Derivation of the Time Dependent Two Dimensional Focusing NLS Equation

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Abstract

We present a microscopic derivation of the two-dimensional focusing cubic nonlinear Schrödinger equation starting from an interacting N-particle system of Bosons. The interaction potential we consider is given by $W_\beta(x) = N^{-1+2\beta} W(N^\beta x)$ for some spherically symmetric and compactly supported potential $W \in L^\infty(\mathbb{R}^2, \mathbb{R})$. The class of initial wave functions is chosen such that the variance in energy is small. Furthermore, we assume that the Hamiltonian $H_{W_\beta,t} = -\sum_{j=1}^N \Delta_j + \sum_{1\leq j<k\leq N} W_\beta(x_j - x_k) + \sum_{j=1}^N A_t(x_j)$ fulfills stability of second kind, that is $H_{W_\beta,t} \geq -CN$. We then prove the convergence of the reduced density matrix corresponding to the exact time evolution to the projector onto the solution of the corresponding nonlinear Schrödinger equation in either Sobolev trace norm, if $\|A_t\|_p < \infty$ for some $p > 2$, or in trace norm, for more general external potentials. For trapping potentials of the form $A(x) = C|x|^s$, $C > 0$, the condition $H_{W_\beta,t} \geq -CN$ can be fulfilled for a certain class of interactions W_β, for all $0 < \beta < \frac{s+1}{s+2}$, see [26].

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1 Introduction

During the last decades the experimental realization and the theoretical investigation of Bose-Einstein condensation (BEC) regained a considerable amount of attention. Mathematically, there is a steady effort to describe both the dynamical as well as the statical properties of such condensates. While the principal mechanism of BEC is similar for many different systems, the specific effective description of such a system depends strongly on the model one studies. In this paper we will focus on a dilute, two dimensional system of bosons with attractive interaction.
Let us first define the N-body quantum problem we have in mind. The evolution of N interacting bosons is described by a time-dependent wave-function $\Psi_t \in L^2_s(\mathbb{R}^{2N}, \mathbb{C})$, $\|\Psi_t\| = 1$ (Here and below norms without index $\| \cdot \|$ always denote the L^2-norm on the appropriate Hilbert space.). $L^2_s(\mathbb{R}^{2N}, \mathbb{C})$ denotes the set of all $\Psi \in L^2(\mathbb{R}^{2N}, \mathbb{C})$ which are symmetric under pairwise permutations of the variables $x_1, \ldots, x_N \in \mathbb{R}^2$. Ψ_t solves the N-particle Schrödinger equation

$$\mathrm{i}\partial_t \Psi_t = H_{W_\beta,t} \Psi_t \ , \tag{1}$$

where the time-dependent Hamiltonian $H_{W_\beta,t}$ is given by

$$H_{W_\beta,t} = -\sum_{j=1}^{N} \Delta_j + \sum_{1 \le j < k \le N} W_\beta(x_j - x_k) + \sum_{j=1}^{N} A_t(x_j) \ . \tag{2}$$

The scaled potential $W_\beta(x) = N^{-1+2\beta} W(N^\beta x)$, $W \in L^\infty_c(\mathbb{R}^2, \mathbb{R})$ describes a strong, but short range potential acting on the length scale of order $N^{-\beta}$ (we assume W to be compactly supported). The external potential $A_t \in L^p(\mathbb{R}^2, \mathbb{R})$ for some $p > 1$ is used as an external trapping potential. Below, we will comment on different choices for A_t in more detail. In general, even for small particle numbers N, the evolution equation (1) cannot be solved directly nor numerically for Ψ_t. Nevertheless, for a certain class of initial conditions Ψ_0 and certain interactions W, which we will make precise in a moment, it is possible to derive an approximate solution of (1) in the trace class topology of reduced density matrices.
Define the one particle reduced density matrix $\gamma^{(1)}_{\Psi_0}$ of Ψ_0 with integral kernel

$$\gamma^{(1)}_{\Psi_0}(x, x') = \int_{\mathbb{R}^{2N-2}} \Psi_0^*(x, x_2, \ldots, x_N) \Psi_0(x', x_2, \ldots, x_N) d^2x_2 \ldots d^2x_N \ .$$

To account for the physical situation of a Bose-Einstein condensate, we assume complete condensation in the limit of large particle number N. This amounts to assume that, for $N \to \infty$, $\gamma^{(1)}_{\Psi_0} \to |\varphi_0\rangle\langle\varphi_0|$ in trace norm for some $\varphi_0 \in L^2(\mathbb{R}^2, \mathbb{C})$, $\|\varphi_0\| = 1$. Define $a = \int_{\mathbb{R}^2} d^2x W(x)$ (throughout this paper, a will always denote the integral over W). Let φ_t solve the nonlinear Schrödinger equation

$$\mathrm{i}\partial_t \varphi_t = (-\Delta + A_t)\, \varphi_t + a|\varphi_t|^2 \varphi_t =: h^{\mathrm{NLS}}_{a,t} \varphi_t \tag{3}$$

with initial datum φ_0. Our main goal is to show the persistence of condensation over time. In particular, we prove that the time evolved reduced density matrix $\gamma^{(1)}_{\Psi_t}$ converges to $|\varphi_t\rangle\langle\varphi_t|$ in trace norm as $N \to \infty$ with convergence rate of order $N^{-\eta}$ for some explicitly computable

$\eta > 0$, see Lemma 3.8. Assuming $W \in L^\infty_c(\mathbb{R}^2, \mathbb{R})$, such that W is spherically symmetric and $W \geq 0$, we were possible to show convergence $\gamma^{(1)}_{\Psi_t} \to |\varphi_t\rangle\langle\varphi_t|$ in trace norm for all $\beta > 0$, under suitable conditions on Ψ_0 and φ_0 [18], see also [21] for a prior result. The problem becomes more delicate for interactions which are not nonnegative, especially if (3) is focusing, which means $a < 0$. For strong, attractive potentials, it is known that the condensate collapses in the limit of large particle number. To prevent this behavior, our proof needs stability of second kind for the Hamiltonian $H_{W_\beta,t}$, that is, we assume $H_{W_\beta,t} \geq -CN$. If W_β is partly or purely nonpositive, this assumption gets highly nontrivial for higher β. For $\beta \leq 1/2$, the inequality

$$\begin{aligned}
&\inf_{\Psi\in L^2(\mathbb{R}^{2N},\mathbb{C}),\|\Psi\|=1} \frac{\langle\!\langle \Psi, H_{W_\beta,t}\Psi\rangle\!\rangle}{N} \\
&\geq \inf_{\varphi\in L^2(\mathbb{R}^2,\mathbb{C}),\|\varphi\|=1} \left(\int_{\mathbb{R}^2} d^2x \left(|\nabla\varphi(x)|^2 + A_t(x)|\varphi(x)|^2 + \frac{1}{2}\int_{\mathbb{R}^2} d^2x |\varphi(x)|^2 (NW_\beta * |\varphi|^2)(x) \right)\right) \\
&-\mathcal{O}(1) - CN^{2\beta-1},
\end{aligned} \tag{4}$$

which was proven in [24], shows $H_{W_\beta,t} \geq -CN$, if (4), which is the ground state energy of the nonlinear Hartree functional, is bounded from below uniformly in N (Here and in the following, $\langle\!\langle \cdot,\cdot \rangle\!\rangle$ denotes the scalar product on $L^2(\mathbb{R}^{2N},\mathbb{C})$.). Assuming $A_t \geq -C$, this is the case if

$$\inf_{\varphi\in H^1(\mathbb{R}^2,\mathbb{C})} \left(\frac{\int_{\mathbb{R}^2} d^2x |\varphi(x)|^2 (|\varphi|^2 * W)(x)}{\|\varphi\|^2 \|\nabla\varphi\|^2} \right) > -1 \tag{5}$$

holds [26]. Assuming condition (5) together with $A_t \in L^1_{\text{loc}}(\mathbb{R}^2,\mathbb{R})$, $A_t(x) \geq C|x|^s$, $s > 0$, stability of second kind was subsequently proven for all $0 < \beta < \frac{s+1}{s+2}$ [26]. In particular, it was shown that the ground state energy per particle of $H_{W_\beta,t}$ is then given (in the limit $N \to \infty$) by the corresponding nonlinear Schrödinger functional; see also [25] for an earlier result which also treats the one- and three-dimensional cases.

Condition (5) thus restricts the set of interactions W. Indeed, stability of the second kind fails if

$$\inf_{\varphi\in H^1(\mathbb{R}^2,\mathbb{C})} \left(\frac{\int_{\mathbb{R}^2} d^2x |\varphi(x)|^2 (|\varphi|^2 * W)(x)}{\|\varphi\|^2 \|\nabla\varphi\|^2} \right) < -1, \tag{6}$$

see [25, 26] for a nice discussion. Let W^- denote the negative part of W and let $a^* > 0$ denote the optimal constant of the Gagliardo-Nirenberg inequality

$$\left(\int_{\mathbb{R}^2} d^2x |\nabla u(x)|^2 \right) \left(\int_{\mathbb{R}^2} d^2y |u(y)|^2 \right) \geq \frac{a^*}{2} \left(\int_{\mathbb{R}^2} d^2x |u(x)|^4 \right) .$$

It is then easy to prove that $\int_{\mathbb{R}^2} d^2x |W^-(x)| < a^*$ implies condition (5). On the other hand, (5) implies $a > -a^*$. While (5) is in general a weaker condition than $\int_{\mathbb{R}^2} d^2x |W^-(x)| < a^*$, for $W \leq 0$, they are equivalent. Consequently, for nonpositive W and for $a < -a^*$, the nonlinear Hartree functional is not bounded from below in the limit $N \to \infty$, which in particular implies that $H_{W_\beta,t}$ is not stable of the second kind. It is also known that a^* is the critical threshold for blow-up solutions, that is, for $a \leq -a^*$ the solution of (3) may blow up in finite time

[6, 7, 8, 20, 42, 44].
The condition $H_{W_\beta,t} \geq -CN$ is needed in our proof to control the kinetic energy of those particles which leave the condensate, see Lemma 3.9. In prior works, it was necessary to control the quantity $\|\nabla_1 q_1 \Psi_t\|$ [1] sufficiently well in order to show convergence of the reduced density matrices for large β, using the method introduced in [40]. While it is possible to obtain an a priori estimate of $\|\nabla_1 q_1 \Psi_t\|$ for repulsive interactions, it is not obvious how one could generalize this estimate for nonpositive W. Our strategy to overcome this difficulty is to control the quantity $\|q_2 \nabla_1 \Psi_t\|$ instead. Under some natural assumptions (see (A2), (A4) and (A5) below), it is possible to obtain a sufficient bound of $\|q_2 \nabla_1 \Psi_t\|$, if initially the variance of the energy

$$\mathrm{Var}_{H_{W_\beta,0}}(\Psi_0) = \frac{1}{N^2} \langle\!\langle \Psi_0, \left(H_{W_\beta,0} - \langle\!\langle \Psi_0, H_{W_\beta,0} \Psi_0 \rangle\!\rangle\right)^2 \Psi_0 \rangle\!\rangle \tag{7}$$

is small and $H_{W_\beta,t}$ is stable of second kind. For product states $\Psi_0 = \varphi^{\otimes k}$, with φ regular enough, a straightforward calculation yields $\mathrm{Var}_{H_{W_\beta,0}}(\Psi_0) \leq C(1 + N^{-1+\beta} + N^{-2+2\beta})$, see Lemma 3.9. Therefore, at least for $\beta < 1$, there exist initial states Ψ_0, for which the variance is small. The strategy to control $\|q_2 \nabla_1 \Psi_t\|$ instead of $\|\nabla_1 q_1 \Psi_t\|$ by means of the energy variance was already used in [23] where the derivation of the Maxwell-Schrödinger equations from the Pauli-Fierz Hamiltonian was shown[2]. Adopting this method, we are able to prove convergence of $\gamma^{(1)}_{\Psi_t}$ to $|\varphi_t\rangle\langle\varphi_t|$ in trace norm as $N \to \infty$ with convergence rate of order $N^{-\eta}$, $\eta > 0$, if the assumptions (A1)-(A5) (see below) are fulfilled. We like to remark that it is unclear if the assumptions (A1) (stability of second kind of $H_{W_\beta,t}$) and (A3) (smallness of $\mathrm{Var}_{H_{W_\beta,0}}(\Psi_0)$) can be fulfilled for $\beta \geq 1$.
A stronger statement than convergence in trace norm is convergence in Sobolev trace norm. For external potentials $A_t \in L^p(\mathbb{R}^2, \mathbb{R})$, with $p \in]2, \infty]$, we are able to show

$$\lim_{N\to\infty} \mathrm{Tr} \left| \sqrt{1-\Delta} (\gamma^{(1)}_{\Psi_t} - |\varphi_t\rangle\langle\varphi_t|) \sqrt{1-\Delta} \right| = 0, \tag{8}$$

if initially the energy per particle $N^{-1} \langle\!\langle \Psi_0, H_{W_\beta,0} \Psi_0 \rangle\!\rangle$ is close to the NLS energy $\langle \varphi_0, \left(-\Delta + \frac{a}{2}|\varphi_0|^2 + A_0\right) \varphi_0 \rangle$. To obtain this type of convergence, we adapt some recent results of [1, 34], where a similar statement was proven.
The rigorous derivation of effective evolution equations has a long history, see e.g. [3, 4, 5, 12, 13, 14, 15, 22, 35, 36, 39, 40, 41] and references therein. In particular, for the two-dimensional case we consider, it has been proven, for $0 < \beta < 3/4$ and W nonnegative, that $\gamma^{(1)}_{\Psi_t}$ converges to $|\varphi_t\rangle\langle\varphi_t|$ as $N \to \infty$ [21]. This was later extended by us to hold for all $\beta > 0$ [18]. For $A(x) = |x|^2$ and $W \leq 0$ sufficiently small such that $H_{W_\beta,t} \geq -CN$, the respective convergence has been proven in [9] for $0 < \beta < 1/6$. One key estimate of the proof was to show the stability condition $H_{W_\beta,t} \geq -CN$. The authors note that their proof actually works for all $0 < \beta < 3/4$, if $H_{W_\beta,t} \geq -CN$ holds. As mentioned, this was subsequently proven by [26] in a more general setting. Recently, the validity of the Bogoloubov approximation for the two-dimensional attractive bose gas was shown in [37] for $0 < \beta < 1$. In contrast to our result, the authors were actually able to achieve norm convergence and did not need to impose the

[1] Here, q_1 denotes the complement of the projection onto $\varphi_t(x_1)$, see Definition 3.2 below.
[2] We would like to thank Nikolai Leopold for pointing out to us the idea to use the variance of the energy $\mathrm{Var}_{H_{W_\beta,t}}(\Psi_t)$ in our estimates and how to include time-dependent external potentials.

stability condition $H_{W_\beta,t} \geq -CN$, but only required the bound $\int_{\mathbb{R}^2} d^2x |W^-(x)| < a^*$. They then use some refined localization method on the number of particles. This strategy enables them to analyze the dynamics without any external field. It would be nice to achieve a better understanding on the relationship between stability of second kind and the bound on the negative part of W, as stated above. We plan to come back to this point in the future. We want to emphasize that norm convergence is a stronger statement than convergence in the topology of reduced densities. However, convergence in Sobolev trace norm as defined in (8) does in general not follow from norm convergence.

It is also possible to consider the two dimensional Bose gas in the regime where the short scale correlation structure is important for the dynamics. The scaling to consider in such a case is given by $e^{2N}W(e^N x)$. We refer the reader to [18, 30].

For $0 < \beta < 1/4$, it can be verified that the methods presented in [38], where the attractive three dimensional case is treated, can be applied, assuming some regularity conditions on φ_t (the corresponding conditions for the three dimensional system were proven in [11]). Interestingly, this proof does not restrict the strength of the nonpositive potential nor does it require stability of second kind, but rather assumes a sufficiently fast convergence of $\gamma^{(1)}_{\Psi_0}$ to $|\varphi_0\rangle\langle\varphi_0|$. Therefore, one can prove BEC in two dimensions for $\beta < 1/4$ and arbitrary strong attractive interactions for times for which the solution φ_t exists and is regular enough, that is, before some possible blow-up.

2 Main result

We will bound expressions which are uniformly bounded in N and t by some constant $C > 0$. We will not distinguish constants appearing in a sequence of estimates, i.e. in $X \leq CY \leq CZ$ the constants may differ. We denote by $\langle\langle \cdot, \cdot \rangle\rangle$ the scalar product on $L^2(\mathbb{R}^{2N}, \mathbb{C})$ and by $\langle \cdot, \cdot \rangle$ the scalar product on $L^2(\mathbb{R}^2, \mathbb{C})$.

