STOCHASTIC OPTIMIZATION IN MARKET DESIGN AND INCENTIVE MANAGEMENT PROBLEMS

by

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Dissertation submitted in partial fulfillment of the
requirements for the degree of Doctor of Philosophy
in the Department of Business Administration
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ABSTRACT

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Abstract

This dissertation considers practical operational settings, in which a decision maker needs to either coordinate preferences or to align incentives among different parties. We formulate these issues into stochastic optimization problems and use a variety of techniques from the theories of applied probability, queueing and dynamic programming.

First, we study matching over time with short- and long-lived players who are very sensitive to mismatch, using a novel method to characterize the mismatch. In particular, players' preferences are uniformly distributed on a circle, so the mismatch between two players is characterized by the one-dimensional circular angle between them. This framework allows us to capture matching processes in applications ranging from ride sharing to job hunting. Our analytical framework relies on threshold matching policies, and is focused on a limiting regime where players demonstrate low tolerance towards mismatch. This framework yields closed-form optimal matching thresholds. If the matching process is controlled by a centralized social planner, the matching threshold reflects the trade-off between matching rate and matching quality. The corresponding optimal matching threshold is smaller than myopic matching threshold, which helps building market thickness. We further compare the centralized system with decentralized systems, where players decide their matching partners. We find that matching controlled by either side of the market may achieve or almost achieve optimal social welfare. However, if long-lived players are match makers, the matching system hurts short-lived player greatly by taking away their entire surplus in each match.

Second, we consider a dynamic incentive management problem in which a principal induces effort from an agent to reduce the arrival rate of a Poisson process of adverse events. The effort is costly to the agent, and unobservable to the principal, unless the principal is monitoring the agent. Monitoring ensures effort but is costly to the principal. The optimal contract involves monetary payments and monitoring sessions that depend on past arrival times. We formulate the problem as a stochastic optimal control model and solve the problem analytically. The optimal schedules of payment and monitoring demonstrate different structures depending on model parameters. Overall, the optimal dynamic contracts are simple to describe, easy to compute and implement, and intuitive to explain.

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Chapter 1

Introduction

It is important to coordinate preferences and to align incentives among participates in any business setting as they have great impacts. Thus, achieving the aforementioned goals in an optimal way is the key to success of a firm. This dissertation considers two applications in stochastic matching and dynamic incentive management, respectively. We build analytical models and search for optimal solutions in the two applications.

1.1 Matching Supply and Demand with Mismatch-Sensitive Players

The recent development of online matching platforms draws lots of attention from the operations community. The easy accessibility of these platforms lead to diverse user groups having different preferences. Take matching on ride-sharing platforms as an example. In the past, hitchhiking riders can only stand on the side of a road waiting to be picked up by a kind driver who may go to similar direction that the rider is hoping for. Nowadays, a rider can simply check her phone for a real-time match with a driver. However, easier accessibility leads to pickier users. Both drivers and riders are very selective on their matching parters.

While collaborating with DiDi Chuxing, the largest Chinese ride-sharing company, we realized that there was a lack of theoretical support in matching policies. There are two main questions desperately need to be answered: What is the relationship between a matching policy and the market density? Which side of the market, riders or drivers, should be match makers?

In order to fulfill this gap between the industrial practice and academic wisdom, in Chapter 2 we come up with a novel method to characterize customers' preferences over the quality of matches. We model all incoming players' preferences as uniformly distributed on the boundary of a circle. Therefore, the mismatch between two players can be characterized by the one dimensional circular angle between them, which we refer as the mismatch angle. Then we focus on threshold matching policies on the mismatch angle. As long as two players of different types have a mismatch angle less than such a threshold, a match is purposed by the platform to players. However, in this two-sided market, each customer from either side can reject the proposal. We provide a complete methodology for analyzing the matching model on a limiting regime where customers have low tolerance towards mismatch angles. By leveraging on similarities with queueing systems, we first bound the birth-death process of customers by two $M/M/\infty$ queues, which greatly simplifies our analysis. Next, we use Taylor expansions on customers' limited mismatch tolerance to obtain the optimal matching threshold with closed-form expression, which reveals its relationship with market density. We also analyze decentralized matching systems where riders or drivers are match makers, respectively. It turns out, optimal social welfare can always be achieved if short-lived players are match makers. However, if long-lived players are match makers, optimal social can be close but never reach the optimal centralized value. Furthermore, the platform should be cautious letting long-lived players act as match makers because it induces extreme utility distributions between short- and long-lived players.

Although our circular model can draw apparent connections with spatial features, our model is not specifically tailored towards ride-sharing platforms. For example, in a job hunting scenario, the mismatch angle can be interpreted as the discrepancy between a candidate's skill and the position requirement.

1.2 Optimal Monitoring Schedule in Dynamic Contracts

Adverse events often bring significant damages to firms and the society. In many applications, better effort can reduce the arrival rate of such events to a minimal. Moreover, an agent who is in charge of the effort cannot bear all consequences caused by an adverse event due to limited liability. Thus, a principal, be it a firm or a government, may choose to monitor the agent, which reduces the arrival rate of adverse events, and ensures that should they happen, they are not caused by lack of effort. However, monitoring is expensive and too costly to conduct all the time. Furthermore, payments contingent on arrivals can also motivate the agent to always exert effort. Thus, what is the optimal schedule for a principal to induce effort from an agent with minimum payment and monitoring cost?

In Chapter 2, we formulate this problem as an stochastic optimal control problem in continuous time. A risk-neutral principal hires an agent who can reduce the instantaneous arrival rate of costly adverse events by exerting effort. The effort is costly to the agent and unobservable to the principal without monitoring. As expected, if the monitoring cost is lower than a threshold, the principal should monitor all the time. In this case, the agent's total future utility or promised utility is always kept at 0. Interesting structures emerge when the monitoring cost is above the threshold. In this case, the promised utility serves as a sufficient statistic of the entire history of arrival times, on which the optimal monitoring and payment schedules critically depend. In general, the agent needs to be penalized for each arrival when not being monitored. Because we assume that the agent has limited liability and cannot pay the principal, the penalty takes the form of a downward jump of the promised utility upon each arrival whenever the agent is not being monitored. Between arrivals, the promised utility gradually increases. When downward jumps due to arrivals bring the promised utility below a threshold, the principal starts monitoring. Thus, whenever downward jumps due to arrivals bring the promised utility below a threshold, the principal starts monitoring. Monitoring stops only after the promised utility climbs back to the threshold, during which arrivals do not matter. A flow of payment starts only when the promised utility reaches and stays at an upper bound. As soon as another arrival occurs, the promised utility takes a downward jump from the upper bound, which stops the payment.

1.3 Structure of the Dissertation

In the remainder of this dissertation, we first analyze a stochastic matching system in Chapter 2. We consider both centralized and decentralized matching in this chapter and provide corresponding insights. Second, we consider a dynamic contract design problem involving monitoring and payment schedules. We identify the optimal schedules and explain the intuition behind them. All proofs and supplementary materials are presented in Chapter 4. Finally, in Chapter 5, we conclude this dissertation with a few final words.

Chapter 2

Matching Supply and Demand with Mismatch-Sensitive Players

2.1 Introduction

The booming development of online matching platforms draw lots of attention from the operations community. Easily accessible to public, these matching platforms gather millions of potential clients every day. The sheer size of user base means diverse preferences among users. Players' preferences mainly has two dimensions: quality and timing of matches. Take carpooling in ride-sharing platforms as an example. During morning(evening) rush hours, riders and drivers depart from the same neighborhood(business district), but post different individual destinations on the platform. Obviously, players prefer to be matched with others who have similar destinations. Moreover, riders typically operate on tight time constraints, and may actively look for alternatives elsewhere if not being matched immediately on the platform. Drivers can be more patient but need to depart eventually after some time. DiDi, the largest ride-sharing company in China, offers a peer-to-peer car-sharing platform to serve riders and drivers in the example above, called DiDi Hitch. It is based on a two-sided search process that involves both riders' and drivers' inputs of destinations and mutual acceptance. In this service, drivers are mainly commuters instead of professional drivers. Each rider submits her final destination to the platform and receives a list of potential drivers who are going to similar directions in return. Then the rider proposes to a driver of her choice. The corresponding driver may accept or reject it. Blablacar in Europe has similar services as well. Moreover, based on our conversation with DiDi Hitch, two questions need to be answered. First of all, how to does platform's pricing decision impact the matching process? Second, as the carpooling service is mostly decentralized at the moment, which side of market should be the decision maker? In DiDi Hitch's situation, should riders or drivers be the side of the market who has the final say on whether a potential match can move forward?

To answer these questions, we consider a two-sided matching market where a platform serves as an intermediary who matches players arriving following Poisson processes. Building upon the observation from DiDi Hitch, one side of players (riders) are short-lived, who leave the platform immediately if not being matched upon arrival. The other side of players (drivers) are long-lived, who stay on the platform for an exponentially distributed period of time without a match. When matching players, the match maker needs to consider their individual preferences. Otherwise, players may reject the match.

A salient feature of this paper is that we provide a novel method to characterize players' individual preferences over the quality of matches. In other words, we construct a measure for mismatch between players that directly affects matching outcomes. We have all incoming players' preferences uniformly distributed on the boundary of a circle. The mismatch between two players can be characterized by the one dimensional circular angle between them, which we refer to as the *mismatch angle*. Thus, throughout our model and analysis, we can use this one-dimensional scaler to describe the compatibility between any two players. Our characterization of players' mismatch is natural in ride sharing and others scenarios with spatial features. In the example of DiDi Hitch, mismatch is the difference between destination of a rider's and a drivers' destinations. Closer destinations translate to smaller mismatch angles in our model. Furthermore, in each match, each individual may not need to bear the entire mismatch. Consider the carpooling example. Only drivers bear mismatch angles as riders are always dropped off at their requested destinations.

Before going into results and analytical schematics, it is worth noting that our model is not restrictive to carpooling in ride-sharing. On a gig job hunting platform, service seekers want to find suitable candidates as soon as possible. A candidate may be more patient but will not settle with a job requiring very different skills than what she has already possessed. In this context, a mismatch angle of our model captures the difference between the skill a candidate possesses and the talent a seeker is looking for. The smaller the angle is, the more compatible two players are.

We consider a setting where the platform can set a *price* that a short-lived player needs to pay to a long-lived player in each match. This money transfer direction is consistent with our motivating examples. In DiDi Hitch's example, riders, who are short-lived players need to pay drivers, who are long-lived players, since drivers bear the cost of detour (mismatch angle) as well as cost of providing the service. Our results reveal clear relationship between matching policy and the market thickness. Since our model captures mismatch between players using a circular angle, we focus on *threshold policies* that match two players if their mismatch angle is small enough. By designing the price, the platform can affect players' tolerance and thresholds on mismatch angles. First, we show that the platform should not match myopically (match as many players as possible) when acting as the match maker. Instead, we provide a closed-form expression to approximate the optimal price, which is smaller than the myopic price. Here the intuition is that a smaller price induces players to be picker, which improves matching quality. Although it may decrease overall matching rate, a smaller price thickens the market of long-lived players. Thus, the slight drop in matching rate is compensated by more potential matching parters and better matching quality. Second, the optimal price has an inverse relationship to the thickness of the market. That is, if either side of the market becomes thicker, the optimal price is always smaller.

We also study two decentralized systems where the platform allows short-lived and long-lived players to be match makers. Both scenarios are formulated as games with potentially infinite number of players. In the first scenario, we consider a game among short-lived players. In equilibrium, they match myopically. The platform can exploit this myopic behavior of short-lived players by simply using the optimal centralized price to recover the optimal social welfare. If long-lived players are match makers, on the other hand, we provide a numerical procedure that allows us to compute the effective mismatch angle in equilibrium. We find that long-lived players' equilibrium threshold on the mismatch angle is smaller than what the platform desires. Thus, if the platform keeps using the centralized price, long-lived players would be too picky and give up many matching opportunities that generate positive utilities. Therefore, in order to improve the social welfare, the platform needs to inflate the price, which hurts short-lived players to the point that all their surplus are extracted and paid to long-lived players. Our results indicate that this price inflation can recover most of the loss in social welfare comparing to that of the centralized matching system. However, the platform should caution the extremely unbalanced utility distributions between short- and long-lived players in this decentralized matching system.

Our analytical framework that yields the aforementioned results tackles many technical challenges in this problem. Note that since short-lived players never stay on the platform, their arrival rate represents their side of market thickness. However, describing the market thickness for long-lived players is non-trivial. We model the number of long-lived players on the platform as a Continuous Time Markov Chain (CTMC), to be more specific, a Birth-Death process. We perform all analysis of the matching system under the steady state of this CTMC. As mentioned in the results, an aggressive matching policy that tries to match as many players as possible may generate good short-term revenue at the moment but leads to a thinner market. Intuitively, matching in a thin market is difficult and hurts long-term revenue. Utilizing a CTMC helps us answer the question: "What is a good matching policy that balances the market thickness and the immediate matching rate?" both qualitatively and quantitatively.

Aiming to characterize the matching system with closed-form expressions, we consider the case where players who have really small mismatch angle. Under this limiting regime, we introduce appropriate $M/M/\infty$ queues to bound the original CTMC. The classic result on the stationary distribution of $M/M/\infty$ queues (see e.g., [Igl65]) greatly simplifies our analysis. Moreover, using Taylor expansions on players' tolerance towards mismatch angles helps us further obtain various results in closed-form.

Recent literature on matching intermediaries paves the way for our work. A stream of related papers study matching using fixed matching probability for all players. In this literature, players are homogeneous in the sense that they are indifferent to whom they are matched with (see e.g., [UÏ0, AAYG17, BC17, ALG19]). In particular, [ALG19] consider a dynamic matching system in a networked market with departure. They show that market thickness is important if the platform can identify which player is about to leave and only match these players. Many papers (see e.g., [UÏ0, AAYG17, ALG19]) show that myopic policy is near optimal. In comparison, our problem deals with players with heterogeneity in preferences and our results indicate that strategically delayed matching can be beneficial.

Other papers consider heterogeneous players who can only be matched (or have higher matching probability) if matching partners' (discrete) types are compatible. [ABJM19] study a dynamic matching market with easy and hard to match players. In their model, hard to match players have significantly lower matching probability compared with that of easy to match players, and all players want to be matched as soon as possible. They analyze the performance of myopic matching policies involving bilateral and chain matching. [OW16] consider a matching problem for ride-sharing. They use players' origin or destination as types. Riders only accept drivers who can arrive in a certain time window. They take advantage of a large market, where players are considered as a continuum, and identify policies that match the most players. In other applications, although any pair of players can be matched, the matching reward depends on players' types. [HZ19] consider a two-sided matching problem in a discrete-time finite horizon setting. Players from both supply and demand side can abandon the system in any period. [CHZ19] consider a two-sided market with short-lived demand and long-lived supply. Players from each side has two types and matching rewards has a supermodular structure according to players' types. Our work differs from the papers in our circular model to connect players' individual preferences(types) with matching outcomes. This innovation leads us to obtain concise and informative results.

Another steam of research study two-sided matching via queueing methods. [ADM14] study trading systems using double-sided queues. They also consider short and long-lived players similar to our model without circular preference. They provide performance measures of queues under First-Come-First-Serve policy such as expected waiting time, etc. Many other papers analyze matching policies under fluid limits (see e.g., [ZCW00, SZ06, AAAE12, GW14, OW16, KS19]). Our work exploits the similarity between our matching system with $M/M/\infty$ queues.

Our circular modeling approach also resembles the Hotelling's circular city model in economics. The original model appears in [Sal79], which uses a circular model as a geographical representation of a city. Suppliers in that model sit at fixed locations on a circle, and consumers have preferences over their relative locations to the suppliers. Recently, [PG19] extend this model to a three dimensional space as a cylinder, representing two dimensional preferences. However, there is no arrivals or departures in either of these two papers. Circular city model has also been applied in the Operations literature. [FKW20] consider a ride-hailing scenario in a circular city. In their model, riders follow a Poisson process with origins and destinations distributed on the circle. The number of taxi on the circle is fixed and always travel clock/counterclockwise with constant speed. They compare efficiency between the traditional taxi services and that of the ride hailing services. Both of their mechanisms are comparable to our myopic matching policy as a rider is always matched (or is picked up) with the nearest available taxi (or by the first taxi passing by) immediately.

The rest of this paper is organized as follows. We introduce the model and formulate the matching system in Section 3.2. In Section 2.3, we introduce results when the platform is the match maker, and compare with results on decentralized systems in Section 2.4. We conclude our discussion in Section 3.7 and highlight some future directions. All proofs and some minor derivations are presented in the Appendix.

2.2 Model Setup

Consider two classes $\{L, S\}$ of players arriving to the matching platform. Type L players are *long-lived* (patient). They follow a Poisson arrival process with rate λ_L. Each arrival has a "life time" that is exponentially distributed with rate γ; if no match occurs by the end of its life time, the player disappears from the platform at that time. Type S players are *short-lived* (impatient) with exponential arrival rate λ_S and leave the platform immediately if not matched upon arrival. Two players are matchable only if they are from different classes. Therefore, matches can only be made upon arrival of short-lived players.

We use a "circular" model to describe compatibility between players. Upon arrival, each player from either class uniformly and independently claims a random spot on the edge of a circle. Between players from two classes, their mismatch is measured by the arc, or, equivalently, the central angel between their spots on the circle. To simplify notations in future sections, we define $\phi \in [0, 1]$ as the *mismatch angle*, which is the ratio between the actual angle of two locations to the maximum possible angle π. For example, two players with angle $\pi/4$ (or 45 degrees) between their locations on the circle have a mismatch angle 0.25. Mismatch angle effectively normalizes the actual angle on the circle into a variable inside the interval $[0, 1]$. Furthermore, we assume that the long-lived player in each match bears the entire cost of mismatch. In the carpooling example, a rider needs to be sent to her requested location and her driver needs to drive the entire detour.

We assume that each short-lived player receives a benefit of u per match while each long-lived player is subjected to mismatch cost with coefficient c per unit of mismatch angle. Furthermore, the platform sets a payment P to be transfered from a short-lived player to a long-lived player in each successful match. Therefore, we define player's utility in a match with mismatch angel ϕ and payment P as,

$$W_S(\phi) = u - P, \quad W_L(\phi) = P - c\phi, \quad \forall \phi \in [0, 1]. \tag{2.1}$$

As we can see from (2.1), short-lived players' utility is fixed in each match despite the quality of the match as they do not bear any mismatch angle. However, long-lived players' utility function is decreasing *w.r.t.* the mismatch angle.

Before moving on, it is worth pointing out we can define the social welfare generated in each match as the summation of utilities from both sides. In other words, given mismatch angle of

$\phi \in [0,1]$ in a match, the social welfare is

$$W_{SW}(\phi) = u - c\phi, \quad \forall \phi \in [0,1]. \tag{2.2}$$

Since the transfer between players are internal to the system, the social welfare in (2.2) is simply the difference between the benefit generated for a short-lived player and the mismatch penalty for a long-lived player. Recall the utility functions in (2.1). A short-lived player pays $P \leq u$ and a long-lived player shall participate in a match only if the mismatch angle is no greater than P/c. Therefore, define mismatch *tolerance* as

$$\epsilon := \frac{u}{c}. \tag{2.3}$$

The larger ϵ is, the more tolerant players are towards mismatch angles. In this paper, we consider players as very sensitive to mismatch. That is, ϵ approaches 0. Furthermore, define normalized *price* as

$$\rho := \frac{P}{u}, \tag{2.4}$$

representing the fraction of a short-lived player's benefit transfered to a long-lived player in each match. For example, if the price $\rho = 1$, the platform leaves a short-lived player 0 surplus. If ρ is small, a short-lived player can keep most of the benefit generated in a match.

2.3 Centralized Matching

In this section, the platform is the match maker. Whenever a short-lived player arrives, there are $x \geq 0$ number of long-lived players on the platform. Denote ϕ_i, $i \in \{1, ..., x\}$ as the mismatch ratio between each of the x long-lived players and the focal short-lived player. Moreover, let $\underline{\phi} = \min\{\phi_i \,|\, i = 1, ..., x\}$ to represent the minimum mismatch ratio between the short-lived player and the x long-lived players. After observing $\underline{\phi}$, the platform decides whether to match between the short-lived player and the closest long-lived player. In this section, we assume that the platform is the match maker. That is, players from both sides have to accept the match as long as both utilities are non-negative. This means that a match is successful only if $\underline{\phi} \leq \epsilon\rho \leq \epsilon$, or

$$\frac{\underline{\phi}}{\epsilon} \leq \rho \leq 1, \tag{PC}$$

when the minimum mismatch angle is $\underline{\phi}$. The first inequality in (PC) follows (2.1), (2.3) and (2.4), representing the maximum mismatch angle for a long-lived player; the second inequality in (PC) represents a short-lived player never pays a price more than her benefit. We refer (PC) as players' *participation constraint*. Note that (PC) is similar to players' ex-post individual rationality constraints in the economics literature. Immediately, following (PC), we observe that any feasible price ρ shall induce a threshold $\epsilon\rho$ such that matching happens only if $\underline{\phi} \leq \epsilon\rho$.

2.3.1 Matching probability and Birth-Death process

As matches can only occur with arrivals of short-lived players, in this paper, we use the term *matching probability* to represent the probability that *upon the arrival* of a short-lived player, she can be matched with a long-lived player.

When there are x long-lived players available and a price ρ, the matching probability is

$$p_\epsilon(x, \rho) = 1 - (1 - \epsilon\rho)^x, \tag{2.5}$$

where inside the parentheses is the probability that the short-lived player cannot be matched with a long-lived player, whose location is uniformly distributed on the circle. Since long-lived players are located independently, $(1-\epsilon\rho)^x$ is the probability that all x number of long-lived players cannot be matched with the short-lived player.

Next we derive the distribution function of minimum mismatch angle $\underline{\phi}$, which is a random variable that depends on the number of current long-lived players. Recall ϕ_i is the mismatch angle between the arriving short-lived player and a long-lived player $i \in \{1, 2, ..., x\}$ when there are x number of long-lived players in total. The probability that the minimum mismatch angle $\underline{\phi}$ is no greater than $\phi \in [0, 1]$ is

$$H_x(\phi) = 1 - \Pi_{i=1}^{x} \mathbb{P}(\phi_i > \phi) = 1 - (1 - \phi)^x, \quad \forall\, 0 \leq \phi \leq 1, \tag{2.6}$$

which is the C.D.F. of the random variable $\underline{\phi}$. By differentiating $H(\cdot)$, we obtain its P.D.F.

$$g_x(\underline{\phi}) = x(1 - \underline{\phi})^{x-1}, \quad \forall\, 0 \leq \underline{\phi} \leq 1. \tag{2.7}$$

Note that both the C.D.F. and the P.D.F. of the minimum mismatch angle is parametrized by the number of long-lived players on the platform, x. Recall that we focus on threshold policies on the minimum mismatch angle. The distribution of $\underline{\phi}$ helps us characterize the quality of matching outcomes.

As we can see, both matching probability and distribution of the minimum mismatch angle critically depend on the number x of long-lived players on the platform. Therefore, in order to characterize the dynamics of the long-lived players, we formulate the arrival and departure of long-lived players as a Continuous-Time Markov Chain (CTMC), which is a "Birth-Death" process. Denote $f_\epsilon(x, \rho)$ to represent the density function of the stationary distribution of this "Birth-Death" process, which solves the following system of equations,

$$\begin{aligned} \lambda_L f_\epsilon(0,\rho) &= (\lambda_s p_\epsilon(1,\rho)+\gamma) f_\epsilon(1,\rho), \\ \lambda_L f_\epsilon(x-1,\rho) + (\lambda_s p_{\epsilon\rho}(x+1) + \gamma(x+1)) f_\epsilon(x+1,\rho) &= (\lambda_L + \lambda_s p_\epsilon(x,\rho) + \gamma x) f_\epsilon(x,\rho), \quad x = 1, 2, ... \end{aligned}$$

This system of difference equations has the following unique solution: [1]

$$f_\epsilon(x,\rho) = \frac{{\lambda_L}^x f_\epsilon(0,\rho)}{\Pi_{k=1}^x (\lambda_s p_\epsilon(k,\rho)+\gamma k)}, \quad \forall x \geq 1, \tag{2.8}$$

and

$$f_\epsilon(0,\rho) = \left(1 + \sum_{x=1}^{\infty} \frac{{\lambda_L}^x}{\Pi_{k=1}^x (\lambda_s p_\epsilon(k,\rho)+\gamma k)}\right)^{-1}. \tag{2.9}$$

We denote $X_\epsilon(\rho)$ as the random variable according to the steady state distribution $f_\epsilon(\cdot, \rho)$. It is worth noting that $X_\epsilon(\rho)$ reflects the *market thickness* of long-lived players. More players available leads to a thicker market. Thickness of short-lived players is characterized by their arrival rate λ_S since they never stay on the market.

In the following sections, we conduct analysis under steady state of the matching system where the number of long-lived players follows the distribution $f_\epsilon(\cdot, \rho)$ in every match. As a result, any utility functions are evaluated by taking expectations over the random variable $X_\epsilon(\rho)$.

2.3.2 Social welfare rate

We consider the platform maximizes the social welfare rate by designing the price ρ and subjected to players' (PC) condition. That is, the platform's (centralized) problem is

$$\max_{0 \leq \rho \leq 1} \mathbb{E} h_\epsilon(X_\epsilon(\rho), \rho) \tag{2.10}$$

where

$$h_\epsilon(x,\rho) = \mathbb{E}_{\underline{\phi}}\left[W_{SW}(\underline{\phi})\mathbb{I}\{\underline{\phi} \le \epsilon\rho\} \,|\, x\right] = \mathbb{E}_{\underline{\phi}}\left[c(\epsilon - \underline{\phi})\mathbb{I}\{\underline{\phi} \le \epsilon\rho\} \,|\, x\right], \tag{2.11}$$

and function $W_{SW}(\cdot)$ is the social welfare generated per match in (2.2). Function $h_\epsilon(x,\rho)$ represents the expected social welfare generated in a match under price ρ when there are x number of long-lived players on the platform. The expectation in (2.11) is taken *w.r.t.* random variable $\underline{\phi}$, representing the minimum mismatch angle, which follows distribution function (2.7). It is worth pointing out that the objective function in (2.10) represents the expected social welfare generated per match under steady state. The social welfare rate is $\lambda_S \mathbb{E} h_\epsilon(X_\epsilon(\rho),\rho)$ per unit of time because matching can only happen upon their arrivals.

Before searching for the optimal price, it is worth noting a very simple *myopic* price $\rho_M := 1$, which matches as many players as possible in a myopic fashion. The myopic price ρ_M extracts all surplus from short-lived players to long-lived players. By doing so, this myopic price also induces long-lived players to be as tolerant as possible. In other words, the myopic price ρ_M maximizes matching rate as long as players generate non-negative social welfare. Later, we show that the platform can design a better price than using the myopic one.

2.3.3 Approximations

It is hard to obtain analytical results with closed-form expressions from the matching system proposed before. Even though (2.11) has a closed-form expression, its expectation *w.r.t.* $X_\epsilon(\rho)$ is hard to compute in closed-form due to complicated expressions of its steady state distribution. Therefore, we propose an approximation approach in this section when ϵ approaches 0.

Before going into the limiting regime, we first consider an $M/M/\infty$ queue related to our original CTMC, with arrival rate $\lambda = \lambda_L$ and service rate $\mu = \gamma + \lambda_S \epsilon\rho$. Note that this $M/M/\infty$ queue mimics the original Birth-Death process. The x active servers resemble x long-lived players on the platform. The matching rate corresponds to system departure rate is $x(\epsilon\theta + \gamma)$. Thus, the matching probability in this system is simply $\epsilon\rho x$, when there are x long-lived players. The next lemma shows the relationship between matching probabilities under the original process and the $M/M/\infty$ queue.

Lemma 2.1. *Fixing* $0 \le \epsilon\theta \le 1$, *we have*

$$\epsilon\rho x \ge p_\epsilon(x,\rho) = 1 - (1-\epsilon\rho)^x, \quad \forall\, x \ge 1.$$

The intuition behind this upper bound of matching probability is straightforward. Note $\epsilon\rho x$ is the matching probability when the x long-lived players have no overlap in their tolerance angles on the circle. Thus, their total tolerance covers the maximum possible angle on the circle, which leads to a greater matching probability. Therefore, $\epsilon\rho x$ is an upper bound of $p_\epsilon(x,\rho)$ and this $M/M/\infty$ system's departure rate is higher or equal to the one in the original process.

Following classic results on $M/M/\infty$ queue [Igl65], the stationary distribution of the aforementioned $M/M/\infty$ queue is a Poisson random variable, with parameter $\dfrac{\lambda_L}{\lambda_S\epsilon\rho+\gamma}$. Denote $Y_\epsilon(\rho)$ to represent this Poisson random variable. Similarly, we can consider another Poisson random variable $Y_\epsilon(0)$ to be the stationary distribution of another $M/M/\infty$ queue with departure rate γ ($\rho=0$ in $\lambda_S\epsilon\rho+\gamma$), which is no greater than the one in the Birth-Death process for all states. The next proposition formalizes the relationship between $Y_\epsilon(0)$, $Y_\epsilon(\rho)$ and $X_\epsilon(\rho)$.

Proposition 2.1. *Fixing* $0 \le \rho \le 1$ *and* $0 \le \epsilon < 1$*, we have*

$$Y_\epsilon(0) \succeq_1 X_\epsilon(\rho) \succeq_1 Y_\epsilon(\rho). \tag{2.12}$$

Moreover, we have

$$\mathbb{E}h_\epsilon(Y_\epsilon(\rho),\rho) \le \mathbb{E}h_\epsilon(X_\epsilon(\rho),\rho) \le \mathbb{E}h_\epsilon(Y_\epsilon(0),\rho). \tag{2.13}$$

We use the notation $\succeq_1$ representing the first order stochastic dominance between two random variables [2]. Proposition 2.1 shows that replacing $X_\epsilon(\rho)$ with $Y_\epsilon(\rho)$ and $Y_\epsilon(0)$ lead to *lower* and *upper* bounds for the platform's objective function, respectively.

From this point forwards, we use $Y_\epsilon(\rho)$ to replace $X_\epsilon(\rho)$ and let ϵ approaches 0 when seeking analytical results. The next proposition justifies this approximation.

Proposition 2.2. *Consider function* h_ϵ *in* (2.11) *and fix* $0 \le \rho \le 1$*. We have*

$$\lim_{\epsilon\to 0} |\mathbb{E}h_\epsilon(Y_\epsilon(\rho),\rho) - \mathbb{E}h_\epsilon(X_\epsilon(\rho),\rho)| = 0. \tag{2.14}$$

Proposition 2.2 states that using random variable $Y_\epsilon(\rho)$ instead of $X_\epsilon(\rho)$ in the objective function is a good approximation when ϵ approaches 0. It sheds light on searching for analytical results for the social welfare rate in (2.10) because $\mathbb{E}_{Y_\epsilon(\rho)}h_\epsilon(Y_\epsilon(\rho),\rho)$ has closed-form expressions. From Proposition 2.2, we know that $Y_\epsilon(0)$ is also a good approximation for $X_\epsilon(\rho)$ when ϵ goes to 0. The reason that we do not use $Y_\epsilon(0)$ is because such an approximation effectively assumes that players depart only by reneging, with no matching aspect at all.

Furthermore, note that the expression $\mathbb{E}h_\epsilon(Y_\epsilon(\rho),\rho)$ (provided in the appendix) is still complex. Thus, we perform a Taylor expansion over $\mathbb{E}_{Y_\epsilon(\rho)}h_\epsilon(Y_\epsilon(\rho),\rho)$ around $\epsilon = 0$. The next proposition provides a simple approximation when letting ϵ approach 0.

Proposition 2.3. *Consider the Poisson random variable $Y_\epsilon(\rho)$ with rate parameter $\frac{\lambda_L}{\lambda_S\epsilon\rho+\gamma}$. There exists a third order polynomial J of ρ and ϵ such that,*

$$\lim_{\epsilon\to 0}\frac{1}{\epsilon^3}\left|\mathbb{E}h_\epsilon(Y_\epsilon(\rho),\rho) - J(\rho,\epsilon)\right| = 0, \ \ or \ \ \mathbb{E}h_\epsilon(Y_\epsilon(\rho),\rho) = J(\rho,\epsilon) + o(\epsilon^3). \tag{2.15}$$

We provide the closed-form expression of function $J(\rho,\epsilon)$ in the appendix. Here $o(\epsilon)$ represents terms such that $\lim_{\epsilon\to 0}\frac{o(\epsilon)}{\epsilon} = 0$.

Proposition 2.3 greatly simplifies expressions of social welfare rate in (2.10) under the limiting regime when ϵ goes to 0. In particular, the expected utility can be approximated by $J(\rho,\epsilon)$, which is a simple polynomial of ρ. Thus, we have set up an approximation method for analytical results. Proposition 2.2 and 2.3 are the building blocks for solving platform's problems formulated in (2.10).

With the help of our approximation framework under the limiting regime where ϵ approaches 0, we seek analytical result of the optimal matching threshold. Note that using our approximation, social welfare rate in (2.10) can be approximated as $J(\rho,\epsilon)$, which is a simple polynomial of ρ according to Proposition 2.3. Furthermore, denote $\rho^*_\epsilon \in [0,1]$ to be a maximizer of $J(\rho,\epsilon)$ for a given ϵ. We present the approximated optimal price in the next Theorem.

Theorem 2.1. *We have*

$$\rho^*_\epsilon = \hat{\rho}_\epsilon + o(\epsilon), \tag{2.16}$$

in which

$$\hat{\rho}_\epsilon = \rho_M - \frac{\lambda_S}{2\gamma}\epsilon \tag{2.17}$$

and $\rho_M = 1$. Furthermore, there exists a $\bar{\epsilon} \geq 0$ such that for all $\epsilon < \bar{\epsilon}$,

$$\mathbb{E}h_\epsilon(Y_\epsilon(\hat{\rho}_\epsilon),\hat{\rho}_\epsilon) \geq \mathbb{E}h_\epsilon(Y_\epsilon(\rho_M),\rho_M). \tag{2.18}$$

It is worth pointing out that function $J(\rho, \epsilon)$ is concave in ρ if ϵ is small (we provide a sufficient condition in the proof of Theorem 2.1 in appendix). Therefore, solving for $\hat{\rho}_\epsilon$ and ρ^*_ϵ is very easy.

In Theorem 2.1, (2.17) provides an approximate optimal solution that maximizes the approximate objective function J. The inequity in (2.18) confirms that despite the approximations, solution $\hat{\rho}_\epsilon$ indeed out-performs the myopic price ρ_M for the original objective function J.

