Simplicial Homology and De Rham's Theorem

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April 20, 2009

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Submitted in partial fulfillment of the requirements for
Graduation with Distinction

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Abstract

After giving the necessary background in simplicial homology and cohomology, we will state Stokes's theorem and show that integration of differential forms on a smooth, triangulable manifold M provides us with a homomorphism from the De Rham cohomology of M to the simplicial cohomology of M. De Rham's theorem, which claims that this homomorphism is in fact an isomorphism, will then be stated and proved.

Contents

1 Acknowledgments

I would like to gratefully acknowledge the supervision and guidance of Prof. William Allard during this work. I never would have thought that I could have learned and accomplished as much as is encompassed in this project had it not been for his confidence in me. I also thank Prof. Lenhard Ng both for agreeing to evaluate my work and for his encouragement during my classroom experience with him. Additionally, I thank Prof. Kraines for his constant support for all of the undergraduates in the department.

I extend my gratitude to Dr. Joshua Davis and Dr. Amir Jafari for their encouragement during my first years at Duke, their inspiration, and the wonderful chats during office hours that first sparked my passion for pure mathematics and introduced me to the beauty of geometry. Their influence and interaction with me has been invaluable.

Next, I would like to thank my friends for listening to me and giving me advice throughout the course of this project. I especially thank Barry Wright III and Damien Wilburn for their constant support and feedback.

Finally, I thank my family; this undertaking would not have been possible without their unconditional love and their guidance.

2 Introduction

In an elementary course in vector calculus, one is exposed to the gradient of a scalar field and the curl of vector field in two-dimensional Euclidean space in terms of the operator

$$\nabla = (\frac{\partial}{\partial x}, \frac{\partial}{\partial y})$$

where x, y are the cartesian coordinates of $\mathbb{R}^2$. For a vector field $F(x,y) = (P(x,y), Q(x,y)) : U \subset \mathbb{R}^2 \to \mathbb{R}^2$ we have

$$\text{curl } F = \nabla \times F = \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y},$$

and for a scalar field $f : U \subset \mathbb{R}^2 \to \mathbb{R}$, we have

$$\text{grad } f = \nabla f = (\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}).$$

From these definitions, we can see clearly that curl (grad f) $= \nabla \times \nabla f = 0$. Hence, if $F = \nabla g$ for some $g \in C^\infty(U, \mathbb{R})$, then curl $F = 0$. But given a smooth vector field F, it is not necessarily true that curl $F = 0$ implies that there exists a smooth scalar field $g \in C^\infty(U, \mathbb{R})$ such that $F = \nabla g$. This is the case because of the topological obstructions. For example, consider the vector field F defined on $U = \mathbb{R}^2 \setminus \{(0,0)\}$ given by

$$F(x,y) = (\frac{-y}{x^2+y^2}, \frac{x}{x^2+y^2}).$$

An easy calculation reveals that curl $F = 0$, and one could naively expect that Green's theorem would imply that the integral $\int_\gamma F \cdot dr = 0$ for any closed curve γ in U. But if we consider the parametrization γ given by $(x,y) \mapsto (\cos t, \sin t)$ where $t \in [0, 2\pi]$, one easily calculates that $\int_\gamma F \cdot dr = 2\pi$. This implies that there does not exist $g \in C^\infty(U, \mathbb{R})$ such that $F = \nabla g$. In the language of differential forms, the smooth 1-form $\omega = F \cdot dr$ is closed (i.e. $d\omega = 0$) but there does not exist a smooth 0-form τ such that $\omega = d\tau$. This is because U is not simply connected; it is missing the origin. The topology of U obstructs $F(x,y)$ from being the gradient of a smooth scalar function.

Define Z to be the vector space of all smooth vector fields v defined on U such that curl $v = 0$, and define B to be the vector space of the gradients of all smooth scalar fields defined on U. It is clear from the foregoing that $B \subset Z$, but for topological reasons $B \neq Z$. We can think of the quotient space Z/B as a measurement of the topological obstructions in U. The space Z/B is not trivial in this example, but if it were, then every smooth vector field define on U with a curl equal to 0 would be the gradient of a smooth potential function defined on U.

We can generalize this problem to n-dimensional manifolds and differential forms defined on them: Given a smooth k-form ω on a smooth manifold M such that $d\omega = 0$, when will $\omega = d\tau$, where τ is a smooth $(k-1)$-form? In order understand this problem, we first develop tools to study simplices (which can be thought of as the building blocks of manifolds) and their topology, particularly simplicial homology. Once we do this, we relate simplicial homology to the De Rham cohomology of manifolds, which gives us a framework in which we can tackle this problem formally.

3 Simplices

We begin our discussion of topology of manifolds with developing some machinery to discuss simplices. A simplex can be discussed in complete abstraction, but its geometric realization is (intuitively) the arbitrary-dimensional analogue of a triangle in $\mathbb{R}^2$.[1]

Definition 3.1. *A **simplicial complex** K is an ordered pair (V, S) consisting of a finite nonempty set V (whose elements are called **vertices**) and a set S of finite nonempty subsets of V (whose elements are called **simplices**) such that*

(1) Any set consisting of exactly one vertex is a simplex.

(2) Any nonempty subset of a simplex is a simplex.

A **p-dimensional simplex**, also called a **p-simplex**, is a simplex s containing $p+1$ vertices. If t is a (proper) subset of s, then t is called a **(proper) face** of s. The union of the proper faces of s is called the **boundary** of s, written as Bd s; the **interior** of s is $s\backslash\text{Bd } s$. If t has dimension $q < p$, then t is called a **q-face** of s. The **p-skeleton** of K (written as $K^{(p)}$) is set of all p-simplices of K for a given p. Condition 1 implies that there exists a bijection from $K^{(0)}$ to V, and Condition 2 implies that any simplex is determined by its 0-faces. If K is a simplicial complex, its **dimension**, denoted by $\dim K$, is equal to $\sup\{\dim s : s \in K\}$. Since we will be considering only the case where V has finitely many vertices, $\dim K = \max\{\dim s : s \in K\}$.

3.1 Geometric realization of an abstract simplicial complex

Given a simplicial complex K and its vertex set $V = \{v_1, \ldots, v_m\}$ consider the set of all functions $b : V \to I = [0, 1]$ such that for $j \in \{1, \ldots, m\}$,

(a) For any such b, $\{v_j \in V : b(v_j) \neq 0\}$ is a simplex of K

(b) For any such b, $\displaystyle\sum_{j=1}^{m} b(v_j) = 1$.

The real number $b(v_j)$ is called the **j-th barycentric coordinate** of b. We write the set of all such functions b as $|K|$. Because b is a function from V to I, $|K| \subset \mathbb{R}^V$. Given $b_1, b_2 \in |K|$, $|K|$ has the metric

$$d(b_1, b_2) = \sqrt{\sum_{j=1}^{m} [b_1(v_j) - b_2(v_j)]^2}.$$

Using barycentric coordinates, we define for any $s \in K$ the **closed simplex** $|s|$ by

$$|s| = \{b \in |K| : b(v_j) \neq 0 \Rightarrow v_j \in s\}.$$

If s is a p-simplex, $|s|$ is in one-to-one correspondence with the set $\{x \in \mathbb{R}^{p+1} | 0 \leq x_i \leq 1, \sum x_i = 1\}$. Furthermore, the metric topology on $(|K|, d)$ induces on $|s|$ a topology that makes $(|s|, d)$ a topological space homeomorphic to the above compact convex subset of $\mathbb{R}^{p+1}$. If $s_1, s_2 \in K$, then clearly $s_1 \cap s_2$ is either empty (in which case $|s_1| \cap |s_2| = \emptyset$) or a face of both s_1 and s_2 (in which case $|s_1 \cap s_2| = |s_1| \cap |s_2|$). Therefore, in either case $(|s_1| \cap |s_2|, d)$ is a closed set in both $(|s_1|, d)$ and $(|s_2|, d)$. Therefore, a subset $A \subset |K|$ is closed in $|K|$ if and only if $A \cap |s|$ is closed for every $s \in K$. Since we are only considering the case where K is a finite simplicial complex, it follows that $|K|$ is compact.

For $s \in K$, the **open simplex** $(s) \subset |K|$ is defined by

$$(s) = \{b \in |K| : b(v_j) \neq 0 \Rightarrow v_j \in s \text{ for all } v_j \in V\}.$$

[1]A more complete treatment of simplices can be found in [2], [4], [5], and [6].

Note that this is consistent with the definition of Int s.

Given a vertex $v \in V$, the **star** of v, written as St v, is defined by

$$\text{St } v = \{b \in |K| : b(v) \neq 0\}.$$

The reader can verify that it follows from this that St $v = \bigcup\{\text{Int } s : v \in s \in K\}$.

Because $b \to b(v)$ is a continuous map from $(|K|, d)$ to I, St v is open in $(|K|, d)$, and hence also in $|K|$. This definition can also be stated as such: St v is the union of all open simplices that contain v as a vertex.

Let K be a simplicial complex and let $v_0, \ldots, v_p$ be the points of a closed p-simplex $|s|$. Given real numbers $\lambda_0, \ldots, \lambda_p$ such that $0 \leq \lambda_i \leq 1$ for $i = 0, \ldots, p$ and such that $\sum_{i=0}^{p} \lambda_i = 1$, the function $x = \sum_{i=0}^{p} \lambda_i v_i$ is again a point of $|s|$. Therefore, each closed simplex has a linear structure such that convex combinations of its points are again points of the closed simplex.

If X is a topological space which is a subset of some real vector space, then a continuous map $f : |K| \to X$ is **linear in** K if for every $b \in |K|$, $\sum_{v_j \in V} b(v_j) f(v_j)$ is a point of X and $f(b) = \sum_{v_j \in V} b(v_j) f(v_j)$. We now want to consider linear imbeddings of $|K|$ in Euclidean space.

Definition 3.2. *Given a set $\{v_0, \ldots, v_n\}$ of points in $\mathbb{R}^n$, this set is said to be* ***affinely independent*** *if for any real scalars λ_k, where $k \in \{0, \ldots, n\}$, the equations*

$$\sum_{k=0}^{n} \lambda_i = 0 \quad \text{and} \quad \sum_{k=0}^{n} \lambda_i v_i = 0$$

imply that $\lambda_k = 0$ for $k = 0, \ldots, n$. An ***affine transformation*** *is a mapping α from a vector space V to a vector space W that has coefficients in the same division ring as V such that $\alpha(v) = T(v) + w$, where $v \in V$, for some linear transformation $T : V \mapsto W$ and some vector $w \in W$.*

It is easy to verify that T is injective if and only if α is injective, and T is surjective if and only if α is surjective. An **affine isomorphism** is a bijective affine transformation.