We will require the following assumptions:

(A1) For $\beta > 0$, let W_β be given by $W_\beta(x) = N^{-1+2\beta} W(N^\beta x)$, for $W \in L^\infty_c(\mathbb{R}^2, \mathbb{R})$, W spherically symmetric. We assume that there exist constants $0 < \epsilon, \mu < 1$ such that

$$H^{(\epsilon,\mu)}_{W_\beta,t} = (1-\epsilon)\sum_{k=1}^N (-\Delta_k) + \sum_{i<j} W_\beta(x_i - x_j) + (1-\mu)\sum_{k=1}^N A_t(x_k) \geq -CN \ .$$

(A2) For any real-valued function f, decompose $f(x) = f^+(x) - f^-(x)$ with $f^+(x), f^-(x) \geq 0$, such that the supports of f^+ and f^- are disjoint. We assume that $A^-_t \in L^\infty(\mathbb{R}^2, \mathbb{R})$. Furthermore, we assume that $A_t \in H^2(\mathbb{R}^2, \mathbb{R})$ is differentiable with respect to t and fulfills

$$\dot{A}_t \in L^\infty(\mathbb{R}^2, \mathbb{R}) \ , \nabla \dot{A}_t \in L^\infty(\mathbb{R}^2, \mathbb{R}) \ , \Delta \dot{A}_t \in L^\infty(\mathbb{R}^2, \mathbb{R})$$

for all $t \in \mathbb{R}$.

(A3) For any $s \in \mathbb{R}$, we denote for $k \in \mathbb{N}$ the domain of the self-adjoint operator $(H_{W_\beta,s})^k$ by $\mathcal{D}((H_{W_\beta,s})^k)$. Define the energy variance $\mathrm{Var}_{H_{W_\beta,s}} : \mathcal{D}((H_{W_\beta,s})^2) \to \mathbb{R}^+$ as

$$\mathrm{Var}_{H_{W_\beta,s}}(\Psi) = \frac{1}{N^2} \langle\langle \Psi, \left(H_{W_\beta,s} - \langle\langle \Psi, H_{W_\beta,s}\Psi \rangle\rangle\right)^2 \Psi \rangle\rangle \ .$$

We then require $\mathrm{Var}_{H_{W_\beta,0}}(\Psi_0) \leq CN^{-\delta}$ for some $\delta > 0$.

(A4) Let φ_t the solution to $i\partial_t \varphi_t = h^{\mathrm{NLS}}_{a,t}\varphi_t$, $\|\varphi_0\| = 1$. We assume that $\varphi_t \in H^4(\mathbb{R}^2, \mathbb{C})$.

(A5) Assume that the energy per particle

$$N^{-1}|\langle\!\langle \Psi_0, H_{W_\beta,0}\Psi_0\rangle\!\rangle| \leq C$$

and the NLS energy

$$\left|\langle \varphi_0, \left(-\Delta + \frac{a}{2}|\varphi_0|^2 + A_0\right)\varphi_0\rangle\right| \leq C$$

are bounded uniformly in N initially.

(A5)' Assume that there exists a $\delta > 0$, such that

$$\left|N^{-1}\langle\!\langle \Psi_0, H_{W_\beta,0}\Psi_0\rangle\!\rangle - \langle \varphi_0, \left(-\Delta + \frac{a}{2}|\varphi_0|^2 + A_0\right)\varphi_0\rangle\right| \leq CN^{-\delta}.$$

Remark 2.1 *(a) Note that (A1) together with (A2) directly implies* $H^{(\epsilon,0)}_{W_\beta,t} \geq -CN$, $H^{(0,\mu)}_{W_\beta,t} \geq -CN$ *and* $H_{W_\beta,t} = H^{(0,0)}_{W_\beta,t} \geq -CN$. *As mentioned in the introduction, (A1) and (A2) are fulfilled for* $A(x) \geq C|x|^s$, $s > 0$ *for any* $0 < \beta < \frac{s+1}{s+2}$, *assuming* (5). *[26].*

(b) Assuming $\Psi_0 = \varphi_0^{\otimes N}$ *with* $\varphi_0 \in W^{2,\infty}(\mathbb{R}^2,\mathbb{C}) \cap H^1(\mathbb{R}^2,\mathbb{C})$, $\|\varphi_0\| = 1$ *such that* $\langle \varphi_0, A_0\varphi_0\rangle + \langle \varphi_0, A_0^2\varphi_0\rangle \leq C$, *it follows that* $Var_{H_{W_\beta,0}}(\Psi_0) \leq C(N^{-1+\beta} + N^{-2+2\beta})$, *see Lemma 5.1; and hence (A3) is then valid for all* $0 < \beta < 1$.

(c) For $A_t \in \{0, |x|^2\}$, *(A4) follows from the persistence of regularity of solutions, assuming* $\varphi_0 \in H^4(\mathbb{R}^2,\mathbb{C})$, $\|A_0^2\varphi_0\| < \infty$, *see Appendix 5.2. However, for regular enough external potentials,* $a > -a^*$ *and regular enough* φ_0 *we believe (A4) to be valid, too.*

(d) It is interesting to note that both (A1) and (A3) can be fulfilled for $0 < \beta < 1$, *while it is unclear if they hold for* $\beta \geq 1$.

We now state our main Theorem:

Theorem 2.2 *Let* $\Psi_0 \in L^2_s(\mathbb{R}^{2N},\mathbb{C}) \cap \mathcal{D}((H_{W_\beta,0})^2)$ *with* $\|\Psi_0\| = 1$. *Let* $\varphi_0 \in L^2(\mathbb{R}^2,\mathbb{C})$ *with* $\|\varphi_0\| = 1$ *and assume* $\lim_{N\to\infty}\left(N^\delta\, Tr|\gamma^{(1)}_{\Psi_0} - |\varphi_0\rangle\langle\varphi_0||\right) = 0$ *for some* $\delta > 0$. *Let* Ψ_t *the unique solution to* $i\partial_t\Psi_t = H_{W_\beta,t}\Psi_t$ *with initial datum* Ψ_0. *Let* φ_t *the unique solution to* $i\partial_t\varphi_t = h^{NLS}_{a,t}\varphi_t$ *with initial datum* φ_0.

(a) (Convergence in trace norm) Assume (A1)-(A5). Then, for any $t > 0$

$$\lim_{N\to\infty}\gamma^{(1)}_{\Psi_t} = |\varphi_t\rangle\langle\varphi_t| \tag{9}$$

in trace norm.

(b) (Convergence in Sobolev trace norm) Assume (A1)-(A5) and (A5)'. Furthermore, assume that $A_t \in L^p(\mathbb{R}^2,\mathbb{R})$ *holds for some* $p \in]2,\infty]$ *and for all* $t \in \mathbb{R}$. *Then, for any* $t > 0$

$$\lim_{N\to\infty} Tr\left|\sqrt{1-\Delta}(\gamma^{(1)}_{\Psi_t} - |\varphi_t\rangle\langle\varphi_t|)\sqrt{1-\Delta}\right| = 0. \tag{10}$$

Remark 2.3 *(a) It is well known that convergence of $\gamma^{(1)}_{\Psi_t}$ to $|\varphi_t\rangle\langle\varphi_t|$ in trace norm is equivalent to the respective convergence in operator norm since $|\varphi_t\rangle\langle\varphi_t|$ is a rank-1-projection, see Remark 1.4. in [41]. Other equivalent definitions of asymptotic 100% condensation can be found in [33]. Furthermore, the convergence of the one-particle reduced density matrix $\gamma^{(1)}_{\Psi_t} \to |\varphi_t\rangle\langle\varphi_t|$ in trace norm implies convergence of any k-particle reduced density matrix $\gamma^{(k)}_{\Psi_t}$ against $|\varphi_t^{\otimes k}\rangle\langle\varphi_t^{\otimes k}|$ in trace norm as $N\to\infty$ and k fixed, see for example [22].*

(b) In our proof we will give explicit error estimates in terms of the particle number N. We shall show that the rate of convergence is of order $N^{-\delta}$ for some $\delta > 0$, assuming that initially $Tr|\gamma^{(1)}_{\Psi_0} - |\varphi_0\rangle\langle\varphi_0|| \le CN^{-2\delta}$ holds. See (21) for the precise error estimate.

(c) Under assumption (A2), the domains $\mathcal{D}(H_{W_\beta,t})$ and $\mathcal{D}((H_{W_\beta,t})^2)$ of the time dependent Hamiltonian $H_{W_\beta,t}$ are time-invariant, see Appendix 5.3. Therefore, the condition $\Psi_0 \in \mathcal{D}((H_{W_\beta,0})^2)$ is sufficient to define and to differentiate the variance of the energy $Var_{H_{W_\beta,t}}(\Psi_t)$.

(d) For $A_t(x) = |x|^2$, $0 < \beta < 3/4$ and under condition (5), the assumptions (A1)-(A5) can be fulfilled by choosing $\Psi_0 = \varphi_0^{\otimes N}$ with φ_0 regular enough. We are therefore able to reproduce the result presented in [9] under slightly different assumptions, using the result of [26] which implies (A1).

(e) For external potentials A_t which are bounded from below, assumption (A1) has been proven for all $0 < \beta \le 1/2$, under the condition (5) [24]. We are therefore able to control the convergence of $\gamma^{(1)}_{\Psi_t}$ to $|\varphi_t\rangle\langle\varphi_t|$ in Sobolev trace norm as $N\to\infty$ for $0 < \beta \le 1/2$.

(f) In our estimates, we need the regularity conditions

$$\|\Delta\varphi_t\|_\infty < \infty,\ \|\nabla\varphi_t\|_\infty < \infty,\ \|\varphi_t\|_\infty < \infty,\ \|\nabla\varphi_t\| < \infty,\ \|\Delta\varphi_t\| < \infty\ .$$

That is, we need $\varphi_t \in H^2(\mathbb{R}^2,\mathbb{C}) \cap W^{2,\infty}(\mathbb{R}^2,\mathbb{C})$. Then, $\|\Delta|\varphi_t|^2\|$ which also appears in our estimates, can be bounded by

$$\begin{aligned}\Delta|\varphi_t|^2 =&\varphi_t^*\Delta\varphi_t + \varphi_t\Delta\varphi_t^* + 2(\nabla\varphi_t^*)\cdot(\nabla\varphi_t)\\ \|\Delta|\varphi_t|^2\| \le& 2\|\Delta\varphi_t\|\|\varphi_t\|_\infty + 2\|\nabla\varphi_t\|\|\nabla\varphi_t\|_\infty\end{aligned}$$

Recall the Sobolev embedding Theorem, which implies in particular $H^k(\mathbb{R}^2,\mathbb{C}) = W^{k,2}(\mathbb{R}^2,\mathbb{C}) \subset C^{k-2}(\mathbb{R}^2,\mathbb{C})$. If $\varphi \in C^2(\mathbb{R}^2,\mathbb{C}) \cap H^2(\mathbb{R}^2,\mathbb{C})$, then $\varphi \in W^{2,\infty}(\mathbb{R}^2,\mathbb{C})$ follows since both φ and $\nabla\varphi$ have to decay at infinity. Thus, $\varphi_t \in H^4(\mathbb{R}^2,\mathbb{C})$ implies $\varphi_t \in H^2(\mathbb{R}^2,\mathbb{C}) \cap W^{2,\infty}(\mathbb{R}^2,\mathbb{C})$, which suffices for our estimates [3].

3 Proof of Theorem 2.2 (a)

We fix the notation we are going to employ during the rest of the paper.

[3] Actually, $\varphi_t \in H^{3+\epsilon}(\mathbb{R}^2,\mathbb{C})$ for some $\epsilon > 0$ would suffice for our estimates. Note that it is reasonable to expect persistence of regularity of φ_t assuming $\varphi_t \in L^\infty(\mathbb{R}^2,\mathbb{C})$, see also Appendix 5.2.

Notation 3.1 *(a) We will denote the operator norm defined for any linear operator* $f : L^2(\mathbb{R}^{2N}, \mathbb{C}) \to L^2(\mathbb{R}^{2N}, \mathbb{C})$ *by*

$$\|f\|_{op} = \sup_{\psi \in L^2(\mathbb{R}^{2N},\mathbb{C}), \|\Psi\|=1} \|f\Psi\| \ .$$

(b) We will denote for any multiplication operator $F : L^2(\mathbb{R}^2, \mathbb{C}) \to L^2(\mathbb{R}^2, \mathbb{C})$ *the corresponding operator*

$$\mathbb{1}^{\otimes(k-1)} \otimes F \otimes \mathbb{1}^{\otimes(N-k)} : L^2(\mathbb{R}^{2N}, \mathbb{C}) \to L^2(\mathbb{R}^{2N}, \mathbb{C})$$

acting on the N*-particle Hilbert space by* $F(x_k)$*. In particular, we will use, for any* $\Psi, \Omega \in L^2(\mathbb{R}^{2N}, \mathbb{C})$ *the notation*

$$\langle\!\langle \Omega, \mathbb{1}^{\otimes(k-1)} \otimes F \otimes \mathbb{1}^{\otimes(N-k)} \Psi \rangle\!\rangle = \langle\!\langle \Omega, F(x_k)\Psi \rangle\!\rangle \ .$$

In analogy, for any two-particle multiplication operator $K : L^2(\mathbb{R}^2, \mathbb{C})^{\otimes 2} \to L^2(\mathbb{R}^2, \mathbb{C})^{\otimes 2}$*, we denote the operator acting on any* $\Psi \in L^2(\mathbb{R}^{2N}, \mathbb{C})$ *by multiplication in the variable* x_i *and* x_j *by* $K(x_i, x_j)$*. In particular, we denote*

$$\langle\!\langle \Omega, K(x_i, x_j)\Psi \rangle\!\rangle = \int_{\mathbb{R}^{2N}} K(x_i, x_j)\Omega^*(x_1, \dots, x_N)\Psi(x_1, \dots, x_N) d^2x_1 \dots d^2x_N \ .$$

(c) We will denote by $\mathcal{K}(\varphi_t)$ *a constant depending on time, via* $\|\dot{A}_t\|_\infty$, $\|A_t^-\|_\infty$, $\int_0^t ds \|\dot{A}_s\|_\infty$ *and* $\|\varphi_t\|_{H^4}$*. As mentioned above, we make use of the embedding* $H^2(\mathbb{R}^2, \mathbb{C}) \cap W^{2,\infty}(\mathbb{R}^2, \mathbb{C}) \subseteq H^4(\mathbb{R}^2, \mathbb{C})$*.*

The method we use in this paper is introduced in detail in [40] and was generalized to derive various mean-field equations. Our proof is based on [18, 39]. In [18] we proved the equivalent of Theorem 2.2 for nonnegative potentials and for all $\beta > 0$. However, since we are not covering the two dimensional Gross-Pitaevskii regime where one considers an exponential scaling of the interaction, our estimates are less involved. Furthermore, we adopt some ideas which were first presented in [23]. Heuristically speaking, the method we are going to employ is based on the idea of counting for each time t the relative number of those particles which are not in the state φ_t. It is then possible to show that the rate of particles which leave the condensate is small, if initially almost all particles are in the state φ_0. In order to compare the exact dynamic, generated by $H_{W_\beta,t}$, with the effective dynamic, generated by $h^{\text{NLS}}_{a,t}$, we define the projectors p_j^φ and q_j^φ.

Definition 3.2 *Let* $\varphi \in L^2(\mathbb{R}^2, \mathbb{C})$ *with* $\|\varphi\| = 1$*. For any* $1 \le j \le N$ *the projectors* $p_j^\varphi : L^2(\mathbb{R}^{2N}, \mathbb{C}) \to L^2(\mathbb{R}^{2N}, \mathbb{C})$ *and* $q_j^\varphi : L^2(\mathbb{R}^{2N}, \mathbb{C}) \to L^2(\mathbb{R}^{2N}, \mathbb{C})$ *are defined as*

$$p_j^\varphi \Psi = \varphi(x_j) \int_{\mathbb{R}^2} \varphi^*(\tilde{x}_j)\Psi(x_1, \dots, \tilde{x}_j, \dots, x_N) d^2\tilde{x}_j \quad \forall\, \Psi \in L^2(\mathbb{R}^{2N}, \mathbb{C}) \tag{11}$$

and $q_j^\varphi = 1 - p_j^\varphi$*. We shall also use, with a slight abuse of notation, the bra-ket notation* $p_j^\varphi = |\varphi(x_j)\rangle\langle\varphi(x_j)|$*.*

For ease of notation, we will often omit the upper index φ on p_j, q_j, except where their φ-dependence plays an important role. Our key strategy is to define a convenient functional α, depending on Ψ_t and φ_t, such that $\lim_{N\to\infty} \alpha(\Psi_t,\varphi_t)=0$ implies Theorem 2.2.