The expression of the optimal centralized price in (2.17) shows the fundamental connection between *market thickness* and platform's matching decisions. We can interpret the ratio $\dfrac{\lambda_S}{\gamma}$ as the average number of short-lived players that a long-lived player shall encounter during the time on the platform. Intuitively, the larger the value of λ_S is, the thicker the side of short-lived players is. As a result, each long-lived player encounters more potential matching partners on average, which leads to better quality matches. Similarly, as γ decreases, long-lived players stay for longer periods of time on average, which leads to thicker market as well. Under a thicker market, the benefit of better matching quality outweighs the slight reduction in matching rate by using a smaller price. The thicker the market is, the smaller $\hat{\rho}_\epsilon$ is.

Furthermore, as it turns out, $\hat{\rho}_\epsilon$ is no greater than ρ_M. That is, the platform does not extract all surplus from short-lived players. Recall that a smaller price means smaller matching rate. The reason that the platform is pickier than matching myopically is to build market thickness. This is consistent with strategic delay/waiting identified in the matching literature [ALG19]. We can see the effect of this move from looking at the Poisson random variable $Y_\epsilon(\rho)$, which represents the number of long-lived players on the platform. Note $Y_\epsilon(\rho)$ has mean $\dfrac{\lambda_L}{\lambda_S \epsilon \rho + \gamma}$, which increases as ρ decreases. (The same result holds for random variable $X_\epsilon(\rho)$ as well.) In other words, by building a thicker market of long-lived players, the platform gives up some marginal welfare from low profit matches but gains greater future welfare. Furthermore, we observe that the gap between the optimal and myopic thresholds disappears when γ goes to infinity, representing the case where long-lived customers become more like short-lived. This effect demonstrates the importance of having long-lived players on the platform.

2.4 Decentralized Matching

In this section, we study situations, under which the platform sets the price and let either side to decide the matching threshold. That is, the platform designs the price anticipating players' behaviors, in order to maximize social welfare rate. We investigate the impact on social welfare rate if either short- or long-lived players act as match makers.

We first consider the case where short-lived players choose their matching threshold. Long-lived players have to accept the match as long the utility generated is positive. Since short-lived players never stay, their game is simple and leads to myopic decisions. That is, a short-lived player proposes a match to the long-lived player who she has the minimum mismatch angle with, if this match generates a non-negative utility for her. Therefore, the matching process is exactly the same as centralized matching in Section 2.3. As a result, the platform can simply uses the price in Theorem 2.1 and achieve the optimal social welfare rate when short-lived players are match makers. Therefore, in this section, we focus on long-lived players as match makers.

Suppose the platform still announces the price ρ but allows each long-lived player to screen her matching parters by committing to a threshold $\epsilon\theta \leq \epsilon\rho$ on mismatch angle prior to joining the platform. While on the matching platform, a long-lived player is matched with the first arriving short-lived player with whom their mismatch angle is the smallest and is no greater than $\epsilon\theta$. For notational convenience, we scale a long-lived player's threshold from $\epsilon\theta$ to θ, and consider θ as a long-lived players' acceptable cost. This is a game with potentially infinite number of players, and each long-lived player's utility depends on other players' thresholds on acceptable cost. In this section, we first study θ_ρ, which is the equilibrium matching threshold of long-lived players when platform sets price ρ. Recall that there is a one-to-one correspondence between price and mismatch angle/cost. For example, in centralized matching, a price ρ induces the platform to match players with mismatch angle no greater than $\epsilon\rho$. More generally, we have cost no more than price or, $\theta_\rho \leq \rho$. We still denote random variable $X_\epsilon(\theta)$ taking the stationary distribution of the Birth-Death process for the *total* number of long-lived players on the platform. Anticipating that each price ρ induces a cost θ_ρ among long-lived players, the platform aims to maximize social welfare by designing ρ. That is, the platform's problem is

$$\max_{0 \leq \rho \leq 1} \lambda_S \mathbb{E} h_\epsilon(X_\epsilon(\theta_\rho), \theta_\rho), \tag{2.19}$$

where function h_ϵ is defined in (2.11). For the rest of this section, we first derive the value function for each long-lived player and provide a heuristic method to evaluate it efficiently. Afterwards, we identify the equilibrium cost θ_ρ in numerically. Furthermore, we can also calculate the platform's optimal price ρ_L solves (2.19), and compare it with the centralized optimal price ρ^*, which is optimal to (2.10).

We consider threshold policies in this setting. That is, when there are x long-lived players on the platform, and let $\{\theta_i\}_{i=1,\dots,x}$ represent each long-lived player's reported cost threshold. A threshold policy matches the arriving short-lived player with an existing long-lived player i if and only if $\phi_i = \underline{\phi}$ and $\phi_i \leq \epsilon\theta_i$. Recall that ϕ_i represents the mismatch angle between long-lived player i and a short-lived player, and $\underline{\phi} = \min\{\phi_i \,|\, i = 1, ..., x\}$.

Next, we derive long-lived players' equilibrium cost threshold θ_ρ induced by price ρ, so that no individual player would unilaterally deviate. Consider the situation where all other x long-lived players commit to threshold θ and the focal player uses a threshold $\hat{\theta}$. It is clear that without loss of generality, we can focus on $\leq \theta, \hat{\theta} \in [0, \rho]$.

We note there are two immediate outcomes upon an arrival of a short-lived player. First, the focal player is matched with the short-lived player and claims expected utility. Recall the distribution function of minimum mismatch angle in (2.7), and define function

$$\mathcal{A}(x, \hat{\theta}, \theta, \rho) := \int_0^{\epsilon\theta} \int_0^{\phi} c(\epsilon\rho - \hat{\phi}) d\hat{\phi} g_x(\phi) d\phi + \int_{\epsilon\theta}^{1} \int_0^{\epsilon\hat{\theta}} c(\epsilon\rho - \hat{\phi}) d\hat{\phi} g_x(\phi) d\phi, \quad \forall \leq \theta, \hat{\theta} \in [0, \rho] \text{ and } x \geq 0, \tag{2.20}$$

which represents the expected utility of a long-lived player whose threshold is $\hat{\theta}$ while all other x long-lived players on the market use threshold θ. Then we have

$$\mathbb{E}_{\hat{\phi}, \underline{\phi}} \Big[W_L(\hat{\phi}) \mathbb{I}\{\hat{\phi} \leq \min\{\underline{\phi}, \epsilon\hat{\theta}\}\} \,|\, x \Big] = \begin{cases} \mathcal{A}(x, \hat{\theta}, \hat{\theta}, \rho), & \text{if } \hat{\theta} \leq \theta, \\ \mathcal{A}(x, \hat{\theta}, \theta, \rho), & \text{if } \hat{\theta} > \theta, \end{cases} \tag{2.21}$$

where function W_L is defined in (2.1); random variable $\hat{\phi}$ represents the mismatch angle between the focal player and the short-lived player, following uniform distribution on [0,1]; random variable $\underline{\phi}$ represents the minimum mismatch angle among all other players, following distribution in (2.7); and the indicator function inside the expectation states the condition that the focal player shall be matched. Together, the expression in (2.21) represents the focal player's expected utility when she faces x other long-lived players upon the time a short-lived player arrives.

Second, the focal player may be unmatched when competing with other x long-lived players. Define functions

$$\mathcal{B}(x,\hat{\theta},\theta) = \int_0^{\epsilon\hat{\theta}} \int_{\underline{\phi}}^1 d\hat{\phi} h_x(\underline{\phi}) d\underline{\phi} + \int_{\epsilon\hat{\theta}}^{\epsilon\theta} \int_{\epsilon\hat{\theta}}^1 d\hat{\phi} h_x(\underline{\phi}) d\underline{\phi}, \text{ and} \tag{2.22}$$

$$\mathcal{C}(x,\hat{\theta},\theta) = \int_{\epsilon\theta}^1 \int_{\epsilon\hat{\theta}}^1 d\hat{\phi} h_x(\underline{\phi}) d\underline{\phi}, \quad \forall \leq \theta, \hat{\theta} \in [0,\rho] \text{ and } x \geq 1. \tag{2.23}$$

Then we have

$$\mathbb{P}\left(\hat{\phi} > \min\{\underline{\phi}, \epsilon\hat{\theta}\} \text{ and } \underline{\phi} \leq \epsilon\theta \,\middle|\, x\right) = \begin{cases} \mathcal{B}(x,\hat{\theta},\theta), & \text{if } \hat{\theta} \leq \theta, \\ \mathcal{B}(x,\theta,\theta), & \text{if } \hat{\theta} > \theta, \end{cases} \tag{2.24}$$

representing the probability that the focal player is unmatched while another long-lived player is matched. Furthermore, we have

$$\mathbb{P}\left(\hat{\phi} > \min\{\underline{\phi}, \epsilon\hat{\theta}\} \text{ and } \underline{\phi} > \epsilon\theta \,\middle|\, x\right) = \mathcal{C}(x,\hat{\theta},\hat{\theta}), \tag{2.25}$$

representing the probability that no player is matched.

Now we define $V(x,\hat{\theta},\theta,\rho)$ as the expected utility function for a focal player when the platform's price is ρ, all other x long-lived players commit to cost threshold θ, and the focal one uses threshold $\hat{\theta}$. The utility function $V(x)$ (we drop variables $\hat{\theta},\theta$ and ρ in function V when there is no confusion) solves the following difference equation for $x \geq 1$:

$$V(x) = \begin{cases} \dfrac{\lambda_S \mathcal{A}(x,\hat{\theta},\hat{\theta},\rho) + \left[\lambda_S \mathcal{B}(x,\hat{\theta},\theta) + x\gamma\right] V(x-1) + \lambda_L V(x+1)}{\lambda_L + \lambda_S(1-\mathcal{C}(x,\hat{\theta},\theta)) + (x+1)\gamma}, & \text{if } \hat{\theta} \leq \theta, \\ \dfrac{\lambda_S \mathcal{A}(x,\hat{\theta},\theta,\rho) + \left[\lambda_S \mathcal{B}(x,\theta,\theta) + x\gamma\right] V(x-1) + \lambda_L V(x+1)}{\lambda_L + \lambda_S(1-\mathcal{C}(x,\hat{\theta},\theta)) + (x+1)\gamma}, & \text{if } \hat{\theta} > \theta, \end{cases} \tag{2.26}$$

with boundary condition

$$\lambda_S \mathcal{A}(0,\hat{\theta},\theta,\rho) = (\lambda_L + \lambda_S \mathcal{C}(0,\hat{\theta},\theta) + \gamma) V(0) - \lambda_L V(1), \tag{2.27}$$

where functions $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{C}$ are defined in (2.20), (2.22) and (2.23), respectively. We leave the detailed derivation of the difference equations in Appendix 4.1.3.

Assume that upon each long-lived player's arrival, the matching system already reaches its steady state. Define a long-lived player's expected utility upon joining the platform if she uses threshold $\hat{\theta}$

while all other long-lived players use θ as

$$\mathbb{E}[V(X_\epsilon(\theta), \hat{\theta}, \theta, \rho)], \quad \theta, \hat{\theta} \in [0, \rho], \tag{2.28}$$

where the expectation follows P.A.S.T.A (Possion Arrival See Time Average). That is, the number of long-lived players follows the same steady state distribution ([Wol82]). Furthermore, we formally define the long-lived players' equilibrium cost threshold as

$$\theta_\rho \in \underset{\hat{\theta} \in [0,\rho]}{\operatorname{argmax}} \, \mathbb{E}[V(X_\epsilon(\theta_\rho), \hat{\theta}, \theta_\rho, \rho)]. \tag{2.29}$$

Therefore, in order to evaluate the effective price of long-lived players, we need to compute function V first.

It is very hard to find a closed-form solution to the difference equation in (2.26) even with our limiting regime. Note that x can go to infinity in (2.26). Thus, solving this system of difference equations (in exact) numerally also appears challenging. Here, we provide a good approximation method to solve for function V heuristically.

Proposition 2.4. *Fix $\theta, \hat{\theta} \in [0, \rho]$.*

(i) There exists an upper bound B such that $V(x, \hat{\theta}, \theta, \rho) \leq B < \infty$ for all $x \geq 0$.

(ii) Fix$\bar{x} \in \mathbb{Z}_+$. Denote $V_{\bar{x}}(x, \hat{\theta}, \theta, \rho)$ as the solution to the system of equations that solves (2.27) *and* (2.26) *for $x < \bar{x}$ with $V_{\bar{x}}(\bar{x}, \hat{\theta}, \theta, \rho) = 0$. We have*

$$0 \leq V(x, \hat{\theta}, \theta, \rho) - V_{\bar{x}}(x, \hat{\theta}, \theta, \rho) \leq \frac{B}{\left(1 + \frac{\gamma}{\lambda_L}\right)^{\bar{x}-x}}, \quad \forall\, 0 \leq x \leq \bar{x}. \tag{2.30}$$

Proposition 2.4(ii) suggests a very efficient heuristic to compute the value function V numerically, which only involves solving a system of sparse linear equations with $\bar{x}$ variables. Furthermore, Proposition 2.4(ii) also indicates the error introduced by this heuristic calculation decreases exponentially with the choice of $\bar{x}$. In fact, as Figure 2.1 suggests, the improvement on the value function of using $\bar{x} = 5000$ instead of $\bar{x} = 500$ is negligible for all $x < 490$. So we do not need to choose a very large $\bar{x}$. Moreover, according to Proposition 2.1, we have $X_\epsilon(\theta) \succeq_1 Y_\epsilon(0)$, where $Y_\epsilon(0)$ is a Poisson random variable with load $\dfrac{\lambda_L}{\gamma}$. Thus, the distribution function of $X_\epsilon(\theta)$ is also "light-tailed". Therefore, following Proposition 2.4(ii), our heuristic evaluation of function V also has little impact on long-lived players' expected utility upon arrival, $\mathbb{E}[V(X_\epsilon(\theta), \hat{\theta}, \theta, \rho)]$.

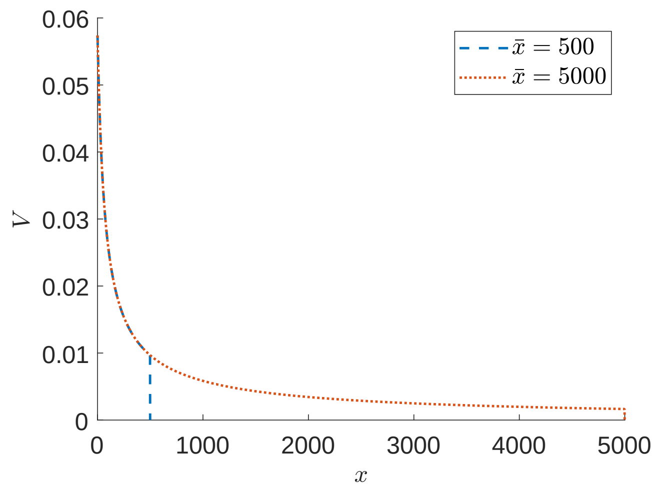


Figure 2.1: Function $V_{\bar{x}}$ with $\bar{x} = 500$ and $\bar{x} = 5000$

In the following numerical examples, we focus on value function $V_{\bar{x}}$ instead of function V. That is, we compute the market effect price θ_ρ as

$$\theta_\rho \in \underset{\hat{\theta} \in [0,\rho]}{\operatorname{argmax}} \mathbb{E}[V_{\bar{x}}(X_\epsilon(\theta_\rho), \hat{\theta}, \theta_\rho, \rho)]. \tag{2.31}$$

As mentioned earlier in this section, if the platform sets a price ρ, long-lived players will in general choose the equilibrium cost threshold $\theta \leq \rho$. That is, in order to solve the maximization problem in (2.19), if the platform could set a price such that the equilibrium cost θ_ρ is as close as possible to the centralized price ρ^*, in order to maximize the social welfare rate. Therefore, the platform needs to inflate the price $\rho \geq \rho^*$ in the hope of achieving social optimal when long-lived players are match makers.

In our numerical calculation, we find that the platform needs to always set price at $\rho_L = 1$, the highest possible value. Given this price, the long-lived players' equilibrium cost is still lower than the centralized optimal price (and cost) ρ^*. Therefore, when long-lived players are the match maker, the system cannot achieve optimal social welfare. Figure 2.2 provides examples on the relationship between the prices/cost and players' tolerance, departure rate, or players' arrival rates. As we can see, in Figure 2.2 (a), as the tolerance ϵ increases, both the centralized optimal price ρ^* and the equilibrium cost θ_{ρ_L} among players are decreasing. Figure 2.2 (b) shows that as the departure rate γ (or patience level) of long-lived players are increasing, ρ^* and θ_{ρ_L} are both increasing. Next,

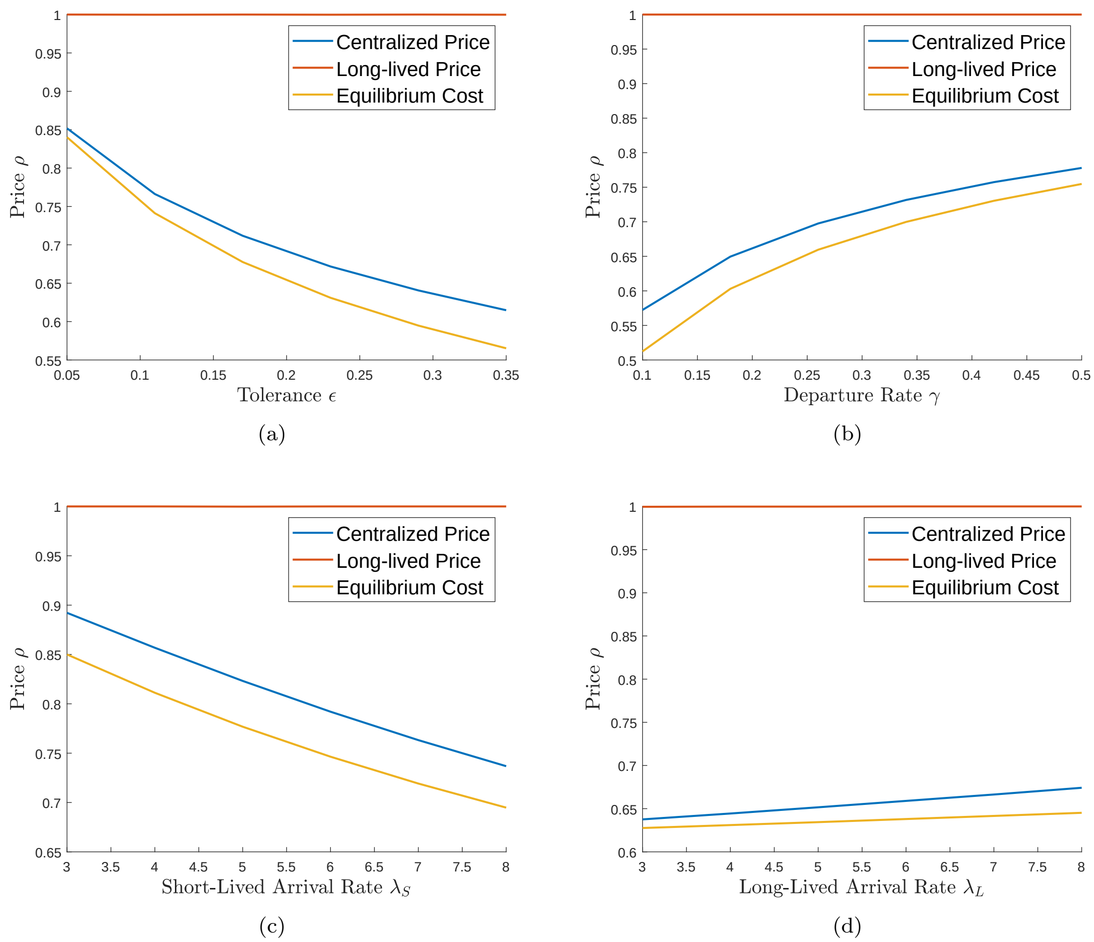


Figure 2.2: Centralized price v.s. Long-lived Price v.s. Equilibrium Cost

Figure 2.2 (c) demonstrates that as short-live players' arrival rate λ_S increases, ρ^* and θ_{ρ_L} both decrease. Finally, Figure 2.2 (d) illustrates that as long-lived players' arrival rate λ_L increases, ρ^* and θ_{ρ_L} both increase. These results are fairly intuitive. The more tolerate long-lived players are (larger ϵ), the less surplus from short-lived players needs to transfer to long-lived players to induce participation. The less patient long-lived players are (bigger γ), the more they behave like short-lived players and choose to match at higher cost. In terms of arrival rates, if short-lived players arriving more frequently, long-lived players can be picker. However, if long-lived players arrive more frequently, the competition level increases and they have to be more tolerant. Furthermore, the gap between the resulting equilibrium cost θ_{ρ_L} and the centralized optimal price ρ^* appears to be increasing as player's tolerance ϵ or arrival rates λ_S, λ_L increases. However, the gap appears to be decreasing as players' departure rate γ (patience) increases.

ϵ	ρ^*	ρ_L	$\theta\rho_L$	Social Welfare Loss	Long-lived Utility Increment
0.05	0.5724	1	0.5124	0.28%	283.41%
0.11	0.6496	1	0.6030	0.17%	228.54%
0.17	0.6976	1	0.6596	0.11%	201.34%
0.23	0.7316	1	0.6998	0.08%	184.57%
0.29	0.7574	1	0.7304	0.06%	173.02%
0.35	0.7780	1	0.7548	0.04%	164.46%

Table 2.1: Summary for $\epsilon = 0.05$ to 0.35, $\lambda_L = 10$, $\lambda_S = 10$ and $\gamma = 1$

γ	ρ^*	ρ_L	$\theta\rho_L$	Social Welfare Loss	Long-lived Utility Increment
0.1	0.8518	1	0.8402	0.01%	137.95%
0.18	0.7662	1	0.7412	0.05%	169.18%
0.26	0.7118	1	0.6776	0.09%	193.76%
0.34	0.6718	1	0.6310	0.13%	214.43%
0.42	0.6406	1	0.5948	0.16%	232.34%
0.5	0.6148	1	0.5652	0.19%	248.15%

Table 2.2: Summary for $\gamma = 0.05$ to 0.5, $\lambda_L = 10$, $\lambda_S = 10$ and $\epsilon = 0.35$

Tables 2.1, 2.2, 2.3, and 2.4 summarize the results when we change long-lived players' tolerance ϵ, their patience level (departure rate) γ, short-lived players' arrival rate λ_S and long-lived players' arrival rate λ_L, respectively. In particular, the tables show the loss of social welfare and the increment of Long-lived players' utilities if they are match makers instead of the platform. Although the optimal social welfare cannot be achieved if long-lived players are match makers, the loss in social welfare is minimal. Furthermore, because $\rho_L^* = 1$, short-lived players' utility is also kept at 0 when

λ_S	ρ^*	ρ_L	$\theta\rho_L$	Social Welfare Loss	Long-lived Utility Increment
3	0.8922	1	0.8500	0.07%	120.70%
4	0.8568	1	0.8112	0.10%	129.97%
5	0.8232	1	0.7768	0.12%	140.29%
6	0.7920	1	0.7464	0.12%	151.64%
7	0.7632	1	0.7192	0.12%	163.81%
8	0.7368	1	0.6948	0.12%	176.74%

Table 2.3: Summary for $\lambda_S = 3$ to 8, $\lambda_L = 10$, $\gamma = 1$ and $\epsilon = 0.2$

λ_L	ρ^*	ρ_L	$\theta\rho_L$	Social Welfare Loss	Long-lived Utility Increment
3	0.6376	1	0.6276	0.01%	267.37%
4	0.6444	1	0.6310	0.02%	257.42%
5	0.6516	1	0.6344	0.03%	247.61%
6	0.6590	1	0.6380	0.04%	238.25%
7	0.6664	1	0.6416	0.06%	229.34%
8	0.6742	1	0.6452	0.07%	220.68%

Table 2.4: Summary for $\lambda_L = 3$ to 8, $\lambda_S = 10$, $\gamma = 1$ and $\epsilon = 0.2$

long-lived players are match makers. Thus, comparing to centralized matching processes, long-lived players' utilities are improved significantly if they are match makers.

To conclude our study on decentralized matching systems, we have considered letting short- and long-lived players to be match makers, respectively. We find that if short-lived players are match makers, the optimal social welfare can be recovered, because the platform's optimal price can be enforced by myopic match makers. However, if long-lived players are in charge, the platform's price cannot be enforced anymore, because players are forward-looking and use a cost $\theta_\rho \leq \rho$ instead. As a result, in equilibrium, long-lived players are picker and choose short-lived players using a smaller threshold on the mismatch angle than the one induced by the platform's price. To counter long-lived player's behavior, the platform needs to inflate its price purposely to recover as much social welfare as possible. However, this price inflation hurts short-lived players by taking away all their surplus in each match and also dramatically increases long-lived players' utility. Although, the social welfare when long-lived players are match makers are very close to the optimal centralized social welfare, the platform should exercise caution with this matching system due to its extreme welfare distributions between short- and long-lived players.

2.5 Conclusion

We study optimal threshold pricing in a matching market with short and long-lived players. Players' individual preferences are characterized by a circular model, which links preferences of players directly with different matching probabilities. Under the limiting regime where players are very mismatch-sensitive, ϵ approaches 0, we are able to obtain optimal price in closed-form for various scenarios. The closed-form expressions of optimal price provide clear intuition on the relationship between matching policies and market thickness. We also compare two decentralized matching systems where either short- or long-lived players are match makers. We show that the optimal social welfare can be obtained if short-lived players are match makers. However, if long-lived players are match makers, the corresponding social welfare can be close but never reaches the optimal social welfare. Furthermore, the platform has to inflate the price in this scenario and extracts a short-lived player's surplus entirely in each match. This price inflation leads to extreme utility distributions between short- and long-lived players.

Chapter 3

Optimal Monitoring Schedule in Dynamic Contracts

3.1 Introduction

Adverse events often bring significant damages to an organization or the society. In many situations, better efforts in maintaining and safeguarding a system can reduce the chance of such adverse events. The challenge is that these events may still occur, albeit less frequently, despite the best effort. And efforts are often hard to verify. Furthermore, people in charge of the effort (an agent) often cannot bear the full consequence of an adverse event due to limited liability. In practice, an agent is often a hired employee or subcontractor, who can be paid one way or another, but cannot compensate damages. In order to ensure efforts, a principal, be it a firm or a government, may decide to "keep an eye" on the agent, which ensures that adverse events occur at a lower frequency, and are not due to lack of effort should they happen. For example, Accenture served as an outside vendor to support the IT systems for Kbank of Thailand. According to conversations with Accenture, once in a while, the in-house IT team at Kbank would show up to work together with the Accenture team. Such monitoring activities are often too costly to conduct at all the time. The principal can also schedule payments that are contingent on arrivals to motivate effort. How do we induce effort from the agent with minimum payments and monitoring costs? In a dynamic setting where adverse events occur stochastically over time, what is the optimal schedule to pay and to monitor the agent?

To answer these questions, we study an optimal contract design problem in a dynamic setting, in which a risk-neutral principal faces a Poisson process of costly adverse events. (Think of adverse events as system breakdowns or production defects.) The instantaneous rate of the Poisson process can be reduced by a risk-neutral agent, if the agent exerts effort at that moment. Effort is costly to the agent and observable to the principal only when the principal monitors the agent. The principal, who can commit to a long term contract over an infinite horizon in continuous time, needs to trade-off direct payments to the agent, versus costly monitoring, in order to induce effort.

We formulate this optimal dynamic contract design problem as a continuous time stochastic optimal control model, and are able to provide complete characterizations of the optimal monitoring and payment schedules, which vary depending on the monitoring cost. As expected, if the monitoring cost is lower than a threshold, the principal should monitor all the time. In this case, the agent's total future utility [SS87] is always kept at 0. Interesting structures emerge when the monitoring cost is above the threshold. In this case, the promised utility serves as a sufficient statistic of the entire history of arrival times, on which the optimal monitoring and payment schedules critically depend.

Generally speaking, the agent needs to be penalized for each arrival when not being monitored. Because we assume that the agent has limited liability and cannot pay the principal, the penalty takes the form of a downward jump of the promised utility upon each arrival whenever the agent is not being monitored. Between arrivals, the promised utility gradually increases. When downward jumps due to arrivals bring the promised utility below a threshold, the principal starts monitoring. Monitoring stops only after the promised utility climbs back to the threshold, during which arrivals do not matter. A flow of payment starts only when the promised utility reaches and stays at an upper bound. As soon as another arrival occurs, the promised utility takes a downward jump from the upper bound, which stops the payment.

The aforementioned movements of the promised utility and payment schedule is not completely new to our model. In fact, both [BMRV10] and [Mye15] study similar models as ours, without monitoring. [BMRV10] consider the agent as a firm, and the principal as an investor, who can change the firm size when the promised utility becomes too low. [Mye15], on the other hand, considers a political economy setting, in which the principal can dynamically replace an agent with a new one. A fundamental difference between [BMRV10] and [Mye15] is the time discount rate. In [BMRV10], the principal is strictly more patient than the agent, while [Mye15] assumes the two players' time discount rates are the same. Equal time discount rate in this setting introduces an "infinite back-loading" issue. That is, the principal always prefers to delay the cash payment to the future while promising to pay the corresponding interest. In order to prevent the problem to become unbounded, [Mye15] introduces an exogenous upper bound on the promised utility. For the different discount rate case, [BMRV10] obtain an endogenous upper bound on the promised utility. We study both the different and equal time discount cases. For the different discount case, although it is also not necessary to introduce an exogenous upper bound, we also describe the optimal contract under such a bound, in case the endogenous one is high for the agent to stomach in practice.

Our optimal contract demonstrates subtle and important features that do not arise in [BMRV10], even though the aforementioned movements of the promised utility above the monitoring threshold also appear in [BMRV10] and [Mye15]. When the principal is more patient than the agent, in particular, the promised utility may always be strictly positive, which means that the individual rationality (IR) constraint may never be binding. This may be counter intuitive to people familiar with the mechanism design literature, starting from [Mye81]. As we explain in the paper, raising the promised utility threshold for monitoring allows the principal to shorten the monitoring time, which is preferable when the monitoring cost is high. From a technical point of view, the optimal value function demonstrates the "smooth pasting" phenomenon that often arises in optimal stopping problems [Dix94]. Smooth pasting does not arise in [BMRV10]. Finally, we identify a subtle connection between monitoring and allowing the agent to shirk. Essentially, the optimal contract that allows the agent to shirk can be solved as a special case of the monitoring problem with a specific monitoring cost.

Optimal scheduling of monitoring in a dynamic environment is fundamentally an operations problem. There is a recent stream of papers in the operations research/management science literature that study incentive issues related to auditing/monitoring/inspecting. Most of these papers compare a few classes of practically useful mechanisms, or focus on static settings. [BT12], for example, study three mechanisms (deferred payment mechanism, inspection mechanism, and a mechanism that combines the two) for dealing with product adulteration issues when manufacturers cannot control the suppliers' actions. [RL15] study a similar sets of mechanisms in a similar problem setting, with endogenous procurement decisions and more general arrival discovery processes. [Kim15] studies environmental disclosure and inspection policies in a dynamic setting, and compares deterministic versus random inspection schedules. [PT16] and [PT17] also study environmental monitoring and disclosure issues, using static models. [WSdV16] study monetary and inspection instruments to induce the agent to report the occurrence of an adverse environmental event. The paper models this dynamic adverse selection problem as an optimal control model in continuous time and identifies optimal policies.

Monitoring is a way to conduct *costly state verification* under asymmetric information in the economics literature, started from [Tow79] for adverse selection issues in a static setting. [Dye86] extends the idea to moral hazard problems, also in a static setting. In continuous time dynamic settings, [PW16] study a model in which the underlying uncertainty is a Brownian motion, and the principal checks the agent following a Poisson process, the rate of which is the design issue. If the agent is found shirking, the principal may terminate the contract. [VMS17] study a two state hidden Markov model, whose instantaneous transition rates are affected by the agent's effort. The principal decides the schedule of inspecting the true state of the Markov chain in order to induce effort.

Our model and analysis is rooted in the continuous time optimal contracting literature. [San08] provides the analytical foundation for these types of models. In [San08], the agent's effort affects the drift of a Brownian motion. And the optimal contract is solved as the solution of a stochastic optimal control problem. The Brownian motion setup is natural for corporate finance applications [DS06, BMPR07, Fu17].

[BMRV10] build upon this framework and study continuous time optimal contracting based on Poisson processes, instead of Brownian motions. One important advantage of Poisson process based models is that the optimal control policies are often easier to describe and implement. More recently, [ST17] study how to induce an agent to increase the arrival rate of a Poisson process in a continuous time infinite horizon setting. More broadly, a sequence of recent papers also study optimal contracting problems related to Poisson arrivals (see. e.g.[MV15, Var17, GT16, Sha17, Hid17]). Our paper differs from the aforementioned continuous time dynamic contracting literature in the monitoring component.

More generally, dynamic moral hazard problem has also been the focus of some recent papers published in Operations Research. [PZ03] study continuous time control of incentive issues in a make-to-stock production system. [LZF12] investigate how to motivate multiple agents (suppliers) in a discrete time dynamic setting. Their model is also based on the promised utility framework originated from [SS87].

The rest of this paper is organized as follows. We introduce the model and supporting concepts in Section 3.2, and focus on the equal discount rate case in Section 3.3, where we introduce general structures of optimal contracts and value functions. Built upon these concepts, Section 3.4 further investigates the case where the principal is more patient than the agent. We then present theories that supports a computational algorithm for the optimal value function and contract in Section 3.5, study a simple cyclic monitoring schedule in Section 3.6, and conclude the paper with a discussion of allowing shirking in Section 3.7. All the proofs are presented in the Appendix. In addition,

Appendix 4.2.5 contains a number of extensions and some potential directions for future research.

3.2 The Model

We consider a principal-agent model in a continuous time setting. The principal faces a Poisson process with arrival rate $\bar{\lambda}$ of adverse events (arrivals), each costing the principal a value K.[3] The principal hires an agent, who can bring down the instantaneous arrival rate to $\lambda = \bar{\lambda} - \Delta\lambda$, if the agent exerts effort at this point in time. The principal does not observe the effort, unless *monitoring* the agent, at a cost rate m per unit of time. Denote a left-continuous counting process $\{N_t\}_{t\geq 0}$ to represent the total number of arrivals up to time t, the rate of which depends on the agent's effort process. Further denote $\Lambda = \{\lambda_t\}_{t\geq 0}$ to represent the agent's effort process. That is, $\lambda_t \in \{\lambda, \bar{\lambda}\}$ at each time epoch t. The principal and the agent are both risk-neutral and discount future cash flows with discount rates r and ρ, respectively. As is often assumed in the literature [BMRV10], $\rho \geq r > 0$; i.e., the principal is no less patient than the agent. We start with $\rho = r$ in Section 3.3 before moving to consider $\rho > r$ in Section 3.4.