Proposition 3.3. *Given a set of unique points $A = \{a_0, \ldots, a_n\}$, the following are equivalent:*

(1) A is an affinely independent set.
(2) The set $\{a_i - a : a \neq a_i\}$ is linearly independent for all $a_i \in A$.
(3) The set $\{a_i - a : a \neq a_i\}$ is linearly independent for any $a_i \in A$.

Proposition 3.4. *Given a transformation f, the following are equivalent:*

(1) α is an affine transformation.
(2) The mapping $x \mapsto \alpha(x) - \alpha(x_0)$ is linear for all $x_0 \in \mathbb{R}^n$.
(3) The mapping $x \mapsto \alpha(x) - \alpha(x_0)$ is linear for any $x_0 \in \mathbb{R}^n$.

Lemma 3.5. *Given a simplex s with the vertex set $\{v_0, \ldots, v_p\}$, a linear map $f : |s| \to \mathbb{R}^n$ is an imbedding if and only if it maps the vertex set of s to an affinely independent set in $\mathbb{R}^n$, where $n \geq p$.*

(These proofs are left to the reader.) A **geometric realization** of a simplicial complex K in $\mathbb{R}^n$ is a linear imbedding of $|K|$ in $\mathbb{R}^n$. Our look at simplicial homology will focus on geometric realizations of abstract simplicial complexes. From now on, for some imbedding f, an abstract simplicial complex K will be identified with $f(|K|)$, and a simplex s with a vertex set $\{v_0, \ldots, v_p\}$ will be identified with

$$\sigma = f(|s|) = \{x \in U \supset \mathbb{R}^p | x = \sum_{i=0}^{p} \lambda_i v_i\},$$

(where λ_i is exactly the barycentric coordinate of x with respect to the vertex v_i), and $|K|$ will be identified with $\bigcup_{s\in K} f(|s|)$.

3.2 The Simplicial Approximation Theorem.

Definition 3.6. *If $\sigma =< v_0 \dots v_p >$ is a simplicial complex, then the **barycenter** of σ is defined to be the point*

$$\hat{\sigma} = \sum_{i=0}^{p} \frac{1}{p+1} v_i.$$

It is the point of Int σ all of whose barycentric coordinates with respect to the vertices of σ are equal. If general, $\hat{\sigma}$ is the centroid of σ.

If K is a simplicial complex, we can define a sequence of subdivisions of the skeletons of K as follows: Let $L_0 = K^{(0)}$. In general, if L_p is a subdivision of $K^{(p)}$, let L_{p+1} be the subdivision of the $K^{(p+1)}$ obtained by starring L_p from the barycenters of the $(p+1)$-simplices of K. The union of the complexes L_p is called the is called the **first barycentric subdivision** of K, denoted $sd(K)$. We can define the second barycentric subdivision of K as $sd(sd(K)) = sd^2(K)$; in general, the **n-th barycentric subdivision** of K is defined to be $sd^n(K)$.

We now will show that if $h : |K| \to |L|$ is a continuous map, then there is a subdivision K' of K such that h has a simplicial approximation $f : K' \to L$. The proof when K is finite follows easily from the use of barycentric subdivisions.

Theorem 3.7 (The Simplicial Approximation Theorem)**.** *Let K and L be simplicial complexes; let K be finite. Given a continuous map $h : |K| \to |L|$, there is an N such that h has a simplicial approximation $f : sd^N(K) \to L$.*

PROOF: Cover $|K|$ by the open sets $h^{-1}(\text{St } w)$ for all vertices $w \in L$. Given this open covering $\mathcal{A}$ of the compact metric space K, there is a number λ such that any set of diameter less than λ lies in one of the elements of $\mathcal{A}$. If there is no such λ, one could choose a sequence S_n of sets with diameter less than $1/n$ but does not lie in any element of $\mathcal{A}$. Choose $x_n \in S_n$; by compactness, some subsequence x_{n_i} converges, say to x. Now $x \in A$ for some $A \in \mathcal{A}$. Because A is open, it contains C_{n_i} for sufficiently large i, contrary to our construction.

Choose N so that each simplex in $sd^N(K)$ has diameter less than $\lambda/2$. Then each star of a vertex in $sd^N(K)$ has diameter less than λ, so it lies in one of the sets $h^{-1}(\text{St } w)$. Then $h : |K| \to |L|$ satisfies the star condition relative to $sd^N(K)$ and L; that is, for each vertex $v \in sd^N(K)$ there is a vertex $w \in L$ such that $h(\text{St } v) \subset \text{St } w$. Therefore, the desired simplicial approximation exists. □

4 Homology of a Simplicial Complex

Since we will be integrating over simplices, discussion of orientation is critical. If σ is a p-simplex with a vertex set $\{v_0, \ldots, v_n\}$, define two orderings of its vertex set to be equivalent if they differ by an even permutation. If $p > 0$, then this equivalence relation has two equivalence classes, each of which is called an **orientation** of σ. An **oriented simplex** is a simplex σ paired with an orientation. If $v_0, \ldots, v_p$ are affinely independent, let $v_0 \ldots v_p$ denote the simplex they span, and let $< v_0 \ldots v_p >$ denote the oriented simplex consisting of the simplex $v_0, \ldots, v_p$ and the equivalence class of the particular ordering of $(v_0, \ldots, v_p)$.

Definition 4.1. *Let K be a simplicial complex, and let $\mathcal{O}_p$ the oriented p-simplices of K. A* ***p*-chain** *on K is a function $c : \mathcal{O}_p \to \mathbb{Z}$ such that*

(1) $c(-\sigma) = -c(\sigma)$, where σ and $-\sigma$ are opposite orientations of the same simplex.
(2) $c(\sigma) = 0$ for all but finitely many oriented simplices σ.

If we write the law of composition in $\mathbb{Z}$ additively, then we add p-chains by adding their values, forming the **group of (oriented) p-chains of K** (denoted $C_p(K, \mathbb{Z})$). If $p < 0$ or $p > \dim K$, then $C_p(K, \mathbb{Z})$ is the trivial group. If $\sigma \in \mathcal{O}$, the **elementary chain** c corresponding to σ is the function defined as follows:

(1) $c(\sigma) = 1$

(2) $c(-\sigma) = -1$

(3) $c(\tau) = 0$ when $\tau \in \mathcal{O} \setminus \{\sigma\}$.

By abuse of notation, σ can be a simplex, an oriented simplex, or the elementary p-chain c corresponding to the oriented simplex σ. We will try to maintain the notation that σ is an oriented simplex.

$C_p(K, \mathbb{Z})$ is a free abelian group with a basis obtained by orienting each p-simplex and using the corresponding chains as a basis, for once all p-simplices of K are (arbitrarily) oriented, each p-chain can be written uniquely as a finite linear combination $c = \sum n_i \sigma_i$ of each elementary chain σ_i. c then assigns the value n_i to each σ_i, $-n_i$ to each $-\sigma_i$, and 0 to each oriented p-simplex not appearing in the summation. In fact, given an arbitrary abelian group G, the group $C_p(K, G)$ of p-chains of K with coefficients in G can be defined as the set of all formal linear combinations

$$\sum_{\sigma \in U \subset K} g_\sigma \sigma, \qquad g_\sigma \in G,$$

subject to the relation $g_\sigma(-\sigma) = -g_\sigma(\sigma)$. (We are writing the group operation in G additively.) However, for our purposes, we are only interested in the cases where G is $\mathbb{Z}$ or $\mathbb{R}$.

Definition 4.2. *Let $\sigma =< v_0 \ldots v_p >$ be an oriented p-simplex. The* **boundary** *of σ, denoted $\partial_p \sigma$, is the $(p-1)$-chain defined by*

$$\partial_p \sigma = \sum_{i=0}^{p} (-1)^i < v_0, \ldots, \hat{v}_i, \ldots, v_p >,$$

where $\hat{x}$ means that x is deleted.

Since ∂_p is obviously linear, the boundary of a p-chain is a $(p-1)$-chain. For an arbitrary abelian group G and a set of oriented simplices $U \subset K$, we have that

$$\partial_p(\sum_{\sigma \in U} g_\sigma \sigma) = \sum_{\sigma \in U} g_\sigma(\partial_p \sigma),$$

making the boundary operator a group homomorphism from $\partial_p : C_p(K, G) \to C_{p-1}(K, G)$.

Lemma 4.3. $\partial_{p-1} \circ \partial_p = 0$.

PROOF: Since $\partial_{p-1} \circ \partial_p$ is linear, it suffices to check this on generators $< v_0 \dots v_p >$ as follows:

$$\begin{aligned}
\partial_{p-1}(\partial_p < v_0 \dots v_p >) &= \partial_{p-1}(\sum_{i=0}^{p}(-1)^i < v_0 \dots \hat{v}_i \dots v_p >) \\
&= \sum_{i=0}^{p}(-1)^i \partial_{p-1} < v_0 \dots \hat{v}_i \dots v_p > \\
&= \sum_{j<i}^{p}(-1)^{i+j} < \dots \hat{v}_j \dots \hat{v}_i \dots > + \sum_{j>i}^{p}(-1)^{i+j-1} < \dots \hat{v}_i \dots \hat{v}_j \dots > \\
&= \sum_{i<j}^{p}((-1)^{i+j} + (-1)^{i+j-1}) < \dots \hat{v}_i \dots \hat{v}_j \dots > \\
&= 0. \quad \square
\end{aligned}$$

Now that we have a series of free abelian groups $C_p(K,G)$ and a homomorphism ∂_p between them such that $\partial_{p-1} \circ \partial_p = 0$, we can look at the chain complex

$$\dots \xrightarrow{\partial_2} C_1(K,G) \xrightarrow{\partial_1} C_0(K,G) \xrightarrow{\partial_0} C_{-1}(K,G) \xrightarrow{\partial_{-1}} \dots$$

As earlier, if $p < 0$ or $p > \dim K$, then we let $C_p(K)$ denote the trivial group, making the nontrivial part of the chain complex the exact sequence

$$0 \xrightarrow{\partial_{p+1}} C_p(K,G) \xrightarrow{\partial_p} C_{p-1}(K,G) \xrightarrow{\partial_{p-1}} \dots \xrightarrow{\partial_1} C_0(K,G) \xrightarrow{\partial_0} 0.$$

Definition 4.4. *The set* $\ker(\partial_p)$*, also denoted as* $Z_p(K,G)$*, is the set of* ***p-cycles*** *of* K*. The set* $im(\partial_{p+1})$*, also denoted as* $B_p(K,G)$*, is the set of* ***p-boundaries****. Because* $\partial_p \circ \partial_{p+1} = 0$*, each boundary of a* $(p+1)$*-chain is a p-cycle, so* $B_p(K,G) \subset Z_p(K,G)$*. We define* $H_p(K,G) = Z_p(K,G)/B_p(K,G)$ *and call it the* ***p-th homology group of*** K*. Two p-chains* c_1 *and* c_2 *are considered* ***homologous*** *if* $c_1 - c_2 = \partial_{p+1}c$ *for some* $(p+1)$*-chain* c*. If* $c_1 = \partial_{p+1}c$*, then* c_1 *is* ***homologous to zero****.*

Theorem 4.5. *If* K *and* K' *are two simplicial complexes and there exists a homeomorphism between* $|K|$ *and* $|K'|$*, then* $H_p(K,G)$ *is isomorphic to* $H_p(K',G)$*, making homology groups a topological invariant.*[2]

[2]Proof of this landmark theorem of algebraic topology can be found in [2].

