eta : N \to B$, it is an integral manifold of the EDS $\mathcal{I}_0$ generated by the three closed 3-forms $\Upsilon_{ij} = \mathrm{d}(\omega_i \wedge \omega_j)$ with $1 \le i < j \le 3$. Computation shows that this ideal has $(s_1, s_2, s_3) = (0, 2, 1)$, while an integral manifold on which $\Omega = \omega_1 \wedge \omega_2 \wedge \omega_3$ is non-vanishing (an obvious requirement for a section $\eta(M) \subset B$) must satisfy

$$\eta^*\omega_{ij} = \epsilon_{ijk} S_{kl}\, \eta^*\omega_l\,, \tag{3.18}$$

where ϵ_{ijk} is fully antisymmetric in its indices, with $\epsilon_{123} = 1$, and $S_{kl} = S_{lk}$. In particular, the space of admissible integral elements of $(\mathcal{I}_0, \Omega)$ at each point of B has dimension $6 < s_1{+}2s_2{+}3s_3$. Hence, the system $(\mathcal{I}_0, \Omega)$ is not involutive.

However, the equations (3.18) show that any integral of $(\mathcal{I}_0, \Omega)$ is also an integral of the 2-form

$$\Upsilon = \omega_1 \wedge \omega_{23} + \omega_2 \wedge \omega_{31} + \omega_3 \wedge \omega_{12} = \tfrac{1}{2}\epsilon_{ijk}\, \omega_i \wedge \omega_{jk}\,. \tag{3.19}$$

Moreover, since

$$\mathrm{d}(\omega_i \wedge \omega_j) = \epsilon_{ijk}\, \omega_k \wedge \Upsilon, \tag{3.20}$$

the differential ideal $\mathcal{I}$ generated by Υ contains $\mathcal{I}_0$. Calculation using (3.17) yields

$$2\,\mathrm{d}\Upsilon = \epsilon_{ijk}\, \omega_i \wedge \omega_{lj} \wedge \omega_{lk} - R\,\Omega \tag{3.21}$$

where R is the scalar curvature of the metric g. Thus, the ideal $\mathcal{I}$ is generated algebraically by Υ and $\mathrm{d}\Upsilon$. An integral element of $(\mathcal{I}, \Omega)$ is now cut out by equations of the form

$$\omega_{ij} - \epsilon_{ijk}\, S_{kl}\, \omega_l = 0 \tag{3.22}$$

where $S = (S_{kl})$ is a 3-by-3 symmetric matrix that satisfies $\sigma_2(S) = \frac{1}{2}R$ (where $\sigma_2(S)$ is the second elementary function of the eigenvalues of S). The characteristic variety of such an integral element consists of the null (co-)vectors of the quadratic form

$$Q_S = \mathrm{tr}(S)\, g - S_{ij}\, \omega_i\, \omega_j\,. \tag{3.23}$$

In particular, except for the case $S = 0$ (which can only occur when $R = 0$), this integral element is Cartan-regular, with Cartan character sequence $(s_1, s_2, s_3) = (1, 2, 0)$. In particular, the system $(\mathcal{I}, \Omega)$ is involutive, and, in the real-analytic case, the general integral depends on two functions of 2 variables.

Note, by the way, that when Q_S is positive (or negative) definite, the linearization around such a solution is elliptic and hence such coclosed coframings are as regular as the metric g. In particular, this always happens when $R > 0$, i.e., when the scalar curvature is positive.[6] Thus, for a real-analytic metric with positive scalar curvature, all of its coclosed coframings are real-analytic.

4. G_2-manifolds

For background on the group $\mathrm{G}_2 \subset \mathrm{SO}(7)$ and G_2-manifolds, the reader can consult [3, 4, 9, 10]. I will generally follow the notation in [4].