We assume that the principal has commitment power to issue a long term contract with the agent. This assumption allows us to formulate the strategic interaction between the principal and the agent as a dynamic optimization/optimal control problem, in which the contract is a contingency plan that both players understand that the principal would follow through.[4] The contract specifies a payment and monitoring schedule over time, which depends on past arrival times. More generally, the contract should also specify when to exert effort and when to shirk. For the most part of the paper we restrict attention to contracts that always induce effort from the agent, until in the very end of the paper when we show that the shirking problem is a just a special case.

The agent has limited liability, which means that monetary transfer is from the principal to the agent at any point in time. Therefore, the agent cannot buy out the principal to mitigate misalignment of incentives. Under this assumption, the principal needs to compensate the agent's effort, which costs a constant rate b per unit of time. Therefore, the agent *benefits* from shirking at the rate b. Beyond this flow payment b in the background, denote L_t to represent the principal's cumulative payment to the agent up to time t, such that $dL_t = dI_t + \ell_t dt$, in which the pure jump process I_t represents the cumulative instantaneous payment by time t, and ℓ_t is a flow payment at time t, with $dI_t \geq 0$ and $\ell_t \geq 0$. (Again, ℓ_t does not include the payment b.)

Denote process $\mathcal{M} = \{m_t\}_{t\geq 0}$ to represent the monitoring schedule under the contract, in which $m_t \in \{m, 0\}$ captures the monitoring cost at time t. Monitoring during a time interval $(t, t+\delta]$ guarantees the agent's effort during this time interval. Monitoring may start in one of two ways. First, at any point in time t when an arrival occurs, the principal may decide to start monitoring with a probability $y_t \in [0, 1]$. Second, monitoring may also start at time t in a "deterministic" fashion with respect to realizations of past uncertainties. In order to formally characterize past uncertainties, it is convenient to split the counting process $\{N_t\}_{t\geq 0}$ into two processes $\{N_t^s\}_{t\geq 0}$ and $\{N_t^n\}_{t\geq 0}$, which represent the total numbers of arrivals which have and have not triggered monitoring up to time t, respectively. Therefore, $N_t^s + N_t^n = N_t$. In our setting, there are two sources of randomness. One is from nature (the Poisson arrival), and the other from control (random start of monitoring upon arrival). The counting processes N_t^s and N_t^n fully capture these two sources of uncertainties in the history. Consequently, we define filtration $\mathcal{F} = \{\mathcal{F}_t\}_{t\geq 0}$ such that $\mathcal{F}_t$ captures the entire historical information up to time t specified by the 2-variate counting process $\{N_t^s, N_t^n\}_{t\geq 0}$. Overall, a contract Γ consists of $\mathcal{F}_t$-predictable payment and monitoring processes, L_t and m_t, respectively.[5]

Before proceeding, it is worth noting that in order to start monitoring randomly upon arrival, the outcome of the probability process y_t need to follow a device that is agreed upon and commonly observable by the principal and the agent. In practice, players may use devices such as the last two digits of a stock market index to generate a commonly observed random outcome. It is also worth explaining at this point why we need this randomness in the control policy space. As we will show in Section 3.4, when the two players' discount rates are different and the monitoring cost is relatively low, in order to show that always monitoring is optimal, we need to specify optimal control policies for all positive promised utility values. Such an optimal control policy critically depends on random start of monitoring.

3.2.1 Agent's Utility

Given contract Γ and the agent's effort process Λ, the agent's total utility is defined as

$$u(\Gamma, \Lambda) = \mathbb{E}^{\Gamma,\Lambda}\Big[\int_0^\infty e^{-\rho\tau}(dL_\tau + b\,\mathbb{I}_{\lambda_\tau=\bar{\lambda}}\,d\tau)\Big]. \tag{3.1}$$

It is standard and convenient to work with the agent's continuation utility [**?**, also referred to as the *promised utility*, see, for example,]]SpearSrivastava1987. That is, the total discounted utility starting from time t, defined as the following left-continuous process,

$$W_t(\Gamma, \Lambda) = \mathbb{E}^{\Gamma,\Lambda}\Big[\int_t^{\infty} e^{-\rho(\tau-t)}(dL_\tau + b\,\mathbb{I}_{\lambda_\tau=\bar{\lambda}}\,d\tau)\,\Big|\,\mathcal{F}_t\Big]. \tag{3.2}$$

When there is no confusion, we omit (Γ, Λ) and refer to the agent's continuation utility at time t as W_t. Clearly, $W_0 = u(\Gamma, \Lambda)$. As is often assumed in the literature, the agent does not commit to staying in the contract. Therefore, we require the following participation (also referred to as the *individual rationality*, or, IR) constraint

$$W_t \geq 0, \text{ for all } t \geq 0. \tag{IR}$$

Later in the paper, the optimal contract and the principal's value function are all expressed as functions of the agent's promised utility, so that the optimal contract design problem is essentially a stochastic optimal control with W_t being the state variable.

Under any contract, the agent's promised utility W_t must satisfy the following dynamics.

Lemma 3.1. *For any contract Γ, there exits $\mathcal{F}_t$-predictable process $\{(H_t^s, H_t^n)\}_{t\geq 0}$, such that*

$$dW_t = \{\rho W_t - b\mathbb{I}_{\lambda_\tau=\bar{\lambda}} + \lambda_t[y_t H_t^s + (1-y_t)H_t^n]\}dt - H_t^s dN_t^s - H_t^n dN_t^n - dL_t. \tag{PK}$$

In order to satisfy the agent's continued participation (IR)*, H_t^s and H_t^n are less than or equal to W_t.*

The condition (PK) stands for "promise keeping," which ensures that W_t is indeed the agent's continuation utility starting from time t. Lemma 3.1 follows directly from the Martingale Representation Theorem (Theorem T9,page 64 [Bré81]), and extends Lemma 1 in [BMRV10] and Lemma 6 in [ST17] to our setting with a multi-variate counting process. Here we provide an heuristic derivation of (PK) following discrete time approximation, which offers an intuitive illustration.

Consider the promised utility at the beginning of a small time interval $[t, t+\delta)$ to be W_t. With probability $\lambda_t \delta y_t$, there is an arrival that triggers monitoring to start. In this case, the promised utility moves to $W_t - H_t^s$. With probability $\lambda_t\delta(1-y_t)$, on the other hand, there is an arrival that does not trigger monitoring. In this case, the promised utility moves to $W_t - H_t^n$. Finally, with probability $1 - \lambda_t\delta$, no arrival occurs and the promised utility moves to $W_{t+\delta}$. Taking into consideration the potential benefit from shirking, and ignoring payment for simplicity, we have

$$W_t = b\delta\mathbb{I}_{\lambda_\tau=\bar{\lambda}} + e^{-\rho\delta}\left\{\lambda_t\delta\left[y_t(W_t - H_t^s) + (1-y_t)(W_t - H_t^n)\right] + (1-\lambda_t\delta)W_{t+\delta}\right\}.$$

Following the standard procedure of subtracting W_t from and dividing δ on both sides, and letting δ approach zero, we obtain

$$\lim_{\delta \downarrow 0} \frac{W_{t+\delta} - W_t}{\delta} = \rho W_t - b\mathbb{I}_{\lambda_\tau = \bar{\lambda}} + \lambda_t \left[y_t H_t^s + (1 - y_t) H_t^n \right],$$

which explains the continuously changing part of (PK). The additional terms in (PK), $H_t^s dN_t^s$ and $H_t^n dN_t^n$, further capture the jumps in the promised utility due to arrivals as mentioned just now. Finally, payment dL_t at time t naturally brings down total future payments.

3.2.2 Incentive Compatibility

In this paper we mainly focus on contracts that always induce effort from the agent. We first argue that assuming

$$m \leq K\Delta\lambda - b, \tag{3.3}$$

the principal should always motivate the agent to exert effort. For any contract that allows shirking, we can always improve it by replacing a shirking period with monitoring. This is because monitoring costs the principal m plus the effort cost b, while shirking costs the principal $K\Delta\lambda$ from higher arrival rate. (The principal does not reimburse effort when the agent shirks under the contract.) In the end of the paper, we consider the situation where (3.3) is violated and shirking is allowed.

Condition (3.3) allows us to transform our problem into an optimal control model over contracts that must always induce effort. Correspondingly, the counting process $\{N_t\}_{t \geq 0}$ admits intensity $\lambda_t = \lambda$ for all $t \geq 0$. If a contract Γ induces the agent to always exert effort, that is,

$$u(\Gamma, \underline{\Lambda}) \geq u(\Gamma, \Lambda), \text{ for } \underline{\Lambda} := \{\lambda_t = \lambda\}_{t \geq 0} \text{ and } \forall \Lambda, \tag{3.4}$$

then we call contract Γ *incentive compatible*.

Incentive compatibility critically depends on the following ratio between the agent's private benefit b and the difference in arrival rates $\Delta\lambda$,

$$\beta := \frac{b}{\Delta\lambda}. \tag{3.5}$$

Intuitively, should the principal be able to charge the agent an amount β for each arrival, the agent would be indifferent between exerting effort or not. – Shirking in a small time interval δ brings the agent a benefit $b\delta$, which is offset by the higher penalty cost $\Delta\lambda\beta\delta$. – Because charging the agent is not allowed in our setting, the principal instead reduces the agent's promised utility by at least β for each arrival in order to induce effort.[6] Therefore, the value β is the minimum penalty in the promised utility that induces effort from the agent, as mentioned in the introduction. This is formalized in the following result.

Lemma 3.2. *Contract* Γ *is incentive compatible, i.e., it satisfies* (3.4)*, if and only if the following condition holds:*

$$y_t H_t^s + (1 - y_t) H_t^n \geq \beta, \quad \textit{if } m_t = 0. \tag{IC}$$

The term $y_t H_t^s + (1 - y_t) H_t^n$ is the expected downward jump when there is an arrival at time t. Constraint (IC) implies that this expected downward jump is at least β when the principal does not monitor.

Constraint (IC) further implies that either H_t^s or H_t^n has to be at least β. Therefore, we can simultaneously satisfy constraints (IC) and (IR) after the potential downward jump only if the promised utility W_t is at least β. This implies that, in order to induce effort, the principal has to monitor the agent, instead of relying on (IC), whenever the promised utility $W_t < \beta$. We summarize this in the following corollary.

Corollary 3.1. *Under any incentive compatible contract, the agent is monitored* ($m_t = m$) *when* $W_t < \beta$.

When $W_t \geq \beta$, the principal trades off monitoring and enforcing (IC), and should monitor the agent if and only if the shadow cost of the (IC) constraint is higher than the monitoring cost m.

3.2.3 Principal's Utility

Assume that the principal receives a constant revenue flow of R. Under an incentive compatible contract, the agent always exerts full effort. The corresponding principal's total discounted utility under an incentive compatible contract Γ is

$$\mathbb{E}^{\Gamma,\underline{\Delta}}\Big[\int_0^\infty e^{-rt}\Big((R - m_t)dt - K dN_t - dL_t\Big)\Big] = \frac{R - K\lambda}{r} - \mathbb{E}^{\Gamma,\underline{\Delta}}\Big[\int_0^\infty e^{-rt}\Big(m_t dt + dL_t\Big)\Big] .$$

Because the term R only shifts the principal's total utility by a constant and is independent of the contract design, without loss of generality, we set

$$R = K\lambda. \tag{3.6}$$

Therefore, the principal's total discounted utility under an incentive compatible contract Γ is

$$U(\Gamma) = -\mathbb{E}^{\Gamma,\underline{\Delta}}\Big[\int_0^\infty e^{-rt}\Big(m_t dt + dL_t\Big)\Big]. \tag{3.7}$$

Before finishing this section, we present the following verification result, which provides an upper bound of the principal's utility $U(\Gamma)$ over all incentive compatible contracts. This result forms the foundation of proving optimality under various parameter regimes.

Lemma 3.3. *Suppose $F(w)$ is a continuous, concave, and upper-bounded function, with $F'(w) \geq -1$. (If $F(w)$ is not differentiable at a point w, we denote $F'(w)$ to be the average between its left and right derivatives.) Consider any incentive compatible contract Γ, which yields the agent's expected utility $u(\Gamma, \underline{\Delta}) = w = W_0$, followed by the promised utility process $\{W_t\}_{t\geq 0}$ according to* (PK)*. Define a stochastic process $\{\Psi_t\}_{t\geq 0}$, where*

$$\begin{aligned}\Psi_t \quad := \quad & F'(W_t)\rho W_t - rF(W_t) - m_t + \lambda y_t[F'(W_t)H_t^s + F(W_t - H_t^s) - F(W_t)] \\ & +\lambda(1-y_t)[F'(W_t)H_t^n + F(W_t - H_t^n) - F(W_t)].\end{aligned} \tag{3.8}$$

If the process $\{\Psi_t\}_{t\geq 0}$ is non-positive almost surely, then we have $F(w) \geq U(\Gamma)$.

3.3 Equal Discount Rate

In this section, we consider the case in which $r = \rho$. That is, the principal shares the same discount rate with the agent. In this setting, we need to introduce un upper bound for the agent's continuation utility, $\bar{w}$, as an additional model parameter. This is due to the "infinite back-loading" problem identified in [Mye15] for the equal discount moral hazard problem without monitoring. Without such an upper bound, the principal would keep increasing the agent's promised utility to infinity without payment. We will provide a comprehensive discussion on the intuitive reason for such an upper bound towards the end of this section. To avoid triviality, the upper bound $\bar{w}$ is set to be above β.[7] It is worth noting that we only need such an upper bound when $r = \rho$. When $r < \rho$, the principal is more patient than the agent, it is no longer necessary to introduce $\bar{w}$, as we will discuss in the next section.

When $r = \rho$, the optimal contract structure is not unique. Interestingly, when $r < \rho$ to be discussed in the next section, either contract structure may be optimal, depending on how high the monitoring cost is.

3.3.1 Optimal Contract Structure: Deterministic and randomized

For the equal discount case, the optimal contract may take two different forms with the same performance. We first describe the evolution of the promised utility under the "deterministic" optimal contract (that is, probability y_t in (PK) is always 0). As discussed in the previous section, the (IC) condition implies that the principal has to monitor the agent *if* the promised utility W_t is lower than β. In this section, we establish that when $r = \rho$, it is optimal to monitor *if and only if* $W_t < \beta$. Moreover, the principal does not penalize the agent for any arrivals (i.e., $H_t^n = H_t^s = 0$) and does not pay the agent while monitoring (i.e., $dL_t = 0$). Therefore, when $W_t < \beta$, (PK) implies that the promised utility evolves according to

$$dW_t = \rho W_t dt. \tag{3.9}$$

Intuitively, the term $\rho W_t dt$ is the "interest" accrued from the promised utility due to time discounting. Furthermore, whenever $W_t < \beta$ at time t, condition (3.9) implies that the promised utility increases deterministically following the simple exponential curve

$$W_{t+\tau} = W_t e^{\rho\tau}, \tag{3.10}$$

until $W_{t+\tau}$ reaches the threshold β.

When the promised utility $W_t \in [\beta, \bar{w})$, the principal no longer monitors the agent, and the promised utility takes a downward jump of β upon each arrival. That is, following the optimal contract, $H_t^s = H_t^n = \beta$ in (PK). In this case, the principal still does not pay the agent (i.e., $dL_t = 0$), and (PK) reduces to

$$dW_t = (\rho W_t + \beta\lambda)dt - \beta dN_t. \tag{3.11}$$

Compared with (3.9), the rate of increase in (3.11) is higher. Besides the interest term $\rho W_t dt$, the term $\beta\lambda dt$ is the "information rent" that the agent receives, in the form of faster increase in the promised utility when there is no arrival. To see this intuitively, remember that in order to motivate effort, the principal shall charge the agent utility β for each arrival, which occurs with probability λdt. Because the agent cannot actually pay the principal money upon arrivals, when there is no arrival during a period dt, the principal increases the agent's promised utility by $\beta\lambda dt$. This exactly equals the expected decrease of the agent's utility for an arrival during this time period, reflected in the downward jump term βdN_t.

Therefore, if there is no arrival during the time interval $[t, t+\tau]$, then, again, the promised utility increases following the curve

$$W_{t+\tau} = W_t e^{\rho\tau} + \frac{\beta\lambda}{\rho}\left(e^{\rho\tau} - 1\right), \tag{3.12}$$

as long as $W_{t+\tau} < \bar{w}$. Compared with (3.10), the additional term involving $\beta\lambda$ is the information rent as discussed earlier.

Without an arrival, the agent's promised utility keeps increasing according to a rate $\rho W_t + \beta\lambda$ until W_t reaches the upper bound $\bar{w}$. At this point, the promised utility cannot increase any more, and stays at $\bar{w}$ until the next arrival. That is, when $W_t = \bar{w}$, the promised utility evolves according to

$$dW_t = -\beta dN_t. \tag{3.13}$$

In order to keep W_t at $\bar{w}$, the principal has to pay the agent a flow rate

$$\ell_t = \rho\bar{w} + \beta\lambda, \tag{3.14}$$

to release the upward pressure that would otherwise occur to the promised utility.

Now we are ready to present a formal definition for a class of contracts that includes the optimal one. Note that in the following definition, we allow the monitoring threshold to be a more general level s, although it is just β as described before. This is because the optimal threshold can be higher than β in the next section when $r < \rho$.

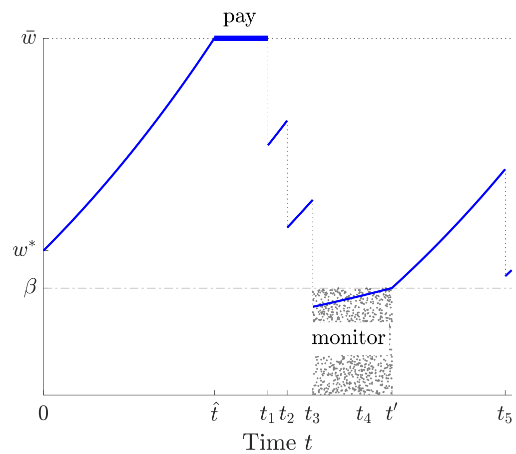
Definition 3.1. *Contract $\Gamma_d(w; s, \bar{w})$ is defined as:*

(i) The dynamics of the agent's promised utility, W_t, follows (3.9) *for $W_t \in [0, s)$,* (3.11) *for $W_t \in [s, \bar{w})$, and* (3.13) *for $W_t = \bar{w}$, starting from $W_0 = w$.*

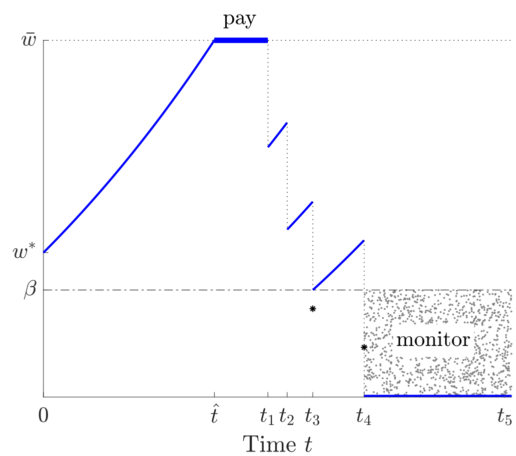
(ii) In terms of payments, the agent is not paid when $W_t < \bar{w}$, and is paid at a flow rate (3.14) *when $W_t = \bar{w}$.*

(iii) Regarding monitoring, $m_t = m$ if and only if $W_t < s$.

In Definition 3.1, the subscript "*d*" in $\Gamma_d(w; s, \bar{w})$ stands for *deterministic*, because monitoring starts in a "deterministic" fashion with respect to the arrival times. That is, the promised utility W_t is adapted to the filtration generated from the arrival process. The parameter s represents the monitoring threshold. Clearly, in this section we are only interested in contract $\Gamma_d(w; \beta, \bar{w})$. In the



(a) Deterministic Contract $\Gamma_d(w^*; \beta, \bar{w})$



(b) Randomized Contract $\Gamma_r(w^*; \bar{w})$

Figure 3.1: Sample Trajectories of W_t under the Deterministic and Randomized Contracts

following section when we discuss $\rho > r$, however, the threshold s could be strictly higher than β, and the upper bound $\bar{w}$ may be replaced with an endogenous one.

Figure 3.1(a) provides a sample trajectory under contract $\Gamma_d(w^*; \beta, \bar{w})$. On this sample trajectory, there are a total of five arrivals, labeled as time epochs t_1, t_2, ... t_5. In the beginning, the promised utility keeps increasing with slope $\rho w_t + \beta\lambda$ until reaching the upper bound $\bar{w}$ at time $\hat{t}$. Then three arrivals cause three downward jumps of magnitude β that eventually bring the promised utility below the threshold β. Monitoring starts from time t_3 and lasts until time t', during which the arrival at t_4 has no impact. The kink at t' reflects the difference between the slopes ρw_t on the left and $\rho w_t + \lambda\beta$ on the right.

The above description shows that it is fairly easy for the principal to implement this contract over time, keeping track of a single number in a simple way. Furthermore, the contract guarantees incentive compatibility on an intuitive level. A Monitoring session starts when arrivals occur rather frequently over a period of time. Under monitoring, the promised utility grows rather slowly, which means that payment can only happen far into the future. Therefore, monitoring not only ensures effort at the moment, but also serves as a threat to the agent before its use.

Furthermore, the contract motivates the agent to exert effort besides using the threat of monitoring. When the promised utility is at a level w above β and below $\bar{w}$, payment starts if there is no arrivals for the next $\frac{1}{\rho}\ln\frac{\rho\bar{w}+\beta\lambda}{\rho w+\beta\lambda}$ period of time (following (3.12) with $W_t = w$ and $W_{t+\tau} = \bar{w}$). Exerting effort increases the chance of the promised utility reaching $\bar{w}$ and payment. Once payment has started, an arrival brings the promised utility down from $\bar{w}$ to $\bar{w}-\beta$, and pauses the flow payment for at least a period of time of $\frac{1}{\rho}\ln\frac{\rho\bar{w}+\beta\lambda}{\rho\bar{w}+\beta(\lambda-\rho)}$ (again, following (3.12)). Therefore, once being paid, the agent is willing to exert effort in order to prolong the payment period before it pauses.

Figure 3.1(b) depicts an alternative contract with the same performance that involves randomization upon arrivals, facing the same sample trajectory as in Figure 3.1(a). In particular, whenever the promised utility drops to below β, it either jumps to 0 or β. For example, at time t_3, the randomization brings the promised utility to β, while at t_4, the promised utility lands at 0, which triggers monitoring ever after.

More formally, if an arrival occurs when W_{t-} is in $[\beta, 2\beta)$, the next moment's promised utility lands on 0 with probability $2-W_{t-}/\beta$, and on β with probability $W_{t-}/\beta-1$. Technically, this means that in (PK), the jumps are $H_t^s = W_{t-}$ and $H_t^n = W_{t-}-\beta$, and the randomization probability is $y_t = 2 - W_{t-}/\beta$. Therefore, we define the following class of contracts $\Gamma_r(w;\bar{w})$, in which the subscript "r" stands for *randomized*.

Definition 3.2. *Define contract $\Gamma_r(w;\bar{w})$ the same as contract $\Gamma_d(w;\beta,\bar{w})$ in Definition 3.1, except that the dynamics of the agent's promised utility W_t follows*

$$dW_t = \begin{cases} 0, & \textit{if } W_t = 0, \\ (\rho W_t + \beta\lambda)dt - W_t dN_t^s - (W_t - \beta)dN_t^n, & \textit{if } W_t \in [\beta, \min\{\bar{w}, 2\beta\}), \\ (\rho W_t + \beta\lambda)dt - \beta dN_t, & \textit{if } W_t \in [\min\{\bar{w}, 2\beta\}, \bar{w}), \\ -\beta dN_t, & \textit{if } W_t = \bar{w}, \end{cases} \tag{3.15}$$

starting from $W_0 = w$.

The reason that both deterministic and randomized contracts are optimal is revealed in the next subsection, where we show that the principal's value function is linear in the interval $[0, \beta]$. For the different discount rate case of Section 3.4, however, whether the value function is linear or not depends on how high the monitoring cost is. In particular, the deterministic contract is optimal when the monitoring cost is high, and the randomized contract is optimal when the monitoring cost is low.

3.3.2 Principal's Value Function

Following the evolution of the promised utility described above, next we heuristically derive the dynamics of the principal's utility as a function of the agent's promised utility using discrete time approximation. Specifically, denote $F(w)$ to represent the principal's total discounted utility when the agent's promised utility is w.

First, for any $w \in [0, \beta)$, over a small time interval with length δ, the principal incurs a monitoring cost $m\delta$, and, following the deterministic contract dynamics (3.9), the agent's promised utility increases to $we^{\rho\delta}$. Therefore, we have the following expression for the principal's utility function,

$$F(w) = -m\delta + e^{-r\delta} F\left(we^{\rho\delta}\right) + o(\delta). \tag{3.16}$$

Assuming $F(w)$ is differentiable on $[0, \beta)$, following standard procedures of subtracting $F(w)$ from and dividing δ on both sides, and by letting δ approach 0, we obtain,

$$rF(w) = \rho w F'(w) - m. \tag{3.17}$$

We keep both r and ρ in (3.17) and all the equations in this section, because the value function takes the same expressions when $r < \rho$ in the next section.

Differential equation (3.17) has the following standard solution,

$$F_\theta(w) = \theta w^{\frac{r}{\rho}} - \frac{m}{r}, \tag{L}$$

parameterized with a scalar θ. The tag (L) indicates that the promised utility is *lower* than β. Later in this section, we specify the choice of θ to complete the description of the value function $F_\theta(w)$. When $r = \rho$, (L) reduces to

$$F_\theta(w) = \theta w - \frac{m}{r}, \tag{L$_l$}$$

which is a linear function of w, and hence the subscript l in the tag.

Next, for any $w \in [\beta, \bar{w})$, the principal no longer monitors the agent. Following similar heuristic derivations for (3.16), we reach a delay differential equation (DDE),

$$(\lambda + r)F(w) = \lambda F(w - \beta) + (\rho w + \lambda\beta)F'(w), \tag{H}$$

where the tag (H) indicates that the promised utility is *higher* than β. We denote function $F_\theta(w)$ to be the solution to the DDE (H) on $w \in [\beta, \bar{w})$ with boundary condition (L_l) on $w \in [0, \beta)$.

Finally, for $w = \bar{w}$, the principal pays the agent a flow payment according to (3.14) and keeps the promised utility at $\bar{w}$, until the next arrival. Standard arguments imply that the principal's value function takes the form of,

$$(\lambda + r)F(\bar{w}) = \lambda F(\bar{w} - \beta) - (\rho\bar{w} + \beta\lambda), \tag{U}$$

where the tag (U) stands for *upper* bound. The combination of (H) and (U) implies that $F'(\bar{w}) = -1$. Intuitively, when the slope of the principal's value function is -1, increasing the promised utility further by an amount costs the principal the same amount. This is consistent with the fact that at this point delaying payment while letting the promised utility increase does not yield further benefit to the principal any more.

In order to specify the optimal value function, we need to determine θ in (L_l). To that end, we introduce function $J(w)$ to be the solution of DDE (H) for $w \geq \beta$ with boundary condition $J(w) = 1$, instead of (L_l), on $w \in [0, \beta)$. Therefore, function $J(w)$ is independent of θ and the monitoring cost m. It is easy to verify that when $\rho = r$, function $F_\theta(w)$, which is the solution to (H) with boundary condition (L_l), can be expressed in terms of $J(w)$ as

$$F_\theta(w) = \theta w - \frac{m}{r}J(w), \quad \text{for } w \leq \bar{w}. \tag{3.18}$$

We further extend the function to $w \geq \bar{w}$ with slope -1, i.e.,

$$F_\theta(w) = F_\theta(\bar{w}) + \bar{w} - w, \ \text{for } w > \bar{w}. \tag{3.19}$$

Furthermore, if we define

$$\theta(\bar{w}) = \frac{m}{r}J'(\bar{w}) - 1, \tag{3.20}$$

it is easy to verify that $F'_{\theta(\bar{w})}(\bar{w}) = -1$ so function $F'_{\theta(\bar{w})}(\bar{w})$ is differentiable at $\bar{w}$ with slope -1.

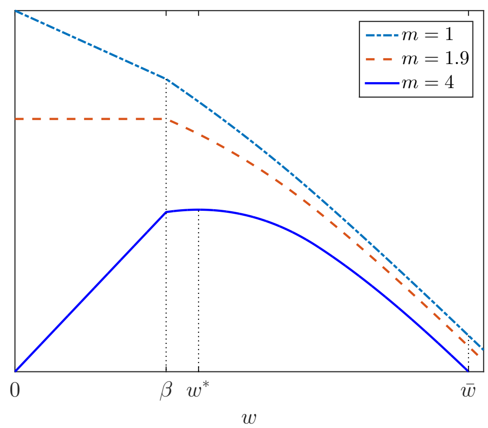
Proposition 3.1. *We have the following properties regarding the value function $F_{\theta(\bar{w})}(w)$ defined according to* (3.18), (3.19) *and* (3.20)*:*

(i) The value $\theta(\bar{w})$ is bounded. Specifically,

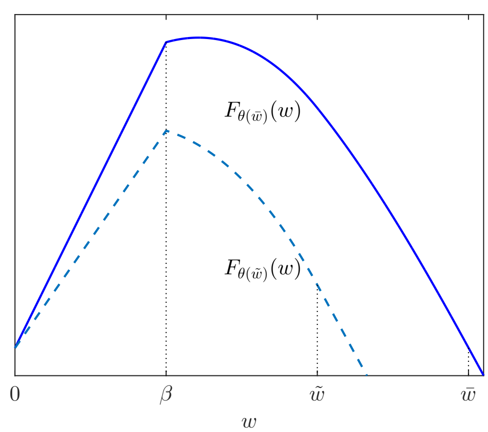
$$-1 \leq \theta(\bar{w}) < \frac{m}{\beta r}. \tag{3.21}$$

(ii) Function $F_{\theta(\bar{w})}(w)$ is linear for $w \in [0, \beta)$ and strictly concave on $[0, \bar{w}]$ with $F'_{\theta(\bar{w})}(\bar{w}) = -1$.

(iii) Moreover, for any $\bar{w}$ and $\tilde{w}$ such that $\beta \leq \tilde{w} < \bar{w}$, we have $F_{\theta(\tilde{w})}(w) < F_{\theta(\bar{w})}(w)$ for any $w \geq 0$.



(a) Different Monitoring Costs



(b) Different Upper Bounds ($\tilde{w} < \bar{w}$)

Figure 3.2: Principal's Value Function $F_{\theta(\bar{w})}(w)$

Figure 3.2 depicts the value function with different model parameters. As we can see, all the functions plotted in the two sub-figures are concave, as described in Proposition 3.1(ii). Therefore, it is easy to find its maximizer

$$w^* = \arg\max_{w \geq 0} F_{\theta(\bar{w})}(w),$$

as depicted in Figure 3.2(a) for the case of $m = 4$. In order to maximize the total future expected utility, the principal should use contract $\Gamma_d(w^*; \beta, \bar{w})$, starting the contract from promised utility w^*.

Figure 3.2(a) also demonstrates that the value function decreases with the monitoring cost m, which is intuitive, and consistent with (3.18). According to (3.18) and (3.20), the slope $\theta(\bar{w})$ increases in m, and is positive if and only if $m > r/J'(\bar{w})$. In Figure 3.2(a), $\theta(\bar{w})$ is positive when $m = 4$, zero when $m = 1.9$, and negative when $m = 1$. If $m < r/J'(\bar{w})$, the slope $\theta(\bar{w}) < 0$ and the maximizer of $F_{\theta(\bar{w})}(w)$ is $w^* = 0$. In this case, the monitoring cost is low enough such that it is optimal for the principal to always monitor while keeping the agent's promised utility at 0. This is depicted in the curve with $m = 1$ in Figure 3.2(a).

Figure 3.2(b) further depicts the value function under different exogenous upper bounds on the promised utility. Consistent with Proposition 3.1(iii), the value function increases with the upper bound. This is also intuitive. From an optimization point of view, the upper bound puts a constraint on the optimal control problem. Relaxing it improves the objective function. From an economic point of view, a higher upper bound allows the principal to delay payments further into the future, and, therefore, improves the principal's utility. This explains the infinite back-loading problem: if allowed, the principal would choose the upper bound to approach infinity.

It is worth discussing the underlying reason why infinite back-loading arises in the equal discount setting. First of all, let us consider the first-best/efficient outcome, one that maximizes the societal utility without private information. In the equal discount setting, the efficient outcome corresponds to the low arrival rate λ with no monitoring, since payments have no impact on the total utility of the two players. It is important to realize that in our setting with arrivals of adverse events, any incentive compatible contract (including the optimal one) cannot induce the first-best outcome. This is because the (IC) constraint implies that no matter how high the promised utility is, there is always a positive probability with which the promised utility drops below the threshold β following a number of frequent arrivals, triggering monitoring, and losing efficiency. On the other hand, the higher the promised utility, the longer it takes to start monitoring (i.e., lose efficiency). This explains why the higher the upper bound $\bar{w}$, the better the objective function. If $\bar{w}$ is infinity, however, the principal essentially does not pay the agent in any finite time, which is no longer a meaningful contract. In fact, a contract becomes meaningless if $\bar{w}$ is too high (say higher than the total wealth of the world), since the principal does not have the credibility of delivering such a promised utility to the agent.

With different discount rate as will be discussed in the next section, this infinite back-loading problem does not appear. This is because the cost of early payments to the principal is lower than its benefit to the agent. As a result, it is no longer beneficial for the principal to always delay payment. The corresponding upper bound of promised utility at which payment starts becomes finite.

3.3.3 Proof of Optimality

So far we have not formally established the connection between the value function $F_{\theta(\bar{w})}(w)$ and either contract structure. Now we establish that $F_{\theta(\bar{w})}(w)$ is indeed the optimal value function.

First, the following proposition states that $F_{\theta(\bar{w})}(w)$ is indeed the value function of both contracts $\Gamma_d(w;\beta,\bar{w})$ and $\Gamma_r(w;\bar{w})$.