5 Applications of Homology Groups

Homology groups are among the most useful topological invariants of a space because of their computational tractability, making it convenient to define other topological properties of a space in terms of the space's homology groups.

5.1 The Euler Characteristic

For example, given a simplicial complex, consider the Euler characteristic χ of $|K|$. Originating with Euler and Descartes, the polyhedral formula $\chi = V - E + F = 2$ relates the number of vertices V, the number of edges E, and the number of faces F of a convex polyhedron. This can be generalized using a surface of genus g to the Poincare formula $\chi \equiv V - E + F = \chi(g)$, where $\chi(g) = 2 - 2g$ is the Euler characteristic for oriented compact surfaces ($g = 0$ yields the polyhedral formula). Since $|K|$ is clearly an oriented compact surface, it makes sense to compute its Euler characteristic. In terms of simplices, V is the number of 0-simplices in K, E is the number of 1-simplices in K, and F is the number of 2-simplices in K. Generalizing this for surfaces of dimension greater than 2 gives us the formula

$$\chi(|K|) = \sum_{p=0}^{\dim K} (-1)^p (\text{number of } p\text{-simplices in } K).$$

Theorem 5.1. $\chi(|K|) = \sum_{p=0}^{\dim K} (-1)^p \dim H_p(K, \mathbb{Z})$.

PROOF: If we look further, noting that $\dim B_{-1}(K, \mathbb{R}) = \dim B_{\dim K}(K, \mathbb{R}) = 0$, we can use the fact that the number of p-simplices of $C_p(K, \mathbb{R})$ is equal to $\dim C_p(K, \mathbb{R}) = \dim \ker \partial_p + \dim$ Im $\partial_p = \dim Z_p(K, \mathbb{R}) + \dim B_{p-1}(K, \mathbb{R})$ to show that

$$\begin{aligned}
\chi(|K|) &= \sum_{p=0}^{\dim K} (-1)^p (\text{number of } p\text{-simplices in } K) \\
&= \sum_{p=0}^{\dim K} (-1)^p (\dim Z_p(K, \mathbb{R}) - \dim B_{p-1}(K, \mathbb{R})) \\
&= \sum_{p=0}^{\dim K} (-1)^p \dim Z_p(K, \mathbb{R}) + \sum_{p=0}^{\dim K} (-1)^p \dim B_{p-1}(K, \mathbb{R}) \\
&= \sum_{p=0}^{\dim K} (-1)^p \dim Z_p(K, \mathbb{R}) + \sum_{p=0}^{\dim K} (-1)^{p+1} \dim B_p(K, \mathbb{R}) \\
&= \sum_{p=0}^{\dim K} (-1)^p (\dim Z_p(K, \mathbb{R}) - \dim B_p(K, \mathbb{R})) \\
&= \sum_{p=0}^{\dim K} (-1)^p \dim H_p(K, \mathbb{R}). \quad \square
\end{aligned}$$

Since homology is a topological invariant, the Euler characteristic is a topological invariant.

5.2 The Fundamental Group

Homology has a special relationship with the fundamental group of an arcwise-connected topological space.

Definition 5.2. *Let X be a topological space. A* ***path*** *in X with origin x_0 and endpoint x_1 is a continuous map $\alpha : [0,1] \to X$ such that $\alpha(0) = x_0$ and $\alpha(1) = x_1$. We let λ_x denote some path from x_0 to x; we shall take λ_{x_0} to be the constant path.*

Definition 5.3. *For $x_0, x_1, x_2 \in X$, let α be a path from x_0 to x_1, and let β be a path from x_1 to x_2. The* ***product*** *of two paths α and β, denoted $\alpha * \beta$, is the path from x_0 to x_2 defined by*

$$\alpha\beta(t) = \begin{cases} \alpha(2t), & 0 \leq t \leq \frac{1}{2} \\ \beta(2t-1), & \frac{1}{2} \leq t \leq 1 \end{cases}.$$

The ***inverse of*** *α is the path α^{-1} from x_1 to x_0 defined by $\alpha^{-1}(t) = \alpha(1-t)$.*

Two paths α and β from x_0 to x_1 are **homotopic** (written as $\approx$) if there exists a continuous map $F : [0,1] \times [0,1] \to X$ such that $F(0, t_1) = x_0$ and $F(1, t_1) = x_1$ for all $t_1 \in [0,1]$; and $F(t_2, 0) = \alpha(t_2)$ and $F(t_2, 1) = \beta(t_2)$ for all $t_2 \in [0,1]$. The relation $\approx$ is clearly an equivalence relation; it is reflexive, symmetric, and transitive. If we denote the class of paths that are homotopic to a path α under the relation $\approx$ as $\langle\alpha\rangle$, then we can define the product of two homotopy classes $\langle\alpha\rangle$ and $\langle\beta\rangle$ to be $\langle\alpha * \beta\rangle$ wherever the path $\alpha * \beta$ is defined. We can also define the inverse of a homotopy class as $\langle\alpha\rangle^{-1} = \langle\alpha^{-1}\rangle$.

If X is a topological space and $x_0 \in X$, the set of $\approx$ equivalence classes of paths where the starting point and endpoint are both x_0 forms a group under the operations of multiplication and inversion as previously defined. This group, denoted by $\pi_1(X, x_0)$, is called the **fundamental group**, or **first homotopy group**, of the pair (X, x_0). We now introduce some group theory that will help us relate the fundamental group to the first homology group.

Definition 5.4. *Let g, h be elements of a group G. The* ***commutator*** *of g and h is $[g,h] = ghg^{-1}h^{-1}$.*

Lemma 5.5. *$[g,h] = 1$ if and only if g and h commute.*

Proof: Multiply $[g,h]$ and 1 by hg on the right. $[g,h]hg$ simplifies to gh, and we have $gh = hg$. □

Definition 5.6. *Let G be a group. The* ***commutator subgroup*** *of G is the smallest subgroup of G containing all of the commutators of G. This subgroup is denoted by $[G,G]$. Using this, one can define an abelian group as a group with a trivial commutator subgroup.*

Lemma 5.7. *The commutator subgroup of a group G is a normal subgroup of G.*

Proof: Let $k, g, h \in G$. We then compute

$$\begin{aligned} k[g,h]k^{-1} &= k(ghg^{-1}g^{-1})k^{-1} \\ &= (kgk^{-1})(khk^{-1})(kg^{-1}k^{-1})(kh^{-1}k^{-1}) \\ &= (kgk^{-1})(khk^{-1})(kgk^{-1})^{-1}(khk^{-1})^{-1} \\ &= [kgk^{-1}, khk^{-1}] \in [G,G], \end{aligned}$$

making $[G,G]$ closed under conjugation of elements in G. Therefore, $[G,G]$ is normal in G. □

Specifically, the left and right cosets of $[G,G]$ agree.

Lemma 5.8. *If G is any group, then $G/[G,G]$ is an abelian group.*

Proof: For any $g, h \in G$, we compute

$$\begin{aligned} [g[G,G], h[G,G]] &= [[G,G]g, h[G,G]] \\ &= [G,G]gh[G,G][G,G]^{-1}g^{-1}h^{-1}[G,G]^{-1} \\ &= [G,G]ghg^{-1}h^{-1}[G,G] \\ &= [G,G][g,h][G,G] \\ &= [G,G]. \end{aligned}$$

Therefore, any two elements in $G/[G,G]$ commute, making $G/[G,G]$ abelian. $\square$

We call $G/[G,G]$ the **abelianization** of G, denoted $\tilde{G}$.

Theorem 5.9. *By regarding loops as 1-cycles, we obtain a homomorphism $h : \pi_1(|K|, x_0) \to H_1(K, \mathbb{Z})$. If $|K|$ is path-connected, then h is surjective and $\ker h = [\pi_1(|K|, x_0), \pi_1(|K|, x_0)]$. Therefore, h induces an isomorphism from $\tilde{\pi}_1(|K|, x_0)$ onto $H_1(K, \mathbb{Z})$*[3].

PROOF: We can regard loops as 1-cycles by applying the simplicial approximation theorem. We denote $f \approx g$ for a relation of homotopy, fixing endpoints, between paths f and g. Regarding f and g as chains, $f \sim g$ will mean that f is homologous to g, meaning that $f - g$ will be the boundary of a 2-chain. Here are some facts about this relation.

(i) If f is a constant path, then $f \sim 0$. Namely, f is a cycle since it is a loop, and since $H_1(point) = 0$, f must then be a boundary.

(ii) If $f \approx g$, then $f \sim g$. To see this, consider a homotopy $F : [0,1] \times [0,1] \to |K|$ from f to g. This yields a pair of 2-simplices in $|K|$ by subdividing the square $[0,1] \times [0,1]$ with orientation $< v_0v_1v_2v_3 >$ into two triangles $\sigma_1 =< v_0v_1v_3 >$ and $\sigma_2 =< v_0v_2v_3 >$. Let f be the loop moving along σ_1 and let g be the loop moving along σ_2. Then $\partial(\sigma_1 - \sigma_2) = f - g$.