Definition 3.3 *Let* $\Psi \in L^2(\mathbb{R}^{2N},\mathbb{C}) \cap \mathcal{D}((H_{W_\beta,\cdot})^2)$ *and let* $\varphi \in L^2(\mathbb{R}^2,\mathbb{C})$, $\|\varphi\|=1$. *Define*

$$\begin{aligned}&\alpha : L^2(\mathbb{R}^{2N},\mathbb{C}) \times L^2(\mathbb{R}^2,\mathbb{C}) \to \mathbb{R}^+,\\ &\alpha(\Psi,\varphi) = \langle\!\langle \Psi, q_1^\varphi \Psi \rangle\!\rangle \ + Var_{H_{W_\beta,\cdot}}(\Psi) \ .\end{aligned} \tag{12}$$

Using a general strategy, we will estimate the time derivative $\frac{d}{dt}\alpha(\Psi_t,\varphi_t)$. In particular, we show that

$$\frac{d}{dt}\alpha(\Psi_t,\varphi_t) \le \mathcal{K}(\varphi_t)\left(\alpha(\Psi_t,\varphi_t)+N^{-\delta}\right)$$

holds for some $\delta>0$. By a Grönwall estimate, which precise form can be found below, we then obtain $\alpha_t(\Psi_t,\varphi_t)\to 0$ as $N\to\infty$, if $\alpha(\Psi_0,\varphi_0)$ converges to zero.

Lemma 3.4 *Let* $\Psi \in L_s^2(\mathbb{R}^{2N},\mathbb{C}) \cap \mathcal{D}((H_{W_\beta,\cdot})^2)$, $\|\Psi\|=1$ *and let* $\varphi \in L^2(\mathbb{R}^2,\mathbb{C})$, $\|\varphi\|=1$. *Let* $\alpha(\Psi,\varphi)$ *be defined as above. Then,*

$$\begin{aligned}\lim_{N\to\infty}\alpha(\Psi,\varphi)=0 \quad \Leftrightarrow \qquad &\lim_{N\to\infty}\gamma_\Psi^{(1)} = |\varphi\rangle\langle\varphi| \ \textit{in trace norm}\\ &\textit{and} \lim_{N\to\infty} Var_{H_{W_\beta,\cdot}}(\Psi)=0 \ .\end{aligned} \tag{13}$$

Proof: $\lim_{N\to\infty}\langle\!\langle\Psi,q_1^\varphi\Psi\rangle\!\rangle = 0 \Leftrightarrow \lim_{N\to\infty}\gamma_\Psi^{(1)} = |\varphi\rangle\langle\varphi|$ in trace norm follows from the inequality $\langle\!\langle\Psi,q_1^\varphi\Psi\rangle\!\rangle \le \mathrm{Tr}|\gamma_\Psi^{(1)}-|\varphi\rangle\langle\varphi|| \le \sqrt{8\langle\!\langle\Psi,q_1^\varphi\Psi\rangle\!\rangle}$, see [22].

□

Definition 3.5 *Let*

$$Z_\beta^\varphi(x_j,x_k) = W_\beta(x_j-x_k) - \frac{a}{N-1}|\varphi|^2(x_j) - \frac{a}{N-1}|\varphi|^2(x_k) \ . \tag{14}$$

Define the functional $\gamma : L^2(\mathbb{R}^{2N},\mathbb{C}) \times L^2(\mathbb{R}^2,\mathbb{C}) \to \mathbb{R}_0^+$ *by*

$$\gamma(\Psi,\varphi) = 2N\left|\langle\!\langle\Psi, p_1p_2Z_\beta^\varphi(x_1,x_2)q_1p_2\Psi\rangle\!\rangle\right| \tag{15}$$

$$+2N\left|\langle\!\langle\Psi, p_1p_2Z_\beta^\varphi(x_1,x_2)q_1q_2\Psi\rangle\!\rangle\right| \tag{16}$$

$$+2N\left|\langle\!\langle\Psi, q_1p_2Z_\beta^\varphi(x_1,x_2)q_1q_2\Psi\rangle\!\rangle\right| \ . \tag{17}$$

Lemma 3.6 *Let* Ψ_t *the unique solution to* $i\partial_t\Psi_t = H_{W_\beta,t}\Psi_t$ *with initial datum* $\Psi_0 \in L_s^2(\mathbb{R}^{2N},\mathbb{C}) \cap \mathcal{D}((H_{W_\beta,0})^2)$, $\|\Psi_0\|=1$. *Let* φ_t *the unique solution to* $i\partial_t\varphi_t = h_{a,t}^{NLS}\varphi_t$ *with initial datum* $\varphi_0 \in H^2(\mathbb{R}^2,\mathbb{C})$, $\|\varphi_0\|=1$. *Let* $\alpha(\Psi_t,\varphi_t)$ *be defined as in Definition 3.3. Then*

$$\frac{d}{dt}\alpha(\Psi_t,\varphi_t) \le \gamma(\Psi_t,\varphi_t) + \left|\frac{d}{dt}Var_{H_{W_\beta,t}}(\Psi_t)\right| \ . \tag{18}$$

Remark 3.7 *The three different contributions of $\gamma(\Psi,\varphi)$ can be identified with three distinct transitions of particles out of the condensate described by φ. The first line can be identified as the interaction of two particles in the state φ, causing one particle to leave the condensate. The second line estimates the evaporation of two particles. The last contribution describes the interaction of one particle in the condensate with one particle outside the condensate, causing the particle in the state φ to leave the condensate.*

Proof: For the proof of the Lemma we restore the upper index φ_t in order to pay respect to the time dependence of $p_1^{\varphi_t}$ and $q_1^{\varphi_t}$. The proof is a straightforward calculation of

$$\begin{aligned}
&\frac{d}{dt}\langle\!\langle\Psi_t, q_1^{\varphi_t}\Psi_t\rangle\!\rangle \\
=&i\langle\!\langle H_{W_\beta,t}\Psi_t, q_1^{\varphi_t}\ \Psi_t\rangle\!\rangle - i\langle\!\langle\Psi_t, q_1^{\varphi_t}\ H_{W_\beta,t}\Psi_t\rangle\!\rangle - i\langle\!\langle\Psi_t,[-\Delta_1 + a|\varphi_t|^2(x_1) + A_t(x_1), q_1^{\varphi_t}\ ]\Psi_t\rangle\!\rangle \\
=&i(N-1)\langle\!\langle\Psi_t,[Z_\beta^{\varphi_t}(x_1,x_2), q_1^{\varphi_t}\ ]\Psi_t\rangle\!\rangle = -2(N-1)\mathrm{Im}\left(\langle\!\langle\Psi_t, Z_\beta^{\varphi_t}(x_1,x_2)q_1^{\varphi_t}\Psi_t\rangle\!\rangle\right)\ .
\end{aligned}$$

Using the identity $1 = p_1^{\varphi_t} + q_1^{\varphi_t}$, we obtain

$$\begin{aligned}
\frac{d}{dt}\langle\!\langle\Psi_t, q_1^{\varphi_t}\Psi_t\rangle\!\rangle &= -2(N-1)\mathrm{Im}\left(\langle\!\langle\Psi_t, p_1^{\varphi_t}Z_\beta^{\varphi_t}(x_1,x_2)q_1^{\varphi_t}\Psi_t\rangle\!\rangle\right) \\
&= -2(N-1)\mathrm{Im}\left(\langle\!\langle\Psi_t, p_1^{\varphi_t}p_2^{\varphi_t}Z_\beta^{\varphi_t}(x_1,x_2)q_1^{\varphi_t}p_2^{\varphi_t}\Psi_t\rangle\!\rangle\right) \\
&\quad -2(N-1)\mathrm{Im}\left(\langle\!\langle\Psi_t, p_1^{\varphi_t}p_2^{\varphi_t}Z_\beta^{\varphi_t}(x_1,x_2)q_1^{\varphi_t}q_2^{\varphi_t}\Psi_t\rangle\!\rangle\right) \\
&\quad -2(N-1)\mathrm{Im}\left(\langle\!\langle\Psi_t, p_1^{\varphi_t}q_2^{\varphi_t}Z_\beta^{\varphi_t}(x_1,x_2)q_1^{\varphi_t}p_2^{\varphi_t}\Psi_t\rangle\!\rangle\right) \\
&\quad -2(N-1)\mathrm{Im}\left(\langle\!\langle\Psi_t, p_1^{\varphi_t}q_2^{\varphi_t}Z_\beta^{\varphi_t}(x_1,x_2)q_1^{\varphi_t}q_2^{\varphi_t}\Psi_t\rangle\!\rangle\right)\ .
\end{aligned}$$

Note that $\mathrm{Im}\left(\langle\!\langle\Psi_t, p_1^{\varphi_t}q_2^{\varphi_t}Z_\beta^{\varphi_t}(x_1,x_2)q_1^{\varphi_t}p_2^{\varphi_t}\Psi_t\rangle\!\rangle\right) = 0$, which concludes the proof.

□

We now establish the Grönwall estimate.

Lemma 3.8 *Let Ψ_t the unique solution to $i\partial_t\Psi_t = H_{W_\beta,t}\Psi_t$ with initial datum $\Psi_0 \in L_s^2(\mathbb{R}^{2N},\mathbb{C})\cap \mathcal{D}(H^2_{W_\beta,0})$, $\|\Psi_0\| = 1$. Let φ_t the unique solution to $i\partial_t\varphi_t = h_{a,t}^{NLS}\varphi_t$ with initial datum $\varphi_0 \in L^2(\mathbb{R}^2,\mathbb{C})$, $\|\varphi_0\| = 1$. Assume (A1)-(A5). Then,*

$$\gamma(\Psi_t,\varphi_t) \leq \mathcal{K}(\varphi_t)\ln(N)^{1/2}\left(\alpha(\Psi_t,\varphi_t) + N^{-2\beta}\ln(N)^{1/2} + N^{-1/3}\ln(N)^{3/2}\right)\ . \tag{19}$$

The proof of this Lemma can be found in Section 3.2.

Proof of Theorem 2.2 (a): Once we have proven Lemma 3.8, we obtain with Grönwall's Lemma that

$$\begin{aligned}
&\alpha(\Psi_t,\varphi_t) \leq N^{\frac{\int_0^t ds\mathcal{K}(\varphi_s)}{\ln(N)^{1/2}}}\alpha(\Psi_0,\varphi_0) \\
&+\int_0^t ds\mathcal{K}(\varphi_s)N^{\frac{\int_s^t d\tau\mathcal{K}(\varphi_\tau)}{\ln(N)^{1/2}}}\left(N^{-2\beta}\ln(N) + N^{-1/3}\ln(N)^2\right)\ .
\end{aligned} \tag{20}$$

Note that under the assumptions (A2) and (A4) there exists a time-dependent constant $C_t < \infty$, such that $\int_0^t ds\mathcal{K}(\varphi_s) \leq C_t$. Furthermore, the assumption $\lim_{N\to\infty}\left(N^{\delta}\mathrm{Tr}|\gamma^{(1)}_{\Psi_0} - |\varphi_0\rangle\langle\varphi_0||\right) = 0$ for some $\delta > 0$ then implies together with (A3)

$$\lim_{N\to\infty}\left(N^{\frac{\int_0^t ds\mathcal{K}(\varphi_s)}{\ln(N)^{1/2}}}\alpha(\Psi_0,\varphi_0)\right) = 0\ ,$$

since $\langle\langle\Psi, q_1^{\varphi}\Psi\rangle\rangle \leq \mathrm{Tr}|\gamma^{(1)}_{\Psi} - |\varphi\rangle\langle\varphi|| \leq \sqrt{8\langle\langle\Psi, q_1^{\varphi}\Psi\rangle\rangle}$, see [22]. Therefore,

$$\begin{aligned}\mathrm{Tr}|\gamma^{(1)}_{\Psi_t} - |\varphi_t\rangle\langle\varphi_t|| \leq & CN^{\frac{C_t}{2\ln(N)^{1/2}}-\delta/2} \\ & + C\sqrt{C_t N^{\frac{\sup_{s\in[0,t]}|C_t - C_s|}{\ln(N)^{1/2}}}\left(N^{-2\beta}\ln(N) + N^{-1/3}\ln(N)^2\right)}\ . \end{aligned} \tag{21}$$

This proves Theorem 2.2 (a).

□

3.1 Energy estimates

Lemma 3.9 *Let $\Psi_0 \in L^2_s(\mathbb{R}^{2N},\mathbb{C}) \cap \mathcal{D}((H_{W_\beta,0})^2)$ with $\|\Psi_0\| = 1$. Let Ψ_t the unique solution to $i\partial_t\Psi_t = H_{W_\beta,0}\Psi_t$ with initial datum Ψ_0, $\|\Psi_0\| = 1$. Let φ_t the unique solution to $i\partial_t\varphi_t = h^{NLS}_{a,t}\varphi_t$ with initial datum $\varphi_0 \in L^2(\mathbb{R}^2,\mathbb{C})$, $\|\varphi_0\| = 1$. Assume (A1),(A2), (A4) and (A5). Then,*

(a)

$$\|\nabla_1\Psi_t\| \leq \mathcal{K}(\varphi_t)\ . \tag{22}$$

(b)

$$\|q_2^{\varphi_t}\nabla_1\Psi_t\|^2 \leq \mathcal{K}(\varphi_t)\left(\alpha(\Psi_t,\varphi_t) + N^{-1/2}\right)\ . \tag{23}$$

(c) For any $p \in \mathbb{N}$, there exists a constant C_p, depending on p, such that

$$\left\|\sqrt{|NW_\beta(x_1 - x_2)|}q_1^{\varphi_t}q_2^{\varphi_t}\Psi_t\right\|^2 \leq \mathcal{K}(\varphi_t)C_pN^{\beta/p}\left(\alpha(\Psi_t,\varphi_t) + N^{-1/2}\right)\ . \tag{24}$$

Proof:

(a) Using (A1) together with (A2), we directly obtain the operator inequality

$$-\sum_{k=1}^{N}\epsilon\Delta_k \leq H_{W_\beta,t} + CN.$$

Using $\frac{d}{dt}N^{-1}\langle\langle\Psi_t, H_{W_\beta,t}\Psi_t\rangle\rangle \leq \|\dot{A}_t\|_\infty$ together with (A5), the energy per particle $N^{-1}\langle\langle\Psi_t, H_{W_\beta,t}\Psi_t\rangle\rangle \leq \mathcal{K}(\varphi_t)$ is uniformly bounded in N. Since Ψ_t is symmetric, we obtain

$$N\epsilon\langle\langle\Psi_t, -\Delta_1\Psi_t\rangle\rangle = \langle\langle\Psi_t, \left(-\sum_{k=1}^{N}\epsilon\Delta_k\right)\Psi_t\rangle\rangle \leq \mathcal{K}(\varphi_t)N.$$

(b) (see also [23].) We estimate

$$
\begin{aligned}
&\epsilon\|q_2\nabla_1\Psi_t\|^2 = \frac{1}{N-1}\langle\!\langle\Psi_t, q_2\epsilon\sum_{k=1}^{N}(-\Delta_k)q_2\Psi_t\rangle\!\rangle - \frac{\epsilon}{N-1}\langle\!\langle\Psi_t, q_2(-\Delta_2)q_2\Psi_t\rangle\!\rangle\\
\leq&\frac{1}{N-1}\langle\!\langle\Psi_t, q_2 H_{W_\beta,t}q_2\Psi_t\rangle\!\rangle + C\langle\!\langle\Psi_t, q_1\Psi_t\rangle\!\rangle\\
=&\frac{N}{N-1}\langle\!\langle\Psi_t, q_2\left(\frac{H_{W_\beta,t}}{N} - N^{-1}\langle\!\langle\Psi_t, H_{W_\beta,t}\Psi_t\rangle\!\rangle\right)\Psi_t\rangle\!\rangle + \frac{1}{N-1}\langle\!\langle\Psi_t, H_{W_\beta,t}\Psi_t\rangle\!\rangle\langle\!\langle\Psi_t, q_2\Psi_t\rangle\!\rangle\\
-&\frac{1}{N-1}\langle\!\langle\Psi_t, q_2 H_{W_\beta,t}p_2\Psi_t\rangle\!\rangle + C\langle\!\langle\Psi_t, q_1\Psi_t\rangle\!\rangle\\
\leq&C\mathrm{Var}_{H_{W_\beta,t}}(\Psi_t) + \frac{1}{N-1}|\langle\!\langle\Psi_t, q_2 H_{W_\beta,t}p_2\Psi_t\rangle\!\rangle| + 2\mathcal{K}(\varphi_t)\|q_1\Psi_t\|^2.
\end{aligned}
$$