Proposition 3.2. *For $F_{\theta(\bar{w})}(w)$ defined according to* (3.18), (3.19) *and* (3.20), *we have*

$$F_{\theta(\bar{w})}(w) = U\big(\Gamma_d(w;\beta,\bar{w})\big) = U\big(\Gamma_r(w;\bar{w})\big).$$

Therefore, starting from w^*, contracts $\Gamma_d(w^*;\beta,\bar{w})$ and $\Gamma_r(w^*;\bar{w})$ both yields the maximum $F_{\theta(\bar{w})}(w^*)$ for the principal.

According to both (3.9) and (3.15), the promise utility stays at 0 forever whenever it falls to 0, according to both contracts $\Gamma_d(w;\beta,\bar{w})$ and $\Gamma_r(w;\bar{w})$. That is, once $W_t = 0$, the principal needs to monitor the agent (and endure the monitoring cost) forever. Following the optimal contract $\Gamma_d(w^*;\beta,\bar{w})$, however, the promised utility hitting zero is a zero measure event. And, starting from a promised utility $w \in (0,\beta)$, dynamics (3.10) implies that the length of a monitoring episode under the optimal contract $\Gamma_d(w^*;\beta,\bar{w})$ is

$$T_m(w) := \frac{1}{\rho}\left(\ln\beta - \ln w\right), \tag{3.22}$$

which is finite. Superficially, the randomized contract $\Gamma_r(w^*;\bar{w})$ may appear worse due to the possibility of monitoring forever. In fact, when the promised utility before an arrival is in the interval

$(\beta, 2\beta)$, under the deterministic contract $\Gamma_d(w^*; \beta, \bar{w})$, the principal has to pay the monitoring cost for a period of time after each arrival. Under the randomized contract $\Gamma_r(w^*; \bar{w})$, however, there is a chance that monitoring does not happen at all, which balances the chance of monitoring ever after. This explains, intuitively, why these two contracts are equivalent to the risk-neutral principal.

The following theorem, together with Proposition 3.2, establishes the optimality of value function $F_{\theta(\bar{w})}(w)$ and both contracts $\Gamma_d(w^*; \beta, \bar{w})$ and $\Gamma_r(w^*; \bar{w})$.

Theorem 3.1. *For any incentive compatible contract* Γ *which yields an agent's utility* $w \leq \bar{w}$*, we have*

$$U(\Gamma) \leq F_{\theta(\bar{w})}(w) \leq F_{\theta(\bar{w})}(w^*). \tag{3.23}$$

Therefore, contracts $\Gamma_d(w^*; \beta, \bar{w})$ *and* $\Gamma_r(w^*; \bar{w})$ *are both optimal, which yield expected utility* $F_{\theta(\bar{w})}(w^*)$ *to the principal.*

The first inequality in (3.23) follows from Lemma 3.3. The second inequality simply follows from w^* being the maximizer of $F_{\theta(\bar{w})}(w)$. Therefore, Theorem 3.1 and Proposition 3.2 imply that the principal's expected utility generated from contracts $\Gamma_d(w^*; \beta, \bar{w})$ and $\Gamma_r(w^*; \bar{w})$ is higher than those generated from any other incentive compatible contracts. Hence, these two contracts are both optimal. Finally, it is worth pointing out that various combinations of the contracts $\Gamma_d(w^*; \beta, \bar{w})$ and $\Gamma_r(w^*; \bar{w})$ are also optimal. That is, whenever in the interval $(0, \beta)$, the promised utility w can either continuously increase following (3.9), or randomly jump between 0 and β, following (3.15), no matter how it behaved in this interval previously.

3.4 Different Discount Rates

In this section, we consider the case of $\rho > r$. That is, the principal is more patient than the agent. One important distinction compared with the case of $\rho = r$ is that there exists a finite upper bound $\bar{w}^*$ on the promised utility, which is implied endogenously under the optimal contract. Therefore, we no longer need to introduce the exogenous upper bound as in the previous section. Nevertheless, when $\rho > r$ we can still include an exogenous upper bound $\bar{w}$ for the promised utility in the model, which is discussed in Section 3.4.3.

For the main parts of this section (Sections 3.4.1 and 3.4.2), we show that the structure of the optimal contract changes with model parameters, especially the monitoring cost, as illustrated in

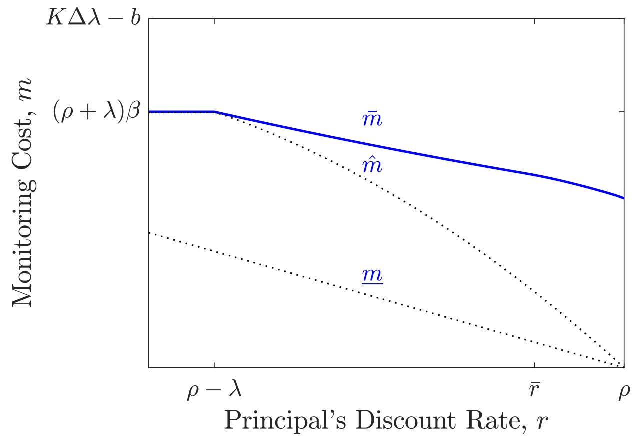


Figure 3.3: Split of the Low and High Monitoring Cost

Figure 3.3. In this figure, we vary the principal's discount rate r (as the x-axis) and the monitoring cost m (as the y-axis), while keeping other model parameters fixed. In the following two subsections, we show that if the monitoring cost m is above a threshold $\bar{m}$ (the solid curve), the optimal contract takes a structure similar to the deterministic contract defined in the previous section. If m is below $\bar{m}$, on the other hand, it is optimal for the principal to always monitor the agent. Figure 3.3 depicts two additional dotted curves $\hat{m}$ and $\underline{m}$ in this region. We defer the detailed discussion on them to Section 3.4.2.

Here, we first define the threshold $\bar{m}$ as

$$\bar{m} := \inf_{w>\beta} \frac{r}{J'(w)}. \tag{3.24}$$

For the case of $\rho = r$, equation (3.20) implies that $\theta(\bar{w}) < 0$ for any $\bar{w} > \beta$ when $m < \bar{m}$, which implies that the value function is decreasing, and, therefore, it is optimal to always monitor the agent. Later in Section 3.4.2, we show that for the case $\rho > r$, it is still optimal to always monitor the agent. Next, we first study the case of $m \geq \bar{m}$.

3.4.1 High Monitoring Cost

In this subsection, we investigate the case in which the monitoring cost is above the threshold $\bar{m}$.

Recall Corollary 3.1, the principal needs to monitor when the promised utility w is lower than β. When $m \geq \bar{m}$, the principal may still need to monitor the agent even if w is higher than β. That is, the optimal contract is similar to the deterministic contract Γ_d in the previous section, in which monitoring occurs whenever the promised utility is below a threshold $\alpha \geq \beta$.

Following the same heuristic derivation in Section 3.3.2, define function $F_{\theta,\alpha}(w)$ to be the solution of DDE (H) for $w \in [\alpha, \infty)$ with boundary condition (L) for $w \in [0, \alpha)$. Next, we identify the optimal value function in a two step procedure. In the first step, we specify α for a given parameter θ. In the second step, we establish θ and the endogenous upper bound $\bar{w}^*$.

First, we identify the threshold α for a given parameter θ. A key property of the value function $F_{\theta,\alpha}(w)$ is "smooth pasting" [DP94] at $w = \alpha$. That is, the left and right derivatives, $F'_{\theta,\alpha}(\alpha_-)$ and $F'_{\theta,\alpha}(\alpha_+)$, respectively, are set to be equal to each other if possible.[8] To this end, it is convenient to define the following function,

$$f(\alpha) := \left(F'_{\theta,\alpha}(\alpha_-) - F'_{\theta,\alpha}(\alpha_+)\right)(\rho\alpha + \beta\lambda) = m - \lambda\theta\alpha^{\frac{r}{\rho}}\left[\left(1 - \frac{r\beta}{\rho\alpha}\right) - \left(1 - \frac{\beta}{\alpha}\right)^{\frac{r}{\rho}}\right], \tag{3.25}$$

in which $F'_{\theta,\alpha}(\alpha_-)$ and $F'_{\theta,\alpha}(\alpha_+)$ are obtained from (L) and (H) with switching point α, respectively. Therefore, we may set $f(\alpha) = 0$ to achieve $F'_{\theta,\alpha}(\alpha_-) = F'_{\theta,\alpha}(\alpha_+)$.

Lemma 3.4. *Function $f(\alpha)$ is increasing in α on $[\beta, \infty)$, and $\lim_{\alpha\to\infty} f(\alpha) = m$.*

In order to find the threshold α by solving the equation $f(\alpha) = 0$, denote f^{-1} to represent the inverse function of the monotone function f, and, for any θ, define

$$\alpha_\theta := \begin{cases} \beta, & \text{if } f(\beta) \geq 0, \\ f^{-1}(0), & \text{if } f(\beta) < 0. \end{cases} \tag{3.26}$$

Proposition 3.3. *(i) We have*

$$f(\alpha_\theta) \geq 0 \text{ and } (\alpha_\theta - \beta) f(\alpha_\theta) = 0. \tag{3.27}$$

Therefore, if $\alpha_\theta = \beta$, we have $F'_{\theta,\alpha_\theta}(\alpha_{\theta-}) \geq F'_{\theta,\alpha_\theta}(\alpha_{\theta+})$, while if $\alpha_\theta > \beta$, we have $F'_{\theta,\alpha_\theta}(\alpha_{\theta-}) = F'_{\theta,\alpha_\theta}(\alpha_{\theta+})$.

(ii) Furthermore, if $\alpha_\theta > \beta$, we have $F''_{\theta,\alpha_\theta}(\alpha_{\theta+}) < F''_{\theta,\alpha_\theta}(\alpha_{\theta-}) < 0$.

(iii) Finally, for any $\alpha \in [\beta, \alpha_\theta)$, $F'_{\theta,\alpha}(\alpha_-) < F'_{\theta,\alpha}(\alpha_+)$; for $\alpha \in (\alpha_\theta, \infty)$, $F'_{\theta,\alpha}(\alpha_-) > F'_{\theta,\alpha}(\alpha_+)$.

Proposition 3.3(i) and (ii) imply that function $F_{\theta,\alpha_\theta}(w)$ is locally concave at α. This is important for us to later show (global) concavity, and optimality, of this function. Proposition 3.3(iii) further helps us to establish Proposition 3.4(ii) to be presented later.

After characterizing α, we now describe the optimal θ and the endogenous upper bound $\bar{w}^*$.

Lemma 3.5. *(i) Function $F_{\theta,\alpha_\theta}(w)$ is super-modular in $(\theta, w) \in \mathbb{R} \times \mathbb{R}^+$. Therefore, derivative $F'_{\theta,\alpha_\theta}(w)$ increases in θ for any w.*

(ii) For any given parameter $m \geq \bar{m}$, there exist positive quantities $\bar{\theta}$ and $\bar{w}^*$, such that

$$\inf_{w>\alpha_{\bar{\theta}}} F'_{\bar{\theta},\alpha_{\bar{\theta}}}(w) = -1 \text{ and } \bar{w}^* := \inf\left\{\arg\inf_{w>\alpha_{\bar{\theta}}} F'_{\bar{\theta},\alpha_{\bar{\theta}}}(w)\right\}, \text{ respectively.} \tag{3.28}$$

Furthermore, we have

$$0 \leq \bar{\theta} < \frac{m}{r}\beta^{-\frac{r}{\rho}} \quad \text{and} \quad \bar{w}^* \in [\alpha_{\bar{\theta}}, \infty). \tag{3.29}$$

Lemma 3.5(i) implies that the value $\bar{\theta}$ as defined in (3.28) is unique, and (ii) further indicates that it is upper and lower bounded. Therefore, the value of $\bar{\theta}$ can be identified using binary search. The lower bound 0 implies that the value function is non-decreasing on $[0, \alpha_{\bar{\theta}}]$. Therefore, the maximizer of the value function is non-negative. This further implies that if the monitoring cost is higher than $\bar{m}$, it would be too costly for the principal to always monitor the agent. The upper bound can be used in a binary search algorithm to find $\bar{\theta}$. Finally, (3.29) also indicates that the endogenous upper bound $\bar{w}^*$ is indeed finite.

Now we are ready to define the following value function $F(w)$ based on $F_{\bar{\theta},\alpha_{\bar{\theta}}}(w)$,

$$F(w) := \begin{cases} F_{\bar{\theta},\alpha_{\bar{\theta}}}(w), & \text{if } w \leq \bar{w}^*, \\ F_{\bar{\theta},\alpha_{\bar{\theta}}}(\bar{w}^*) - (w - \bar{w}^*), & \text{otherwise.} \end{cases} \tag{3.30}$$

Here are some key properties of the value function that are essential for proving its optimality.

Proposition 3.4. *For $m \geq \bar{m}$, we have:*

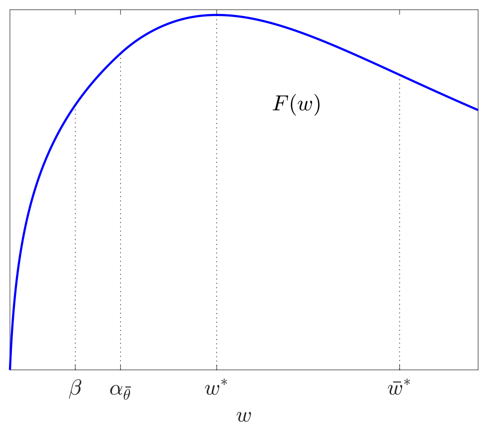
(i) Function $F(w)$ is strictly concave on $w \in [0, \bar{w}^*]$, with $F'(w) = -1$ for $w \geq \bar{w}^*$.

(ii) For any $w \in (\alpha_{\bar{\theta}}, \bar{w}^*]$, we have $rF(w) > \rho w F'(w) - m$.

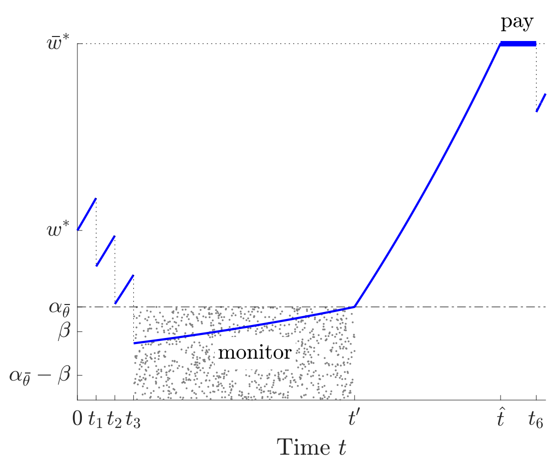
Proposition 3.4(i) is a standard property that often arises in the dynamic contracting literature; it is the foundation for proving that $F(w)$ is the optimal value function. Proposition 3.4(ii), however, appears unique to our setting, and requires a novel proof based on Proposition 3.3. Comparing this differential inequality with the differential equation (3.17), it is clear that for any promised utility

w above the threshold $\alpha_{\bar{\theta}}$, the principal is better off not to monitor. This condition is critical in proving optimality of the threshold structure in our contract.

Based on the calculation of threshold $\alpha_{\bar{\theta}}$ and upper bound $\bar{w}^*$ in Lemma 3.5, we can establish that the optimal contract is $\Gamma_d(w^*; \alpha_{\bar{\theta}}, \bar{w}^*)$ following Definition 3.1, in which w^* is a maximizer of function $F(w)$ defined in (3.30).



(a) Function $F(w)$



(b) A Sample Trajectory of W_t under $\Gamma_d(w; \alpha_{\bar{\theta}}, \bar{w}^*)$

Figure 3.4: High Monitoring Cost (i.e., $m \geq \bar{m}$)

Figure 3.4(a) provides a sample sketch of the principal's value function. Under this particular parameter setting, we have $\alpha_{\bar{\theta}} > \beta$, and, therefore, according to Proposition 3.3, the value function demonstrates the "smooth pasting" property at $\alpha_{\bar{\theta}}$.

Figure 3.4(b) presents a sample trajectory of the promised utility according to the optimal contract, using the same parameter values as in Figure 3.4(a). As we can see, in this particular example, we have $\alpha_{\bar{\theta}} > \beta$ and $w^* > \alpha_{\bar{\theta}}$. Therefore, the principal starts the contract with the initial promised utility w^*, without monitoring the agent. As long as W_t is above $\alpha_{\bar{\theta}}$, the promised utility W_t takes a downward jump of β for each arrive (at time t_1, t_2, t_3, and t_6 in the figure). In this sample trajectory, the promised utility drops below $\alpha_{\bar{\theta}}$ at time t_3. The principal starts monitoring the agent at this point while the promised utility W_t cumulates interest and increases along the exponential curve (3.10) until it reaches $\alpha_{\bar{\theta}}$, regardless of arrivals (t_4 and t_5) in the interval $[t_3, t')$. When the promised utility climbs back to $\alpha_{\bar{\theta}}$ (at time t'), it keeps increasing along the other exponential curve (3.12) as long as there is no arrival. A flow payment starts when W_t reaches $\bar{w}$ at time $\hat{t}$, and stops when another arrival (at t_6 in this figure) drops W_t to below $\bar{w}^*$ again.

Similar to Proposition 3.2 and Theorem 3.1 for the equal discount case, the following result establishes the optimality for the case of $\rho > r$.

Theorem 3.2. *For $m \geq \bar{m}$ and any incentive compatible contract Γ that yields an agent's utility w, we have $U(\Gamma) \leq F(w) \leq F(w^*) = U\big(\Gamma_d(w^*; \alpha_{\bar{\theta}}, \bar{w}^*)\big)$. Therefore, contract $\Gamma_d(w^*; \alpha_{\bar{\theta}}, \bar{w}^*)$ is the optimal contract, which yields utilities $F(w^*)$ for the principal and w^* for the agent.*

It is worth noting that under the optimal contract $\Gamma_d(w^*; \alpha_{\bar{\theta}}, \bar{w}^*)$, it is possible that the agent's promised utility never reaches 0. That is, constraint (IR) is never binding. In fact, as long as $\alpha_{\bar{\theta}} > \beta$, a downward jump induced by an arrival at most brings the promised utility W_t down to $\alpha_{\bar{\theta}} - \beta$, and never lower. (Figure 3.4 depicts such a case.) This phenomenon contrasts sharply with long held insights in the optimal mechanism/contract design literature, where the individual rationality constraint is generally binding. Therefore, a curious reader may wonder if the agent is over paid under our definition of contract $\Gamma_d(w^*; \alpha_{\bar{\theta}}, \bar{w}^*)$ when the monitoring threshold $\alpha_{\bar{\theta}} > \beta$.

In fact, similar to (3.22), for any monitoring threshold $\alpha > \beta$, starting from the lowest possible promised utility level $\alpha - \beta$, it takes time $\dfrac{1}{\rho} \ln \dfrac{\alpha}{\alpha - \beta}$ for monitoring to stop. This time increases as α decreases, and approaches infinity as α decreases to β. Therefore, the higher the value of α, the shorter the monitoring time period, during which the principal has to endure the monitoring cost at rate m. This explains why for high monitoring cost m (as in the current case), the principal is willing to set a threshold α higher than β, in order to avoid long episodes of monitoring the agent. Even though the agent's promised utility is maintained at strictly positive levels, this strategy yields lower monitoring costs than keeping the threshold at β. This phenomenon highlights the tradeoff that the principal faces between payments to the agent and monitoring costs.

3.4.2 Low Monitoring Cost

We now consider the case when the monitoring cost m is lower than $\bar{m}$. First, it is helpful to consider the case of $m = \bar{m}$. Following the previous subsection, it is easy to verify that $\bar{\theta} = 0$ and $\alpha_{\bar{\theta}} = \beta$. In this case, the function $F(w)$ is linear (in fact, constant) for $w \in [0, \beta)$. If we decrease m further and still follow contract Γ_d, then Lemma 3.5 yields a negative $\bar{\theta}$. Loosely speaking, the corresponding value function is decreasing, and, therefore, the optimal contract is to always monitor the agent and keep the promised utility at 0. Indeed, this is the structure of the optimal contract.

More rigorously, contract Γ_d with a starting promised utility 0 and monitoring threshold 0 is, in fact, not optimal. A value function following (L) with a negative $\bar{\theta}$ is convex, instead of concave. A non-concave value function cannot be optimal, because it can be improved through concavification with randomization.

In fact, the exact form of the optimal value function varies with model parameters when $m < \bar{m}$. In Figure 3.3, two dotted curves, $\hat{m}$ and $\underline{m}$, further divide the $m < \bar{m}$ area into three regions, each corresponding to a distinct value function form. Here we provide the expressions for $\hat{m}$ and $\underline{m}$ as

$$\underline{m} := (\rho - r)\beta \quad \text{and} \quad \hat{m} := \begin{cases} \beta(\rho - r)(2\lambda + r)/\lambda, & \text{if } r > \rho - \lambda, \\ (\rho + \lambda)\beta, & \text{if } r \le \rho - \lambda. \end{cases} \tag{3.31}$$

The simplest value function is a linear function,

$$F(w) = -\frac{m}{r} - w, \tag{3.32}$$

which is optimal when $m < \underline{m}$. In this case, the monitoring cost is so low that the principal should simply pay off any positive promised utility immediately and start monitoring the agent forever.

If $m \in [\underline{m}, \hat{m})$, the optimal value function is a slightly more complex piecewise linear function of the following form,

$$F(w) = \begin{cases} -\dfrac{m}{r} - \left[1 - \dfrac{(\rho - r)\beta - m}{(\lambda + r)\beta}\right] w, & \text{if } w \le \beta, \\ F(\beta) - (w - \beta), & \text{if } w > \beta. \end{cases} \tag{3.33}$$

The randomized contract $\Gamma_r(w; \beta)$ following Definition 3.2 achieves the value function. That is, similar to Proposition 3.2, we can show that $F(w) = U\big(\Gamma_r(w; \beta)\big)$. In other words, if, for whatever reason, the initial promised utility is $w > \beta$, then the principal pays the agent $w - \beta$ to bring the promised utility down to β, and then keeps it there while paying a flow of interest and information rent, $(\rho + \lambda)\beta$, to the agent, until the first arrival. Upon the arrival, the principal start monitoring the agent forever and stops the payment. Because the value function is decreasing, its maximizer is 0. Therefore, the optimal contract $\Gamma_r(0; \beta)$ effectively starts monitoring from the very beginning.

If $m \in [\hat{m}, \bar{m})$, however, the optimal value function is more complex. It is the solution to DDE (H) for $w \in [\beta, \bar{w}^*]$ with boundary condition (L_l) for $w \in [0, \beta)$, where θ in (L_l) and $\bar{w}^*$ are defined as the following,

$$\inf_{w > \beta} F'_{\bar{\theta}}(w) = -1 \quad \text{and} \quad \bar{w}^* = \inf\Big\{ \arg \inf_{w>\beta} F'_{\bar{\theta}}(w) \Big\}. \tag{3.34}$$

Therefore, function $F(w)$, define as

$$F(w) = \begin{cases} F_{\bar{\theta}}(w), & \text{if } w \le \bar{w}^*, \\ F_{\bar{\theta}}(\bar{w}^*) + \bar{w}^* - w, & \text{if } w > \bar{w}^*, \end{cases} \tag{3.35}$$

is linear on $[0, \beta)$ and nonlinear on $[\beta, \bar{w}^*)$, and takes a slope of -1 on $[\bar{w}^*, \infty)$. Furthermore, the next result characterizes $\bar{w}^*$ and $\bar{\theta}$.

Proposition 3.5. *For $m \in [\hat{m}, \bar{m})$, and $\bar{\theta}$ and $\bar{w}^*$ defined in* (3.34)*, we have*

$$\bar{w}^* \ge 2\beta \quad \textit{and} \quad -1 + \frac{\rho - r}{\lambda} < \bar{\theta} \le 0. \tag{3.36}$$

The randomized contract $\Gamma_r(w; \bar{w}^*)$ achieves the value function. That is, the proof of Proposition 3.2 already establishes that $F(w) = U\big(\Gamma_r(w; \bar{w}^*)\big)$. Again, the value function is decreasing. Therefore, following contract $\Gamma_r(0; \bar{w}^*)$, the principal monitors the agent from the beginning forever.

Figure 3.5 depicts the optimal value functions for different monitoring costs. As we can see from the figure, for the monitoring cost $m_3 \in [0, \underline{m})$, the value function is a straight line with slope -1. If we further increase the monitoring cost to $m_2 \in [\underline{m}, \hat{m})$, the value function becomes a piece-wise linear function. If the monitoring cost further increases to $m_1 \in [\hat{m}, \bar{m})$, the value function is non-linear in the interval $[\beta, \bar{w}^*)$, with an endogenous $\bar{w}^* > 2\beta$. Finally, the value function decreases with the monitoring cost.

Now we are ready to show our main result of this section.

Theorem 3.3. *For $m < \bar{m}$ and any incentive compatible contract Γ that yields an agent's utility w, we have $U(\Gamma) \le F(w) \le F(0) = -m/r$, in which concave function $F(w)$ is defined as* (3.32)

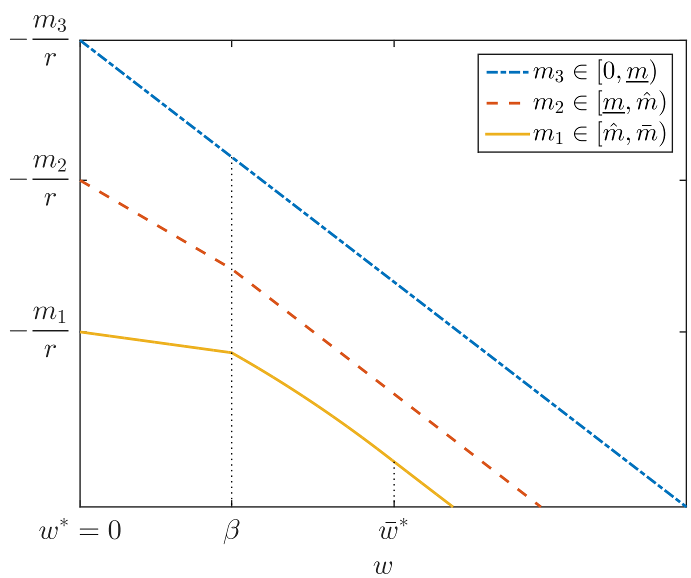


Figure 3.5: Value Functions with Low Monitoring Costs

for $m \in [0, \underline{m})$, (3.33) *for* $m \in [\underline{m}, \hat{m})$, *and* (3.35) *for* $m \in [\hat{m}, \bar{m})$, *where* $\bar{\theta}$ *and* $\bar{w}^*$ *are defined in* (3.34). *Therefore, it is optimal for the principal to always monitor the agent.*

3.4.3 Exogenous Upper Bound

In certain practical settings, the principal may not be able to allow the promised utility to grow too high before paying the agent. This is especially true if the agent's discount rate is close to the principal's, in which case the endogenous upper bound $\bar{w}^*$, although finite, tends to be very large. Therefore, in this subsection we allow the model to include an exogenous upper bound $\bar{w}$ on the promised utility. That is, our optimization problem has an additional constraint $w \leq \bar{w}$, similar to the case of $\rho = r$.

It is clear that if the exogenous upper bound $\bar{w}$ is higher than the endogenous $\bar{w}^*$, then the constraint $w \leq \bar{w}$ is not binding and it has no effect on the optimal contract. Therefore, we focus on the situation where, after computing $\bar{w}^*$ without considering $\bar{w}$, the principal realizes that $\bar{w} < \bar{w}^*$.

An immediate observation is that the threshold $\bar{m}$, which separates the high and low monitoring cost regions, needs to change from (3.24) to the following,

$$\bar{m}(\bar{w}) := \inf_{w \in (\beta, \bar{w}]} \frac{r}{J'(w)}. \tag{3.37}$$

Obviously, this new threshold $\bar{m}(\bar{w})$ increases in the upper bound $\bar{w}$, and, therefore, is greater than or equal to $\bar{m}$ defined in (3.24). Therefore, the principal may choose to always monitor the agent for higher monitoring costs comparing with the base model without $\bar{w}$. This is intuitive not only mathematically, but also practically. The upper bound pushes the principal to start payments "prematurely." Given the trade-off between payments and monitoring costs, such a pressure makes monitoring more favorable.

Finally, thresholds $\hat{m}$ and $\underline{m}$ do not change with the upper bound $\bar{w}$. The main results of this section only require slight changes to accommodate the upper bound $\bar{w}$. For example, in specifying the monitoring threshold $\alpha_{\bar{\theta}}$ and optimal value functions, (3.28) and (3.34) are changed to $F'_{\bar{\theta},\alpha_{\bar{\theta}}}(\bar{w}) = -1$ and $F'_{\bar{\theta}}(\bar{w}) = -1$, respectively. The optimal contract for high monitoring cost is $\Gamma_d(w^*; \alpha_{\bar{\theta}}, \bar{w})$, in which w^* is the maximizer of the corresponding updated value function. For the low monitoring cost case, contract $\Gamma_r(0; \bar{w})$ achieves the optimal value function in place of $\Gamma_r(0; \bar{w}^*)$.

3.5 Computation

It is worth pointing out that the optimal value functions and contracts presented in Sections 3.3 and 3.4 are very easy to compute. For the value function $F_{\theta(\bar{w})}(w)$ of Section 3.3, we only need to first solve the function $J(w)$ following DDE (H) using, for example, the standard shooting method starting from $J(w) = 1$ for $w \in [0, \beta)$. After obtaining the function $J(w)$, we obtain the slope $\theta(\bar{w})$ using (3.20). Then the value function $F_{\theta(\bar{w})}(w)$ is readily available following (3.18). The exact definition of the optimal contract follows the initial promised utility w^*, which is a maximizer of $F_{\theta(\bar{w})}(w)$. After obtaining the optimal contract, implementing it over time becomes very easy, as we have already discussed in Section 3.3.

When the principal is more patient than the agent ($\rho > r$), threshold $\bar{m}$ defined in (3.24) is easy to compute. In fact, it has closed form expressions if r and p do not differ too much, as shown in the following result.

Proposition 3.6. *(a) For $r \in (0, \rho - \lambda]$, we have $\bar{m} = (\rho + \lambda)\beta$;*

(b) For $r \in (\rho - \lambda, \bar{r}]$, we have

$$\bar{m} = (\rho + \lambda)\beta\Big[1 - \frac{\beta\rho}{\beta(2\rho + \lambda)}\Big]^{\frac{\lambda + r}{\rho} - 1},$$

in which $\bar{r}$ is the unique solution to the following equation on $[\rho - \lambda, \rho]$,

$$\left[1 - \frac{\beta\rho}{\beta(2\rho+\lambda)}\right]^{\frac{\lambda+\bar{r}}{\rho}-1} = 1 - \frac{\rho - \bar{r}}{\lambda}. \tag{3.38}$$

If the monitoring cost is higher than the threshold $\bar{m}$, the computation is slightly more complex than the case with equal discount. In order to specify the optimal value function $F_{\theta,\alpha_{\bar{\theta}}}(w)$, we also need to search for the slope $\bar{\theta}$ through a binary search. In Algorithm 1 we provide a pseudo code for the arguably more complex case of $\rho > r$ and $m > \bar{m}(\bar{w})$ with an exogenous upper bound $\bar{w}$.

Algorithm 1

1: Let *Stopping* $\leftarrow 0$, $\theta_l \leftarrow 0$, and $\theta_h \leftarrow m\frac{\beta^{-r/\rho}}{r}$ following (3.29)
2: **while** *Stopping* $= 0$ **do**
3: Let $\theta \leftarrow (\theta_l + \theta_h)/2$
4: Compute α_θ according to (3.26), in which the function f is defined in (3.25)
5: Use the shooting method to compute function $F_\theta(w)$ following DDE (H) for $w \geq \alpha_\theta$ with boundary condition (L) on $w \in [0, \alpha_\theta)$, until a point $\hat{w} \in [\alpha_\theta, \bar{w}]$ that must satisfies one of the following cases:
6: **if** $F'_{\theta,\alpha_\theta}(\hat{w}) < -1$ **then**
7: Let $\theta_l \leftarrow \theta$
8: **else if** ($\hat{w} < \bar{w}$ **and** $F''_{\theta,\alpha_\theta}(\hat{w}) \geq 0$) **or** $\hat{w} = \bar{w}$ **then**
9: **if** ($F'_{\theta,\alpha_\theta}(\hat{w}) > -1$ **and** $\hat{w} < \bar{w}$) **then**
10: Let $\theta_h \leftarrow \theta$
11: **else if** ($F'_{\theta,\alpha_\theta}(\hat{w}) = -1$ **or** $\hat{w} = \bar{w}$) **then**
12: Let $\bar{w}^* \leftarrow \hat{w}$, $\bar{\theta} \leftarrow \theta$ and *Stopping* $\leftarrow 1$
13: **end if**
14: **end if**
15: **end while**

The logic behind Steps 6 and 7 of Algorithm 1 follows from Lemma 3.5 and Proposition 3.7 below.

Proposition 3.7. *Function $F_{\theta,\alpha_\theta}(w)$ is strictly concave on $w \in [0, \hat{w}]$ for $\hat{w}$ defined as the following,*

$$\hat{w} := \begin{cases} \inf\{\arg\inf_{w>\alpha_\theta} F'_{\theta,\alpha_\theta}(w)\}, & \text{if } \theta \geq \bar{\theta}, \\ \inf\{w : w \geq \alpha_\theta \text{ and } F'_{\theta,\alpha_\theta}(w) < -1\}, & \text{if } \theta < \bar{\theta}. \end{cases} \tag{3.39}$$

Following the definition of $\bar{\theta}$ in (3.28), Lemma 3.5(i) implies that if $\theta \geq \bar{\theta}$, we must have $F_{\theta,\alpha_\theta}(w) \geq -1$ for all $w \geq \alpha_\theta$. Therefore, the existence of a point $\hat{w}$ such that $F_{\theta,\alpha_\theta}(\hat{w}) < -1$ must imply that $\theta < \bar{\theta}$. Consequently, value θ serves as a lower bound θ_l for $\bar{\theta}$. Furthermore, Proposition 3.7 guarantees that if $\theta < \bar{\theta}$, for any $w \leq \hat{w}$, we must have $F''_{\theta,\alpha_\theta}(w) < 0$. Therefore, the search does not stop prematurely at a point following Steps 8 and 9.

The logic behind Steps 8 and 9 also follows from Proposition 3.7 together with Lemma 3.5(i). In particular, Proposition 3.7 implies that for $\theta > \bar{\theta}$, as soon as we observe a point $\bar{w}$ with $F''_{\theta,\alpha_\theta}(\hat{w}) = 0$ for the first time, the point $\hat{w}$ must be the minimum of derivative $F'_{\theta,\alpha_\theta}(w)$ over the entire interval $[\alpha_\theta, \infty)$. Hence, if $F'_{\theta,\alpha_\theta}(\hat{w}) > -1$, we must have $\theta > \bar{\theta}$.