The crucial point is that the group G_2 can be defined as the stabilizer in $\mathrm{GL}(7, \mathbb{R})$ of the 3-form

$$\phi = e^{123} + e^{145} + e^{167} + e^{246} - e^{257} - e^{347} - e^{356}. \tag{4.1}$$

[6]It is interesting to note that, when $R > 0$, the ideal $\mathcal{I}$ on the 6-manifold B is algebraically equivalent at each point to the special Lagrangian ideal on $\mathbb{C}^3$.

where $e^{ijk} = e^i \wedge e^j \wedge e^k$ and the e^i are a basis of linear forms on $\mathbb{R}^7$. The Lie group G_2 is connected, has dimension 14, preserves the metric and orientation for which the e^i are an oriented orthonormal basis, and acts transitively on the unit sphere $S^6 \subset \mathbb{R}^7$. The G_2-stabilizer of e^1 is the subgroup $SU(3) \subset SO(6)$ that preserves the 2-form $e^{23} + e^{45} + e^{67}$ and the 3-form

$$e^{246} - e^{257} - e^{347} - e^{356} = \mathrm{Re}\big((e^2 + \mathrm{i}\, e^3) \wedge (e^4 + \mathrm{i}\, e^5) \wedge (e^6 + \mathrm{i}\, e^7)\big). \tag{4.2}$$

The $GL(7,\mathbb{R})$-orbit of ϕ in $\Lambda^3(\mathbb{R}^7)$ is an open (but not convex) cone $\Lambda^3_+(\mathbb{R}^7) \subset \Lambda^3(\mathbb{R}^7)$ that consists precisely of the 3-forms on $\mathbb{R}^7$ whose stabilizers are isomorphic to G_2. Consequently, for any smooth 7-manifold M^7, there is a well-defined open subset $\Lambda^3_+(T^*M) \subset \Lambda^3_+(T^*M)$ consisting of the 3-forms whose stabilizers are isomorphic to G_2.

A G_2-structure on M is thus specified by a 3-form σ on M with the property that σ_x lies in $\Lambda^3_+(T^*_xM)$ for all $x \in M$. Such a 3-form σ will be said to be *definite*. Explicitly, the G_2-structure $B = B_\sigma$ consists of the linear isomorphisms $u : T_x \to \mathbb{R}^7$ that satisfy $u^*\phi = \sigma_x$. Conversely, given a G_2-structure $B \to M$, there is a unique definite 3-form σ on M such that $B = B_\sigma$. Put another way, the G_2-structure bundle $\mathcal{F}(M)/G_2$ is naturally identified with $\Lambda^3_+(M)$ by identifying $[u] = u \cdot G_2$ with $u^*\phi \in \Lambda^3_+(T^*_{\pi(u)}M)$ for all $u \in \mathcal{F}(M)$.

A $\sigma \in \Omega^3_+(M)$ determines a unique metric g_σ and orientation $*_\sigma$ by requiring that the corresponding coframings $u \in B_\sigma$ be oriented isometries.

It is a fact [3] that B_σ is torsion-free if and only if σ is g_σ-parallel, which, in turn, holds if and only if

$$\mathrm{d}\sigma = 0 \qquad \text{and} \qquad \mathrm{d}(*_\sigma\sigma) = 0. \tag{4.3}$$

Thus, a G_2-manifold can be regarded as a pair (M^7, σ) where $\sigma \in \Omega^3_+(M)$ satisfies the nonlinear system of PDE (4.3).

By a theorem of Bonan (see [1, Chapter X]), for any G_2-manifold (M, σ), the associated metric g_σ has vanishing Ricci tensor. In particular, by a result of DeTurck and Kazdan [7], the metric g_σ is real-analytic in g_σ-harmonic coordinates. Since σ is g_σ-parallel, it, too, must be real-analytic in g_σ-harmonic coordinates.

There is a natural differential ideal on $\Lambda^3_+(T^*M) = \mathcal{F}(M)/G_2$ defined as follows: For $[u] \in \mathcal{F}(M)/G_2$, define

$$\boldsymbol{\sigma}_{[u]} = \pi^*(u^*\phi) \qquad \text{and} \qquad \boldsymbol{\tau}_{[u]} = \pi^*\big(u^*(*_\phi\phi)\big), \tag{4.4}$$

where $\pi : \mathcal{F}(M)/G_2 \to M$ is the natural basepoint projection. Let $\mathcal{I}$ be the differential ideal on $\mathcal{F}(M)/G_2 = \Lambda^3_+(T^*M)$ generated by $\mathrm{d}\boldsymbol{\sigma}$ and $\mathrm{d}\boldsymbol{\tau}$. The following result is proved in [3]:

