The spherical manifold realization problem

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Abstract

The Lickorish-Wallace theorem states that any closed, orientable, connected 3-manifold can be obtained by integral Dehn surgery on a link in S^3. The spherical manifold realization problem asks which spherical manifolds (i.e., those with finite fundamental groups) can be obtained through integral surgery on a knot in S^3. The problem has previously been solved by Greene [Gre13] and Ballinger et al. [Bal+16] for lens space and prism manifolds, respectively. In this project, we determine which of the remaining three types of spherical manifolds (tetrahedral, octahedral, and icosahedral) can be obtained by positive integral surgery on a knot in S^3. We follow methods inspired by those presented by Greene [Gre13].

1 Introduction and previous results

One of the most basic characterizations of manifolds is by fundamental group. In this paper, we explore spherical manifolds in particular, those 3-manifolds with finite fundamental groups, through the lens of Dehn surgery. One open question about these 3-manifolds is the spherical manifold realization problem.

Question 1. Which spherical manifolds can be realized by performing nontrivial integral Dehn surgery on a knot in S^3?

An important result to understanding this question is the following theorem from [Lic62] and [Wal60]:

Theorem 1.1 (Lickorish-Wallace Theorem). Every closed, orientable, connected 3-manifold can be realized by integral Dehn surgery on a link in S^3.

Five families of spherical manifolds exist: lens spaces, prism manifolds, tetrahedral manifolds, octahedral manifolds, and icosahedral manifolds, the latter three named for the structure of their fundamental groups and abbreviated **T**-, **O**-, and **I**-type manifolds. Greene [Gre13] and Ballinger et al. [Bal+16] have provided analyses of the spherical manifold realization problem for lens spaces and prism manifolds, respectively. Gu [Gu14] provided preliminary solutions for the remaining three types of spherical manifolds (tetrahedral, octahedral, and icosahedral) using a Floer homological approach. Our study uses an alternate combinatorial approach to solve the problem for these latter three types of manifolds.

Our main result is as follows.

Theorem 1.2. If p, q are relatively prime negative integers and $Y(p, q)$ is a **O**-type manifold, **T**-type manifold, or **I**-type manifold obtained by positive integral Dehn surgery on a knot in S^3, then the pair (p, q) is one of those listed in Tables 2, 3, and 4, respectively.

Remark. While each entry in Tables 2-4 necessarily corresponds to a spherical 3-manifold obtained by surgery on a knot in S^3, these correspondences are not necessarily one-to-one; multiple (p, q) may correspond to the same manifold.

We take an approach inspired by [Gre13] and [Bal+16], applying changemaker lattices, a combinatorial tool described in more detail in section 3. As discussed in section 2, if $Y(p, q)$ is a spherical 3-manifold, then $Y = \partial X$, where X is negative definite. This allows us to, with some work, associate to Y a negative Euclidean integer lattice $\mathbb{Z}^{m+n+4}$, where m and n are specified by the Y's link surgery coefficients. From there, we apply some results of [Gre13] to determine the list of possible (p, q) referenced in Theorem 1.2. Some questions arise from these results. In particular, our work focuses on positive integral surgery (surgery in which $p, q < 0$), so one future direction will be to investigate negative integral surgery (when p or $q > 0$), which should be achievable using methods similar to those presented in this paper. Moreover, we must determine whether the list of (p, q) we found can be further refined. This can be done by individually examining invariants the knots determined by those (p, q).

1.1 Organization of the paper

In section 2, we provide background on spherical manifolds and Dehn surgery. In section 3, we explore changemaker lattices, an important combinatorial tool we use to analyze the manifolds in this paper. Section 4 describes our main methods and results.

1.2 Acknowledgements

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2 Spherical manifolds and Dehn surgery

To unpack our results, we first provide some details and definitions. This paper focuses on surgery producing spherical manifolds.

Definition 2.1. If $Y(p,q)$ is a 3-manifold and takes the form S^3/G, where $G \leq SO(4)$ acts on S^3, and $Z(G) = \pi_1(M)$ is finite, then Y is a *spherical* (or *elliptic*) 3-manifold.

There are five main families of spherical 3-manifolds, classified by the structure of their fundamental groups: prism manifolds, lens spaces, octahedral (**O**-type) manifolds, tetrahedral (**T**-type) manifolds, and icosahedral (**I**-type) manifolds. We focus on the latter three types as solutions already exist for the former two.

Definition 2.2. M is said to be an ***O**-type* manifold if

$$\pi_1(M) \cong \langle x, y \mid (xy)^2 = x^3 = y^4 \rangle.$$

Definition 2.3. M is said to be a ***T**-type* manifold if

$$\pi_1(M) \cong \langle x, y, z \mid (xy)^2 = x^2 = y^2, zxz^{-1} = y, zyz^{-1} = xy, z^{3^k} = 1 \text{ for some } k \in \mathbb{Z}_+ \rangle.$$

Definition 2.4. M is said to be an ***I**-type* manifold if

$$\pi_1(M) \cong \langle x, y \mid (xy)^2 = x^3 = y^5 \rangle.$$

The Lickorish-Wallace Theorem tells us that all spherical manifolds can be obtained by integral Dehn surgery on a framed link in S^3. Moreover, these manifolds can be viewed as Seifert fibered spaces, which are decompositions of the manifold into disjoint unions of fibers with certain conditions on them. In this way, we can represent spherical manifolds as a sequence of integers, the Seifert invariants: if $Y(p,q)$ is an oriented spherical manifold, then we have $(-1;(2,1),(a,p),(b,q))$ as the Seifert invariants. In the **O**-type case, $a = 3$ and $b = 4$; in the **T**-type case, $a = b = 3$, and in the **I**-type case, $a = 3$ and $b = 5$. Thus, we have the surgery diagram for a spherical manifold as in Figure 1.

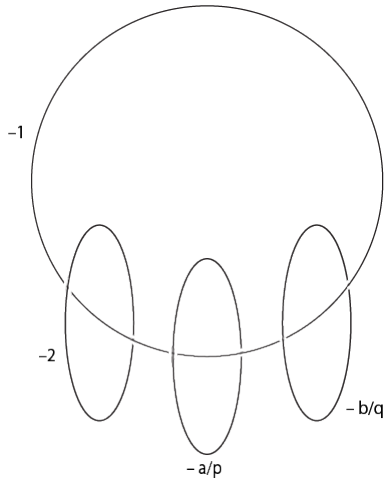

Figure 1: Surgery diagram for an orientable spherical manifold with Seifert invariants $(-1;(-2,1),(-a,p),(-b,q))$

3 Changemaker lattices

We now discuss changemaker lattices, which play an important role in our methods. If $Y(p,q)$ is a spherical 3-manifold of type **O**, **T**, or **I** with $p,q<0$, then Y bounds a smooth, negative definite 4-manifold $Y(p,q)$. See section 4 for a proof. Moreover, we apply [Gre13](§2) and [Bal+16](§1) to see that if $Y(p,q)$ is realizable by integral surgery on a knot $K\subset S^3$, then Y bounds a 2-handle cobordism $W(K)$ so that $A:=X(p,q)\cup -W(K)$ is a smooth, compact, negative definite 4-manifold. We now state a theorem of Donaldson.

Theorem 3.1 (Donaldson's Theorem A). If the definite intersection form of a compact, orientable, smooth 4-manifold is negative definite, then it is diagonalizable over the integers [Don87].

As A satisfies the conditions in the theorem, we have that the intersection pairing on $H_2(A;\mathbb{Z})$ is isomorphic to $-\mathbb{Z}^m$, where m is the second Betti number of A. Then, if we let $\Delta(p,q)$ denote the negative intersection pairing, we have that $\Delta(p,q)$ embeds into $\mathbb{Z}^m$ as a sublattice with codimension 1.

Definition 3.1. A vector $\sigma=(\sigma_0,\dots,\sigma_n)\in\mathbb{Z}^{n+1}$ with $\sigma_0\le\cdots\le\sigma_n$ is a *changemaker vector* if for every $0<k\le\sigma_0+\sigma_1+\cdots+\sigma_n$, there exists $S\subset\{0,\dots,i-1\}$ such that $k=\sum_{s\in S}\sigma_s$ [Gre13](Def. 1.5).

As shown in [Gre12](Thm. 3.3), if $Y(p,q)$, a spherical 3-manifold, is the result of integral surgery on a knot $K\subset S^3$, then $\Delta(p,q)$ embeds into $\mathbb{Z}^m$ as $(\sigma)^\perp\subset\mathbb{Z}^m$, where σ is a changemaker vector.

Definition 3.2. If σ is a changemaker vector, then $(\sigma)^\perp\subset\mathbb{Z}^{n+1}$ is called a *changemaker lattice*.

It is shown in [Gre13](Def. 3.11) that we can find a basis for a changemaker lattice $(\sigma)^\perp$ given by three types of vectors, called *tight*, *gappy*, and *just right*.

Definition 3.3. Let v_i be a basis element of $(\sigma)^\perp$. Then v_i is called *tight* if

$$v_i=-e_i+e_{i-1}+e_{i-2}+\cdots+e_1+2e_0,$$

where the e_js are the standard basis vectors for $\mathbb{Z}^{n+1}$. We call v_i *gappy* if

$$v_i=-e_i+e_{i-1}+\cdots+e_{j+1}+e_{j-1}+\cdots+e_k,$$

where the index j is called a gappy index, $k\ge 0$, and there may be multiple gappy indices. Finally, v_i is *just right* if

$$v_i=-e_i+e_{i-1}+e_{i-2}+\cdots+e_k,$$

where $k>0$ and no indices between $i-1$ and k are skipped.