(iii) $f * g \sim f + g$. On $< v_0v_1v_2 >$, put f on the edge $< v_0v_1 >$ and g on the edge $< v_1v_2 >$. Then define a 2-simplex σ to be constant on the lines perpendicular to the edge $< v_0v_2 >$, resulting in $f * g$ being on the edge $< v_0v_2 >$. Then $\partial\sigma = g - f * g + f$ as desired.

(iv) $f^{-1} \sim -f$. This follows from (i)-(iii), which give $f + f^{-1} \sim f * f^{-1} \sim 0$.

By (ii), we have a well defined function

$$\phi : \pi_1(|K|, x_0) \to H_1(K, \mathbb{Z})$$

taking homotopy classes $[f]$ to homology classes $[[f]]$. This is a homomorphism; to see this, let f, g be loops and note that $\phi([f][g]) = \phi([f * g]) = [[fg]] = [[f]] + [[g]]$ by (iii). Our goal is to prove that ϕ is an isomorphism. To do this, we first define the function that will be the inverse of ϕ. Let f be a path, and put $\hat{f} = \lambda_{f(0)} * f * \lambda_{f(1)}^{-1}$, which is a loop at x_0. Define $\psi(f) = [\hat{f}] \in \tilde{\pi}_1(|K|, x_0)$. This extends to a homomorphism

$$\psi : C_1(K, \mathbb{Z}) \to \tilde{\pi}_1(|K|, x_0).$$

We need two more lemmas in order to complete the proof of the theorem.

Lemma 5.10. *ϕ takes the group $B_1(K, \mathbb{Z})$ of 1-boundaries into $1 \in \tilde{\pi}_1(|K|, x_0)$.*

PROOF: Let $\sigma =< v_0v_1v_2 >$ and let f be the path along $< v_0v_1 >$, g the path along $< v_1v_2 >$, and h the path along $< v_2v_0 >$. Then

$$\begin{aligned}
\psi(\partial\sigma) &= \psi(< v_0v_1 > + < v_1v_2 > + < v_2v_0 >) \\
&= \psi(f + g + h) \\
&= \psi(f)\psi(g)\psi(h) \\
&= [\hat{f}][\hat{g}][\hat{h}] \\
&= [\hat{f}\hat{g}\hat{h}] \\
&= [\lambda_{v_0} * f * \lambda_{v_1}^{-1} * \lambda_{v_1} * g * \lambda_{v_2}^{-1} * \lambda_{v_2} * h * \lambda_{v_0}^{-1}] \\
&= [\lambda_{v_0} * f * g * h * \lambda_{v_0}^{-1}] \\
&= [constant] = 1. \quad \square
\end{aligned}$$

[3]This follows the proof found in [1]

If f is a loop, then $\psi \circ \phi([f]) = \psi([[f]]) = [\lambda_{x_0} * f * \lambda_{x_0}^{-1}] = [f]$ since λ_{x_0} was chosen to be a constant path. Therefore, $\psi \circ \phi$ is the identity function. We have yet to show that $\phi \circ \psi$ is the identity function.

The assignment $x \mapsto \lambda_x$ takes 0-simplices into 1-simplices and thus extends to a homomorphism $\lambda : C_0(K, \mathbb{Z}) \to C_1(K, \mathbb{Z})$ by $\lambda_{\sum n_x x} = \lambda(\sum_x n_x x) = \sum_x n_x \lambda_x$.

Lemma 5.11. *If σ is a 1-simplex in $|K|$, then the class $\phi \circ \psi(\sigma)$ is represented by the cycle $\sigma + \lambda_{\partial\sigma}$. Also, if c is a 1-chain, then $\phi \circ \psi(c) = [[c - \lambda_{\partial c}]]$.*

PROOF: We compute

$$\phi \circ \psi(\sigma) = \phi[\lambda_{v_0} * \sigma * \lambda_{v_1}^{-1}] = [[\lambda_{v_1} * \sigma * \lambda_{v_0}^{-1}]] = [[\lambda_{v_1} + \sigma + \lambda_{v_0}^{-1}]] = [[\lambda_{v_1} + \sigma - \lambda_{v_0}]]$$

by (iii) and (iv). The rest follows immediately.

If c is a 1-cycle, then by the preceding lemma, $\phi \circ \psi[[c]] = [[c]]$, finishing the proof of the theorem.

Theorem 5.12. *Let $|K|$ be any $(n-1)$-connected based space. Then*

$$\phi : \pi_n(|K|, x_0) \to H_n(K, \mathbb{Z})$$

is the Abelianization homomorphism if $n = 1$ and is an isomorphism if $n > 1$.[4]

[4]A proof of this theorem can be found in [3].

6 Cohomology of a Simplicial Complex

Definition: Let K be a simplicial complex. For $0 \leq p \leq \dim K$, let

$$C^p(K,G) = \mathrm{Hom}(C_p(K,G),G) = (C_p(K,G))^*,$$

where V^* is the notation for the dual of V. Let $\delta^p : C^p(K,G) \to C^{p+1}(K,G)$ be the adjoint of the map $\partial_{p+1} : C_{p+1}(K,G) \to C_p(K,G)$. Thus δ^p is defined by

$$\langle \delta^p c^p, d_{p+1} \rangle = \langle c^p, \partial_{p+1} d_{p+1} \rangle,$$

where c^p is a p-cochain and d_p is a p-chain. (Here, $\langle c^p, c_p \rangle$ is used to denote the value of c^p evaluated at c_p.) As such, we have the relation $\delta^{p+1} \circ \delta^p = 0$, and we can look at the chain complex

$$0 \xrightarrow{\delta^0} C^1(K,G) \xrightarrow{\delta^1} C^2(K,G) \xrightarrow{\delta^3} \ldots \xrightarrow{\delta^{p-1}} C^p(K,G) \xrightarrow{\delta^p} 0$$

As we did with homology, we define $\ker \delta^p = Z^p(K,G)$ (whose elements are called **cocycles**), im $\delta^p = B^{p+1}(K,G)$ (whose elements are called **coboundaries**), and we note that $\delta^{p+1} \circ \delta^p = 0$ because $\partial_p \circ \partial_{p+1} = 0$. As such, $B^p(K,G) \subset Z^p(K,G)$, and we can define the ***p*-th cohomology group of** K as $H^p(K,G) = Z^p(K,G)/B^p(K,G)$.

We now derive an explicit formula exhibiting the effect of δ^p on elements of $C_p(K,G)$. For oriented p-simplices $\sigma_i, \sigma_j \in K$, let $\varphi_\sigma \in C^p(K,G)$ be defined by

$$\varphi_{\sigma_i}(\sigma_j) = \delta^i_j,$$

where δ^i_j is the Kronecker delta, and $\varphi_\sigma(-\sigma) = -\varphi_\sigma(\sigma) = -1$. Thus, if $\{\sigma_1, \ldots, \sigma_m\}$ is a basis for $C_p(K,G)$, then $\{\varphi_{\sigma_1}, \ldots, \varphi_{\sigma_m}\}$ is the dual basis for $C^p(K,G)$. Since δ^p is linear, we need only compute the effects of δ^p on the generators of a chain. So for oriented simplices $\sigma =< v_0 \ldots v_p >$ and $\tau =< w_0 \ldots w_{p+1} >$, we have that

$$\begin{aligned}
(\delta^p \varphi_\sigma)(\tau) &= \varphi_\sigma(\partial_p \tau) \\
&= \varphi_\sigma(\sum_{i=0}^{p+1} (-1)^i < w_0 \ldots \hat{w}_i \ldots w_{p+1} >) \\
&= \sum_{i=0}^{p+1} (-1)^i \varphi_\sigma(< w_0 \ldots \hat{w}_i \ldots w_{p+1} >) \\
&= \sum_{k \in \mathcal{V}} \varphi_{< v_k v_0 \ldots v_p >} \tau
\end{aligned}$$

where $\mathcal{V} = \{k :< v_k v_0 \ldots v_p > \in K\}$.

From this point forward, we will not include a subscript or superscript when referring to the boundary or coboundary operators for more tidy notation; the appropriate subscript or superscript will be made clear by the context in which each operator is used. Additionally, it will be assumed that chains and cochains will take coefficients in $G = \mathbb{R}$, so we will write $C_p(K,G)$ as $C_p(K)$ and so on.

7 Duality between $H_p(K)$ and $H^p(K)$

Definition 7.1. *If A and B are real vector spaces over a field F, then $Hom_F(A, B)$ is the set of linear transformations from A into B, which is also a vector space over F.*

Lemma 7.2. *Let $\mathcal{C}$ be a chain complex, and let F be a field. Let*

$$\phi : Hom(C_p, F) \to Hom_F(C_p \otimes F, F)$$

be defined by the equation

$$\langle \phi(f), c_p \otimes \alpha \rangle = \langle f, c_p \rangle \cdot \alpha,$$

where $f \in Hom(C_p, F)$, $c_p \in C_p$, and $\alpha \in F$. Then ϕ is a vector space isomorphism that commutes with δ.

Proof: We first check that ϕ is linear:

$$\begin{aligned}
\langle \phi(f), \alpha(c_p \otimes \beta) \rangle &= \langle \phi(f), c_p \otimes \alpha\beta \rangle \\
&= \langle f, c_p \rangle \cdot (\alpha\beta) \\
&= \alpha \cdot (\langle f, c_p \rangle \cdot \beta) \\
&= \alpha \cdot \langle \phi(f), c_p \otimes \beta \rangle.
\end{aligned}$$

To prove injectivity, suppose that $\phi(f)$ is the zero linear transformation. Then

$$\langle \phi(f), c_p \otimes 1 \rangle = 0 = \langle f, c_p \rangle \cdot 1 = \langle f, c_p \rangle$$

for all $c_p \in C_p$. This implies that f is the zero homomorphism.

To prove surjectivity, let $\tilde{\phi} : C_p \otimes F \to F$ be a linear transformation. Let us define $f : C_p \to F$ by the equation $f(c_p) = \tilde{\phi}(c_p \otimes 1)$. It follows that f is a homomorphism of abelian groups because $f(0) = 0$ and

$$\begin{aligned}
f(c_p + d_p) &= \tilde{\phi}((c_p + d_p) \otimes 1) \\
&= \tilde{\phi}(c_p \otimes 1 + d_p \otimes 1) \\
&= \tilde{\phi}(c_p \otimes 1) + \tilde{\phi}(d_p \otimes 1) \\
&= f(c_p) + f(d_p).
\end{aligned}$$

Furthermore, $\phi(f) = \tilde{\phi}$ since

$$\langle \phi(f), c_p \otimes \alpha \rangle = \langle f, c_p \rangle \cdot \alpha = \tilde{\phi}(c_p \otimes 1) \cdot \alpha = \tilde{\phi}((c_p \otimes 1) \cdot \alpha) = \tilde{\phi}(c_p \otimes \alpha),$$

where the last two equalities hold because $\tilde{\phi}$ is a linear transformation.