It remains to estimate

$$
\begin{aligned}
&\frac{1}{N-1}|\langle\!\langle\Psi_t, q_2 H_{W_\beta,t}p_2\Psi_t\rangle\!\rangle|\\
\leq&\frac{1}{N-1}|\langle\!\langle\Psi_t, q_2(-\Delta_2)p_2\Psi_t\rangle\!\rangle| + |\langle\!\langle\Psi_t, q_2 W_\beta(x_1-x_2)p_2\Psi_t\rangle\!\rangle| + \frac{1}{N-1}|\langle\!\langle\Psi_t, q_2 A_t(x_1)p_2\Psi_t\rangle\!\rangle|\\
\leq&\frac{1}{N-1}\left(\|\nabla_1\Psi_t\|\|\nabla\varphi_t\| + \|\nabla\varphi_t\|^2\right) + \|W_\beta\|\|\varphi_t\|_\infty\|\mathbb{1}_{B_{CN^{-\beta}}(0)}(x_1-x_2)\Psi_t\| + \|\varphi_t\|_\infty^2\|W_\beta\|_1\\
+&\frac{1}{N-1}(\|A_t^-\|_\infty + \|\sqrt{A_t^+}\Psi_t\|\|\sqrt{A_t^+}\varphi_t\| + \langle\varphi_t, A_t\varphi_t\rangle),
\end{aligned}
$$

where we used $q_2 = 1 - p_2$ for all three contributions in the last inequality. Recall the two-dimensional Sobolev's inequality as presented in e.g. [27], Theorem 8.5. For any $\rho \in H^1(\mathbb{R}^2, \mathbb{C})$ and for any $2 \leq p < \infty$, there exists a constant C_p, depending on p, such that

$$
\|\rho\|_p^2 \leq C_p\left(\|\rho\|^2 + \|\nabla\rho\|^2\right) \tag{25}
$$

holds. It is shown in Theorem 8.5. in [27] that C_p fulfills $C_p \leq Cp$. We use this inequality in the x_1 variable and obtain together with Hölder's inequality, to obtain

$$
\begin{aligned}
&\|\mathbb{1}_{B_{CN^{-\beta}}(0)}(x_1-x_2)\Psi_t\|^2\\
\leq&\|\mathbb{1}_{B_{CN^{-\beta}}(0)}\|_{\frac{N}{N-1}}\int d^2x_2\ldots d^2x_N\left(\int d^2x_1|\Psi_t(x_1,\ldots,x_N)|^{2N}\right)^{1/N}\\
\leq&CN^{1-2\beta}\int d^2x_2\ldots d^2x_N\left(\int d^2x_1|\nabla_1\Psi_t(x_1,\ldots,x_N)|^2 + \int d^2x_1|\Psi_t(x_1,\ldots,x_N)|^2\right).\\
\leq&CN^{1-2\beta}\left(\|\nabla_1\Psi_t\|^2 + \|\Psi_t\|^2\right).
\end{aligned}
$$

With $\|W_\beta\| = CN^{-1+\beta}$, we obtain together with (a)

$$
\|W_\beta\|\|\mathbb{1}_{B_{CN^{-\beta}}(0)}(x_1-x_2)\Psi_t\| \leq \mathcal{K}(\varphi_t)N^{-1/2}. \tag{26}
$$

Next, we show that $\|\sqrt{A_t^+}\Psi_t\|$ and $\|\sqrt{A_t^+}\varphi_t\|$ are uniformly bounded in N. Using the operator inequality (A1) together with (A2) and (A5) directly implies

$$
\epsilon\langle\!\langle\Psi_t\sum_{k=1}^{N}A_t^+(x_k)\Psi_t\rangle\!\rangle \leq \langle\!\langle\Psi_t, H_{W_\beta,t}\Psi_t\rangle\!\rangle + \mathcal{K}(\varphi_t)N \leq \mathcal{K}(\varphi_t)N\ .
$$

To control $\langle\varphi_t, A_t^+\varphi_t\rangle$, let $\Omega_t = \varphi_t^{\otimes N}$. Then

$$\begin{aligned}
&\epsilon\langle\varphi_t, A_t^+\varphi_t\rangle \leq N^{-1}\langle\!\langle\Omega_t, H_{W_\beta,t}\Omega_t\rangle\!\rangle + \mathcal{K}(\varphi_t)\\
=&\langle\varphi_t, \left(-\Delta + \frac{a}{2}|\varphi_t|^2 + A_t\right)\varphi_t\rangle + \mathcal{K}(\varphi_t)\\
+&\langle\varphi_t, \left(\frac{1}{2}(N-1)W_\beta * |\varphi_t|^2 - \frac{a}{2}|\varphi_t|^2\right)\varphi_t\rangle\ .
\end{aligned}$$

Note that

$$\left|\frac{d}{dt}\langle\varphi_t, \left(-\Delta + \frac{a}{2}|\varphi_t|^2 + A_t\right)\varphi_t\rangle\right| \leq \|\dot{A}_t\|_\infty\ .$$

which implies together with (A5)

$$\langle\varphi_t, \left(-\Delta + \frac{a}{2}|\varphi_t|^2 + A_t\right)\varphi_t\rangle \leq \mathcal{K}(\varphi_t)\ .$$

Furthermore, we obtain as in (43)

$$\begin{aligned}
&\left|\langle\varphi_t, \left((N-1)W_\beta * |\varphi_t|^2 - a|\varphi_t|^2\right)\varphi_t\rangle\right|\\
\leq&\|\varphi_t\|_\infty^2\left(\left\|NW_\beta * |\varphi_t|^2 - a|\varphi|^2\right\| + \|W_\beta\|_1\|\varphi_t\|_\infty^2\right)\\
\leq&\mathcal{K}(\varphi_t)(N^{-2\beta}\ln(N) + N^{-1})\ .
\end{aligned}$$

This concludes the proof of (b).

(c) First, we like to recall the following Gagliardo-Nirenberg interpolation inequality: for any $\varrho \in H^1(\mathbb{R}^2, \mathbb{C})$ any for any $2 < m < \infty$, there exists a constant C_m, depending only on m, such that $\|\varrho\|_m \leq C_m\|\nabla\varrho\|^{\frac{m-2}{m}}\|\varrho\|^{\frac{2}{m}}$ holds. For $1 < p < \infty$, we estimate, using Hölder's - and the inequality above

$$\begin{aligned}
&\left\|\sqrt{|NW_\beta(x_1 - x_2)|}q_1q_2\Psi_t\right\|^2 \leq \|NW_\beta\|_\infty\left\|\mathbb{1}_{B_{CN^{-\beta}}(0)}(x_1 - x_2)q_1q_2\Psi_t\right\|^2\\
\leq&CN^{2\beta}\int_{\mathbb{R}^{2N-2}} d^2x_2\dots d^2x_N\int_{\mathbb{R}^2} d^2x_1|(q_1q_2\Psi_t)(x_1,\dots,x_N)|^2\mathbb{1}_{B_{CN^{-\beta}}(0)}(x_1 - x_2)\\
\leq&C_pN^{2\beta}\|\mathbb{1}_{B_{CN^{-\beta}}(0)}\|_{\frac{p}{p-1}}\int_{\mathbb{R}^{2N-2}} d^2x_2\dots d^2x_N\left(\int_{\mathbb{R}^2} d^2x_1|(q_1q_2\Psi_t)(x_1,\dots,x_N)|^{2p}\right)^{1/p}\\
\leq&C_pN^{2\beta\left(1-\frac{p-1}{p}\right)}\int_{\mathbb{R}^{2N-2}} d^2x_2\dots d^2x_N\left(\int_{\mathbb{R}^2} d^2x_1|(\nabla_1q_1q_2\Psi_t)(x_1,\dots,x_N)|^2\right)^{\frac{p-1}{p}}\\
\times&\left(\int_{\mathbb{R}^2} d^2\tilde{x}_1|(q_1q_2\Psi_t)(\tilde{x}_1,\dots,x_N)|^2\right)^{1/p}
\end{aligned} \tag{27}$$

We use Hölder's inequality with respect to the $x_2,\dots x_N$-integration with the conjugate pair $r = \frac{p}{p-1}$ and $s = p$ to obtain

$$(27) \leq C_pN^{\frac{2\beta}{p}}\|\nabla_1q_1q_2\Psi_t\|^{2\frac{p-1}{p}}\|q_1q_2\Psi_t\|^{\frac{2}{p}}.$$

Note that

$$\|\nabla_1 q_1 q_2 \Psi_t\|^2 \leq 2\|\nabla_1 p_1 q_2 \Psi_t\|^2 + 2\|\nabla_1 q_2 \Psi_t\|^2 \leq \mathcal{K}(\varphi_t) \left(\alpha(\Psi_t, \varphi_t) + N^{-1/2}\right).$$

Renaming p, we thus obtain with part (b), that there exists a constant depending on p such that

$$\left\| \sqrt{|NW_\beta(x_1 - x_2)|} q_1 q_2 \Psi_t \right\|^2 \leq C_p \mathcal{K}(\varphi_t) N^{\beta/p} \left(\alpha(\Psi_t, \varphi_t) + N^{-1/2}\right).$$

□

3.2 Proof of Lemma 3.8

Following a general strategy, we will smear out the potential W_β in order to use smoothness properties of Ψ. For this, we define

Definition 3.10 *For any $0 \leq \beta_1 \leq \beta$, we define*

$$U_{\beta_1}(x) = \begin{cases} \frac{a}{\pi} N^{-1+2\beta_1} & \text{for } |x| < N^{-\beta_1}, \\ 0 & \text{else.} \end{cases} \tag{28}$$

and

$$h_{\beta_1,\beta}(x) = \frac{1}{2\pi} \int_{\mathbb{R}^2} \ln|x-y| (W_\beta(y) - U_{\beta_1}(y)) d^2y \,. \tag{29}$$

Lemma 3.11 *For any $0 \leq \beta_1 \leq \beta$, we obtain with the above definition*

(a)

$$\Delta h_{\beta_1,\beta} = W_\beta - U_{\beta_1}. \tag{30}$$

(b)

$$\|h_{\beta_1,\beta}\| \leq C N^{-1-\beta_1} \ln(N) \text{ for } \beta_1 > 0 \,, \tag{31}$$

$$\|h_{0,\beta}\| \leq C N^{-1} \text{ for } \beta > 0 \,, \tag{32}$$

$$\|\nabla h_{\beta_1,\beta}\| \leq C N^{-1} (\ln(N))^{1/2} \,, \tag{33}$$

Proof: The lemma has been proven in [18], Lemma 7.2. for nonnegative potentials W_β. It is easy to verify that this proof also works without specifying the sign of W_β. We thus refer the reader to [18] for the details of the proof.

□

Next, we prove some well known properties of the projectors p_j, q_j.

Lemma 3.12 *Let $f \in L^1(\mathbb{R}^2, \mathbb{C})$, $g \in L^2(\mathbb{R}^2, \mathbb{C})$. Then,*

$$\|p_j f(x_j - x_k) p_j\|_{op} \leq \|f\|_1 \|\varphi\|_\infty^2 \,, \tag{34}$$

$$\|p_j g^*(x_j - x_k)\|_{op} = \|g(x_j - x_k) p_j\|_{op} \leq \|g\| \, \|\varphi\|_\infty \tag{35}$$

$$\||\varphi(x_j)\rangle\langle\nabla_j \varphi(x_j)| h^*(x_j - x_k)\|_{op} = \|h(x_j - x_k) \nabla_j p_j\|_{op} \leq \|h\| \|\nabla\varphi\|_\infty \,. \tag{36}$$

Proof: First note that, for bounded operators A, B, $\|AB\|_{\text{op}} = \|B^* A^*\|_{\text{op}}$ holds, where A^* is the adjoint operator of A. To show (34), note that

$$p_j f(x_j - x_k) p_j = p_j (f \star |\varphi|^2)(x_k) \; . \tag{37}$$

It follows that

$$\|p_j f(x_j - x_k) p_j\|_{\text{op}} \leq \|f\|_1 \|\varphi\|_\infty^2 \; .$$

For (35) we write

$$\begin{aligned}
\|g(x_j - x_k) p_j\|_{\text{op}}^2 = &\sup_{\|\Psi\|=1} \|g(x_j - x_k) p_j \Psi\|^2 = \\
= &\sup_{\|\Psi\|=1} \langle\!\langle \Psi, p_j |g(x_j - x_k)|^2 p_j \Psi \rangle\!\rangle \\
\leq &\|p_j |g(x_j - x_k)|^2 p_j\|_{\text{op}} \; .
\end{aligned}$$

With (34) we get (35). For (36) we use

$$\begin{aligned}
\|g(x_j - x_k) \nabla_j p_j\|_{\text{op}}^2 = &\sup_{\|\Psi\|=1} \langle\!\langle \Psi, p_j (|g|^2 * |\nabla\varphi|^2)(x_k) \Psi \rangle\!\rangle \leq \||g|^2 * |\nabla\varphi|^2\|_\infty \\
\leq &\|g\|^2 \|\nabla\varphi\|_\infty^2
\end{aligned}$$

□

We further need the following

Lemma 3.13 *Let $\Omega, \chi \in L^2(\mathbb{R}^{2N}, \mathbb{C})$ such that Ω and χ are symmetric w.r.t. to the exchange of the variables $x_2, \ldots, x_N$. Let $O_{i,j}$ be an operator acting on the i^{th} and j^{th} coordinate. Then*

$$|\langle\!\langle \Omega, O_{1,2}\chi \rangle\!\rangle| \leq \|\Omega\|^2 + |\langle\!\langle O_{1,2}\chi, O_{1,3}\chi \rangle\!\rangle| + \frac{1}{N-1} \|O_{1,2}\chi\|^2 \; . \tag{38}$$

Proof: Using symmetry and Cauchy Schwarz

$$\begin{aligned}
|\langle\!\langle \Omega, O_{1,2}\chi \rangle\!\rangle| = &\frac{1}{N-1} |\langle\!\langle \Omega, \sum_{j=2}^N O_{1,j}\chi \rangle\!\rangle| \leq \frac{1}{N-1} \|\Omega\| \left\| \sum_{j=2}^N O_{1,j}\chi \right\| \\
\leq &\|\Omega\|^2 + \frac{1}{(N-1)^2} \left\| \sum_{j=2}^N O_{1,j}\chi \right\|^2 \; .
\end{aligned}$$

Note that

$$\begin{aligned}
\left\| \sum_{j=2}^N O_{1,j}\chi \right\|^2 = &\langle\!\langle \sum_{j=2}^N O_{1,j}\chi, \sum_{k=2}^N O_{1,k}\chi \rangle\!\rangle \\
\leq &\sum_{j=2}^N |\langle\!\langle O_{1,j}\chi, O_{1,j}\chi \rangle\!\rangle| + |\sum_{j\neq k, j,k=2}^N \langle\!\langle O_{1,j}\chi, O_{1,k}\chi \rangle\!\rangle| \\
\leq &(N-1) |\langle\!\langle O_{1,2}\chi, O_{1,2}\chi \rangle\!\rangle| + (N-1)(N-2) |\langle\!\langle O_{1,2}\chi, O_{1,3}\chi \rangle\!\rangle| \; .
\end{aligned}$$

□

We now prove Lemma 3.8.