Theorem 3. *The ideal $\mathcal{I}$ on $\Lambda^3_+(T^*M)$ is involutive. A section $\sigma \in \Omega^3_+(M)$ is an integral of $\mathcal{I}$ and only if it is g_σ-parallel. Modulo diffeomorphisms, the general $\mathcal{I}$-integral σ depends on 6 functions of 6 variables.*

4.1. **Hypersurfaces.** The group G_2 acts transitively on $S^6 \subset \mathbb{R}^7$, with stabilizer $SU(3)$. Hence, an oriented $N^6 \subset M$ inherits a canonical $SU(3)$-structure, which is determined by the $(1,1)$-form ω and $(3,0)$-form $\Omega = \phi + \mathrm{i}\,\psi$ defined by

$$\omega = \mathbf{n} \lrcorner \sigma \qquad \text{and} \qquad \Omega = \phi + \mathrm{i}\,\psi = N^*\sigma - \mathrm{i}\,(\mathbf{n} \lrcorner *_\sigma\sigma). \tag{4.5}$$

In fact, if one defines $f : \mathbb{R} \times N \to M$ by

$$f(t,p) = \exp_p\big(t\,\mathbf{n}(p)\big), \tag{4.6}$$

then

$$f^*\sigma = \mathrm{d}t \wedge \omega + \mathrm{Re}(\Omega) \qquad \text{and} \qquad f^*(*_\sigma\sigma) = \tfrac{1}{2}\,\omega^2 - \mathrm{d}t \wedge \mathrm{Im}(\Omega). \tag{4.7}$$

where, now, ω and Ω are forms on N that depend on t.

For each fixed $t = t_0$, the induced $\mathrm{SU}(3)$-structure on N satisfies

$$\mathrm{d}\,\mathrm{Re}(\Omega) = \mathrm{d}(f_{t_0}^*\sigma) = 0 \qquad \text{and} \qquad \mathrm{d}(\tfrac{1}{2}\,\omega^2) = \mathrm{d}\big(f_{t_0}^*(*_\sigma\sigma)\big) = 0, \tag{4.8}$$

so these are necessary conditions on the $\mathrm{SU}(3)$-structure on N that it be induced by immersion into a G_2-holonomy manifold M.

Theorem 4. *A real-analytic $\mathrm{SU}(3)$-structure on N^6 is induced by embedding into a G_2-manifold if and only if its defining forms ω and Ω satisfy*

$$\mathrm{d}\,\mathrm{Re}(\Omega) = 0 \qquad \textit{and} \qquad \mathrm{d}(\tfrac{1}{2}\,\omega^2) = 0. \tag{4.9}$$

Proof. The necessity of (4.9) has already been demonstrated, so I will just prove the sufficiency.

Define a tautological 2-form $\boldsymbol{\omega}$ and 3-form $\boldsymbol{\Omega}$ on $\mathcal{F}(N)/\,\mathrm{SU}(3)$ as follows: For a coframe $u : T_xN \to \mathbb{C}^3$, define these forms at $[u] = u \cdot \mathrm{SU}(3) \in \mathcal{F}(N)/\,\mathrm{SU}(3)$ by

$$\boldsymbol{\omega}_{[u]} = \pi^*\big(u^*(\tfrac{\mathrm{i}}{2}({}^t\mathrm{d}z \wedge \mathrm{d}\bar{z}))\big) \qquad \text{and} \qquad \boldsymbol{\Omega}_{[u]} = \pi^*\big(u^*(\mathrm{d}z^1 \wedge \mathrm{d}z^2 \wedge \mathrm{d}z^3)\big) \tag{4.10}$$

where $\pi : \mathcal{F}(N)/\,\mathrm{SU}(3) \to N$ is the basepoint projection.

On $X = \mathbb{R} \times \mathcal{F}(N)/\,\mathrm{SU}(3)$, consider the 3-form and 4-form defined by

$$\begin{aligned} \boldsymbol{\sigma} &= \mathrm{d}t \wedge \boldsymbol{\omega} + \mathrm{Re}(\boldsymbol{\Omega}) \\ \boldsymbol{\tau} &= \tfrac{1}{2}\,\boldsymbol{\omega}^2 - \mathrm{d}t \wedge \mathrm{Im}(\boldsymbol{\Omega}). \end{aligned} \tag{4.11}$$

Let $\mathcal{I}$ be the EDS generated by the closed 4-form $\mathrm{d}\boldsymbol{\sigma}$ and 5-form $\mathrm{d}\boldsymbol{\tau}$. Then the calculation used to prove Theorem 3 (see [3]) shows that $\mathcal{I}$ is involutive, with characters

$$(s_1, \ldots, s_7) = (0, 0, 1, 4, 10, 13, 0). \tag{4.12}$$

Since $\mathrm{d}\big(\mathrm{Re}(\Omega)\big) = \mathrm{d}(\tfrac{1}{2}\,\omega^2) = 0$, the given $\mathrm{SU}(3)$-structure on N defines a regular integral manifold $L \subset X$ of $\mathcal{I}$ lying in the hypersurface $t = 0$.

