EXISTENCE AND COMPUTATION OF GENERALIZED WANNIER FUNCTIONS FOR NON-PERIODIC SYSTEMS IN TWO DIMENSIONS AND HIGHER

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Abstract. Exponentially-localized Wannier functions (ELWFs) are a basis of the Fermi projection of a material consisting of functions which decay exponentially fast away from their maxima. When the material is insulating and crystalline, conditions which guarantee existence of ELWFs in dimensions one, two, and three are well-known, and methods for constructing the ELWFs numerically are well-developed. We consider the case where the material is insulating but not necessarily crystalline, where much less is known. In one spatial dimension, Kivelson and Nenciu-Nenciu have proved ELWFs can be constructed as the eigenfunctions of a self-adjoint operator acting on the Fermi projection. In this work, we identify an assumption under which we can generalize the Kivelson-Nenciu-Nenciu result to two dimensions and higher. Under this assumption, we prove that ELWFs can be constructed as the eigenfunctions of a sequence of self-adjoint operators acting on the Fermi projection. We conjecture that the assumption we make is equivalent to vanishing of topological obstructions to the existence of ELWFs in the special case where the material is crystalline. We numerically verify that our construction yields ELWFs in various cases where our assumption holds and provide numerical evidence for our conjecture.

1. Introduction

The starting point for understanding electronic properties of materials is the many-body ground state of the material's electrons. In the independent electron approximation, electrons in the ground state occupy the eigenstates of the single-electron Hamiltonian with energy up to the Fermi level. The subspace of the single-electron Hilbert space occupied by electrons in the ground state is known as the Fermi projection [3]. It is often desirable to find orthonormal bases of the Fermi projection which are as spatially localized as possible. Such bases are important both for theoretical and numerical studies of materials. For example, they form the basis of the modern theory of polarization [24, 49, 35], and can dramatically speed up numerical calculations [19, 30, 54, 35].

For insulating *crystalline* materials a natural family of spatially localized bases are the Wannier bases. The elements of a Wannier basis are known as Wannier functions and are constructed (in the simplest case) by integrating the Bloch functions of the occupied Bloch bands with respect to the quasi-momentum over the Brillouin zone. Since each Bloch function is only defined up to a complex phase, or "gauge", Wannier functions are not unique. By changing the gauge, one can change the spatially localization of the corresponding Wannier functions.

In pioneering work, Kohn found that for non-degenerate Bloch bands of inversion-symmetric crystals in one spatial dimension it is always possible to choose the gauge of the Bloch functions such that the associated Wannier functions decay *exponentially fast* in space [29]. In the years since many authors have worked to generalize Kohn's result. A summary of these results is as follows.

- In one spatial dimension, the Fermi projection of an insulating crystalline material can always be represented by exponentially-localized Wannier functions [8, 13, 40, 23, 39].
- In two dimensions, the same result holds if and only if the Chern number, a topological invariant associated with the occupied Bloch functions, vanishes [8, 13, 40, 23, 39, 7, 44, 36].
- In three dimensions, the result holds as long as three "Chern-like" numbers all vanish [8, 13, 40, 23, 39, 7, 44].

An important special case of these results is that exponentially-localized Wannier bases always exist whenever the insulating crystal is time-reversal symmetric, since this implies that the Chern and Chern-like numbers vanish [44].

For insulating materials without crystalline atomic structure, Bloch functions do not exist and hence cannot be used to construct a spatially localized basis. In spite of this, Nenciu-Nenciu [37], following up an idea of Kivelson [27], proved by construction that exponentially-localized bases of the Fermi projection always exist in one spatial dimension. When the material is crystalline and inversion-symmetric, the elements of the basis constructed in this way reduce to the Wannier functions originally constructed by Kohn [29, 27]. Exponentially-localized bases of the Fermi projection are conjectured to exist more generally, at least whenever time-reversal symmetry holds [38, 41]. Beyond one spatial dimension, their existence has been proven in a few special cases [9, 38, 18, 46] (see Section 1.3 for discussion of these works); for more details of the one-dimensional case see [48, 28]. In what follows, we will refer to any localized basis of the Fermi projection of a non-periodic insulator as a generalized Wannier basis, and to the elements of such a basis as generalized Wannier functions.

In this work we introduce and prove validity of a new construction of exponentially localized generalized Wannier functions for periodic and non-periodic insulators in two dimensions and higher. Our results can be thought of as generalizing Kivelson and Nenciu-Nenciu's ideas to higher dimensions, although our method relies on a novel assumption which is unnecessary in one dimension. We conjecture that in two dimensions this assumption is equivalent to triviality of the Chern number in the case where the material is periodic, and provide numerical evidence to support this. We provide full details of the proof in the case of an infinite material in two dimensions described by a continuum PDE model.

Our proof involves technical innovations compared with the proofs of Nenciu-Nenciu. In general, our proof is more operator-theoretic, which allows us to extend our proof to discrete and/or finite models in a straightforward way (up to details which we explain in each case). Our operator-theoretic proof of exponential localization in particular requires several new ideas compared with that of Nenciu-Nenciu.

Our construction implies a new algorithm for numerically computing generalized Wannier functions in finite systems, both with periodic boundary conditions and otherwise. We find that this algorithm indeed yields exponentially localized Wannier functions for the Haldane model in its non-topological phase with periodic and Dirichlet boundary conditions, even with weak disorder. The algorithm fails in the topological (Chern) insulator phase because our basic assumption does not hold in this case. We remark that in general, methods for numerical computation of *generalized* Wannier functions are significantly less developed than methods for computing Wannier functions [34, 53, 35, 14, 11, 12], although see Appendix A of [34] and [52]. Numerical methods for computing generalized Wannier functions in finite systems can be viewed as Boys localization schemes [6].

1.1. **Paper Organization.** The remainder of our paper is organized as follows. We explain our main theoretical result without making our assumptions completely explicit, and explain in what sense our result generalizes Kivelson-Nenciu-Nenciu's one dimensional result, in Section 1.2. We review previous literature on constructing generalized Wannier functions for non-periodic materials in Section 1.3. We show results of implementing our numerical algorithm in Section 2, where we also show numerical evidence supporting our conjecture that our assumption is equivalent to vanishing of the Chern number. We state the main theorem we will prove precisely in Section 4, after reviewing some notations in Section 3. The proof of our main theorem is presented across Sections 5, 6, and 7. We sketch the generalizations of our results to higher dimensional systems in Section 8, and to discrete systems in Section 9. We summarize our conclusions and highlight some future directions in Section 10. We defer proofs of key estimates required for the proof of our main theorem to Appendices A, B, and C, and proofs of estimates needed for the discrete case to Appendix D.

1.2. **The Kivelson-Nenciu-Nenciu Idea and our Main Theorem.** In this section we begin by reviewing Kivelson-Nenciu-Nenciu's construction of exponentially localized generalized Wannier functions in one spatial dimension. We will then present our main theorem without making our assumptions completely precise. We will make our assumptions precise, and then re-state our main result, in Section 4.

Consider the Hilbert space $\mathcal{H} = L^2(\mathbb{R})$ and let H be a Hamiltonian of the form

$$H = -\Delta + V(x),$$

where V is a real potential, not necessarily periodic, satisfying certain regularity conditions. For example, we can take $V \in L^\infty(\mathbb{R})$ (see (2.1) of [37] for a weaker condition). Assume that the Fermi level lies in a spectral gap of H, let P denote the Fermi projection, and let X denote the position operator $Xf(x) = xf(x)$. The Kivelson-Nenciu-Nenciu idea is that *in one spatial dimension, the eigenfunctions of the operator PXP form an exponentially localized generalized Wannier basis.* A sketch of Nenciu-Nenciu's rigorous proof that this proposal works is as follows.

Using exponential decay of P (Lemma 1 of [37]), PXP is well-defined on the domain $\mathcal{D}(X) \cap \text{range}(P)$, and extends to an unbounded self-adjoint operator range(P) $\rightarrow$ range(P). Because of decay induced by X (in the resolvent), PXP has compact resolvent and hence only real, discrete eigenvalues. Since the spectral theorem implies that the eigenfunctions of PXP form an orthonormal basis of range(P), it remains only to prove that the eigenfunctions of PXP exponentially decay. This can be verified by a direct calculation from the eigenequation $PXPf = \lambda f$ which again relies on exponential decay of P.

In this work, we focus on the Hilbert space $\mathcal{H} = L^2(\mathbb{R}^2)$ and let H be a Hamiltonian of the form

$$H = (-i\nabla + A(x, y))^2 + V(x, y). \tag{1.1}$$

Suppose V is a real scalar function and A is a real vector function, not necessarily periodic, satisfying certain regularity conditions. For example, we can take $V \in L^\infty(\mathbb{R}^2)$, $A \in L^\infty(\mathbb{R}^2;\mathbb{R}^2)$, and $\mathrm{div} A \in L^\infty(\mathbb{R}^2;\mathbb{R})$. Assume again that the Fermi level lies in a spectral gap of H, and let P denote the Fermi projection.

The Kivelson-Nenciu-Nenciu idea does not generalize to this case in a straightforward way for the following reason. Suppose we let X denote a two-dimensional position operator acting as $Xf(x,y) = xf(x,y)$ with respect to some choice of coordinate axes. Then the decay induced by X is not enough for the resolvent of PXP to be compact in two dimensions. To generalize the Kivelson-Nenciu-Nenciu idea to two dimensions, we make an additional assumption. The additional assumption we make is that the operator PXP has *uniform spectral gaps*, a notion we make precise in Assumption 4.3. Numerical simulations on the Haldane model [21] suggest this assumption is equivalent to vanishing of the Chern number when the system is periodic: see Figure 1.1.

Sorted Non-Zero Eigenvalues of PXP

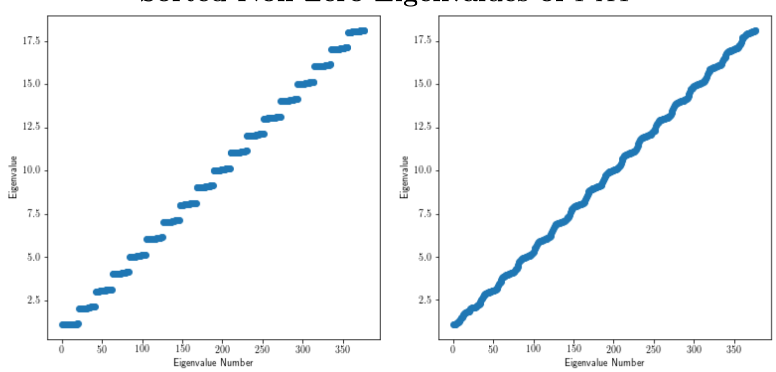

Figure 1.1. Detail from plot of the sorted non-zero eigenvalues of PXP where P is the Fermi projection and X is the lattice position operator $[X\psi]_{m,n} = [m\psi]_{m,n}$ for the Haldane model on a 21×21 lattice with periodic boundary conditions. Parameters for the Haldane model are defined in Section 2.1. The left plot corresponds to a non-topological phase (Chern $= 0$) with parameters $(t, t', v, \phi) = (1, 0, 1, 0)$. The right plot corresponds to topological phase (Chern $= 1$) with parameters $(t, t', v, \phi) = (1, \frac{1}{4}, 1, \frac{\pi}{2})$. In the non-topological phase, the spectrum of PXP shows clear gaps, while in the topological phase, the spectrum does not have clear gaps.

The uniform spectral gap assumption allows us to reduce the original problem of finding an exponentially-localized basis of range(P) to the problem of finding exponentially-localized bases of the set of subspaces range(P_j), where P_j denotes the spectral projection onto each separated component of the spectrum of PXP. Crucially, functions in range(P_j) are quasi-one dimensional in the sense that they decay with respect to x away from lines $x = \eta_j$, where η_j is a real constant, for each j. Using this property, we can apply the Kivelson-Nenciu-Nenciu idea to each range(P_j) in turn, and thereby build up a generalized Wannier basis of all of range(P).

We consider the family of operators $P_j Y P_j$, where Y is a position operator acting in a non-parallel direction to X as $Yf(x,y) = yf(x,y)$. We first prove, using exponential decay of P_j (proved in Appendix B), these operators are well-defined on the domain $\mathcal{D}(Y)$, and extend to unbounded self-adjoint operators on all of $L^2(\mathbb{R}^2)$. We then prove, using decay induced by Y combined with the fact that functions in P_j decay in x, that each P_jYP_j has compact resolvent and hence only real, discrete eigenvalues. We finally prove that the eigenfunctions of P_jYP_j exponentially decay. We prove this by a direct calculation from the eigenequation $P_jYP_jf = \lambda f$, again using exponential decay of P_j. It now follows immediately that the set of eigenfunctions of each of the P_jYP_j operators forms an exponentially localized basis of range(P).

In summary, the main results of this paper are as follows:

Main Theorem. *Let $\mathcal{H} = L^2(\mathbb{R}^2)$, and let H be the continuum Hamiltonian* (1.1)*, where the potentials A and V satisfy certain regularity assumptions but are not necessarily periodic. Suppose that H has a spectral gap containing the Fermi level, and let P be the Fermi projection. Let X and Y denote position operators $Xf(x,y) = xf(x,y)$ and $Yf(x,y) = yf(x,y)$ with respect to a choice of two-dimensional axes. Then, if PXP has uniform spectral gaps, there exist functions $\{\psi_{j,m}\}_{(j,m)\in\mathcal{J}\times\mathcal{M}}$ and points $\{(a_j,b_m)\}_{(j,m)\in\mathcal{J}\times\mathcal{M}} \in \mathbb{R}^2$ such that*

(1) The collection $\{\psi_{j,m}\}_{(j,m)\in\mathcal{J}\times\mathcal{M}}$ is an orthonormal basis of range(P).

