Temperature Consideration in the Shallow Lake Model and Its Policy Implications for Eutrophication Governance

By

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Executive summary

Eutrophication is a significant environmental issue affecting shallow lakes and is closely related to human activities. The shallow lake model serves as an environmental economic model for studying this problem. In this study, we first reviewed the scientific rationale of this economic model, subsequently, analyzed the original shallow lake model proposed by Mäler et al., discussing market failure issues in static optimization based on previous research. We then introduced the factor of temperature to enable the model to consider the effects of seasonal temperature changes and long-term climate warming on eutrophication processes. We conducted an analysis of the shallow lake model incorporating temperature. Analysis of the state equation indicated that temperature variation significantly influences the internal phosphorus release in the water body, with increased temperature leading to the transition of the shallow lake to hysteresis or irreversible states. Analysis of the static optimization problem of shallow lake utility revealed that temperature increase makes it more likely for utility maximization to occur in states with high phosphorus content. Additionally, we explored the existence of emission control strategies under temperature variation scenarios. Finally, based on our study of this environmental economic model, practical policy implications were provided.

Contents

Introduction

Have you ever seen a pond or river covered by green floating matter? This foul-smelling green floating debris is the result of algal bloom, which is one of the most direct manifestations of eutrophication (Figure 1). Eutrophication is an ecological process occurring in water bodies. This process initiates with the accumulation of nutrients such as phosphates and nitrates in the water. The accumulation of nutrients accelerates microbial growth, which in turn depletes oxygen in the water, ultimately resulting in other harmful consequences (Khan & Mohammad, 2014; Smith & Schindler, 2009). Eutrophication is considered a primary water body issue globally. 63.1% of freshwater bodies worldwide are believed to be experiencing eutrophication problems (Wang et al., 2018; Zhang et al., 2021).

Eutrophication has wide-ranging impacts. Primarily, it can danger the stability of ecosystems. Oxygen depletion caused by eutrophication may directly kill fish in water bodies, especially during winter months when fish demand more oxygen (Schindler, 1972). Additionally, the proliferation of algae can disrupt the food web, leading to long-term modifications in ecosystem structure, such as altering the population of fish-eating predators (Carpenter & Kitchell, 1996; Elser et al., 2000). Secondly, eutrophication can influence pollutant concentrations in water bodies. Previous studies suggest that eutrophic waters exhibit higher rates of biodegradation for petrochemicals, aromatic hydrocarbons, and pesticides (Graham et al., 2000). Meanwhile, other research indicates that eutrophication may increase the accumulation of organic chlorides in water bodies (Berglund, 2003). Moreover, eutrophication is closely associated with human health hazards as well. Water bodies are important carriers of pathogens. Eutrophication can increase the load of pathogens such as Escherichia coli and waterborne viruses in water (Hofmann, 2004; Wilson et al., 1996). Whether used for domestic or recreational purposes, eutrophic waters can potentially cause widespread gastrointestinal, respiratory, ocular, nasal, aural, and cutaneous infections, even, according to some study, eutrophication can trigger an outbreak of cholera (Cottingham et al., 2003; Griffin et al., 2003).

Figure 1 A serious algae bloom in the US[1]

Among all freshwater bodies, shallow lakes, being the ones most closely associated with human activities, bear the highest risk of eutrophication and consequently pose the greatest potential for harm. There is no strict definition about the depth of a shallow lake, with some studies using 6 m as the threshold and others considering lakes with a depth of less than 13.8 m as shallow lakes (Scheffer & van Nes, 2007). Actually, many of the well-known lakes such as Lake Taihu and Lake Poyang in eastern China are considered as shallow lakes. Shallow lakes, in the concept of ecology, are lakes that

[1] Horn Point Laboratory, University of Maryland: Rapid Expansion of Industrial Farming in U.S. Contributes to Increased Pollution, Harmful Algal Blooms. Available at: https://www.umces.edu/news/rapid-expansion-of-industrial-farming-in-us-contributes-to-increased-pollution-harmful-algal

contain only epilimnion, a surface layer, without deeper stratification like metalimnion and hypolimnion. In shallow lakes, the water is able to circulate between different depths, which means that the substances in the water can be fully exchanged (Scheffer, 2004). Shallow lakes are more vulnerable to eutrophication processes than deeper lakes. This stems partly from its ecological characteristics that internal flow keeps bring nutrients back to lake water; and partly from the fact that shallow lakes are usually located in areas with fertile land and large populations, where more phosphorus-containing substances are discharged (Zhou et al., 2022). In essence, shallow lakes are closely entwined with human activities, particularly in the context of eutrophication caused by phosphate discharges, making them a focal point for environmental attention (Scheffer & van Nes, 2007).

Phosphorus is considered to be the main cause of eutrophication in shallow lakes. Of course, this does not deny that the synergistic effects of nitrates and phosphates also contribute to the eutrophication of shallow water systems (Qin et al., 2020). Reviewing the causes of eutrophication, under natural conditions, nutrients such as animal waste and phosphate minerals are released into water, leading to natural eutrophication. However, human activities, such as the discharge of domestic sewage and the use of agricultural fertilizers, release large amounts of phosphates and nitrates, causing cultural eutrophication. Natural eutrophication occurs slowly over geological time scales, while cultural eutrophication becomes increasingly common with the industrialization of human society (Callisto et al., 2014). Therefore, addressing eutrophication issues should focus on reducing anthropogenic nutrient emissions. Given that shallow water bodies have a lower capacity to absorb nutrients and are more susceptible to human activities, the management of shallow lakes has long been a topic of concern for researchers and policy makers (Abell et al., 2022; Lau & Lane, 2002; Qin et al., 2006, p. 20; Zhou et al., 2022).

Despite a clear understanding of the cause of eutrophication in shallow lakes, it has always been a challenging issue in environmental governance. To address this problem, there are typically two approaches: reducing existing nutrients as well as phytoplankton, and controlling nutrient inputs (Qin et al., 2006). Reducing existing nutrients and phytoplankton involves using non-toxic chemical agents to neutralize phosphates and nitrates, stocking herbivorous fish such as grass carp, and manually removing phytoplankton. However, these methods are costly and often only address symptoms rather than the root cause. Controlling nutrient inputs usually relies on strict regulation of wastewater discharge, delineation of discharge prohibited areas, and levying discharge fees or taxes as policy tools. These policy tools can prevent or reverse the eutrophication process through market or non-market approaches. However, in practice, the governance of shallow lakes often fails to achieve the expected results. Taking Lake Taihu as an example, as the third largest freshwater lake in China, it has suffered from severe nutrient enrichment issues since the 1980s, driven by rapid industrial and agricultural development in the Yangtze River Delta region. Following a major cyanobacterial bloom event in 2007, a series of remediation projects, as well as national and local level policies including strict land use regulations, restrictions on fertilizer use, and emissions trading, were implemented (X. Yan et al., 2024). However, after a decade of comprehensive management and an investment exceeding $14 billion USD, by 2017, there was no significant improvement in the water quality of Lake Taihu. Instead, on May 16, 2017, the area covered by cyanobacteria in Lake Taihu reached its historical peak (Qin et al., 2019; ZHANG et al., 2023).

There are a few reasons why efforts to manage eutrophication often yield little return. Firstly, similar to many environmental issues, there exists market failure. This is manifested in the fact that the negative externalities of nutrient emissions are not internalized or are underestimated; furthermore, as shallow lakes are public goods, local entities lack the incentive for proactive management (Chandrasekaran & Ajaz, 2019). Secondly, the eutrophication problem in shallow lakes exhibits ecological complexity. The concentration of phosphates in shallow lakes exhibits hysteresis and irreversibility, making many management measures difficult to implement or requiring greater effort to be effective (Carpenter et al., 1999; Carpenter & Lathrop, 2008). Moreover, the eutrophication problem in shallow lakes shows significant temperature dependence. Research indicates that higher temperatures can increase phosphate concentrations in freshwater bodies (Zhao et al., 2022). Statistical data suggests that higher temperatures are more conducive to algal blooms (Fang et al., 2022). For this reason, establishing an environmental economics model that focuses on phosphorus discharge and accumulation in shallow lakes, while integrating human behavior with environmental science background, is crucial for shallow lake management.

In their seminal work in 2003, Mäler, Xepapadeas, and de Zeeuw integrated eutrophication processes in shallow lakes into economic models, offering a comprehensive analysis ranging from steady-state conditions to open-loop Nash equilibrium. This groundbreaking study unveiled two equilibria within the shallow lake problem: one characterized as an oligotrophic state and the other as a eutrophic state. The complexity of this phenomenon is compounded by hysteresis and irreversibility issues, leading to the existence of a delicate governance threshold, as Carpenter, Ludwig, and Brock (1999) have pointed out.

Building upon this foundation, subsequent research in environmental economics has explored various dimensions of the shallow lake problem. Dechert and O'Donnell (2006) introduced random shocks to phosphorus inflow, simulating additional discharges occurring sporadically due to factors such as rainwater scouring or ice thawing. Their numerical simulations demonstrated that even in the presence of random shocks, a long-term dynamic game could result in the persistence of a eutrophic equilibrium state. Kossioris et al. (2019) introduced stochastic disturbances to phosphorus in algae, delving into common solutions for stochastic considerations. Meanwhile, Grass et al. (2017) developed a two-dimensional model incorporating not only the state function of lake water but also that of mud. By solving the optimal problem for this more complex system, they identified intermediate states deviating from the original Skiba points, termed weak Skiba points. Within these weak Skiba points, the shallow lake system followed diverse evolutionary paths, ultimately converging to a long-term steady state characterized by the same level of welfare.