Since the given $\mathrm{SU}(3)$-structure is assumed to be real-analytic, the system $\mathcal{I}$ and the integral manifold $L \subset X$ are real-analytic by constuction. Hence the Cartan-Kähler Theorem can be applied to conclude that L lies in a unique $\mathcal{I}$-integral $M^7 \subset X$. The pullback of $\boldsymbol{\sigma}$ to M is then a closed definite 3-form σ on M while the pullback of $\boldsymbol{\tau}$ to M is closed and equal to $*_\sigma\sigma$. Thus, (M, σ) is a G_2-manifold. By construction, N is imbedded into M as the locus $t = 0$ and the $\mathrm{SU}(3)$-structure on N induced by σ is the given one. □

Theorem 5. *There exist non-real-analytic $\mathrm{SU}(3)$-structures on N^6 whose associated forms (ω, Ω) satisfy (4.9) but that are not induced from an immersion into a G_2-manifold (M, σ).*

In fact, if a non-analytic $\mathrm{SU}(3)$-structure satisfies (4.9) and

$$*\,\big(\omega \wedge \mathrm{d}\big(\mathrm{Im}(\Omega)\big)\big) = C \tag{4.13}$$

for some constant C, then it cannot be G_2-immersed.

Non-analytic $\mathrm{SU}(3)$-structures satisfying (4.9) and (4.13) do exist.

Proof. When an SU(3)-structure on N^6 with defining forms (ω, Ω) is induced via a G_2-immersion $N^6 \hookrightarrow M^7$, the mean curvature H of N in M is given by

$$-12H = *\big(\omega \wedge \mathrm{d}\big(\mathrm{Im}(\Omega)\big)\big). \tag{4.14}$$

Thus, when the right hand side of this equation is constant, it follows by elliptic regularity that N^6 is a real-analytic submanifold of the real-analytic (M^7, σ).

Thus, if the given SU(3)-structure on N satisfying (4.9) and (4.13) is not real-analytic, it cannot be induced by an embedding into a G_2-holonomy manifold M.

It remains to construct a non-analytic example satisfying (4.9) and (4.13). Here is why it is somewhat delicate: Since $\dim\big(\mathrm{GL}(6,\mathbb{R})/\,\mathrm{SU}(3)\big) = 28$, a choice of an SU(3)-structure (ω, Ω) on N^6 depends on 28 functions of 6 variables. Modulo diffeomorphisms, this leaves 22 functions of 6 variables. On the other hand, the equations

$$\mathrm{d}\big(\mathrm{Re}(\Omega)\big) = 0, \qquad \mathrm{d}\big(\tfrac{1}{2}\,\omega^2\big) = 0, \qquad *\big(\omega \wedge \mathrm{d}\big(\mathrm{Im}(\Omega)\big)\big) = C \tag{4.15}$$

constitute $15 + 6 + 1 = 22$ equations for the SU(3)-structure.

Thus, the equations to be solved are 'more determined' than in the analogous SU(2) case. Nevertheless, their diffeomorphism invariance still allows one to construct the desired example, as will now be shown.

Say that a 3-form $\phi \in \Omega^3(N^6)$ is *elliptic* if, at each point, it is linearly equivalent to $\mathrm{Re}\big(\mathrm{d}z^1 \wedge \mathrm{d}z^2 \wedge \mathrm{d}z^3\big)$. This is a open pointwise condition on ϕ (i.e., it is *stable* in Hitchin's sense [8]): The elliptic 3-forms are sections of an open subbundle $\Lambda^3_e(T^*N) \subset \Lambda^3(T^*N)$. I will denote the set of elliptic 3-forms on N by $\Omega^3_e(N)$.