(2) Each $\psi_{j,m}$ is exponentially localized at $(a_{j,m}, b_{j,m})$ in the sense that

$$\int_{\mathbb{R}^2} e^{2\gamma\sqrt{1+(x-a_{j,m})^2+(y-b_{j,m})^2}}|\psi_{j,m}(x,y)|^2 \,\mathrm{d}x\,\mathrm{d}y \le C, \tag{1.2}$$

where (C,γ) denote finite positive constants which are independent of j and m.

(3) The set of $\{\psi_{j,m}\}_{(j,m)\in\mathcal{J}\times\mathcal{M}}$ are the set of eigenfunctions of the operators P_jYP_j, where P_j are the band projectors defined by Definition 4.1.

Here $\mathcal{J}$ and $\mathcal{M}$ are the countable sets which index the projectors P_j, and the eigenfunctions of P_jYP_j for fixed j, respectively.

We break up the proof of our main theorem into a series of lemmas to be proved, and make our theorem more precise, in Section 4. We prove these lemmas in Sections 5, 6, and 7, using estimates on P and P_j proved in Appendices A, B, and C.

The generalization of our main theorem to discrete models is relatively simple because of the operator-theoretic structure of our proof. As long as the off-diagonal entries of the discrete Hamiltonian decay exponentially, we can establish identical operator bounds on the Fermi projector P as in the continuum case. Using these properties, we can directly derive properties of the spectral projectors P_j corresponding to spectral subspaces of PXP and generalize our result to this setting. For details, see Section 9 and Appendix D.

In finite systems with periodic or Dirichlet boundary conditions, the operators P_jYP_j which appear in the proof of our main theorem can be constructed and diagonalized numerically. Hence our theorem and its proof imply a simple algorithm for generating a generalized Wannier basis for a given H. Note that in our main theorem we allow for a possibly non-zero magnetic potential A so that we can justify applying our construction to the Haldane model even with complex hopping. We test the effectiveness of this algorithm on the Haldane model in Section 2.

A sketch of the generalization of our main result to three dimensions is as follows. Consider position operators X, Y, and Z associated with a three-dimensional basis acting on $\mathcal{H} := L^2(\mathbb{R}^3)$, and consider the operator PXP. Assume PXP has uniform spectral gaps, and let P_j denote spectral projections onto each of the separated components of the spectrum of PXP. Now assume the operators $P_j Y P_j$ *also* have uniform spectral gaps, and let $P_{j,k}$ denote spectral projections onto each of the separated components of the spectrum of $P_j Y P_j$. By analogous reasoning to the two dimensional case, functions in range$(P_{j,k})$ are quasi-one dimensional. We therefore claim that the set of eigenfunctions of the operator $P_{j,k} Z P_{j,k}$ will form an exponentially localized basis of range$(P_{j,k})$ for each j, k, and that the union of all of these eigenfunctions over j and k will form an exponentially-localized basis of range(P). We conjecture that the existence of uniform spectral gaps for PXP *and* $P_j Y P_j$ for each $j \in \mathcal{J}$ is equivalent to triviality of topological obstructions in 3d in the case where the structure is periodic [44, 22]. For more detail, see Section 8.

We have already conjectured that PXP having uniform spectral gaps is equivalent to vanishing of the Chern number in the case of an infinite periodic system. It would be interesting to understand whether PXP having uniform spectral gaps can be related to formulations of the Chern number for non-periodic systems such as the non-commutative Chern number [4], see also [32, 25, 5, 45, 20, 51, 33].

It is natural to ask whether our result is sensitive to the choice of position operators X and Y. In particular, it is natural to ask whether the validity of the uniform gap assumption on PXP depends on this choice. We observe numerically (see Section 2.6), but cannot prove, that for a given model if the uniform spectral gap assumption holds for one particular choice of X and Y it holds for all possible X and Y.

1.3. **Previous Works on generalized Wannier functions.**

We pause to discuss existing literature on generalized Wannier functions, other than the works of Kivelson and Nenciu-Nenciu we have already mentioned [27, 37]. Niu [42] pointed out that eigenfunctions of PXP should decay at least polynomially. Geller and Kohn have studied generalized Wannier functions in “nearly periodic” materials [18, 17]. Nenciu and Nenciu have proved existence of generalized Wannier functions for materials whose atomic potential is related to that of a crystal with exponentially-localized Wannier functions via an interpolation which does not close the spectral gap at the Fermi level [38].

More recent work by Cornean, Nenciu, and Nenciu [9] showed that the one-dimensional result of Nenciu-Nenciu [37] can be generalized to the case where $H = -\Delta + V$ and the potential V is concentrated along a single axis. Hastings and Loring have introduced the concept that Wannier functions in two dimensions could be defined as “simultaneous approximate eigenvectors” of the operators PXP and PYP [22]. E and Lu proved existence and exponential localization of Wannier functions in smoothly deformed crystals in the limit where the deformation length-scale tends to infinity [15]. Prodan [46] showed that by diagonalizing $Pe^{-R}P$ (where R denotes the radial position operator $\sqrt{X^2+Y^2}$) one can construct an orthogonal basis of functions which are concentrated on spherical shells in arbitrary dimension.

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2. Numerical Results

In this section we present results of implementing the numerical scheme suggested by our main theorem for generating exponentially localized generalized Wannier functions. The scheme is as follows:

(1) Choose position operators X and Y acting in orthogonal directions.
(2) Compute the Fermi projector P by diagonalizing the Hamiltonian H.
(3) Diagonalize the operator PXP, and inspect $\sigma(PXP)$ for clusters of eigenvalues separated from other eigenvalues by spectral gaps.
(4) Form band projectors P_j onto each cluster of eigenvalues.
(5) Diagonalize the operators P_jYP_j to obtain exponentially localized eigenvectors which span the Fermi projection.

As numerically, one can only deal with a finite system, it is necessary to clarify two points compared with the infinite case.

First, note that any vector in a finite system is trivially exponentially-decaying by taking $C > 0$ sufficiently large and $\gamma > 0$ sufficiently small in (1.2). It is necessary to clarify, therefore, that the algorithm presented above yields exponentially-decaying eigenvectors with $C > 0$ and $\gamma > 0$ which are *independent of system size*. In this sense, our algorithm yields a non-trivial result.

Second, in finite systems, all operators have purely discrete spectrum and hence there will be a spectral gap between *any* pair of eigenvalues. However, to obtain localized eigenvectors it is not enough to simply form band projectors for each eigenvalue of PXP alone. Hence the clarification in the algorithm that we must form band projectors from clusters of nearby eigenvalues separated from the remainder of the spectrum by clear spectral gaps. This point is clarified by our rigorous analysis in the following sections, where we show that the localization of the generalized Wannier functions produced by our scheme is related to the minimal gap between the bands of $\sigma(PXP)$ (see Section 7).

We choose to test our scheme on the Haldane model [21] at half-filling, a simple two-dimensional model whose Fermi projection, in the crystalline setting, may or may not have non-zero Chern number depending on model parameters. For this reason, the Haldane model is a natural model for testing our hypothesis that gaps of PXP are equivalent to topological triviality of P in the case where the material is periodic. Historically, the Haldane model was the first model of a Chern insulator: a material exhibiting quantized Hall response without net magnetic flux through the material. We now briefly recap the essential features of this model.

2.1. **The Haldane Model.**

The Haldane model describes electrons in the tight-binding limit hopping on a honeycomb lattice. In addition to real nearest-neighbor hopping terms, the model allows for a real on-site potential difference between the A and B sites of the lattice, and for *complex* next-nearest-neighbor hopping terms which break time-reversal symmetry without introducing net magnetic flux.

In the crystalline case, the action of the Haldane tight-binding Hamiltonian acting on wave-functions $\psi \in \mathcal{H} := l^2(\mathbb{Z}^2; \mathbb{C}^2)$ is:

$$\begin{aligned} \begin{bmatrix} (H\psi)^A_{m,n} \\ (H\psi)^B_{m,n} \end{bmatrix} = v \begin{bmatrix} \psi^A_{m,n} \\ -\psi^B_{m,n} \end{bmatrix} + t \begin{bmatrix} \psi^B_{m,n} + \psi^B_{m,n-1} + \psi^B_{m-1,n} \\ \psi^A_{m,n} + \psi^A_{m+1,n} + \psi^A_{m,n+1} \end{bmatrix} \\ + t' e^{i\phi} \begin{bmatrix} \psi^A_{m,n+1} + \psi^A_{m-1,n} + \psi^A_{m+1,n-1} \\ \psi^B_{m,n-1} + \psi^B_{m+1,n} + \psi^B_{m-1,n+1} \end{bmatrix} \\ + t' e^{-i\phi} \begin{bmatrix} \psi^A_{m,n-1} + \psi^A_{m+1,n} + \psi^A_{m-1,n+1} \\ \psi^B_{m,n+1} + \psi^B_{m-1,n} + \psi^B_{m+1,n-1} \end{bmatrix}. \end{aligned} \tag{2.1}$$

Here, t, v, t', and ϕ are real parameters expressing the magnitude of nearest-neighbor hopping, the magnitude of on-site potential difference, the magnitude of complex next-nearest neighbor hopping, and the complex argument of the nearest-neighbor hopping, respectively.

By definition, at half-filling the Fermi level is at 0. An explicit calculation using Bloch theory [21] (see also [16]) shows that H has a spectral gap (and hence describes an insulator) at 0 whenever

$$v \neq \pm 3\sqrt{3} t' \sin\phi.$$

Further calculation shows that the Fermi projection has a non-trivial Chern number (equal to 1 or -1) whenever

$$|v| < 3\sqrt{3} |t' \sin\phi|. \tag{2.2}$$

In this case, exponentially-localized Wannier functions do not exist [36, 33]. Whenever the parameters t, v, t', ϕ are such that (2.2) holds, we say the Haldane model is in its *topological phase*.

For some of our experiments, we add a perturbation to the Hamiltonian (2.1) which models disorder. We replace the on-site potential v in (2.1) by a spatially varying on-site potential $v + \eta(m, n)$, where $\eta(m, n)$ is drawn for each m, n from independent Gaussian distributions with mean 0 and variance σ^2:

$$\eta(m, n) \sim \mathcal{N}(0, \sigma^2) \text{ for each } m, n. \tag{2.3}$$

We refer to this kind of disorder as "onsite" disorder. Assuming H (2.1) has a spectral gap with $\sigma^2 = 0$ (i.e. without disorder), then for sufficiently small σ^2, the spectral gap will persist almost surely and our method can be applied.

To implement our method, we have to make a choice of position operators on the space $l^2(\mathbb{Z}^2; \mathbb{C}^2)$. The simplest choice is to define X and Y consistently with the crystal lattice by:

$$\begin{bmatrix} (X\psi)^A_{m,n} \\ (X\psi)^B_{m,n} \end{bmatrix} = \begin{bmatrix} m\psi^A_{m,n} \\ m\psi^B_{m,n} \end{bmatrix} \quad \begin{bmatrix} (Y\psi)^A_{m,n} \\ (Y\psi)^B_{m,n} \end{bmatrix} = \begin{bmatrix} n\psi^A_{m,n} \\ n\psi^B_{m,n} \end{bmatrix}.$$

We refer to this choice of X and Y as the standard position operators. A couple of remarks are in order. First, note that X and Y do not distinguish between A and B sites. Second, the crystal lattice vectors are not orthogonal hence eigenvalues of X and Y do not represent co-ordinates with respect to orthogonal axes. Since the lattice vectors are linearly independent our method can nonetheless be applied.

2.2. **Parameters for numerical tests and further remarks.** For our numerical tests, we consider the Haldane model just described truncated to a 21×21 lattice (hence $\mathcal{H}$ has dimension $2 \times 21 \times 21$) under the following conditions:

- Dirichlet boundary conditions with standard position operators, without disorder (Section 2.3).
- Dirichlet boundary conditions with standard position operators, with weak disorder which does not close the spectral gap of H (Section 2.4).
- Dirichlet boundary conditions with standard position operators, with strong disorder (Section 2.5). Note that in this case the spectral gap assumption on H is no longer valid; though P will still be exponentially localized due to the Anderson localization.
- Dirichlet boundary conditions with non-standard (rotated) position operators, without disorder (Section 2.6).
- Periodic boundary conditions with standard position operators, without disorder. We consider parameter values such that the system is in a non-topological phase and values such that the system is in a topological phase (Sections 2.7 and 2.8).

Note that we do not consider any examples with Dirichlet boundary conditions in the topological phase. This is because H does not have a spectral gap in this case due to edge states.

In each case we will display plots of the generalized Wannier functions generated by our algorithm. Specifically, given a generalized Wannier functions $\psi \in \mathcal{H} = l^2(\{1, ..., 21\}^2; \mathbb{C}^2)$, we will plot the following matrix in a 3D surface plot:

$$\begin{bmatrix} \sqrt{|\psi^A_{1,1}|^2 + |\psi^B_{1,1}|^2} & \sqrt{|\psi^A_{1,2}|^2 + |\psi^B_{1,2}|^2} & \cdots & \sqrt{|\psi^A_{1,21}|^2 + |\psi^B_{1,21}|^2} \\ \sqrt{|\psi^A_{2,1}|^2 + |\psi^B_{2,1}|^2} & \sqrt{|\psi^A_{2,2}|^2 + |\psi^B_{2,2}|^2} & \cdots & \sqrt{|\psi^A_{2,21}|^2 + |\psi^B_{2,21}|^2} \\ \vdots & \vdots & \ddots & \vdots \\ \sqrt{|\psi^A_{21,1}|^2 + |\psi^B_{21,1}|^2} & \sqrt{|\psi^A_{21,2}|^2 + |\psi^B_{21,2}|^2} & \cdots & \sqrt{|\psi^A_{21,21}|^2 + |\psi^B_{21,21}|^2} \end{bmatrix}.$$

To make the exponential decay of ψ as clear as possible, we will also show 2D plots of the elementwise logarithm of this matrix.