However, neither of the previous research has investigated temperature impact on the shallow lake model. Remarkably, the Hill–Langmuir equation used to describe the ecological mechanism has the potential to include temperature variations. Seizing this opportunity, this paper introduces the temperature impact into the shallow lake model. By doing so, we aimed to demonstrate the seasonal characteristics of eutrophication management in shallow lakes and to illustrate the potential impact of climate change on the management of shallow water bodies.

The original shallow lake model

A typical environmental economics model consists of two parts: a state equation and a utility function. The state equation explains how the state variables in the environmental problem change over time and with control variables (Conrad, 1999). In Mäler et al. (2003)'s shallow lake model, the state variable is the phosphorus amount in the lake water, while the control variable is the amount of phosphorus discharged into the lake. The utility function indicates the value people can derive from the shallow lake, which includes both economic value and potential ecological value.

The state equation

The state equation is designed align with the real science theory, but a simplified one. According to theories in environmental science, the phosphorus process involves several key steps, shown in Figure 2 (Sondergaard et al., 2001). Firstly, the use of fertilizers in agricultural production and emissions from chemical factories can lead to the release of phosphorus into water in the form of dissolved substances or particles. These sources of phosphorus are widespread in human activities around lakes, and the phosphorus in these dissolved substances and particles becomes a primary nutrient source in lake ecosystems. Secondly, not all the released phosphorus will persist in the water indefinitely. A portion of the phosphorus may be permanently absorbed, depositing in the sediment at the lake bottom. The phosphorus in these sediment deposits forms a "hidden reservoir", which may have an impact on the water body over an extended period. However, a concerning aspect is that some phosphorus may be released back into the water. This can occur due to hydrodynamic processes in the lake, such as the mixing of lake water or the resuspension of sediment, causing previously adsorbed phosphorus to re-enter the water column (Pettersson, 1998). Additionally, some phosphorus may be lost to the surrounding areas of the lake, especially in the case of shallow water lakes. This loss can occur through runoff, rainfall runoff, or other hydrological processes (Sondergaard et al., 2001). Therefore, a comprehensive understanding of the sources, fate, and transformation processes of phosphorus in lakes is crucial for the effective management of lake ecosystems and the prevention of eutrophication issues.

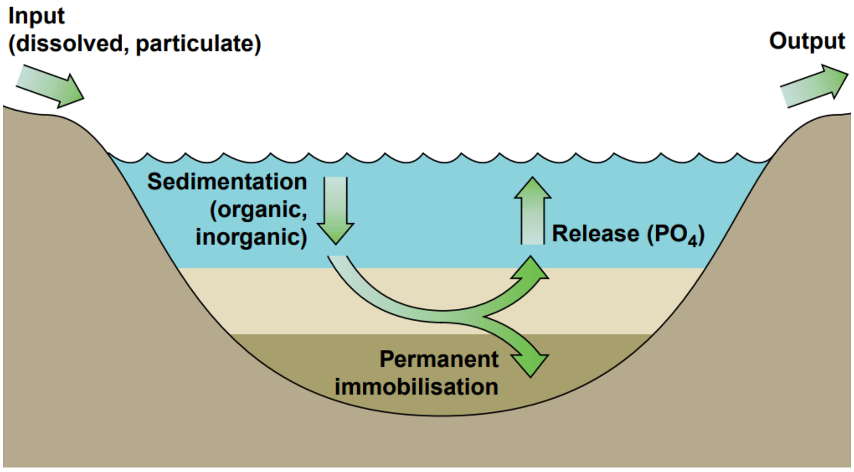

Figure 2 Conceptual diagram showing phosphorus process in a shallow lake

According to the above analysis, the state equation describing the phosphorate process in shallow lakes is depicted as Equation (1) (Mäler et al., 2003).

$$\dot{P}(t) = L(t) - sP(t) + r\left(\frac{P^2(t)}{P^2(t) + m^2}\right), P(0) = P_0 \tag{1}$$

It is a differential equation about the amount of phosphorus in algae, where $P(t)$ is the amount of phosphorus along with time, $L(t)$ is the input of phosphorus, s is the rate of loss consisting of sedimentation, outflow, and sequestration in other biomass, r is the maximum rate of internal loading and m is the anoxic level.

The left-hand side of Equation (1) represents the rate of change of phosphorus content in lake water, which is the rate of change of the state variable in the model. The right-hand side of the equation represents the simplified phosphorus processes depicted in Figure 2. The first term represents the amount of phosphorus input into the water body, which is determined by the agents' behavior, i.e., the control variable in the environmental economics model. The second term represents the amount of phosphorus permanently deposited and discharged, which is directly proportional to the phosphorus content in algae. The third term represents internal flow, which includes phosphorus released from sediments, etc. This is described using the Hill–Langmuir equation, which is widely used to describe adsorption and release processes in ecology and biochemistry. In this equation, the Hill coefficient is taken as 2, and the dissociation constant is considered to be determined by the anoxic level (Gromov & Upmann, 2021; Mäler et al., 2003).

Equation (1) can be transformed into a dimensionless version through some algebra substitution. Using $x = \frac{P}{m}, a = \frac{L}{r}$, and $b = \frac{sm}{r}$, the state equation is transformed into Equation (2). In the new equation, x refers to the phosphorus amount in the shallow lake water, a is the relative emission from all emitters, and b is the only systemic parameter. In addition, this state equation only makes sense when all of x, a, and b are positive real numbers.

$$\dot{x} = a(t) - b * x(t) + \frac{x^2(t)}{x^2(t) + 1} \tag{2}$$

Steady states can be figured out through Equation (2). Set $\dot{x} = 0$, the phosphorus inputs will have a stock-dependent relationship for all the steady states, shown in Equation (3).

$$0 = a - b * x + \frac{x^2}{x^2 + 1} \tag{3}$$

To solve the relationship between a and x, we treated Equation (3) as a cubic equation for x. According to the analysis of the discriminate and first order condition, when $b \geq \frac{3\sqrt{3}}{8}$, a has a monotonic relationship with x, that is, given a phosphorus state of the lake, there is one single input value can keep this state a steady state, shown in Figure 3(a). When $\frac{1}{2} < b < \frac{3\sqrt{3}}{8}$, the steady states always exist however the relationship between a and x is no longer monotonic, which means, the same input value can result in different phosphorus level, shown in Figure 3(b). If $b < \frac{1}{2}$, the equation yields some negative a, which is not acceptable for explanation. In this scenario, the relationship between a and x is discrete, shown in Figure 3(c).

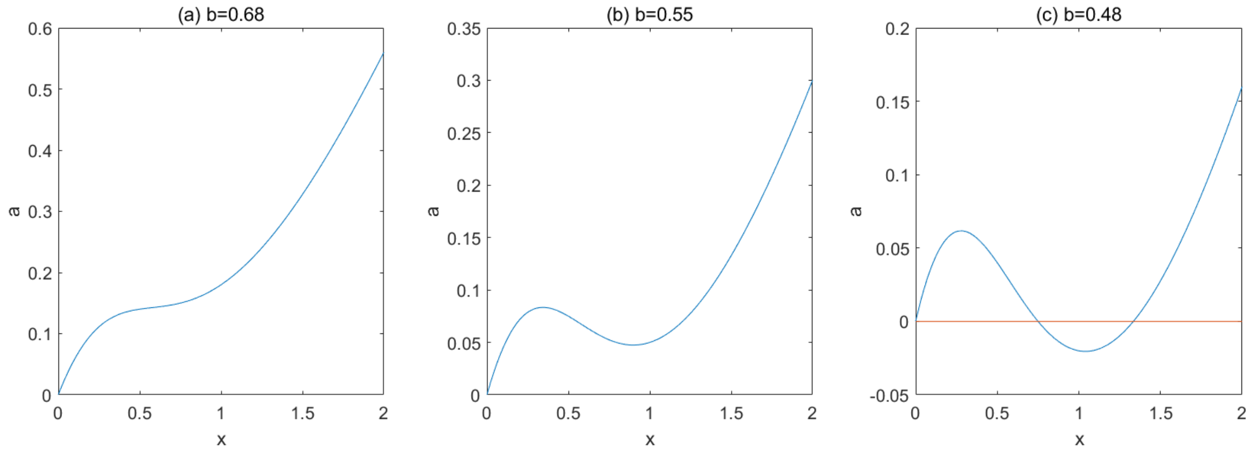

Figure 3 Steady states for different b value

Different values of b represent the steady state situations in the context of shallow lake management, showcasing the hysteresis and irreversibility phenomena of concern. Figure 3(a) illustrates a healthy state of the shallow lake, where each level of discharge corresponds to a unique total phosphorus state, so that reducing discharge levels can lower the phosphorus content in the water body. Figure 3(b) shows a case of hysteresis, since x is not decrease along a monotonically. Hysteresis refers to the situation where, in some cases, even if the input of nutrients is reduced, the system may not immediately return to its original healthy state. This is because the changes in shallow lakes caused by eutrophication have inertia. Specifically, a high-nutrient state implies a high internal flow, so even if the input is reduced, the phosphorus content will not immediately decrease to a low-nutrient level.

Figure 3(c) is a perfect presentation of irreversibility, as a cannot achieve a negative value, high phosphorus concentrate state will never go to the lower states. Irreversibility refers to the situation where once a lake or water body receives excessive nutrient inputs and undergoes eutrophication, the ecosystem may not be able to fully return to a healthy, low-nutrient state. This may be because the changes in phosphorus processes in the water body are permanent.