Lemma 3.14 *Let* $\Psi \in L^2_s(\mathbb{R}^{2N},\mathbb{C}) \cap \mathcal{D}(H_{W_\beta,\cdot}), \|\Psi\| = 1$ *and let* $\varphi \in L^2(\mathbb{R}^2,\mathbb{C}),\ \|\varphi\| = 1$. *Assume (A1), (A2) and (A5). Then,*

(a)

$$N\left|\langle\!\langle \Psi p_1 p_2 Z^{\varphi}_{\beta}(x_1,x_2) q_1 p_2 \Psi\rangle\!\rangle\right| \leq \mathcal{K}(\varphi)\left(N^{-1} + N^{-2\beta}\ln(N)\right) . \tag{39}$$

(b)

$$N|\langle\!\langle \Psi, p_1 p_2 Z^{\varphi}_{\beta}(x_1,x_2) q_1 q_2 \Psi\rangle\!\rangle| \leq \mathcal{K}(\varphi)\left(\langle\!\langle \Psi, q_1\Psi\rangle\!\rangle + N^{-1/3}\ln(N)^2\right) . \tag{40}$$

(c)

$$N\left|\langle\!\langle \Psi, q_1 p_2 Z^{\varphi}_{\beta}(x_1,x_2) q_1 q_2 \Psi\rangle\!\rangle\right| \leq \mathcal{K}(\varphi)\ln(N)^{1/2}\left(\alpha(\Psi,\varphi) + N^{-1/2}\right) . \tag{41}$$

(d) Let φ_t *the solution to* $i\partial_t\varphi_t = h^{NLS}_{a,t}\varphi_t,\ \|\varphi_0\| = 1$. *Let* Ψ_t *the solution to* $i\partial\Psi_t = H_{W_\beta,t}\Psi_t$ *with* $\Psi_0 \in L^2_s(\mathbb{R}^{2N},\mathbb{C}) \cap \mathcal{D}((H_{W_\beta,0})^2), \|\Psi_0\| = 1$ *Then,*

$$\left|\frac{d}{dt} Var_{H_{W_\beta,t}}(\Psi_t)\right| \leq \mathcal{K}(\varphi_t)\left(\alpha(\Psi_t,\varphi_t) + N^{-1}\right) . \tag{42}$$

Remark 3.15 *(a) and (b) have essentially been proven in [18] for a slightly different definition of* $\alpha(\Psi,\varphi)$. *It is (c) where our old estimates fail. In [18] we we able to control (c) by estimating* $\|\nabla_1 q_1\Psi\|$ *in terms of* $\alpha(\Psi,\varphi)$ *and some small error. In this estimate, it was crucial that* $(1-p_1p_2)W_\beta(x_1-x_2)(1-p_1p_2) \geq 0$ *holds as an operator inequality and therefore forces* W_β *to be nonnegative (cf. the definition of* $Q_\beta(\Psi,\varphi)$ *below Equation (100) in [18]). In this paper, we make use of a strategy which was developed in [23] to derive the Maxwell-Schrödinger equations from the Pauli-Fierz Hamiltonian. Instead of estimating* $\|\nabla_1 q_1\Psi\|$, *we now control* $\|\nabla_1 q_2\Psi\|$, *see Lemma 3.9.*

Proof:

(a) We estimate

$$N\left|\langle\!\langle \Psi, p_1 p_2 Z^{\varphi}_{\beta}(x_1,x_2) q_1 p_2 \Psi\rangle\!\rangle\right| \leq N\|p_1p_2 Z^{\varphi}_{\beta}(x_1,x_2) q_1 p_2\|_{\text{op}} .$$

$\|p_1p_2 Z^{\varphi}_{\beta}(x_1,x_2) q_1 p_2\|_{\text{op}}$ can be estimated using $p_1q_1 = 0$ and (37).

$$\begin{aligned}
&N\left\|p_1p_2\left(W_\beta(x_1-x_2) - \frac{a}{N-1}|\varphi(x_1)|^2 - \frac{a}{N-1}|\varphi(x_2)|^2\right) q_1p_2\right\|_{\text{op}} \\
&\leq \|p_1p_2(NW_\beta(x_1-x_2) - a|\varphi(x_1)|^2)p_2\|_{\text{op}} + C\|\varphi\|^2_\infty N^{-1} \\
&\leq \|\varphi\|_\infty\ \|N(W_\beta \star |\varphi|^2) - a|\varphi|^2\| + C\|\varphi\|^2_\infty N^{-1} .
\end{aligned}$$

Let h be given by

$$h(x) = \frac{1}{2\pi}\int_{\mathbb{R}^2} d^2y \ln|x-y| N W_\beta(y) - \frac{a}{2\pi}\ln|x| \; .$$

It then follows

$$\Delta h(x) = N W_\beta(x) - a\delta(x)$$

in the sense of distributions. Since $a = \int_{\mathbb{R}^2} d^2x W(x)$, this implies (see Lemma 3.11), $h(x) = 0$ for $x \notin B_{RN^{-\beta}}(0)$, where $RN^{-\beta}$ is the radius of the support of W_β. Thus,

$$\begin{aligned}\|h\|_1 \leq & \frac{1}{2\pi}\int_{\mathbb{R}^2} d^2x \int_{\mathbb{R}^2} d^2y |\ln|x-y|| \mathbb{1}_{B_{RN^{-\beta}}(0)}(x) N W_\beta(y) \\ & + \frac{|a|}{2\pi}\int_{\mathbb{R}^2} d^2x \ln(|x|)\mathbb{1}_{B_{RN^{-\beta}}(0)}(x) \\ & \leq C N^{-2\beta}\ln(N) \; .\end{aligned}$$

Integration by parts and Young's inequality then imply

$$\begin{aligned}& \|N(W_\beta \star |\varphi|^2) - a|\varphi|^2\| = \|(\Delta h)\star|\varphi|^2\| \\ \leq & \|h\|_1 \|\Delta|\varphi|^2\| \leq \mathcal{K}(\varphi) N^{-2\beta}\ln(N) \; .\end{aligned} \tag{43}$$

Thus, we obtain the bound

$$N\left|\langle\!\langle \Psi, p_1p_2 Z^{\varphi}_{\beta}(x_1,x_2) q_1 p_2 \widehat{w}\Psi\rangle\!\rangle\right| \leq \mathcal{K}(\varphi)\left(N^{-1} + N^{-2\beta}\ln(N)\right) \; , \tag{44}$$

which then proves (a).

(b) We will first consider the case $0 < \beta \leq 1/3$. Note that $p_1p_2 Z^{\varphi}_{\beta}(x_1,x_2)q_1q_2 = p_1p_2 W_\beta(x_1 - x_2)q_1q_2$. We estimate

$$\begin{aligned} N|\langle\!\langle \Psi, q_1q_2 W_\beta(x_1-x_2)p_1p_2\Psi\rangle\!\rangle| = & \frac{N}{N-1}|\langle\!\langle q_1\Psi, \sum_{k=2}^{N} q_k W_\beta(x_1-x_k)p_1p_k\Psi\rangle\!\rangle| \\ \leq & \frac{N}{N-1}\|q_1\Psi\| \left\|\sum_{k=2}^{N} q_k W_\beta(x_1-x_k)p_1p_k\Psi\right\| \\ \leq & \langle\!\langle \Psi, q_1\Psi\rangle\!\rangle + \left\|\sum_{k=2}^{N} q_k W_\beta(x_1-x_k)p_1p_k\Psi\right\|^2 \\ = & \langle\!\langle \Psi, q_1\Psi\rangle\!\rangle + (N-1)\left\|q_2 W_\beta(x_1-x_2)p_1p_2\Psi\right\|^2 \\ & + (N-1)(N-2)\langle\!\langle \Psi, p_1p_2 W_\beta(x_1-x_2)q_2q_3 W_\beta(x_1-x_3)p_1p_3\Psi\rangle\!\rangle \\ \leq & \langle\!\langle \Psi, q_1\Psi\rangle\!\rangle + N\|W_\beta\|^2\|\varphi\|_\infty^2 \\ & + N^2\|p_2 W_\beta(x_1-x_2)p_1\|_{\mathrm{op}}^2 \|q_1\Psi\|^2 \\ \leq & \mathcal{K}(\varphi)\left(\langle\!\langle \Psi, q_1\Psi\rangle\!\rangle + N^{-1+2\beta}\right) \; .\end{aligned}$$

In the last estimate, we used Lemma 3.11 together with Lemma 3.12 to estimate $\|p_1 W_\beta(x_1-x_2)p_2\|_{\mathrm{op}} \leq \|p_1\sqrt{|W_\beta|}\|_{\mathrm{op}}^2 \leq \|\varphi\|_\infty^2 \|\sqrt{|W_\beta|}\|^2 \leq \mathcal{K}(\varphi)N^{-1}$.

This proves (b) for the case $\beta \leq 1/3$.

(b) for $1/3 < \beta$: We use U_{β_1} from Definition 3.10 for some $0 < \beta_1 \leq 1/3$. We then obtain

$$\begin{aligned} &N\langle\!\langle \Psi, p_1 p_2 W_\beta(x_1-x_2) q_1 q_2 \Psi \rangle\!\rangle \\ =&N\langle\!\langle \Psi, p_1 p_2 U_{\beta_1}(x_1-x_2) q_1 q_2 \Psi \rangle\!\rangle \qquad (45)\\ +&N\langle\!\langle \Psi, p_1 p_2 \left(W_\beta(x_1-x_2) - U_{\beta_1}(x_1-x_2)\right) q_1 q_2 \Psi \rangle\!\rangle . \qquad (46) \end{aligned}$$

Term (45) has been controlled above. So we are left to control (46).

Let $\Delta h_{\beta_1,\beta} = W_\beta - U_{\beta_1}$, as in Lemma 3.11. Integrating by parts and using $\nabla_1 h_{\beta_1,\beta}(x_1 - x_2) = -\nabla_2 h_{\beta_1,\beta}(x_1-x_2)$ gives

$$\begin{aligned} &N\left|\langle\!\langle \Psi, p_1 p_2 \left(W_\beta(x_1-x_2) - U_{\beta_1,\beta}(x_1-x_2)\right) q_1 q_2 \Psi \rangle\!\rangle\right| \\ &\leq N\left|\langle\!\langle \nabla_1 p_1 \Psi, p_2 \nabla_2 h_{\beta_1,\beta}(x_1-x_2) q_1 q_2 \Psi \rangle\!\rangle\right| \qquad (47)\\ &+ N\left|\langle\!\langle \Psi, p_1 p_2 \nabla_2 h_{\beta_1,\beta}(x_1-x_2) \nabla_1 q_1 q_2 \Psi \rangle\!\rangle\right| . \qquad (48) \end{aligned}$$

Let $t_1 \in \{p_1, \nabla_1 p_1\}$ and let $\Gamma \in \{q_1\Psi, \nabla_1 q_1 \Psi\}$.

For both (47) and (48), we use Lemma 3.13 with $O_{1,2} = N^{1+\eta/2} q_2 \nabla_2 h_{\beta_1,\beta}(x_1-x_2) p_2$, $\chi = t_1\Psi$ and $\Omega = N^{-\eta/2}\Gamma$. This yields

$$\begin{aligned} (47) + (48) \leq 2 &\sup_{t_1 \in \{p_1, \nabla_1 p_1\}, \Gamma \in \{q_1\Psi, \nabla_1 q_1 \Psi\}} \Big(N^{-\eta} \|\Gamma\|^2 \qquad (49)\\ &+ \frac{N^{2+\eta}}{N-1} \|q_2 \nabla_2 h_{\beta_1,\beta}(x_1-x_2) t_1 p_2 \Psi\|^2 \qquad (50)\\ &+ N^{2+\eta} \left|\langle\!\langle \Psi, t_1 p_2 q_3 \nabla_2 h_{\beta_1,\beta}(x_1-x_2) \nabla_3 h_{\beta_1,\beta}(x_1-x_3) t_1 q_2 p_3 \Psi \rangle\!\rangle\right| \Big) . \qquad (51) \end{aligned}$$

The first term can be bounded using $\|\nabla_1 q_1 \Psi\| \leq \mathcal{K}(\varphi)$.

Using $\|t_1\Psi\|^2 \leq \mathcal{K}(\varphi)$, we obtain

$$\begin{aligned} (50) \leq& \mathcal{K}(\varphi) \frac{N^{2+\eta}}{N-1} \|\nabla_2 h_{\beta_1,\beta}(x_1-x_2) p_2\|_{\text{op}}^2 \leq \mathcal{K}(\varphi) \frac{N^{2+\eta}}{N-1} \|\varphi\|_\infty^2 \|\nabla h_{\beta_1,\beta}\|^2 \\ \leq& \mathcal{K}(\varphi) N^{\eta-1} \ln(N) , \end{aligned}$$

where we used Lemma 3.11 in the last step.

Next, we estimate

$$\begin{aligned} (51) \leq& N^{2+\eta} \|p_2 \nabla_2 h_{\beta_1,\beta}(x_1-x_2) t_1 q_2 \Psi\|^2 \\ \leq& 2N^{2+\eta} \|p_2 h_{\beta_1,\beta}(x_1-x_2) t_1 \nabla_2 q_2 \Psi\|^2 \\ &+ 2N^{2+\eta} \||\varphi(x_2)\rangle\langle\nabla\varphi(x_2)| h_{\beta_1,\beta}(x_1-x_2) t_1 q_2 \Psi\|^2 \\ \leq& 2N^{2+\eta} \|p_2 h_{\beta_1,\beta}(x_1-x_2)\|_{\text{op}}^2 \|t_1 \nabla_2 q_2 \Psi\|^2 \\ &+ 2N^{2+\eta} \||\varphi(x_2)\rangle\langle\nabla\varphi(x_2)| h_{\beta_1,\beta}(x_1-x_2)\|_{\text{op}}^2 \|t_1 q_2 \Psi\|^2 \\ \leq& \mathcal{K}(\varphi) N^{2+\eta} \|h_{\beta_1,\beta}\|^2 \leq \mathcal{K}(\varphi) N^{\eta - 2\beta_1} \ln(N)^2 . \end{aligned}$$

Thus, for all $\eta \in \mathbb{R}$

$$\begin{aligned} &N\langle\!\langle \Psi, p_1 p_2 \left(W_\beta(x_1-x_2) - U_{\beta_1,\beta}(x_1-x_2)\right) q_1 q_2 \Psi \rangle\!\rangle \\ \leq& \mathcal{K}(\varphi, A_t) \left(N^{-\eta} + N^{\eta-1} \ln(N) + N^{\eta-2\beta_1} \ln(N)^2 \right) . \end{aligned}$$

Hence, we obtain, using $N^{\eta-1}\ln(N) < N^{\eta-2\beta_1}\ln(N)$,

$$\begin{aligned}&N\langle\!\langle\Psi, p_1p_2W_\beta(x_1-x_2)q_1q_2\Psi\rangle\!\rangle\\ &\leq \mathcal{K}(\varphi, A_t)\left(\langle\!\langle\Psi, q_1\Psi\rangle\!\rangle + \inf_{\eta>0}\inf_{\frac{1}{3}\geq\beta_1>0}\left(N^{\eta-2\beta_1}\ln(N)^2 + N^{-1+2\beta_1} + N^{-\eta}\right)\right) .\end{aligned}$$

and we get (b) in full generality by choosing $\eta = \beta_1 = 1/3$.

(c) First note that

$$N\left|\langle\!\langle\Psi, q_1p_2\frac{a}{N-1}|\varphi(x_1)|^2q_1q_2\Psi\rangle\!\rangle\right| \leq C\|\varphi\|_\infty^2\langle\!\langle\Psi, q_1\Psi\rangle\!\rangle .$$

Let U_0 be given by Definition 3.10. Using Lemma 3.12 and integrating by parts we get

$$\begin{aligned}&N\,|\langle\!\langle\Psi, q_1p_2W_\beta(x_1-x_2)q_1q_2\Psi\rangle\!\rangle|\\ \leq&N\,|\langle\!\langle\Psi, q_1p_2U_0(x_1-x_2)q_1q_2\Psi\rangle\!\rangle| + N\,|\langle\!\langle\Psi, q_1p_2(\Delta_1h_{0,\beta}(x_1-x_2))q_1q_2\Psi\rangle\!\rangle|\\ \leq&N\|q_1\Psi\|\,\|U_0\|_\infty\|q_1q_2\Psi\| \qquad (52)\\ +&N\,|\langle\!\langle\nabla_2p_2q_1\Psi, (\nabla_2h_{0,\beta}(x_1-x_2))q_1q_2\Psi\rangle\!\rangle| \qquad (53)\\ +&N\,|\langle\!\langle\Psi, q_1p_2(\nabla_2h_{0,\beta}(x_1-x_2))\nabla_2q_1q_2\Psi\rangle\!\rangle| . \qquad (54)\end{aligned}$$

The first contribution is bounded by

$$(52) \leq C\langle\!\langle\Psi, q_1\Psi\rangle\!\rangle .$$

The second term (53) can be estimated as

$$\begin{aligned}(53) =&N\,|\langle\!\langle\Delta_2p_2q_1\Psi, h_{0,\beta}(x_1-x_2)q_1q_2\Psi\rangle\!\rangle| \qquad (55)\\ +&N\,|\langle\!\langle\nabla_2p_2q_1\Psi, h_{0,\beta}(x_1-x_2)q_1\nabla_2\Psi\rangle\!\rangle| \qquad (56)\\ +&N\,|\langle\!\langle\nabla_2p_2q_1\Psi, h_{0,\beta}(x_1-x_2)q_1\nabla_2p_2\Psi\rangle\!\rangle| . \qquad (57)\end{aligned}$$

The last contribution (54) can be rewritten as

$$\begin{aligned}(54) \leq&N\,|\langle\!\langle\Psi, q_1p_2(\nabla_2h_{0,\beta}(x_1-x_2))\nabla_2q_1\Psi\rangle\!\rangle| \qquad (58)\\ +&N\,|\langle\!\langle\Psi, q_1p_2(\nabla_2h_{0,\beta}(x_1-x_2))\nabla_2p_2q_1\Psi\rangle\!\rangle| . \qquad (59)\end{aligned}$$

We estimate each contribution separately, using Lemma 3.11 together with Lemma 3.12. We obtain

$$\begin{aligned}(55) \leq&N\|q_1\Psi\|\|h_{0,\beta}(x_1-x_2)\Delta_2p_2\|_{\text{op}}\|q_1q_2\Psi\|\\ \leq&C\|\Delta\varphi\|_\infty\langle\!\langle\Psi, q_1\Psi\rangle\!\rangle.\end{aligned}$$

Analogously,

$$\begin{aligned}(57) \leq&N\|q_1\Psi\|\|h_{0,\beta}(x_1-x_2)\nabla_2p_2\|_{\text{op}}\|\nabla\varphi\|_\infty\|q_1\Psi\|\\ \leq&C\|\nabla\varphi\|_\infty^2\langle\!\langle\Psi, q_1\Psi\rangle\!\rangle.\end{aligned}$$

Next, we control

$$\begin{aligned}(59)\ \leq & N\|q_1\Psi\|\|p_2\nabla_2 h_{0,\beta}(x_1-x_2)\nabla_2 p_2\|_{\mathrm{op}}\|q_1\Psi\| \\ \leq & C\|\nabla\varphi\|_\infty\|\varphi\|_\infty N\|\nabla h_{0,\beta}\|_1\|q_1\Psi\|^2 \\ \leq & C\|\nabla\varphi\|_\infty\|\varphi\|_\infty\langle\!\langle\Psi, q_1\Psi\rangle\!\rangle.\end{aligned}$$

To control (56) and (58), we estimate for $t_2 \in \{p_2, |\varphi(x_2)\rangle\langle(\nabla\varphi)(x_2)|\}$ and $s \in \{h_{0,\beta}, \nabla_2 h_{0,\beta}\}$

$$\begin{aligned}& N\,|\langle\!\langle q_1\Psi, t_2 s(x_1-x_2)\nabla_2 q_1\Psi\rangle\!\rangle| \\ \leq & \ln(N)^{1/2}\|q_1\Psi\|^2 + \ln(N)^{-1/2}N^2\|t_2 s(x_1-x_2)\|_{\mathrm{op}}^2\|\nabla_2 q_1\Psi\|^2 \\ \leq & \mathcal{K}(\varphi)\ln(N)^{1/2}\left(\mathrm{Var}_{H_{W_\beta,\cdot}}(\Psi) + \langle\!\langle\Psi, q_1\Psi\rangle\!\rangle + N^{-1/2}\right).\end{aligned}$$

In the last line, we used Lemma 3.9 (b) together with Lemma 3.11.