We remark that while our theoretical results hold equally well in the both periodic and non-periodic cases for infinite systems, we find for finite systems our algorithm works better for systems with Dirichlet boundary conditions. This is not entirely surprising given that the position operators X and Y do not respect periodic boundary conditions. It is possible that a better choice (see [50, 56, 2] for potentially related ideas) would improve the results in the case of periodic boundary conditions. Exploring these modifications is the subject of ongoing work.

2.3. **Dirichlet Boundary Conditions using Standard Position Operators.** We consider the Haldane model with Dirichlet boundary conditions and parameters $(t, t', v, \phi) = (1, \frac{1}{10}, 1, \frac{\pi}{2})$, which correspond to the non-topological phase. For this choice of parameters the Hamiltonian H has a gap of ~ 1.02. We plot the eigenvalues of PXP in Figure 2.1, where we see $\sigma(PXP)$ shows clear gaps. We plot the eigenvectors of PXP in Figure 2.2. We see that these eigenvectors are concentrated along lines $x = c$ for constants c. We finally plot the eigenfunctions

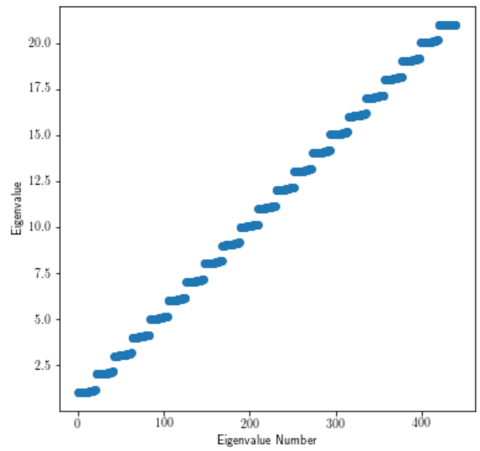

Figure 2.1. Plot of sorted non-zero eigenvalues of PXP where P is the Fermi projection and X is the lattice position operator for the Haldane model on 21×21 system with Dirichlet boundary conditions. Parameters chosen are $(t, t', v, \phi) = (1, \frac{1}{10}, 1, \frac{\pi}{2})$. The spectrum shows clear gaps.

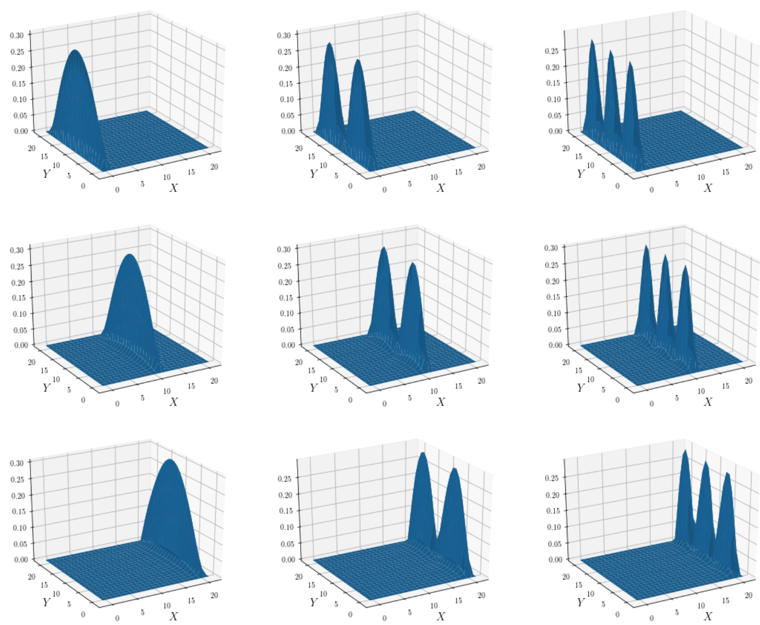

Figure 2.2. Plot of eigenfunctions of the operator PXP where P is the Fermi projection and X is the lattice position operator for the Haldane model on 21×21 system with Dirichlet boundary conditions. Parameters chosen are $(t, t', v, \phi) = (1, \frac{1}{10}, 1, \frac{\pi}{2})$. Each eigenvector of PXP is localized along a line $x = c$ for constant c.

of $P_j Y P_j$, which are localized with respect to x and y, for a few different values of j in Figure 2.3.

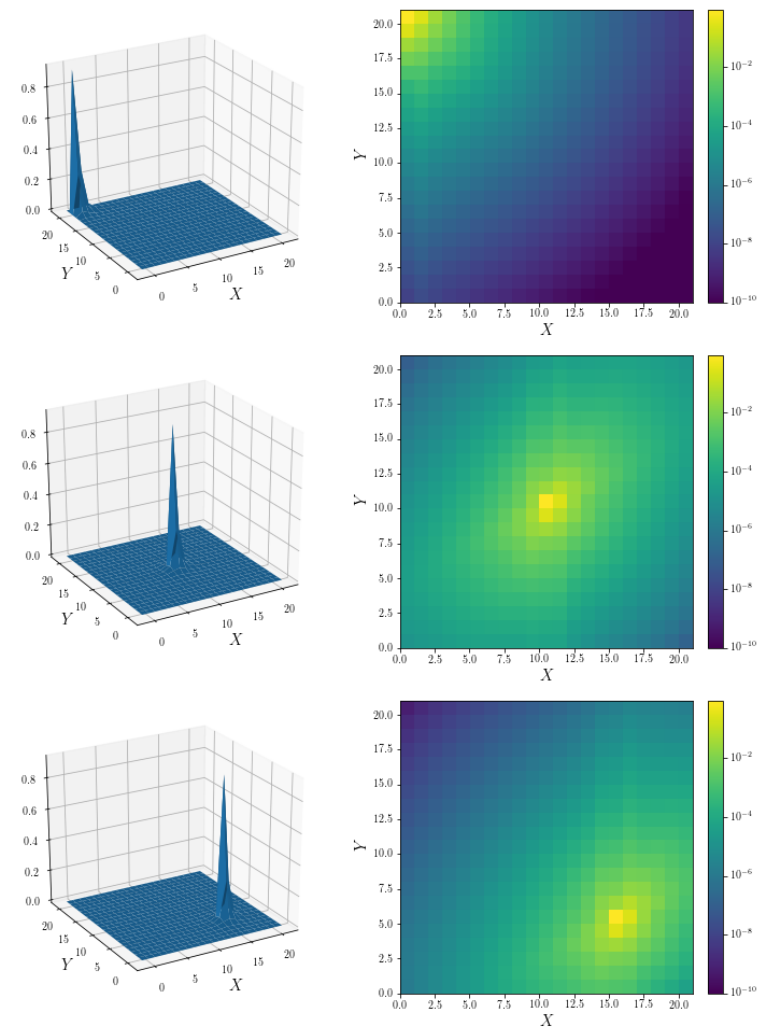

Figure 2.3. Plot of eigenfunctions of the operator P_jYP_j for different values of j where $\{P_j\}_{j\in\mathcal{J}}$ are the band projectors for PXP, P is the Fermi projection, and X, Y are the lattice position operators. The projection P comes from the Haldane model on 21×21 system with Dirichlet boundary conditions. Parameters chosen are $(t, t', v, \phi) = (1, \frac{1}{10}, 1, \frac{\pi}{2})$. Each eigenfunction shows clear exponential localization in line with our theoretical results.

2.4. **Dirichlet Boundary Conditions with Weak Disorder.** We now consider a case where translational symmetry is broken even away from the edge of the material. Starting with the same parameters as in Section 2.3, we add onsite disorder as in (2.3), with $\sigma^2 = \frac{1}{4}$. We plot results for a realization of the onsite disorder such that H has a clear gap $\sim .46$. We find that the eigenvalues of PXP show clear gaps despite the disorder, see Figure 2.4. We can therefore form projectors P_j, and the operators P_jYP_j. We plot the eigenfunctions of P_jYP_j in Figure 2.5. We observe that they are again exponentially localized, just as in the case without disorder (Figure 2.3), in line with our theoretical results.

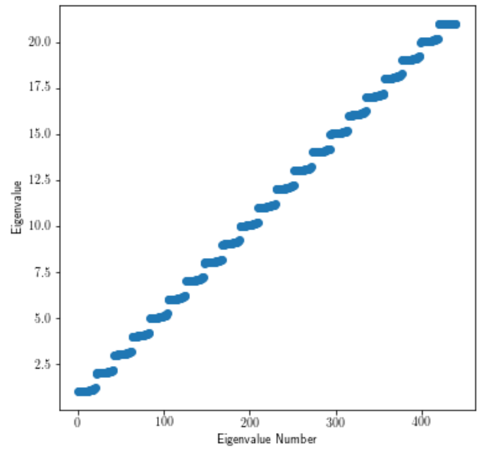

Figure 2.4. Plot of sorted non-zero eigenvalues of PXP where P is the Fermi projection and X is the lattice position operator for the Haldane model on 21×21 system with Dirichlet boundary conditions. Parameters chosen are $(t, t', v, \phi) = (1, \frac{1}{10}, 1, \frac{\pi}{2})$, with onsite disorder drawn from a mean zero normal distribution with variance $\frac{1}{4}$. Despite the disorder the spectrum still shows clear gaps.

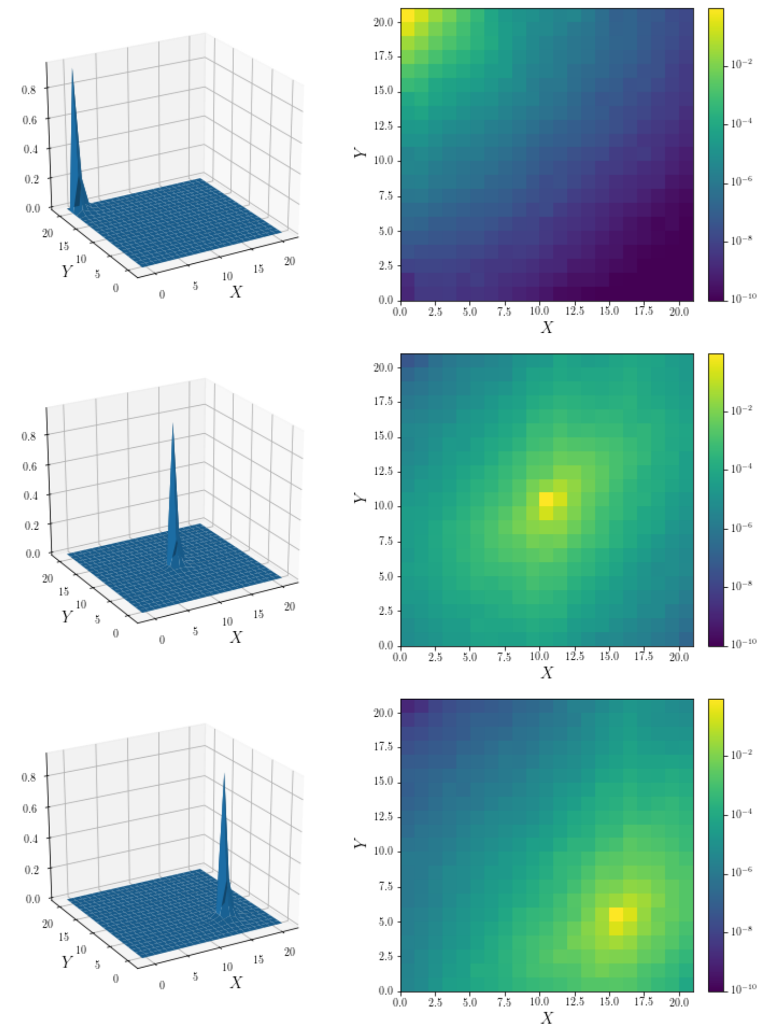

Figure 2.5. Plot of eigenfunctions of the operator $P_j Y P_j$ for different values of j where $\{P_j\}_{j \in \mathcal{J}}$ are the band projectors for PXP, P is the Fermi projection, and X, Y are the lattice position operators. Parameters chosen are $(t, t', v, \phi) = (1, \frac{1}{10}, 1, \frac{\pi}{2})$, with onsite disorder drawn from a mean zero normal distribution with variance $\frac{1}{4}$. Despite the disorder our algorithm yields exponentially-localized generalized Wannier functions.

2.5. **Dirichlet Boundary Conditions with Strong Disorder.** We consider the same setup as the previous section, but with disorder strong enough ($\sigma^2 = 100$) to close the gap of H (for the results shown in Figure 2.6, the gap of $H \approx .07$). Although our results do not directly apply to this case, the eigenfunctions of H are themselves localized because of Anderson localization [1]. It is therefore plausible that PXP may have gaps and that the eigenfunctions of P_jYP_j are localized nonetheless.

We plot the non-zero eigenvalues in PXP in Figure 2.6. We find that $\sigma(PXP)$ shows clear gaps, and hence we may define projectors P_j and operators P_jYP_j.

We observe that the eigenfunctions of P_jYP_j are well localized. In Figure 2.7, we plot the eigenfunctions of H in order of increasing energy value and plot an eigenfunction of P_jYP_j which has the same center. We observe that as the energy level increases, the corresponding eigenfunction of H becomes less localized. In comparison, the eigenfunctions of P_jYP_j have similar rates of decay for all values of $j \in \mathcal{J}$.

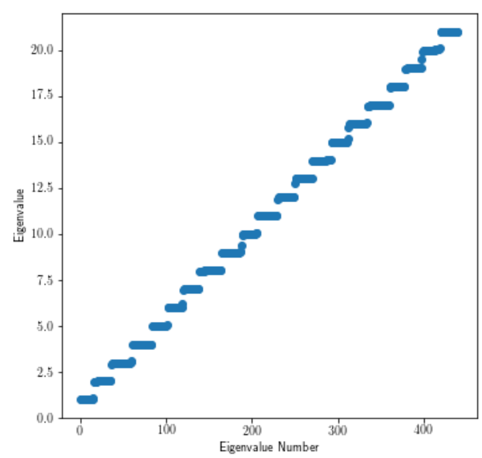

Figure 2.6. Plot of sorted non-zero eigenvalues of PXP where P is the Fermi projection and X is the lattice position operator for the Haldane model on 21×21 system with Dirichlet boundary conditions. Parameters chosen are $(t, t', v, \phi) = (1, \frac{1}{10}, 1, \frac{\pi}{2})$, with disorder is drawn from a mean zero normal distribution with variance 100.