<u>The utility function</u>

Apart from the state equation, the utility function is another core element of the shallow lake model. In Mäler et al. (2003), the state function is defined as Equation (4). The utility function represents the benefits and losses that the shallow lake provides to the surrounding communities. For these communities, discharging phosphorus into the shallow lake means they can engage in agricultural production and use the shallow lake as a waste sink, which can bring profits, represented in the utility function as $\ln a_i$. On the other hand, excessive phosphorus content in the lake can lead to eutrophication, reducing the ecological rent of the shallow lake and harming the surrounding communities, represented in the utility function as $-cx^2$. The coefficient c here measures the trade-off between economic value and ecological value. A larger c indicates greater emphasis on ecological rent, and consequently, greater losses due to eutrophication in shallow lakes. Conversely, a smaller c suggests that the economic value derived from communities around shallow lakes discharging phosphorus-rich wastewater into the lakes is relatively higher, resulting in smaller losses from eutrophication. It is worth noting that the value of the waste sink that each community obtains from the shallow lake is determined by their respective behaviors, choices of control inputs, while the loss of ecological rent due to eutrophication of the shallow lake is determined by the total phosphorus content.

$$U_i(a_i, x) = \ln a_i - cx^2, \sum_{i=1}^{n} a_i = a \tag{4}$$

When considering the infinite horizon time scale, there is integral version for the utility function, shown in Equation (5). This function includes a continuous discount rate ρ and calculates the overall utility over an infinite time scale.

$$W_i = \int_0^\infty e^{-\rho t}[\ln a_i(t) - c * x^2(t)]dt\,, i = 1 \dots n \tag{5}$$

<u>Static economic analysis</u>

With the state equation and utility function, economic analysis can be applied to determine the maximization of utility. When considering utility maximization at a single point in time, it becomes a static optimization problem. Considering that there may be multiple communities settled around the shallow lake, different scenarios may exist among these communities. When these n communities cooperate in the governance of the shallow lake, or when they are managed by a unified government, it is possible to maximize the total utility of all communities. This represents the optimal management scenario, shown in Equation (6).

$$\begin{gathered} maximize\ U(a_i, x) = \sum_{i=1}^{n} \ln a_i - n * cx^2 \\ s.t.\, 0 = a - bx + \left(\frac{x^2}{x^2+1}\right) \end{gathered} \tag{6}$$

This problem can be solved by using first order condition.

$$\frac{d(\sum_{i=1}^{n} \ln a_i - n * cx^2)}{dx} = 0 \tag{7}$$

Considering the homogeneity among different communities in this model, a_i can be regarded as $\frac{a}{n}$. Thus, Equation (7) becomes a simpler formation:

$$\frac{d\left(n * \ln\frac{a}{n} - n * cx^2\right)}{dx} = 0 \tag{8}$$

$$\Rightarrow n * \frac{1}{n} * \frac{n}{a} * \frac{da}{dx} - 2cnx = 0 \tag{9}$$

Where, according to Equation (3):

$$\frac{da}{dx} = b - \frac{2x}{(x^2+1)^2} \tag{10}$$

The optimal amount of phosphorus in the shallow lake is determined by Equation (11).

$$b - \frac{2x}{(x^2+1)^2} - 2cx * \left(bx - \frac{x^2}{x^2+1}\right) = 0 \tag{11}$$

Since the parameters in the shallow lake model are dimensionless, the optimal management may not necessarily lead to a eutrophic state of the lake. If the weight c assigned to ecological services in the utility function is relatively low, choosing a eutrophic state with high agricultural activity levels might be optimal. As shown in Figure 4, taking $b = 0.68$, for larger values of c, the optimal management problem has a maximum value for x below the inflection point. As the value of c decreases, initially, there is a local maximum for high x, while low x s till achieves the global maximum (Mäler et al., 2003).

Furthermore, this curve approaches the vertical and horizontal axes in two directions respectively. In other words, when c, representing ecological value preference, tends towards infinity, the phosphorus level x approaches 0; conversely, when the phosphorus level x maximizing utility tends towards infinity, the corresponding ecological preference tends towards 0 rather than a negative value. This property confirms that the model is consistent with reality and avoids making undefined predictions in extreme scenarios.

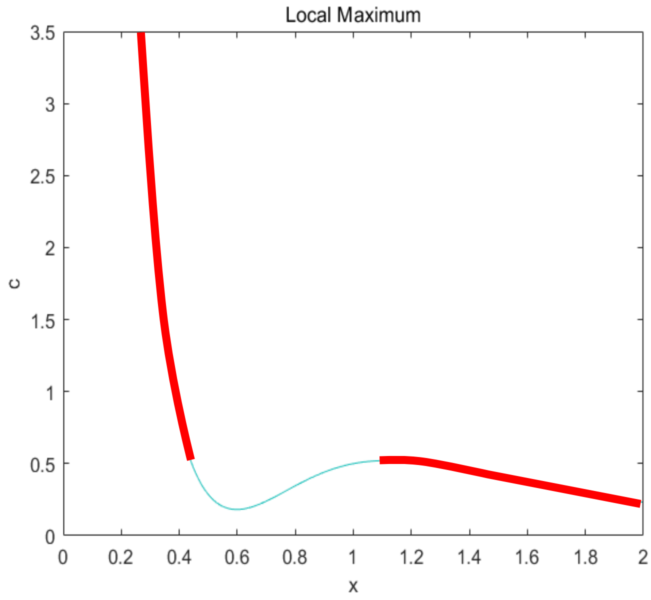

Figure 4 The $c - x$ relationship for static optimizations

Another scenario apart from the optimal control is when each community acts independently without cooperation, maximizing their own benefits. This leads to the existence of Nash equilibria that vary with the number of communities. The optimization problem corresponding to this scenario is depicted in Equation (12).

$$maximize\ U_i(a_i, x) = \ln a_i - cx^2, a = \sum_{i=1}^{n} a_i$$

$$s.t. 0 = a - bx + \left(\frac{x^2}{x^2 + 1}\right) \tag{12}$$

For this problem, the Nash Equilibrium occurs when every player is making the best decision they can, given the decision of the other players. The first order condition leads:

$$\frac{d(\ln a_i - cx^2)}{da_i} = 0 \tag{13}$$

Assuming all the communities have taken their strategies, then one of the players attempts to make a change on the input. Thus, here, the relationship between a and a_i follows:

$$\frac{da}{da_i} = \frac{1}{n} \tag{14}$$

Plug (14) into the first order condition (13):

$$\frac{1}{a} - \frac{2c}{n} x * \frac{dx}{da} = 0 \tag{15}$$

The optimal amount of phosphorus in the shallow lake is determined by Equation (16).

$$b - \frac{2x}{(x^2 + 1)^2} - \frac{2c}{n} x * \left(bx - \frac{x^2}{x^2 + 1}\right) = 0 \tag{16}$$

In Figure 5, it can be observed that, similar to optimal control, the equilibrium phosphorus level x is dependent on the preference for ecological value c.

As the number of communities n participating in the game increases, the gap between these game curves and the optimal control curve becomes larger. This gap between the game curves and optimal control illustrates that, in competitive scenarios, there is an increasing necessity to prioritize ecological values in order to maintain the shallow lake at the same phosphorus level. In other words, as more communities share the lake, the phosphorus content in the lake will tend to be higher, which means a higher potential of eutrophication.

Furthermore, as n increases, the inflection point is significantly raised. In the case of optimal control, the inflection point is located at $c = 0.52$. As n increases, the inflection point is raised by a factor of n, reaching $c = 3.12$ when $n = 6$. This implies that with more players participating in this game, the global maximum of the shallow lake is more likely to be in a high-phosphorus state, even under high ecological value preferences.

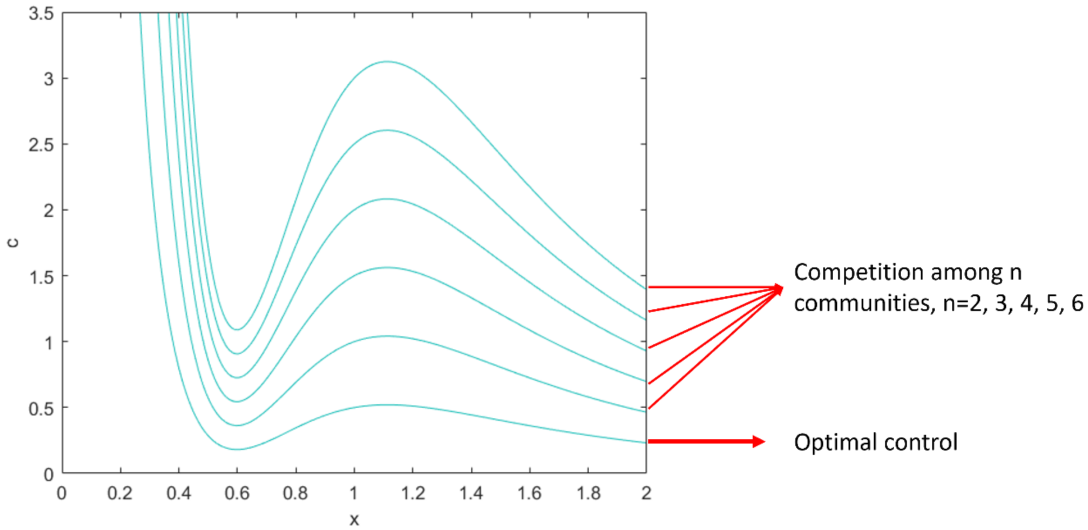

Figure 5 The $c - x$ relationship for optimal control and multiparty Nash equilibria.