Fix an orientation of N^6. An elliptic $\phi \in \Omega^3_e(N)$ then defines a unique, orientation-preserving almost-complex structure J_ϕ on N^6 such that

$$\Omega_\phi = \phi + \mathrm{i}\, J^*_\phi(\phi) \tag{4.16}$$

is of J_ϕ-type $(3,0)$.

Now assume that $\phi \in \Omega^3_e(N)$ is closed. Then $\mathrm{d}\Omega_\phi$ is purely imaginary and yet must be a sum of terms of J_ϕ-type $(3,1)$ and $(2,2)$. Thus, $\mathrm{d}\Omega_\phi$ is purely of J_ϕ-type $(2,2)$.

Let $\Lambda^{1,1}_+(N, J_\phi)$ denote the set of real 2-forms that are of J_ϕ-type $(1,1)$ and that are positive on all J_ϕ-complex lines. The squaring map

$$\sigma : \Lambda^{1,1}_+(N, J_\phi) \to \Lambda^{2,2}(N, J_\phi)$$

given by $\sigma(\omega) = \omega^2$ is a diffeomorphism onto the open set $\Lambda^{2,2}_+(N, J_\phi) \subset \Lambda^{2,2}(N, J_\phi)$ that consists of the real 4-forms of J_ϕ-type $(2,2)$ that are positive on all J_ϕ-complex 2-planes.

Now fix a constant $C \neq 0$. One sees from the above discussion that it is a C^1-open condition on ϕ that

$$\mathrm{d}\Omega_\phi = \tfrac{\mathrm{i}}{6} C\,(\omega_\phi)^2 \qquad \text{for some} \qquad \omega_\phi = \overline{\omega_\phi} \in \Omega^{1,1}_+(N, J_\phi). \tag{4.17}$$

Now, the pair $(\omega_\phi, \Omega_\phi)$ are the defining forms of an SU(3)-structure on N if and only if

$$\tfrac{1}{6}(\omega_\phi)^3 - \tfrac{1}{8}\mathrm{i}\,\Omega_\phi \wedge \overline{\Omega_\phi} = 0. \tag{4.18}$$

This is a single, first-order scalar equation on the closed 3-form ϕ. It is easy to see that there are non-analytic solutions. (For example, if one starts with the standard structure induced on the 6-sphere in flat $\mathbb{R}^7$ endowed with its flat G_2-structure, then small perturbations of the corresponding closed ϕ can be made that solve (4.18) but for which the induced SU(3)-structure has constant curvature on a proper open subset of S^6. Such an SU(3)-structure clearly cannot be real-analytic everywhere.)

Assuming (4.18) is satisfied, we have

$$\mathrm{d}(\mathrm{Re}\,\Omega_\phi) = \mathrm{d}\phi = 0, \tag{4.19}$$

and

$$\mathrm{d}\big(\tfrac{1}{2}(\omega_\phi)^2\big) = \mathrm{d}\big(-3\mathrm{i}\,\tfrac{1}{C}\,\mathrm{d}\Omega_\phi\big) = 0, \tag{4.20}$$

and, finally

$$*_\phi\big(\omega_\phi\wedge\mathrm{d}(\mathrm{Im}\,\Omega_\phi)\big) = *_\phi\big(\omega_\phi\wedge\tfrac{1}{6}C\,(\omega_\phi)^2\big) = C. \tag{4.21}$$

□

4.2. **The flow interpretation.** On $N^6\times\mathbb{R}$, with (ω,Ω) defining an SU(3)-structure on N^6 depending on $t\in\mathbb{R}$, consider the equations

$$\mathrm{d}\big(\mathrm{d}t\wedge\omega + \mathrm{Re}(\Omega)\big) = 0 \qquad \text{and} \qquad \mathrm{d}\big(\tfrac{1}{2}\,\omega^2 - \mathrm{d}t\wedge\,\mathrm{Im}(\Omega)\big) = 0, \tag{4.22}$$

which assert that $\sigma = \mathrm{d}t\wedge\omega + \mathrm{Re}(\Omega)$, which is a definite 3-form on $M = N^6\times\mathbb{R}$, is both closed and co-closed (since $*_\sigma\sigma = \frac{1}{2}\,\omega^2 - \mathrm{d}t\wedge\,\mathrm{Im}(\Omega)$).