(d) We estimate

$$\begin{aligned}& \left|\frac{d}{dt}\mathrm{Var}_{H_{W_\beta,t}}(\Psi_t)\right| = N^{-2}\left|\frac{d}{dt}\langle\!\langle\Psi_t, \left(H_{W_\beta,t} - \langle\!\langle\Psi_t, H_{W_\beta,t}\Psi_t\rangle\!\rangle\right)^2\Psi_t\rangle\!\rangle\right| \\ \leq & 2N^{-1}\left|\langle\!\langle\Psi_t, \left(H_{W_\beta,t} - \langle\!\langle\Psi_t, H_{W_\beta,t}\Psi_t\rangle\!\rangle\right)\left(\dot{A}_t(x_1) - \langle\!\langle\Psi_t, \dot{A}_t(x_1)\Psi_t\rangle\!\rangle\right)\Psi_t\rangle\!\rangle\right| \\ \leq & 2N^{-1}\left|\langle\!\langle\Psi_t, \left(H_{W_\beta,t} - \langle\!\langle\Psi_t, H_{W_\beta,t}\Psi_t\rangle\!\rangle\right)\left(p_1\dot{A}_t(x_1)p_1 - \langle\!\langle\Psi_t, p_1\dot{A}_t(x_1)p_1\Psi_t\rangle\!\rangle\right)\Psi_t\rangle\!\rangle\right| \\ + & 2N^{-1}\left|\langle\!\langle\Psi_t, \left(H_{W_\beta,t} - \langle\!\langle\Psi_t, H_{W_\beta,t}\Psi_t\rangle\!\rangle\right)\left(p_1\dot{A}_t(x_1)q_1 - \langle\!\langle\Psi_t, p_1\dot{A}_t(x_1)q_1\Psi_t\rangle\!\rangle\right)\Psi_t\rangle\!\rangle\right| \\ + & 2N^{-1}\left|\langle\!\langle\Psi_t, \left(H_{W_\beta,t} - \langle\!\langle\Psi_t, H_{W_\beta,t}\Psi_t\rangle\!\rangle\right)\left(N^{-1}\sum_{k=1}^N q_k\dot{A}_t(x_k)p_k - \langle\!\langle\Psi_t, q_1\dot{A}_t(x_1)p_1\Psi_t\rangle\!\rangle\right)\Psi_t\rangle\!\rangle\right| \\ + & 2N^{-1}\left|\langle\!\langle\Psi_t, \left(H_{W_\beta,t} - \langle\!\langle\Psi_t, H_{W_\beta,t}\Psi_t\rangle\!\rangle\right)\left(q_1\dot{A}_t(x_k)q_1 - \langle\!\langle\Psi_t, q_1\dot{A}_t(x_k)q_1\Psi_t\rangle\!\rangle\right)\Psi_t\rangle\!\rangle\right| \\ \leq & 4\mathrm{Var}_{H_{W_\beta,t}}(\Psi_t) + \left\|\left(p_1\dot{A}_t(x_1)p_1 - \langle\!\langle\Psi_t, p_1\dot{A}_t(x_1)p_1\Psi_t\rangle\!\rangle\right)\Psi_t\right\|^2 \\ + & \left\|\left(p_1\dot{A}_t(x_1)q_1 - \langle\!\langle\Psi_t, p_1\dot{A}_t(x_1)q_1\Psi_t\rangle\!\rangle\right)\Psi_t\right\|^2 \\ + & \left\|\left(N^{-1}\sum_{k=1}^N q_k\dot{A}_t(x_k)p_k - \langle\!\langle\Psi_t, q_1\dot{A}_t(x_1)p_1\Psi_t\rangle\!\rangle\right)\Psi_t\right\|^2 \\ + & \left\|\left(q_1\dot{A}_t(x_1)q_1 - \langle\!\langle\Psi_t, q_1\dot{A}_t(x_1)q_1\Psi_t\rangle\!\rangle\right)\Psi_t\right\|^2\end{aligned}$$

Note that

$$\begin{aligned}& \left\|\left(p_1\dot{A}_t(x_1)p_1 - \langle\!\langle\Psi_t, p_1\dot{A}_t(x_1)p_1\Psi_t\rangle\!\rangle\right)\Psi_t\right\|^2 \\ = & \left(\int_{\mathbb{R}^2} d^2x\,\dot{A}_t(x)|\varphi_t(x)|^2\right)^2\langle\!\langle\Psi_t, p_1\Psi_t\rangle\!\rangle\left(1 - \langle\!\langle\Psi_t, p_1\Psi_t\rangle\!\rangle\right) \\ \leq & \mathcal{K}(\varphi_t)\langle\!\langle\Psi_t, q_1\Psi_t\rangle\!\rangle\ .\end{aligned}$$

Furthermore

$$
\begin{aligned}
&\left\| \left(N^{-1} \sum_{k=1}^{N} q_k \dot{A}_t(x_k) p_k - \langle\!\langle \Psi_t, q_1 \dot{A}_t(x_1) p_1 \Psi_t \rangle\!\rangle \right) \Psi_t \right\|^2 \\
=& N^{-2} \sum_{k,l=1}^{N} \langle\!\langle \Psi_t, p_l \dot{A}_t(x_l) q_l q_k \dot{A}_t(x_k) p_k \Psi_t \rangle\!\rangle - \left| \langle\!\langle \Psi_t, q_1 \dot{A}_t(x_1) p_1 \Psi_t \rangle\!\rangle \right|^2 \\
\leq& \langle\!\langle \Psi_t, p_1 \dot{A}_t(x_1) q_1 q_2 \dot{A}_t(x_2) p_2 \Psi_t \rangle\!\rangle + \frac{1}{N} \langle\!\langle \Psi_t, p_1 \dot{A}_t(x_1) q_1 \dot{A}_t(x_1) p_1 \Psi_t \rangle\!\rangle + \|\dot{A}_t\|_\infty \langle\!\langle \Psi_t, q_1 \Psi_t \rangle\!\rangle \\
\leq& \mathcal{K}(\varphi_t) \left(\langle\!\langle \Psi_t, q_1 \Psi_t \rangle\!\rangle + \frac{1}{N} \right) .
\end{aligned}
$$

To control the two remaining terms, let $s_1 \in \{p_1, q_1\}$. Then, we need to estimate

$$
\begin{aligned}
&\left\| \left(s_1 \dot{A}_t(x_1) q_1 - \langle\!\langle \Psi_t, s_1 \dot{A}_t(x_1) q_1 \Psi_t \rangle\!\rangle \right) \Psi_t \right\|^2 \\
=& \langle\!\langle \Psi_t, q_1 \dot{A}_t(x_1) s_1 \dot{A}_t(x_1) q_1 \Psi_t \rangle\!\rangle - \left| \langle\!\langle \Psi_t, s_1 \dot{A}_t(x_1) q_1 \Psi_t \rangle\!\rangle \right|^2 \\
\leq& 2 \|\dot{A}_t\|_\infty^2 \langle\!\langle \Psi_t, q_1 \Psi_t \rangle\!\rangle .
\end{aligned}
$$

In total, we obtain

$$
\left| \frac{d}{dt} \mathrm{Var}_{H_{W_\beta,t}}(\Psi_t) \right| \leq \mathcal{K}(\varphi_t) \left(\alpha(\Psi_t, \varphi_t) + N^{-1} \right) .
$$

Combining the estimates (a)-(d), Lemma 3.8 is then proven.

□

4 Proof of Theorem 2.2 (b)

Proof: We make use of the inequality

$$
\mathrm{Tr} \left| \sqrt{1-\Delta} (\gamma^{(1)}_{\Psi_t} - |\varphi_t\rangle\langle\varphi_t|) \sqrt{1-\Delta} \right| \leq C(1 + \|\nabla_1 \varphi_t\|)^2 (\|q_1 \Psi_t\| + \|q_1 \Psi_t\|^2 + \|\nabla_1 q_1 \Psi_t\| + \|\nabla_1 q_1 \Psi_t\|^2),
$$

which was proven in [34], see also [1]. Using Theorem 2.2 (a), we are left to show $\lim_{N\to\infty} \|\nabla_1 q_1 \Psi_t\| = 0$. In general, this does not follow from $\lim_{N\to\infty} \|\nabla_1 q_2 \Psi_t\| = 0$. To see this, consider the symmetrized wave-function

$$
\Gamma(x_1, \ldots, x_N) = \frac{1}{\sqrt{N}} \sum_{k=1}^{N} \varphi(x_1) \ldots \eta(x_k) \ldots \varphi(x_N)
$$

for $\eta, \varphi \in L^2(\mathbb{R}^2, \mathbb{C})$, $\|\eta\| = \|\varphi\| = 1$, $\langle \eta, \varphi \rangle = 0$. Then

$$
\|q_1^\varphi \Gamma\|^2 = N^{-1}, \qquad \|\nabla_1 q_2^\varphi \Gamma\|^2 = N^{-1} \|\nabla \varphi\|^2, \qquad \|\nabla_1 q_1^\varphi \Gamma\|^2 = N^{-1} \|\nabla \eta\|^2 .
$$

Note that $\|\nabla \eta\|$ can be chosen arbitrarily. However, for $A_t \in L^p(\mathbb{R}^2, \mathbb{R})$, with $p > 2$, it is possible to control $\|\nabla_1 q_1 \Psi_t\|$ in terms of $\|q_1 \Psi_t\|$, $\|\nabla_1 q_2 \Psi_t\|$ and the energy difference

$\left|N^{-1}\langle\!\langle\Psi_0, H_{W_\beta,0}\Psi_0\rangle\!\rangle - \langle\varphi_0, \left(-\Delta + \frac{a}{2}|\varphi_0|^2 + A_0\right)\varphi_0\rangle\right|$, assuming conditions (A2) and (A4). Together with assumptions (A1), (A3), (A5) and (A5)' and Theorem 2.2, part (a), it is then possible to bound $\|\nabla_1 q_1 \Psi_t\|$ sufficiently well. First, we consider

$$\left|\|\nabla_1\Psi_t\|^2 - \|\nabla\varphi_t\|^2\right| \leq \left|\frac{1}{N}\langle\!\langle\Psi_0, H_{W_\beta,0}\Psi_0\rangle\!\rangle - \langle\varphi_0, \left(-\Delta + \frac{a}{2}|\varphi_0|^2 + A_0\right)\varphi_0\rangle\right| \tag{60}$$

$$+ \int_0^t ds \left|\langle\!\langle\Psi_s, \dot{A}_s(x_1)\Psi_s\rangle\!\rangle - \langle\varphi_s, \dot{A}_s\varphi_s\rangle\right| \tag{61}$$

$$+\frac{1}{2}\left|\langle\!\langle\Psi_t, p_1p_2(N-1)W_\beta(x_1-x_2)p_1p_2\Psi_t\rangle\!\rangle - a\langle\varphi_t, |\varphi_t|^2\varphi_t\rangle\right| \tag{62}$$

$$+N\left|\langle\!\langle\Psi_t, p_1p_2W_\beta(x_1-x_2)(1-p_1p_2)\Psi_t\rangle\!\rangle\right| \tag{63}$$

$$+N\left|\langle\!\langle\Psi_t, (1-p_1p_2)W_\beta(x_1-x_2)(1-p_1p_2)\Psi_t\rangle\!\rangle\right| \tag{64}$$

$$+\left|\langle\!\langle\Psi_t, A_t(x_1)\Psi_t\rangle\!\rangle - \langle\varphi_t, A_t\varphi_t\rangle\right|. \tag{65}$$

We estimate each line separately. From condition (A5)', it follows that (60) $\leq CN^{-\delta}$. Using $\dot{A}_t \in L^\infty(\mathbb{R}^2, \mathbb{R})$, we estimate

$$\begin{aligned}(61) \leq& t \sup_{s\in[0,t]} \left(|\langle\varphi_s, \dot{A}_s\varphi_s\rangle|\langle\!\langle\Psi_s, q_1^{\varphi_s}\Psi_s\rangle\!\rangle + 2\left|\langle\!\langle\Psi_s, p_1^{\varphi_s}\dot{A}_s(x_1)q_1^{\varphi_s}\Psi_s\rangle\!\rangle\right| + \left|\langle\!\langle\Psi_s, q_1^{\varphi_s}\dot{A}_s(x_1)q_1^{\varphi_s}\Psi_s\rangle\!\rangle\right|\right)\\ \leq& t \sup_{s\in[0,t]} \left(\|\dot{A}_s\|_\infty(\|q_1^{\varphi_s}\Psi_s\| + \|q_1^{\varphi_s}\Psi_s\|^2)\right).\end{aligned}$$

Next,

$$\begin{aligned}(62) \leq& \left|\langle\varphi_t, (N-1)W_\beta \star |\varphi_t|^2\varphi_t\rangle\langle\!\langle\Psi_t, p_1p_2\Psi_t\rangle\!\rangle - a\langle\varphi_t, |\varphi_t|^2\varphi_t\rangle\right|\\ \leq& \mathcal{K}(\varphi_t)\|NW_\beta\|_1\|q_1\Psi_t\|^2 + \left|\langle\varphi_t, ((N-1)W_\beta \star |\varphi_t|^2 - a|\varphi_t|^2)\varphi_t\rangle\right|\\ \leq& \mathcal{K}(\varphi_t)\left(\langle\!\langle\Psi_t, q_1\Psi_t\rangle\!\rangle + N^{-2\beta}\ln(N) + N^{-1}\right).\end{aligned}$$

Note that

$$\begin{aligned}(63) + (64) \leq& C\left\|\sqrt{N|W_\beta(x_1-x_2)|}p_1p_2\Psi_t\right\|\left(\left\|\sqrt{N|W_\beta(x_1-x_2)|}q_1q_2\Psi_t\right\| + \left\|\sqrt{N|W_\beta(x_1-x_2)|}q_1p_2\Psi_t\right\|\right)\\ &+ C\left\|\sqrt{N|W_\beta(x_1-x_2)|}p_1q_2\Psi_t\right\|^2 + C\left\|\sqrt{N|W_\beta(x_1-x_2)|}q_1q_2\Psi_t\right\|^2.\end{aligned}$$

Using Lemma 3.9 (c), we obtain

$$\left\|\sqrt{|NW_\beta(x_1-x_2)|}q_1q_2\Psi_t\right\|^2 \leq \mathcal{K}(\varphi_t)C_pN^{\beta/p}\left(\alpha(\Psi_t,\varphi_t) + N^{-1/2}\right).$$

Furthermore,

$$\left\|\sqrt{N|W_\beta(x_1-x_2)|}p_1\right\|_{\text{op}} \leq \mathcal{K}(\varphi_t).$$