The difference in phosphorus levels between optimal management and game scenarios can be explained by market failure. Considering governance of the shallow lake rather than the act of discharging wastewater into it, which are essentially equivalent. In this context, the ecological value in the utility function represents the benefits each community can derive from governing the shallow lake, while the first term represents the costs of governance. Clearly, governance costs are borne individually by each community, but the benefits of governance are the same, non-excludable, and non-rivalrous. From this perspective, the shallow lake behaves as a public good. When communities are not considered as a whole, free riders emerge. Given that each community makes rational decisions in the game scenario, each community's strategy involves contributing less governance costs compared to optimal management. This results in a greater risk of eutrophication for the shallow lake and, consequently, a loss in overall societal utility.

Dynamic economic analysis

Dynamic economic analysis aims to maximize utility under an infinite horizontal time scale. Similar to static economic analysis, it is divided into discussions of optimal management and game theory. However, dynamic game theory is more complex because the strategies employed by each participant are not fixed but rather depend on time. In practical discussions, dynamic games can be categorized into open-loop and closed-loop strategies (Weber & Kryazhimskiy, 2011). In open-loop strategies, players formulate decisions without considering the reactions of other players or changes in the environment. In other words, their decisions are not influenced by opponents or environmental changes but are based on predetermined information and strategies, making them static in nature and not accounting for dynamic changes. In closed-loop strategies, players consider the reactions of opponents and changes in the environment when formulating decisions. They adjust their strategies based on feedback from the environment to adapt and be more effective in dynamic conditions. To better align with the subsequent discussion on temperature effects, we only focused on the simpler open-loop game here.

Optimal management is to maximize the sum of infinite-time-scale utility, defined by Equation (5), which yields this maximum problem shown in Equation (17).

$$maximize\ W(a_i, x) = \sum_{i=1}^{n} \int_0^{\infty} e^{-\rho t}[\ln a_i(t) - c * x^2(t)]dt$$
$$s.t.\, \dot{x}(t) = a(t) - bx(t) + \left(\frac{x^2(t)}{x^2(t)+1}\right), x(0) = x_0 \tag{17}$$

This maximization problem involves the issue of finding the extremum of a functional, and we solved it with current value Hamiltonian, shown in Equation (18), where p is the coefficient for the co-state equations (Weber & Kryazhimskiy, 2011).

$$H(p, a, x) = \ln a - cx^2 + p\left(a - bx + \frac{x^2}{x^2+1}\right) \tag{18}$$

First order conditions:

$$\begin{cases} \dot{x} = \dfrac{\partial H}{\partial p} \\ \dot{p} = -\dfrac{\partial H}{\partial x} + \rho p \\ \dfrac{\partial H}{\partial a} = 0 \end{cases} \tag{19}$$

The first order condition yields a group of differential equations for a and x:

$$\begin{cases} \dot{x} = a(t) - b * x(t) + \dfrac{x^2(t)}{x^2(t)+1} \\ \dot{a} = -\left[(b+\rho) - \dfrac{2x}{(x^2+1)^2}\right] * a + 2cx * a^2 \end{cases} \tag{20}$$

The solution of this system of differential equations depends on the choice of parameters and initial conditions, but overall, a solution exists. This implies the existence of an optimal management strategy that maximizes utility over an infinite time horizon.

For the case of open-loop games, the objective of maximization shifts from the overall utility of all communities to the utility of each individual community. In this scenario, the maximization problem is as shown in Equation (21).

$$maximize\ W_i(a_i, x) = \int_0^{\infty} e^{-\rho t}[\ln a_i(t) - c * x^2(t)]dt\, , a = \sum_{i=1}^{n} a_i$$
$$s.t.\, \dot{x}(t) = a(t) - bx(t) + \left(\frac{x^2(t)}{x^2(t)+1}\right), x(0) = x_0 \tag{21}$$

Similarly, using the Hamiltonian method for solving, we can obtain the system of differential equations, which also has critical solutions that depend on parameters and initial conditions.

$$\begin{cases} \dot{x} = a(t) - b * x(t) + \frac{x^2(t)}{x^2(t)+1} \\ \dot{a} = -\left[(b+\rho) - \frac{2x}{(x^2+1)^2}\right] * a + \frac{2c}{n} x * a^2 \end{cases} \tag{22}$$

Temperature consideration in the shallow lake model (Static)

As previously mentioned, temperature plays a critical role in influencing eutrophication processes. From a practical standpoint, it's essential to consider both the seasonal fluctuations and the long-term shifts in temperature, which are increasingly attributed to climate change. Taking Lake Taihu as an example, it experiences the most significant instances and impacts of eutrophication during the spring and summer seasons (C. Yan et al., 2016). During spring, agricultural activities are at their peak, leading to higher volumes of phosphorus-laden wastewater being discharged into the lake. This influx of nutrients exacerbates the eutrophication process. Similarly, in summer, the rising temperatures further escalate the risk of eutrophication. The combination of warmer temperatures and increased nutrient load creates favorable conditions for algal blooms and other harmful phenomena. Moreover, experimental studies conducted on sediment and water samples from shallow lakes suggest a direct correlation between temperature and phosphorus release (Cheng et al., 2020). Specifically, as temperatures rise, there is a corresponding increase in the release of phosphorus from internal flows within the lake. This mechanism provides a plausible explanation for the heightened eutrophication risk observed during the warmer summer months.

Taking into consideration the experimental results of Cheng et al. (2020), we revisited the components of the state equation, in Equation (2). To represent the third term accounting for internal release, which corresponds to the Hill-Langmuir equation, we introduced an exponential temperature term, $\theta^T * \theta_0$, shown in Equation (23). Here, θ and θ_0 are constants, T is the magnitude of temperature. As the temperature changes, the value of this term exhibits exponential variation. Unless otherwise specified, in subsequent sections, the baseline temperature is set at 15 degrees Celsius, which approximates the annual average temperature in the Yangtze River Delta region of China. The value of θ and θ_0 are to be referenced in the subsequent section about time-dependent temperature. This modification to the shallow lake model aligns with both the experimental findings and the thermodynamic mechanisms of the Hill-Langmuir equation (Chao et al., 2007; Hipsey et al., 2005).

$$\dot{x}(t) = a(t) - bx(t) + \theta^T * \theta_0 * \frac{x^2(t)}{x^2(t)+1} \tag{23}$$

Temperature impact on steady states

Regarding the influence of temperature on the performance of the shallow lake model, we first qualitatively explored its effect on steady state. Figure 6 illustrates how a slight increase in temperature can transform the monotonic relationship between input and phosphorus levels into a situation with hysteresis effects. Here, the parameter b is set to 0.65. At the reference temperature of 15 degrees Celsius, there is a monotonic relationship between a and x, without any hysteresis effects. However, when the temperature rises by 3.3%, equivalent to global warming in 50 years later, hysteresis effects occur when x is in the range of approximately 0.4 to 0.8. Similarly, with a temperature increase of 6.7%, equivalent to global warming in 100 years later, hysteresis effects occur when x is in the range of approximately 0.35 to 1. This indicates that, under identical conditions, an increase in temperature, even if small in magnitude, can exacerbate the difficulty of controlling eutrophication in shallow lakes.

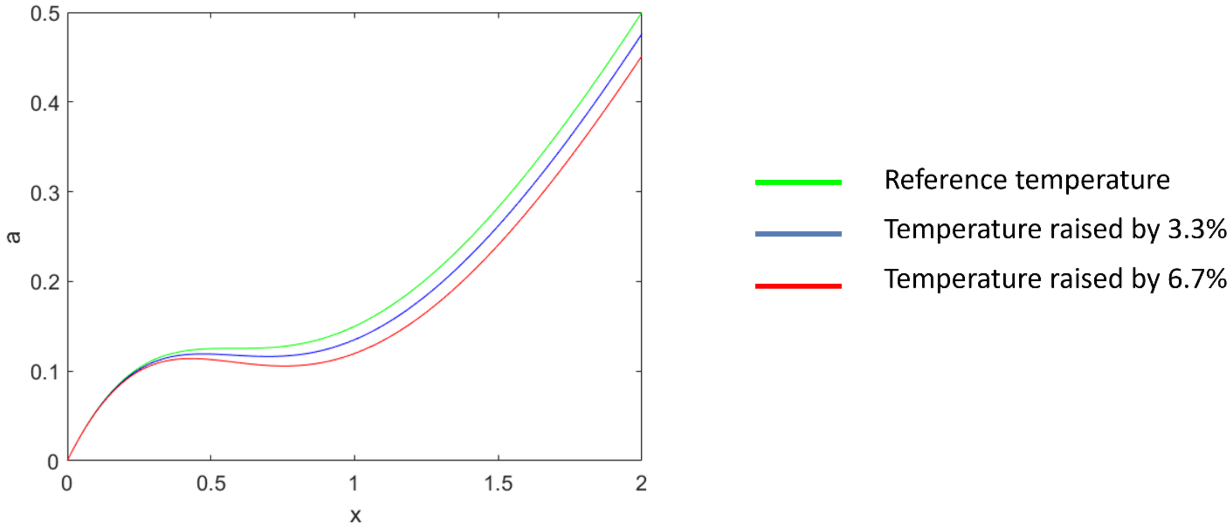

Figure 6 Impact from slight temperature change on steady states

In Figure 7, the range of temperature variation is expanded. With parameter b set to 0.68, we plotted the input and phosphorus level images for both a 66.7% increase and a 66.7% decrease in temperature. It is worth noting that we simulated seasonal temperature changes here. The reference temperature can be considered as representing spring and autumn conditions. A 66.7% increase in temperature implies an average temperature of 25 degrees Celsius, representing summer, while a 66.7% decrease in temperature results in an average temperature of 5 degrees Celsius, representing winter. Significant shifts in the curves can be observed under these conditions. With a 66.7% increase in temperature, most of the curve is below zero, indicating that the shallow lake is prone to irreversible eutrophication. According to the parameters in the graph, when x is greater than 0.5, even without any further discharge of phosphorus-containing wastewater, the phosphorus level in the shallow lake will not decrease. Conversely, with a 66.7% decrease in temperature, the relationship between a and x approaches linearity. Under these conditions, compared to the reference temperature scenario, lakes in winter can tolerate higher levels of phosphorus discharge, approximately 60% higher at the same phosphorus level.