Think of Ω as $\phi + \mathrm{i}J^*_\phi(\phi)$, so that the SU(3)-structure is determined by (ω,ϕ) where $\phi = \mathrm{Re}(\Omega)$. The conditions (4.22) for fixed t are then

$$\mathrm{d}\phi = 0 \qquad \text{and} \qquad \mathrm{d}(\omega^2) = 0, \tag{4.23}$$

while the G_2-evolution equations implied by (4.22) for such (ω,ϕ) are then

$$\frac{\mathrm{d}}{\mathrm{d}t}(\phi) = \mathrm{d}\omega \qquad \text{and} \qquad \frac{\mathrm{d}}{\mathrm{d}t}(\omega) = -{L_\omega}^{-1}\left(\mathrm{d}\big(J^*_\phi(\phi)\big)\right), \tag{4.24}$$

where $L_\omega : \Omega^2(N)\to\Omega^4(N)$ is the invertible map $L_\omega(\beta) = \omega\wedge\beta$.

Theorems 4 and 5 show that solutions to the 'G_2-flow' (4.24) do exist for analytic initial SU(3)-structures satisfying the closure conditions (4.23), but need not exist for non-analytic initial SU(3)-structures satisfying these closure conditions.

5. Spin(7)-manifolds

For background on the group Spin(7) ⊂ SO(8) and Spin(7)-manifolds, the reader can consult [3, 9, 10]. I will generally follow the notation in [3].

The main point is that the group Spin(7) ⊂ SO(8) is the $\mathrm{GL}(8,\mathbb{R})$-stabilizer of the 4-form $\Phi_0\in\Lambda^4(\mathbb{R}^8)$, defined by

$$\Phi_0 = e^0\wedge\phi + *_\phi\phi \tag{5.1}$$

where ϕ is defined by (4.1) and $e^0,\dots,e^7$ is a basis of linear forms on $\mathbb{R}^8 = \mathbb{R}\oplus\mathbb{R}^7$. The group Spin(7) is a connected Lie group of dimension 21 that is the double cover of SO(7). It acts transitively on the unit sphere in $\mathbb{R}^8$ and the Spin(7)-stabilizer of e^0 is G_2. The $\mathrm{GL}(8,\mathbb{R})$-orbit of Φ_0 in $\Lambda^4(\mathbb{R}^8)$ will be denoted by $\Lambda^4_s(\mathbb{R}^8)$. This orbit has dimension 43, so it is not open in $\Lambda^4(\mathbb{R}^8)$.

A Spin(7)-structure on M^8 is thus specified by a 4-form $\Phi\in\Omega^4(M)$ that is linearly equivalent to Φ_0 at each point of M. The set of such 4-forms will be denoted $\Omega^4_s(M)$. They are the sections of the bundle $\mathcal{F}(M)/\,\mathrm{Spin}(7)\to M$, which has a natural embedding into $\Lambda^4(T^*M)$.

Given a 4-form $\Phi \in \Omega^4_s(M)$, the corresponding Spin(7)-structure $B = B_\Phi$ is defined to be the set of coframings $u : T_xM \to \mathbb{R}^8$ that satisfy $u^*\Phi_0 = \Phi_x$. Conversely, every Spin(7)-structure $B \to M$ is of the form $B = B_\Phi$ for a unique 4-form $\Phi \in \Omega^4_s(M)$.

In particular, each $\Phi \in \Omega^4_s(M)$ determines a metric g_Φ and orientation $*_\Phi$ on M by requiring that the elements $u \in B_\Phi$ be oriented isomorphisms.

It is a fact [3] that B_Φ is torsion-free if and only if Φ is g_Φ-parallel, which, in turn, holds if and only if

$$\mathrm{d}\Phi = 0. \tag{5.2}$$

Thus, a Spin(7)-manifold can be regarded as a pair (M^8, Φ) where $\Phi \in \Omega^4_s(M)$ satisfies the nonlinear system of PDE (5.2).

By a theorem of Bonan (see [1, Chapter X]), for any Spin(7)-manifold (M, Φ), the associated metric g_Φ has vanishing Ricci tensor. In particular, by a result of DeTurck and Kazdan [7], the metric g_Φ is real-analytic in g_Φ-harmonic coordinates. Since Φ is g_Φ-parallel, it, too, must be real-analytic in g_Φ-harmonic coordinates.