In [Gre13], Greene provides a number of restrictions on these basis vectors that allow us to rule out a number of cases in our analysis. To determine which particular values of p and q produce a graph lattice isomorphic to a changemaker lattice, we must provide some preliminary definitions and propositions related to the structure of lattices in general.

Proposition 1. If a changemaker lattice contains a tight vector, then it is unique. Moreover, this tight vector must have norm ≥ 3 [Gre13](Lem. 4.2).

Definition 3.4. A vector v is said to be *reducible* if there exist x,y with $\langle x,y\rangle\ge 0$ such that $v=x+y$, and *irreducible* otherwise. If $v=x+y$ with $|x|,|y|\ge 3$ and $\langle x,y\rangle=-1$, then v is said to be *breakable*; otherwise, v is *unbreakable*. As in [Gre13](Lem. 3.15), if v is a breakable basis vector, then it must be tight.

Definition 3.5. If $\mathcal{B}=\{v_1,\dots,v_n\}$ is a basis for a lattice Λ, we define the *pairing graph* of Λ to be $G(\mathcal{B})=(\mathcal{B},E)$, where $(v_i,v_j)\in E$ iff $v_i\cdot v_j\neq 0$ with $i\neq j$.

Definition 3.6. A set of three distinct basis vectors $v_1,v_2,v_3\in\mathcal{B}$ such that the pairwise dot products are nonzero form a *heavy triple* if $|v_1|,|v_2|,|v_3|\ge 3$. As in [Gre13](Def. 4.9, Lem. 4.10), no such triple can exist in the pairing graph of a changemaker lattice.

Definition 3.7. A set of four basis vectors, none of which are tight, are said to form a *claw* if they form the subgraph in Figure 2. According to [Gre13](Lem. 4.8), if no basis vectors are tight, then there can be no claws. If a subgraph of the shape Figure 2 exists, then the vertex of degree 3 must be tight.

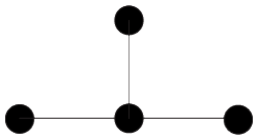

Figure 2: Claw graph.

4 Results

In this paper, we determine which of three types of spherical manifolds are realizable by positive integral Dehn surgery on a knot. We proceed on a case-by-case basis, examining each of the **O**-, **T**-, and **I**-type manifolds separately. Recall that all these manifolds can be obtained by $\frac{p}{q}$-surgery on a link as in Figure 1, where $\gcd(p,q) = 1$.

4.1 O-type manifolds

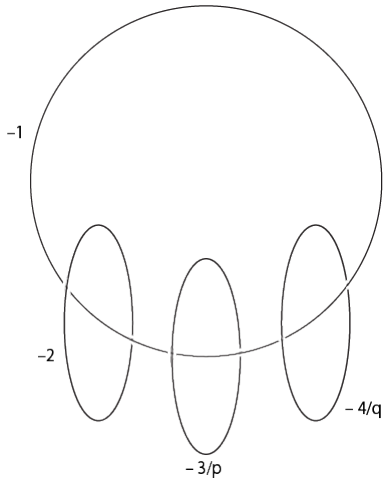

Figure 3: Surgery on the link with framings as shown produces a **O**-type manifold.

Figure 3 shows the link surgery diagram for an **O**-type manifold, where $p, q < 0$ are coprime integers. Inspired by the work in [Gre13], we will use continued fractions to view this surgery in terms of a lattice. We first consider the Hirzebruch-Jung continued fractions with all entries integers ≥ 2 for $-\frac{3}{p}$ and $-\frac{4}{q}$. The general form of such a fraction is

$$b/c = [a_0, \ldots, a_n]^- = a_0 - \frac{1}{a_1 - \frac{1}{a_2 - \cdots \frac{1}{a_n}}},$$

where each $a_i \geq 2$. For some values of p, such as $p = -4$ and $p = -5$, the continued fraction for $-\frac{3}{p}$ does not terminate, and similarly for some values of q in the fraction $-\frac{4}{q}$. Therefore, we aim to replace p

and q with integers p' and q' through Kirby moves that preserve the structure of the manifold but allow us to have finite continued fractions. By [GS99](§5) and [Sav99](Prop. 3.2), we can replace p with p' such that $|pp'| \equiv 1 \pmod 3$ and q with q' such that $|qq'| \equiv 1 \pmod 4$. With a Rolfsen twist, we can replace p and q with $p' = p + 3n$ and $q' = q + 3n$, respectively, which increases the framing of the central knot by n. In particular, we can let $p' = 1$ or 2 and $q' = 1$ or 3, which will result in the framing of the central component becoming $k = 1 + \frac{|p|-p'}{3} + \frac{|q|-q'}{4}$. This replacement allows us to examine only four cases for $p, q < 0$:

$$\begin{aligned}
-p &\equiv 1 \pmod 3, -q \equiv 1 \pmod 4 \\
-p &\equiv 1 \pmod 3, -q \equiv 3 \pmod 4 \\
-p &\equiv 2 \pmod 3, -q \equiv 1 \pmod 4 \\
-p &\equiv 2 \pmod 3, -q \equiv 3 \pmod 4
\end{aligned}$$

As in [Gre13] and [Bal+16], we can associate to our manifold a pairing graph $\Lambda(p, q)$ as defined in Def. 3.5. To each vertex x_i in this graph we can associate a weight a_i or b_j where for $0 \leq i \leq m$, a_i is the i^{th} entry of the continued fraction for $-\frac{3}{p}$, and where $0 \leq j \leq n$, b_j is the j^{th} entry of the continued fraction for $-\frac{4}{p}$ as in Figure 4. To the two central vertices we assign the weights k and 2, respectively, as those are the framings of the components of the original link corresponding to these vertices.

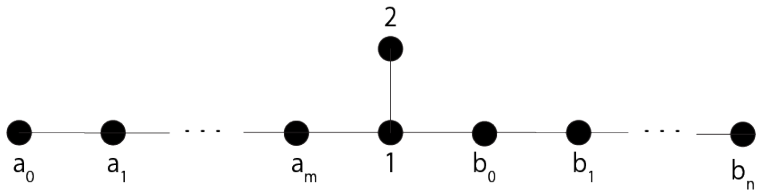

Figure 4: General lattice structure for an **O**-type manifold.

In order to show that $Y(p, q)$ in this case is realizable by positive integral surgery on a knot in S^3, we will utilize some methodology partially due to Ozsváth and Szabó in [OS03]. In Heegaard Floer homology, the invariant called a *correction term* $d(Y, \mathfrak{t})$ for an oriented rational homology sphere Y along with a Spin^c structure $\mathfrak{t}$ is a rational number associated to Y. The correction term for Y with reversed orientation is $-d(Y, \mathfrak{t})$.

Definition 4.1. Suppose $Y = \partial X$ for a negative definition 4-manifold X. If for every $\mathfrak{t} \in \text{Spin}^c(Y)$ there exists $\mathfrak{s} \in \text{Spin}^c(X)$ an extension of $\mathfrak{t}$ such that

$$c_1(\mathfrak{s})^2 + b_2(X) = 4d(Y, \mathfrak{t}),$$

then X is said to be *sharp*.

Here, $Y(p, q)$ must bound a sharp and negative definite 4-manifold in order to be realizable by integral surgery on a knot in S^3 [Gre13](§2). To prove these, we utilize the following theorems and definitions.

Theorem 4.1. If the intersection pairing Q_x given by the entries of the continued fractions for $-\frac{3}{p'}$ and $-\frac{4}{q'}$ is negative definite, then $Y(p, q)$ bounds a negative definite 4-manifold.

Proof. See [Gre12](§2). □

Proposition 2. When $Y(p, q)$ is an **O**-type manifold, it bounds a sharp 4-manifold and its intersection pairing Q_x is negative definite.

Proof. By [Gre12](§2), a 3-manifold $Y(p,q)$ bounds a sharp 4-manifold if in the continued fraction expansion $\frac{p}{q} = [a_0, a_1, \ldots, a_n]^-$, $-a_i \leq -\deg v$, where v is the i^{th} vertex of the pairing graph. Moreover, we may have up to two vertices that do not satisfy this criterion but still have a sharp manifold. In the continued fraction expansions for $-\frac{3}{p'}$ and $-\frac{4}{q'}$, each entry is at least 2, and the weights of the constant vertices of the pairing graph are -1 and -2, and the degree of each vertex other than the central vertex in the graph is either 1 or 2. Therefore the criterion is satisfied and $Y(p,q)$ bounds a sharp 4-manifold.