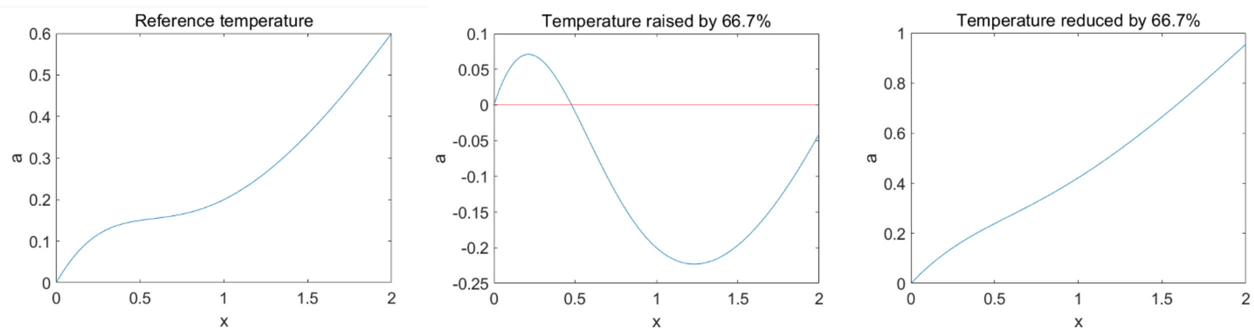

Figure 7 Impact from expanded temperature change on steady states

Temperature impact on static optimization

Following that, we further investigated the influence of temperature on static optimization scenarios. Similar to earlier discussions, static optimization is divided into cooperative and competitive (game-theoretic) scenarios. For the optimization problem under cooperative conditions, the temperature effect is incorporated onto Equation (6), resulting in Equation (24).

$$maximize\ U(a_i, x) = \sum_{i=1}^{n} \ln a_i - n * cx^2$$
$$s.t. 0 = a - bx + \theta^T * \theta_0 * \left(\frac{x^2}{x^2+1}\right) \tag{24}$$

The problem can be solved with the first order condition. The result is given by an equation:

$$b - \theta^T * \theta_0 * \frac{2x}{(x^2+1)^2} - 2cx * \left(bx - \theta^T * \theta_0 * \frac{x^2}{x^2+1}\right) = 0 \tag{25}$$

Basing on the above differential equation, we plot the relationship between phosphorus level x and ecological preference factor c, shown in Figure 8. The parameter b is chosen as 0.7. Similar with Figure 6, there are three scenarios: reference temperature, temperature raised by 3.3%, and temperature raised by 6.7%. As ecological preference factor c decreases from high to low, the graph may exhibit one, two, or three equilibrium points for x. These equilibrium points represent either global maximum or local extremum points. Considering all equilibrium situations, there is a turning point at $x = 1$, where all the curves intersect. For $x < 1$, as temperature increases, the curve shifts downwards. This suggests that, under the same ecological preference, optimal management at higher temperatures requires lower phosphorus levels. This phenomenon arises because when the phosphorus content in the shallow lake is already low, higher temperatures lead to a greater increase in phosphorus levels for the same emissions. To maximize social utility, emissions need to be reduced, resulting in optimal management at lower phosphorus levels. For $x > 1$, an increase in temperature causes the curve to shift upwards. According to previous sections, on the right side of the inflection point, the global maximum occurs in a high-phosphorus state. It can be observed that an increase in temperature raises the inflection point value of c. This implies that with higher temperatures, the ecological value preference requirement for the global maximum to occur in a high-phosphorus state becomes lower.

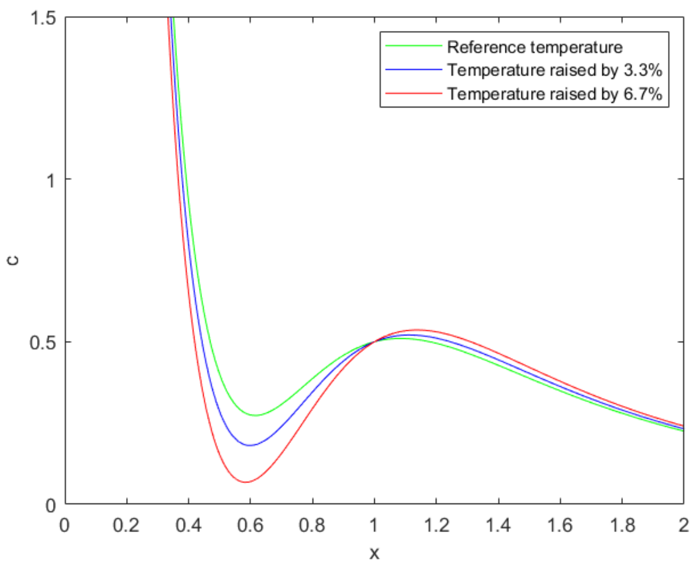

Figure 8 Temperature’s impact on static optimal management

Next comes the static optimization of the game scenario. The temperature influence is incorporated into Equation (12), resulting in Equation (26).

$$maximize\ U_i(a_i, x) = \ln a_i - cx^2, a = \sum_{i=1}^{n} a_i$$

$$s.t. 0 = a - bx + \left(\frac{x^2}{x^2+1}\right) \tag{26}$$

Similarly, employing the first-order condition for solving yields results that can be expressed as the following equation.

$$b - \theta^T * \theta_0 * \frac{2x}{(x^2+1)^2} - \frac{2c}{n} x * \left(bx - \theta^T * \theta_0 * \frac{x^2}{x^2+1}\right) = 0 \tag{27}$$

In Figure 4, we examined the curve differences between the game and cooperation scenarios without considering temperature effects. This disparity represents the existence of market failure. Here, we plotted the cooperation and game curves for two scenarios: a 3.3% temperature rise and a 6.7% temperature rise. By observing these curves, we can draw some qualitative conclusions, as shown in Figure 9.

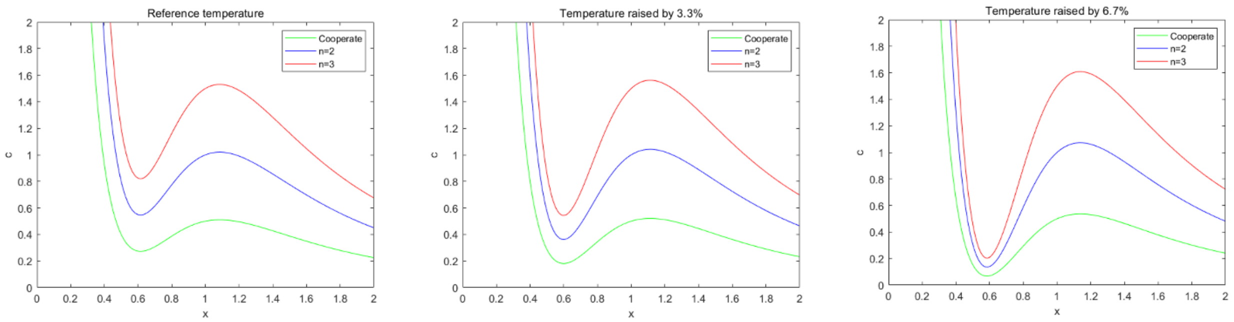

Figure 9 Temperature's impact on static optimization for optimal control and competition

Figure 9 uses $b = 0.7$ as the parameter and 15 degrees Celsius as the reference temperature. It can be observed that as the temperature rises, the gap between the equilibrium of optimal management and competition at lower phosphorus levels gradually decreases. In mathematical terms, by examining the difference between Equation (27) and Equation (25), we can see that the additional coefficient n in Equation (27) amplifies the temperature's effect on the downward shift of the curves. In other words, as the temperature increases, all curves shift downward, and the more communities participate in the game, the more the curves shift downward, thereby reducing the gap with optimal management. Intuitively, although each community attempts to reduce its investment in shallow lake governance, the increase in temperature increases the risk of eutrophication, forcing them to choose to invest more in governance. This is why the increase in temperature reduces the difference between the optimal management and competition scenarios.

Temperature consideration in the shallow lake model (Dynamic)

Next, we explored the temperature effects in dynamic problems. Clearly, the difference between temperature effects in dynamic problems and static problems lies in the consideration of temperature variation over time. In reality, temperature changes at different time scales. Within a day, it is cold in the morning and evening while hot at noon; over a year, it is cold in winter and hot in summer. There are also variations in average temperatures between years. In our study of dynamic temperature changes, we mainly consider seasonal temperature variations and long-term average temperatures resulting from climate changes.