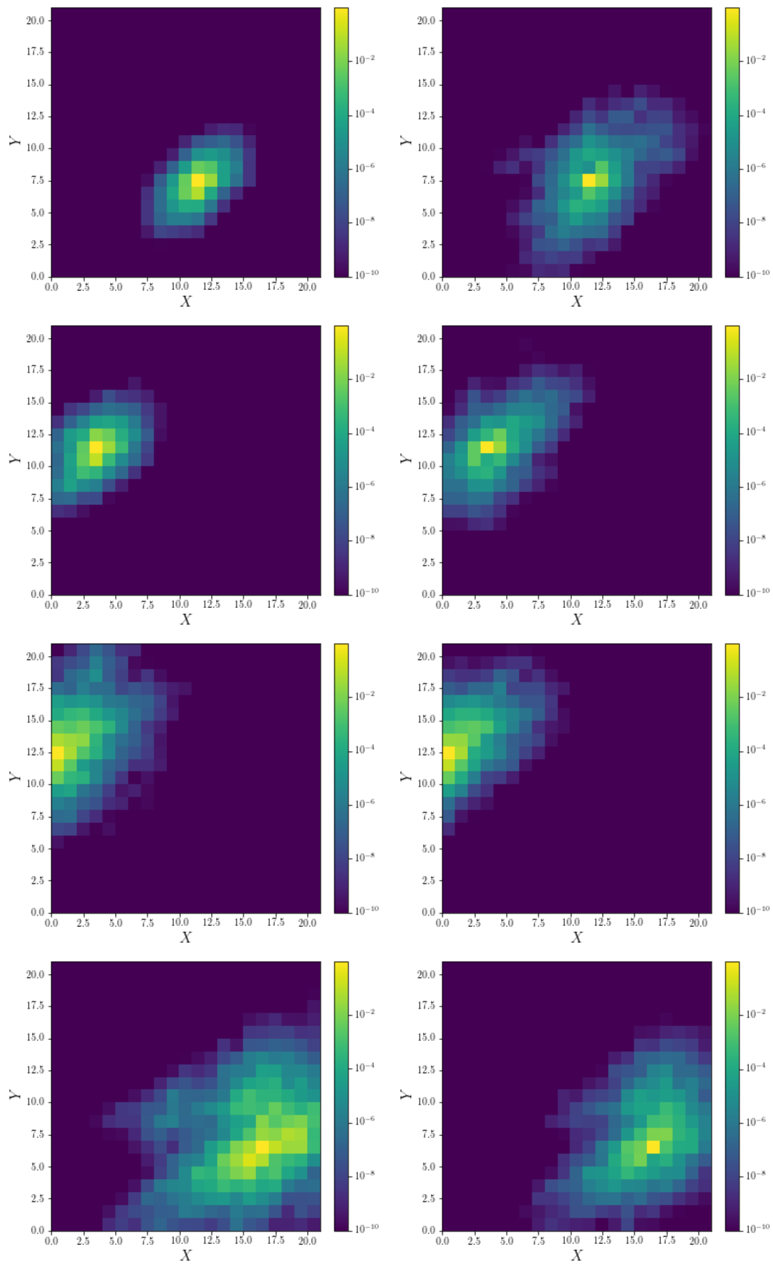

Figure 2.7. Plot of eigenfunctions of H (left) and $P_j Y P_j$ (right) for different values of j where $\{P_j\}_{j\in\mathcal{J}}$ are the band projectors for PXP, P is the Fermi projection, and X, Y are the lattice position operators. Parameters chosen are $(t, t', v, \phi) = (1, \frac{1}{10}, 1, \frac{\pi}{2})$, while onsite disorder is drawn from a mean zero normal distribution with variance 100. The eigenfunctions of H are sorted in order of increasing energy (top $\rightarrow$ low energy, bottom $\rightarrow$ high energy) and eigenfunctions of $P_j Y P_j$ were chosen to have the same center as the corresponding H eigenfunction.

2.6. **Dirichlet Boundary Conditions using Rotated Position Operators.** We now consider how our results change when we choose to work with different two-dimensional position operators (equivalently, different two-dimensional axes). Note that, although our proofs are independent of any particular choice of position operators, we cannot rule out the possibility that the uniform spectral gap assumption on PXP (Assumption 4.3) holds only for particular choices. We also expect that different choices of position operators will yield different exponentially-localized generalized Wannier functions.

We consider the same Haldane model with Dirichlet boundary conditions but without disorder as in Section 2.3, and introduce rotated position operators

$$\tilde{X} := \frac{X+Y}{\sqrt{2}}, \; \tilde{Y} := \frac{X-Y}{\sqrt{2}}. \tag{2.4}$$

The eigenvalues of $P\tilde{X}P$ are shown in Figures 2.8. We find that, just like the eigenvalues of PXP in Figure 2.1, the spectrum shows clear gaps. The eigenfunctions of $P\tilde{X}P$ are shown in Figure 2.9. They are clearly localized along lines $x+y=c$ for constant c. Since $P\tilde{X}P$ has gaps (Figure 2.8), we can define the band projectors P_j as before. The eigenfunctions of $P_j\tilde{Y}P_j$ are shown in Figure 2.10 and clearly exponentially decay similarly to those in Figure 2.3.

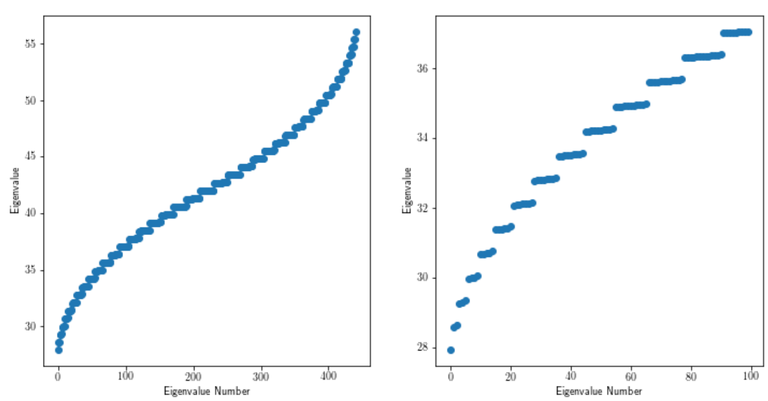

Figure 2.8. Plot of sorted non-zero eigenvalues of $P\tilde{X}P$ where P is the Fermi projection and $\tilde{X}$ is the lattice position operator rotated by $45°$ (see Equation (2.4)) for the Haldane model on 21×21 system with Dirichlet boundary conditions. Parameters chosen are $(t, t', v, \phi) = (1, \frac{1}{10}, 1, \frac{\pi}{2})$. Full non-zero spectrum (left), zoom-in for the first 100 eigenvalues (right). The spectrum shows clear gaps.

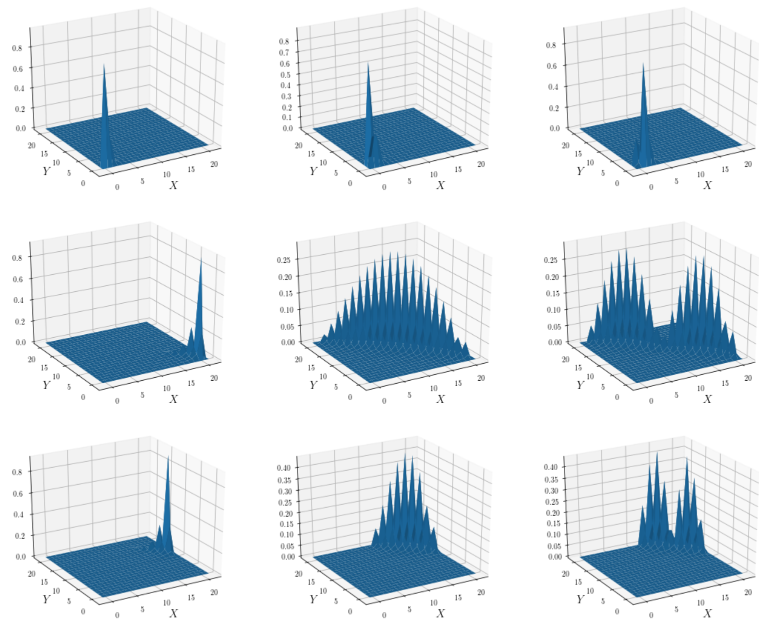

Figure 2.9. Plot of eigenfunctions of the operator $P\tilde{X}P$ where P is the Fermi projection and $\tilde{X}$ is the rotated lattice position operator for the Haldane model on 21×21 system with Dirichlet boundary conditions. Parameters chosen are $(t, t', v, \phi) = (1, \frac{1}{10}, 1, \frac{\pi}{2})$. Each eigenfunction is localized along a line $x + y = c$ for some constant c.

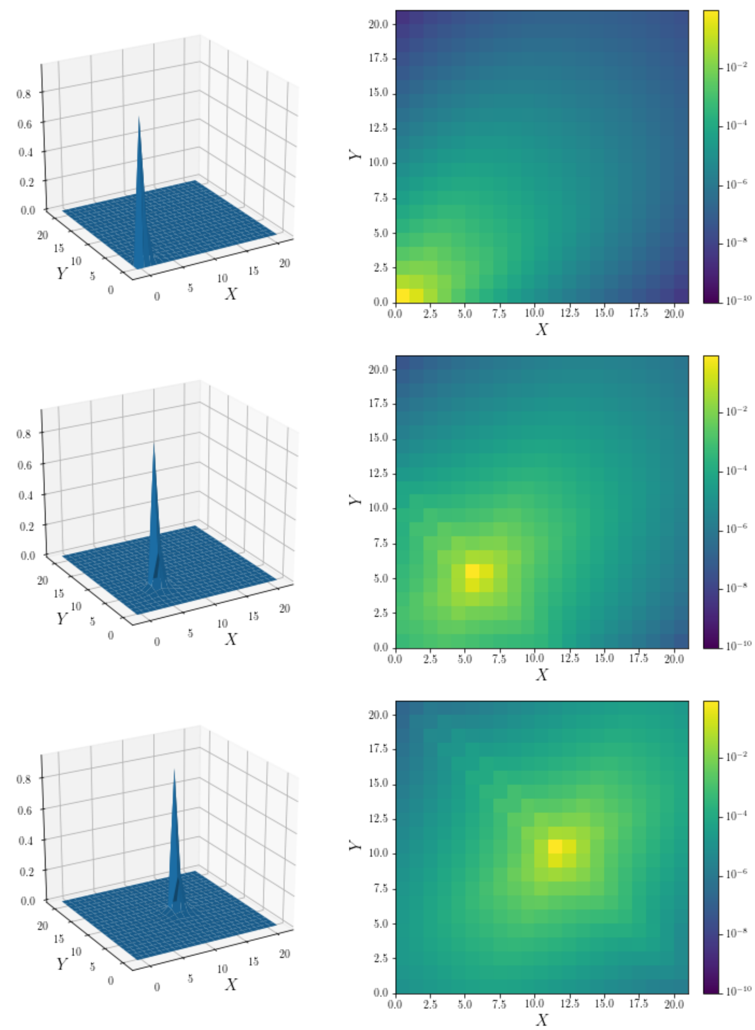

Figure 2.10. Plot of eigenfunctions of the operator $P_j \tilde{Y} P_j$ for different values of j where $\{P_j\}_{j \in \mathcal{J}}$ are the band projectors for $P\tilde{X}P$, P is the Fermi projection, and $\tilde{X}$, $\tilde{Y}$ are the rotated lattice position operators (see Equation (2.4)). The projection P comes from the Haldane model on 21×21 system with Dirichlet boundary conditions. Parameters chosen are $(t, t', v, \phi) = (1, \frac{1}{10}, 1, \frac{\pi}{2})$.

2.7. **Periodic Boundary Conditions, Topological versus Non-Topological.** Recall (Section 1) we conjecture that for periodic systems, PXP having spectral gaps is equivalent to triviality of the Chern number of the Fermi projection. In this section we numerically test this conjecture by forming the Fermi projector P from the Haldane Hamiltonian with periodic boundary conditions and numerically computing the spectrum of PXP for different values of the Haldane model parameters. Our results are shown in Figure 2.11. We find that for model parameters such that the model is in a non-topological phase, $\sigma(PXP)$ shows clear gaps. For model parameters such that the model is in a topological phase, every gap of $\sigma(PXP)$ closes. This conclusion holds even when we choose model parameters such that the spectral gap of H is approximately equal in either case (≈ 2).

Note that in the case where every gap of $\sigma(PXP)$ closes, our construction is technically well defined since the spectrum of PXP is bounded on a finite domain. On the other hand, it is totally ineffective because we can only define one band projector P_j, which equals P. Hence the eigenfunctions of P_jYP_j in this case are the eigenfunctions of PYP, which do not decay in x.