To show that $Y(p,q)$ bounds a negative definite 4-manifold, observe that the intersection pairing Q_x can be written as

$$Q_x = \begin{bmatrix} -a_0 & 0 & 0 & & & \cdots & & & 0 \\ 0 & -a_1 & 0 & & & \cdots & & & 0 \\ \vdots & \vdots & \ddots & & & & & & \vdots \\ & & & -a_m & 1 & 0 & \cdots & & \\ & & & & -1 & 1 & 1 & 0 & \cdots \\ & & & & 1 & -2 & 0 & 0 & \cdots \\ & & & & 1 & 0 & -b_0 & 0 & \cdots \\ & & & & 0 & \vdots & 0 & \ddots & \\ & & & & & & & -b_{n-1} & 0 \\ & & & & & & & & b_n \end{bmatrix}$$

Note that Q_x is block diagonal. All the entries on the diagonal are negative, so the diagonal block matrices are all negative definite. Therefore, the entire matrix is negative definite, so $Y(p,q)$ bounds a sharp, negative definite manifold. □

Theorem 4.2. If $Y(p,q)$ with p,q coprime is realizable by integral surgery on a knot in S^3, then the lattice given by the continued fractions is isomorphic to $(\sigma)^\perp$ for some changemaker vector $\sigma \in \mathbb{Z}^{n+m+4}$ [Gre13](Thm. 1.6), [Bal+16](Thm. 2.9), [Gre12](Thm. 3.3).

Proof. See [Gre12](Thm. 3.3). □

Thus, we aim to categorize all possible changemaker lattices isomorphic to the weighted pairing graph given by the continued fractions, and in doing so, enumerate all possible (p,q) that yield a realizable **O**-type manifold.

When $-\frac{3}{p} = [a_0, a_1, \ldots, a_m]^-$ and $-\frac{4}{q} = [b_0, b_1, \ldots, b_n]^-$, we obtain the lattice structure seen in Figure 4. Specifically, we will work with the four cases in Figure 5, where k is as on the previous page.

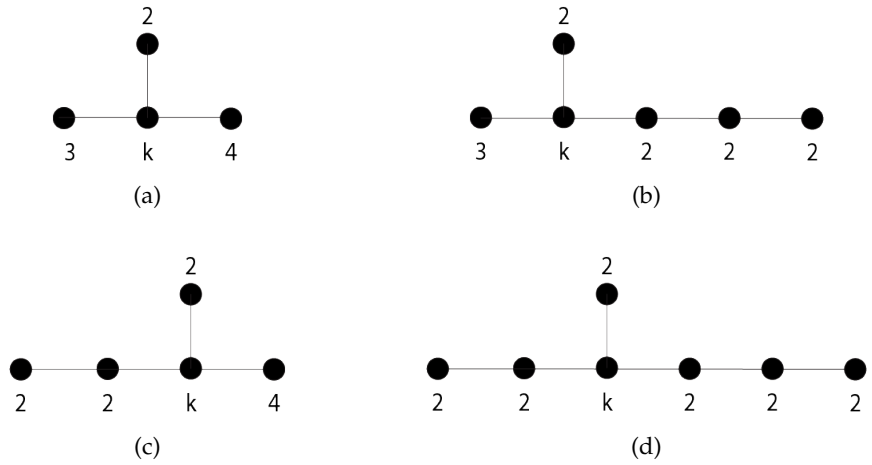

Figure 5: The four cases for **O**-type manifolds with $p, q < 0$.

4.2 Changemaker lattices

To determine the possible values for p and q for a changemaker lattice, we pick a basis $\{x_i\}$ for a linear lattice and assume that it is isomorphic to a changemaker lattice. Then we know that the resultant graph from the v_is must be isomorphic to the lattice structure for that particular case, as in figure 4. Note that without a tight vector, this structure contains a claw shape for every case, so we must assume that the vertex with degree 3 is tight. Therefore all other basis vectors are gappy or just right. In the following section, we outline the procedure for determining all possible bases satisfying these requirements. We then compute all possible values of p and q corresponding to these bases.

4.2.1 O-type manifolds

Recall that there are four possible structures for the **O**-type manifold lattice, as in figure 5. We examine each case, beginning with the one with three vertices. A basis for this lattice takes the form $\{v_1, v_2, v_3, v_4\}$ where $v_1 = -e_1 + e_0$ and exactly one of the remaining three vertices is tight.

Lemma 4.3. A changemaker basis of this lattice (Figure 5(a)) must take the form

$$\begin{aligned} v_1 &= -e_1 + e_0 \\ v_2 &= -e_2 + e_1 + e_0 \\ v_3 &= -e_3 + e_2 + e_1 + 2e_0 \\ v_4 &= -e_4 + e_3 + e_1 + e_0. \end{aligned}$$

Proof. Given that $v_1 = -e_1 + e_0$, it is clear that v_1 corresponds to the weight 2 vertex (as there can only be one vertex of weight 2, as in Figure 5(a)), and therefore $v_1 \cdot v_i = 0$ when v_i is not tight. We also know that v_2 must be $-e_2 + e_1$, $-e_2 + e_1 + e_0$, or $-e_2 + e_1 + 2e_0$.

The first case is impossible since there is only one vertex in the lattice with weight 2. The second case appears to work, since $v_1 \cdot v_2 = 0$ and the weight 3 vertex is not connected to the weight 2 vertex. The third case does not work because then $v_1 \cdot v_2 = 1$. Therefore $v_2 = -e_2 + e_1 + e_0$.

Then, either v_3 or v_4 must be tight. If v_3 is tight, then $v_3 = -e_3 + e_2 + e_1 + 2e_0$, so $v_3 \cdot v_1 = -1$ and $v_3 \cdot v_2 = 2$. Also, v_4 must be $-e_4 + e_3 + e_2 + e_1$, $-e_4 + e_3 + e_2 + e_0$, or $-e_4 + e_3 + e_1 + e_0$. Only the final case provides the correct dot products. It can be shown similarly that no other possible values of e_3 and e_4 provide the desired result. Therefore, in the four-vector case, we have

$$
\begin{aligned}
v_1 &= -e_1 + e_0 \\
v_2 &= -e_2 + e_1 + e_0 \\
v_3 &= -e_3 + e_2 + e_1 + 2e_0 \\
v_4 &= -e_4 + e_3 + e_1 + e_0
\end{aligned}
$$

□

Using a similar method, we move on to the case with five vertices.

Lemma 4.4. A changemaker basis of this lattice (Figure 5(b)) takes one of the following forms:

$$
\begin{aligned}
v_1 &= -e_1 + e_0 \\
v_2 &= -e_2 + e_1 \\
v_3 &= -e_3 + e_2 + e_1 + e_0 \\
v_4 &= -e_4 + e_3 + e_2 + e_1 + 2e_0 \\
v_5 &= -e_5 + e_4
\end{aligned}
$$

or

$$
\begin{aligned}
v_1 &= -e_1 + e_0 \\
v_2 &= -e_2 + e_1 + 2e_0 \\
v_3 &= -e_3 + e_2 \\
v_4 &= -e_4 + e_3 \\
v_5 &= -e_5 + e_4 + e_3 + e_2.
\end{aligned}
$$

Proof. We proceed as in the previous proof. We know that $v_1 = -e_1 + e_0$ and that v_1 must correspond to one of the vertices with weight 2. There are three possibilities for $v_2 : -e_2 + e_1 + e_0$, $-e_2 + e_1$, and $-e_2 + e_! + 2e_0$. We can eliminate the first possibility, since it is not tight and none of the vertices in the lattice have weight 3. We consider two cases now.

Case 1: $v_2 = -e_2 + e_1$

If $v_2 = -e_2 + e_1$, then $v_2 \cdot v_1 = -1$, and those vertices are connected, so they must correspond to x_1 and x_2. Then $v_3 = -e_3 + e_2, -e_3 + e_2 + e_1 + e_0, -e_3 + e_2 + e_1, -e_3 + e_2 + e_0$, or $-e_3 + e_2 + e_1 + 2e + 0$. The first case is impossible since then $v_3 \cdot v_2 = -1$, but there is only one vertex of weight 2 connected to v_2. The second case appears to satisfy the lattice requirements, since $v_3 \cdot v_2 = v_3 \cdot v_1 = 0$ and $|v_3| = 4$, so v_3 would correspond to x_3. The third and fourth cases are impossible since there are no non-tight vertices of weight 3. Finally, the fourth case also appears to satisfy the lattice requirements, since then v_3 is the unique tight vector corresponding to x_*, and $v_3 \cdot v_2 = 0$ while $v_3 \cdot v_1 = 1$, which agrees with v_1 corresponding to x_2 and v_2 corresponding to x_1.

Case 1.i: $v_3 = -e_3 + e_2 + e_1 + e_0$

If $v_3 = -e_3 + e_2 + e_1 + e_0$, then one of v_4 and v_5 must have weight 2 while the other is tight. If v_4 has weight 2 then $v_4 = -e_4 + e_3$ and $v_5 = -e_5 + e_4 + e_3 + e_2 + e_1 + 2e_0$, with v_4 corresponding to x_{**} and v_5 to x_*. But then $v_4 \cdot v_5 = 0$, a contradiction, since $x_* \cdot x_{**} \neq 0$. Therefore we must have $v_4 = -e_4 + e_3 + e_2 + e_1 + 2e_0$ and $v_5 = -e_5 + e_4$. It is easy to check that this configuration agrees with the lattice.

Case 1.ii: $v_3 = -e_3 + e_2 + e_1 + 2e_0$

In this case, one of v_4 and v_5 has weight 2, corresponding to x_{**}, and the other has weight 4, corresponding to x_3. If $v_4 = -e_4 + e_3$, then $v_5 = -e_5 + e_4 + z$, where z is the some of any two elements of $\{e_3, e_2, e_1, e_0\}$. Unless one of the summands of z is e_3 we have a contradiction, since we must have $v_4 \cdot v_5 = 0$.