Note that it was shown in part (a) that $\alpha(\Psi_t,\varphi_t)+ \leq \mathcal{K}(\varphi_t)N^{-\delta}$ for some $\delta > 0$. Choosing $p \in \mathbb{N}$ large enough, we then obtain (63) + (64) $\leq \mathcal{K}(\varphi_t)N^{-\gamma}$, for some $\gamma > 0$. We estimate, using $|\langle\langle \Psi_t, q_1 A_t(x_1) q_1 \Psi_t \rangle\rangle| \leq \|q_1\Psi_t\|(\|A_t\varphi_t\| + \|A_t(x_1)\Psi_t\|)$,

$$\begin{aligned}|(65)| \leq& |\langle \varphi_t, A_t\varphi_t\rangle| \langle\langle \Psi_t, q_1\Psi_t\rangle\rangle + 2\,|\langle\langle \Psi_t, p_1 A_t(x_1) q_1\Psi_t\rangle\rangle| + |\langle\langle \Psi_t, q_1 A_t(x_1) q_1 \Psi_t\rangle\rangle| \\ \leq& |\langle \varphi_t, A_t\varphi_t\rangle| \langle\langle \Psi_t, q_1\Psi_t\rangle\rangle + \|q_1\Psi_t\|(\|A_t\varphi_t\| + \|A_t(x_1)\Psi_t\|).\end{aligned}$$

If $A_t \in L^\infty(\mathbb{R}^2, \mathbb{R})$ holds, we obtain

$$|(65)| \leq \mathcal{K}(\varphi_t)(\|q_1\Psi_t\| + \|q_1\Psi_t\|^2).$$

On the other hand, using Sobolev and Hölder inequality (see the proof of Lemma 3.9), together with $|\langle \varphi_t, A_t\varphi_t\rangle| + \|\nabla_1\Psi_t\| + \|\nabla\varphi_t\| \leq \mathcal{K}(\varphi_t)$, we obtain, for any $1 < p < \infty$

$$|(65)| \leq \mathcal{K}(\varphi_t)\left(1 + \|A_t\|_{\frac{2p}{p-1}}\right)(\|q_1\Psi_t\| + \|q_1\Psi_t\|^2).$$

Therefore, if $A_t \in L^p(\mathbb{R}^2, \mathbb{C})$ holds for some $p \in]2, \infty]$ and for all $t \in \mathbb{R}$, we obtain

$$|\|\nabla_1\Psi_t\|^2 - \|\nabla\varphi_t\|^2| \leq t \sup_{s\in[0,t]} \left(\mathcal{K}(\varphi_s)\left(\alpha(\Psi_s, \varphi_s) + \sqrt{\alpha(\Psi_s, \varphi_s)} + N^{-\delta} + N^{-1}\right)\right).$$

Since

$$\|\nabla_1 q_1\Psi_t\|^2 \leq |\|\nabla_1\Psi_t\|^2 - \|\nabla\varphi_t\|^2| + \|\nabla\varphi_t\|^2 \langle\langle \Psi_t, q_1\Psi_t\rangle\rangle + 2\|\nabla\varphi_t\|\|q_1\Psi_t\|\|\Psi_t\|$$

holds, we obtain with part (a) of Theorem 2.2, part (b) of Theorem 2.2.

□

5 Appendix

5.1 Energy variance of a product state

Lemma 5.1 *Let $\Psi = \varphi^{\otimes N}$ and assume that $\|\varphi\|_\infty + \|-\Delta\varphi\|_\infty + \|-\Delta\varphi\| + \|\nabla\varphi\| + \langle\varphi, A_s\varphi\rangle + \langle\varphi, A_s^2\varphi\rangle \leq C$. Then,*

$$Var_{H_{W_\beta,s}}(\Psi) \leq C(N^{-1} + N^{-1+\beta} + N^{-2+2\beta})\,. \tag{66}$$

Proof: The proof is a direct calculation using the product structure of $\Psi = \varphi^{\otimes N}$. We first calculate, denoting $T = \sum_{k=1}^N (-\Delta_k)$, $\mathcal{W} = \sum_{i<j}^N W_\beta(x_i - x_j)$ and $\mathcal{A} = \sum_{k=1}^N A_s(x_k)$,

$$\begin{aligned}\frac{1}{N^2}\langle\langle \Psi, H_{W_\beta,s}\Psi\rangle\rangle^2 =& \frac{1}{N^2}\langle\langle \Psi, (T + \mathcal{W} + \mathcal{A})\,\Psi\rangle\rangle^2 \\ =& \frac{1}{N^2}(N\langle\varphi, -\Delta\varphi\rangle + \frac{N(N-1)}{2}\langle\varphi, W_\beta * |\varphi|^2\varphi\rangle + N\langle\varphi, A_s\varphi\rangle)^2 \\ =& \langle\varphi, -\Delta\varphi\rangle^2 + \frac{(N-1)^2}{4}\langle\varphi, W_\beta * |\varphi|^2\varphi\rangle^2 + \langle\varphi, A_s\varphi\rangle^2 \\ &+ (N-1)\langle\varphi, -\Delta\varphi\rangle\langle\varphi, W_\beta * |\varphi|^2\varphi\rangle + 2\langle\varphi, -\Delta\varphi\rangle\langle\varphi, A_s\varphi\rangle \\ &+ (N-1)\langle\varphi, A_s\varphi\rangle\langle\varphi, W_\beta * |\varphi|^2\varphi\rangle.\end{aligned}$$

It then follows that

$$
\begin{aligned}
\mathrm{Var}_{H_{W_\beta,s}}(\Psi) =& \langle\!\langle \Psi, \frac{H^2_{W_\beta,s}}{N^2}\Psi\rangle\!\rangle - \frac{1}{N^2}\langle\!\langle \Psi, H_{W_\beta,s}\Psi\rangle\!\rangle^2 \\
=& \frac{1}{N^2}\langle\!\langle \Psi, T^2\Psi\rangle\!\rangle - \langle \varphi, -\Delta\varphi\rangle^2 \qquad (67)\\
+& \frac{1}{N^2}2\mathrm{Re}(\langle\!\langle T\Psi, \mathcal{W}\Psi\rangle\!\rangle) - (N-1)\langle \varphi, -\Delta\varphi\rangle\langle \varphi, W_\beta * |\varphi|^2\varphi\rangle \qquad (68)\\
+& \frac{1}{N^2}\langle\!\langle \Psi, \mathcal{W}^2\Psi\rangle\!\rangle - \frac{(N-1)^2}{4}\langle \varphi, W_\beta * |\varphi|^2\varphi\rangle^2 \qquad (69)\\
+& \frac{1}{N^2}\langle\!\langle \Psi, \mathcal{A}^2\Psi\rangle\!\rangle - \langle \varphi, A_s\varphi\rangle^2 \qquad (70)\\
+& \frac{1}{N^2}2\mathrm{Re}(\langle\!\langle \mathcal{A}\Psi, T\Psi\rangle\!\rangle) - 2\langle \varphi, -\Delta\varphi\rangle\langle \varphi, A_s\varphi\rangle \qquad (71)\\
+& \frac{1}{N^2}2\mathrm{Re}(\langle\!\langle \mathcal{A}\Psi, \mathcal{W}\Psi\rangle\!\rangle) - (N-1)\langle \varphi, W_\beta * |\varphi|^2\varphi\rangle\langle \varphi, A_s\varphi\rangle. \qquad (72)
\end{aligned}
$$

We estimate each line separately.

$$
\begin{aligned}
|(67)| =& \left| \frac{1}{N}\langle\!\langle \Psi, (-\Delta_1)^2\Psi\rangle\!\rangle + \frac{N-1}{N}\langle\!\langle \Psi, (-\Delta_1)(-\Delta_2)\Psi\rangle\!\rangle - \langle \varphi, -\Delta\varphi\rangle^2 \right| \\
\le& \frac{\|-\Delta\varphi\|^2 + \|\nabla\varphi\|^4}{N} \,.
\end{aligned}
$$

Note that

$$
\begin{aligned}
& \frac{1}{N^2}2\mathrm{Re}(\langle\!\langle T\Psi, \mathcal{W}\Psi\rangle\!\rangle) \\
=& \frac{1}{N^2}\sum_{k=1}^{N}\sum_{i\neq j=1}^{N}\mathrm{Re}(\langle\!\langle (-\Delta_k)\Psi, W_\beta(x_i - x_j)\Psi\rangle\!\rangle) \\
=& \frac{2(N-1)}{N}\mathrm{Re}(\langle\!\langle (-\Delta_1)\Psi, W_\beta(x_1 - x_2)\Psi\rangle\!\rangle) \\
+& \frac{(N-1)(N-2)}{N}\mathrm{Re}(\langle\!\langle (-\Delta_1)\Psi, W_\beta(x_2 - x_3)\Psi\rangle\!\rangle) \\
\le& \frac{2(N-1)}{N}\|\Delta\varphi\|_\infty \|W_\beta(x_1 - x_2)\|_1 \|\varphi\|_\infty \\
+& \frac{(N-1)(N-2)}{N}\|\nabla\varphi\|^2\langle \varphi, W_\beta * |\varphi|^2\varphi\rangle \\
\le& CN^{-1}\|\Delta\varphi\|_\infty\|\varphi\|_\infty + \frac{(N-1)(N-2)}{N}\langle \varphi, -\Delta\varphi\rangle\langle \varphi, W_\beta * |\varphi|^2\varphi\rangle,
\end{aligned}
$$

which immediately implies

$$
|(68)| \le C\frac{\|\nabla\varphi\|_\infty\|\varphi\|_\infty + \|\nabla\varphi\|^2\|\varphi\|_\infty^4}{N} \,.
$$

Next, we calculate

$$
\begin{aligned}
\frac{1}{N^2}\langle\!\langle \Psi, \mathcal{W}^2\Psi\rangle\!\rangle =&\frac{1}{4N^2}\sum_{i\neq j=1}^{N}\sum_{k\neq l=1}^{N}\langle\!\langle \Psi, W_\beta(x_i-x_j)W_\beta(x_k-x_l)\Psi\rangle\!\rangle \\
=&\frac{N-1}{2N}\langle\!\langle \Psi, W_\beta(x_1-x_2)^2\Psi\rangle\!\rangle \\
&+\frac{(N-1)(N-2)}{N}\langle\!\langle \Psi, W_\beta(x_1-x_2)W_\beta(x_2-x_3)\Psi\rangle\!\rangle \\
&+\frac{(N-1)(N-2)(N-3)}{4N}\langle\!\langle \Psi, W_\beta(x_1-x_2)W_\beta(x_3-x_4)\Psi\rangle\!\rangle \; .
\end{aligned}
$$

The first term is bounded by

$$
\frac{N-1}{2N}\langle\!\langle \Psi, W_\beta(x_1-x_2)^2\Psi\rangle\!\rangle \leq \|\varphi\|_\infty^2\|W_\beta\|^2 \leq CN^{-2+2\beta}\|\varphi\|_\infty^2 \; .
$$

The second term can be bounded using

$$
\begin{aligned}
f(x_2) =&N^{-1+2\beta}\left|\int_{\mathbb{R}^2} dx_1|\varphi(x_1)|^2 W(N^\beta(x_1-x_2))\right| \\
\leq&N^{-1}\int_{\mathbb{R}^2} dx_1|\varphi(N^{-\beta}x_1)|^2|W(x_1-N^\beta x_2)| \leq N^{-1}\|W\|_1\|\varphi\|_\infty^2
\end{aligned}
$$

by

$$
\begin{aligned}
&\frac{(N-1)(N-2)}{N}\langle\!\langle \Psi, W_\beta(x_1-x_2)W_\beta(x_2-x_3)\Psi\rangle\!\rangle \\
=&\frac{(N-1)(N-2)}{N}\int_{\mathbb{R}^2} dx_2|\varphi(x_2)|^2 f(x_2)^2 \leq \frac{1}{N}\|W\|_1^2\|\varphi\|_\infty^4 .
\end{aligned}
$$

It therefore follows that

$$
|(69)| \leq CN^{-2+2\beta}\|\varphi\|_\infty^2 + CN^{-1}(\|\varphi\|_\infty^2+\|\varphi\|_\infty^4) \; .
$$

(70) is estimated by

$$
\begin{aligned}
|(70)| =&\left|\frac{1}{N}\langle\!\langle \Psi, A_s(x_1)^2\Psi\rangle\!\rangle + \frac{N-1}{N}\langle\!\langle \Psi, A_s(x_1)A_s(x_2)\Psi\rangle\!\rangle - \langle\varphi, A_s\varphi\rangle^2\right| \\
\leq&\frac{\langle\varphi, A_s^2\varphi\rangle + \langle\varphi, A_s\varphi\rangle^2}{N} \; .
\end{aligned}
$$

Furthermore,

$$
\begin{aligned}
|(71)| \leq&\left|\frac{2}{N}|\langle\!\langle \Psi, A_s(x_1)(-\Delta_1)\Psi\rangle\!\rangle| + 2\frac{N-1}{N}\langle\!\langle \Psi, A_s(x_1)(-\Delta_2)\Psi\rangle\!\rangle - 2\langle\varphi, -\Delta\varphi\rangle\langle\varphi, A_s\varphi\rangle\right| \\
\leq&C\frac{\|-\Delta\varphi\|\|A_s\varphi\| + \|\nabla\varphi\|^2\langle\varphi, A_s\varphi\rangle}{N} \; .
\end{aligned}
$$

Finally,

$$
\begin{aligned}
|(72)| \leq&C\left(|\langle\!\langle A_s(x_1)\Psi, W_\beta(x_1-x_2)\Psi\rangle\!\rangle| + |\langle\!\langle \Psi, A_s(x_1)W_\beta(x_2-x_3)\Psi\rangle\!\rangle|\right) \\
\leq&C(\|A_s\varphi\|\|W_\beta\|\|\varphi\|_\infty + \langle\varphi, A_s\varphi\rangle\|W_\beta\|_1\|\varphi\|_\infty^2) \\
\leq&C(N^{-1+\beta}\|A_s\varphi\|\|\varphi\|_\infty + N^{-1}\langle\varphi, A_s\varphi\rangle\|\varphi\|_\infty^2) \; .
\end{aligned}
$$

□

5.2 Persistence of regularity of φ_t

We study the nonlinear Schrödinger equation in two spatial dimensions (3) with a harmonic potential

$$i\partial_t \varphi_t = (-\Delta + a|\varphi_t|^2 + |x|^2)\varphi_t$$

under the conditions $a > -a^*$ and $\|\varphi_0\| = 1$. The solution theory of (3) is well studied in absence of external fields. There, the global existence and persistence of regularity of $\varphi_t \in H^k(\mathbb{R}^2, \mathbb{C})$ was established, assuming φ_0 regular enough [8]. The condition $a > -a^*$ is known to be optimal, that is, for $a < -a^*$, there exist blow-up solutions. It is interesting to note that global existence of solutions in $L^\infty(\mathbb{R}^2, \mathbb{C})$ directly implies persistence of higher regularity of solutions in $H^k(\mathbb{R}^2, \mathbb{C})$, see [8] and below.

Lemma 5.2 *Let $\varphi_0 \in H^1(\mathbb{R}^2, \mathbb{C}), \|\varphi_0\| = 1$ such that $\|\nabla\varphi_0\|^2 + \||x|\varphi_0\|^2 + \frac{a}{2}\langle\varphi_0, |\varphi_0|^2\varphi_0\rangle \leq C$. Let $a > -a^*$.*

(a) The nonlinear Schrödinger equation

$$i\partial_t \varphi_t = (-\Delta + a|\varphi_t|^2 + |x|^2)\varphi_t$$

admits a solution $\varphi_t \in H^1(\mathbb{R}^2, \mathbb{C})$ globally in time.

(b) Define the norm $\|u\|_{\Sigma,m} = \sqrt{\sum_{k=0}^m (\|\nabla^k u\|^2 + \||x|^k u\|^2)}$. Then

$$\|\varphi_t\|_{\Sigma,4} \leq \|\varphi_0\|_{\Sigma,4} e^{C\int_0^t ds \|\varphi_s\|_\infty^2} .$$

(c) Assume $\|\varphi_0\|_{\Sigma,4} < \infty$. Then, there exist a time dependent constant C_t, also depending on $\|\varphi_0\|_{\Sigma,4}$, such that $\|\varphi_t\|_{\Sigma,4} \leq C_t$.