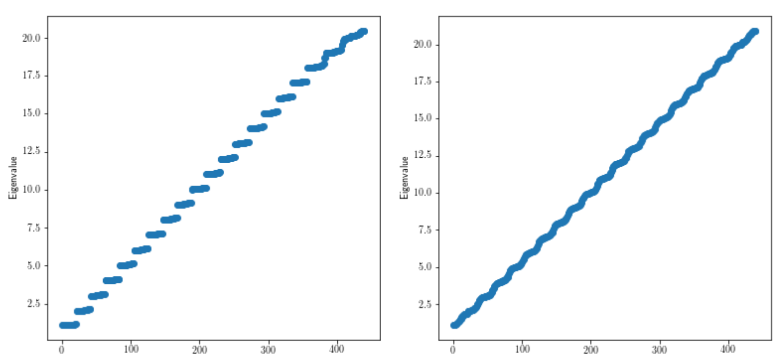

Figure 2.11. Plot of sorted non-zero eigenvalues of PXP where P is the Fermi projection and X is the lattice position operator for the Haldane model on 21×21 system with periodic boundary conditions. The left plot corresponds to parameters $(t, t', v, \phi) = (1, 0, 1, \frac{\pi}{2})$ (non-topological phase) and the right plot corresponds to parameters $(t, t', v, \phi) = (1, \frac{1}{4}, 0, \frac{\pi}{2})$. The gap in H for non-topological phase is 2, whereas the gap in H for the topological phase is ≈ 2.043.

2.8. **Periodic Boundary Conditions, Standard Position Operators.** In this section we implement our algorithm in the non-topological phase of Haldane with periodic boundary conditions, when $\sigma(PXP)$ shows clear gaps (Figure 2.11). Note that when we take periodic boundary conditions the last three bands of PXP appear to merge together. Since this does not occur in the case of Dirichlet boundary conditions (see Figure 2.1), we conjecture that this behavior is because the operator X does not respect translation symmetry with respect to x.

Despite this, our theory still applies since we can enclose the last three bands by a single contour when we define the collection $\{P_j\}_{j \in \mathcal{J}}$. For all bands but the last one, we find the eigenfunctions of $P_j Y P_j$ are exponentially localized like before. These results are shown in Figure 2.12. For the last band, we find that instead of the eigenfunctions of $P_j Y P_j$ being localized along a single line $x = c$ for constant c, they are somewhat spread across an interval of x values: see Figure 2.13.

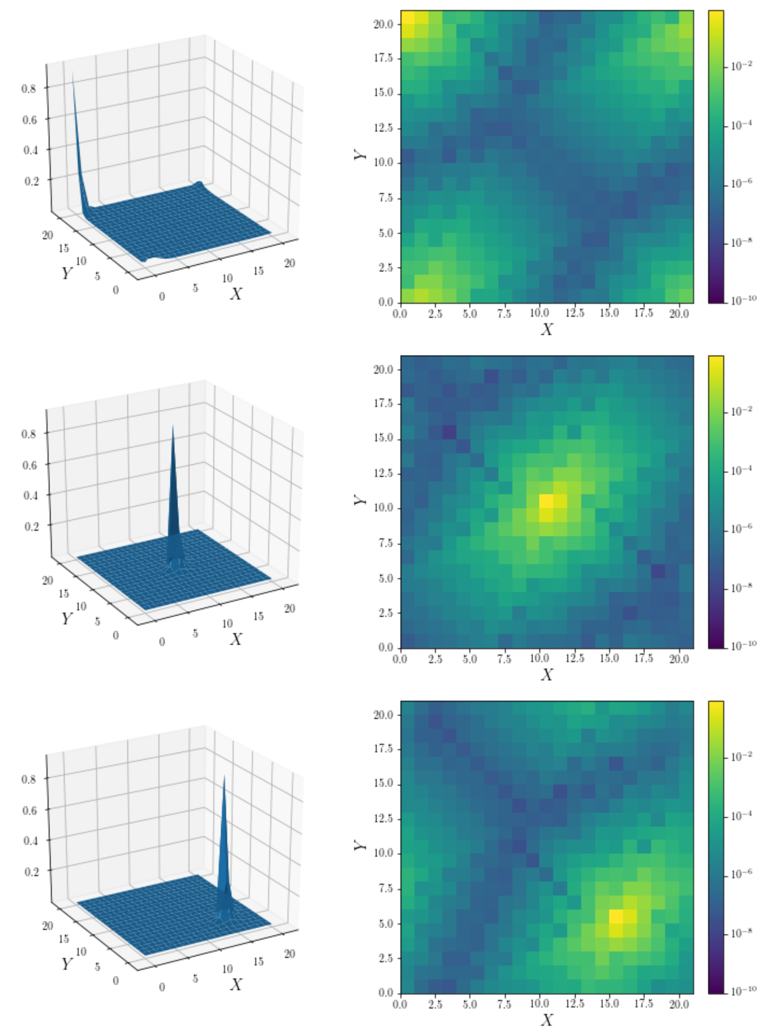

Figure 2.12. Plot of eigenfunctions of the operator $P_j Y P_j$ for different values of j where $\{P_j\}_{j\in\mathcal{J}}$ are the band projectors for PXP, P is the Fermi projection, and X, Y are the lattice position operators. The projection P comes from the Haldane model on 21×21 system with periodic boundary conditions. Parameters chosen are $(t, t', v, \phi) = (1, 0, 1, \frac{\pi}{2})$. For these figures we avoid the P_j where a few bands of the spectrum of PXP have clumped together (see Figure 2.11).

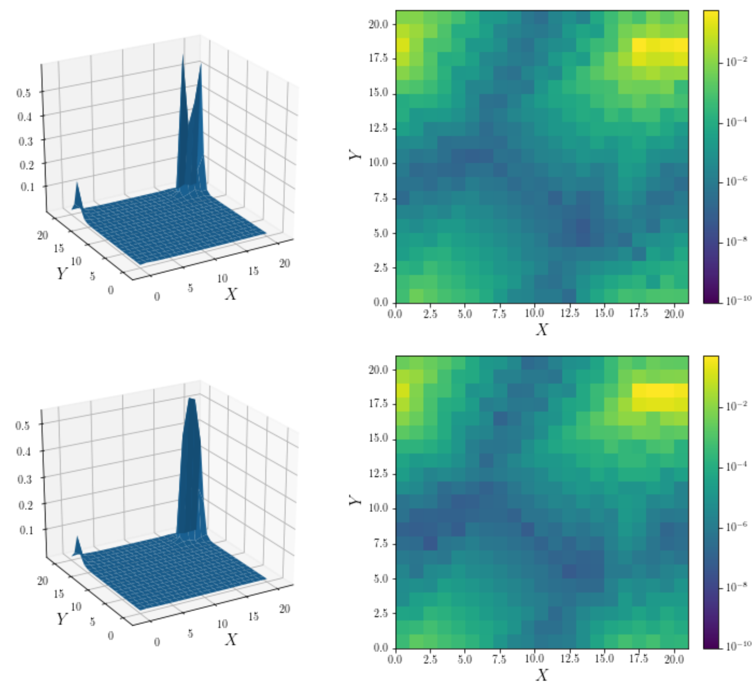

Figure 2.13. Plot of eigenfunctions of the operator $P_j Y P_j$ for different values of j where $\{P_j\}_{j\in\mathcal{J}}$ are the band projectors for PXP, P is the Fermi projection, and X, Y are the lattice position operators. The projection P comes from the Haldane model on 21×21 system with Dirichlet boundary conditions. Parameters chosen are $(t, t', v, \phi) = (1, 0, 1, \frac{\pi}{2})$. For these figures we consider the P_j where a few bands of the spectrum of PXP have clumped together (see Figure 2.11). Note that the generalized Wannier function generated by our method in this case has a relatively large spread in x relative to those plotted in Figure 2.12.

3. Notation and Conventions

Before moving on to our proofs, we pause to review some notation. For any $f : \mathbb{R}^2 \to \mathbb{C}$, we will use $\|f\|$ to denote the L^2-norm of f defined as follows:

$$\|f\| := \left(\int_{\mathbb{R}^2} |f(x,y)|^2 \mathrm{d}x\mathrm{d}y \right)^{1/2}.$$

Similarly, for any linear operator we will use $\|A\|$ to denote the induced norm when we view A as a mapping $L^2(\mathbb{R}^2) \to L^2(\mathbb{R}^2)$. That is,

$$\|A\| := \sup_{\substack{f \in L^2(\mathbb{R}^2) \\ f \neq 0}} \frac{\|Af\|}{\|f\|}.$$

We define the L^∞-norm of a function $f : \mathbb{R}^2 \to \mathbb{C}$ as follows

$$\|f\|_{L^\infty} := \inf \left\{ C \geq 0 : |f(x)| \leq C \text{ almost everywhere} \right\}.$$

Given two sets $A, B \subseteq \mathbb{R}$ we define their diameter and distance as follows:

$$\begin{aligned} \operatorname{diam}(A) &:= \sup\{|a_1 - a_2| : a_1, a_2 \in A\} \\ \operatorname{dist}(A, B) &:= \inf\{|a - b| : a \in A, b \in B\} \end{aligned}$$

For any contour in the complex plane, $\mathcal{C}$, we will use $\ell(\mathcal{C})$ to denote the length of $\mathcal{C}$.

Given a point $(a,b) \in \mathbb{R}^2$, and a non-negative constant $\gamma \geq 0$, we define an exponential growth operator by

$$B_{\gamma,(a,b)} := \exp\left(\gamma \sqrt{1 + (X-a)^2 + (Y-b)^2} \right).$$

Given a linear operator A, we define

$$A_{\gamma,(a,b)} := B_{\gamma,(a,b)} A B^{-1}_{\gamma,(a,b)}. \tag{3.1}$$

We refer to $A_{\gamma,(a,b)}$ as "exponentially-tilted" relative to A. We will often prove estimates where we use the notation (3.1) but omit the point (a,b). In this case the estimate should be understood as uniform in the choice of point (a,b). As a note, per our convention, when $\gamma = 0$, $A_{\gamma,(a,b)} = A$.

4. Precise Statement of Main Theorem

In this section, we present our assumptions in full detail and re-state our main theorem precisely. The details of the proof will be presented in Sections 5, 6, and 7, with proofs of key estimates postponed until Appendices A, B, and C.

We start with a basic regularity assumption.

Assumption 4.1 (Regularity). *We consider the Hilbert space $\mathcal{H} := L^2(\mathbb{R}^2)$ acted on by the Hamiltonian*

$$H = (-i\nabla + A(x,y))^2 + V(x,y).$$

We assume that the vector function $A \in L^\infty(\mathbb{R}^2;\mathbb{R}^2)$ and $\operatorname{div}(A) \in L^\infty(\mathbb{R}^2;\mathbb{R})$. We assume that $V \in L^\infty(\mathbb{R}^2;\mathbb{R})$.

Assumption 4.1 ensures that H is essentially self-adjoint (see Chapter 1 of [10]) and is sufficient to prove e.g. exponential localization of the Fermi projection (see Appendix A). We expect Assumption 4.1 could we weakened at the cost of making our proofs more complex. It would be interesting, for example, to see if our proofs could be generalized to the weaker assumptions given in Remark 3.2 of [36]. Note that we do not make any assumptions about periodicity of H and hence our results apply for both periodic and non-periodic systems.

Our first major assumption is that H has a spectral gap. More formally:

Assumption 4.2 (Spectral gap)**.** *We assume that we can write*

$$\sigma(H) = \sigma_0 \cup \sigma_1,$$

where $\operatorname{dist}(\sigma_0, \sigma_1) > 0$ *and* $\operatorname{diam}(\sigma_0) < \infty$.

Since H is essentially self-adjoint, by the spectral theorem there exists an orthogonal projector P associated with σ_0. Because of the spectral gap of H, by the Riesz projection formula we can find a contour $\mathcal{C}$ in the complex plane enclosing σ_0 so that:

$$P = \frac{1}{2\pi i} \int_{\mathcal{C}} (\lambda - H)^{-1} \, \mathrm{d}\lambda. \tag{4.1}$$

Furthermore, we may choose $\mathcal{C}$ so that $\mathcal{C}$ has finite length and

$$\sup_{\lambda \in \mathcal{C}} \|(\lambda - H)^{-1}\| < \infty.$$

We define two-dimensional position operators X and Y with respect to a choice of two-dimensional non-parallel axes by

$$Xf(x,y) = xf(x,y), \text{ and } Yf(x,y) = yf(x,y), \tag{4.2}$$

We now claim the following Lemma:

Lemma 4.1. *Let H satisfy the regularity and spectral gap assumptions (Assumptions 4.1 and 4.2). Then, with P as in* (4.1) *and X as in* (4.2)*, the operator PXP is well-defined on $\mathcal{D}(X)$ and essentially self-adjoint.*

We prove Lemma 4.1 in Section 5.1. Lemma 4.1 generalizes part (i) of Theorem 1 of Nenciu-Nenciu [37] to two dimensions. The proof is essentially identical, relying on exponential localization of P, proved in Appendix A.

We are now in a position to give our precise assumption on PXP. When it holds, we say that PXP has *uniform spectral gaps.*

Assumption 4.3 (Uniform Spectral Gaps)**.** *We assume there exist constants* (d, D) *such that:*

(1) There exists a countable set, $\mathcal{J}$, such that:

$$\sigma(PXP) = \bigcup_{j \in \mathcal{J}} \sigma_j.$$

(2) The distance between σ_j, σ_k ($j \neq k$) is uniformly lower bounded:

$$d := \min_{j \neq k} \Big(\operatorname{dist}(\sigma_j, \sigma_k) \Big) > 0.$$

(3) The diameter of each σ_j is uniformly bounded:

$$D := \max_{j \in \mathcal{J}} \Big(\operatorname{diam}(\sigma_j) \Big) < \infty.$$

If PXP has uniform spectral gaps in the sense of Assumption 4.3, we can define spectral projections associated with each subset $\{\sigma_j\}_{j\in\mathcal{J}}$ of $\sigma(PXP)$. We will refer to these projections as *band projectors*. Note that our use of "band" in this context should not be confused with its use in the context of Bloch eigenvalue bands of periodic operators.