Define a 4-form $\boldsymbol{\Phi}$ on $\mathcal{F}(M)/\operatorname{Spin}(7)$ by the following rule: For $u : T_xM \to \mathbb{R}^8$ and $[u] = u \cdot \operatorname{Spin}(7)$, set

$$\boldsymbol{\Phi}_{[u]} = \pi^*\big(u^*\Phi_0\big) \tag{5.3}$$

where $\pi : \mathcal{F}(M) \to M$ is the basepoint projection. Let $\mathcal{I}$ be the ideal generated by $\mathrm{d}\boldsymbol{\Phi}$ on $\mathcal{F}(M)/\operatorname{Spin}(7)$. The following result is proved in [3]:

Theorem 6. *A section $\Phi \in \Omega^4_s(M)$ satisfies* (5.2) *if and only if it is an integral of $\mathcal{I}$. The ideal $\mathcal{I}$ is involutive. Modulo diffeomorphisms, the general $\mathcal{I}$-integral Φ depends on* 12 *functions of* 7 *variables.*

5.1. **Hypersurfaces.** Spin(7) acts transitively on S^7 and the stabilizer of a point is G_2. An oriented hypersurface $N^7 \subset M^8$ inherits a G_2-structure $\sigma \in \Omega^3_+(M)$ that is defined by the rule

$$\sigma = \mathbf{n} \lrcorner \Phi \tag{5.4}$$

where $\mathbf{n}$ is the oriented normal vector field along N. It also satisfies

$$*_\sigma\sigma = N^*\Phi. \tag{5.5}$$

The structure equations show that

$$*_\sigma\big(\sigma \wedge \mathrm{d}\sigma\big) = 28H \tag{5.6}$$

where H is the mean curvature of N in (M, g_Φ).

Theorem 7. *If $\sigma \in \Omega^3_+(N^7)$ is real-analytic and satisfies $\mathrm{d}(*_\sigma\sigma) = 0$, then σ is induced by an immersion of N into a* Spin(7)*-manifold (M, Φ).*

Proof. The argument in this case is entirely analogous to the SU(2) and G_2 cases already treated:

Recall the definitions of $\boldsymbol{\sigma}$ and $\boldsymbol{\tau}$ on $\mathcal{F}(N^7)/\mathrm{G}_2$ and, on $X = \mathbb{R} \times \mathcal{F}(N)/\mathrm{G}_2$, define

$$\boldsymbol{\Phi} = \mathrm{d}t \wedge \boldsymbol{\sigma} + \boldsymbol{\tau}. \tag{5.7}$$

Let $\mathcal{I}$ be the ideal on $\mathbb{R} \times \mathcal{F}(N)/\mathrm{G}_2$ generated by $\mathrm{d}\boldsymbol{\Phi}$. The same calculation used to prove Theorem 6 (see [3]) then yields that $\mathcal{I}$ is involutive, with character sequence

$$(s_1, \ldots, s_8) = (0, 0, 0, 1, 4, 10, 20, 0). \tag{5.8}$$

Since $\mathrm{d}(*_\sigma\sigma) = 0$, the G_2-structure σ defines a regular $\mathcal{I}$-integral $L \subset X$ within the locus $t = 0$.

Since σ is assumed to be real-analytic with respect to some underlying analytic structure on N, it follows that X, $\mathcal{I}$, and L are real-analytic with respect to the obvious induced analytic structures on the appropriate underlying manifolds. Thus, the Cartan-Kähler theorem applies to show that L lies in a unique $\mathcal{I}$-integral $M^8 \subset X$. The form $\mathbf{\Phi}$ then pulls back to M to be a closed $\Phi \in \Omega^4_s(M)$ which induces the given σ on $N \simeq L \subset M$ defined by $t = 0$. □

Theorem 8. *There exist non-real-analytic* G_2*-structures* $\sigma \in \Omega^3_+(N^7)$ *that satisfy*

$$\mathrm{d}(*_\sigma\sigma) = 0 \tag{5.9}$$

but that are not induced from a $\mathrm{Spin}(7)$*-immersion.*

In fact, if a non-analytic G_2*-structure satisfies* (5.9) *and*

$$*_\sigma\,\big(\sigma \wedge \mathrm{d}\sigma\big) = C \tag{5.10}$$

where C is a constant, then it cannot be $\mathrm{Spin}(7)$*-immersed.*

Non-analytic G_2*-structures* $\sigma \in \Omega^3_+(N^7)$ *satisfying* (5.9) *and* (5.10) *do exist.*

Proof. The first claim follows from the second and third.