To show that ϕ commutes with δ, we observe that

$$\begin{aligned}
\langle \delta\phi(f), c_{p+1} \otimes \alpha \rangle &= \langle \phi(f), (\partial \otimes i_F)(c_{p+1} \otimes \alpha) \rangle \\
&= \langle f, \partial c_{p+1} \rangle \cdot \alpha \\
&= \langle \delta f, c_{p+1} \rangle \cdot \alpha \\
&= \langle \phi(\delta f), c_{p+1} \otimes \alpha \rangle. \quad \square
\end{aligned}$$

Theorem 7.3. *Let $\mathcal{C}$ be a chain complex, and let F be a field. Then there is a natural vector space isomorphism*

$$H^p(\mathcal{C}) \to Hom_F(H_p(\mathcal{C}), F).$$

Proof: We first note that if

$$0 \to A \to B \to C \to 0$$

is a short exact sequence of linear transformations between vector spaces over F, then for any vector space V over F, the dual sequence

$$0 \to Hom_F(C, V) \to Hom_F(B, V) \to Hom_F(A, V) \to 0$$

is also exact (proof of this is left to the reader).

Let $\mathcal{E}$ denote the chain complex $\mathcal{C} \otimes F$. Then $E_p = C_p \otimes F$ is a vector space over F, as are the boundaries (B_p) and cycles (Z_p) in the chain complex $\mathcal{E}$. Consider the short exact sequence of vector spaces

$$0 \to Z_p \to E_p \to B_{p-1} \to 0.$$

This gives rise to a short exact dual sequence

$$0 \to Hom_F(B_{p-1}, F) \to Hom_F(E_p, F) \to Hom_F(Z_p, F) \to 0.$$

The cochain complex in the middle is isomorphic to the cochain complex $Hom(C_p, F)$ by the preceding lemma. If $j_p : B_p \to Z_p$ is the inclusion map and its dual is written as j^*, then we obtain the exact sequence

$$0 \to \text{coker } j^*_{p-1} \to H^p(\mathcal{C}, F) \to \ker j^*_p \to 0.$$

We now consider the sequence

$$0 \to B_p \xrightarrow{j_p} Z_p \to H_p(\mathcal{E}) \to 0.$$

Because it is a sequence of vector spaces and linear transformations, the dual sequence is exact:

$$0 \to Hom_F(H_p(\mathcal{E}), F) \to Hom_F(Z_p, F) \xrightarrow{j^*_p} Hom_F(B_p, F) \to 0.$$

Therefore coker $j^*_p = 0$ and $\ker j^*_p \cong Hom_F(H_p(\mathcal{C}), F)$, proving the theorem. □

This theorem shows that if K is a simplicial complex, then $H^p(K)$ can be identified in a natural way with the dual vector space of $H_p(K)$. Since we are only considering finite-dimensional simplicial complexes, this means that $H_p(K)$ and $H^p(K)$ are isomorphic, but the isomorphism depends on the choice of basis one uses.

8 Differential Forms on Manifolds

We will now discuss some preliminaries about smooth manifolds. For x in a smooth manifold X, let $T(X,x)$ denote the tangent space at $x \in X$. The cotangent space at $x \in X$ is defined as $T^*(X,x)$. Define

$$T(X) = \bigcup_{x \in X} T(X,x) \qquad \text{and} \qquad T^*(X) = \bigcup_{x \in X} T^*(X,x).$$

$T(X)$ is called the tangent bundle of X, and $T^*(X)$ is called the cotangent bundle of X. A projection map $\pi : T(X) \to X$ is defined as follows. If $v \in T(X)$, then $v \in T(X,x)$ for some unique $x \in X$. Set $\pi(v) = x$. Similarly, there is a projection map from $T^*(X)$ to X that shall also be denoted as π.

If V is an n-dimensional real vector space, then $\mathcal{G}(V^*)$, the exterior algebra of V^*, is equal to $\Lambda^0(V^*) \oplus \Lambda^1(V^*) \oplus \cdots \oplus \Lambda^n(V^*)$, where $\Lambda^k(V^*)$ is the space of all skew-symmetric k-linear forms on V. From these definitions, we have

$$\mathcal{G}(X) = \bigcup_{x \in X} \mathcal{G}(T^*(X,x)) \qquad \text{and} \qquad \Lambda^k(X) = \bigcup_{x \in X} \Lambda^k(T^*(X,x))$$

A k-form on X is a mapping $\mu : X \to \Lambda^k(X)$ such that $\pi \circ \mu = i_X$. A k-form on X is smooth if whenever $V_1, \ldots, V_k$ are smooth vector fields, then

$$\mu(V_1, \ldots, V_k) \in C^\infty(X, \mathbb{R})$$

where

$$\mu(V_1, \ldots, V_k)(x) = \mu(x)(V_1(x), \ldots, V_k(x)).$$

A differential form on X is a mapping $\omega : X \to \mathcal{G}$ such that $\pi \circ \omega = i_X$; it is smooth if its component in $\Lambda^k(X)$ is smooth for each k. The set of smooth k-forms on X is denoted by $C^\infty(X, \Lambda^k(X))$. The set of all smooth differential forms is denoted by $C^\infty(X, \mathcal{G}(X))$. Note that $C^\infty(X, \Lambda^k(X))$ is a free abelian group under point-wise addition and scalar multiplication.

Theorem 8.1. *Let X be a smooth manifold. There exists a unique linear map $d : C^\infty(X, \mathcal{G}(X)) \to C^\infty(X, \mathcal{G}(X))$, called the exterior differential, such that the following properties hold:*

(1) $d : C^\infty(X, \Lambda^k(X)) \to C^\infty(X, \Lambda^{k+1}(X))$;
(2) $d(f) = df$ (the ordinary differential) for $f \in C^\infty(X, \Lambda^0(X))$;
(3) if $\mu \in C^\infty(X, \Lambda^k(X))$ and $\tau \in C^\infty(X, \mathcal{G}(X))$, then $d(\mu \wedge \tau) = (d\mu) \wedge \tau + (-1)^k \mu \wedge \tau$;
(4) $d^2 = 0$.

We can now look at the chain complex

$$C^\infty(X, \Lambda^0(X)) \xrightarrow{d} C^\infty(X, \Lambda^1(X)) \xrightarrow{d} C^\infty(X, \Lambda^2(X)) \xrightarrow{d} \ldots$$

which closely resembles the coboundary operator δ.

Definition 8.2. *Let X be a smooth manifold. A smooth differential form ω on X is closed if $d\omega = 0$. A form ω is exact if $\omega = d\tau$ for some smooth form τ. Let $Z^k(X,d)$ denote the vector space of closed k-forms on X. Let $B^k(X,d)$ denote the space of exact k-forms. Noting that $d^2 = 0$, we can see that exact forms are closed forms as well. If $\omega = d\tau$, then $d\omega = d(d\tau) = d^2\tau = 0$, implying that $B^k(X,d) \subset Z^k(X,d)$. Let $H^k(X,d) = Z^k(X,d)/B^k(X,d)$. $H^k(X,d)$ is called the* ***k-th De Rham cohomology group of*** *X.*

Roughly speaking, De Rham cohomology measures the extent to which the fundamental theorem of calculus fails in higher dimensions and on general manifolds. We now proceed to classify $H^p(\mathbb{R}^n, d)$ for all $p \geq 0$. First, let $p = 0$. It is clear that $Z^0(\mathbb{R}^n, d) = \{\text{constant functions on } \mathbb{R}^n\} \cong \mathbb{R}$. Since there are no p-forms when $p < 0$, it follows that $B^0(\mathbb{R}^n, d) = 0$. We now have that $H^0(\mathbb{R}^n, d) \cong \mathbb{R}$. (More generally, if a smooth manifold X is n-connected, then similar arguments show that $H^0(X,d) \cong \mathbb{R}^n$.) The following landmark theorem completes our classification.

Theorem 8.3 (Poincare's Lemma). [5] *If an open set $U \subset \mathbb{R}^n$ is homotopic to a point $x \in U$ (i.e., U is contractible), then $H^p(U, d) = 0$ for all $p > 0$.*

[5] A proof of this theorem can be found in [8].

9 Stokes's Theorem: Integration and Cohomology

Definition 9.1. *A* ***smoothly triangulated manifold*** *is a triple* (X, K, h) *where* X *is a* C^∞ *manifold,* K *is a simplicial complex, and* $h : |K| \to X$ *is a homeomorphism such that for each* $\sigma \in K$*, there is an open subset* $U \subset X$ *with a coordinate system* $\chi : U \to \mathbb{R}^n$ *such that* $h(supp(\sigma))$ *(which makes sense when we look at* σ *as a simplicial complex in its own right) is a subset of* U *and* $\chi \circ h|supp(\sigma)$ *is equal to* $A|supp(\sigma)$*, where* A *is an affine transformation. Here,* A *is an affine transformation from the vector space associated with* σ*, denoted by* $\mathbb{V}(\sigma) = \{b \in |K| : v \notin \sigma \Rightarrow b(v) = 0\}$*, to* $\mathbb{R}$*.*