Therefore $v_5 = -e_5 + e_4 + e_3 + e_2$, $-e_5 + e_4 + e_3 + e_1$, or $-e_5 + e_4 + e_3 + e_0$. Only the first case is possible, since $x_3 \cdot x_1 = x_3 \cdot x_2 = 0$, and we have that $v_1 = -e_1 + e_0$ corresponds to either x_1 or x_2. But if $v_5 = -e_5 + e_4 + e_3 + e_2$, then $v_5 \cdot v_3 = 0$, a contradiction, since $x_3 \cdot x_* \neq 0$. Therefore $v_3 \neq -e_3 + e_2 + e_1 + 2e_0$.

Case 2: $v_2 = -e_2 + e_1 + 2e_0$

If v_2 is the unique tight vector, then $v_2 \cdot v_1 = 1$, so v_1 must correspond to either x_2 or x_{**}. We must also have $v_3 = -e_3 + e_2$ or $-e_3 + e_2 + e_1 + e_0$, since v_3 cannot be tight or have norm 3, as the only vertex in the lattice that could possibly have norm 3 is the one with weight k, which by assumption corresponds to v_2.

Case 2.i: $v_3 = -e_3 + e_2$

In this case, $v_3 \cdot v_1 = 0$, so if v_1 corresponds to x_2 then v_3 corresponds to x_{**}, and if v_1 corresponds to x_{**} then v_3 corresponds to x_2. (Note that v_3 cannot correspond to x_1 since $v_3 \cdot v_2 = -1$.) In either case, one of v_4 has norm 2 and the other has norm 4. If $v_4 = -e_4 + e_3$, then $v_4 \cdot v_3 = -1$, so v_4 corresponds to x_1 and v_3 to x_2. In that case, $v_5 = -e_5 + e_4 + e_3 + \eta$, where $\eta \in \{v_2, v_1, v_0\}$. We must have e_3 as a summand so $v_5 \cdot v_4 = 0$. We require $v_5 \cdot v_3 = v_5 \cdot v_1 = 0$, so $\eta = e_2$. From here, it can easily be checked using inner products that values of $v_1, \ldots, v_5$ agree with the lattice structure.

On the other hand, if v_4 has norm 4 and $v_5 = -e_5 + e_4$, then $v_5 \cdot v_1 = v_5 \cdot v_3 = 0$, a contradiction since v_5 must correspond to x_1.

Case 2.ii: $v_3 = -e_3 + e_2 + e_1 + e_0$

In this case, v_1 corresponds to either x_2 or x_{**}, and we must have that v_5 and v_4 both have norm 2. Then $v_5 = -e_5 + e_4$ and $v_4 = -e_4 + e_3$ corresponding to x_1 and x_2 in no particular order, but $v_5 \cdot v_2 = v_4 \cdot v_2 = 0$, a contradiction.

Thus, the two possible configurations for the changemaker lattice are as postulated. □

Lemma 4.5. A changemaker basis corresponding to an **O**-type lattice with six vertices (as in Figure 5(c)) takes one of the following forms:

$$\begin{aligned} v_1 &= -e_1 + e_0 \\ v_2 &= -e_2 + e_1 \\ v_3 &= -e_3 + e_2 \\ v_4 &= -e_4 + e_3 + e_2 + e_1 + 2e_0 \\ v_5 &= -e_5 + e_4 \\ v_6 &= -e_6 + e_5 + e_4 \end{aligned}$$

or

$$\begin{aligned} v_1 &= -e_1 + e_0 \\ v_2 &= -e_2 + e_1 + e_0 \\ v_3 &= -e_3 + e_2 + e_1 + 2e_0 \\ v_4 &= -e_4 + e_3 \\ v_5 &= -e_5 + e_4 \\ v_6 &= -e_6 + e_5 \end{aligned}$$

The proof of Lemmas 4.5 is very similar to the proofs of Lemmas 4.3 and 4.4.

Lemma 4.6. No **O**-type lattices have seven vertices (as in Figure 5(d)).

Proof. Suppose such a seven-vertex lattice exists. Then every vertex has weight 2 except for the vertex of degree 3, so $v_i = -e_i + e_{i-1}$ for all i except for one, which is tight. It is simple to check that no choice of tight basis vector provides the right inner products to correspond to a lattice with the correct edges. □

These lemmas lead us to the following theorem.

Theorem 4.7. If $Y(p,q)$ is an **O**-type manifold that can be obtained by integral surgery on a knot in S^3, then (p,q) of the link surgery diagram for Y is in the following table for n a nonnegative integer.

k	$-p$	$-q$
7	3n + 1	25 - 4n
8	3n + 1	31 - 4n
7	3n + 1	27 - 4n
8	3n + 2	29 - 4n
6	3n + 2	21 - 4n

Table 1: Possible values of $-p$ and $-q$ for an **O**-type manifold.

Proof. To determine p and q, recall that $k = |v_t|$, where v_t is the unique tight basis vector in a changemaker basis. From Lemmas 2.3 - 2.5, we have that when the lattice has four vertices, $k = 7$, and

$$k = 1 - p' - q' = 1 + \frac{|p|+1}{3} + \frac{|q|+1}{4},$$

and rearranging gives us $|p| = 3n+1$, $|q| = 25-4n$.

We proceed similarly for the rest of the cases, recalling that when the lattice has five vertices, $-p' = \frac{|p|+2}{3}$ and $-q' = \frac{|q|+1}{4}$, and when the lattice has six vertices, $-p' = \frac{|p|+1}{3}$ and $-q' = \frac{|q|+3}{4}$. □

4.2.2 T-type manifolds

Recall that **T**-type manifolds are obtained by surgery on the link in figure 6, where as before $p, q < 0$ and $\gcd(p,q) = 1$.

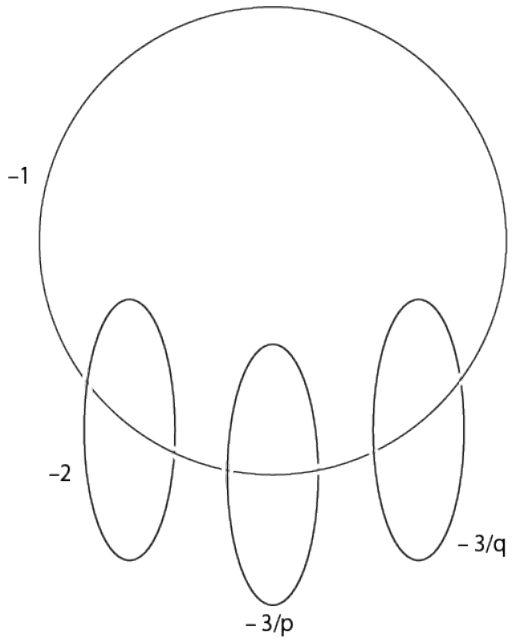

Figure 6: Link surgery for a **T**-type manifold.

Observe that the general lattice structure for this type of manifold is the same as that for **O**-type manifolds, as in Figure 4, but where $-\frac{3}{p} = [a_0, \ldots, a_m]^-$ and $-\frac{3}{q} = [b_0, \ldots, b_n]^-$. Moreover, T-type manifolds bound sharp, negative definite manifolds, proof of which is nearly identical to that in section 4.1. We can use a similar method of replacing p and q by p' and q', this time where $-p' = (-p) \pmod 3$ and $-q' = (-q) \pmod 3$. Then we define $k = 1 + \frac{|p|+|p'|}{3} + \frac{|q|+|q'|}{3}$. We again have four cases for $(-p', -q')$, but this time, some are isomorphic ((b) and (c) in Figure 7).

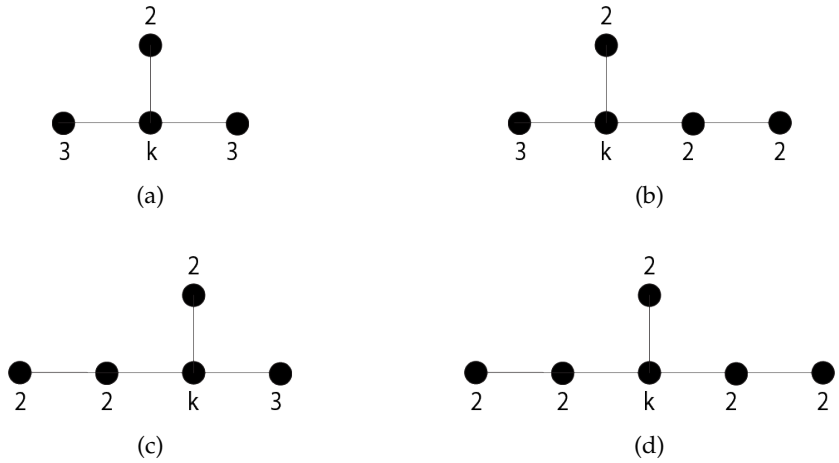

Figure 7: The four cases for **T**-type manifolds with $p, q < 0$.

This isomorphism will allow us to consider 3 cases instead of 4, but for the two isomorphic cases we will consider p and q interchangeable. We proceed similarly to the **O**-type case.

Lemma 4.8. No changemaker basis for a **T**-type lattice with four vertices exists.

Proof. Suppose such a basis exists. Then we have $v_1 = -e_1 + e_0$ corresponding to x_{**} since no other vertex has weight 2. There are two possibilities for v_2 : $-e_2 + e_1 + e_0$ and $-e_2 + e_1 + 2e_0$.