If $\sigma \in \Omega^3_+(N^7)$ is induced from at $\mathrm{Spin}(7)$-immersion $(N, \sigma) \hookrightarrow (M^8, \Phi)$ and has $*_\sigma\big(\sigma\wedge\mathrm{d}\sigma\big)$ equal to a constant, then the hypersurface $N \subset M$ has constant mean curvature. Since (M, g_Φ) is real-analytic, constant mean curvature hypersurfaces in M are also real-analytic, so it follows that σ must be real-analytic. This proves the second claim.

It remains now to verify the third claim by explaining how to construct non-analytic G_2-structures σ satisfying (5.9) and (5.10). To save space, I will only sketch the argument, the details of which are somewhat involved, though the idea is the same as for $\mathrm{SU}(2)$:

If such a σ is to be real-analytic, it will have to be real-analytic in g_σ-harmonic coordinates. Now, the system of first-order equations

$$\mathrm{d}(*_\sigma\sigma) = 0, \qquad *_\sigma\big(\sigma\wedge\mathrm{d}\sigma\big) = C, \qquad \mathrm{d}(*_\sigma\mathrm{d}x) = 0$$

for $\sigma \in \Omega^3_+(\mathbb{R}^7)$ is only $21+1+7 = 29$ equations for 35 unknowns. This underdetermined system is not elliptic, but its symbol mapping has constant rank and it can be embedded into an appropriate sequence to show that it has non-real-analytic solutions. □

5.2. **The flow interpretation.** Finally, let us consider the 'flow' interpretation: If $f : N \hookrightarrow M$ is an oriented smooth hypersurface in a $\mathrm{Spin}(7)$-manifold (M, Φ) with oriented unit normal $\mathbf{n} : N \to TM$, then the normal exponential mapping can be used to embed a neighborhood U of N in M into $\mathbb{R} \times N$ in such a way that, on this neighborhood U, one can write

$$\Phi = \mathrm{d}t \wedge \sigma + *_\sigma\sigma \tag{5.11}$$

where $\sigma \in \Omega^3_+(N)$ now depends on t (in the case that N is noncompact, the domain of σ might depend on t). The closure of Φ implies that

$$\frac{\mathrm{d}}{\mathrm{d}t}\big(*_\sigma\sigma\big) = \mathrm{d}\sigma. \tag{5.12}$$

This equation is enough to determine the time derivative of σ as well because of the following observation: For any real vector space V of dimension 7, the map S :

$\Lambda^3_+(V^*) \to \Lambda^4(V^*)$ defined by $\mathsf{S}(\phi) = *_\phi \phi$ is a 2-to-1 smooth local diffeomorphism of $\Lambda^3_+(V^*)$ onto an open cone $\Lambda^4_+(V^*) \subset \Lambda^4(V^*)$. In fact, if $\mathsf{S}(\phi) = \mathsf{S}(\psi)$, then $\phi = \pm\psi$. Both ϕ and $-\phi$ are definite, but they each determine opposite orientations, i.e., $*_{-\phi}1 = - *_\phi 1$. In particular, if one fixes an orientation on V, then for any $\tau \in \Lambda^4_+(V^*)$, there is a unique element $\phi \in \Lambda^3_+(V^*)$ such that $*_\phi\phi = \tau$ and $*_\phi 1$ is a positive volume form on V. I will denote this element by $\mathsf{S}^{-1}(\tau) = \phi$. Using this notation and the assumption that N is oriented, the above equation can be written in the more obviously 'evolutionary' form

$$\frac{\mathrm{d}}{\mathrm{d}t}\big(\tau\big) = \mathrm{d}\left(\mathsf{S}^{-1}(\tau)\right) \tag{5.13}$$

where

$$\Phi = \mathrm{d}t \wedge \mathsf{S}^{-1}(\tau) + \tau \tag{5.14}$$

with $\tau \in \Omega^4_+(N)$ depending on t.

The content of Theorems 7 and 8 is then that (5.13) has a solution when the initial τ_0 is closed and real-analytic, but need not have a solution for a non-analytic closed τ_0.

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