Overall, the algorithm involves a binary search for $\bar{\theta}$, and solving for the value function given any current choice of θ. This computation, again, is very easy to implement. Overall, simple computation and contract structures make our results easily implementable in practice.

3.6 A Simple Cyclic Monitoring Schedule

The optimal monitoring and payment schedules are dynamically adjusted following changes of the promised utility. In certain situations the principal may prefer an even simpler, more "regular" schedule. For example, one may think of a "periodic review" contract, which is determined by a set of parameters (T, N_d, π). Under this contract, the principal reviews the performance of the agent every T time units. If the number of arrivals during this period is less than or equal to N_d, the agent collects an amount of payment π; otherwise the agent is not compensated for this cycle. While payment to the agent is based on the actual performance, such a contract may not be incentive compatible. Imagine that the number of arrivals is already N_d in the middle of the cycle. In this case the agent has no incentive to continue the effort for the remaining time in the cycle. Similarly, the agent may not want to bother exerting effort towards the end of a cycle when the number of arrivals is still far below N_d. Lack of full effort also implies that it may be hard to determine the optimal values for the contract parameters, because the contract design is no longer an optimization problem, but involves a differential game.

Here we propose a different, incentive compatible, cyclic contract that is very easy to compute and manage. Each cycle starts with a flow payment, until an arrival, which starts a monitoring episode of a fixed period of time without payment. After the monitoring episode a new cycle starts. This schedule is not only very easy to implement, its optimal parameters (payment level and length of monitoring episode) are also very easy to compute. For the equal discount case, they are even in closed forms.

Now we derive the optimal parameters for this contract. Denote ℓ to be the flow payment, T the length of each monitoring period, and $\hat{w}$ the agent's continuation utility while being paid. An arrival brings down this continuation utility by β, as long as $\hat{w} > \beta$. Therefore, we have

$$\hat{w} = \int_0^\infty \lambda e^{-\lambda t} \left[\int_0^t e^{-\rho\tau} \ell d\tau + e^{-\rho t} (\hat{w} - \beta) \right] dt \quad \text{and} \quad \hat{w} - \beta = e^{-\rho T} \hat{w},$$

which imply that

$$\hat{w} = \frac{1}{\rho}(\ell - \lambda\beta) \quad \text{and} \quad e^{-\rho T} = \frac{\ell - (\lambda + \rho)\beta}{\ell - \lambda\beta}. \tag{3.40}$$

Clearly, to make the cyclic monitoring schedule meaningful, the flow payment rate ℓ must be no less than $(\lambda + \rho)\beta$. Equation (3.40) reveals the one-to-one correspondence between the payment ℓ, maximal utility $\hat{w}$, and monitoring period length T, from binding incentive constraints. Next, denote $C(\ell)$ to be the principal's cost of this simple contract as a function of the payment ℓ. It is easy to verify that function $C(\ell)$ follows a recursive formulation,

$$C(\ell) \quad = \quad \int_0^\infty \lambda e^{-\lambda t} \left[\int_0^t e^{-r\tau} \ell d\tau + e^{-rt} \int_0^T e^{-r\tau} m d\tau + e^{-r(T+t)} C(\ell) \right] dt.$$

Therefore, following (3.40),

$$C(\ell) = \frac{m}{r} - \frac{m - \ell}{\lambda + r - \lambda \left(\frac{\ell - (\lambda+\rho)\beta}{\ell - \lambda\beta} \right)^{\frac{r}{\rho}}},$$

from which we can compute the optimal ℓ^* that minimizes $C(\ell)$ following a simple one-dimensional search, and then obtain the optimal value following (3.40).

For the equal discount case (i.e., $r = \rho$), we may express the principal's cost as a simple convex function of payment rate ℓ,

$$C(\ell) = \frac{\ell - \lambda\beta}{r} + \frac{\lambda m \beta}{r\ell},$$

which is minimized at $\ell^* = \sqrt{\lambda m \beta}$ when $m > (\lambda + \rho)^2\beta/\lambda$. The corresponding monitoring time T^* is

$$T^* = \frac{1}{r} \ln \frac{\sqrt{\lambda m \beta} - \lambda\beta}{\sqrt{\lambda m \beta} - \beta(\lambda + r)}.$$

The principal's optimal cost under this contract is $C(\ell^*) = (2\sqrt{\lambda\beta m} - \lambda\beta)/r$. When $m \leq (\lambda+\rho)^2\beta/\lambda$, on the other hand, the optimal flow payment is $\ell^* = (\rho+\lambda)\beta < (\lambda+\rho)\beta$, and the first arrival triggers monitoring forever ($T^* = \infty$). The corresponding principal's value is $C(\ell^*) = \beta + \lambda m/[r(r + \lambda)]$.

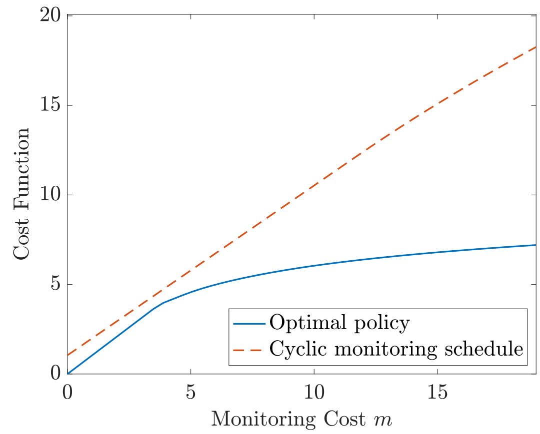
Figure 3.6 provides numerical comparisons between the optimal policy and the cyclic monitoring schedule proposed above. As we can see, the sub-optimality of the cyclic monitoring schedule is substantial when m is high and/or the discount rate of the agent is close to that of the principal. The intuition is that, comparing to the cyclic monitoring schedule, under which each arrival triggers a costly monitoring episode, the optimal policy leads to relatively infrequent monitoring in the two scenarios above. To be more specific, when m is high, the principal reduces monitoring frequency in the optimal policy to avoid high monitoring cost. When the discount rate of the agent is close to that of the principal, the principal sets a high threshold in the promised utility for payment (back-loading). As a result, in the optimal policy, it may take many arrivals before a monitoring episode is triggered.

3.7 Concluding Remarks and Further Discussion

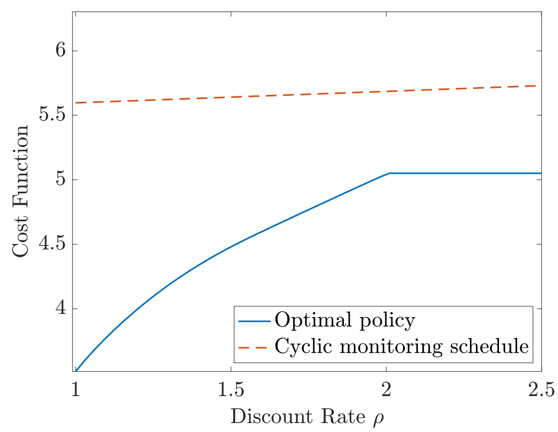
This paper studies the optimal monitoring and payment mechanism to induce an agent's effort in order to reduce the arrival rate of adverse events. Under condition (3.3), the contract design problem can be formulated as a continuous time optimal control model. The structures of its optimal solution depend on model parameters, in particular, the monitoring cost. The variations of the contract structures highlight the trade off between the monitoring cost and direct payment to the agent. The optimal contract structures are simple to describe, easy to compute and implement, and intuitive to explain. In particular, the key for computing the optimal contract only involves numerically solving a delay differential equation combined with a single dimensional search.

Our results easily extends to discrete time settings. The only issue worth mentioning is that towards the very end of a monitoring episode, it may be optimal to randomly stop monitoring either in the current period, or in the next period. This is because in discrete time settings, the switch between monitoring and non-monitoring may not be a single threshold any more. Instead, there could be an interval in which the optimal value function is linear. The corresponding control policy when the promised utility w falls in this interval is to randomize it between either end of the interval. The upper bound of the interval corresponds to stop monitoring, while the lower bound of the interval corresponds to continue monitoring for one more period, after which the promised utility would increase to the upper bound.

Recall that our paper has focused on the case in which the monitor cost is below $K\Delta\lambda - b$, such



(a)



(b)

Figure 3.6: Performance of the Cyclic Monitoring Schedule, with parameters $\lambda = 10$, $\beta = 1$, $r = 0.99$, and $\rho = 1.5$ in (a) and $m = 5$ in (b).

that the principal should always induce full effort from the agent. When condition (3.3) does not hold, we argue that the principal should never monitor but allow the agent to shirk instead. In fact, consider any contract with monitoring, and construct another contract during which the monitoring periods are replaced with periods that the agent shirks. (When the agent is shirking the principal does not pay the effort cost b.) Path-wise, the dynamics of the agent's promised utility remain the same, while the principal's value improves, because $K\Delta\lambda - b < m$. Therefore, in a problem where the principal allows the agent to shirk, the optimal shirking and payment schedule can be solved exactly the same as we have done in the paper, using a monitoring cost of $m = K\Delta\lambda - b$.

Although the shirking problem appears quite challenging for the Brownian motion [Zhu13] and the good arrival Poisson [ST17] settings, it is readily solved in the bad arrival Poisson setting as a special case of our paper.

Chapter 4

Proofs and Supplementary Materials

4.1 Supplementary Materials for Chapter 2

4.1.1 Existence of the Stationary Distribution

It is well understood that a Birth-Death process has stationary distribution if and only if

$$\sum_{x=1}^{\infty} \Pi_{k=1}^{x} \frac{\lambda(k-1)}{\mu(k)} \leq \infty,$$

where $\lambda(x)$ is the system birth rate and $\mu(x)$ is the system death rate when system state is x. For the Birth-Death process in this paper, we have

$$\sum_{x=1}^{\infty} \frac{\lambda_L^x}{\Pi_{k=1}^{x}(\lambda_S p_\epsilon(k,\rho)+\gamma k)} \leq \sum_{x=1}^{\infty} \frac{\lambda_L^x}{\Pi_{k=1}^{x}\gamma k} = \sum_{x=1}^{\infty} \left(\frac{\lambda_L^x}{\gamma}\right)^k \frac{1}{x!} = \exp\left(\frac{\lambda_L}{\gamma}\right) - 1 < \infty, \tag{4.1}$$

where the last equality follows the *p.m.f.* of a Poisson random variable with load λ_L/γ. Thus, the stationary distribution of this Birth-Death process always exists for positive and finite λ_L and γ.

4.1.2 Proofs

Proof of Lemma 2.1. We prove this result by showing that $g_1(x) := \epsilon\rho x - [1-(1-\epsilon\rho)^x]$ is non-negative for all $x \geq 1$.

By taking the first and second order derivatives of $g_1(x)$, we reach

$$\frac{d\,g_1(x)}{d\,x} = \epsilon\rho + (1-\epsilon\rho)^x \ln(1-\epsilon\rho), \text{ and } \frac{d^2\,g_1(x)}{d\,x^2} = (1-\epsilon\rho)^x(\ln(1-\epsilon\rho))^2.$$

Since $0 \leq \epsilon\rho \leq 1$, we have $\frac{d^2\,g_1(x)}{d\,x^2} \geq 0$ and therefore

$$\frac{d\,g_1(x)}{d\,x} \geq \left.\frac{d\,g_1(x)}{d\,x}\right|_{x=1} = \epsilon\rho + (1-\epsilon\rho)\ln(1-\epsilon\rho) =: g_2(\rho).$$

We can verify that $g_2(\cdot)$ is a decreasing function by checking

$$\frac{d\,g_2(\rho)}{d\,\rho} = -\epsilon \ln(1-\epsilon\rho) \geq 0,$$

since $0 \leq \epsilon\rho \leq 1$. Therefore, we have

$$\frac{d\,g_1(x)}{d\,x} \geq g_2(0) = 0.$$

As the result, we have $g_1(\cdot)$ is an increasing function. Since $g_1(1) = 0$, we reach the final result that $g_1(x) \geq 0$ for all $x \geq 1$. $\square$

We show a more general result in order to prove Proposition 2.1.

Lemma 4.1. *Consider two Birth-Death Processes $X_1 = \{X_1(t), t \geq 0\}$ and $X_2 = \{X_2(t), t \geq 0\}$ with the same arrival rate λ. Process X_1 and X_2 have departure rates $\mu_1(x)$ and $\mu_2(x)$ such that $\mu_1(x) \geq \mu_2(x)$ for all states $x \geq 1$. Denote $\bar{X}_1$ and $\bar{X}_2$ as the random variables that take stationary distributions (suppose they exist) of processes X_1 and X_2. We have*

$$\bar{X}_2 \succeq_1 \bar{X}_1. \tag{4.2}$$

Proof of Lemma 4.1. Denote the *P.M.F.* (C.M.F.) of $\bar{X}_1$ and $\bar{X}_2$ as $f_1(\cdot)$ and $f_2(\cdot)$ ($F_1(\cdot)$ and $F_2(\cdot)$), respectively. We show that $F_1(x) \geq F_2(x)$ for all $x \geq 0$.

First, note that for Birth-Death processes, one can verify that

$$\frac{f_1(x)}{f_2(x)} = \frac{f_1(0)}{f_2(0)} \Pi_{i=1}^{x} \frac{\mu_2(x)}{\mu_1(x)}. \tag{4.3}$$

Since we have $\mu_1(x) \geq \mu_2(x)$ for all states $x \geq 1$, there is $\Pi_{i=1}^{x} \dfrac{\mu_2(x)}{\mu_1(x)} \leq 1$, which implies $f_1(0) \geq f_2(0)$ since $f_1(\cdot)$ and $f_2(\cdot)$ are well-defined *p.m.f.*

Next, we prove the desired result by contradiction. Suppose there exists some $x \geq 1$ such that $F_1(x) < F_2(x)$ and let $k := \min\{x \,|\, F_1(x) < F_2(x)\}$. By the definition of k, we have $F_1(k) < F_2(k)$ and $F_1(k-1) \geq F_2(k-1)$. These two inequalities imply that $f_1(k) < f_2(k)$.

Note by (4.3), we have that

$$\frac{f_1(x+1)}{f_2(x+1)} = \frac{f_1(x)\mu_2(x+1)}{f_2(x)\mu_1(x+1)},$$

which implies that $f_1(x) < f_2(x)$ for all $x > k$ since $\frac{\mu_2(x+1)}{\mu_1(x+1)} \leq 1$ for all $x \geq 1$. Therefore, since both $f_1(\cdot)$ and $f_2(\cdot)$ are well-defined *p.m.f.*, we reach the following inequalities

$$\sum_{i=k+1}^{\infty} f_1(i) < \sum_{i=k+1}^{\infty} f_2(i), \text{ and } \sum_{i=0}^{k} f_1(i) < \sum_{i=0}^{k} f_2(i).$$

By adding up the two inequalities above, we reach contradiction as $1 < 1$. Therefore, we have $F_1(x) \geq F_2(x)$ for all $x \geq 0$. □

Proof of Proposition 2.1. The proof of the first statement follows the result of Lemma 4.1 directly. In addition, for Poisson random variables $Y_\epsilon(\rho)$ and $Y_\epsilon(0)$, first order stochastic dominance can be shown directly by comparing the expressions of their *C.M.F.*

In order to prove the second statement, we show that $h_\epsilon(x, \rho)$ is an increasing function of $x \geq 0$ if $0 \leq \rho \leq 1$ and $0 \leq \epsilon < 1$. Then the desired result follows by the property of first order stochastic dominance.

Fix $0 \leq \rho \leq 1$, $0 \leq \epsilon < 1$ and denote auxiliary function $g(x) := \frac{h_\epsilon(x,\rho)}{c}$. That is,

$$g(x) = \epsilon - \frac{1}{1+x} + \frac{[1 - \epsilon(1 + (1-\rho)x)]}{1+x}, \quad x \geq 0. \tag{4.4}$$

Furthermore, one can verify that

$$\begin{aligned} \frac{dg(x)}{dx} &= \frac{1}{(1+x)^2}\{1 - (1-\epsilon\rho)^{1+x} + (1+x)[1 - \epsilon(1 + (1-\rho)x)] \ln(1-\epsilon\rho)(1-\epsilon\rho)^x\} \\ &\geq \frac{1}{(1+x)^2}\{1 - [1 + (1+x)\ln(1-\epsilon\rho)](1-\epsilon\rho)^{1+x}\}, \end{aligned} \tag{4.5}$$

where the inequity follows $\epsilon < 1$ and $\rho \leq 1$. By taking the first order derivatives of (4.5) *w.r.t.* ρ, we have

$$-\epsilon(1-\epsilon\rho)^x \ln(1-\epsilon\rho) > 0,$$

as $\epsilon\rho < 1$. Therefore, we have

$$\frac{dg(x)}{dx} \geq \frac{1}{(1+x)^2}\{1 - [1 + (1+x)\ln(1-\epsilon\rho)](1-\epsilon\rho)^{1+x}\} > 0, \tag{4.6}$$

where we use $\rho = 0$ in the second inequality. Thus, we have shown that $g(x)$ is an increasing function, so is $h_\epsilon(x, \rho)$ *w.r.t.* x. $\square$

Proof of Proposition 2.2. First, recall that $Y_\epsilon(\rho)$ is a Poisson random variable with parameter $\frac{\lambda_L}{\gamma + \lambda_S \epsilon \rho}$. Therefore, as $\epsilon \to 0$, $Y_\epsilon(\rho)$ converges to $Y_\epsilon(0)$ in distribution by definition.

Next, note function $h_\epsilon(x, \cdot)$ is strictly increasing in x. Recall Proposition 2.1, we have

$$\mathbb{E}h_\epsilon(Y_\epsilon(\rho), \rho) \leq \mathbb{E}h_\epsilon(X_\epsilon(\rho), \rho) \leq \mathbb{E}h_\epsilon(Y_\epsilon(0), \rho),$$

which gives

$$|\mathbb{E}h_\epsilon(Y_\epsilon(\rho), \rho) - \mathbb{E}h_\epsilon(Y_\epsilon(\rho), \rho)| \quad \leq \quad |\mathbb{E}h_\epsilon(Y_\epsilon(\rho), \rho) - \mathbb{E}h_\epsilon(Y_\epsilon(0), \rho)|.$$

Consider another $M/M/\infty$ queue with service rate $\lambda_S \rho$ and denote the random variable takes its stationary distribution as $Z_\epsilon(\rho)$. By Lemma 4.1, we have that $Y_\epsilon(\rho) \succeq_1 Z_\epsilon(\rho)$ since $\epsilon < 1$ where $Z_\epsilon(\rho)$ is the Poisson random variable with load $\frac{\lambda_L}{\gamma + \lambda_S \rho}$.

Next, consider function $h_\epsilon(x, \cdot)$ in (2.11), which is an increasing function *w.r.t.* x according to the proof of Proposition 2.1. One can easily verify that $\lim_{x \to \infty} h_\epsilon(x, \cdot) = \epsilon$. Thus, we have function h_ϵ is upper bounded by $\epsilon < \infty$.

Thus we have

$$\begin{aligned}
\lim_{\epsilon \to 0} |\mathbb{E}h_\epsilon(Y_\epsilon(\rho), \rho) - \mathbb{E}h_\epsilon(Y_\epsilon(0), \rho)| \quad &\leq \quad \lim_{\epsilon \to 0} |\mathbb{E}h_\epsilon(Z_\epsilon(\rho), \rho) - \mathbb{E}h_\epsilon(Y_\epsilon(0), \rho)| \\
&= \quad |\mathbb{E} \lim_{\epsilon \to 0} h_\epsilon(Z_\epsilon(\rho), \rho) - \mathbb{E} \lim_{\epsilon \to 0} h_\epsilon(Y_\epsilon(0), \rho)| = 0,
\end{aligned}$$

where the first equality follows the fact $\lim_{x \to \infty} h_\epsilon(x, \cdot) = \epsilon < \infty$ together with Dominated Convergence Theorem, and the last equality follows simple algebra.

$\square$

Proof of Proposition 2.3. Note $Y_\epsilon(\rho)$ is a Poisson random variable with load factor $\frac{\lambda_L}{\gamma + \lambda_S \epsilon \rho}$, one can verify that

$$\mathbb{E}h_\epsilon(Y_\epsilon(\rho), \rho) \quad = \quad \frac{1}{\lambda_L} \left[\epsilon(\lambda_L - \lambda_S \rho) - \gamma + [\gamma + \epsilon(\rho(\lambda_L + \lambda_S) - \lambda_L)] \exp\left(-\frac{\epsilon \lambda_L \rho}{\epsilon \lambda_S \rho + \gamma} \right) \right]. \tag{4.7}$$

Next, we perform third order Taylor expansions over (4.7) and denote

$$J(\rho,\epsilon) \quad := \quad \lambda_L \left[\frac{\rho(2-\rho)}{2\gamma}\epsilon^2 - \frac{\rho^2[3\lambda_S(2-\rho)+\lambda_L(3-2\rho)]}{6\gamma^2}\epsilon^3 \right]. \tag{4.8}$$

One can verify that

$$\lambda_S \mathbb{E} h_\epsilon(Y_\epsilon(\rho),\rho) \quad = \quad J(\rho,\epsilon) + o(\epsilon^3),$$

which implies the result according to the definition of "*little-o*" notations.

Note that one can also perform Taylor expansion over function h_ϵ directly before taking the expectation *w.r.t.* $Y_\epsilon(\rho)$. However, the "higher-order" terms ($o(\epsilon^3)$ and higher) contain the random variable $Y_\epsilon(\rho)$, which can be infinity. Then one need to verify that these "higher-order" terms are indeed going to zero as $\epsilon \to 0$, which can be shown using the "light-tail" property of Poisson random variables. We omit details for this procedure.

□

Proof of Theorem 2.1. First, we provide a sufficient condition for function $J(\rho,\epsilon)$ to be concave in ρ:

$$\epsilon < \bar{\epsilon} := \min\{\frac{\gamma}{\lambda_s+\lambda_L}, \frac{3\gamma}{9\lambda_S+\lambda_L}\}. \tag{4.9}$$

By taking the second order derivatives of function $\hat{J}$ *w.r.t.* ρ, we have

$$\frac{d^2 J(\rho,\epsilon)}{d\rho^2} = \lambda_L \epsilon^2 \left[-\frac{1}{\gamma} + \frac{\lambda_L(2p-1)+\lambda_S(3p-2)}{\gamma^2} \right]. \tag{4.10}$$

One can verify that as long as

$$\epsilon < \bar{\epsilon}, \tag{4.11}$$

the expression in (4.10) is always negative. Thus, if $\epsilon < \bar{\epsilon}$, function $J(\rho,\epsilon)$ is strictly concave in ρ so it has a unique maximizer $0 \le \rho^*_\epsilon \le 1$, which solves $\dfrac{dJ(\rho,\epsilon)}{d\rho} = 0$. Thus, we have

$$\rho^*_\epsilon = \frac{\epsilon(\lambda_L+2\lambda_S)+\gamma-\sqrt{\epsilon^2(\lambda_L+2\lambda_S)^2-2\epsilon(\lambda_L+\lambda_S)\gamma+\gamma^2}}{\epsilon(2\lambda_L+3\lambda_S)}. \tag{4.12}$$

By performing a first order Taylor expansion over ρ_ϵ^* around $\epsilon = 0$ and denoting

$$\hat{\rho}_\epsilon = 1 - \frac{\lambda_S}{2\gamma},$$

we have

$$\rho_\epsilon^* = \hat{\rho}_\epsilon + o(\epsilon).$$

In order to prove the second statement in Theorem 2.1, we take the difference between using $\rho = 1$ and $\rho = \hat{\rho}_\epsilon$ in the objective function:

$$\begin{aligned}
\mathbb{E}h_\epsilon(Y_\epsilon(\hat{\rho}_\epsilon), \hat{\rho}_\epsilon) - \mathbb{E}h_\epsilon(Y_\epsilon(1), 1) &= \frac{1}{\lambda_L^2}\Bigg[\epsilon\lambda_S + \frac{\epsilon\lambda_S(\epsilon\lambda_S + 2\gamma)}{2\gamma} - (\epsilon\lambda_S + \gamma)\exp\left(-\frac{\epsilon\lambda_L}{\epsilon\lambda_S + \gamma}\right) \\
&\quad + \left(\epsilon\lambda_S + \gamma - \frac{\epsilon^2\lambda_S(\lambda_L + \lambda_S)}{2\gamma}\right)\exp\left(-\frac{\epsilon\lambda_L(\epsilon\lambda_S + 2\gamma)}{\epsilon^2\lambda_S^2 - 2\epsilon\lambda_S\gamma - 2\gamma^2}\right)\Bigg] \\
&= \frac{\lambda_S^2}{8\gamma^2}\epsilon^4 - \frac{\lambda_S^2(\lambda_L + 9\lambda_S)}{24\gamma^4}\epsilon^5 + o(\epsilon^5). \qquad (4.13)
\end{aligned}$$

Therefore, we conclude that

$$\lim_{\epsilon \to 0} \frac{1}{\epsilon^4}[\mathbb{E}h_\epsilon(Y_\epsilon(\hat{\rho}_\epsilon), \hat{\rho}_\epsilon) - \mathbb{E}h_\epsilon(Y_\epsilon(1), 1)] = \frac{\lambda_S^2}{8\gamma^2}\epsilon^4 \geq 0. \qquad (4.14)$$

Note that as long as $\epsilon < \bar{\epsilon}$, the sum of the forth and fifth order term in (4.13) is non-negative, which completes this proof. □

Proof of Proposition 2.4. Before going into the actual proof, we define the following notations. We use $X_{\hat{\theta},\theta} = \{X_{\hat{\theta},\theta}(t), t \geq 0\}$ to denote the process describing the total number of *other* long-lived players on the platform from a focal player's perspective. Moreover, denote $N = \{N^L, N^S, N^\gamma\}$ as a three-dimensional counting process describing the total number of arrival of long/short-lived players and departure of long-lived players, respectively. Furthermore, we split $N^S(t) = N^{Sm}(t) + N^{Sn}(t)$ representing the total numbers of short-lived arrivals that are matched and not matched by time t, respectively. Thus, we have

$$X_{\hat{\theta},\theta}(t) = X_{\hat{\theta},\theta}(0) + N^L(t) - N^\gamma(t) - N^{Sm}(t), \quad \forall\, t \geq 0. \qquad (4.15)$$

Therefore, we can write the utility function as an integral over time as

$$V(x,\hat{\theta},\theta,\rho) \;=\; \begin{cases} \mathbb{E}^N\left[\displaystyle\int_0^\infty e^{-\gamma t}\mathcal{A}(X_{\hat{\theta},\theta}(t),\hat{\theta},\hat{\theta},\rho)dN^S(t)\,\Big|\, X_{\hat{\theta},\theta}(0)=x\right], & \text{if } \hat{\theta}\le\theta, \\ \mathbb{E}^N\left[\displaystyle\int_0^\infty e^{-\gamma t}\mathcal{A}(X_{\hat{\theta},\theta}(t),\hat{\theta},\theta,\rho)dN^S(t)\,\Big|\, X_{\hat{\theta},\theta}(0)=x\right], & \text{if } \hat{\theta}>\theta, \end{cases} \tag{4.16}$$

where function $\mathcal{A}$ defined in (2.20) represents the expected utility of the focal player upon arrival of a short-lived player. Intuitively, $e^{-\gamma t}$ comes from the P.D.F. of the exponential distribution for reneging. It is equivalent to a discount factor.

(i) We show the result for $\hat{\theta}\le\theta$. First we show that $\mathcal{A}(x,\hat{\theta},\hat{\theta},\rho)$ is decreasing *w.r.t.* x. Note that ee have

$$\frac{\partial^2\,\mathcal{A}(x,\hat{\theta},\hat{\theta},\rho)}{\partial\, x\,\partial\,\hat{\theta}} = -\epsilon^2(\hat{\theta}-\rho)(1-\epsilon\hat{\theta})^x\ln(1-\epsilon\hat{\theta})\le 0,$$

since $\hat{\theta}\le\rho\le 1$. Thus, we have

$$\frac{\partial\,\mathcal{A}(x,\hat{\theta},\hat{\theta},\rho)}{\partial\, x}\le\frac{\partial\,\mathcal{A}(x,0,0,\rho)}{\partial\, x}=0.$$

Therefore, we reach the result that for any $x\ge 1$,

$$\mathcal{A}(x,\hat{\theta},\hat{\theta},\rho)\le\mathcal{A}(0,\hat{\theta},\hat{\theta},\rho). \tag{4.17}$$

Recall the expression of the value function as an integral in (4.16). We have

$$\begin{aligned} V(x,\epsilon\hat{\theta},\epsilon\theta,\rho) &= \mathbb{E}^N\left[\int_0^\infty e^{-\gamma t}\mathcal{A}(x,\hat{\theta},\hat{\theta},\rho)dN^S(t)\right] \\ &\le \mathbb{E}^N\left[\int_0^\infty e^{-\gamma t}\mathcal{A}(0,\hat{\theta},\hat{\theta},\rho)dN^S(t)\right] \\ &\le \mathcal{A}(0,\hat{\theta},\hat{\theta},\rho)\int_0^\infty e^{-\gamma t}\,dt=\frac{\mathcal{A}(0,\hat{\theta},\hat{\theta},\rho)}{\gamma}=:B,\quad \forall\, x\ge 0, \end{aligned} \tag{4.18}$$

where the first inequality follows (4.17) and the second inequality follows the definition of N_{λ_S}.

The proof for the case where $\hat{\theta}>\theta$ follows the same logic so it is omitted.

(ii) Again, we only show the result for $\hat{\theta}\le\theta$ as the counterpart follows the same steps.

Consider a pure birth process with $\tilde{X}=\{\tilde{X}(t),t\ge 0\}$ with arrival rate λ_L. Define $\tau=\min\{t\ge 0\,|\,X_{\hat{\theta},\theta}(t)=\bar{x}\}$ and $\tilde{\tau}=\min\{t\ge 0\,|\,\tilde{X}(t)=\bar{x}\}$. Note that $\tilde{\tau}$ follows Erlang distribution with parameter λ_L and $\bar{x}-\tilde{X}(0)$ since $\tilde{X}$ is a pure Birth (Poisson) process.

We show that if two processes have $X_{\hat{\theta},\theta}(0) = \tilde{X}(0) \leq \bar{x}$, then there is $\mathbb{P}(\tilde{\tau} < t) \geq \mathbb{P}(\tau < t)$ for all $t \geq 0$ through coupling. As $\tilde{X}$ is a counting process, define left-continuous jump process Y such that

$$Y(t) = \tilde{X}(t) - Z(Y(t_-)), \quad t \geq 0, \tag{4.19}$$

where $Z(Y) = \{Z(Y(t_-)) \,|\, t \geq 0\}$ is also a counting process with arrival rate $\gamma y + \lambda_S \mathcal{B}(\hat{\theta}, \theta, y)$ if $Y(t_-) = y$. Denote $\hat{\tau} = \min\{t \geq 0 \,|\, Y(\hat{t}) = \bar{x}\}$. Thus, by construction, $Y(t) \stackrel{D}{=} X(t)$ (equal in distribution), which implies that

$$\hat{\tau} \stackrel{D}{=} \tau. \tag{4.20}$$

Since $\gamma > 0$, we have $\tilde{X}(t) \geq Y(t)$ almost surely for all $t \geq 0$, which implies that

$$\tilde{\tau} \leq \hat{\tau}, \quad a.s., \tag{4.21}$$

which gives

$$\mathbb{P}(\tilde{\tau} < t) \geq \mathbb{P}(\hat{\tau} < t) = \mathbb{P}(\tau < t), \quad \forall\, t \geq 0, \tag{4.22}$$

where the inequality follows (4.21) and the equality follows (4.20).

By the definition of first order stochastic dominance and the fact that $e^{-\gamma t}$ is strictly decreasing *w.r.t.* t, we reach

$$\mathbb{E}_{\tau}\left[e^{-\gamma\tau}\right] \leq \mathbb{E}_{\tilde{\tau}}\left[e^{-\gamma\tilde{\tau}}\right]. \tag{4.23}$$

Now, we can write the utility functions $V(x)$, $V_{\bar{x}}(x)$ as an integrals over time similar to (4.16). For $X_{\hat{\theta},\theta}(0) = x$, there are

$$V(x, \hat{\theta}, \theta, \rho) = \mathbb{E}^N\left[\int_0^{\tau} e^{-\gamma t}\mathcal{A}(X_{\hat{\theta},\theta}(t), \hat{\theta}, \hat{\theta}, \rho)dN^S(t) + e^{-\gamma\tau}V(\bar{x}, \hat{\theta}, \theta, \rho)\right], \tag{4.24}$$

$$V_{\bar{x}}(x, \hat{\theta}, \theta, \rho) = \mathbb{E}^N\left[\int_0^{\tau} e^{-\gamma t}\mathcal{A}(X_{\hat{\theta},\theta}(t), \hat{\theta}, \hat{\theta}, \rho)dN^S(t) + 0\right]. \tag{4.25}$$

Therefore,

$$V(x, \hat{\theta}, \theta, \rho) - V_{\bar{x}}(x, \hat{\theta}, \theta, \rho) = \mathbb{E}_{\tau}\left[e^{-\gamma\tau}\right] V(\bar{x}, \hat{\theta}, \theta, \rho) \leq \mathbb{E}_{\tilde{\tau}}\left[e^{-\gamma\tilde{\tau}}\right] V(\bar{x}, \hat{\theta}, \theta, \rho) \leq \mathbb{E}_{\tilde{\tau}}\left[e^{-\gamma\tilde{\tau}}\right] B = B\left(1 + \frac{\gamma}{\lambda_L}\right)^{-(\bar{x}-x)}$$

where the first inequality follows (4.23), second inequality follows part(i) and last equality follows the moment generating function of Erlang random variables. □

4.1.3 Long-lived Players' Utility Function

Then we can write out value functions recursively in a heuristic manner. Fix $\hat{\theta} \leq \theta$ and consider an infinitesimal time period $[t, t+\delta)$,

$$\begin{aligned} V(x) &= (1-\gamma\delta)\Big\{\lambda_L\delta V(x+1) + x\gamma\delta V(x-1) + (1-\lambda_L\delta - \lambda_S\delta - x\gamma\delta)V(x) \\ &\qquad + \lambda_S\delta\left[\mathcal{A}(x,\hat{\theta},\hat{\theta},\rho) + \mathcal{B}(x,\hat{\theta},\theta)V(x-1) + \mathcal{C}(x,\hat{\theta},\theta)V(x)\right]\Big\} \\ &= (1-\gamma\delta)\Big\{\lambda_S\delta\mathcal{A}(x,\hat{\theta},\hat{\theta},\rho) + \lambda_L\delta V(x+1) + \left[1-\lambda_L\delta - \lambda_S\delta[1-\mathcal{C}(x,\hat{\theta},\theta)] - x\gamma\delta\right]V(x) \\ &\qquad + \left[\lambda_S\mathcal{B}(x,\hat{\theta},\theta) + x\gamma\right]\delta V(x-1)\Big\}. \end{aligned}$$

By dividing both sides with δ and then take $\delta \to 0$, we reach,

$$V(x) = \frac{\lambda_S\mathcal{A}(x,\hat{\theta},\hat{\theta},\rho) + \left[\lambda_S\mathcal{B}(x,\hat{\theta},\theta) + x\gamma\right]V(x-1) + \lambda_L V(x+1)}{\lambda_L + \lambda_S(1-\mathcal{C}(x,\hat{\theta},\theta)) + (x+1)\gamma}.$$

The derivation for $\hat{\theta} > \theta$ follows the same steps and thus it is omitted.