Case 1: $v_2 = -e_2 + e_1 + e_0$

In this case, we have that one of v_3 and v_4 has norm 3 and the other is tight. If v_3 has norm 3 then $v_3 = -e_3 + e_2 + e_1$ or $-e_3 + e_2 + e_0$, both of which are contradictions since we require $v_3 \cdot v_1 = 0$. If v_4 has norm 3 then $v_4 = -e_4 + e_3 + e_2$, $-e_4 + e_3 + e_1$, or $-e_4 + e_3 + e_0$. Only the first is possible since we require $v_4 \cdot v_1 = 0$. But then we have $v_4 \cdot v_2 = -1 \neq 0$, a contradiction.

Case 2: $v_2 = -e_2 + e_1 + 2e_0$

In this case, both v_3 and v_4 have norm 3. Then $v_3 = -e_3 + e_2 + e_1$ or $-e_3 + e_2 + e_0$, both contradictions since we require $v_3 \cdot v_1 = 0$.

□

Lemma 4.9. A changemaker basis for case (b) or (c) takes one of the following forms

$$
\begin{aligned}
v_1 &= -e_1 + e_0 \\
v_2 &= -e_2 + e_1 + e_0 \\
v_3 &= -e_3 + e_2 + e_1 + 2e_0 \\
v_4 &= -e_4 + e_3 \\
v_5 &= -e_5 + e_4,
\end{aligned}
$$

$$
\begin{aligned}
v_1 &= -e_1 + e_0 \\
v_2 &= -e_2 + e_1 \\
v_3 &= -e_3 + e_2 + e_1 \\
v_4 &= -e_4 + e_3 + e_2 + e_1 + 2e_0 \\
v_5 &= -e_5 + e_4,
\end{aligned}
$$

or

$$
\begin{aligned}
v_1 &= -e_1 + e_0 \\
v_2 &= -e_2 + e_1 \\
v_3 &= -e_3 + e_2 + e_1 + 2e_0 \\
v_4 &= -e_4 + e_3 \\
v_5 &= -e_5 + e_4 + e_3.
\end{aligned}
$$

Proof. We proceed labeling the x_is as in (c), but the same reasoning applies if we switch the order x_1 and x_2 as in (b). We have $v_1 = -e_1 + e_0$ corresponding to one of x_1, x_2, or x_{**}. Then $v_2 = -e_2 + e_1 + e_0$, $-e_2 + e_1$, or $-e_2 + e_1 + 2e_0$.

Case 1: $v_2 = -e_2 + e_1 + e_0$

In this case, v_2 corresponds to x_3, and $v_2 \cdot v_1 = 0$, agreeing with the lattice structure. Then $v_3 = -e_3 + e_2$ or $-e_3 + e_2 + e_1 + 2e_0$, since no non-tight vertex of weight greater than 2 exists.

Case 1.i: $v_3 = -e_3 + e_2$

In this case, $v_1 \cdot v_3 = 0$, so v_1 corresponds to x_1 or x_2 and v_3 to x_{**}, or vice-versa. Moreover, we have that one of v_4 and v_5 is tight and the other has norm 2. If v_4 is tight, then $v_5 = -e_5 + e_4$, but $v_5 \cdot v_3 = v_5 \cdot v_1 = 0$, and we require one of those to be nonzero since x_1 and x_2 are connected. Therefore v_5 must be tight and $v_4 = -e_4 + e_3$, so $v_4 \cdot v_3 = -1$ and $v_4 \cdot v_1 = 0$, yielding v_3 and v_4 corresponding in some order to x_1 and x_2, and v_1 to x_{**}. Then $v_5 = -e_5 + e_4 + e_3 + e_2 + e_1 + 2e_0$, but $v_5 \cdot v_4 = v_5 \cdot v_3 = 0$, a contradiction.

Case 1.ii: $v_3 = -e_3 + e_2 + e_1 + 2e_0$

In this case, v_4 and v_5 must both have norm 2. Then $v_4 = -e_4 + e_3$ and $v_5 = -e_5 + e_4$, so $v_5 \cdot v_5 = -1, v_5 \cdot v_3 = 0$, and $v_4 \cdot v_3 = -1$, so v_5 corresponds to x_1, v_4 to x_2, and v_1 to x_{**}. It is easy to check that the rest of the inner products agree with the lattice structure.

Case 2: $v_2 = -e_2 + e_1$

In this case, $v_1 \cdot v_2 = -1$, so v_1 and v_2 correspond to x_1 and x_2. Then v_3 must be one of $-e_3 + e_2$, $-e_3 + e_2 + e_1$, $-e_3 + e_2 + e_0$, or $-e_3 + e_2 + e_1 + 2e_0$. The first case is impossible, however, since then $v_3 \cdot v_2 = -1$. The third case is also impossible since then $v_3 \cdot v_2, v_3 \cdot v_1 \neq 0$.

Case 2.i: $v_3 = -e_3 + e_2 + e_1$

In this case, one of v_4 and v_5 has norm 2 and the other is tight. Suppose $v_4 = -e_4 + e_3$. Then $v_4 \cdot v_3 = -1$, a contradiction since x_{**} and x_3 are disconnected. Therefore $v_5 = -e_5 + e_4$ and $v_4 = -e_4 + e_3 + e_2 + e_1 + 2e_0$, which gives us $v_4 \cdot v_1 = 1$, $v_4 \cdot v_2 = 0$, $v_4 \cdot v_3 = 1$, and $v_4 \cdot v_5 = -1$, so v_1 corresponds to x_2, v_2 to x_1, v_3 to x_3, v_4 to x_*, and v_5 to x_{**}. It is easy to check that the rest of the inner products satisfy the lattice structure.

Case 2.ii: $v_3 = -e_3 + e_2 + e_1 + 2e_0$

In this case, one of v_4 and v_5 has norm 3 and the other has norm 2. Suppose v_4 has norm 2. Then $v_4 = -e_4 + e_3$ and $v_5 = -e_5 + e_4 + \eta$, where $\eta \in \{e_3, e_2, e_1, e_0\}$. Observe that η must be e_3 since $v_5 \cdot v_4$ must be 0. It is easy to check from here that the rest of the lattice requirements are satisfied.

Case 3: $v_2 = -e_2 + e_1 + 2e_0$

In this case, one of v_3, v_4, and v_5 must have norm 3 and the others have norm 2. Suppose v_3 has norm 3. Then $v_3 = -e_3 + e_2 + e_1$ or $-e_3 + e_2 + e_0$, but both are contradictions, since we require $v_3 \cdot v_1 = 0$.

Therefore $v_3 = -e_3 + e_2$. Suppose now that v_4 has norm 3. Then $v_4 = -e_4 + e_3 + e_2$ since $v_4 \cdot v_1$ must be 0, but $v_5 = -e_5 + e_4$, a contradiction since $v_4 \cdot v_5$ must also be 0. Therefore v_5 must have norm 3, so $v_5 = -e_5 + e_4 + \eta$, where $\eta \in \{e_3, e_2, e_1, e_0\}$. Since we must have $v_5 \cdot v_4 = v_5 \cdot v_3 = 0$ and $v_5 \cdot 2 \neq 0$, we must have $\eta = e_1$ or e_0. But then $v_5 \cdot v_1 \neq 0$, a contradiction.

□

Lemma 4.10. No changemaker basis for a **T**-type lattice of type (d) exists.

Proof. Suppose such a lattice did exist. Then all the basis vectors have norm 2 except for one tight vector. As in the proof of Lemma 2.7, it is simple to compute inner products for the six possibilities of tight vectors, none of which correspond to a lattice of the correct structure. □

As in the previous section, we apply these lemmas to prove the following theorem.

Theorem 4.11. If $Y(p,q)$ is a **T**-type manifold that can be obtained by integral surgery on a knot in S^3, then (p,q) of the link surgery diagram for Y is in the following table for n a nonnegative integer.

k	$-p$	$-q$
7	n	27 - n
8	n	24 - n

Table 2: Possible values of $-p$ and $-q$ for an **T**-type manifold.

Proof. As in the proof of Theorem 4.8, we determine p and q by examining the possible values of $k = 1 + \frac{|p|-|p'|}{3} + \frac{|q|-|q'|}{3}$. As the only possible $(-p', -q')$ are $(1,2)$ and $(2,1)$ from Lemmas 2.9-2.11, we have that the only values of k are the possible values of the norm of the unique tight vector in Lemma 2.10, which is either 7 or 8. Substituting for k yields the values of p and q in Table 2. □

4.2.3 I-type manifolds

We proceed similarly to the previous two sections to determine all possible values of (p,q) for icosahedral manifolds. Recall that **I**-type manifolds are obtained by surgery on the link in Figure 8, where $p, q < 0$ and $\gcd(p,q) = 1$.

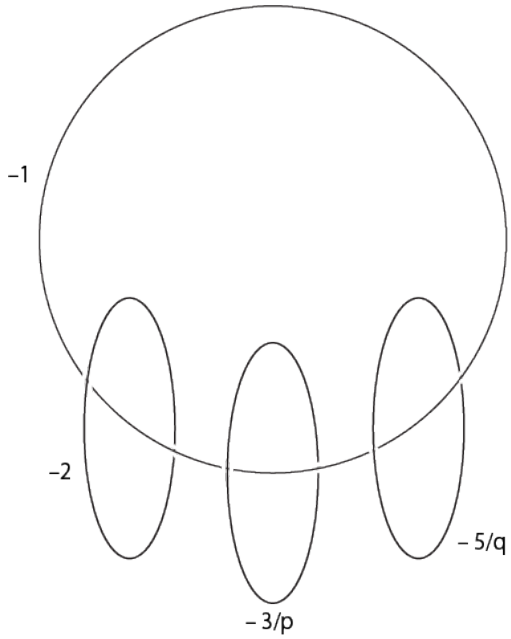

Figure 8: Link surgery for a **I**-type manifold.