Given a smoothly triangulated manifold (X, K, h), we can see that homomorphisms $\tilde{f}_p : H^p(X, d) \to H^p(K)$ are defined whenever there is given a sequence of linear maps $f_p : C^\infty(X, \Lambda^p(X)) \to C^p(K)$ such that $\delta \circ f_p = f_{p+1} \circ d$ for all p. Then, $f_p(Z^p(X, d)) \subset Z^p(K)$, because $d\omega = 0$ (hence $\omega \in C^p(X, d)$ implies that

$$\delta(f_p(\omega)) = f_{p+1}(d\omega) = f_{p+1}(0) = 0.$$

Also, $f_p(B^p(X, d)) \subset B^p(K)$, because $\omega = d\tau$ for some $\tau \in C^{p-1}(X, d)$ implies that

$$f_p(\omega) = f_p(d\tau) = \delta(f_{p-1}\tau) \in \text{Im } \delta$$

We now can proceed to define a sequence of linear maps

$$I_p : C^\infty(X, \Lambda^p(X)) \to C^p(K),$$

where I_p is integration. For $\omega \in C^\infty(X, \Lambda^p(X))$, $I_p(\omega)$ will be linear functional on $C_p(K)$. Thus it suffices to specify the values of $I_p(\omega)$ on the basis elements of $C_p(K)$, that is, on the oriented p-simplices σ, where $p \le \dim X = n$. To do this, consider a coordinate system $x : U \to \mathbb{R}^n$ where and U is an open set containing $supp(\sigma)$. Note that $x(supp(\sigma))$ is a geometric p-simplex in $\mathbb{R}^n$. We then choose an oriented p-simplex η in $\mathbb{R}^p$ and an affine map $P : \mathbb{R}^p \to \mathbb{R}^n$ such that $P(supp(\eta)) = x(supp(\sigma))$ and $x(\sigma)$ and $P(\eta)$ agree in orientation. If we set $h_\sigma = x^{-1} \circ P$, then $h_\sigma^\sharp(\omega)$, (where $^\sharp$ denotes pullback) is a smooth p-form on U. We define $I_p(\omega)(\sigma)$ to be the integral of this p-form over σ:

$$I_p(\omega)(\sigma) = \int_\sigma h_\sigma^\sharp(\omega).$$

In other words, let $(r_1, \dots, r_p)$ denote the coordinates in the plane of σ consistent with the orientation of σ; if $\sigma =< v_0 \dots v_p >$, let $(r_1, \dots, r_p)$ be the coordinates relative to the ordered basis $\{v_1 - v_0, \dots, v_p - v_0\}$. Then

$$h_\sigma^\sharp(\omega) = g dr_1 \wedge \cdots \wedge dr_p$$

for some continuous function g on U, and

$$I_p(\omega)(\sigma) = \int_\sigma g dr_1 \dots dr_p,$$

which is the Riemann integral. Note that this integral is independent of the homeomorphism h.

Theorem 9.2. $\delta \circ I_p = I_{p+1} \circ d$.

This is exactly Stokes's theorem. For any given smooth p-form ω and oriented $(p+1)$-simplex σ,

$$\begin{aligned} I_{p+1}(d\omega)(\sigma) = \int_\sigma (h_\sigma)^\sharp(d\omega) &= \int_\sigma d(h_\sigma^\sharp(\omega)) \\ &= \int_{\partial\sigma} h_\sigma^\sharp(\omega) = I_p(\omega)(\partial\sigma) = \delta \circ I_p(\omega)(\sigma). \quad \square \end{aligned}$$

I_p has now been shown to be a homomorphism from $H^p(X, d)$ to $H^p(K)$.

10 De Rham's Theorem

We now want to show that for smoothly triangulated manifolds (X, K, h), the De Rham cohomology of X is isomorphic to the simplicial cohomology of K, making the De Rham cohomology of X dual to the simplicial homology of K.

Theorem 10.1. *(De Rham's Theorem): Let (X, K, h) be a smoothly triangulated manifold. Then*

$$I_p : H^p(X, d) \to H^p(K)$$

is an isomorphism for each p where $0 \leq p \leq \dim X$.

In order for us to prove De Rham's theorem, we need to prove three lemmas pertaining to extension of forms on simplices, surjectivity of I_p, and injectivity of I_p.[6] This particular approach is motivated by geometric construction rather than algebraic manipulation.

10.1 Extension of Forms

In order to prove De Rham's Theorem theorem, we need to prove a preliminary lemma allowing us to extend forms that are defined near the boundary of a simplex through the neighborhood of a simplex. Let σ be a p-simplex in $\mathbb{R}^n$, $n \geq p$.

Lemma 10.2. (a_r)*: Let ω be a closed smooth r-form near $\partial\sigma$ with $r \geq 0$, $s \geq 1$. Suppose that*

$$\int_{\partial\sigma} \omega = 0 \quad \text{if} \quad p = r + 1.$$

Then there is a closed smooth form ω' near σ which equals ω near $\partial\sigma$.

(b_r)*: Let ω be a closed smooth r-form near σ with $r \geq 1$, $p \geq 1$, and let ξ be a smooth $(r-1)$-form near $\partial\sigma$ such that $d\xi = \omega$ near $\partial\sigma$. Suppose that*

$$\int_{\partial\sigma} \xi = \int_{\sigma} \omega \quad \text{if} \quad r = p.$$

Then there is a smooth form ξ' near σ such that $\xi' = \xi$ near $\partial\sigma$ and $d\xi' = \omega$ near σ.

Proof: We will prove, for all s, that

$$(a_0) \Rightarrow (b_1) \Rightarrow (a_1) \Rightarrow (b_2) \Rightarrow (a_2) \Rightarrow \dots$$

(a_0): First, note that ω is closed. If $d\omega = 0$, then ω is constant on any connected subset of σ. Now, suppose that $p = 1$ and that $\sigma^1 =< v_0 v_1 >$. Then

$$0 = \int_{\partial\sigma} \omega = \omega(v_1) - \omega(v_0)$$

so $\omega(v_0) = \omega(v_1)$. Since σ is connected, ω is constant and $\omega \equiv \omega(v_0)$. We can then let $\omega' = \omega(v_0)$. Now let $p > 1$. Since $d\omega = 0$, ω is constant near $\partial\sigma$, and we can choose ω' be be that constant.

$(a_{r-1}) \Rightarrow (b_r)$: ω is a closed r-form ($r \geq 1$) defined on an open set containing σ. By the Poincare lemma, ω is exact near σ; that is, there exists a smooth $(r-1)$-form ξ_1 defined near σ such that $d\xi_1 = \omega$ near σ. In general, ξ_1 will not be equal to ξ near $\partial\sigma$. Consider the difference $\eta = \xi_1 - \xi$ near $\partial\sigma$. It is closed since, near $\partial\sigma$, $d(\xi_1 - \xi) = \omega - \omega = 0$. Furthermore, if $p = (r-1) + 1 = r$, then

$$\int_{\partial\sigma} \eta = \int_{\partial\sigma} \xi_1 - \int_{\partial\sigma} \xi = \int_{\sigma} d\xi_1 - \int_{\partial\sigma} \xi = \int_{\sigma} \omega - \int_{\partial\sigma} \xi = 0.$$

[6] This proof is based upon the proof of de Rham's theorem found in [5] and [7].

We can now apply (a_{r-1}) to the form η. There exists a smooth closed $(r-1)$-form η' defined near σ such that $\eta' = \eta$ near $\partial\sigma$. Let $\xi' = \xi - \eta'$. Then ξ' is a smooth $(r-1)$-form defined near σ such that $\xi' = \xi_1 - \eta' = \xi$ near $\partial\sigma$, and $d\xi' = d\xi_1 - d\eta' = \omega - 0 = \omega$ near σ.

$(b_r) \Rightarrow (a_r)$ for $r > 0$: Let $\sigma =< v_0 \dots v_p >$; set $\sigma' =< v_1 \dots v_p >$. Let Q be the union of all proper faces of σ with v_0 as a vertex. Any open set containing Q contains a star-shaped neighborhood U of Q. By (b_r), there is a smooth form ξ_0 such that $d\xi_0 = \omega$ in U. In particular, this holds in $\partial\sigma'$.

If $p - 1 = s$, then by letting $A = \partial\sigma - \sigma'$, we have $\partial A = \partial(\partial\sigma - \sigma') = \partial^2\sigma - \partial\sigma' = -\partial\sigma'$. Therefore,

$$\begin{aligned}\int_{\sigma'} \omega - \int_{\partial\sigma'} \xi_0 = \int_{\sigma'} \omega - \int_{-\partial A} \xi_0 &= \int_{\sigma'} \omega + \int_{\partial A} \xi_0 \\ &= \int_{\sigma'} \omega + \int_A d\xi_0 = \int_{\sigma'} \omega + \int_A \omega = \int_{\sigma'+A} \omega = \int_{\partial\sigma} \omega = 0.\end{aligned}$$

We can now use (b_r), using σ', to assume the existence of a smooth form ξ_1 near σ' such that $\xi_1 = \xi_0$ near $\partial\sigma'$ and $d\xi_1 = \omega$ near σ'. There exists a neighborhood U' of $\partial\sigma'$ in which ξ_0 and ξ_1 are defined and equal to each other. Let ξ' be their common value here, and set $\xi' = \xi_0$ near $Q \setminus U'$ and $\xi' = \xi_1$ near $\sigma' \setminus U'$. Then ξ' is a smooth form such that $d\xi' = \omega$ near $\partial\sigma$. Since there is a smooth form ξ near σ which equals ξ' near $\partial\sigma$. By setting $\omega' = d\xi$ near σ, we obtain a form with the required properties. □

10.2 Surjectivity of I_p

We must now show that I_p is both surjective and injective. We first prove surjectivity.

Lemma 10.3. *: There exists a sequence of linear maps*

$$\Phi_p : C^p(K) \to C^\infty(X, \Lambda^p(X)) \qquad (0 \le p \le \dim X)$$

with the following properties:

(1) $d \circ \Phi_p = \Phi_{p+1} \circ \delta$.

(2) $I_p \circ \Phi_p =$ *identity*

(3) If c^0 *denotes the 0-cochain such that* $c^0(v) = 1$ *for each vertex* $v \in K$*, then* $\Phi_0(c^0) = 1$.

(4) If σ *is an oriented p-simplex of* K*, then the 1-form* $\Phi_p(\varphi_\sigma)$ *is identically zero in a neighborhood of* $X \setminus$ *St* σ.

Proof: We will use the following notation throughout this proof: $\mathcal{P} = \{0, \dots, p\}$, $\mathcal{J} = \{j_0, \dots, j_p\}$, $\mathcal{M} = \{1, \dots, m\}$, $\mathcal{V} = \{k \in \mathcal{M} :< v_k v_{j_0} \dots v_{j_p} >\in K\}$, and $\mathcal{M}_i = \mathcal{M} \setminus \{j_i\}$.

We shall identify $|K|$ and X through the homeomorphism h (i.e. $|K| = X$ and $h =$ *identity*). We begin by constructing a partition of unity, subordinate to the covering

$$\{\text{St } v : v \in K^{(0)}\}$$

of X. Let $v_1, \dots, v_m$ denote the vertices of K. For each $j \in \{1, \dots, m\}$, let b_j denote the jth barycentric coordinate function on $|K| = X$ and let

$$F_j = \{x \in X : b_j(x) \ge 1/(n+1)\} \quad \text{and} \quad G_j = \{x \in X : b_j(x) \le 1/(n+2)\},$$

where $n = \dim X$. Then F_j and G_j are closed sets in X with the following properties:

(a) $F_j \subset \text{St } v_j$.