4.2 Supplementary Materials for Chapter 3

4.2.1 Proofs in Section 3.2

Proof of Lemma 3.1. For a generic contract Γ and effort process Λ, following Equations (3.1) and (3.2), we define the agent's total expected utility conditioned on the information available at time t as

$$\begin{aligned} u_t(\Gamma,\Lambda) &:= \mathbb{E}^{\Gamma,\Lambda}\Big[\int_0^\infty e^{-\rho\tau}(dL_\tau + b\,\mathbb{I}_{\lambda_\tau=\bar{\lambda}}\,d\tau)\,\Big|\,\mathcal{F}_t\Big] \\ &= \int_0^t e^{-\rho\tau}(dL_\tau + b\,\mathbb{I}_{\lambda_\tau=\bar{\lambda}}\,d\tau) + e^{-\rho t}W_t(\Gamma,\Lambda). \end{aligned} \tag{4.26}$$

Therefore, $u_0(\Gamma,\Lambda) = u(\Gamma,\Lambda)$. Moreover, it is easy to verify that process $\{u_t\}_{t\geq 0}$ is an $\mathcal{F}_t$-martingale by conditional expectation's tower property. Define processes

$$\begin{aligned} M_t^{s,\Lambda} &:= \int_0^t y_\tau\lambda_\tau\,d\tau - N_t^s \\ \text{and } M_t^{n,\Lambda} &:= \int_0^t (1-y_\tau)\lambda_\tau\,d\tau - N_t^n. \end{aligned} \tag{4.27}$$

Following the Martingale Representation Theorem [Bré81], there exist $\mathcal{F}_t$-predictable processes $\{H_t^s(\Gamma,\Lambda)\}_{t\geq 0}$ and $\{H_t^n(\Gamma,\Lambda)\}_{t\geq 0}$ such that

$$u_t(\Gamma,\Lambda) = u_0(\Gamma,\Lambda) + \int_0^t e^{-\rho\tau}\Big[H_\tau^s(\Gamma,\Lambda)\,dM_\tau^{s,\Lambda} + H_\tau^n(\Gamma,\Lambda)\,dM_\tau^{n,\Lambda}\Big], \quad \forall t \geq 0. \tag{4.28}$$

On the one hand, (4.26) implies

$$du_t = e^{-\rho t}\Big[dL_t + b\,\mathbb{I}_{\lambda_t=\bar{\lambda}}\,dt - \rho W_t(\Gamma,\Lambda)\,dt + dW_t(\Gamma,\Lambda)\Big]. \tag{4.29}$$

On the other hand, (4.28) implies

$$\begin{aligned} du_t &= e^{-\rho t}\Big[H_t^s(\Gamma,\Lambda)\,dM_t^{s,\Lambda} + H_t^n(\Gamma,\Lambda)\,dM_t^{n,\Lambda}\Big] \\ &= e^{-\rho t}\Big[H_t^s(\Gamma,\Lambda)\Big(y_t\lambda_t\,dt - dN_t^s\Big) + H_t^n(\Gamma,\Lambda)\Big((1-y_t)\lambda_t\,dt - dN_t^n\Big)\Big], \end{aligned} \tag{4.30}$$

where the second equality follows from the definitions in (4.27). Combining (4.29) and (4.30) yields (PK). □

Proof of Lemma 3.2. This result corresponds to Proposition 1 in [BMRV10]. Denote $\mathcal{F}_t$-measurable random variable $\tilde{u}_t(\Gamma,\Lambda,\underline{\Lambda})$ to represent the agent's utility under effort process Λ before time t and effort process $\underline{\Lambda}$ afterwards. We have

$$\tilde{u}_t(\Gamma,\Lambda,\underline{\Lambda}) = \int_0^t e^{-\rho\tau}(dL_\tau + b\,\mathbb{I}_{\lambda_\tau=\bar{\lambda}}\,d\tau) + e^{-\rho t}W_t(\Gamma,\underline{\Lambda}). \tag{4.31}$$

Consider any sample trajectory of $\{N_t^s, N_t^n\}_{t\geq 0}$ and effort process Λ and $\underline{\Lambda}$,

$$\begin{aligned} \tilde{u}_t(\Gamma,\Lambda,\underline{\Lambda}) &= u_t(\Gamma,\underline{\Lambda}) + \int_0^t e^{-\rho\tau} b\,\mathbb{I}_{\lambda_\tau=\bar{\lambda}}\,d\tau \\ &= u_0(\Gamma,\underline{\Lambda}) + \int_0^t e^{-\rho\tau}\Big[H_\tau^s(\Gamma,\underline{\Lambda})\,dM_\tau^{s,\underline{\Lambda}} + H_\tau^n(\Gamma,\underline{\Lambda})\,dM_\tau^{n,\underline{\Lambda}} + b\,\mathbb{I}_{\lambda_\tau=\bar{\lambda}}\,d\tau\Big] \\ &= u_0(\Gamma,\underline{\Lambda}) + \int_0^t e^{-\rho\tau}\Big[H_\tau^s(\Gamma,\underline{\Lambda})\,dM_\tau^{s,\Lambda} + H_\tau^n(\Gamma,\underline{\Lambda})\,dM_\tau^{n,\Lambda}\Big] \\ &\quad - \int_0^t e^{-\rho\tau}\Big[y_\tau H_\tau^s(\Gamma,\underline{\Lambda}) + (1-y_\tau)H_\tau^n(\Gamma,\underline{\Lambda}) - \beta\Big]\Delta\lambda\mathbb{I}_{\lambda_\tau=\bar{\lambda}}\,d\tau, \end{aligned}$$

where the first equality follows from (4.26) and (4.31), the second equality from (4.28), the third equality from (4.27) and the definition of β in (3.5).

Consider any two times $t' < t$,

$$\begin{aligned}\mathbb{E}[\tilde{u}_t(\Gamma,\Lambda,\underline{\Lambda})\,|\,\mathcal{F}_{t'}] &= u_0(\Gamma,\underline{\Lambda}) + \int_0^{t'} e^{-\rho\tau}\Big[H^s_\tau(\Gamma,\underline{\Lambda})\,dM^{s,\Lambda}_\tau + H^n_\tau(\Gamma,\underline{\Lambda})\,dM^{n,\Lambda}_\tau\Big] \\ &\quad - \int_0^{t'} e^{-\rho\tau}\Big[y_\tau H^s_\tau(\Gamma,\underline{\Lambda}) + (1-y_\tau)H^n_\tau(\Gamma,\underline{\Lambda}) - \beta\Big]\Delta\lambda\,\mathbb{I}_{\lambda_\tau=\bar{\lambda}}\,d\tau \\ &\quad -\mathbb{E}\Big[\int_{t'}^{t} e^{-\rho\tau}[y_\tau H^s_\tau(\Gamma,\underline{\Lambda}) + (1-y_\tau)H^n_\tau(\Gamma,\underline{\Lambda}) - \beta]\Delta\lambda\,\mathbb{I}_{\lambda_\tau=\bar{\lambda}}\,d\tau\,\Big|\,\mathcal{F}_{t'}\Big] \\ &= \tilde{u}_{t'}(\Gamma,\Lambda,\underline{\Lambda}) - \Delta\lambda\mathbb{E}\Big[\int_{t'}^{t} e^{-\rho\tau}[y_\tau H^s_\tau(\Gamma,\underline{\Lambda}) + (1-y_\tau)H^n_\tau(\Gamma,\underline{\Lambda}) - \beta]\,\mathbb{I}_{\lambda_\tau=\bar{\lambda}}\,d\tau\,\Big|\,\mathcal{F}_{t'}\Big].\end{aligned} \tag{4.32}$$

(i) On the one hand, if (IC) holds under contract Γ, (4.32) suggests that

$$\mathbb{E}[\tilde{u}_t(\Gamma,\Lambda,\underline{\Lambda})\,|\,\mathcal{F}_{t'}] \le \tilde{u}_{t'}(\Gamma,\Lambda,\underline{\Lambda}),$$

which implies that the process $\{\tilde{u}_t\}_{t\ge 0}$ is a super-martingale. Taking $t'=0$ and letting $t\to\infty$, we have

$$u(\Gamma,\Lambda) = \lim_{t\to\infty}\Big\{\mathbb{E}[\tilde{u}_t(\Gamma,\Lambda,\underline{\Lambda})\,|\,\mathcal{F}_0]\Big\} \le \tilde{u}_0(\Gamma,\Lambda,\underline{\Lambda}) = u(\Gamma,\underline{\Lambda}). \tag{4.33}$$

That is, effort process $\underline{\Lambda}$ dominates any other process Λ under contract Γ, or, Γ is incentive compatible if (IC) holds.

(ii) On the other hand, suppose (IC) does not hold, or, $y_\tau H^s_\tau(\Gamma,\underline{\Lambda}) + (1-y_\tau)H^n_\tau(\Gamma,\underline{\Lambda}) < \beta$ for $m_\tau = 0$ over a subset of $[0,t]$ with positive measure. Define an effort process $\Lambda = \{\lambda_\tau\}_{\tau\ge 0}$, such that $\lambda_\tau = \lambda$ for $\forall\tau\in(t,\infty)$, and for $\forall\tau\in[0,t]$:

$$\lambda_\tau = \begin{cases}\bar{\lambda} & \text{if } y_\tau H^s_\tau(\Gamma,\underline{\Lambda}) + (1-y_\tau)H^n_\tau(\Gamma,\underline{\Lambda}) < \beta \text{ and } m_\tau = 0,\\ \lambda & \text{otherwise.}\end{cases}$$

Clearly, we must have

$$-\mathbb{E}\Big[\int_0^{t} e^{-\rho\tau}[y_\tau H^s_\tau(\Gamma,\underline{\Lambda}) + (1-y_\tau)H^n_\tau(\Gamma,\underline{\Lambda}) - \beta]\Delta\lambda\mathbb{I}_{\Lambda_\tau=\bar{\lambda}}\,d\tau\Big|\mathcal{F}_{t'}\Big] > 0.$$

As a result, taking $t'=0$ and letting $t\to\infty$ in (4.32), we obtain

$$u(\Gamma,\Lambda) = \lim_{t\to\infty}\Big\{\mathbb{E}[\tilde{u}_t(\Gamma,\Lambda,\underline{\Lambda})\,|\,\mathcal{F}_0]\Big\} > \tilde{u}_0(\Gamma,\Lambda,\underline{\Lambda}) = u(\Gamma,\underline{\Lambda}).$$

This implies that effort process Λ dominates $\underline{\Lambda}$, or, contract Γ is *not* incentive compatible when (IC) does not hold. □

Proof of Lemma 3.3. Following Itô's change of variable formula with function F [CE15], for any $\tau \geq 0$, we have:

$$\begin{aligned} e^{-r\tau}F(W_\tau) &= F(w) + \int_0^\tau \Big[e^{-rt}dF(W_t) - re^{-rt}F(W_t)dt\Big] \\ &= F(w) + \int_0^\tau e^{-rt}(m_t dt + dL_t) + \int_0^\tau e^{-rt}d\mathcal{A}_t, \end{aligned} \tag{4.34}$$

where

$$\begin{aligned} d\mathcal{A}_t &:= dF(W_t) - rF(W_t)dt - m_t dt - dL_t \\ &= F'(W_t)\Big[\Big(\rho W_t + \lambda(y_t H_t^s + (1-y_t)H_t^n)\Big)dt - \ell_t dt\Big] - rF(W_t)dt \\ &\quad + F(W_t - H_t^s dN_t^s - H_t^n dN_t^n - dI_t) - F(W_t) - m_t dt - dL_t. \end{aligned}$$

Further define

$$\begin{aligned} d\mathcal{B}_t &:= [F(W_t - H_t^s) - F(W_t)](dN_t^s - \lambda y_t dt) \\ &\quad + [F(W_t - H_t^n) - F(W_t)](dN_t^n - \lambda(1-y_t)dt). \end{aligned} \tag{4.35}$$

Because function $F(w)$ is concave and $F'(w) \geq -1$, we have

$$\begin{aligned} d\mathcal{A}_t &\leq F'(W_t)\Big(\rho W_t + \lambda(y_t H_t^s + (1-y_t)H_t^n)\Big)dt + F(W_t - H_t^s dN_t^s - H_t^n dN_t^n) \\ &\quad - F'(W_t)\ell_t dt - F'(W_t - H_t^s dN_t^s - H_t^n dN_t^n)dI_t - F(W_t) - rF(W_t)dt - m_t dt - dL_t \\ &\leq F'(W_t)\Big(\rho W_t + \lambda(y_t H_t^s + (1-y_t)H_t^n)\Big)dt - rF(W_t)dt - m_t dt + F(W_t - H_t^s dN_t^s - H_t^n dN_t^n) - F(W_t) \\ &= F'(W_t)\Big(\rho W_t + \lambda(y_t H_t^s + (1-y_t)H_t^n)\Big)dt - rF(W_t)dt - m_t dt + [F(W_t - H_t^s) - F(W_t)]dN_t^s \\ &\quad + [F(W_t - H_t^n) - F(W_t)]dN_t^n \\ &= d\mathcal{B}_t + \Psi_t dt. \end{aligned}$$

Therefore, if $\Psi_t \leq 0$, we must have $d\mathcal{A}_t \leq d\mathcal{B}_t$ almost surely. Taking the expectation on both sides of (4.34), we immediately have

$$F(w) \geq \mathbb{E}^{\Gamma,\underline{\Lambda}}\Big[e^{-r\tau}F(W_\tau) - \int_0^\tau e^{-rt}\Big(m_t dt + dL_t\Big)\Big],$$

where we use the fact that $\int_0^\tau e^{-rt} d\mathcal{B}_t$ is a martingale.

Taking $\tau \to \infty$, the above inequality reduces to

$$F(w) \geq -\mathbb{E}^{\Gamma,\Delta}\Big[\int_0^\infty e^{-rt}\Big(m_t dt + dL_t\Big)\Big] = U(\Gamma).$$

This completes the proof. □

4.2.2 Proofs in Section 3.3

We first present the following technical lemma.

Lemma 4.2. *For any $\alpha \geq \beta$, starting with any boundary condition $F(w)$ that is continuous for $w \in [0, \alpha]$, DDE* (H) *uniquely determines a continuous function $F(w)$ on $w \in [0, \infty)$. Furthermore, function $F(w)$ is increasing on $[\alpha, \infty)$ if either of the following two conditions holds.*

(i) Function $F(w)$ is positive and non-decreasing on $[0, \alpha]$;

(ii) Function $F(w)$ is increasing within $[0, \alpha]$ and $F(\alpha) \geq 0$.

Proof. Solving DDE (H) with well-defined boundary conditions over $w \in [0, \alpha]$ is equivalent to solving a sequence of initial value problems over interval $[\alpha + k\beta, \alpha + (k+1)\beta]$ where $k = 0, 1, \ldots$. This sequence of initial value problems satisfy the Cauchy-Lipschitz Theorem [Har82]; therefore, a unique and differentiable solution is guaranteed over $w \in (\alpha, \infty)$.

(i) If $F(w)$ is positive and non-decreasing for $w \in [0, \alpha]$, suppose it is non-increasing for some $w \geq \alpha$. Let

$$\hat{w} := \min\{w | F'(w_+) \leq 0 \text{ and } w \geq \alpha\}.$$

DDE (H) implies

$$rF(\hat{w}) + \lambda[F(\hat{w}) - F(\hat{w} - \beta)] \leq 0,$$

which is impossible, because $F(\hat{w}) \geq F(\alpha) > 0$ as assumed, and $F(\hat{w}) \geq F(\hat{w} - \beta)$ for $w \in [\alpha, \hat{w}]$ from the definition of $\hat{w}$. Therefore, we must have $F'(w) > 0$ for $\forall w \in [\alpha, \infty)$. Part (ii) can be proven similarly. □

Lemma 4.3. *Consider the case $r = \rho$. Function $J(w)$, which is the solution of DDE* (H) *with boundary condition $J(w) = 1$ on $w \in [0, \beta)$, is increasing and strictly convex on $w \in [\beta, \infty)$.*

Proof. Note that following DDE (H), function $J(w)$ is differentiable for $w > \beta$, and is twice-differentiable except at $w = 2\beta$. Taking derivatives on both sides of DDE (H) yields

$$J''(w) = \frac{(\lambda + r - \rho)J'(w) - \lambda J'(w-\beta)}{\rho w + \beta\lambda}, \quad \text{for } w \in (\beta, 2\beta) \cup (2\beta, \infty). \tag{4.36}$$

In particular, there is

$$J(w) = \frac{r}{\lambda + r}\Big(\frac{\rho w + \lambda\beta}{\rho\beta + \lambda\beta}\Big)^{\frac{\lambda+r}{\rho}} + \frac{\lambda}{\lambda + r}, \text{ for } w \in (\beta, 2\beta), \tag{4.37}$$

whose first and second derivatives are

$$J'(w) = \frac{r(\rho w + \lambda\beta)^{\frac{\lambda+r-\rho}{\rho}}}{(\rho\beta + \lambda\beta)^{\frac{\lambda+r}{\rho}}}$$

$$J''(w) = \frac{r(\rho w + \lambda\beta)^{\frac{\lambda+r-2\rho}{\rho}}}{(\rho\beta + \lambda\beta)^{\frac{\lambda+r}{\rho}}}(\lambda + r - \rho). \tag{4.38}$$

When $r = \rho$, the closed-form expression in (4.37) is clearly convex, i.e., $J(w)$ is convex for $w \in [\beta, 2\beta)$. Consider the point $w = 2\beta$, (4.36) and (4.38) yield

$$J''(2\beta_+) = \frac{\lambda r[(2\beta\rho + \beta\lambda)^{\frac{\lambda}{\rho}} - (\beta\rho + \beta\lambda)^{\frac{\lambda}{\rho}}]}{(2\beta\rho + \beta\lambda)(\beta\rho + \beta\lambda)^{\frac{\lambda+r}{\rho}}} > 0.$$

Therefore, we only need to show $J''(w) > 0$ for all $w > 2\beta$. We prove by contradiction. Suppose, on the contrary, there exists some $w > 2\beta$, such that $J''(w) \leq 0$. Define

$$\hat{w} := \min\{w | J'(w) \leq J'(w-\beta) \text{ and } w > 2\beta\}.$$

By construction, we have $J''(w) > 0$ for all $w \in (\beta, \hat{w})$. First, $\hat{w}$ cannot be in $(2\beta, 3\beta]$, because otherwise we would have $\hat{w} - \beta \leq 2\beta$, and,

$$J'(\hat{w}) > J'(2\beta) \geq J'(\hat{w} - \beta),$$

which contradicts the definition of $\hat{w}$. Second, $\hat{w}$ cannot be greater than 3β either, because otherwise we have

$$J'(\hat{w}) = J'(\hat{w} - \beta) + \int_0^\beta J''(\hat{w} - \beta + x)dx > J'(\hat{w} - \beta),$$

which, again, contradicts the definition of $\hat{w}$. Therefore, we must have $J''(w) > 0$ for all $w > 2\beta$. Last but not the least, monotonicity of $J(w)$ follows directly from Lemma 4.2, which completes the proof. □

Proof of Proposition 3.1. (i) Given that $\theta(\bar{w}) = \frac{m}{r}J'(\bar{w}) - 1$ and $J'(\bar{w}) \geq 0$ (from Lemma 4.3), we have $\theta(\bar{w}) \geq -1$. We prove $\theta(\bar{w}) < \frac{m}{\beta r}$ by contradiction. Suppose, on the contrary, we have $\theta(\bar{w}) \geq \frac{m}{\beta r}$. It's easy to verify that function $F_{\theta(\bar{w})}(w)$ can be decomposed as

$$F_{\theta(\bar{w})}(w) = \Big(\theta(\bar{w}) - \frac{m}{r\beta}\Big)G_1(w) + \frac{m}{r}G_2(w), \text{ for } w \geq 0,$$

in which functions $G_1(w)$ and $G_2(w)$ are the solution of DDE (H) with boundary conditions being

$$G_1(w) = w \text{ and } G_2(w) = \frac{w}{\beta} - 1, \quad \forall w \in [0, \beta),$$

respectively. Since $G_1(w)$ and $G_2(w)$ are both increasing on $[0, \beta]$ and nonnegative at $w = \beta$, we know that they are both increasing on $[\beta, \infty)$ by Lemma 4.2. As such, $F_{\theta(\bar{w})}(\bar{w})$ is increasing on $[\beta, \infty)$ as well, which contradicts to $F'_{\theta(\bar{w})}(w) = -1$. Therefore, contradiction is established and we must have $\theta_{\bar{w}} < \frac{m}{\beta r}$.

(ii) We only needs to show that $F'_{\theta(\bar{w})}(\beta_+) \leq F'_{\theta(\bar{w})}(\beta_-)$ since for $w > \beta$, concavity of $F_{\theta(\bar{w})}(w)$ follows immediately from the strict convexity of function $J(w)$ in Lemma 4.3 and the decomposition in (3.18). If $r = \rho$, according to (H), we have

$$F'_{\theta(\bar{w})}(\beta_+) = \frac{(\lambda + r)F_{\theta(\bar{w})}(\beta) - \lambda F_{\theta(\bar{w})}(0)}{(r+\lambda)\beta} = \frac{\lambda}{\lambda + r}\theta(\bar{w}) \leq \theta(\bar{w}) = F'_{\theta(\bar{w})}(\beta_-). \tag{4.39}$$

(iii) By the definition of $\theta(\bar{w})$, we know that $\theta(\bar{w})$ is strictly increasing in $\bar{w}$ (recall the convexity of $J(w)$). Therefore, for any $\tilde{w} \in [\beta, \bar{w})$, we have $\theta(\bar{w}) > \theta_(\tilde{w})$. For any $w \geq 0$, decomposition (3.18) implies that

$$F_{\theta(\bar{w})}(w) - F_{\theta(\tilde{w})}(w) = [\theta(\bar{w}) - \theta(\tilde{w})]w > 0.$$

This completes the proof. □

Lemma 4.4. *Consider a concave function $F(w)$ that satisfies equations* (H), (U), *and* (3.19). *For any $w \geq \beta$, the following function $\Phi(w, x)$ is increasing in $x \in (-\infty, 0]$ and decreasing in $x \in [0, \infty)$,*

$$\Phi(w, x) := F'(w)x + F(w - x). \tag{4.40}$$

Proof. Taking the first derivative of $\Phi(w,x)$ with respect to x yields

$$\frac{\partial\Phi(w,x)}{\partial x} = F'(w) - F'(w-x).$$

Because $F(w)$ is concave, we know that $\frac{\partial\Phi(w,x)}{\partial x} \geq 0$ when $x \leq 0$ and $\frac{\partial\Phi(w,x)}{\partial x} \leq 0$ when $x \geq 0$. That is, for any $w \geq \beta$, $\Phi(w,x)$ is increasing in $x \in (-\infty, 0]$ and decreasing in $x \in [0,\infty)$. □

Proof of Proposition 3.2. Starting with any promised utility $W_0 = w \in [0,\bar{w}]$, consider the process $\{W_t\}_{t\geq 0}$ according to (PK) in which the counting processes $\{(N_t^s, N_t^n)\}_{t\geq 0}$ are generated from the effort process $\underline{\Lambda}$ under contracts $\Gamma_d(w;\beta,\bar{w})$ or $\Gamma_r(w;\bar{w})$. Clearly, we must have $0 \leq W_t \leq \bar{w}$ for $\forall t$.

(i) First, we consider contract $\Gamma_d(w;\beta,\bar{w})$ defined in Definition 3.1. Following Itô's Formula for jump processes, we have

$$\begin{aligned} dF_{\theta(\bar{w})}(W_t) \quad &= \quad [F_{\theta(\bar{w})}(W_t - dI_t) - F_{\theta(\bar{w})}(W_t)] + F'_{\theta(\bar{w})}(W_t)\Big\{\rho W_t + \lambda_t[y_t H_t^s + (1-y_t)H_t^n] - \ell_t\Big\}dt \\ &\quad +[F_{\theta(\bar{w})}(W_t - H_t^s) - F_{\theta(\bar{w})}(W_t)]dN_t^s + [F_{\theta(\bar{w})}(W_t - H_t^n) - F_{\theta(\bar{w})}(W_t)]dN_t^n. \quad (4.41) \end{aligned}$$

Following the dynamics of W_t according to Definition 3.1, we have

$$\begin{aligned} dF_{\theta(\bar{w})}(W_t) \quad &= \quad F'_{\theta(\bar{w})}(W_t)\Big(\rho W_t \mathbb{I}_{W_t<\beta} + (\rho W_t + \lambda\beta)\mathbb{I}_{\beta\leq W_t<\bar{w}}\Big)dt \\ &\quad +\Big[F_{\theta(\bar{w})}(W_t - \beta) - F_{\theta(\bar{w})}(W_t)\Big]\mathbb{I}_{\beta\leq W_t\leq\bar{w}} dN_t. \qquad (4.42) \end{aligned}$$

Note that dN_t^s and dN_t^n take values 0 or 1 in the above expressions.

For any $\tau \geq 0$, we have:

$$\begin{aligned} e^{-r\tau}F_{\theta(\bar{w})}(W_\tau) \quad &= \quad F_{\theta(\bar{w})}(w) + \int_0^\tau F_{\theta(\bar{w})}(W_t)\, de^{-rt} + \int_0^\tau e^{-rt} dF_{\theta(\bar{w})}(W_t) \\ &= \quad F_{\theta(\bar{w})}(w) + \int_0^\tau e^{-rt}\Big[F'_{\theta(\bar{w})}(W_t)\Big(\rho W_t \mathbb{I}_{W_t<\beta} + (\rho W_t + \beta\lambda)\mathbb{I}_{\beta\leq W_t<\bar{w}}\Big) - rF_{\theta(\bar{w})}(W_t)\Big]dt \\ &\quad + \int_0^\tau e^{-rt}\Big[F_{\theta(\bar{w})}(W_t - \beta) - F_{\theta(\bar{w})}(W_t)\Big]\mathbb{I}_{\beta\leq W_t\leq\bar{w}} dN_t. \qquad (4.43) \end{aligned}$$

From Equations(3.17) and (H), we know that $F_{\theta(\bar{w})}(w)$ satisfies

$$\begin{aligned} F'_{\theta(\bar{w})}(W_t)\rho W_t \mathbb{I}_{W_t<\beta} \quad &= \quad [rF_{\theta(\bar{w})}(W_t) + m]\mathbb{I}_{W_t<\beta}, \text{ and} \\ F'_{\theta(\bar{w})}(W_t)(\rho W_t + \beta\lambda)\mathbb{I}_{\beta\leq W_t<\bar{w}} \quad &= \quad \Big[(\lambda + r)F_{\theta(\bar{w})}(W_t) - \lambda F_{\theta(\bar{w})}(W_t - \beta)\Big]\mathbb{I}_{\beta\leq W_t<\bar{w}}. \end{aligned}$$

Substituting the above equations into (4.43), we obtain

$$\begin{aligned} e^{-r\tau}F_{\theta(\bar{w})}(W_\tau) &= F_{\theta(\bar{w})}(w) + \int_0^\tau e^{-rt}\Big[F_{\theta(\bar{w})}(W_t-\beta) - F_{\theta(\bar{w})}(W_t)\Big]\mathbb{I}_{\beta\le W_t\le\bar{w}}dN_t \\ &\quad + \int_0^\tau e^{-rt}\Big[(\lambda+r)F_{\theta(\bar{w})}(W_t) - \lambda F_{\theta(\bar{w})}(W_t-\beta)\Big]\mathbb{I}_{\beta\le W_t<\bar{w}}dt \\ &\quad + \int_0^\tau e^{-rt}\Big[(rF_{\theta(\bar{w})}(W_t)+m)\mathbb{I}_{W_t<\beta} - rF_{\theta(\bar{w})}(W_t)\Big]\,dt \\ &= F_{\theta(\bar{w})}(w) + \int_0^\tau e^{-rt}\Big(m\mathbb{I}_{W_t<\beta} + (\rho\bar{w}+\beta\lambda)\mathbb{I}_{W_t=\bar{w}}\Big)dt + \Omega_\tau, \end{aligned} \tag{4.44}$$

where the second equality utilizes Equation (U), and the process $\{\Omega_\tau\}_{\tau\ge 0}$, defined as

$$\Omega_\tau := \int_0^\tau e^{-rt}\Big[F_{\theta(\bar{w})}(W_t-\beta) - F_{\theta(\bar{w})}(W_t)\Big]\mathbb{I}_{\beta\le W_t\le\bar{w}}(dN_t - \lambda dt),$$

is a martingale. Taking expectation on both sides of (4.44) and letting $\tau\to\infty$, we have

$$\begin{aligned} F_{\theta(\bar{w})}(w) &= -\mathbb{E}^{\Gamma_d(w;\beta,\bar{w}),\underline{\Delta}}\Big[\int_0^\infty e^{-rt}\Big(m\mathbb{I}_{W_t<\beta} + (\rho\bar{w}+\beta\lambda)\mathbb{I}_{W_t=\bar{w}}\Big)dt\Big] \\ &= -\mathbb{E}^{\Gamma_d(w;\beta,\bar{w}),\underline{\Delta}}\Big[\int_0^\infty e^{-rt}\Big(m_t\,dt + dL_t\Big)\Big] = U(\Gamma_d(w;\beta,\bar{w})), \end{aligned}$$

where the second equality follows from Definition 3.1.

(ii) Next, we consider contract $\Gamma_r(w;\bar{w})$ defined in Definition 3.2. Following Itô's Formula for jump process, we have

$$\begin{aligned} dF_{\theta(\bar{w})}(W_t) &= F'_{\theta(\bar{w})}(W_t)(\rho W_t + \lambda\beta)\mathbb{I}_{\beta\le W_t<\bar{w}}dt \\ &\quad + \Big\{[F_{\theta(\bar{w})}(\beta) - F(W_t)]\,dN_t^n + [F_{\theta(\bar{w})}(0) - F_{\theta(\bar{w})}(W_t)]dN_t^s\Big\}\mathbb{I}_{\beta\le W_t\le\min\{\bar{w},2\beta\}} \\ &\quad + [F_{\theta(\bar{w})}(W_t-\beta) - F_{\theta(\bar{w})}(W_t)]\mathbb{I}_{\min\{\bar{w},2\beta\}<W_t\le\bar{w}}dN_t^n. \end{aligned} \tag{4.45}$$

For any time $\tau \geq 0$, we have

$$\begin{aligned}
e^{-r\tau}F_{\theta(\bar{w})}(W_\tau) &= F_{\theta(\bar{w})}(w) + \int_0^\tau F_{\theta(\bar{w})}(W_t)de^{-rt} + \int_0^\tau e^{-rt}dF_{\theta(\bar{w})}(W_t) \\
&= F_{\theta(\bar{w})}(w) + \int_0^\tau e^{-rt}\Big(F'_{\theta(\bar{w})}(W_t)(\rho W_t + \lambda\beta)\mathbb{I}_{\beta\leq W_t<\bar{w}} - rF_{\theta(\bar{w})}(W_t)\Big)dt \\
&\quad + \int_0^\tau e^{-rt}\Big\{[F_{\theta(\bar{w})}(\beta) - F_{\theta(\bar{w})}(W_t)]\mathbb{I}_{\beta\leq W_t\leq\min\{\bar{w},2\beta\}}\,dN_t^n \\
&\qquad + [F_{\theta(\bar{w})}(0) - F_{\theta(\bar{w})}(W_t)]\mathbb{I}_{\beta\leq W_t\leq\min\{\bar{w},2\beta\}}\,dN_t^s \\
&\qquad + [F_{\theta(\bar{w})}(W_t-\beta) - F_{\theta(\bar{w})}(W_t)]\mathbb{I}_{\min\{\bar{w},2\beta\}<W_t\leq\bar{w}}\,dN_t\Big\} \\
&= F_{\theta(\bar{w})}(w) + \int_0^\tau e^{-rt}\Big(m\mathbb{I}_{W_t=0}\,dt + (\rho W_t + \beta\lambda)\mathbb{I}_{W_t=\bar{w}}\,dt\Big) + \Omega_\tau, \qquad (4.46)
\end{aligned}$$

where the last equality follows from Equation (L), which implies that $F_{\theta(\bar{w})}(0)(2-\frac{W_t}{\beta})+F_{\theta(\bar{w})}(\beta)(\frac{W_t}{\beta}-1) = F_{\theta(\bar{w})}(W_t-\beta)$, and the process $\{\Omega_\tau\}_{\tau\geq 0}$, defined as

$$\begin{aligned}
\Omega_\tau := \int_0^\tau e^{-rt} &\Big\{\Big(F_{\theta(\bar{w})}(0) - F_{\theta(\bar{w})}(W_t)\Big)\Big(dN_t^s - (2-\frac{W_t}{\beta})\lambda dt\Big)\mathbb{I}_{\beta\leq W_t\leq\min\{\bar{w},2\beta\}} \\
&+\Big(F_{\theta(\bar{w})}(\beta) - F_{\theta(\bar{w})}(W_t)\Big)\Big(dN_t^n - (\frac{W_t}{\beta}-1)\lambda dt\Big)\mathbb{I}_{\beta\leq W_t\leq\min\{\bar{w},2\beta\}} \\
&+\Big(F_{\theta(\bar{w})}(W_t-\beta) - F_{\theta(\bar{w})}(W_t)\Big)\Big(dN_t - \lambda dt\Big)\mathbb{I}_{\min\{\bar{w},2\beta\}<W_t\leq\bar{w}}\Big\},
\end{aligned}$$

is a martingale.