For seasonal temperature variations, we represented them using sine functions, as seen in Equation

(28) and Figure 10. To comply with the discount setting in dynamic optimization, each time unit of t is set to one year. Parameter A represents the seasonal temperature difference, set to 10 degrees Celsius, which is an approximation for the Yangtze River Delta region in China. In Figure 10, it can be seen that the temperature gradually rises from 15 degrees Celsius to 25 degrees Celsius over time, transitioning from spring to summer. Then, the temperature gradually decreases over time, dropping back to 15 degrees Celsius and continuing to decrease to 5 degrees Celsius, representing winter. Overall, this simplification of simulation using simple mathematical functions aligns with actual seasonal temperature variations.

$$T(t)_{seasonal} = A * \sin 2\pi t + T_0 \tag{28}$$

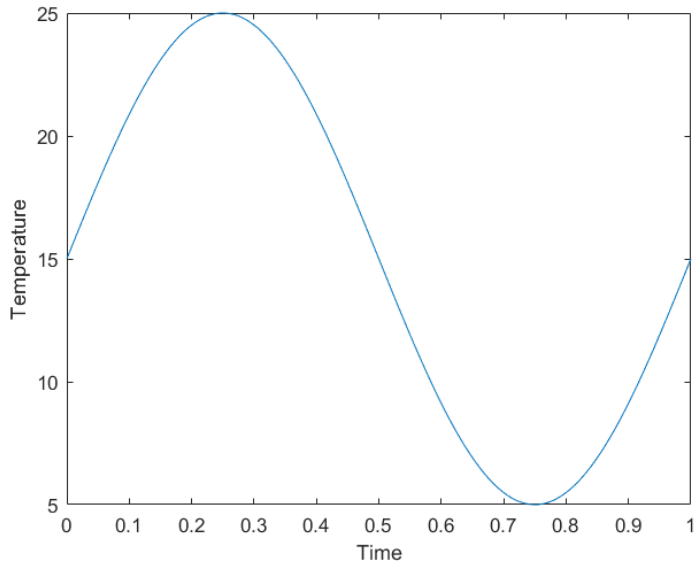

Figure 10 Seasonal temperature change

For long-term global warming, we referred to the IPCC's analysis of climate change trends and data from the World Meteorological Organization[2], that the global average temperature has been rising linearly since the 1970s. Therefore, we chose the linear function to represent the long-term climate change effect, shown in Equation (29). Considering that the global average temperature has risen by approximately 1.2 degrees Celsius since pre-industrial times, the value of k is around 0.01 (Masson-Delmotte et al., 2019).

$$T(t)_{cc} = T_0 + k * t \tag{29}$$

When combing these two parts together, we developed the temperature function depend on time, as shown in Equation (30) or Equation (31) with default values.

$$T(t) = A * \sin 2\pi t + T_0 + k * t \tag{30}$$

$$T(t) = 10 * \sin 2\pi t + 15 + 0.001 * t \tag{31}$$

Of course, we also need to consider the role of temperature in the state equation. As mentioned earlier, we used an exponential temperature factor $\theta^T * \theta_0$. However, we left an unexplained question: how are the parameters set? In fact, we referenced some numerical computational models and environmental science research. Eventually, we decided to use Equation (32) to represent the

[2] World Meteorological Organization: Climate Data and Monitoring. Available at https://community.wmo.int/en/climate-data-and-monitoring

temperature factor in the state equation.

$$temperature\ factor = \theta^* * e^{\delta(A*\sin(2t\pi)+T_0+k*t)} \tag{32}$$

This temperature factor represents the influence of temperature on the internal flow of phosphorus. Within the formula, δ represents the extent of temperature's impact on internal flow, and based on experimental data, $\delta = 0.0588$ (Cheng et al., 2020). θ^* is a normalizing coefficient that ensures this factor equals to 1 when temperature is at its initial value, specifically, $\theta^* = e^{-\delta T(0)}$. In this case, we can finally uncover the mysterious magnitude of θ and θ_0. That is:

$$\begin{cases} \theta = e \\ \theta_0 = \theta^* * e^{\delta} \end{cases} \tag{33}$$

The overall trend of temperature with time described through Equation (31) is shown in Figure 11, and the overall temperature factor value is shown in Figure 12.

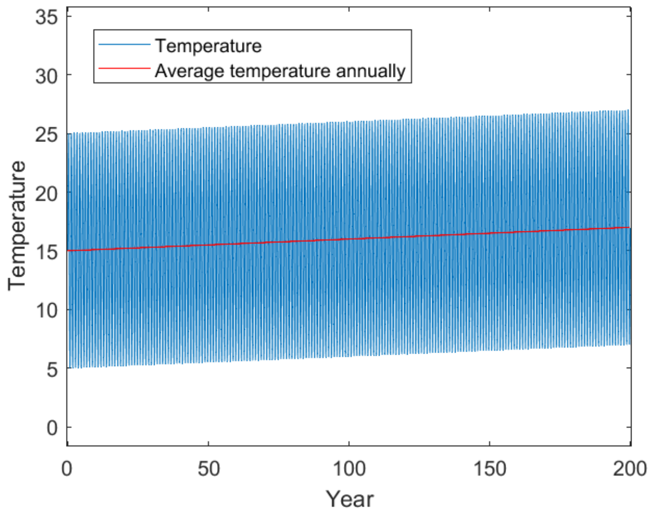

Figure 11 Temperature trend along time

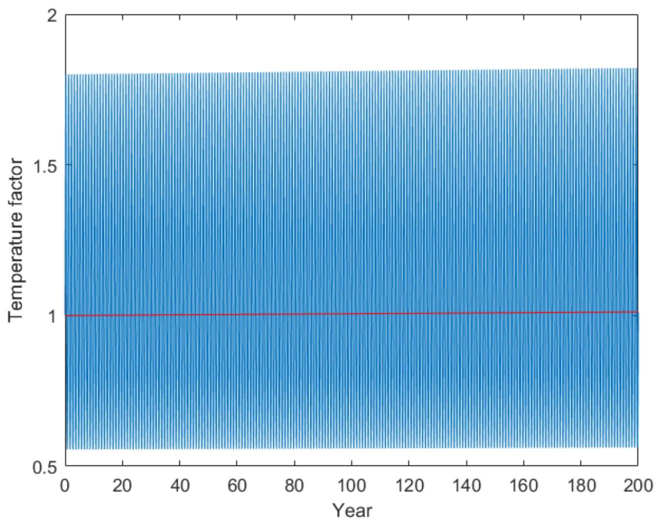

Figure 12 Temperature factor trend along time

Thus, the state equation with dynamic time considered is in the form shown by Equation (34).

$$\dot{x}(t) = a(t) - bx(t) + \theta^* * e^{\delta(A*\sin(2t\pi)+T_0+k*t)} * \left(\frac{x^2(t)}{x^2(t)+1}\right), x(0) = x_0 \tag{34}$$

Dynamic economic analysis

Building upon this state equation that incorporates dynamic time changes, we can once again discuss utility maximization on an infinite horizontal time scale. Cases of cooperation and competition are discussed separately. In the cooperation scenario, we need to solve the dynamic optimization problem presented in Equation (35).

$$maximize\ W(a,x) = \sum_{i=1}^{n} \int_{0}^{\infty} e^{-\rho t} [\ln a_i(t) - c * x^2(t)] dt$$
$$s.t. \dot{x}(t) = a(t) - bx(t) + \theta^* * e^{\delta(A*\sin(2t\pi)+T_0+k*t)} * \left(\frac{x^2(t)}{x^2(t)+1}\right), x(0) = x_0 \tag{35}$$

Here, the Euler-Lagrange equation was performed, instead of the Hamiltonian method as before (Landau et al., 1960). This is because, with the inclusion of a time-dependent temperature factor in the state equation, the equation becomes time-dependent. In such cases, the Euler-Lagrange equation is better suited to handle this optimization problem. Firstly, we established the Lagrangian for the optimization problem defined in Equation (35). Here, homogeneity among different communities is considered, so both terms in the utility equation are scaled by n.

$$\mathcal{L}(x,\dot{x},t) = \ln \frac{\dot{x} + bx - \theta^* * e^{\delta(A*\sin(2t\pi)+T_0+k*t)} * \left(\frac{x^2}{x^2+1}\right)}{n} - cx^2 \tag{36}$$

The Euler-Lagrange equation constructed from this Lagrangian is:

$$\frac{\partial \mathcal{L}}{\partial x} - \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{x}} = 0 \tag{37}$$

Where,

$$\frac{\partial \mathcal{L}}{\partial x} = \frac{n * \left(b - \theta^* * e^{\delta(A*\sin(2t\pi)+T_0+k*t)} * \frac{2x}{(x^2+1)^2}\right)}{\dot{x} + bx - \theta^* * e^{\delta(A*\sin(2t\pi)+T_0+k*t)} * \left(\frac{x^2}{x^2+1}\right)} - 2cx \tag{38}$$

$$\frac{\partial \mathcal{L}}{\partial \dot{x}} = \frac{1}{\dot{x} + bx - \theta^* * e^{\delta(A*\sin(2t\pi)+T_0+k*t)} * \left(\frac{x^2}{x^2+1}\right)} \tag{39}$$

$$\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{x}} = -\frac{\ddot{x} + b - \theta^* * e^{\delta(A*\sin(2t\pi)+T_0+k*t)} * \left(\frac{x^2}{x^2+1}\right) * (2\pi A * \cos 2t\pi + k)}{\left(\dot{x} + bx - \theta^* * e^{\delta(A*\sin(2t\pi)+T_0+k*t)} * \left(\frac{x^2}{x^2+1}\right)\right)^2} \tag{40}$$

$$\frac{n * \left(b - \theta^* * e^{\delta(A*\sin(2t\pi)+T_0+k*t)} * \frac{2x}{(x^2+1)^2}\right)}{\dot{x} + bx - \theta^* * e^{\delta(A*\sin(2t\pi)+T_0+k*t)} * \left(\frac{x^2}{x^2+1}\right)} - 2cx =$$
$$-\frac{\ddot{x} + b - \theta^* * e^{\delta(A*\sin(2t\pi)+T_0+k*t)} * \left(\frac{x^2}{x^2+1}\right) * (2\pi A * \cos 2t\pi + k)}{\left(\dot{x} + bx - \theta^* * e^{\delta(A*\sin(2t\pi)+T_0+k*t)} * \left(\frac{x^2}{x^2+1}\right)\right)^2} \tag{41}$$

We solved the differential equation, Equation (41), using MATLAB's ode15i[3] function. The result shows that this optimization problem does not have critical solutions. The utility can achieve maximum only when there is no phosphorus emitted to the lake.