Again the general lattice structure is the same as those for **O**- and **T**-type manifolds as in Figure 2, but here $-\frac{3}{p} = [a_0, \ldots, a_n]^-$ and $-\frac{5}{q} = [b_0, \ldots, b_m]^-$. We already know the possible a_is, as they are the same as for the previous two cases, but we must compute the b_is specifically for this case since there is a 5 in the numerator rather than a 3 or a 4. We can make an analogous statement to that in Proposition 1 that unless $q = -1, -2, -3$, or -4, then the fraction does not terminate. The proof is the same. Moreover, I-type manifolds bound sharp, negative definite manifolds, proof of which is nearly identical to that in section 4.1. The possible lattices from the continued fractions are shown in Figure 9.

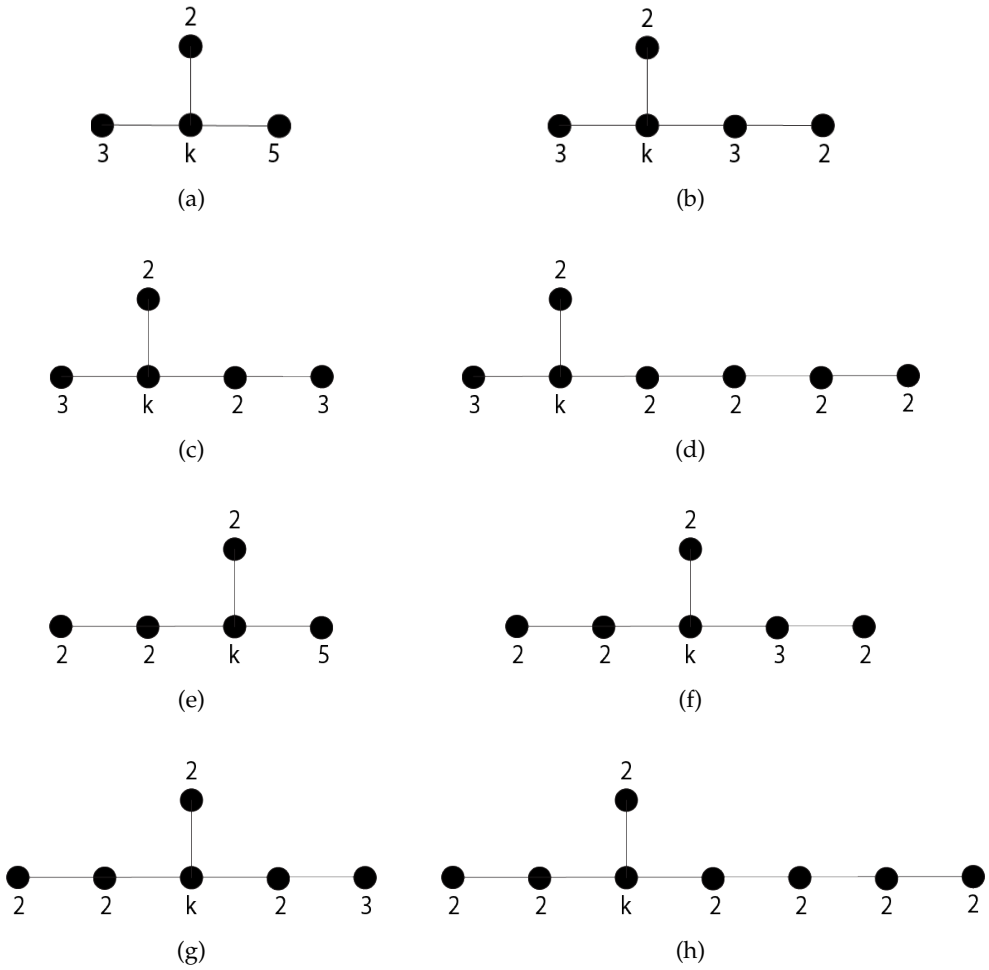

Figure 9: The eight cases for **I**-type manifolds with $p, q < 0$.

Lemma 4.12. No changemaker basis for an **I**-type lattice exists for the lattices pictured in Figure 9(a), (b), (c), (e), (g), or (h).

Proof. Suppose such a basis exists for (a). Then we have $v_1 = -v_1 + v_0$ corresponding to x_{**}, and either $v_2 = -v_2 + v_1 + v_0$ corresponding to x_1 or $v_2 = -v_2 + v_2 + 2v_0$ to x_*. In any case, we must have $v_4 = -e_4 + e_3 + e_2 + e_1 + e_0$ corresponding to x_2 since no other basis elements can have norm 5. If v_2 corresponds to x_1, then $v_2 \cdot v_4 = -1 \neq 0$, a contradiction. On the other hand, if v_2 corresponds to x_*, then $v_3 = -e_3 + e_2 + e_1$ or $-e_3 + e_2 + e_0$, and corresponds to x_1. In either case, $v_3 \cdot v_1 \neq 0$, a contradiction.

Now suppose such a basis exists for (b). Then we have $v_1 = -e_1 + e_0$ corresponding to either x_{**} or x_3. But since $v_1 \cdot v_t \neq 0$ for any tight v_t, we must have v_1 corresponding to x_{**}.

Case 1: $v_2 = -e_2 + e_1$

If $v_2 = -e_2 + e_1$, then v_2 corresponds to x_3. Then if $v_3 = -e_3 + e_2 + e_1$ or $-e_3 + e_2 + e_0$, v_3 must correspond to either x_1 or x_2. But either way, $v_3 \cdot v_1 \neq 0$, a contradiction, so we must have v_3 the unique tight vector, leaving v_4 and v_5 to each have norm 3. Then $v_4 = -e_4 + e_3 + e_i$, where $i = 0, 1$, or 2. But since $v_4 \cdot v_1 = 0$, then $i = 2$. This implies that $v_4 \cdot v_3 = 0$, a contradiction.

Case 2: $v_2 = -e_2 + e_1 + e_0$

Here v_2 corresponds to either x_1 or x_2. We must have one of v_3, v_4, and v_5 with norm 2. Suppose $v_3 = -e_3 + e_2$. Then $v_3 \cdot v_2 = -1$, so v_2 must correspond to x_2. If v_4 is the unique tight vector, then

$v_5 = -e_5 + e_4 + e_i$ where $i = 0, 1, 2$, or 3, a contradiction since $v_5 \cdot v_4 = 0$. Therefore v_5 must be the unique tight vector, and $v_4 = -e_4 + e_3 + e_i$, where $i = 0, 1$, or 2. But since $v_4 \cdot v_1 = 0$, we must have $i = 2$, a contradiction since then $v_4 \cdot v_2 \neq 0$.

Now consider the case when v_3 has norm 3, so $v_3 = -e_3 + e_2 + e_1$ or $-e_3 + e_2 + e_0$. But either way $v_3 \cdot v_1 \neq 0$, a contradiction.

Therefore we must have v_3 the unique tight vector, so one of v_4 and v_5 has norm 2. If $v_4 = -e_4 + e_3$, then $v_4 \cdot v_3 \neq 0$, a contradiction, so $v_5 = -e_5 + e_4$. Then $v_4 = -e_4 + e_3 + e_i$, where $i = 0, 1$, or 2, and $v_4 \cdot v_5 = -1$, so v_4 corresponds to x_3, v_5 to x_2, and v_2 to x_1. But if $i = 0$ or 1, then $v_4 \cdot v_1 \neq 0$, a contradiction, and if $i = 2$, then $v_4 \cdot v_2 = -1$, also a contradiction.

Case 3: $v_2 = -e_2 + e_1 + 2e_0$

Here, v_2 corresponds to the unique tight vector, so $v_3 = -e_3 + e_2$, $-e_3 + e_2 + e_1$, or $-e_3 + e_2 + e_0$. In the first case, $v_3 \cdot v_2 \neq 0$, a contradiction since v_3 must correspond to x_3 if v_3 has norm 2. In both the second and third cases, $v_3 \cdot v_1 \neq 0$, a contradiction.

We now consider the case of (c). Suppose such a basis exists. Similarly to case (b), we consider cases. We must have $v_1 = -e_1 + e_0$ corresponding to x_{**} or x_2.

Case 1: $v_2 = -e_2 + e_1$

In this case, v_2 corresponds to x_2 or x_{**} (whichever x_1 does not), but either way $v_1 \cdot v_2 \neq 0$, a contradiction.

Case 2: $v_2 = -e_2 + e_1 + e_0$

Here we must have v_2 corresponding to x_1, since $v_2 \cdot v_t$ will be nonzero for any tight v_t. Note that we cannot have $v_3 = -e_3 + e_2$, since then $v_3 \cdot v_2 \neq 0$, but v_3 would have to correspond to x_2. Suppose v_3 is tight. Then either v_4 or v_5 has norm 2, but note that if v_5 has norm 2, then $v_5 \cdot v_3 = 0$, so v_4 must have norm 2. Then $v_5 = -e_5 + e_4 + e_i$, where $i = 0, 1, 2$, or 3. Clearly $i \neq 0, 1$, or 2, since then $v_5 \cdot v_1, v_2 \neq 0$, so $i = 3$, still a contradiction since then $v_5 \cdot v_3 \neq 0$. This leaves us with $v_3 = -e_3 + e_2 + e_1$ or $-e_3 + e_2 + e_0$. Neither case works because $v_3 \cdot v_1 \neq 0$.