(b) $X \setminus \text{ St } v_j \subset G_j$.

(c) $F_j \cap G_j = \emptyset$, and $F_j \subset G_j^c$, where $G_j^c = X \setminus G_j$.

(d) Since F_j is a closed set on the compact space X, F_j is compact. Therefore, a smooth function g_j can be found which is greater than 0 on F_j and equal to 0 outside the open set G_j^c.

(e) The closed sets F_j cover X. (Given $x \in X$, then $x \in \text{Int } \sigma$ for some simplex $\sigma =< v_0 \dots v_p >$, where $p \leq n$. Now $b_j(x) = 0$ for $j \notin \mathcal{J}$ and $\sum_{i \in \mathcal{P}} b_{j_i}(x) = 1$. Since $p+1 \leq n+1$, $b_j(x) \geq \frac{1}{n+1}$ for some $j \in \mathcal{J}$. Thus, $x \in F_j$ for this j.) In particular, for each $x \in X$, $g_j(x) \neq 0$ for some j. Furthermore, the open sets G_j^C for an open covering of X.

(f) Since, from property 5, $\sum_{j \in \mathcal{M}} g_j > 0$, we can define functions $\phi_j(x) = \dfrac{g_j(x)}{\sum_{k \in \mathcal{M}} g_k(x)}$ that are defined and smooth on X. Furthermore, $\{\phi_j\}$ is a smooth partition of unity on X subordinate to $\{G_j^C\}$; that is, $\sum_{j \in \mathcal{M}} \phi_j = 1$, and ϕ_j vanishes outside G_j^C. Since $G_j^C \subset \text{St } v_j$, the partition of unity $\{\phi_j\}$ is also subordinate to the open covering $\{\text{St } v_j\}$.

We define Φ_p in terms of $\{\phi_j\}$. Since Φ_p is to be linear, it suffices to specify the values of Φ_p on the generators φ_σ of $C^p(K)$. For $\sigma =< v_{j_0} \dots v_{j_p} >$, we define $\Phi_p(\varphi_\sigma)$ to be the p-form

$$\Phi_p(\varphi_\sigma) = p! \sum_{i \in \mathcal{P}} (-1)^i \phi_{j_i} d\phi_{j_0} \wedge \dots \wedge \widehat{d\phi_{j_i}} \wedge \dots \wedge d\phi_{j_p}.$$

We now verify Properties 1-4.

Property 1: Clearly, $d \circ \Phi_p(\varphi_\sigma) = (p+1)! d\phi_{j_0} \wedge \dots \wedge d\phi_{j_p}$. On the other hand,

$$\begin{aligned}
&\Phi_{p+1} \circ \delta(\varphi_\sigma) \\
&= \Phi_{p+1}\Big(\sum_{k \in \mathcal{V}} \varphi_{<v_k v_{j_0} \dots v_{j_p}>}\Big) \\
&= (p+1)! \sum_{k \in \mathcal{V}} [\phi_k d\phi_{j_0} \wedge \dots \wedge d\phi_{j_p} - \sum_{i \in \mathcal{P}} (-1)^i \phi_{j_i} d\phi_k \wedge d\phi_{j_0} \wedge \dots \wedge \widehat{d\phi_{j_i}} \wedge \dots \wedge d\phi_{j_p}] \\
&= (p+1)!\Big(\sum_{k \in \mathcal{V}} \phi_k d\phi_{j_0} \wedge \dots \wedge d\phi_{j_p} - \sum_{k \in \mathcal{V}} \sum_{i \in \mathcal{P}} (-1)^i \phi_{j_i} d\phi_k \wedge d\phi_{j_0} \wedge \dots \wedge \widehat{d\phi_{j_i}} \wedge \dots \wedge d\phi_{j_p}\Big).
\end{aligned}$$

CLAIM: If the vertices $v_k, v_{j_0}, \dots, v_{j_p}$ are distinct and yet are not the vertices of a $(p+1)$-simplex of K, then $\phi_k d\phi_{j_0} \wedge \dots \wedge d\phi_{j_p} \equiv 0$.

If $x \notin \text{ St } v_k$, then $\phi_k(x) = 0$. If $x \in \text{ St } v_k$, then $b_k(x) \neq 0$. But now $b_{j_i}(x) = 0$ for some $i \in \{0, \dots, p\}$, for otherwise $b_k(x) \neq 0$, $b_{j_0}(x) \neq 0$, $\dots$, $b_{j_p}(x) \neq 0$, making $< v_k v_{j_0} \dots v_{j_p} >$ a $(p+1)$-simplex. This is a contradiction; using this particular value of i, let

$$U = \{y \in X : b_{j_i}(y) = \frac{1}{n+2}\}.$$

Then U is an open set in X containing x, and ϕ_{j_i}, which is identically 0 on U because $U \subset G_{j_i}$. Hence $d\phi_{j_i} \equiv 0$ on U, and, in particular, $d\phi_{j_i}(x) = 0$, verifying the claim.

Applying this result to the expression for $\Phi_{p+1} \circ \delta(\varphi_\sigma)$ yields

$$\sum_{k \in \mathcal{V}} \phi_k d\phi_{j_0} \wedge \dots \wedge d\phi_{j_p} = \sum_{k \in \mathcal{M} \setminus \mathcal{J}} \phi_k d\phi_{j_0} \wedge \dots \wedge d\phi_{j_p} = A,$$

since those terms on the right-hand side which do not appear on the left are identically zero. Furthermore,

$$
\begin{aligned}
&-\sum_{k\in\mathcal{V}}\sum_{i\in\mathcal{P}}(-1)^i\phi_{j_i}d\phi_k\wedge d\phi_{j_0}\wedge\cdots\wedge\widehat{d\phi_{j_i}}\wedge\cdots\wedge d\phi_{j_p}\\
&=-\sum_{i\in\mathcal{P}}(-1)^i\sum_{k\in\mathcal{V}}\phi_{j_i}d\phi_k\wedge d\phi_{j_0}\wedge\cdots\wedge\widehat{d\phi_{j_i}}\wedge\cdots\wedge d\phi_{j_p}\\
&=-\sum_{i\in\mathcal{P}}(-1)^i\sum_{k\in\mathcal{M}\setminus\mathcal{J}}\phi_{j_i}d\phi_k\wedge d\phi_{j_0}\wedge\cdots\wedge\widehat{d\phi_{j_i}}\wedge\cdots\wedge d\phi_{j_p}\\
&=-\sum_{i\in\mathcal{P}}(-1)^i\sum_{k\in\mathcal{M}_i}\phi_{j_i}d\phi_k\wedge d\phi_{j_0}\wedge\cdots\wedge\widehat{d\phi_{j_i}}\wedge\cdots\wedge d\phi_{j_p}\\
&=-\sum_{i\in\mathcal{P}}(-1)^i\phi_{j_i}(\sum_{k\in\mathcal{M}_i}d\phi_k)\wedge d\phi_{j_0}\wedge\cdots\wedge\widehat{d\phi_{j_i}}\wedge\cdots\wedge d\phi_{j_p}\\
&=-\sum_{i\in\mathcal{P}}(-1)^i\phi_{j_i}(-d\phi_{j_i})\wedge d\phi_{j_0}\wedge\cdots\wedge\widehat{d\phi_{j_i}}\wedge\cdots\wedge d\phi_{j_p}\\
&=\sum_{i\in\mathcal{P}}\phi_{j_i}d\phi_{j_0}\wedge\cdots\wedge d\phi_{j_p}\\
&=\sum_{k\in\mathcal{J}}\phi_k d\phi_{j_0}\wedge\cdots\wedge d\phi_{j_p}=B,
\end{aligned}
$$

using the fact that $d(\sum_{k\in\mathcal{M}}\phi_k)=d(1)=0$. We now have that

$$
\begin{aligned}
\Phi_{p+1}\circ\delta(\varphi_\sigma) &= (p+1)!(A+B)\\
&= (p+1)!(\sum_{k\in\mathcal{M}\setminus\mathcal{J}}\phi_k d\phi_{j_0}\wedge\cdots\wedge d\phi_{j_p}+\sum_{k\in\mathcal{J}}\phi_k d\phi_{j_0}\wedge\cdots\wedge d\phi_{j_p})\\
&= (p+1)!(\sum_{k\in\mathcal{M}}\phi_k)d\phi_{j_0}\wedge\cdots\wedge d\phi_{j_p}\\
&= (p+1)!d\phi_{j_0}\wedge\cdots\wedge d\phi_{j_p}\\
&= d\circ\Phi_p(\varphi_\sigma),
\end{aligned}
$$

just as we require.

Property 3: Since $\Phi_0(\varphi_{<v_j>})=\phi_j$,

$$\Phi_0(c^0)=\sum_{j\in\mathcal{M}}\varphi_{<v_j>}=\sum_{j\in\mathcal{M}}\phi_j=1.$$

Property 4: Suppose $\sigma=<v_{j_0}\dots v_{j_p}>$. Then

$$\Phi_p(\varphi_\sigma)=p!\sum_{i\in\mathcal{P}}(-1)^i\phi_{j_i}d\phi_{j_0}\wedge\cdots\wedge\widehat{d\phi_{j_i}}\wedge\cdots\wedge d\phi_{j_p}.$$

Note that if $x\in X$ is such that $b_{j_k}(x)<\frac{1}{n+2}$ for some $k\in\mathcal{P}$, then $x\in G_{j_k}$, so that ϕ_{j_k} and $d\phi_{j_k}$, and hence $\Phi_p(\varphi_\sigma)$, are zero at x. Thus $\Phi_p(\varphi_\sigma)$ is identically zero on

$$\{x\in X|b_{j_k}(x)<\frac{1}{n+2}\text{ for some }k\in\mathcal{P}\},$$

which is an open set containing $X\setminus$ St σ.