The optimization problem in a competitive scenario is shown in Equation (42). Similarly, using the Euler-Lagrange equation, it can be observed that, akin to cooperative optimization, there is no critical solution, but only one maximum point when the emission level is zero.

$$maximize\ W_i(a_i, x) = \int_0^{\infty} e^{-\rho t}[\ln a_i(t) - c * x^2(t)]dt$$
$$s.t.\ \dot{x}(t) = a(t) - bx(t) + \theta^* * e^{\delta(A*\sin(2t\pi)+T_0+k*t)} * \left(\frac{x^2(t)}{x^2(t)+1}\right), x(0) = x_0 \tag{42}$$

These results are not surprising. On one hand, seasonal temperature fluctuations are significant, especially during hot summers, making shallow lakes prone to irreversible eutrophication. Since our optimization model does not allow for a negative value of parameter a, the optimal scenario easily degenerates to a being zero. On the other hand, in the long term, with the continuous rise in temperature, the amount of internal release gradually increases. This may lead to the total utility over an infinite time horizon not converging. In other words, utility losses due to temperature effects remain significant even after an infinite time horizon, even when considering discounting. This is why a maximum value can be obtained at zero emission levels.

Emission control strategy

The conclusion that zero emissions represent the optimal solution from a long-term perspective aligns with rational understanding. However, there are also points worthy of criticism. For instance, in real-world scenarios, achieving complete zero emission is not feasible, and the utility function may not perfectly capture actual emission demands. Therefore, we further explored emission control strategies aimed at maintaining phosphorus levels by utilizing a temperature-aware state equation, shown in Equation (34).

Generally, to maintain stable phosphorus levels, this corresponds to the rate of change of x being zero in the state equation. Consequently, we can derive a time-dependent equation, Equation (43), concerning a and x.

$$0 = a(t) - bx(t) + \theta^* * e^{\delta(A*\sin(2t\pi)+T_0+k*t)} * \left(\frac{x^2(t)}{x^2(t)+1}\right) \tag{43}$$

According to Figure 3, in the absence of temperature variations, when $b = 0.68$, there is a monotonic relationship between a and x. This means that under this parameter setting, reducing emissions can lead to a direct decrease in phosphorus levels. Therefore, we chose $b = 0.68$ for simulating emission control strategies. In Figure 12, assuming we aimed to control phosphorus levels x at 0.5, it can be observed that emission levels a exhibit significant fluctuations and almost reach negative values every summer. According to our model assumptions, negative a values are not allowed. Hence, under $b = 0.68$ and $x = 0.5$, there is no perfect emission control strategy available.

[3] Ode15i is the variable order method aiming to solve fully implicit differential equations. Available at https://www.mathworks.com/help/matlab/ref/ode15i.html

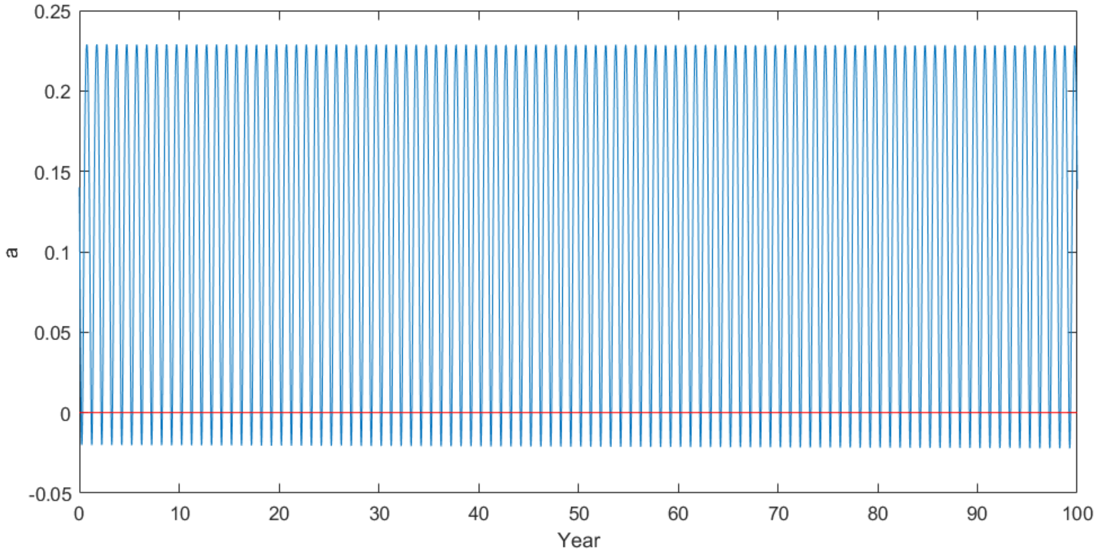

Figure 13 Emission control strategy to achieve $x = 0.5$ when $b = 0.68$

Due to the existence of the emission control strategy depends on the relative magnitudes of b and x, we have established a test equation to explore the conditions under which such a strategy can exist. The considered time frame matches that of Figure 13, spanning from Year 0 to Year 100. Within this timeframe, the temperature reaches its maximum value at $t = 99.25$. Therefore, our test equation is constructed as Equation (44).

$$0 = -bx + \theta^* * e^{\delta(A*\sin(2\pi*99.25)+T_0+k*99.25)} * \left(\frac{x^2}{x^2+1}\right) \tag{44}$$

This equation can provide a specific relationship between x and b. In Figure 13, there is a curve given by this equation, which divides the phase diagram into two parts. On the left side of the curve corresponds to the absence of a perfect emission control strategy. In these cases, within the span of one hundred years, x cannot be maintained at a stable level by non-negative a. On the right side of the curve corresponds to the existence of a perfect emission control strategy. In these regions, there always exists a positive a to maintain x at a stable level. In Figure 15, we have plotted two perfect emission control strategies and marked the positions of the scenarios from Figure 13 in this phase diagram.

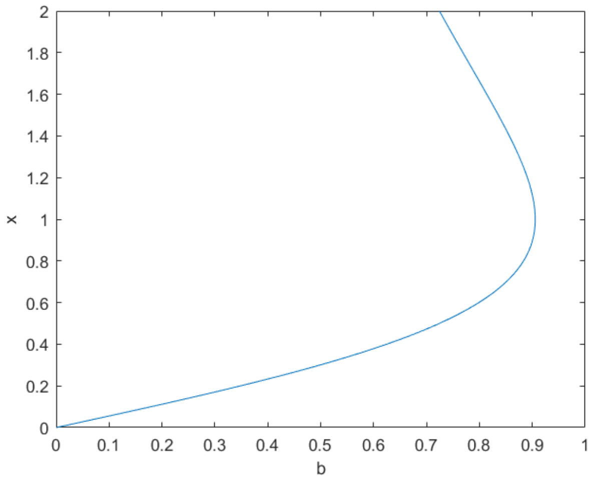

Figure 14 x-b relationship that allows emission control strategy

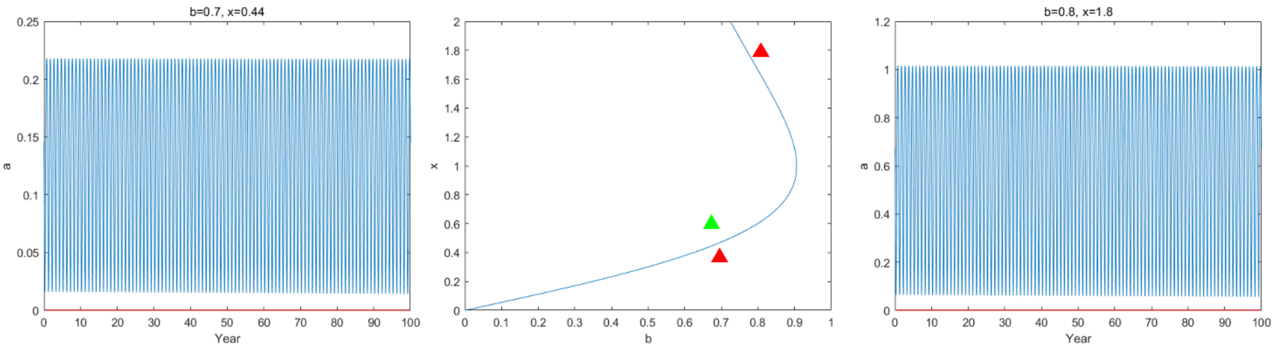

Figure 15 Existence emission control strategy in the phase diagram

Intuitively, the curve that divides the regions where perfect emission control strategies exist and do not exist can be divided into two bundles by a reversal point. In the lower bundle, there is a positive correlation between x and b. Considering that the magnitude of b corresponds to the relative sizes of outflow and the amount permanently sequestered in phosphorus changes, this implies that the larger the outflow and the permanent sequestration, the easier it is to maintain stable phosphorus levels by controlling emissions. In contrast, in the upper bundle, there is a negative correlation between x and b. In these cases, x takes larger values, indicating that the shallow lake is already in a eutrophic state. Here, the changes in internal release with the increase of x are relatively small, while outflow and permanent sequestration still increase linearly with x. Therefore, the demand for b decreases. In other words, for eutrophic lakes, the impact of internal release is relatively low.

Policy implications

As mentioned at the beginning of this article, this study explored the shallow lake model to address three challenges encountered in eutrophication management. These challenges include the hysteresis and irreversible characteristics of eutrophication in shallow lakes, the significant influence of temperature on the eutrophication process in shallow lakes, and the presence of market failure issues in shallow lake management. Now is the time to address these questions.