Case 3: $v_2 = -e_2 + e_1 + 2e_0$

Now v_2 is the unique tight vector. Then v_3 must have norm 2, since if v_4 or v_5 did then their inner products with v_2 would be zero, so v_3 and v_1 correspond to x_2 and x_{**} in either order. Therefore both v_4 and v_5 have norm 3, so $v_4 = -e_4 + e_3 + e_i$, where $i = 0, 1$, or 2. If v_1 corresponds to x_{**}, $v_4 \cdot v_1 = 0$, so $i \neq 0, 1$, and because $v_4 \cdot v_2 = 0$, $i \neq 2$, so this case is not possible. Therefore v_3 must correspond to x_{**}, a contradiction unless $i = 2$, but then $v_4 \cdot v_5 = 0$, still a contradiction.

We consider case (e) now. As usual, we have 3 possibilities for v_2.

Case 1: $v_2 = -e_2 + e_1$

Here, $v_2 \cdot v_1 = -1$, a contradiction if v_1 corresponds to x_{**}, so v_1 must correspond to one of x_1 and x_2, and v_2 to the other. Note that $v_1 \cdot v_t \neq 0$ for any tight v_t, so v_1 corresponds to x_2 and v_2 to x_1. Then observe that v_3 cannot have norm 2 since then $v_3 \cdot v_2 \neq 0$, and no vertex of weight 3 exists, so v_3 must be tight. Then v_4 or v_5 has norm 2 and the other norm 5. If v_4 has norm 2 then $v_5 = -e_5 + e_4 + e_i + e_j + e_k$, where $i, j, k \in \{0, 1, 2, 3\}$. By the pigeonhole principle, one of i, j, and k must be 0 or 1, but then another must be 1 or 0 respectively sine otherwise $v_5 \cdot v_1 \neq 0$. Then $v_5 = -e_5 + e_4 + e_3 + e_1 + e_0$ or $-e_5 + e_4 + e_2 + e_1 + e_0$. The first case is a contradiction because $v_5 \cdot v_2 \neq 0$, and the second case is a contradiction because $v_5 \cdot v_4 \neq 0$.

Case 2: $v_2 = -e_2 + e_1 + e_0$

This is not possible since none of the vertices have weight 3.

Case 3: $v_2 = -e_2 + e_1 + 2e_0$

Here v_2 is the unique tight vector, so we must have one of v_3, v_4, and v_5 with norm 5 and the others with norm 2. Clearly only v_4 and v_5 can have norm 5, so $v_3 = -e_3 + e_2$, yielding $v_3 \cdot v_2 = -1$, which means that v_3 must correspond to x_2 or x_{**}. If v_3 corresponds to x_2, then we must have v_4 with norm 2, since otherwise v_5 would have norm 2 and $v_5 \cdot v_3 = 0$. Thus $v_5 = -e_5 + e_4 + e_i + e_j + e_k$, where $i, j, k \in \{0, 1, 2, 3\}$. We require

$v_5 \cdot v_1 = 0$, so if one of $i, j, k = 0$ or 1, then another must be 1 or 0, respectively. By the pigeonhole principle, one of i, j, and k must be either 0 or 1, so another must be 1 or 0. Therefore $v_5 = -e_5 + e_4 + e_3 + e_1 + e_0$ or $-e_5 + e_4 + e_2 + e_1 + e_0$. Since $v_5 \cdot v_4 = 0$, only the first case is possible. But then $v_5 \cdot v_3 \neq 0$, a contradiction. Therefore v_3 must correspond to x_{**}, so v_1 corresponds to x_2, a contradiction since $v_1 \cdot v_4 = 0$ if v_4 has norm 2 and $v_1 \cdot v_5 = 0$ if v_5 has norm 2.

We proceed similarly in the case of (g).

Case 1: $v_2 = -e_2 + e_1$

In this case, $v_2 \cdot v_1 = -1$, so v_2 corresponds to x_1 and v_1 to x_2. Suppose v_3 is the unique tight vector. Then one of v_4, v_5, and v_6 has norm 3 and the others have norm 2. If v_4 has norm 3 then $v_4 = -e_4 + e_3 + e_i$, where $i = 0, 1$, or 2. The first two are impossible since then $v_4 \cdot v_1 \neq 0$, and the last is also impossible since that $v_4 \cdot v_2 \neq 0$. For the same reason, if v_5 has norm 3, then we must have $v_5 = -e_5 + e_4 + e_3$, but this is a contradiction, since then $v_5 \cdot v_3 \neq 0$. Therefore v_6 has norm 3, and likewise $v_6 = -e_6 + e_5 + e_4$, and since $v_6 \cdot v_4 = -1$, v_4 corresponds to x_3 and v_5 to x_{**}, a contradiction since $v_4 \cdot v_5 \neq 0$.

Therefore v_3 cannot be tight. Suppose instead that v_4 is tight. Then v_3 cannot have norm 3 since then $v_3 \cdot v_4 \neq 0$, so $v_3 = -e_3 + e_2$, a contradiction, since x_2 is the only vertex connected to x_2.

Now suppose that v_5 is tight. Then $v_4 \cdot v_5 \neq 0$ if v_4 has norm 3, but if v_4 has norm 2, then $v_4 \cdot v_5 = 0$, both contradictions.

Finally, if v_6 is tight, then $v_5 \cdot v_6 \neq 0$ if v_5 has norm 3, but if v_5 has norm 2, then $v_5 \cdot v_6 = 0$, again both contradictions.

Case 2: $v_2 = -e_2 + e_1 + e_0$

In this case, v_2 must correspond to x_4. Then all other basis elements except one unique tight vector have norm 2. But note that $v_2 \cdot v_t \neq 0$ for any tight v_t, a contradiction since x_4 does not connect to x_*.

Case 3: $v_2 = -e_2 + e_1 + 2e_0$

Now all other vectors must have norm 2 except one, which has norm 3. Suppose v_3 has norm 3. Then $v_3 = -e_3 + e_2 + e_1$ or $e_3 + e_2 + e_0$. The latter is a contradiction since then $v_3 \cdot v_2 \neq 0$, so $v_3 = -e_3 + e_2 + e_1$, also a contradiction, since then $v_3 \cdot v_1 \neq 0$. Therefore one of v_4, v_5, and v_6 has norm 3. If v_4 has norm 3, then $v_4 = -e_4 + e_3 + e_i$, where $i = 0, 1$, or 2. The first two are impossible since then $v_4 \cdot v_1 \neq 0$, and $i = 2$ is also a contradiction since then $v_4 \cdot v_2 \neq 0$. If v_5 has norm 3, then $v_5 = -e_5 + e_4 + e_i$, where $i = 0, 1, 2$, or 3. For the same reasons, $i \neq 0, 1$, or 2, so $i = 3$. But then $v_6 = -e_6 + e_5$, so $v_6 \cdot v_5 \neq 0$, a contradiction, since we must have $v_5 \cdot v_3 \neq 0$, and x_4 has degree 1. Therefore we have v_6 as the unique vector with norm 3. But in that case, $v_4 \cdot v_3 = v_4 \cdot v_5 = -1$ with $v_5 \cdot v_3 = 0$, implying a linear sublattice with all vertices of norm 2. This structure is not present in Figure 9(g).

Finally, we turn to the case of (h). Here, every vector must have norm 2 except for one, which is tight. Because no vector of norm 3 exists, we have $v_2 = -e_2 + e_1$ or $v_2 = -e_2 + e_1 + 2e_0$. The first case is impossible since then $v_2 \cdot v_1 \neq 0$. Therefore v_2 is the unique tight vector. But then $v_3 \cdot v_4 = v_4 \cdot v_5 = v_5 \cdot v_6 = v_6 \cdot v_7 = v_7 \cdot v_8 = -1$ with all other pairings zero, indicating a linear sublattice with six elements of degree 2. Such a structure is not present in Figure 9(h). □

We now analyze the remaining possible lattice structures.

Lemma 4.13. A changemaker basis for case (d) (as in Figure 9(d)) takes the following form:

$$
\begin{aligned}
v_1 &= -e_1 + e_0 \\
v_2 &= -e_2 + e_1 + e_0 \\
v_3 &= -e_3 + e_2 + e_1 + 2e_0 \\
v_4 &= -e_4 + e_3 \\
v_5 &= -e_5 + e_4 \\
v_6 &= -e_6 + e_5 \\
v_7 &= -e_7 + e_6
\end{aligned}
$$

Proof. Observe that v_2 must be either $-e_2 + e_1 + e_0$ or $-e_2 + e_1 + 2e_0$, since otherwise, $v_2 \cdot v_1$ and no non-tight vector can correspond to a vertex connected to v_1. If $v_2 = -e_2 + e_1 + 2e_0$, then all other basis vectors have norm 2, except one with norm 3. The unique norm 3 vertex must correspond to v_3 since it must connect with the vertex corresponding to v_2. Then $v_3 = -e_3 + e_2 + e_1$ or $-e_3 + e_2 + e_0$, both contradictions since then $v_3 \cdot v_1 \neq 0$.