Remark 5.3 *Part (c) directly implies that $\varphi_t \in H^4(\mathbb{R}^2, \mathbb{C})$. Our proof relies on the works of [6, 7, 8, 20, 42, 44], see also the references therein. It also might be possible to show a polynomial growth in t of the constant C_t, using the refined estimates presented in [6, 7].*

Proof:

(a) The global existence in $H^1(\mathbb{R}^2, \mathbb{C})$ is well known, see Remark 3.6.4 in [8]. We sketch the proof for completeness. Let U_t denote the generator of the time evolution of the linear Schrödinger equation $i\partial_t u_t = (-\Delta + |x|^2)u_t$. For any $\varphi_0 \in L^2(\mathbb{R}^2, \mathbb{C})$, we consider the Duhamel formula

$$\varphi_t = U_t\varphi_0 - ia\int_0^t ds U_{t-s}|\varphi_s|^2\varphi_s \, . \tag{73}$$

Note that is is known that there exists a nonempty open interval I, $0 \in I$ such that (73) has a unique solution φ_t, provided the initial datum φ_0 fulfills $\|\varphi_0\|_{\Sigma,1} \leq C$ (see Proposition 1.5. in [7]). Furthermore, for any $t \in I$, $\|\varphi_t\| = \|\varphi_0\| = 1$. We may assume that I is the maximal interval on which a solution of (73) exists. Assume now that φ_t blows up in finite time, i.e. I is bounded. It is then known that $\int_0^{\sup I} dt \|\varphi_t\|_4^4 = \infty$ [20].

Assume $t \in I$ and consider the NLS energy

$$\mathcal{E}_{\mathrm{NLS}}(\varphi_t) = \|\nabla \varphi_t\|^2 + \frac{a}{2}\langle \varphi_t, |\varphi_t|^2 \varphi_t \rangle + \||x|\varphi_t\|^2 \,.$$

Under the conditions $a > -a^*, \|\varphi_0\| = 1$, the two dimensional Gagliardo-Nirenberg inequality $\frac{a^*}{2}\|u\|_4^4 \le \|\nabla u\|^2 \|u\|^2$, $u \in H^1(\mathbb{R}^2, \mathbb{C})$ implies that $\mathcal{E}_{\mathrm{NLS}}(\varphi_t) > 0$. Furthermore $\frac{d}{dt}\mathcal{E}_{\mathrm{NLS}}(\varphi_t) = 0$, see Proposition 1.6. in [7]. This directly implies that there exists an $\epsilon > 0$ such that

$$\epsilon \|\nabla \varphi_t\|^2 \le C \,.$$

The two dimensional Gagliardo-Nirenberg inequality implies, together with $\|\varphi_t\| = \|\varphi_0\|\,, \forall t \in I$,

$$\int_0^{\sup I} dt \|\varphi_t\|_4^4 \le C \int_0^{\sup I} dt \|\nabla \varphi_t\|^2 \le C \sup I < \infty \,.$$

Therefore, the solution φ_t of (73) exists globally in time and fulfills $\varphi_t \in H^1(\mathbb{R}^2, \mathbb{C})$, $\||x|\varphi_t\| < \infty$.

(b) Let $A(x) = |x|^2$ and define, for any $u \in L^2(\mathbb{R}^2, \mathbb{C})$, the norm $\|u\|_{k,A} = \sqrt{\sum_{m=0}^k \|(-\Delta + A)^m u\|^2}$. Note that $\|\cdot\|_{k,A}$ is invariant under U_t, that is $\|U_t u\|_{k,A} = \|u\|_{k,A}$. We will first show that $\|u\|_{2,A}$ and $\|u\|_{\Sigma,4}$ are equivalent norms. Let $u \in H^4(\mathbb{R}^2, \mathbb{C})$. Note that

$$\begin{aligned}
\|u\|_{2,A}^2 =& \|u\|^2 + \|(-\Delta + A)u\|^2 + \|(-\Delta + A)^2 u\|^2 \\
\le& \|u\|^2 + 2\| - \Delta u\|^2 + 2\|Au\|^2 \\
&+ \| \left((-\Delta)^2 + A^2 + (-\Delta A) + 2A(-\Delta) - 2(\nabla A)\cdot \nabla\right) u\|^2 \\
\le& C\big(\|u\|^2 + \| - \Delta u\|^2 + \|Au\|^2 + \|(-\Delta)^2 u\|^2 + \|A^2 u\|^2 \\
&+ \|A(-\Delta)u\|^2 + \|(\nabla A)\cdot \nabla u\|^2\big) \,.
\end{aligned}$$

Since $\nabla A^2 = 4|x|^2 x,\ \Delta A^2 = 12A$, we obtain,

$$\begin{aligned}
\|A(-\Delta)u\|^2 =& \langle u, (-\Delta)A^2(-\Delta)u\rangle \\
=& \langle u, (-\Delta A^2)(-\Delta)u\rangle + 2\langle u, (-\nabla A^2)\cdot \nabla(-\Delta)u\rangle + \langle u, A^2(-\Delta)^2 u\rangle \\
\le& C(\|Au\|\| - \Delta u\| + \||x|^3 u\|\|\nabla \Delta u\| + \|A^2 u\|\|(-\Delta)^2 u\|) \\
\le& C(\|Au\|^2 + \| - \Delta u\|^2 + \|(-\Delta)^2 u\|^2 + \|A^2 u\|^2) \,.
\end{aligned}$$

For the last inequality, we used $\||x|^3 u\|^2 = \langle |x|^2 u, |x|^4 u\rangle \le \|Au\|^2 + \|A^2 u\|^2$, as well as $\|\nabla \Delta u\| \le \| - \Delta u\|^2 + \|(-\Delta)^2 u\|^2$. We use polar coordinates (r, φ). Then, $(\nabla A)\cdot \nabla = 2r\partial_r$. Hence,

$$\begin{aligned}
\|(\nabla A)\cdot \nabla u\|^2 =& -4\langle u, \partial_r(r^2 \partial_r u)\rangle = -4\langle u, \left(2r\partial_r + r^2\partial_r^2\right) u\rangle \\
=& -4\langle u, r^2\left(r^{-1}\partial_r + \partial_r^2\right) u\rangle - 4\langle u, r\partial_r u\rangle \\
\le& 4\langle r^2 u, -\left(r^{-1}\partial_r + \partial_r^2 + \frac{1}{r^2}\partial_\varphi^2\right) u\rangle - 4\left\langle |x|\frac{x}{|x|}u, \nabla u\right\rangle \\
\le& C(\|Au\|^2 + \| - \Delta u\|^2 + \||x|u\|^2 + \|\nabla u\|^2) \,.
\end{aligned}$$

Therefore, $\|u\|_{2,A} \leq C\|u\|_{\Sigma,4}$ holds. To show the converse, first note that $\|u\|_{\Sigma,4}^2 \leq C(\|u\|^2 + \|Au\|^2 + \|-\Delta u\|^2 + \|A^2 u\|^2 + \|\Delta^2 u\|^2)$. Since $-\Delta \leq -\Delta + |x|^2$ and $|x|^2 \leq -\Delta + |x|^2$ holds as an operator inequality, we directly obtain $\|u\|_{\Sigma,4} \leq C\|u\|_{2,A}$.

By $\|uv\|_{H^k} \leq \|u\|_\infty \|v\|_{H^k} + \|u\|_{H^k}\|v\|_\infty$, $\|\cdot\|_{2,A}$ fulfills the generalized Leibniz rule

$$\begin{aligned}\|uv\|_{2,A} \leq& C\|uv\|_{\Sigma,4} \leq C(\|u\|_\infty \|v\|_{\Sigma,4} + \|u\|_{\Sigma,4}\|v\|_\infty)\\ \leq& C(\|u\|_{2,A}\|v\|_\infty + \|u\|_\infty\|v\|_{2,A}) \ .\end{aligned}$$

From (73), we obtain

$$\begin{aligned}\|\varphi_t\|_{2,A} \leq& \|U_t\varphi_0\|_{2,A} + |a| \int_0^t ds \|U_{t-s}|\varphi_s|^2\varphi_s\|_{2,A}\\ =& \|\varphi_0\|_{2,A} + |a|\int_0^t ds \||\varphi_s|^2\varphi_s\|_{2,A}\\ \leq& \|\varphi_0\|_{2,A} + C\int_0^t ds \|\varphi_s\|_\infty^2 \|\varphi_s\|_{2,A} \ .\end{aligned}$$

By a Grönwall inequality, we obtain (b).

(c) We show that $\varphi_t \in H^2(\mathbb{R}^2, \mathbb{C})$ globally in time. Recall the existence of global in time solutions of

$$i\partial_t u_t = (-\Delta + a|u_t|^2)u_t \ . \tag{74}$$

in $H^2(\mathbb{R}^2,\mathbb{C})$, provided that $a > -a^*$ and $u_0 \in H^2(\mathbb{R}^2,\mathbb{C}), \|u_0\| = 1$ holds. Using the lens transform [6, 42], for $|t| < \pi/2$

$$\varphi_t(x) = \frac{1}{\cos(t)} u_{\tan(t)}\left(\frac{x}{\cos(t)}\right) e^{-i\frac{|x|^2}{2}\tan(t)} \ ,$$

φ_t then solves $i\partial_t\varphi_t = (-\Delta + a|\varphi_t|^2 + |x|^2)\varphi_t$ with initial datum $\varphi_0 = u_0$. We therefore see that the existence of a global-in-time solution of (74) in $H^2(\mathbb{R}^2,\mathbb{C})$ implies existence of a solution φ_t in $H^2(\mathbb{R}^2,\mathbb{C})$ locally in $t \in]-\pi/2, \pi/2[$. By translation invariance of time, the solution φ_t then exists globally in $H^2(\mathbb{R}^2,\mathbb{C})$. By the embedding $L^\infty(\mathbb{R}^2,\mathbb{C}) \subset H^2(\mathbb{R}^2,\mathbb{C})$, we obtain, together with (b), (c).

□

5.3 Self-Adjointness

Lemma 5.4 *Let*

$$H_{W_\beta,t} = \sum_{k=1}^N (-\Delta_k) + \sum_{i<j=1}^N W_\beta(x_i - x_j) + \sum_{k=1}^N A_t(x_k)$$

and assume (A1) and (A2). Then, for all $t \in \mathbb{R}$,

(a) $H_{W_\beta,t}$ *is selfadjoint with domain* $\mathcal{D}(H_{W_\beta,t}) = \mathcal{D}\left(\sum_{k=1}^N(-\Delta_k + A_0(x_k))\right)$.

(b) $(H_{W_\beta,t})^2$ *is selfadjoint with domain* $\mathcal{D}((H_{W_\beta,t})^2) = \mathcal{D}((H_{W_\beta,0})^2)$. *If, in addition,* $W \in C^2(\mathbb{R}^2,\mathbb{R})$, *then* $\mathcal{D}((H_{W_\beta,t})^2) = \mathcal{D}\left((\sum_{k=1}^N(-\Delta_k + A_0(x_k))^2\right)$ *holds.*

Proof:

(a) First note that $\mathcal{D}(H_{W_\beta,0}) = \mathcal{D}\left(\sum_{k=1}^N(-\Delta_k + A_0(x_k))\right)$, since $W_\beta \in L_c^\infty(\mathbb{R}^2,\mathbb{R})$. We write

$$H_{W_\beta,t} = H_{W_\beta,0} + \sum_{k=1}^N \int_0^t ds \dot{A}_s(x_k) \,.$$

Abbreviate $\mathcal{A}_t = \sum_{k=1}^N \int_0^t ds \dot{A}_s(x_k)$. Since $\|\mathcal{A}_t\Psi\| \leq N\int_0^t ds\|\dot{A}_s\|_\infty\|\Psi\|$ holds for all $\Psi \in L^2(\mathbb{R}^2,\mathbb{C})$, $\mathcal{A}_t$ is infinitesimal $H_{W_\beta,0}$ bounded, which implies by Kato-Rellich that $\mathcal{D}(H_{W_\beta,0}) = \mathcal{D}(H_{W_\beta,t})$.

(b) Note that $(H_{W_\beta,0})^2$ is self-adjoint on $\mathcal{D}((H_{W_\beta,0})^2)$. Consider

$$(H_{W_\beta,t})^2 = (H_{W_\beta,0})^2 + H_{W_\beta,0}\mathcal{A}_t + \mathcal{A}_t H_{W_\beta,0} + \mathcal{A}_t^2 \,.$$

Under assumption (A2) $H_{W_\beta,0}\mathcal{A}_t + \mathcal{A}_t H_{W_\beta,0} + \mathcal{A}_t^2$ is a symmetric operator on $\mathcal{D}((H_{W_\beta,0})^2)$. We estimate, for $\Psi \in \mathcal{D}((H_{W_\beta,0})^2)$, $\Psi \neq 0$,

$$\begin{aligned}
&\left\|\left(H_{W_\beta,0}\mathcal{A}_t + \mathcal{A}_t H_{W_\beta,0} + (\mathcal{A}_t)^2\right)\Psi\right\| \\
\leq& 2N\int_0^t ds\|\dot{A}_s\|_\infty\|H_{W_\beta,0}\Psi\| + N^2\left(\int_0^t ds\|\dot{A}_s\|_\infty\right)^2\|\Psi\| \\
&+ N\int_0^t ds\|\Delta\dot{A}_s\|_\infty\|\Psi\| + 2\left\|\sum_{k=1}^N\int_0^t ds\nabla_k\dot{A}_s(x_k)\nabla_k\Psi\right\|
\end{aligned}$$

Note that

$$\begin{aligned}
2N\int_0^t ds\|\dot{A}_s\|_\infty\|H_{W_\beta,0}\Psi\| &= 2N\int_0^t ds\|\dot{A}_s\|_\infty\sqrt{\langle\!\langle\Psi,\left(H_{W_\beta,0}\right)^2\Psi\rangle\!\rangle} \\
\leq& \sqrt{2N^2\left(\int_0^t ds\|\dot{A}_s\|_\infty\right)^2\|\Psi\|^2 + \frac{1}{2}\left\|\left(H_{W_\beta,0}\right)^2\Psi\right\|^2} \\
\leq& \sqrt{2}N\int_0^t ds\|\dot{A}_s\|_\infty\|\Psi\| + \frac{1}{\sqrt{2}}\left\|\left(H_{W_\beta,0}\right)^2\Psi\right\| \,.
\end{aligned}$$

Furthermore, for $\epsilon > 0$

$$\begin{aligned}
2\left\|\sum_{k=1}^N\int_0^t ds\nabla_k\dot{A}_s(x_k)\nabla_k\Psi\right\| &\leq 2\sum_{k=1}^N\int_0^t ds\|\nabla\dot{A}_s\|_\infty\|\nabla_k\Psi\| \\
\leq& \frac{2N}{\epsilon}\left(\int_0^t ds\|\nabla\dot{A}_s\|_\infty\right)^2\|\Psi\| + \frac{\epsilon}{2\|\Psi\|}\sum_{k=1}^N\|\nabla_k\Psi\|^2 \\
\leq& \frac{2N}{\epsilon}\left(\int_0^t ds\|\nabla\dot{A}_s\|_\infty\right)^2\|\Psi\| + \frac{1}{2\|\Psi\|}\langle\Psi, H_{W_\beta,0}\Psi\rangle + CN\|\Psi\| \,.
\end{aligned}$$

Since $\|\Psi\|^{-1}\langle\!\langle\Psi, H_{W_\beta,0}\Psi\rangle\!\rangle \leq \frac{1}{2}\|\Psi\| + \frac{1}{2\|\Psi\|}\|H_{W_\beta,0}\Psi\|^2 \leq \|\Psi\| + \frac{1}{2}\|(H_{W_\beta,0})^2\Psi\|$, we obtain

$$\begin{aligned}&\left\|\left(H_{W_\beta,0}\mathcal{A}_t + \mathcal{A}_t H_{W_\beta,0} + (\mathcal{A}_t)^2\right)\Psi\right\| \leq \left(\frac{1}{\sqrt{2}} + \frac{1}{4}\right)\|(H_{W_\beta,0})^2\Psi\| \\ &+ \left(\sqrt{2}N\int_0^t ds\|\nabla\dot{A}_s\|_\infty + \frac{2N}{\epsilon}\left(\int_0^t ds\|\nabla\dot{A}_s\|_\infty\right)^2 + CN\right)\|\Psi\| \\ &+ N\int_0^t ds\|\Delta\dot{A}_s\|_\infty\|\Psi\| \,.\end{aligned}$$

Thus, $H_{W_\beta,0}\mathcal{A}_t + \mathcal{A}_t H_{W_\beta,0} + (\mathcal{A}_t)^2$ is relatively $(H_{W_\beta,0})^2$ bounded with bound $\frac{1}{\sqrt{2}} + \frac{1}{4} < 1$. By Kato-Rellich, $(H_{W_\beta,t})^2$ is self-adjoint with domain $\mathcal{D}((H_{W_\beta,t})^2) = \mathcal{D}((H_{W_\beta,0})^2)$, for all $t \in \mathbb{R}$. By a similar estimate, we also obtain $\mathcal{D}((H_{W_\beta,0})^2) = \mathcal{D}\left((\sum_{k=1}^N(-\Delta_k + A_0(x_k))^2\right)$ if $W \in C^2(\mathbb{R}^2, \mathbb{C})$.

□

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