Definition 4.1 (Band projectors)**.** *When PXP has uniform spectral gaps with constants (d, D) and decomposition $\{\sigma_j\}_{j\in\mathcal{J}}$ in the sense of Assumption 4.3, we let*

$$P_j := P\left(\frac{1}{2\pi i}\int_{\mathcal{C}_j}(\lambda - PXP)^{-1}d\lambda\right)P \quad j \in \mathcal{J}, \tag{4.3}$$

where $\mathcal{C}_j$ encloses σ_j and satisfies

$$\sup_{\lambda\in\mathcal{C}_j}\|(\lambda - PXP)^{-1}\| \leq Cd^{-1} \text{ and } \ell(\mathcal{C}_j) \leq C'(D+d) \tag{4.4}$$

for some absolute constants C, C' independent of j. In particular, P_j is an orthogonal projection onto the spectral subspace associated with σ_j.

Note that the existence of a contour $\mathcal{C}_j$ satisfying (4.4) is clearly guaranteed by the uniform spectral gap assumption (Assumption 4.3).

The addition of P in the definition (4.3) of the band projectors, P_j, is to ensure that $\text{range}\,(P_j) \subseteq \text{range}\,(P)$. This is necessary to avoid the trivial null space of PXP consisting of functions in the null space of P. Since P commutes with PXP, we have that

$$P_j = \left(\frac{1}{2\pi i}\int_{\mathcal{C}_j}(\lambda - PXP)^{-1}\mathrm{d}\lambda\right)P.$$

Our aim is to apply the Kivelson-Nenciu-Nenciu construction to each of the projections range(P_j). We start with the following Lemma:

Lemma 4.2. *Let H satisfy the regularity and spectral gap assumptions (Assumptions 4.1 and 4.2) and PXP satisfy the uniform spectral gap assumption (Assumption 4.3). Let P_j, with $j \in \mathcal{J}$, be the band projectors as in Definition 4.1, and let Y be as in* (4.2)*. Then the operators P_jYP_j are each well-defined on the domain $\mathcal{D}(Y)$ and extend to unbounded self-adjoint operators on $L^2(\mathbb{R}^2)$.*

We prove Lemma 4.2 in Section 5.2. The proof is very similar to the proof of Lemma 4.1, except that it relies on exponential localization of each P_j, proved in Appendix B.

We now claim that each of the operators P_jYP_j has compact resolvent:

Lemma 4.3. *Let P_jYP_j, $j \in \mathcal{J}$, be as in Lemma 4.2. Then each of the operators P_jYP_j has compact resolvent.*

We prove Lemma 4.3 in Section 6. Lemma 4.3 generalizes part (ii) of Theorem 1 of [37] to two dimensions (we sketch the generalization to higher dimensions in Section 8). The generalization is non-trivial since the decay induced by Y alone is not sufficient for the resolvent of P_jYP_j to be compact. To prove compactness, we make use of the fact that functions in each subspace range(P_j) decay with respect to x.

Our final Lemma states that eigenfunctions of the operators P_jYP_j are exponentially localized:

Lemma 4.4. *Let P_jYP_j, $j \in \mathcal{J}$, be as in Lemma 4.2. Then there exists a $\gamma'' > 0$, independent of j, such that if $\psi \in \operatorname{range}(P_j)$ and $P_jYP_j\psi = \eta'\psi$, then for all $\eta \in \sigma_j$*

$$\int e^{2\gamma''\sqrt{1+(x-\eta)^2+(y-\eta')^2}}|\psi(x,y)|^2 \,\mathrm{d}x\,\mathrm{d}y \leq 16e^{\gamma''\sqrt{1+2b^2}}.$$

Here b is a finite positive constant (independent of j and ψ) which depends only on the collection $\{P_j\}_{j\in\mathcal{J}}$.

We prove Lemma 4.4 in Section 7. Lemma 4.4 generalizes part (iii) of Theorem 1 of Nenciu-Nenciu [37] to two dimensions (we sketch the generalization to higher dimensions in Section 8). Although the proof of Lemma 4.4 has a similar structure to that of [37], our proof involves technical innovations which are necessary to (1) generalize their proof to two dimensions and higher (2) give a proof which does not refer to the original Hamiltonian H, requiring only operator norm estimates on the projectors P_j (proven in Appendix B). The fact that the proof of Lemma 4.4 does not make reference to the form of the original Hamiltonian significantly simplifies the generalization of our proofs to discrete models (see Section 9).

We are now in a position to state our main result in full detail.

Theorem 4.1. *Let H satisfy the regularity and spectral gap assumptions (Assumptions 4.1 and 4.2), and PXP satisfy the uniform spectral gaps assumption (Assumption 4.3). Then there exist functions $\{\psi_{j,m}\}_{(j,m)\in\mathcal{J}\times\mathcal{M}}$ and points $\{(a_j, b_m)\}_{(j,m)\in\mathcal{J}\times\mathcal{M}} \in \mathbb{R}^2$ such that*

1. *The collection $\{\psi_{j,m}\}_{(j,m)\in\mathcal{J}\times\mathcal{M}}$ is an orthonormal basis of* $\operatorname{range}(P)$.
2. *Each $\psi_{j,m}, (j,m) \in \mathcal{J}\times\mathcal{M}$ is exponentially localized in the sense that*

$$\int_{\mathbb{R}^2} e^{2\gamma\sqrt{1+(x-a_{j,m})^2+(y-b_{j,m})^2}}|\psi_{j,m}(x,y)|^2 \,\mathrm{d}x\,\mathrm{d}y \leq C, \tag{4.5}$$

   where (C,γ) denote finite positive constants which are independent of j and m.
3. *The set of $\{\psi_{j,m}\}_{(j,m)\in\mathcal{J}\times\mathcal{M}}$ are the set of eigenfunctions of the operators P_jYP_j, where P_j are the band projectors defined by Definition 4.1.*

Here $\mathcal{J}$ and $\mathcal{M}$ are the countable sets which index the band projectors as in Definition 4.1 and the eigenfunctions of P_jYP_j for fixed j, respectively.

Proof. The proof follows immediately from Lemmas 4.1, 4.2, 4.3, and 4.4 and the spectral theorem for self-adjoint operators. For each $(j,m) \in \mathcal{J}\times\mathcal{M}$, we define the points $(a_{j,m}, b_{j,m})$ by the (η,η') appearing in Lemma 4.4, i.e. $a_{j,m}$ can be taken as any $\eta \in \sigma_j$, while $b_{j,m}$ is the associated eigenvalue of $\psi_{j,m}$ considered as an eigenfunction of P_jYP_j. □

In addition to the particular innovations in the proofs of Lemmas 4.3 and 4.4 already mentioned, we remark that the overall structure of our proofs is more operator-theoretic than [37]. In particular, the precise form of the Hamiltonian H is essentially only used to prove operator norm bounds on P (Appendix A). These operator norm bounds are then used to prove the operator norm bounds on each P_j (Appendix B) which are necessary for the proofs of Lemmas 4.1-4.4. This structure clarifies necessary technical hypotheses and makes generalizing our results to higher dimensions (Section 8) and discrete models (Section 9) straightforward.

5. Proof that PXP and P_jYP_j are essentially self-adjoint (Proofs of Lemmas 4.1 and 4.2)

The proofs that PXP and P_jYP_j are essentially self-adjoint in two dimensions are almost exactly the same and they both use the same argument as given in [37]. Because the proofs are so similar, we will first prove that PXP is essentially self-adjoint and then note the changes which need to be made to prove that P_jYP_j is essentially self-adjoint.

5.1. **Proof of essential self-adjointness of PXP.** To prove PXP is well-defined on $\mathcal{D}(X)$ and essentially self-adjoint we require two lemmas regarding norm estimates for the projector P.

To state these lemmas, we have to introduce a short-hand notation. Given a point $(a,b) \in \mathbb{R}^2$, and a non-negative constant $\gamma \geq 0$, we define an exponential growth operator by

$$B_{\gamma,(a,b)} := \exp\left(\gamma\sqrt{1 + (X-a)^2 + (Y-b)^2}\right).$$

Furthermore, we use the convention that when $\gamma = 0$, $B_{\gamma,(a,b)} = 1$.

Given an operator A, we can then define an "exponentially-tilted" version of A as

$$A_{\gamma,(a,b)} := B_{\gamma,(a,b)} A B^{-1}_{\gamma,(a,b)}. \tag{5.1}$$

We will often prove estimates where we use the notation (5.1) but omit the point (a,b). In this case the estimate should be understood as uniform in of the choice of point (a,b). As a note, that per our convention, when $\gamma = 0$, $A_{\gamma,(a,b)} = A$.

We require the following two lemmas.

Lemma 5.1. *Let H satisfy the regularity and spectral gap assumptions (Assumptions 4.1 and 4.2). Then there exist constants $C > 0$ and $\gamma_0 > 0$ such that for all $0 \leq \gamma \leq \gamma_0$, the spectral projection P defined by* (4.1) *satisfies*

$$\|P_\gamma\| \leq C. \tag{5.2}$$

Proof. Given in Appendix A. □

Lemma 5.2. *Let P be as in Lemma 5.1. Then the operator $[P,X]$ is bounded,i.e.*

$$\|[P,X]\| \leq K_2',$$

where K_2' is a finite, positive constant.

Proof. Given in Appendix A. □

We can now prove that PXP is essentially self-adjoint.

Since X is essentially self-adjoint, we know that for all $\mu > 0$, $(X \pm i\mu)^{-1}$ is well defined and $\|(X \pm i\mu)^{-1}\| \leq \mu^{-1}$. Therefore, since P is a projection,

$$(PXP \pm i\mu)P(X \pm i\mu)^{-1}P = P(X \pm i\mu)P(X \pm i\mu)^{-1}P,$$

where both sides of this equation are well-defined using Lemma 5.1. Commuting $(X \pm i\mu)$ and P now gives

$$\begin{aligned}(PXP \pm i\mu)P(X \pm i\mu)^{-1}P &= P(X \pm i\mu)P(X \pm i\mu)^{-1}P \\ &= P + P[X,P](X \pm i\mu)^{-1}P \\ &= P(I + P[X,P](X \pm i\mu)^{-1}P).\end{aligned} \tag{5.3}$$

Since $\|[X,P]\| \leq K_2' < \infty$ (Lemma 5.2), we can pick $\mu > 2\|[X,P]\|$ to conclude that

$$\|P[X,P](X \pm i\mu)^{-1}P\| \leq \frac{1}{2}.$$

Therefore, we may invert $I + P[X,P](X \pm i\mu)^{-1}P$ in Equation (5.3) to get

$$(PXP \pm i\mu)P(X \pm i\mu)^{-1}P\Big(I + P[X,P](X \pm i\mu)^{-1}P\Big)^{-1} = P.$$

Hence, $(PXP \pm i\mu)$ is surjective on range (P). Since PXP acts trivially on range $(P)^\perp$ and $\mu > 0$ we can therefore conclude that $(PXP \pm i\mu)$ is surjective on its domain. Hence by the fundamental criterion of self-adjointness (see Chapter VIII of [47]), PXP is essentially self-adjoint.

5.2. **Proof of essential self-adjointness of $P_j Y P_j$.** In this section we require analogous estimates on P_j as we required on P in section 5.1.

Lemma 5.3. *Let H satisfy the regularity and spectral gap assumptions (Assumptions 4.1 and 4.2), and PXP satisfy the uniform gap assumption (Assumption 4.3). Let $P_j, j \in \mathcal{J}$ denote the band projectors defined by Definition 4.1. Then there exist constants $C > 0$ and $\gamma_0 > 0$ such that for all $0 \leq \gamma \leq \gamma_0$, each spectral projection P_j defined by* (4.3) *satisfies*

$$\|P_{j,\gamma}\| \leq C.$$

Proof. Given in Appendix B. □

Lemma 5.4. *Let P_j be as in Lemma 5.3. Then the operator $[P_j, Y]$ is bounded, i.e.*

$$\|[P_j, Y]\| \leq K_2'',$$

where K_2' is a finite, positive constant.

Proof. Given in Appendix B. □

We can now prove essential self-adjointness of $P_j Y P_j$ where $j \in \mathcal{J}$ by an identical calculation to that given in Section 5.1.

Using Lemma 5.3, the operator

$$(P_j Y P_j \pm i\tilde{\mu})P_j(Y \pm i\tilde{\mu})^{-1}P_j$$

is well-defined for all $\tilde{\mu} > 0$. Since $\|[P_j, Y]\|$ is bounded (Lemma 5.4), we can choose $\tilde{\mu} > 2\|[P_j, Y]\|$ and repeat the same steps as in Section 5.1 to get that

$$(P_j Y P_j \pm i\tilde{\mu})P_j(Y \pm i\tilde{\mu})^{-1}P_j\Big(I + P_j[Y, P_j](Y \pm i\tilde{\mu})^{-1}P_j\Big)^{-1} = P_j \tag{5.4}$$

which shows that $P_j Y P_j \pm i\tilde{\mu}$ is surjective on range P_j and we are done.

6. Proof that $P_j Y P_j$ has compact resolvent (Proof of Lemma 4.3)

In this section, in addition to Lemmas 5.3 and 5.4, we require the following Lemma regarding P_j.

Lemma 6.1. *Let H satisfy the regularity and spectral gap assumptions (Assumptions 4.1 and 4.2), and PXP satisfy the uniform gap assumption (Assumption 4.3). Let $P_j, j \in \mathcal{J}$ denote the band projectors defined by Definition 4.1. Then, for any $\eta \in \sigma_j$, the operator $(X - \eta)P_j$ is bounded, i.e.*

$$\|(X - \eta)P_j\| \leq K_3''$$

where K_3'' is a positive constant independent of j.