The shallow lake model simulates scenarios of hysteresis and irreversibility in eutrophication through state equations. Among these, when temperature is constant, parameter b directly determines the possibility of these phenomena occurring. Parameter b measures the relative sizes of outflow and permanent sedimentation. When b is large, the occurrence of hysteresis and irreversibility is less likely, whereas when b is small, addressing hysteresis and irreversibility becomes challenging in shallow lake management. Considering practical situations, this implies that shallow lakes with higher outflow of runoff, thus larger outflow, are less likely to suffer from hysteresis and irreversibility issues (Cottingham et al., 2003; Scheffer, 2004). Conversely, shallow lakes with lower outflow of runoff are more prone to encountering hysteresis and irreversibility problems. Additionally, the shallow lake model also indicates that when the phosphorus level in a shallow lake is either low or high, there tends to be a monotonic relationship between phosphorus input and phosphorus level in the lake water. This is because, on one hand, when the phosphorus level is low, the shallow lake is relatively healthy and has not reached a level of hysteresis or irreversibility in eutrophication. On the other hand, when the phosphorus level is high, the change in internal flow with phosphorus level is less than the change in outflow, resulting in a near-linear relationship between phosphorus level and input. From another perspective, the lake has already lost much of its ecological function and is in a poor state of pollution at this point (Alexander et al., 2017; Gromov & Upmann, 2021). In terms of strategies to address hysteresis and irreversibility, emission control should be combined with environmental engineering solutions. This is because when hysteresis or irreversibility occurs, simply reducing emissions has little or no improvement effect on the shallow lake. At this point, it is necessary to neutralize the phosphate content in the water through biological, chemical, and other means. Of course, larger-scale engineering methods such as replacing sediment at the bottom of the lake or increasing outflow can also be employed to avoid hysteresis and irreversibility issues.

As for the influence of temperature on eutrophication in shallow lakes, our model, based on principles of environmental science and experimental data, indicates that temperature has a significant impact. Regarding its effect on steady state, both seasonal fluctuations and long-term temperature increases can potentially lead shallow lakes to evolve into states of hysteresis and irreversibility, thereby increasing the difficulty of management. Particularly, significant temperature differences between seasons necessitate the application of different management strategies in different seasons. Our research suggested that shallow lakes are highly likely to be in a state of irreversibility during summer. This implies that if elevated phosphorus levels or algae blooms are observed during summer, it is essential to address both wastewater discharge control and phosphate neutralization. Furthermore, for shallow lakes surrounded by intensively agricultural areas, spring is the period with the highest phosphorus content in agricultural wastewater, as it corresponds to the peak application period for pesticides and fertilizers (C. Yan et al., 2016). Considering that agricultural wastewater infiltration is often considered non-point source phosphorus discharge, making it difficult to detect and control, it is advisable to impose appropriate restrictions on point source phosphorus emissions during this period. This approach serves to prevent eutrophication during this critical period and reduces the management difficulty during the riskiest summer months.

Regarding market failure, as mentioned earlier, from the perspective of ecological value, we framed the management of shallow lakes as a public good problem. Shallow lakes are considered both non-excludable and non-rivalrous for surrounding communities. Their non-excludable nature arises because the ecological value of shallow lakes, such as air and water quality, and scenic beauty, is enjoyed by all residents living around the lake. Their non-rivalrous nature stems from the fact that the enjoyment of ecological value, including air and water quality, and scenic beauty, by one individual does not diminish the enjoyment for others. Each community can increase the ecological value of the shallow lake by reducing phosphorus wastewater discharge, thereby investing in this public good at a cost. However, this often leads to free-rider behavior. Since the ecological value is shared by all communities, each community tends to contribute less. Consequently, in the presence of multiple communities, the public good is underinvested. Applied to the shallow lake problem, in a situation of multi-party games, the shallow lake may be in a state of higher phosphorus levels than optimal management, and each community may receive less benefit than optimal management would provide.

From a policy design perspective, optimal management can be achieved through top-down management by a unified administrative body. In fact, in China's water governance, systems such as the River Chief System and Lake Chief System, which clearly define water management jurisdiction, can achieve this effect (Cao & Wantzen, 2023; Li et al., 2020). Additionally, addressing market failure issues can involve measures such as emission taxes and emission rights trading to internalize negative externalities. It is worth noting that when discussing the gap between game theory and optimal governance, rising temperatures can reduce this gap. However, this does not mean that decisions under game theory become more ecologically friendly after temperature rises; rather, it means that the change in equilibrium points is not as significant as the change towards optimal governance. Consequently, whether through top-down administrative governance or market-based mechanisms like taxes and trading, adjustments need to be made flexibly in response to changes in temperature.

Discussion

Overall, this study has provided an in-depth exploration of the original shallow lake model and further research by incorporating temperature factors. However, there are still many limitations, and future research is expected to address these shortcomings.

Firstly, in practical eutrophication management, reducing phosphorus emissions and removing phosphorus from water bodies complement each other. However, in the shallow lake model, only non-negative phosphorus input is considered, while the process of phosphorus removal is ignored. This leads to two consequences. On one hand, from the perspective of the shallow lake model, the entry of shallow lakes into an irreversible state is directly declared as a situation to be avoided at all costs. On the other hand, in the discussion of dynamic optimization, especially when considering time, the inability to accommodate negative input values leads to the optimization process terminating when encountering zero emissions. To address this issue, we proposed modifying the utility function to allow for negative phosphorus inputs, representing phosphorus removal. Equation (45) and Figure 15 provide an example of this modification.

$$utility = \max(\ln a\,, -e^{-2a}) - cx^2 \tag{45}$$

In Equation (45), the utility provided by phosphorus input takes two forms. One is $\ln a$, which is the same as in the original model, representing the economic value of the shallow lake as a waste sink. The other form is $-e^{2a}$, which is always a negative real number regardless of the value of a, representing the cost of phosphorus removal. When considering total utility, the larger of the two forms is chosen. In other words, when emissions are positive, each community can choose to reduce emissions or spend money to remove phosphates from the lake. When a is negative, it means that communities can only spend money to remove phosphates from the water. This modification is expected to address the issue encountered in dynamic optimization where the value of a is negative. However, due to space limitations and a lack of appropriate references for the cost of shallow lake management, this study did not delve into this modification.

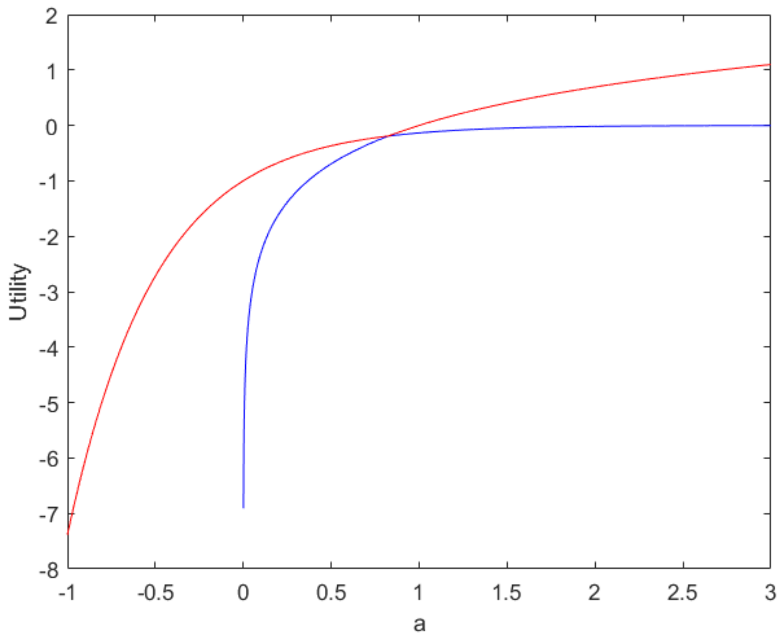

Figure 16 Utility provided by a, based on Equation (45)

Furthermore, it is undeniable that the shallow lake model is an oversimplified model. In reality, there are many factors influencing eutrophication, including phosphorus-nitrogen interactions, ecological structure, light exposure, and many others that have not been incorporated into this model. Therefore, the insights obtained from the shallow lake model can only be qualitative. In this regard, we noted that many researchers are constructing complex and comprehensive environmental science simulation programs, such as the Computational Aquatic Ecosystem Dynamics Model (CAEDYM) (Hipsey et al., 2005). If economic decision-making could be explored based on these simulation programs, significant discoveries may also be made.

Lastly, this paper has another limitation in that it did not include policy discussions in the optimization problem of the shallow lake model. Since Mäler (2003) established the original shallow lake model, researchers have discussed the impact of taxation in the context of dynamic optimization, including how taxation can ensure optimal management and rational tax strategies. It would be meaningful to continue exploring the effects of potential policy tools based on incorporating temperature considerations. However, this discussion is somewhat beyond our scope and may require additional handling of seasonal temperature variations.

Conclusion

In conclusion, this master project first reviewed the rationale behind the establishment of the shallow lake model, highlighting its significance in addressing issues such as hysteresis and irreversibility, temperature effects, and market failure in eutrophication management. Subsequently, we replicated important findings from the original shallow lake model research and conducted a more in-depth analysis, particularly identifying the gap between optimal control and game-theoretic scenarios compared to static optimization results. It is noteworthy that we innovatively incorporated a temperature factor into the shallow lake model to evaluate the impact of seasonal temperature variations and climate change on eutrophication management. The results showed that temperature has a profound impact on the eutrophication process, necessitating seasonal considerations in governance strategies. Moreover, long-term shallow lake management requires phosphorus removal, not just reducing phosphorus emissions, should be considered. Lastly, we discussed the limitations of this study and provided insights for future shallow lake model research.

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