Taking expectation on both sides of (4.46) and letting $\tau \to \infty$, we have

$$F_{\theta(\bar{w})}(w) = -\mathbb{E}^{\Gamma_r(w;\bar{w}),\Lambda}\Big[\int_0^\infty e^{-rt}\Big(m\mathbb{I}_{W_t=0}\,dt + (\rho\bar{w} + \beta\lambda)\mathbb{I}_{W_t=\bar{w}}\,dt\Big)\Big] = U(\Gamma_r(w;\bar{w})).$$

This completes the proof. □

Proof of Theorem 3.1. By Proposition 3.1, we know that $F_{\theta(\bar{w})}(w)$ is concave and $F'_{\theta(\bar{w})}(w) \geq -1$. Therefore, we only need to show $\Psi_t \leq 0$ holds almost surely (recall Lemma 3.3).

From Equation (3.8), we have

$$\begin{aligned}\Psi_t &\leq \lambda\Big[F'_{\theta(\bar{w})}(W_t)\Big(y_t H_t^s + (1-y_t)H_t^n\Big) + y_t F_{\theta(\bar{w})}(W_t - H_t^s) + (1-y_t)F_{\theta(\bar{w})}(W_t - H_t^n) - F_{\theta(\bar{w})}(W_t)\Big] \\ &\quad + F'_{\theta(\bar{w})}(W_t)\rho W_t - rF_{\theta(\bar{w})}(W_t) - m_t \\ &\leq \lambda\Phi\Big(W_t, y_t H_t^s + (1-y_t)H_t^n\Big) + F'_{\theta(\bar{w})}(W_t)\rho W_t - (\lambda + r)F_{\theta(\bar{w})}(W_t) - m_t, \end{aligned} \tag{4.47}$$

in which function Φ is defined in (4.40).

(i) When $W_t < \beta$, we know that the principal monitors the agent (i.e., $m_t = m$). From Equation (3.17), we have

$$\Psi_t \leq \rho W_t F'_{\theta(\bar{w})}(W_t) - rF_{\theta(\bar{w})}(W_t) - m = 0.$$

(ii) When $\beta \leq W_t \leq \bar{w}$, substituting (H) into inequality (4.47) yields

$$\Psi_t \leq \lambda\Phi\Big(W_t, y_t H_t^s + (1-y_t)H_t^n\Big) - \lambda\beta F'_{\theta(\bar{w})}(W_t) - \lambda F_{\theta(\bar{w})}(W_t - \beta) - m_t.$$

- If the principal does not monitor at time t (i.e., $m_t = 0$), we must have $y_t H_t^s + (1-y_t)H_t^n \geq \beta$. Lemma 4.4 implies

$$\Psi_t \leq \lambda\Phi(W_t, \beta) - \lambda\beta F'_{\theta(\bar{w})}(W_t) - \lambda F_{\theta(\bar{w})}(W_t - \beta) = 0.$$

- If the principal monitors at time t (i.e., $m_t = m$), Lemma 4.4 implies

$$\begin{aligned}\Psi_t &\leq \lambda\Phi(W_t, 0) - \lambda\beta F'_{\theta(\bar{w})}(W_t) - \lambda F_{\theta(\bar{w})}(W_t - \beta) - m \\ &= -rF_{\theta(\bar{w})}(W_t) + rW_t F'_{\theta(\bar{w})}(W_t) - m \\ &\leq -rF_{\theta(\bar{w})}(\beta) + r\beta F'_{\theta(\bar{w})}(\beta) - m \\ &= -r\Big(\theta(\bar{w})\beta - \frac{m}{r}\Big) + r\beta\theta\bar{w}\frac{\lambda}{\lambda + r} - m \\ &= r\beta\theta(\bar{w})\Big(\frac{\lambda}{\lambda + r} - 1\Big) \leq 0,\end{aligned}$$

  where the second inequality follows the fact that $-rF(W_t) + rW_t F'(W_t)$ is decreasing in W_t.

To sum up, we must have $\Psi_t \leq 0$. This completes the proof. □

4.2.3 Proofs in Section 3.4

To prove Proposition 3.6 and Lemma 3.5, we first characterize the structural properties of function $J(w)$ for the case $r < \rho$.

Lemma 4.5. *Consider the case $r < \rho$. For any $\alpha \geq \beta$, define function $J(w)$ to be the solution of DDE* (H) *with boundary condition $J(w) = 1$ on $w \in [0, \alpha)$. $J(w)$ exhibits the following properties.*

(i) When $r \leq \rho - \lambda$, $J(w)$ is concave on $[\alpha, \infty)$.

(ii) When $\rho - \lambda < r < \bar{r}$, $J(w)$ is convex on $[\alpha, \alpha + \beta]$ and concave on $[\alpha + \beta, \infty)$.

(iii) We have $\limsup_{w \to \infty} J'(w) = 0$.

Proof. (i) Note that following DDE (H), function $J(w)$ is differentiable for $w > \alpha$, and is twice-differentiable except at $w = \alpha + \beta$. Taking derivatives on both sides of DDE (H) yields

$$J''(w) = \frac{(\lambda + r - \rho)J'(w) - \lambda J'(w - \beta)}{\rho w + \beta\lambda}, \quad \text{for } w \in (\alpha, \alpha + \beta) \cup (\alpha + \beta, \infty). \tag{4.48}$$

Because $J(w)$ is non-decreasing (recall Lemma 4.2), we have $J'(w) \geq 0$ and $J'(w - \beta) \geq 0$ for $w \in (\alpha, \infty)$. As such, we must have $J''(w) \leq 0$ if $r \leq \rho - \lambda$, except $w = \alpha + \beta$, at which point $J(w)$ is differentiable. Therefore, $J(w)$ is concave on (α, ∞) and continuity of function $J(w)$ extends concavity to $w \in [\alpha, \infty)$.

(ii) Starting from boundary conditions $J(w) = 1$ for $w \in [0, \alpha)$, DDE (H) yields the following:

$$J(w) = \frac{r}{\lambda + r}\Big(\frac{\rho w + \lambda\beta}{\rho\beta + \lambda\beta}\Big)^{\frac{\lambda + r}{\rho}} + \frac{\lambda}{\lambda + r}, \text{ for } w \in (\alpha, \alpha + \beta), \tag{4.49}$$

whose first and second derivatives are

$$\begin{aligned} J'(w) &= \frac{r(\rho w + \lambda\beta)^{\frac{\lambda + r - \rho}{\rho}}}{(\rho\beta + \lambda\beta)^{\frac{\lambda + r}{\rho}}} \\ J''(w) &= \frac{r(\rho w + \lambda\beta)^{\frac{\lambda + r - 2\rho}{\rho}}}{(\rho\beta + \lambda\beta)^{\frac{\lambda + r}{\rho}}}(\lambda + r - \rho). \end{aligned} \tag{4.50}$$

Therefore, if $r > \rho - \lambda$, we must have $J''(w) > 0$, i.e., $J(w)$ is convex on $[\alpha, \alpha + \beta]$ since function $J(w)$ is continuous.

Given that the right-hand-side of (3.38) is increasing whereas the left-hand-side is decreasing in $\bar{r}$, it is readily shown that Equation (3.38) has a unique solution within $(\rho-\lambda,\rho)$. Moreover, for $r \in (\rho-\lambda,\bar{r})$, we have

$$\Big(\frac{2\rho+\lambda}{\rho+\lambda}\Big)^{\frac{\lambda+r}{\rho}-1} < \frac{\lambda}{\lambda+r-\rho},$$

which implies that $J''((\alpha+\beta)_+) < 0$, i.e., we have $(\lambda+r-\rho)J'(\alpha+\beta) < \lambda J'(\alpha_+)$.

To prove that $J(w)$ is concave on $[\alpha+\beta,\infty)$, we only need to show $J''(w) \le 0$ for all $w > \alpha+\beta$. Suppose, on the contrary, there exists a $w > \alpha+\beta$ such that $J''(w) > 0$. Define

$$\hat{w} := \min\{w | (\lambda+r-\rho)J'(w) \ge \lambda J'(w-\beta) \text{ and } w > \alpha+\beta\}.$$

- If $\hat{w} \in (\alpha+\beta, \alpha+2\beta]$, we have $J'(\alpha+\beta) \ge J'(\hat{w})$ because $J(w)$ is concave within $[\alpha+\beta,\hat{w}]$. We also have $J'(\beta_+) < J'(\hat{w}-\beta)$ because $J(w)$ is convex on $[\alpha,\alpha+\beta]$. Therefore,

$$(\lambda+r-\rho)J'(\hat{w}) \le (\lambda+r-\rho)J'(\alpha+\beta) < \lambda J'(\beta_+) < \lambda J'(\hat{w}-\beta).$$

- If $\hat{w} > \alpha+2\beta$, then both $J(\hat{w})$ and $J(\hat{w}-\beta)$ are twice continuously differentiable. As such,

$$J'(\hat{w}) = J'(\hat{w}-\beta) + \int_0^\beta J''(\hat{w}-\beta+x)dx < J'(\hat{w}-\beta),$$

  which implies $(\lambda+r-\rho)J'(\hat{w}) < \lambda J'(\hat{w}-\beta)$.

That is, $(\lambda+r-\rho)J'(\hat{w}) \ge \lambda J'(\hat{w}-\beta)$ cannot be true for either case. Therefore, $J(w)$ is concave on $[\alpha+\beta,\infty)$.

(iii) For notational convenience, we let $\ell := \limsup_{w\to\infty} J'(w)$. First we show that ℓ cannot be positive infinity. Suppose, on the contrary, $\ell = \limsup_{w\to\infty} J'(w) = \infty$. Then there exists an increasing divergent sequence $\{w_n\}_{n\ge1}$ in $(\alpha+\beta,\infty)$ such that $\lim_{w\to\infty} J'(w_n) = \infty$ and

$$w_n = \operatorname*{argmax}_{w\in[0,w_n]}\{J'(w)\}.$$

Then for each $n \ge 1$ by mean value theorem, there exists $\hat{w}_n \in (w_n-\beta, w_n)$, such that

$$\begin{aligned}(\rho w_n + \beta\lambda)J'(w_n) &= \lambda[J(w_n) - J(w_n-\beta)] + rJ(w_n)\\ &= \lambda\beta J'(\hat{w}_n) + rJ(w_n).\end{aligned}$$

Rearranging the equation above, one gets

$$J'(\hat{w}_n) = \frac{w_n}{\lambda\beta}[\rho J'(w_n) - \frac{r}{w_n}J(w_n)] + J'(w_n). \tag{4.51}$$

Since $J(0) = 1$ and $\{J'(w_n)\}_n$ is an increasing sequence, there is $J(w_n) - 1 \leq w_n J'(w_n)$ by construction. For such n, from (4.51) and the fact $J(w_n) \leq w_n J'(w_n)$, there is

$$J'(\hat{w}_n) \geq \frac{(\rho - r)w_n J'(w_n)}{\lambda\beta}.$$

Since $J'(w_n) > 0$, an immediate result follows the previous inequality is that

$$\frac{J'(\hat{w}_n)}{J'(w_n)} \geq \frac{(\rho - r)w_n}{\lambda\beta},$$

which goes to infinity as n goes to infinity. Therefore, we can obtain $J'(\hat{w}_n) > J'(w_n)$ eventually, which contradicts to the definition of w_n since $\hat{w}_n < w_n$. Thus, we have that ℓ must be finite.

Consider a new increasing and divergent sequence $\{w_n\}_{n\geq 1}$ in $(\alpha+\beta, \infty)$ such that $\lim_{n\to\infty} J'(w_n) = \ell$. Then for all $n \geq 1$, we can find a constant D such that $J(w_n) \leq \ell w_n + D$. Let $\hat{w}_n \in (w_n - \beta, w_n)$, by substituting $J(w_n) \leq \ell w_n + D$ into the differential equations of $J(w)$ for all $n \geq 1$, we have

$$\rho J'(w_n) - rl \leq \frac{\lambda\beta[J'(\hat{w}_n) - J'(w_n)] + rD}{w_n}.$$

By letting n goes to infinity, there is

$$(\rho - r)l \leq \lambda\beta \liminf_{n\to\infty} \frac{J'(\hat{w}_n)}{w_n}.$$

If $l > 0$, the above inequality implies that ℓ must go to infinity; this contradicts to the fact that l is finite. Therefore, we have $l \leq 0$. Given that $J'(w) \geq 0$ for all w (recall Lemma 4.2), we must have $\ell = 0$. □

Proof of Lemma 3.4. Taking the first derivative with respect to α, we have

$$f'(\alpha) = -\frac{r\lambda\theta}{\rho}\alpha^{\frac{r}{\rho}-1}\Big[1 + \frac{\rho - r}{\rho\alpha}\beta - \Big(1 - \frac{\beta}{\alpha}\Big)^{\frac{r}{\rho}-1}\Big].$$

Consider a function

$$h(x) := 1 + (\rho - r)\beta x - \Big(1 - \rho\beta x\Big)^{\frac{r}{\rho}-1}, \quad x \in \Big[0, \frac{1}{\rho\beta}\Big]. \tag{4.52}$$

We know that $h(0) = 0$ and

$$h'(x) = (\rho - r)\beta\Big[1 - \Big(1 - \rho\beta x\Big)^{\frac{r}{\rho}-2}\Big] < 0,$$

where the inequality holds because $r < \rho$. Therefore, we must have $h(x) < 0,\ \forall x \in \Big(0, 1/(\rho\beta)\Big]$. Consequently,

$$f'(\alpha) = -\frac{r\lambda\theta}{\rho}\alpha^{\frac{r}{\rho}-1}h\Big(\frac{1}{\rho\alpha}\Big) > 0,$$

implying that function $f(\alpha)$ is strictly increasing in $\alpha \in [\beta, \infty)$. Moreover, we have

$$\begin{aligned}
\lim_{\alpha\to\infty} f(\alpha) &= m - \lambda\theta \lim_{\alpha\to\infty} \frac{1 - \frac{r\beta}{\rho\alpha} - \Big(1 - \frac{\beta}{\alpha}\Big)^{\frac{r}{\rho}}}{\alpha^{-\frac{r}{\rho}}} \\
&= m - \lambda\theta \lim_{\alpha\to\infty} \frac{\frac{r\beta}{\rho\alpha^2} - \frac{r\beta}{\rho\alpha^2}\Big(1 - \frac{\beta}{\alpha}\Big)^{\frac{r}{\rho}-1}}{-\frac{r}{\rho}\alpha^{-\frac{r}{\rho}-1}} \\
&= m + \lambda\theta\beta \lim_{\alpha\to\infty} \frac{1 - \Big(1 - \frac{\beta}{\alpha}\Big)^{\frac{r}{\rho}-1}}{\alpha^{1-\frac{r}{\rho}}} = m.
\end{aligned}$$

This completes the proof. □

Proof of Proposition 3.3. (i) The definition of α_θ directly implies that $f(\alpha_\theta) \geq 0$. From the definition of $F_{\theta,\alpha}(w)$, for any $\alpha \geq \beta$, we have

$$F'_{\theta,\alpha}(\alpha_-) = \frac{r\theta}{\rho}\alpha^{\frac{r}{\rho}-1} \text{ and } F''_{\theta,\alpha}(\alpha_-) = \frac{r(r-\rho)}{\rho^2}\theta\alpha^{\frac{r}{\rho}-2} < 0.$$

From DDE (H), we have

$$\begin{aligned}
F'_{\theta,\alpha}(\alpha_+) &= \frac{1}{\rho\alpha + \lambda\beta}\Big[(\lambda + r)F_{\theta,\alpha}(\alpha) - \lambda F_{\theta,\alpha}(\alpha - \beta)\Big], \\
&= \frac{1}{\rho\alpha + \lambda\beta}\Big[(\lambda + r)\theta\alpha^{\frac{r}{\rho}} - \lambda\theta(\alpha - \beta)^{\frac{r}{\rho}} - m\Big].
\end{aligned}$$

As such,

$$F'_{\theta,\alpha}(\alpha_-) - F'_{\theta,\alpha}(\alpha_+) = \frac{1}{\rho\alpha + \beta\lambda}f(\alpha). \tag{4.53}$$

Consider the case $\alpha = \alpha_\theta$. If $f(\beta) \geq 0$, the definition of α_θ implies that $f(\alpha) = f(\beta) \geq 0$, i.e., $F'_{\theta,\alpha_\theta}(\alpha_{\theta-}) \geq F'_{\theta,\alpha_\theta}(\alpha_{\theta+})$. Otherwise we must have $f(\alpha_\theta) = 0$ and $F'_{\theta,\alpha_\theta}(\alpha_{\theta-}) = F'_{\theta,\alpha_\theta}(\alpha_{\theta+})$. Therefore we have (3.27).

(ii) When $\alpha_\theta > \beta$, we must have $f(\alpha_\theta) = 0$. From DDE (H) we have

$$\begin{aligned} F''_{\theta,\alpha_\theta}(\alpha_{\theta+}) &= \frac{1}{\rho\alpha_\theta + \lambda\beta}\Big[(\lambda + r - \rho)F'_{\theta,\alpha_\theta}(\alpha_{\theta+}) - \lambda F'_{\theta,\alpha_\theta}(\alpha_\theta - \beta)\Big], \\ &= \frac{\theta r}{\rho(\rho\alpha_\theta + \lambda\beta)}\Big[(\lambda + r - \rho)\alpha_\theta^{\frac{r}{\rho}-1} - \lambda(\alpha_\theta - \beta)^{\frac{r}{\rho}-1}\Big]. \end{aligned}$$

As such,

$$\begin{aligned} F''_{\theta,\alpha_\theta}(\alpha_{\theta-}) - F''_{\theta,\alpha_\theta}(\alpha_{\theta+}) &= \frac{\theta r\lambda\alpha_\theta^{\frac{r}{\rho}-1}}{\rho(\rho\alpha_\theta + \beta\lambda)}\Big[\Big(1 - \frac{\beta}{\alpha_\theta}\Big)^{\frac{r}{\rho}-1} - 1 - \frac{\beta(\rho - r)}{\rho\alpha_\theta}\Big] \\ &= -\frac{\theta r\lambda\alpha_\theta^{\frac{r}{\rho}-1}}{\rho(\rho\alpha_\theta + \beta\lambda)}h\Big(\frac{1}{\rho\alpha_\theta}\Big) > 0, \end{aligned}$$

where function $h(x)$ is defined in (4.52). The above inequality holds because $h(x) < 0$ for $\forall x \in \Big(0, 1/(\rho\beta)\Big]$.

(iii) Consider an arbitrary $\alpha \geq \beta$. If $\alpha < \alpha_\theta$, we must have $f(\alpha) < f(\alpha_\theta) = 0$; which, together with (4.53), implies $F'_{\theta,\alpha}(\alpha_-) < F'_{\theta,\alpha}(\alpha_+)$. Otherwise if $\alpha > \alpha_\theta$, we must have $f(\alpha) > f(\alpha_\theta) \geq 0$, i.e., $F'_{\theta,\alpha}(\alpha_-) > F'_{\theta,\alpha}(\alpha_+)$. □

Proof of Lemma 3.5. We first show that for any $w \geq 0$, derivative $F'_{\theta,\alpha_\theta}(w_+)$ is increasing in θ. To do so, we define $g(w,\theta) = F_{\theta,\alpha_\theta}(w)$, and show $\dfrac{\partial g(w,\theta)}{\partial\theta}$ is well-defined and strictly increasing in w.

- For $\forall w \in [0, \alpha_\theta)$, we have $\dfrac{\partial g(w,\theta)}{\partial\theta} = w^{\frac{r}{\rho}}$, which is strictly increasing in w.
- For $w = \alpha_\theta$,

$$\begin{aligned} \frac{\partial g(w,\theta_+)}{\partial\theta}\Big|_{w=\alpha_\theta} &= \lim_{\varepsilon\downarrow 0}\frac{F_{\theta+\varepsilon,\alpha_\theta}(\alpha_\theta) - F_{\theta,\alpha_\theta}(\alpha_\theta)}{\varepsilon} + \frac{F_{\theta,\alpha_{\{\theta+\varepsilon\}}}(\alpha_\theta) - F_{\theta,\alpha_\theta}(\alpha_\theta)}{\varepsilon} \times \frac{d\alpha_\theta}{d\theta} \\ &= \lim_{\varepsilon\downarrow 0}\frac{(\theta+\varepsilon)\alpha_\theta^{\frac{r}{\rho}} - \theta\alpha_\theta^{\frac{r}{\rho}}}{\varepsilon} = \alpha_\theta^{\frac{r}{\rho}} = \frac{\partial g(w,\theta_-)}{\partial\theta}\Big|_{w=\alpha_\theta}, \end{aligned}$$

where the second equality follows from $F_{\theta,\alpha_{\{\theta+\varepsilon\}}}(\alpha_\theta) = F_{\theta,\alpha_\theta}(\alpha_\theta)$ because $\alpha_{\theta+\varepsilon} \geq \alpha_\theta$ for any $\varepsilon > 0$.

- For $w > \alpha_\theta$, note that $g(w,\theta) = F_{\theta,\alpha_\theta}(w)$ is a solution to DDE (H), parameterized by θ. That is, the following equality also holds:

$$(\lambda + r)g(w,\theta) = \lambda g(w-\beta,\theta) + (\rho w + \lambda\beta)\frac{\partial g(w,\theta)}{\partial w}.$$

  Taking derivatives $w.r.t$ θ on both sides of the above equation, we have

$$(\lambda + r)\frac{\partial g(w,\theta)}{\partial \theta} = \lambda \frac{\partial g(w-\beta,\theta)}{\partial \theta} + (\rho w + \lambda\beta)\frac{\partial g(w,\theta)}{\partial w \partial \theta},$$

  which implies that $\frac{\partial g(w,\theta)}{\partial \theta}$ satisfies DDE (H) as well. Since $\frac{\partial g(0,\theta)}{\partial \theta} > 0$ and $\frac{\partial g(w,\theta)}{\partial \theta}$ is increasing on $w \in [0, \alpha_\theta]$, we immediately know that $\frac{\partial g(w,\theta)}{\partial \theta}$ is increasing on $w \in [\alpha_\theta, \infty)$ from Lemma 4.2.

Summarizing the above three cases, we conclude that $\frac{\partial g(w,\theta)}{\partial \theta}$ is well-defined and strictly increasing in w. Therefore, derivative $F'_{\theta,\alpha_\theta}(w_+)$ is increasing in θ. As a result, we know that $\inf_w F'_{\theta,\alpha_\theta}(w_+)$ is increasing in θ. On the one hand, when θ is sufficiently large (i.e., $\theta \to \infty$), we know that α_θ approaches ∞ as well. As such,

$$\lim_{\theta\to\infty}\left\{\inf_{w>\beta} F'_{\theta,\alpha_\theta}(w)\right\} = \lim_{\theta\to\infty}\left\{\inf_{w>\beta}\left[\frac{r\theta}{\rho} w^{\frac{r}{\rho}-1}\right]\right\} > 0.$$

On the other hand, when $\theta = 0$, we have $\alpha_\theta = \beta$. It is clear that

$$F_{0,\beta}(w) = -\frac{m}{r}J(w) \quad \text{for} \quad \forall w \geq 0.$$

As such,

$$\inf_{w>\beta} F'_{0,\beta}(w) = -\frac{m}{r}\sup_{w>\beta} J'(w) \leq -1,$$

where the last inequality holds because $J(w)$ is nondecreasing and $m \geq \bar{m}$.

Therefore, there exists a unique $\bar{\theta} \geq 0$, such that $\inf_w F'_{\bar{\theta},\alpha_{\bar{\theta}}}(w_+) = -1$.

Using similar techniques as in the proof of Proposition 3.1(i), we can show that $\bar{\theta} < \frac{m}{r}\beta^{-\frac{r}{\rho}}$. We only need to modify the definition of functions $G_1(w)$ and $G_2(w)$ as:

$$G_1(w) = w^{\frac{r}{\rho}} \text{ and } G_2(w) = \left(\frac{w}{\beta}\right)^{\frac{r}{\rho}} - 1, \text{ for } w \in [0, \alpha_{\bar{\theta}}].$$

We omit the detailed proof to avoid redundancy.

Finally, we show that $\bar{w}^*$, which is determined by (3.28), is finite. Notice that

$$\bar{\theta} = -\frac{1 - mJ'(\bar{w}^*)/r}{G_1'(\bar{w}^*)}.$$

If, on the contrary, $\bar{w}^* \to \infty$, the above equation implies that $\bar{\theta} < 0$ because $\limsup_{w\to\infty} J'(w) = 0$ [recall Lemma 4.5(iii)] and $G_1(w)$ is increasing (recall Lemma 4.2). This contradicts to the fact that $\bar{\theta} > 0$. Therefore, $\bar{w}^*$ must be finite. □

To prove Proposition 3.4, we first show a more general result, as presented in the following Lemma.

Lemma 4.6. *For any $\theta \geq \bar{\theta}$, define $\hat{w} := \inf\{\arg\inf_{w>\beta} F'_{\theta,\alpha_\theta}(w)\}$. Then function $F_{\theta,\alpha_\theta}(w)$ is strictly concave on $w \in [0, \hat{w}]$.*

Proof. Firstly, $F_{\theta,\alpha_\theta}(w)$ is nondecreasing and concave in $[0, \alpha_\theta)$ because $\theta \geq 0$ (recall Lemma 3.5). Next, from Proposition 3.3(i) we know that $F'_{\theta,\alpha_\theta}(\alpha_{\theta-}) \geq F'_{\theta,\alpha_\theta}(\alpha_{\theta+})$. Therefore, in order to show that $F(w)$ is concave, we only need to show that $F_{\theta,\alpha_\theta}(w)$ is concave on $[\alpha_\theta, \hat{w}]$.

Recall that $F_{\theta,\alpha_\theta}(w)$ is continuous on $[0, \hat{w}]$ and $F'_{\theta,\alpha_\theta}(w)$ is continuous on $(\alpha_\theta, \hat{w}]$. By the definition of $\bar{\theta}$, $\bar{w}$ in (3.28) and given the fact that $\theta \geq \bar{\theta}$, we know that $F'_{\theta,\alpha_\theta}(\hat{w}) \geq -1$. Therefore, $F_{\theta,\alpha_\theta}(w) + w$ is increasing on $[0, \hat{w}]$.

We prove by contradiction. Suppose, on the contrary, there exists some $w \in (\alpha_\theta, \hat{w})$, such that $F''_{\theta,\alpha_\theta}(w) \geq 0$ (i.e., $F'_{\theta,\alpha_\theta}(w)$ is increasing at w). Define

$$w_1 := \min\{w | F''_{\theta,\alpha_\theta}(w) \geq 0 \text{ and } \alpha_\theta \leq w < \hat{w}\}.$$

Furthermore, $F'_{\theta,\alpha_\theta}(w)$ must be decreasing on some interval within $(w_1, \hat{w})$, in order for $F'_{\theta,\alpha_\theta}(w)$ to drop to -1 at $\hat{w}$. As such, define

$$w_2 := \inf\{w | F''_{\theta,\alpha_\theta}(w) < 0 \text{ and } w_1 < w < \hat{w}\}.$$

We claim that $F''_{\theta,\alpha_\theta}(w)$ must be continuous at w_2. To show this, recall that only when $\alpha_\theta = \beta$ and $F'_{\theta,\alpha_\theta}(\beta_-) > F'_{\theta,\alpha_\theta}(\beta_+)$, would $F''_{\theta,\alpha_\theta}(w)$ be discontinues at point $w = \beta$ or $w = 2\beta$. Suppose, on the contrary, $F''_{\theta,\alpha_\theta}(w)$ is not continuous at w_2, then we must have $\alpha_\theta = \beta$ and $w_2 = 2\beta$. As such, differentiating (H) would yield

$$\begin{aligned} F''_{\theta,\alpha_\theta}(\{2\beta\}_-) &= \frac{(\lambda + r - \rho)F'_{\theta,\alpha_\theta}(2\beta) - \lambda F'_{\theta,\alpha_\theta}(\beta_-)}{2\rho\beta + \beta\lambda} \\ &< \frac{(\lambda + r - \rho)F'_{\theta,\alpha_\theta}(2\beta) - \lambda F'_{\theta,\alpha_\theta}(\beta_+)}{2\rho\beta + \beta\lambda} = F''_{\theta,\alpha_\theta}(\{2\beta\}_+), \end{aligned}$$

where the inequality follows from Proposition 3.3(i), and the assumption that $F'_{\theta,\alpha_\theta}(w)$ is not continuous at β.

Because $F'_{\theta,\alpha_\theta}(w)$ is increasing on $w \in [w_1, w_2]$, we must have $F''_{\theta,\alpha_\theta}(\{2\beta\}_-) > 0$. Therefore,

$$F''_{\theta,\alpha_\theta}(2\beta_+) > F''_{\theta,\alpha_\theta}(2\beta_-) > 0,$$

which is in contradiction to the definition of w_2. Therefore, $F''_{\theta,\alpha_\theta}(w)$ must be continuous at w_2, and as a result, we must have $F''_{\theta,\alpha_\theta}(w_2) = 0$. Consequently, DDE (H) yields

$$(\rho w_2 + \beta\lambda)F'''_{\theta,\alpha_\theta}(w_2) = -\lambda F''_{\theta,\alpha_\theta}(w_2 - \beta) \le 0,$$

which implies that $F''_{\theta,\alpha_\theta}(w_2 - \beta) \ge 0$. Therefore, we must have $w_2 - \beta \ge w_1$ because $F'_{\theta,\alpha_\theta}(w)$ is decreasing for $w < w_1$. As such, $F'_{\theta,\alpha_\theta}(w)$ is increasing on $[w_2 - \beta, w_2]$. Differentiating (H) at w_2 yields

$$\lambda[F'_{\theta,\alpha_\theta}(w_2) - F'_{\theta,\alpha_\theta}(w_2 - \beta)] = (\rho - r)F'_{\theta,\alpha_\theta}(w_2) \ge 0,$$

which implies that $F'_{\theta,\alpha_\theta}(w_2) \ge 0$.

On the one hand, we have

$$\begin{aligned} \rho w_2 + \lambda\beta[F'_{\theta,\alpha_\theta}(w_2) + 1] &\le (\rho w_2 + \lambda\beta)[F'_{\theta,\alpha_\theta}(w_2) + 1] \\ &= \lambda[F_{\theta,\alpha_\theta}(w_2) - F_{\theta,\alpha_\theta}(w_2 - \beta)] + rF_{\theta,\alpha_\theta}(w_2) + \rho w_2 + \beta\lambda \\ &\le \lambda\beta F'_{\theta,\alpha_\theta}(w_2) + rF_{\theta,\alpha_\theta}(w_2) + \rho w_2 + \beta\lambda, \end{aligned} \tag{4.54}$$

where the second inequality holds because $F_{\theta,\alpha_\theta}(w)$ is convex within $w \in [w_2 - \beta, w_2]$. Rearranging inequality (4.54), yields $F_{\theta,\alpha_\theta}(w_2) \ge 0$, which leads to

$$F_{\theta,\alpha_\theta}(\hat{w}) + \hat{w} > F_{\theta,\alpha_\theta}(w_2) + w_2 \ge w_2 > 0, \tag{4.55}$$

since $F_{\theta,\alpha_\theta}(w) + w$ is increasing on $[0, \hat{w}]$.

On the other hand, Equation (U), in which $\bar{w}$ is set as $\hat{w}$, is equivalent to

$$\lambda[F_{\theta,\alpha_\theta}(\hat{w}) - F_{\theta,\alpha_\theta}(\hat{w} - \beta)] + rF_{\theta,\alpha_\theta}(\hat{w}) + \rho\hat{w} + \beta\lambda = 0.$$

Because $F_{\theta,\alpha_\theta}(\hat{w}) > F_{\theta,\alpha_\theta}(\hat{w} - \beta) - \beta$, we must have $rF_{\theta,\alpha_\theta}(\hat{w}) + \rho\hat{w} < 0$. Consequently,

$$F_{\theta,\alpha_\theta}(\hat{w}) + \hat{w} < -\frac{\rho - r}{r}\hat{w} < 0,$$

which is in contradiction to (4.55). Therefore, $F''_{\theta,\alpha_\theta}(w) < 0$ for $\forall w \in (\alpha_{\bar{\theta}}, \hat{w})$. To summarize, $F_{\theta,\alpha_\theta}(w)$ is strictly concave within $[0, \hat{w}]$. □

Proof of Proposition 3.4. (i) First note that function $F_{\bar{\theta},\alpha_{\bar{\theta}}}(w)$ is strictly concave on $w \in [0, \alpha_{\bar{\theta}})$. The concavity proof on $w \in [\alpha_{\bar{\theta}}, \bar{w}^*]$ is a special case of the concavity proof in Lemma 4.6 with $\theta = \bar{\theta}$ and $\hat{w} = \bar{w}^*$. Therefore, we omit the proof of this part to avoid redundancy.