Proof. Given in Appendix B. □

For the proof of Lemma 4.3, we will start by continuing the main calculation of Section 5.2. We used Lemmas 5.3 and 5.4 to derive equation (5.4), which we re-state here:

$$(P_jYP_j \pm i\tilde{\mu})P_j(Y \pm i\tilde{\mu})^{-1}P_j\Big(I + P_j[Y,P_j](Y \pm i\tilde{\mu})^{-1}P_j\Big)^{-1} = P_j.$$

Since P_jYP_j is essentially self-adjoint on range (P_j) we may invert $(P_jYP_j \pm i\tilde{\mu})$ for $\tilde{\mu} > 0$ to get:

$$P_j(Y \pm i\tilde{\mu})^{-1}P_j\Big(I + P_j[Y,P_j](Y \pm i\tilde{\mu})^{-1}P_j\Big)^{-1} = (P_jYP_j \pm i\tilde{\mu})^{-1}P_j.$$

Since the product of a compact operator and a bounded operator is compact, it follows that to show that $(P_jYP_j \pm i\tilde{\mu})^{-1}$ is compact on range (P_j) it is enough to show that $P_j(Y \pm i\tilde{\mu})^{-1}P_j$ is compact.

Let $\eta \in \sigma_j$, i.e. as in Lemma 6.1. Taking $P_j(Y \pm i\tilde{\mu})^{-1}P_j$ and inserting $(X - \eta + i)^{-1}(X - \eta + i)$ and $(-\Delta + 1)(-\Delta + 1)^{-1}$ gives:

$$\begin{aligned} &P_j(Y \pm i\tilde{\mu})^{-1}P_j \\ &= \Big[P_j(-\Delta+1)\Big]\Big[(-\Delta+1)^{-1}(Y \pm i\tilde{\mu})^{-1}(X-\eta+i)^{-1}\Big]\Big[(X-\eta+i)P_j\Big]. \end{aligned} \tag{6.1}$$

We will prove Lemma 4.3 by proving that the first and last term of Equation (6.1) are bounded (Section 6.1) and the middle term is compact (Section 6.2).

6.1. **Proofs that first and last terms of** (6.1) **are bounded.** The last term of (6.1) is bounded by Lemma 6.1:

$$\begin{aligned} \|(X-\eta+i)P_j\| &\le \|(X-\eta)P_j\| + 1 \\ &\le K_3'' + 1. \end{aligned}$$

As for the first term, since P_j is a spectral projection of PXP, we have

$$\|P_j(-\Delta+1)\| = \|P_jP(-\Delta+1)\| \le \|P(-\Delta+1)\| \le \|P\Delta\| + 1. \tag{6.2}$$

It remains only to bound $P\Delta$. We will prove that ΔP is bounded, from which boundedness of $P\Delta$ follows immediately from duality. Recall that P is defined through the the Riesz projection formula (4.1). Since the contour $\mathcal{C}$ appearing in (4.1) has finite length and Δ and P are both self-adjoint, (6.2) will follow immediately if we can show

$$\sup_{\lambda\in\mathcal{C}} \|\Delta(\lambda - H)^{-1}\| < \infty. \tag{6.3}$$

To do this, we will first prove the following Lemma

Lemma 6.2. *Δ is H-bounded in the sense that $\mathcal{D}(\Delta) \subset \mathcal{D}(H)$ and there exist constants $a, b > 0$ such that for any $\psi \in \mathcal{D}(H)$,*

$$\|\Delta\psi\| \le a\|H\psi\| + b\|\psi\|.$$

Proof. That $\mathcal{D} \subset \mathcal{H}$ is clear under Assumption 4.1. Let Assumption 4.1 on H hold, and let $\psi \in \mathcal{D}(\Delta)$. Then

$$\|\Delta\psi\| = \|(\Delta+H)\psi - H\psi\| \le \|(\Delta+H)\psi\| + \|H\psi\|. \tag{6.4}$$

Explicitly,

$$\Delta + H = -2iA \cdot \nabla - i\text{div}A + A \cdot A + V.$$

Under Assumption 4.1, we have easily that

$$\|(\Delta + H)\psi\| \leq 4\|A\|_{L^\infty} \sup_{j=1,2} \|\partial_j \psi\| + \left(\|\text{div}A\|_{L^\infty} + \|A\|_{L^\infty}^2 + \|V\|_{L^\infty}\right) \|\psi\|. \tag{6.5}$$

Now observe that for any $\epsilon > 0$,

$$\|\partial_j \psi\| \leq \epsilon \|\Delta\psi\| + \frac{1}{\epsilon}\|\psi\| \quad j = 1, 2.$$

Substituting this inequality into (6.5), and then substituting (6.5) into (6.4), we have

$$\begin{aligned} \|\Delta\psi\| \leq &\|H\psi\| + 4\epsilon\|A\|_{L^\infty}\|\Delta\psi\| \\ &+ \left(4\frac{1}{\epsilon}\|A\|_{L^\infty} + \|\text{div}A\|_{L^\infty} + \|A\|_{L^\infty}^2 + \|V\|_{L^\infty}\right)\|\psi\|. \end{aligned} \tag{6.6}$$

Now, by taking ϵ sufficiently small, we can ensure that $1 - 2\epsilon\|A\|_{L^\infty} > 0$. For such ϵ we have that

$$\begin{aligned} \|\Delta\psi\| \leq &\left(1 - 4\epsilon\|A\|_{L^\infty}\right)^{-1} \|H\psi\| \\ &+ \left(1 - 4\epsilon\|A\|_{L^\infty}\right)^{-1} \left(4\frac{1}{\epsilon}\|A\|_{L^\infty} + \|\text{div}A\|_{L^\infty} + \|A\|_{L^\infty}^2 + \|V\|_{L^\infty}\right)\|\psi\|, \end{aligned} \tag{6.7}$$

which proves the Lemma with

$$\begin{aligned} a &= \left(1 - 4\epsilon\|A\|_{L^\infty}\right)^{-1} \\ b &= \left(1 - 4\epsilon\|A\|_{L^\infty}\right)^{-1} \left(4\frac{1}{\epsilon}\|A\|_{L^\infty} + \|\text{div}A\|_{L^\infty} + \|A\|_{L^\infty}^2 + \|V\|_{L^\infty}\right) \end{aligned}$$

and $0 < \epsilon < \frac{1}{4\|A\|_{L^\infty}}$. □

We now prove (6.3), essentially following the proof of Proposition 1.3 of [10]. Let $\phi \in \mathcal{H}$ be arbitrary. Then for $\lambda \in \mathcal{C}$, $(\lambda - H)^{-1}\phi \in \mathcal{D}(\mathcal{H})$. By Lemma 6.2 it follows that

$$\begin{aligned} \|\Delta(\lambda - H)^{-1}\phi\| &\leq a\|H(\lambda - H)^{-1}\phi\| + b\|(\lambda - H)^{-1}\phi\| \\ &= a\|(\lambda - H)(\lambda - H)^{-1}\phi - \lambda(\lambda - H)^{-1}\phi\| + b\|(\lambda - H)^{-1}\phi\| \\ &\leq a\|\phi\| + (a|\lambda| + b)\|(\lambda - H)^{-1}\phi\|. \end{aligned}$$

The bound (6.3) then immediately follows since $\mathcal{C}$ has finite length.

6.2. **Proof that middle term of** (6.1) **is compact.** It remains only to prove that the middle term

$$(-\Delta + 1)^{-1}(Y \pm i\tilde{\mu})^{-1}(X - \eta + i)^{-1} \tag{6.8}$$

is a compact operator. We will do so by proving it is the limit of compact operators in the operator norm topology. To this end, let χ_N denote a cutoff function

$$\chi_N(x, y) := \begin{cases} 1 & \text{if } |x| \leq N \text{ and } |y| \leq N \\ 0 & \text{otherwise.} \end{cases}$$

We can re-write (6.8) using χ_N as

(6.9)
$$(-\Delta+1)^{-1}\chi_N(Y\pm i\tilde{\mu})^{-1}(X-\eta+i)^{-1}+(-\Delta+1)^{-1}(1-\chi_N)(Y\pm i\tilde{\mu})^{-1}(X-\eta+i)^{-1}$$

We first note that since X^{-1} and Y^{-1} decay as $|X|,|Y|\to\infty$, for all N sufficiently large the second term is $O(N^{-1})$ in operator norm. Therefore,

$$\lim_{N\to\infty}(-\Delta+1)^{-1}\chi_N(Y\pm i\tilde{\mu})^{-1}(X-\eta+i)^{-1}=(-\Delta+1)^{-1}(Y\pm i\tilde{\mu})^{-1}(X-\eta+i)^{-1}$$

in the operator norm topology. It remains to prove that the first term in (6.9) is compact for each N. Since $(Y\pm i\tilde{\mu})^{-1}(X-\eta+i)^{-1}$ is bounded, it suffices to prove that

$$(-\Delta+1)^{-1}\chi_N \tag{6.10}$$

is compact for each N.

We will prove (6.10) is compact by showing it is Hilbert-Schmidt, by showing that its integral kernel is in $L^2(\mathbb{R}^2\times\mathbb{R}^2)$. We start by defining, for real and positive z, the "modified Bessel function" $K_0(z)$ through the integral

$$K_0(z):=\int_0^\infty t^{-1}\exp\left(-\frac{z^2}{4t}-t\right)\,\mathrm{d}t,$$

(see (10.32.10) of [43]). Note that $K_0(z)$ has a logarithmic singularity at the origin and decays exponentially in z. The integral kernel of $(-\Delta+1)^{-1}$ in two dimensions can be computed explicitly as (see Section 6.23 of [31]),

$$\frac{1}{2\pi}K_0\left(\sqrt{(x-x')^2+(y-y')^2}\right).$$

Hence (6.10) is an integral operator with kernel

$$K(x,y,x',y'):=K_0\left(\sqrt{(x-x')^2+(y-y')^2}\right)\chi_N(x',y').$$

Integrating $K(x,y,x',y')$ we have

$$\int_{\mathbb{R}}\int_{\mathbb{R}}\int_{\mathbb{R}}\int_{\mathbb{R}}|K(x,y,x',y')|^2\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}x'\,\mathrm{d}y'=\frac{1}{4\pi^2}\int_{\mathbb{R}}\int_{\mathbb{R}}\left(\left|\tilde{K}_0\right|^2*\chi_N\right)(x,y)\,\mathrm{d}x\,\mathrm{d}y,$$

where

$$\tilde{K}_0(x,y):=K_0\left(\sqrt{x^2+y^2}\right).$$

Hence we are done if we can show $|\tilde{K}_0|^2*\chi_N(x,y)\in L^1(\mathbb{R}^2)$. By Young's inequality, this holds as long as $\chi_N(x,y)$ and $|\tilde{K}_0(x,y)|^2$ are both in $L^1(\mathbb{R}^2)$. The first statement is obvious, while the second follows immediately when we change to polar co-ordinates (r,θ) where $x=r\cos\theta$ and $y=r\sin\theta$:

$$\int_{\mathbb{R}}\int_{\mathbb{R}}\left|K_0\left(\sqrt{x^2+y^2}\right)\right|^2\,\mathrm{d}x\,\mathrm{d}y=2\pi\int_0^\infty|K_0(r)|^2\,r\,\mathrm{d}r,$$

and use exponential decay of $K_0(z)$.

7. Eigenfunctions of P_jYP_j are exponentially localized (Proof of Lemma 4.4)

In this section, we will need somewhat stronger bounds on P_j than in Section 6. Let us recall that we define the exponential growth operator for any $(a,b) \in \mathbb{R}^2$ and any $\gamma \geq 0$:

$$B_{\gamma,(a,b)} = \exp\left(\gamma\sqrt{1+(X-a)^2+(Y-b)^2}\right).$$

Since most of the steps in our proof are independent of the choice of (a,b), we will suppress this subscript and simply write B_γ. Recall that for any $\gamma \geq 0$ we define

$$P_{j,\gamma} := B_\gamma P_j B_\gamma^{-1}.$$

With these notations, the following lemma states the bounds on P_j we require which are proved in Appendix B.

Lemma 7.1. *Let H satisfy the regularity and spectral gap assumptions (Assumptions 4.1 and 4.2), and PXP satisfy the uniform gap assumption (Assumption 4.3). Let $P_j, j \in \mathcal{J}$ denote the band projectors associated with the separated components σ_j of $\sigma(PXP)$ defined by Definition 4.1. Then there exist finite, positive constants $(\gamma'', K_1'', K_2'', K_3'')$, independent of j, such that for all $\gamma \leq \gamma''$*

(1) $\|P_{j,\gamma} - P_j\| \leq K_1''\gamma$
(2) $\|[P_{j,\gamma}, Y]\| \leq K_2''$
(3) For all $\eta \in \sigma_j$:

$$\|(X-\eta)P_{j,\gamma}\| \leq K_3'' \text{ and } \|P_{j,\gamma}(X-\eta)\| \leq K_3''.$$

Proof. Given in Appendix B. □

With these estimates in hand, we are now ready to prove Lemma 4.4. The overall strategy, which follows Nenciu-Nenciu [37], is to manipulate the eigenvalue equation $P_jYP_jv = \eta' v$ into the form

$$v = \mathcal{L}v \tag{7.1}$$

for some operator $\mathcal{L}$, multiply both sides of (7.1) by $B_{\gamma,(a,b)}$ for some (a,b), and then use properties of $\mathcal{L}$ to deduce that the left-hand side is bounded. Our proof differs from Nenciu-Nenciu [37] in important details.