Therefore $v_2 = -e_2 + e_1 + e_0$, and v_2 is the unique norm 3 basis vector, and all other vectors have norm 2 except one, which is tight. We require a linear sublattice with four elements of weight 2, so there must be a collection v_i, v_j, v_k, v_l all with norm 2 and i, j, k, l consecutive integers. If v_2 has norm 3, then i, j, k, l can be either $3, 4, 5, 6$ or $4, 5, 6, 7$. The first case gives $v_3 = -e_3 + e_2$, so $v_3 \cdot v_2 \neq 0$, a contradiction. Therefore v_3 is the unique tight vector. It is easy to check that this configuration satisfies all the possible inner products. □

Lemma 4.14. A changemaker basis for case (f) (as in Figure 9(f)) takes the following form:

$$\begin{aligned}
v_1 &= -e_1 + e_0 \\
v_2 &= -e_2 + e_1 \\
v_3 &= -e_3 + e_2 + e_1 + 2e_0 \\
v_4 &= -e_4 + e_3 \\
v_5 &= -e_5 + e_4 + e_3 \\
v_6 &= -e_6 + e_5
\end{aligned}$$

Proof. We consider the usual cases.

Case 1: $v_2 = -e_2 + e_1 + e_0$

v_2 corresponds to x_3. As usual v_1 corresponds to either x_1 or x_{**} since $v_1 \cdot v_t \neq 0$ for tight v_t. Then v_3 either has norm 2 or is tight. If v_3 has norm 2, then $v_3 \cdot v_2 = -1$, so v_3 corresponds to x_4. Then we cannot have v_4 with norm 2, since otherwise $v_4 \cdot v_3 \neq 0$, so v_4 must be tight. Then v_5 and v_6 both have norm 2, but $v_5 \cdot v_4 \neq 0$, so v_5 corresponds to x_{**}. However, $v_6 \cdot v_5 \neq 0$, a contradiction since x_{**} is not connected to any vertex of weight 2.

Suppose instead that v_3 is tight. Then we must have a vertex of norm 2 connected to v_2, but $v_2 \cdot v_4, v_5, v_6 = 0$ if all of the latter have norm 2.

Case 2: $v_2 = -e_2 + e_1 + 2e_0$

Now, one of v_3, v_4, v_5, and v_6 has norm 3 and the rest have norm 2. If v_3 has norm 3, then $v_3 = -e_3 + e_2 + e_1$ or $-e_3 + e_2 + e_0$, both contradictions since then $v_3 \cdot v_1 \neq 0$. Therefore $v_3 = -e_3 + e_2$ and $v_3 \cdot v_2 = -1$, so v_3 corresponds to either x_2 or x_{**}, while v_1 corresponds to the other. If v_4 has norm 3, then $v_4 = -e_4 + e_3 + e_i$ where $i = 0, 1$, or 2. If $i = 0$ or 1 then $v_4 \cdot v_1 \neq 0$, a contradiction, so $v_4 = -e_4 + e_3 + e_2$. Then $v_5 = -e_5 + e_4$ and $v_6 = -e_6 + e_5$, but then $v_5 \cdot v_4, v_6 \neq 0$, impossible since no vertex of weight 2 is connected to one of weight 3 and one of weight 2. Therefore v_4 has norm 2.

Suppose now that v_5 has norm 3. Then $v_5 = -e_5 + e_4 + e_i$, where $i = 0, 1, 2$, or 3. If $i = 0$ or 1, then $v_5 \cdot v_1 \neq 0$, a contradiction. If $i = 3$, then $v_5 \cdot v_2 = 0$, a contradiction. Therefore $v_5 = -e_5 + e_4 + e_2$, but $v_4 = -e_4 + e_3$ and $v_6 = -e_6 + e_5$, so $v_5 \cdot v_4 = v_5 \cdot v_6 \neq 0$, a contradiction since the weight 3 vertex is only connected to one vertex of weight 2.

Therefore, we have $v_6 = -e_6 + e_5 + e_i$, where $i = 0, 1, 2, 3$, or 4. As usual, $i \neq 0, 1$, since then $v_6 \cdot v_1 \neq 0$. If $i = 2$, then $v_6 \cdot v_5 = v_6 \cdot v_4 = -1$, a contradiction since x_3 is only connected to one weight 2 vertex. if $i = 3$ or 4, then $v_6 \cdot v_2 = 0$, a contradiction.

Case 3: $v_2 = -e_2 + e_1$

Here $v_1 \cdot v_2 = -1$, so v_1 corresponds to x_2 and v_2 to x_1. Suppose v_4 is tight. Then one of v_3, v_5, and v_6 has norm 3 and the others have norm 2. If v_3 has norm 3, then $v_3 = -e_3 + e_2 + e_1$ or $-e_3 + e_2 + e_0$, and in either case, $v_3 \cdot v_1 \neq 0$, a contradiction. But if v_3 has norm 2, then $v_3 \cdot v_2 = -1$, a contradiction since x_1 is only connected to one vertex, which corresponds to v_1. Therefore v_4 cannot be tight.

Suppose instead that v_5 is tight. Then one of v_3, v_4, and v_6 has norm 3. If v_3 has norm 3, then again $v_3 \cdot v_1 \neq 0$, and if v_3 has norm 2, then $v_3 \cdot v_2 \neq 0$, both contradictions. The same thing happens if v_6 is tight. Therefore v_3 must be tight, and one of v_4, v_5, v_6 has norm 3 while the others have norm 2. If v_4 has norm 3, then $v_4 = -e_4 + e_3 + e_i$, where $i = 0, 1$, or 2. If $i = 0$ or 1 then $v_4 \cdot v_1 \neq 0$, a contradiction, and if $i = 2$ then $v_4 \cdot v_3 = 0$, also a contradiction. If v_6 has norm 3, then $v_6 = -e_6 + e_5 + e_i$, where $i = 0, 1, 2, 3$, or 4. Again $i = 0$ and 1 fail, and if $i = 2$ then $v_6 \cdot v_2 \neq 0$, a contradiction. If $i = 3$, then $v_6 \cdot v_3 = v_6 \cdot v_5 = -1$, a contradiction since x_3 is only connected to one weight 2 vertex. But if $i = 4$, then $v_6 \cdot v_5 = 0$, a contradiction.

Therefore the lattice must be of the form proposed. It is easy to check that these basis elements produce a lattice of the correct structure.

□

We now prove the main result of this section.

Theorem 4.15. If $Y(p, q)$ is an **I**-type manifold that can be obtained by integral surgery on a knot in S^3, then (p, q) of the link surgery diagram for Y is in the following table for n a nonnegative integer.

k	$-p$	$-q$
7	3n + 1	34 - 5n
7	3n + 2	32 - n

Table 3: Possible values of $-p$ and $-q$ for an **T**-type manifold.

Proof. As usual, we examine the possible values of $k = 1 + \frac{|p|-|p|'}{3} + \frac{|q|-|q'|}{5}$. As the only possible $(-p', -q')$ are $(1, 4)$ and $(2, 2)$, we have from Lemmas 2.13 - 2.15 that the only value of k is 7. Substituting for k yields the values of p and q in Table 3. □

5 Conclusions

In this paper, we used a graph theoretical approach inspired by [Gre13] to address the question of which **O**-, **T**-, and **I**-type manifolds can be obtained by positive integral surgery on knots in S^3. In summary, our methods involved associating lattice structures to each manifold type specified by the manifold's surgery coefficients. From there, we used a combination of Heegaard Floer homology and Donaldson's Theorem A to produce an exhaustive list of surgery coefficients that describe integral surgery on a knot in S^3 producing the desired manifold.

Future directions for this research include determining whether the list of coefficients we found can be further refined, and expanding the proof to include the case when surgery is negative.

References

[Bal+16] William Ballinger et al. "The prism manifold realization problem". Preprint. 2016.

[Don87] S.K. Donaldson. "The orientation of Yang-Mills moduli spaces and 4-manifold topology". In: *Journal of Differential Geometry* 26.3 (1987).

[GS99] Robert E. Gompf and András I. Stipsicz. *4-Manifolds and Kirby Calculus*. American Mathematical Society, 1999.

[Gre12] Joshua Evan Greene. "L-space surgeries, genus bounds, and the cabling conjecture". In: *Journal of Differential Geometry* 100.3 (2012), pp. 491–506.

[Gre13] Joshua Evan Greene. "The lens space realization problem". In: *Annals of Mathematics* 177 (2013), pp. 449–511.

[Gu14] Liling Gu. "Integral Finite Surgeries on Knots in S^3". Unpublished. 2014.

[Lic62] W. B. R. Lickorish. "A representation of orientable combinatorial 3-manifolds". In: *Annals of Mathematics* 76.3 (1962), pp. 531–540.

[OS03] P. Ozaváth and Z. Szabó. "Absolutely graded Floer homologies and intersection forms for four-manifolds with boundary". In: *Adv. Math.* 173.2 (2003), pp. 170–261.

[Sav99] Nikolai Saveliev. *Lectures on the topology of 3-manifolds: an introduction to the Casson invariant*. de Gruyter, 1999.

[Wal60] Andrew H. Wallace. "Modifications and cobounding manifolds". In: *Canadian Journal of Mathematics* 12 (1960), pp. 503–528.