Property 2: This proof will be by induction. For $p=0$, $I_0\circ\Phi_0(\varphi_{<v_j>})$ (where $j\in\mathcal{M}$) is the 0-cochain given by

$$[I_0\circ\Phi_0(\varphi_{<v_j>})](<v_k>)=[I_0(\phi_j)]<v_k>=\phi_j(v_k).$$

But note that $\phi_j(v_k) = 0$ for $k \neq j$ since $v_k \notin \text{St } v_j$ and $\phi_j = 0$ outside St v_j. Furthermore,

$$1 = \sum_{j \in \mathcal{M}} \phi_j(v_k) = \phi_k(v_k) \qquad \text{(for each } k\text{)}.$$

Hence

$$[I_0 \circ \Phi_0(\varphi_{<v_j>})](< v_k >) = \delta_k^j = \varphi_{<v_j>}(< v_k >).$$

Since this holds for all j and k, $I_0 \circ \Phi_0 =$ identity, as required.

Now assume Property 2 for dimension $p-1$. For σ and τ oriented p-simplices of K,

$$[I_p \circ \Phi_p(\varphi_\sigma)](\tau) = \int_\tau \Phi_p(\varphi_\sigma).$$

We must show that this equals 1 if $\sigma = \tau$ and 0 if $\sigma \neq \tau$. That this is zero for $\sigma \neq \tau$ is a consequence of Property 4 since $\Phi_p(\varphi_\sigma)$ is identically zero in a neighborhood of $X \setminus \text{St } \sigma \subset \tau$. So we need only check that $\int_\sigma \Phi_p(\varphi_\sigma) = 1$. For this, let $\sigma =< v_{j_0} \dots v_{j_p} >$ and $\sigma' =< v_{j_1} \dots v_{j_p} >$. Then

$$\int_\sigma \Phi_p(\delta\varphi_{\sigma'}) = \int_\sigma d[\Phi_{p-1}(\varphi_{\sigma'})] = \int_{\partial\sigma} \Phi_{p-1}(\varphi_{\sigma'}).$$

But $\partial\sigma = \sigma'$ plus an alternating sum of other oriented $(p-1)$-simplices, so

$$\int_{\partial\sigma} \Phi_{p-1}(\varphi_\tau) = \int_\tau \Phi_{p-1}(\varphi_\tau) = 1$$

by induction. Hence

$$\int_\sigma \Phi_p(\delta\varphi_\tau) = \int_\sigma \Phi_p(\varphi_\sigma + \text{terms of type } \varphi_\tau \text{ for } \tau \neq \sigma) = \int_\sigma \Phi_p(\varphi_\sigma) = [I_p \circ \Phi_p(\varphi_\sigma)](\sigma),$$

completing the proof. □

10.3 Injectivity of I_p

We now proceed to prove the injectivity of I_p, completing the proof of De Rham's theorem.

Lemma 10.4. *Let ω be a closed p-form on X. Suppose $I_p(\omega) = \delta(c)$ for some $c \in C^{p-1}(K)$. Then there exists a $(p-1)$-form τ on X such that $I_{p-1}(\tau) = c$ and $d\tau = \omega$.*

Proof: We shall construct a sequence

$$\tau_0, \tau_1, \dots, \tau_n \qquad (n = \dim X)$$

of $(p-1)$-forms such that

(1) τ_k is defined in a neighborhood of the k-skeleton $K^{(k)}$ of K,
(2) $d\tau_k = \omega$ near $K^{(k)}$,
(3) $\tau_k = \tau_{k-1}$ near $K^{(k)}$, and
(4) $I_p(\tau_{p-1}) = c$.

Note that this will prove the lemma because for each oriented $(p-1)$-simplex σ of $|K|$ and each $k \geq p-1$,

$$I_{p-1}(\tau_k)(\sigma) = \int_\sigma \tau_k = \int_\sigma \tau_{p-1} = I_{p-1}(\tau_{p-1})(\sigma) = c(\sigma).$$

To construct τ_0, cover $K^{(0)}$ by a collection of mutually disjoint balls. Since ω is closed, ω must be exact in each of these balls because of the Poincare lemma. Hence there exists a smooth $(p-1)$-form τ_0', defined on the union of these balls, such that $d\tau_0' = \omega$ there. If $p \neq 1$, take $\tau_0 = \tau_0'$. If $p = 1$, we want $I_0(\tau_0) = c$. But for a vertex v_j of $|K|$,

$$I_0(\tau_0')(< v_j >) = \int_{< v_j >} \tau_0' = \tau_0'(v_j)$$

Let $a_j = c(v_j) - \tau_0'(v_j)$, and define τ_0 on the ball about v_j by $\tau_0 = \tau_0' + a_j$. Then $d\tau_0 = d\tau_0' = \omega$ near $K^{(0)}$, and $I_0(\tau_0) = c$ as required.

Now assume that τ_{k-1} has been constructed with Properties 1-4. To construct τ_k, note that if we can find, for each k-simplex σ, a smooth $(p-1)$-form $\tau_k(\sigma)$ defined in a neighborhood of σ such that $d(\tau_k(\sigma)) = \omega$ near σ and $\tau_k(\sigma) = \tau_{k-1}$ near σ^{k-1} (a $(k-1)$-simplex), then glueing will yield a smooth $(p-1)$-form τ_k' satisfying Properties 1-3.

To construct $\tau_k(\sigma)$, we shall apply (b_1) from Lemma 8.2. Note that ω is a smooth, closed p-form defined near σ and that τ_{k-1} is a smooth $(p-1)$-form defined near σ^{k-1} such that $d\tau_{k-1} = \omega$ near σ^{k-1}. Furthermore, if $k = p$, then

$$\int_\sigma \omega = I_p(\omega)(\sigma) = \delta^p c(\sigma) = I_{k-1}(\tau_{k-1})(\partial\sigma) = \int_{\partial\sigma} \tau_{k-1}.$$

We can now apply Lemma 8.2, particularly (b_p). There exists a smooth $(p-1)$-form $\tau_k(\sigma)$ near σ such that $\tau_k(\sigma) = \tau_{k-1}$ near σ^{k-1} and $d(\tau_k(\sigma)) = \omega$ near σ. This constructs τ_k' satisfying Properties 1-3. If $k \neq p-1$, set $\tau_k = \tau_k'$. If $k = p-1$, we have τ_{p-1}' satisfying Properties 1-3, and we want τ_{p-1} such that $I_{p-1}(\tau_{p-1}) = c$. Let $c_1 = c - I_{p-1}(\tau_{p-1}')$, and define τ_{p-1} in a neighborhood of $K^{(p-1)}$ by

$$\tau_{p-1} = \tau_{p-1}' + \Phi_{p-1}(c_1),$$

where Φ was defined in Lemma 9.3.

For each r and each oriented r-simplex σ, note that $\Phi_r(\varphi_\sigma)$ is identically zero on a neighborhood of $X\setminus$ St σ. In particular, it is zero near $K^{(r-1)}$ for each r-cochain c. Applying this with $r = p$, then with $r = p-1$, we find

$$d\tau_{p-1} = d\tau_{p-1}' + d \circ \Phi_{p-1}(c_1) = d\tau_{p-1}' + \Phi_p \circ \delta(c_1) = d\tau_{p-1}' = \omega$$

near $K^{(p-1)}$ and

$$\tau_{p-1} = \tau_{p-1}' + \Phi_{p-1}(c_1) = \tau_{p-1}' = \tau_{p-2}$$

near $K^{(p-2)}$. Thus τ_{p-1} satisfies Properties 1-3 with $k = p-1$. Property 4 is also satisfied:

$$I_{p-1}(\tau_{p-1}) = I_{p-1}(\tau_{p-1}) + I_{p-1} \circ \Phi_{p-1}(c_1) = (c - c_1) + c_1 = c. \quad \square$$

11 Conclusion

Lemmas 9.3 and 9.4 are now proven. Therefore, I_p is an isomorphism between the De Rham cohomology $H^p(X,d)$ and the simplicial cohomology $H^p(K)$. In other words, De Rham's theorem shows that the simplicial homology groups (with coefficients in $\mathbb{R}$) of a smoothly triangulated manifold (X,K,h) are dual to the De Rham cohomology groups of X. In particular, these groups are independent of the triangulation (K,h) of X.

Let us now look at the problem posed in the introduction again. We want to know when a closed 1-form in the punctured plane is exact. Let v be a curl free vector field on $U = \mathbb{R}^2 \setminus (0,0)$. Let $\alpha = \int_{\gamma_1} v$, where $\gamma_1(t) = (\cos t, \sin t)$ and $t \in [0, 2\pi]$ is a generating loop for $H_1(U, \mathbb{R})$. Since $\int_{\gamma_1} \omega = 2\pi$, where

$$\omega = \frac{-y}{x^2+y^2}dx + \frac{x}{x^2+y^2}dy,$$

it follows that

$$\int_{\gamma_1} (v - \frac{\alpha}{2\pi}\omega) = 0.$$

Since γ_1 is a generator for $H_1(U,\mathbb{R})$, and the line integral over a sum of loops is the sum of the line integrals over those loops, it follows that

$$\int_{n\gamma_1} (v - \frac{\alpha}{2\pi}\omega) = 0$$

for all $n \in \mathbb{Z}$. Since every loop γ in $\mathbb{R}^2 \setminus (0,0)$ is equivalent to $\gamma_n = n\gamma_1$ for some integer n, and $v - \frac{\alpha}{2\pi}\omega$ is curl free, it follows that $\int_\gamma (v - \frac{\alpha}{2\pi}\omega) = 0$ for every loop γ in U. Thus, by De Rham's theorem, we have that $v - \frac{\alpha}{2\pi}\omega = \nabla\phi$ for some smooth function ϕ on U, and the difference between a curl free vector field on U and the gradient of a smooth function on U is ω multiplied by a real constant. Therefore, the vector space Z/B (as defined in the introduction), which is equal to $H^1(U)$, is a one-dimensional real vector space and is therefore isomorphic to $\mathbb{R}$. Since $H^p(U,d)$ is dual to $H_p(K)$ where K triangulates U, it follows that $H_1(K) \cong \mathbb{R}$. This means that there is only one equivalence class of 1-cycles in U that contains no 1-boundaries. This is class of 1-cycles that encompass the origin, which one could intuitively expect. We can now see that in some cases, it is much easier to analyze homology and cohomology of a triangulation of a manifold rather than looking at differential forms on the manifold itself. In general, for some $p \in \mathbb{R}^n$,

$$H^k(\mathbb{R}^n \setminus \{p\}) = \begin{cases} \mathbb{R}, & k = 0 \\ 0, & 1 \leq k \leq n-2 \\ \mathbb{R}, & k = n-1 \\ 0, & k = n \end{cases}.$$

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