(ii)We prove by contradiction. Suppose, on the contrary, there exists a $w \in (\alpha_{\bar{\theta}}, \bar{w}^*]$, such that

$$rF_{\bar{\theta},\alpha_{\bar{\theta}}}(w) \leq \rho w F'_{\bar{\theta},\alpha_{\bar{\theta}}}(w) - m.$$

Let

$$\hat{w} := \min\{w | rF_{\bar{\theta},\alpha_{\bar{\theta}}}(w) \leq \rho w F'_{\bar{\theta},\alpha_{\bar{\theta}}}(w) - m \text{ and } \alpha_{\bar{\theta}} < w \leq \bar{w}^*\}. \tag{4.56}$$

Therefore, we must have the following relationship,

$$rF_{\bar{\theta},\alpha_{\bar{\theta}}}(w) > \rho w F'_{\bar{\theta},\alpha_{\bar{\theta}}}(w) - m, \quad \text{for } w \in (\alpha_{\bar{\theta}}, \hat{w}). \tag{4.57}$$

For notational convenience, we let $\hat{z} := F_{\bar{\theta},\alpha_{\bar{\theta}}}(\hat{w})$. We first demonstrate that $F_{\bar{\theta},\infty}(\hat{w}) > \hat{z}$ by showing $F'_{\bar{\theta},\alpha_{\bar{\theta}}}(w) < F'_{\bar{\theta},\infty}(w)$ for all $w \in (\alpha_{\bar{\theta}}, \hat{w}]$ using contradiction. Suppose there exists $\tilde{w}$ such that

$$\tilde{w} := \min\{w | F'_{\bar{\theta},\alpha_{\bar{\theta}}}(w) \geq F'_{\bar{\theta},\infty}(w) \text{ and } \alpha_{\bar{\theta}} < w \leq \hat{w}\}.$$

On one hand, when $\alpha_{\bar{\theta}} = \beta$, from Proposition 3.3(i), we have $F'_{\bar{\theta},\alpha_{\bar{\theta}}}(\alpha_{\bar{\theta}+}) \leq F'_{\bar{\theta},\alpha_{\bar{\theta}}}(\alpha_{\bar{\theta}-})$. On the other hand, when $\alpha_{\bar{\theta}} > \beta$, from Proposition 3.3(ii), we have $F''_{\bar{\theta},\alpha_{\bar{\theta}}}(\alpha_{\bar{\theta}+}) < F''_{\bar{\theta},\alpha_{\bar{\theta}}}(\alpha_{\bar{\theta}-}) < 0$. Either of the above cases leads to the result that $F'_{\bar{\theta},\alpha_{\bar{\theta}}}(w) < F'_{\bar{\theta},\infty}(w)$ for all $w \in (\alpha_{\bar{\theta}}, \tilde{w})$, according to the definition of $\tilde{w}$. Thus, the following relationship holds as well:

$$F_{\bar{\theta},\alpha_{\bar{\theta}}}(w) < F_{\bar{\theta},\infty}(w), \quad w \in (\alpha_{\bar{\theta}}, \tilde{w}]. \tag{4.58}$$

Then considering (4.57) at $w = \tilde{w}$, we have

$$rF_{\bar{\theta},\alpha_{\bar{\theta}}}(\tilde{w}) > \rho\tilde{w}F'_{\bar{\theta},\infty}(\tilde{w}) - m = rF_{\bar{\theta},\infty}(\tilde{w}),$$

where the first inequality follows from $F'_{\bar{\theta},\alpha_{\bar{\theta}}}(\tilde{w}) \geq F'_{\bar{\theta},\infty}(\tilde{w})$ and the next equality follows from (3.17). Clearly, this is a contradiction to the relationship in (4.58). Therefore, we must have $F'_{\bar{\theta},\alpha_{\bar{\theta}}}(w) < F'_{\bar{\theta},\infty}(w)$ for all $w \in (\alpha_{\bar{\theta}}, \hat{w})$. As a result, we have $F_{\bar{\theta},\hat{w}}(\hat{w}) = F_{\bar{\theta},\infty}(\hat{w}) > \hat{z}$.

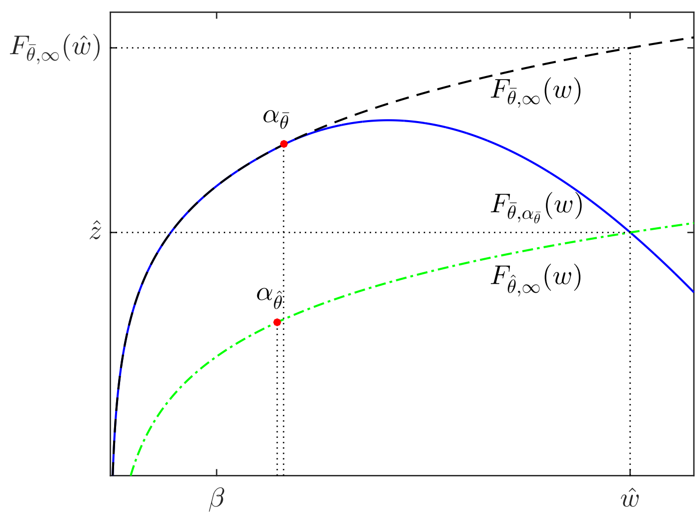


Figure 4.1: Illustration of $F_{\bar{\theta},\alpha_{\bar{\theta}}}(w)$, $F_{\bar{\theta},\infty}(w)$, and $F_{\hat{\theta},\infty}(w)$.

Let

$$\hat{\theta} := \left(\hat{z} + \frac{m}{r}\right)\hat{w}^{-\frac{r}{\rho}}.$$

Then, we must have

$$\hat{\theta} < \left(F_{\bar{\theta},\hat{w}}(\hat{w}) + \frac{m}{r}\right)w^{-\frac{r}{\rho}} = \bar{\theta}.$$

On the one hand, from Equation (4.56), we have:

$$F'_{\bar{\theta},\alpha_{\bar{\theta}}}(\hat{w}) \geq \frac{r\hat{z} + m}{\rho\hat{w}} = F'_{\hat{\theta},\hat{w}}(\hat{w}_-). \tag{4.59}$$

On the other hand, given that function $f(\alpha)$ is strictly increasing, we have $\alpha_{\hat{\theta}} < \alpha_{\bar{\theta}}$. Therefore, $\hat{w} > \alpha_{\bar{\theta}} > \alpha_{\hat{\theta}}$. By Proposition 3.3(i) we have

$$F'_{\hat{\theta},\hat{w}}(\hat{w}_+) < F'_{\hat{\theta},\hat{w}}(\hat{w}_-). \tag{4.60}$$

From DDE (H), we know that

$$\begin{aligned} F'_{\hat{\theta},\hat{w}}(\hat{w}_+) &= \frac{1}{\rho\hat{w}+\beta\lambda}\Big[(\lambda+r)F_{\hat{\theta},\hat{w}}(\hat{w}) - \lambda F_{\hat{\theta},\hat{w}}(\hat{w}-\beta)\Big], \\ F'_{\bar{\theta},\alpha_{\bar{\theta}}}(\hat{w}) &= \frac{1}{\rho\hat{w}+\beta\lambda}\Big[(\lambda+r)F_{\bar{\theta},\alpha_{\bar{\theta}}}(\hat{w}) - \lambda F_{\bar{\theta},\alpha_{\bar{\theta}}}(\hat{w}-\beta)\Big]. \end{aligned}$$

Because $F_{\hat{\theta},\hat{w}}(\hat{w}) = F_{\bar{\theta},\alpha_{\bar{\theta}}}(\hat{w})$ and $F_{\hat{\theta},\hat{w}}(\hat{w}-\beta) < F_{\bar{\theta},\alpha_{\bar{\theta}}}(\hat{w}-\beta)$, we must have

$$F'_{\hat{\theta},\hat{w}}(\hat{w}_+) > F'_{\bar{\theta},\alpha_{\bar{\theta}}}(\hat{w}). \tag{4.61}$$

Combining (4.60) and (4.61) yields,

$$F'_{\hat{\theta},\hat{w}}(\hat{w}_-) > F'_{\hat{\theta},\hat{w}}(\hat{w}_+) > F'_{\bar{\theta},\alpha_{\bar{\theta}}}(\hat{w}),$$

which contradicts to (4.59). Therefore, there does not exist a $w > \alpha_{\bar{\theta}}$ such that (4.56) holds, which completes the proof. □

To prove Theorem 3.2, we first present the following Lemma.

Lemma 4.7. *Consider a concave function $F(w)$ that satisfies* (H) *with $\alpha \geq \beta$ and* (U)*, in which $\bar{w}$ is set as $\bar{w}^*$, at $w = \bar{w}^* \geq \alpha$, and is linear with slope -1 on $[\bar{w}^*, \infty)$. For any $w \geq \bar{w}^*$,*

$$\psi(w) := \lambda F(w-\beta) - (\lambda+r)F(w) - (\rho w + \lambda\beta) \leq 0. \tag{4.62}$$

Proof. Taking the first derivative of $\psi(w)$ yields

$$\begin{aligned} \psi'(w) &= \lambda F'(w-\beta) - (\lambda+r)F'(w) - \rho \\ &= \lambda F'(w-\beta) + \lambda + r - \rho \\ &\leq \lambda F'(\bar{w}-\beta) + \lambda + r - \rho, \end{aligned}$$

where the second equality follows from $F'(w) = -1$ for $w \geq \bar{w}^*$ and the last inequality follows from concavity of $F(w)$. Condition (H) implies that

$$\lambda F'(\bar{w}^*-\beta) = (\lambda+r-\rho)F'(\bar{w}^*) - (\rho\bar{w}^* + \beta\lambda)F''(\bar{w}^*_+).$$

Note that from the definition of $\bar{w}^*$ in (3.28), we have

$$\begin{cases} F''(\bar{w}^*_+) > 0, & \text{if } F''(\bar{w}^*_-) < F''(\bar{w}^*_+), \\ F''(\bar{w}^*_+) = 0, & \text{if } F''(\bar{w}^*_-) = F''(\bar{w}^*_+). \end{cases} \tag{4.63}$$

Therefore,

$$\psi'(w) \le (\lambda + r - \rho)[1 + F'(\bar{w}^*)] - (\rho\bar{w}^* + \beta\lambda)F''(\bar{w}^*_+) \le 0,$$

where the last inequality follows (4.63). Therefore, $\psi(w)$ is decreasing in $w \in [\bar{w}^*, \infty)$. As such, for any $w \ge \bar{w}^*$,

$$\psi(w) \le \psi(\bar{w}^*) = \lambda F(\bar{w}^* - \beta) - (\lambda + r)F(\bar{w}^*) - (\rho\bar{w}^* + \lambda\beta) \ = 0,$$

where we have used equality (U). This completes the proof. □

Proof of Theorem 3.2. The proof is parallel to those of Proposition 3.2 and Theorem 3.1. Note that the proof for $F(w^*) = U(\Gamma_d(w^*; \alpha_{\bar{\theta}}, \bar{w}^*))$ is exactly the same as that of Proposition 3.2 under contract $\Gamma_d(w^*; \beta, \bar{w})$ except that the switching point is $\alpha_{\bar{\theta}}$ instead of β. We omit the proof of this part to avoid redundancy. In the following we only show that $U(\Gamma) \le F(w)$ for any incentive compatible contract Γ.

From Proposition 3.4, we know that $F(w)$ is concave and $F'(w) \ge -1$. Therefore, we only need to show $\Psi_t \le 0$ holds almost surely (recall Lemma 3.3).

Recall the inequality (4.47). We consider the following four cases. (i) When $W_t < \beta$, we know that the principal monitors the agent (i.e., $m_t = m$). Following (3.17), we have

$$\Psi_t \le \rho W_t F'(W_t) - rF(W_t) - m = 0.$$

(ii) When $\beta \le W_t \le \alpha_{\bar{\theta}}$, substituting (3.17) into inequality (4.47) yields

$$\Psi_t \quad \le \quad m - m_t + \lambda\Phi\Big(W_t, y_t H^s_t + (1 - y_t)H^n_t\Big) - \lambda F(W_t).$$

- If the principal does not monitor at time t (i.e., $m_t = 0$), condition (IC) indicates that $y_t H^s_t + (1 - y_t)H^n_t \ge \beta$. Following Lemma 4.4, we have,

$$\begin{aligned}\Psi_t \quad &\le \quad m - m_t + \lambda\Phi\Big(W_t, \beta\Big) - \lambda F(W_t) \\ &= \quad m + \lambda\bar{\theta}W_t^{\frac{r}{\rho}}\Big[\Big(\frac{\beta r}{\rho W_t} - 1\Big) + \Big(1 - \frac{\beta}{W_t}\Big)^{\frac{r}{\rho}}\Big] = f(W_t) < 0,\end{aligned}$$

  in which we have used Equation (L) and the fact that $f(W_t)$ is increasing with $f(\alpha_{\bar{\theta}}) = 0$.

- If the principal conducts monitoring at time t (i.e., $m_t = m$), considering Lemma 4.4, inequality (4.47) yields

$$\Psi_t \quad \leq \quad \lambda\Phi(W_t, 0) - \lambda F(W_t) = \lambda F(W_t) - \lambda F(W_t) = 0.$$

(iii) When $\alpha_{\bar{\theta}} < W_t < \bar{w}^*$, substituting (H) into inequality (4.47) yields

$$\Psi_t \quad \leq \quad \lambda\Phi\Big(W_t, y_t H_t^s + (1-y_t)H_t^n\Big) - \lambda\beta F'(W_t) - \lambda F(W_t - \beta) - m_t.$$

- If the principal does not monitor at time t (i.e., $m_t = 0$), we must have $y_t H_t^s + (1-y_t)H_t^n \geq \beta$. Lemma 4.4 implies

$$\Psi_t \quad \leq \quad \lambda\Phi(W_t, \beta) - \lambda\beta F'(W_t) - \lambda F(W_t - \beta) = 0.$$

- If the principal conducts monitoring at time t (i.e., $m_t = m$), Lemma 4.4 implies

$$\begin{aligned} \Psi_t \quad &\leq \quad \lambda\Phi(W_t, 0) - \lambda\beta F'(W_t) - \lambda F(W_t - \beta) - m \\ &= \quad -rF(W_t) + \rho W_t F'(W_t) - m \;<\; 0, \end{aligned}$$

where the last inequality follows from Proposition 3.4(ii).

(iv) When $W_t \geq \bar{w}^*$, we must have $F'(W_t) = -1$, and inequality (4.47) reduces to

$$\Psi_t \quad \leq \quad \lambda\Phi\Big(W_t, y_t H_t^s + (1-y_t)H_t^n\Big) - \rho W_t - (\lambda + r)F(W_t) - m_t.$$

- If the principal does not monitor at time t (i.e., $m_t = 0$), condition (IC) and Lemma 4.4 imply

$$\Psi_t \quad \leq \quad \lambda\Phi(W_t, \beta) - \rho W_t - (\lambda + r)F(W_t) = \psi(W_t) \leq 0.$$

- If the principal conducts monitoring at time t (i.e., $m_t = m$), Lemma 4.4 implies

$$\begin{aligned} \Psi_t \quad &\leq \quad \lambda\Phi(W_t, 0) - \rho W_t - (\lambda + r)F(W_t) - m \\ &= \quad -rF(W_t) - \rho W_t - m \\ &\leq \quad -rF(\alpha_{\bar{\theta}}) - \rho\alpha_{\bar{\theta}} - m, \end{aligned}$$

where the second inequality holds because $-rF(w) - \rho w$ is decreasing in w. As such, by considering Equation (3.17), we have

$$\Psi_t \quad \leq \quad -\rho\alpha_{\bar{\theta}}[1 + F'(\alpha_{\bar{\theta}-})] \leq 0.$$

To summarize, we know that $\Psi_t \le 0$ holds for all the possible cases. This completes the proof. $\square$

Proof of Proposition 3.5. First, following DDE (H), for any θ, we have the following closed-form solution for $F_\theta(w)$ for $w \in [\beta, 2\beta]$:

$$F_\theta(w) \quad = \quad \mathcal{K}_\theta\Big(\rho w + \beta\lambda\Big)^{\frac{\lambda+r}{\rho}} - \frac{\lambda m}{r(\lambda+r)} + \frac{\theta\lambda\Big[\beta(\rho-r) + (\lambda+r)w\Big]}{(\lambda+r)(\lambda+r-\rho)}, \tag{4.64}$$

where the θ-dependent parameter $\mathcal{K}_\theta$ is defined as

$$\mathcal{K}_\theta = -\frac{1}{\lambda+r}\Big[m + \frac{\theta\beta(\rho-r)(2\lambda+r)}{\lambda+r-\rho}\Big]\Big(\beta(\rho+\lambda)\Big)^{-\frac{\lambda+r}{\rho}}. \tag{4.65}$$

As such, when

$$\theta \ge \underline{\theta} := -\frac{\lambda+r-\rho}{\lambda},$$

we have

$$\mathcal{K}_\theta\Big(\beta(\rho+\lambda)\Big)^{\frac{\lambda+r}{\rho}} \quad \le \quad -\frac{1}{\lambda+r}\Big[m - \frac{(\rho-r)\beta(2\lambda+r)}{\lambda}\Big] < 0, \tag{4.66}$$

where the second inequality holds because $m > \hat{m}$.

Second, we show that for any $w \ge 0$, derivative $F'_\theta(w)$ is increasing in θ. To do so, consider the following decomposition of $F_\theta(w)$,

$$F_\theta(w) = \theta G(w) - \frac{m}{r}J(w),$$

where function $G(w)$ satisfies DDE (H) with boundary condition $G(w) = w$ for all $w \in [0, \beta)$, and function $J(w)$ is defined in Lemma 3.5. Note that because $G'(w) > 0$ for $w \in (0, \beta)$, Lemma 4.2 implies that $G'(w) > 0$ for all $w \in (\beta, \infty)$, which further implies that

$$\frac{\partial F'_\theta(w)}{\partial \theta} = G'(w) > 0.$$

On the one hand, when $\theta = 0$, it is clear that

$$F_0(w) = -\frac{m}{r}J(w), \text{ for } w \ge 0,$$

and, therefore,

$$\inf_{w\ge\beta} F'_0(w) = -\frac{m}{r}\sup_{w\ge\beta} J'(w_+) \ge -\frac{\bar{m}}{r}\sup_{w\ge\beta} J'(w_+) = -1,$$

where the inequality holds because $J(w)$ is nondecreasing and $m \leq \bar{m}$.

On the other hand, when $\theta = \underline{\theta}$, (4.64) implies that

$$F'_{\underline{\theta}}(\beta_+) = \mathcal{K}_{\underline{\theta}}(\lambda + r)(\rho\beta + \lambda\beta)^{\frac{\lambda+r}{\rho}-1} - 1 < -1,$$

where the inequality holds because $\mathcal{K}_{\underline{\theta}} < 0$. Therefore,

$$\inf_{w>\beta} F'_{\theta}(w) \leq F'_{\theta}(\beta_+) < -1.$$

As such, there exists a unique $\bar{\theta} \in (\underline{\theta}, 0]$, such that $\inf_{w>\beta} F'_{\bar{\theta}}(w) = -1$.

From (4.66), we know that $\mathcal{K}_{\bar{\theta}} < 0$. Given that $r > \rho - \lambda$ when $m \in [\hat{m}, \bar{m})$, $F_{\bar{\theta}}(w)$ is strictly concave within $[\beta, 2\beta]$, i.e., $F'_{\bar{\theta}}(w)$ is strictly decreasing in $(\beta, 2\beta]$. By the definitions of $\bar{\theta}$ and $\bar{w}$, we must have $\bar{w} \geq 2\beta$. This completes the proof. □

Proof of Theorem 3.3. We first remark that function $F(w)$ is concave. This is obviously true if $m \in [0, \underline{m})$ or $m \in [\underline{m}, \hat{m})$. For $m \in [\hat{m}, \bar{m})$, the concavity proof of $F(w)$ is similar to those of Proposition 3.1 and Proposition 3.4, except the slight difference in showing that $F'(\beta_-) \geq F'(\beta_+)$. We omit the detailed proof to avoid redundancy.

Next, we show that $U(\Gamma) \leq F(w)$ for any incentive compatible contract Γ and $\forall w \geq 0$. To do so, we only need to show $\Psi_t \leq 0$ holds almost surely (recall Lemma 3.3). Given that $F(w)$ takes different forms, depending on the value of m, we consider three cases in the following.

Case (a) If $m \in [0, \underline{m})$, given that $F(w)$ is linear and $F'(w) = -1$ for all $w \geq 0$ [recall Equation (3.32)], we have

$$\Psi_t = -\rho W_t - rF(W_t) - m_t = -(\rho - r)W_t + (m - m_t).$$

Therefore:

(a.1) If $m_t = m$, we have $\Psi_t \leq 0$ because $\rho \geq r$.

(a.2) If $m_t = 0$, Corollary 3.1 implies $W_t \geq \beta$; consequently,

$$\Psi_t = -(\rho - r)W_t + m \leq -(\rho - r)\beta + m \leq 0,$$

where the inequality holds because $m \leq \underline{m}$.

Therefore, $\Psi_t \leq 0$ holds almost surely. This completes the proof for $m \in [0, \underline{m})$.

Case (b) If $m \in [\underline{m}, \hat{m})$, recall that $F(w)$ takes the piece-wise linear form of (3.33). We consider the following two cases.

(b.1) When $W_t < \beta$, the principal monitors the agent (i.e., $m_t = m$). Following inequality (4.47), we have

$$\begin{aligned}\Psi_t &\leq \rho W_t F'(W_t) - rF(W_t) - m \\ &= -\rho W_t \Big[1 - \frac{(\rho - r)\beta - m}{(\lambda + r)\beta}\Big] + r\Big[1 - \frac{(\rho - r)\beta - m}{(\lambda + r)\beta}\Big] W_t \\ &= -(\rho - r) W_t \Big[1 - \frac{(\rho - r)\beta - m}{(\lambda + r)\beta}\Big] \leq 0,\end{aligned}$$

where the second inequality holds because $m \geq \underline{m}$.

(b.2) When $W_t \geq \beta$, we have $F'(W_t) = -1$. If the principal does not monitor at time t (i.e., $m_t = 0$), considering the (IC) condition and Lemma 4.7, we have,

$$\begin{aligned}\Psi_t &\leq \lambda \Phi(W_t, \beta) + F'(W_t)\rho W_t - (\lambda + r)F(W_t) \\ &= \lambda F(W_t - \beta) - (\lambda + r)F(W_t) - (\rho W_t + \beta\lambda) = \psi(W_t),\end{aligned}$$

where function $\psi(w)$ is decreasing in $w \in [\beta, \infty)$. As such, we have

$$\Psi_t \leq \psi(\beta) = \lambda F(0) - (\lambda + r)F(\beta) - (\rho\beta + \beta\lambda) = 0.$$

If the principal conducts monitoring at time t (i.e., $m_t = m$), Lemma 4.4 implies that

$$\begin{aligned}\Psi_t &\leq \rho W_t F'(W_t) - rF(W_t) - m \\ &= -\rho W_t - rF(W_t) - m \\ &\leq -\rho\beta - rF(\beta) - m < 0.\end{aligned}$$

In summary, we always have $\Psi_t \leq 0$, which completes the proof for $m \in [\underline{m}, \hat{m})$.

Case (c) If $m \in [\hat{m}, \bar{m})$, we consider the following three cases.

(c.1) When $W_t < \beta$, the principal must monitor the agent (i.e., $m_t = m$). From inequality (4.47) we have

$$\begin{aligned}\Psi_t &\leq \rho W_t F'(W_t) - rF(W_t) - m \\ &= \rho W_t \bar{\theta} - r\Big(\bar{\theta} W_t - \frac{m}{r}\Big) - m = \bar{\theta}(\rho - r) W_t \leq 0,\end{aligned}$$

where we have used Equation (L_l) and the fact that $\bar{\theta} \leq 0$.

(c.2) When $\beta \leq W_t < \bar{w}^*$, substituting Equation (H) into inequality (4.47) yields

$$\Psi_t \leq \lambda\Phi\Big(W_t, y_t H_t^s + (1-y_t)H_t^n\Big) - \lambda\beta F'(W_t) - \lambda F(W_t - \beta) - m_t.$$

If the principal does not monitor at time t (i.e., $m_t = 0$), we must have $y_t H_t^s + (1-y_t)H_t^n \geq \beta$. By Lemma 4.4, we have,

$$\Psi_t \leq \lambda\Phi(W_t, \beta) - \lambda\beta F'(W_t) - \lambda F(W_t - \beta) = 0.$$

If the principal conducts monitoring at time t (i.e., $m_t = m$), by Lemma 4.4, we have

$$\begin{aligned}\Psi_t &\leq \lambda\Phi(W_t, 0) - \lambda\beta F'(W_t) - \lambda F(W_t - \beta) - m \\ &= -rF(W_t) + \rho W_t F'(W_t) - m \\ &\leq -rF(\beta) + \rho\beta F'(\beta_+) - m \\ &\leq -rF(\beta) + \rho\beta F'(\beta_-) - m \leq 0.\end{aligned}$$

(c.3) When $W_t \geq \bar{w}^*$, we must have $F'(W_t) = -1$, and inequality (4.47) reduces to

$$\Psi_t \leq \lambda\Phi\Big(W_t, y_t H_t^s + (1-y_t)H_t^n\Big) - \rho W_t - (\lambda + r)F(W_t) - m_t.$$

If the principal does not monitor at time t (i.e., $m_t = 0$), by (IC) condition and Lemma 4.4, we have,

$$\begin{aligned}\Psi_t &\leq \lambda\Phi(W_t, \beta) - \rho W_t - (\lambda + r)F(W_t) \\ &= \lambda F(W_t - \beta) - (\lambda + r)F(W_t) - (\rho W_t + \beta\lambda) \leq 0,\end{aligned}$$

where the last inequality follows from Lemma 4.4.

If the principal conducts monitoring at time t (i.e., $m_t = m$), by Lemma 4.4, we have

$$\begin{aligned}\Psi_t &\leq \lambda\Phi(W_t, 0) - \rho W_t - (\lambda + r)F(W_t) - m \\ &= -rF(W_t) - \rho W_t - m \\ &\leq -rF(\beta) - \rho\beta - m,\end{aligned}$$

where the second inequality holds because $-rF(w) - \rho w$ is decreasing in w. As such, considering (L_l), we have

$$\begin{aligned}\Psi_t &\leq -r\Big(\bar{\theta}\beta - \frac{m}{r}\Big) - \rho\beta - m \\ &\leq -r\beta\frac{\lambda + r - \rho}{\lambda} - \rho\beta \\ &= -(\rho - r)\frac{(r+\lambda)\beta}{\lambda} < 0,\end{aligned}$$

in which the second inequality holds because $\bar{\theta} \geq -1 + (\rho - r)/\lambda$.

In summary, for any contract Γ, we have $\Psi_t \leq 0$ almost surely. This completes the proof. □

4.2.4 Proofs in Section 3.5

Proof of Proposition 3.6. (i) From Lemma 4.5(i), we know that if $r \leq \rho - \lambda$, $J'(w)$ is decreasing on $[\beta, \infty)$. Therefore,

$$\bar{m} = \frac{r}{J'(\beta_+)} = (\rho + \lambda)\beta.$$

From Lemma 4.5(ii), we know that if $\rho - \lambda < r < \bar{r}$, $J'(w)$ is increasing on $[\beta, 2\beta]$ and decreasing on $[2\beta, \infty)$. Therefore,

$$\bar{m} = \frac{r}{J'(2\beta)} = \frac{[(\rho+\lambda)\beta]^{\frac{\lambda+r}{\rho}}}{[(2\rho+\lambda)\beta]^{\frac{\lambda+r}{\rho}-1}}.$$

This completes the proof. □

Proof of Proposition 3.7. This proposition follows the same logic as Lemma 4.6. Therefore, the proof is omitted. □

4.2.5 Further discussions

Fixed Cost to Start Monitoring

In our paper we did not consider a fixed cost of start monitoring. If such a fixed cost exists, we expect that there are two different thresholds. When the promised utility falls below the lower threshold, monitoring starts. And monitoring stops after the promised utility increases to be above the higher threshold. The intuition for such a "control band" structure is similar to the (s, S) policy in inventory control with fixed ordering cost.

General Cost of Arrival

When we introduce the model, we claim our results extend naturally to the case of random cost for each arrival, as long as the random cost is not associated with the effort process. In this case we just need to use K to represent the mean cost per arrival. More generally, however, the effort process may affect the random cost. For example, the agent's effort may affect not only the rate of arrival, but also the distribution of the cost. In this case, the optimal contract needs to take advantage of the information contained in the magnitude of the cost of an arrival. Such a setting is much more complex than the one studied in this paper. Even without monitoring, the dynamic contracting problem with multiple signal types has not been well understood. We suspect that the general optimal contract could be so complex such that a fruitful way to proceed is to explore approximations of the optimal contracts that are easy to compute and implement.

Another way of thinking about arrivals is not to consider them simply as arrivals, as we do in this paper, but as breakdowns of a production process (machine). That is, the agent is a maintenance team, whose effort reduces the arrival rate of breakdowns. The cost therefore corresponds to the lost revenue when the machine is down. The breakdown time can be random. Even without monitoring, such a model has not been studied in the literature, and one of the authors of this paper has been working on a related problem in an on-going research project. It would be interesting to consider combining such a model with monitoring as a potential future research direction.

Agent More Patient than Principal

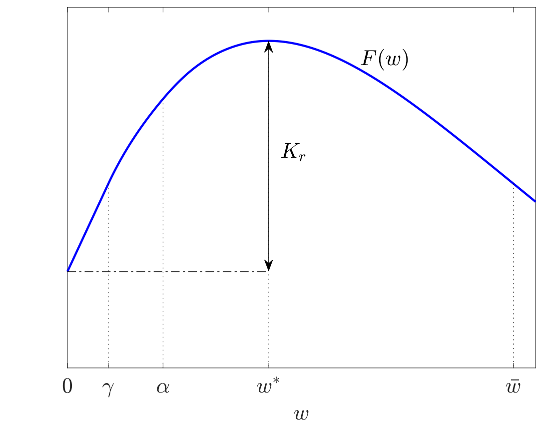
Following long traditions of the dynamic contracting literature, we assume that the principal is no less patient than the agent. This is true in most practical settings, where the principal, as the contract designer, often possesses more resources than the agent. One may wonder what happens if the agent is more patient than the principal, or $\rho < r$. In this case, delaying payments is even more beneficial to the principal, because the interest the principal collects during the delay is higher than what is demanded by the agent. As a result, we believe that one still need to introduce the exogenous upper bound on the promised utility to make sure that an optimal solution exists. In order to prove optimality, we still need to establish concavity of the value function. This appears to be quite challenging when $\rho < r$. Neither the proof techniques in this paper nor the ones in [BMRV10] work when $\rho < r$. We suspect one needs to carefully go through a discrete time model and follow a proof logic of [BMPR07] and [ST17] to show that uniform convergence between the discrete time value function and the continuous time one. Given the length and complexity of this paper, and the relative lack of practical motivation for this technically interesting setting, we consider it outside the scope of this paper.

Imperfect Monitoring

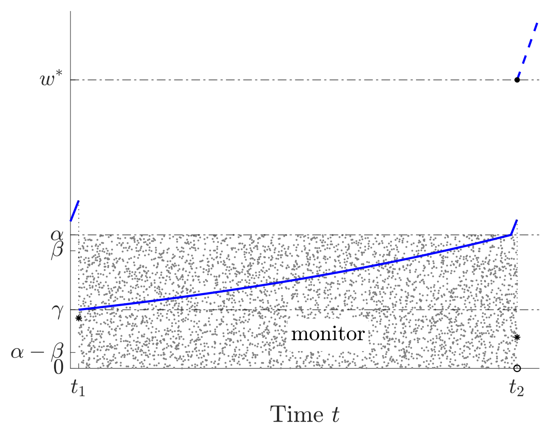
Monitoring in our setting can be perceived as another signal on the agent's effort level that the principal can pay to obtain, besides the arrivals. In this paper we assume that this signal perfectly reflects the effort level. More generally, one can consider imperfect monitoring. That is, the agent's effort changes the statistic of another stochastic process, which is observable to the principal only if the principal pays for it. For example, in the quality control setting mentioned throughout the paper, the arrivals represent customer complains. The principal may choose to monitor by conducting costly customer surveys, and collect praises from customers. The arrivals of praises constitutes the second Poisson process that is observable to the principal only if the principal pays for this information. Our model sheds light on tackling more complex and general incentive systems such as these.

Opportunity of Replacing Agents

In certain practical situations, the principal has the opportunity of replacing an agent with a cost K_r. Intuitively, if such a cost is relatively low, the principal may prefer such an option over monitoring the agent for a long period of time (when the promised utility drops too low). The general idea of replacing agent in other dynamic contracting settings has been discussed in details in [Mye15] and Section 5.2 of [ST17]. Here we provide a description for the case with different discount rates and a high monitoring cost using an example, and leave the complete and detailed results for interested readers to work out.



(a) Function $F(w)$



(b) A Sample Trajectory of W_t

Figure 4.2: Value Function and Sample Trajectory with Agent Replacement

Figure 4.2 provides an example of the value function and a corresponding trajectory considering agent replacement. In Figure 4.2(a), there is an additional threshold γ, compared with Figure 3.4(a). Smooth pasting works again at this threshold. That is, the value function's left and right derivatives are the same at γ. Furthermore, the value function is linear on $[0, \gamma]$, which implies that upon an arrival that decreases an agent's promised utility below γ, the principal immediately randomly reset the promised utility to either 0 (replacing the agent with a new one) or γ (continue monitoring the current agent). Recall that a new contract starts at promised utility w^*, which maximizes the value function F. Therefore, in Figure 4.2 (b), we have $F(w^*) - F(0) = K_r$.

Figure 4.2 (b) provides a partial sample trajectory of the agent's promised utility. Arrivals occurring at time t_1 and t_2 bring the promised utility below the threshold γ. At time t_1, randomization takes the promised utility to γ so the contract continues with a monitoring session. Randomization at time t_2, on the other hand, brings the promised utility to 0 for the current agent, and triggers replacement. The new agent's promised utility starts at w^* and follows the trajectory of the dashed curve in the end.

Chapter 5

Conclusion

This dissertation considers optimization problems in stochastic matching and dynamic incentive management problems. In each application, we develop analytical methods to solve the problem and explain managerial insights beneath the solutions. The topics discussed in this dissertation serve as motivations for me to continue exploring related fields. It is also my hope that the techniques developed in this dissertation shall be applied in a wide range of applications.

Endnotes

1. We show that the stationary distribution always exists in Appendix 4.1.1.

2. For any two random variables X and Y with cumulative probability distributions F_X and F_Y respectively, X first-order stochastically dominates X, written $X \succeq_1 Y$, if and only if $F_X(k) \leq F_Y(k)$ for all k.

3. Assuming a constant cost K for each arrival is for simplicity of exposition. Our results naturally extend to the case where the cost of each arrival is a random variable and K is its mean, as long as this random cost is independent to the effort process.

4. The commitment power assumption follows a long tradition of dynamic contracting literature, [BMRV10, Mye15, San08]. Without this assumption, one may have to use the subgame perfect equilibrium concept, which is very hard to describe. In many practical circumstances in which the principal has much more power and resources than the agent, this is a reasonable assumption.

5. Note that we have considered a very general class of feasible contracts in our models. Mathematically speaking, one may generalize the contract space even more by introducing a switch control at a non-arrival time with a certain probability. Such a control is essentially 'adding points' following the terminology of [Bré81], which involves mathematical tools that are beyond [Bré81]. In particular, it would invoke control of "piecewise-deterministic-Markov-processes" (PDP), a much more sophisticated mathematical framework introduced in [Dav84]. Even if we generalize the class of contract this way, the current optimal contracts remain optimal. Therefore, it is not necessary to introduce additional mathematical complexities with no practical benefits.

6. Without the limited liability constraint, even if the agent cannot buy out the entire enterprise, the principal can simply charge the agent a cash amount of β to induce effort. Therefore, the limited

liability constraint prevents the model from becoming trivial.

7. If $\bar{w} < \beta$, the only (IC) contract is to always monitor the agent.

8. Note that smooth pasting does not arise in Section 3.3 for the equal discount case, because the optimal monitoring threshold is always β, while in the different discount case, the threshold can be higher than β.

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