Suppose that P_jYP_j has an eigenvector v with eigenvalue η' and $v \in \text{range}\,(P_j)$. Since $v \in \text{range}\,(P_j)$ we have that

$$P_jYP_jv = \eta' v \iff P_jYP_jv = \eta' P_jv \iff P_j(Y-\eta')P_jv = 0.$$

Now for any operator O such that $v \in \mathcal{D}(P_jOP_j)$, adding iP_jOP_jv to both sides of the above equation gives

$$P_j(Y-\eta'+iO)P_jv = iP_jOP_jv. \tag{7.2}$$

The main difference between our proof and the proof of Nenciu-Nenciu lies in the choice of the operator O. For now, let's suppose that we have chosen O so that $(Y-\eta'+iO)$ is invertible and multiply both sides of Equation (7.2) by $(Y-\eta'+iO)^{-1}$ to get

$$(Y-\eta'+iO)^{-1}P_j(Y-\eta'+iO)P_jv = i(Y-\eta'+iO)^{-1}P_jOP_jv. \tag{7.3}$$

We can simplify the left hand side of Equation (7.3) by commuting P_j and $(Y - \eta' + iO)$ as follows

$$\begin{aligned}(Y - \eta + iO)^{-1}&P_j(Y - \eta' + iO)P_j \\ &= P_j + (Y - \eta' + iO)^{-1}[P_j, Y - \eta' + iO]P_j \\ &= \Big(I + (Y - \eta' + iO)^{-1}([P_j, Y] + i[P_j, O])\Big)P_j.\end{aligned}$$

Therefore, we can write Equation (7.3) as follows

$$\Big(I + (Y - \eta' + iO)^{-1}([P_j, Y] + i[P_j, O])\Big)P_j v = i(Y - \eta' + iO)^{-1}P_jOP_jv.$$

To reduce the number of terms in the next steps, let's define

$$A := (Y - \eta' + iO)^{-1}([P_j, Y] + i[P_j, O]).$$

With this definition and using that $v \in \text{range}\,(P_j)$ we have that

$$(I + A)v = i(Y - \eta' + iO)^{-1}P_jOv. \tag{7.4}$$

For the next step of the proof, we will want to show that the $(I + A)$ has bounded inverse. To do this, by Neumann series, it is enough to show that

$$||A|| = \|(Y - \eta' + iO)^{-1}([P_j, Y] + i[P_j, O])\| \leq \frac{3}{4}. \tag{7.5}$$

For Equation (7.5) to hold, we require a particular choice of the operator O which differs from the choice made in Nenciu-Nenciu [37]. We require that O satisfies the following properties:

(1) O commutes with B_γ.
(2) O contains a cutoff in both the X and Y directions.
(3) $(Y - \eta' + iO)$ is invertible.

For our proof, we let $b > 0$ be a constant to be chosen later and set O to be the following operator:

$$O = b\Pi^X_{[\eta-b,\eta+b]}\Pi^Y_{[\eta'-b,\eta'+b]} + |X - \eta| \tag{7.6}$$

where Π^X_I (resp. Π^Y) is a spectral projection for X (resp. Y) onto the interval I and $|X - \eta|$ is the polar decomposition of $X - \eta$ defined by $|X - \eta| := \sqrt{(X - \eta)^2}$.

Before continuing we make three important observations about this choice for O:

(1) Since X and Y commute and are essentially self-adjoint, the operator $\Pi^X_{[\eta-b,\eta+b]}\Pi^Y_{[\eta'-b,\eta'+b]}$ is an orthogonal projector.
(2) Due to the properties of the polar decomposition [47] and our bounds on P_j we know that for all γ sufficiently small both $\|P_{j,\gamma}|X-\eta|\|$ and $\||X-\eta|P_{j,\gamma}\|$ are bounded[1].
(3) For all $b > 0$, $\|(Y - \eta' + iO)^{-1}\| \leq b^{-1}$.

---

[1]The polar decomposition gives that there exist partial isometries U and V such that $|X - \eta| = U(X - \eta) = (X - \eta)V$ and so

$$\begin{aligned}\||X - \eta|P_{j,\gamma}\| &= \|U(X - \eta)P_{j,\gamma}\| \leq \|(X - \eta)P_{j,\gamma}\| \\ \|P_{j,\gamma}|X - \eta|\| &= \|P_{j,\gamma}(X - \eta)V\| \leq \|P_{j,\gamma}(X - \eta)\|\end{aligned}$$

In what follows, we will abbreviate $\Pi := \Pi^X_{[\eta-b,\eta+b]}\Pi^Y_{[\eta'-b,\eta'+b]}$.

The key trick which allows us to show that $\|A\| \le \frac{3}{4}$ is the following lemma (see Corollary 8 of [55] for an independent proof, see also [26]):

Lemma 7.2. *Let B, C be two bounded operators. If B is positive semidefinite then*

$$\|[B,C]\| \le \|B\|\|C\|.$$

If both B and C are positive semidefinite then

$$\|[B,C]\| \le \frac{1}{2}\|B\|\|C\|.$$

Proof. Suppose that B is positive semidefinite and define $\tilde{B} := B - \frac{1}{2}\|B\|I$. Since B is a positive semidefinite operator its spectrum lies in the range $[0, \|B\|]$. Therefore, the spectrum of $\tilde{B}$ lies in the range $[-\frac{1}{2}\|B\|, \frac{1}{2}\|B|]$ and hence $\|\tilde{B}\| = \frac{1}{2}\|B\|$. Since the identity commutes with every operator we have

$$\|[B,C]\| = \|[\tilde{B},C]\| = \|\tilde{B}C - C\tilde{B}\| \le 2\|\tilde{B}\|\|C\| = \|B\|\|C\|.$$

If C is also positive semidefinite then we can repeat the same argument using $\tilde{C} := C - \frac{1}{2}\|C\|I$ as well to get $\|[B,C]\| \le \frac{1}{2}\|B\|\|C\|$. ◻

We can now prove that, for b sufficiently large, $\|A\| \le \frac{3}{4}$. The following calculations are clear:

$$\begin{aligned}
\|A\| &= \|(Y-\lambda+iO)^{-1}[P_j, Y+iO]\| \\
&\le \|(Y-\lambda+iO)^{-1}\|\Big(\|[P_j,Y]\| + \|[P_j,O]\|\Big) \\
&\le b^{-1}\Big(\|[P_j,Y]\| + b\|[P_j,\Pi]\| + \|[P_j,|X-\eta|]\|\Big) \\
&\le \|[P_j,\Pi]\| + b^{-1}\Big(\|[P_j,Y]\| + \|P_j|X-\eta|\| + \||X-\eta|P_j\|\Big).
\end{aligned}$$

Since P_j and Π are both orthogonal projectors we can apply Lemma 7.2 to conclude that $\|[P_j,\Pi]\| \le \frac{1}{2}$. It now follows that

$$\|A\| \le \frac{1}{2} + b^{-1}\Big(\|[P_j,Y]\| + \|P_j|X-\eta|\| + \||X-\eta|P_j\|\Big),$$

and if we choose b so that

$$b > 4\Big(\|[P_j,Y]\| + \|P_j|X-\eta|\| + \||X-\eta|P_j\|\Big),$$

we have that

$$\|A\| \le \frac{3}{4}.$$

Note that because of our estimates from Lemma 7.1 we know that we can choose b so that $b < \infty$.

Returning to Equation (7.4), we now know that we can invert $(I+A)$ and get

$$\begin{aligned}
(I+A)v &= i(Y-\eta'+iO)^{-1}P_jOv \\
v &= i(I+A)^{-1}(Y-\eta'+iO)^{-1}P_jOv \qquad (7.7)
\end{aligned}$$

To reduce the number of terms in the next steps, let's define

$$C := (I+A)^{-1}(Y-\eta'+iO)^{-1}.$$

With this definition Equation (7.7) becomes

$$v = iCP_jOv.$$

Recalling that we chose $O := b\Pi + |X - \eta|$, we have that

$$\begin{aligned} v &= iCP_j Ov \\ &= iCP_j(b\Pi + |X-\eta|)v \\ &= ibCP_j\Pi v + iCP_j|X-\eta|v \end{aligned}$$

$$(I - iCP_j|X-\eta|)v = ibCP_j\Pi v. \tag{7.8}$$

Similar to before, we would like to invert the operator $(I - iCP_j|X-\eta|)$. Recall that if

$$b > 4\Big(\|[P_j, Y]\| + \|P_j|X-\eta|\| + \||X-\eta|P_j\|\Big),$$

then $\|A\| \leq \frac{3}{4}$ so we have that

$$\begin{aligned} \|C\| &= \|(I+A)(Y-\eta' + iO)^{-1}\| \\ &\leq \|(I+A)\|\|(Y-\eta'+iO)^{-1}\| \\ &\leq 4b^{-1}. \end{aligned}$$

Therefore,

$$\|iCP_j|X-\eta|\| \leq 4b^{-1}\|P_j|X-\eta|\|.$$

Since we have chosen $b > 4\|P_j|X-\eta|\|$, the operator $(I - iCP_j|X-\eta|)$ is invertible.

Using this fact allows us to rewrite Equation (7.8) as

$$v = ib\big(I - iCP_j|X-\eta|\big)^{-1}CP_j\Pi v. \tag{7.9}$$

After all of these algebraic steps, we have been able to derive an expression for v as the product of a bounded operator and a cutoff function. The final step in this argument will be to multiply both sides of Equation (7.9) by the exponential growth operator B_γ and show that the result is bounded. The inclusion of the cutoff function is what makes it possible to control this multiplication because $B_\gamma\Pi$ is bounded.

At least formally, we can multiply both sides of Equation (7.9) by B_γ and insert copies of $B_\gamma^{-1}B_\gamma$ to get

$$\begin{aligned} B_\gamma v &= ibB_\gamma\big(I - iCP_j|X-\eta|\big)^{-1}CP_j\Pi v \\ &= ibB_\gamma\big(I - iC\Big(B_\gamma^{-1}B_\gamma\Big)P_j|X-\eta|\big)^{-1}\Big(B_\gamma^{-1}B_\gamma\Big)C\Big(B_\gamma^{-1}B_\gamma\Big)P_j\Big(B_\gamma^{-1}B_\gamma\Big)\Pi v \\ &= ib\Big[(I - iC_\gamma P_{j,\gamma}|X-\eta|)^{-1}\Big]\Big[C_\gamma P_{j,\gamma}\Big]\Big[B_\gamma\Pi\Big]v, \end{aligned}$$

where we have used our convention for exponentially tilted operators $P_{j,\gamma} := B_\gamma P_j B_\gamma^{-1}$ and similarly for C. We will now show each of the bracketed terms are bounded.

The easiest term to bound is the last term, $B_\gamma\Pi$. Let's recall the definition of B_γ:

$$B_{\gamma,(a,b)} = \exp\left(\gamma\sqrt{1 + (X-a)^2 + (Y-b)^2}\right).$$

While we have ignored the center point (a,b) thus far in the argument, here we will explicitly choose $(a,b) = (\eta,\eta')$. Since $\Pi = \Pi^X_{[\eta-b,\eta+b]}\Pi^Y_{[\eta'-b,\eta'+b]}$ we clearly have:

$$\|B_{\gamma,(\eta,\eta')}\Pi\| \leq e^{\gamma\sqrt{1+2b^2}}.$$

To show that the first two terms are bounded we will show that, for γ sufficiently small, $\|C_\gamma\| = O(b^{-1})$. Once we show this, the second term, $C_\gamma P_{j,\gamma}$, is clearly bounded and we may pick b sufficiently large so that the first term is also bounded.

By definition, we have:

$$\begin{aligned} C_\gamma &= B_\gamma (I+A)^{-1}(Y-\eta'+iO)^{-1}B_\gamma^{-1} \\ &= B_\gamma\Big(I+(Y-\eta'+iO)^{-1}[P_j, Y+iO]\Big)^{-1}(Y-\eta'+iO)^{-1}B_\gamma^{-1} \\ &= \Big(I+(Y-\eta'+iO)^{-1}[P_{j,\gamma}, Y+iO]\Big)^{-1}(Y-\eta'+iO)^{-1}. \end{aligned}$$

For the above calculations to make sense, it suffices to show that

$$\|(Y-\eta'+iO)^{-1}[P_{j,\gamma}, Y+iO]\| \leq \frac{3}{4}.$$

Since $\|P_{j,\gamma}-P_j\| \leq K_1''\gamma$ we have that:

$$\begin{aligned} &\|(Y-\eta'+iO)^{-1}[P_{j,\gamma}, Y+iO]\| \\ &\quad\leq \|(Y-\eta'+iO)^{-1}\|\Big(\|[P_{j,\gamma}, Y]\| + b\|[P_{j,\gamma}, \Pi]\| + \|[P_{j,\gamma}, |X-\eta|]\|\Big) \\ &\quad\leq b^{-1}\Big(\|[P_{j,\gamma}, Y]\| + b\|[P_{j,\gamma}-P, \Pi]\| + b\|[P, \Pi]\| + \|[P_{j,\gamma}, |X-\eta|]\|\Big) \\ &\quad\leq b^{-1}\Big(\frac{1}{2}b + b\|P_{j,\gamma}-P\| + \|[P_{j,\gamma}, Y]\| + \|P_{j,\gamma}|X-\eta|\| + \||X-\eta|P_{j,\gamma}\|\Big) \\ &\quad\leq \frac{1}{2} + K_1''\gamma + b^{-1}\Big(\|[P_{j,\gamma}, Y]\| + \|P_{j,\gamma}|X-\eta|\| + \||X-\eta|P_{j,\gamma}\|\Big). \end{aligned}$$

Therefore, if we pick $\gamma \leq (8K_1'')^{-1}$ and

$$b \geq 8\Big(\|[P_{j,\gamma}, Y]\| + \|P_{j,\gamma}|X-\eta|\| + \||X-\eta|P_{j,\gamma}\|\Big),$$

we have that

$$\|P_{j,\gamma}(Y-\eta'+iO)^{-1}[P_{j,\gamma}, Y+iO]\| \leq \frac{3}{4}.$$

Therefore, for these choices of b and γ we have that:

$$\begin{aligned} \|C_\gamma\| &\leq \|\Big(I+P_{j,\gamma}(Y-\eta'+iO)^{-1}[P_{j,\gamma}, Y+iO]\Big)^{-1}\|\|(Y-\eta'+iO)^{-1}\| \\ &\leq 4b^{-1}. \end{aligned}$$

Now recall the original equation we wanted to bound:

$$B_\gamma v = i\Big[(I-iC_\gamma P_{j,\gamma}|X-\eta|)^{-1}\Big]\Big[bC_\gamma P_{j,\gamma}\Big]\Big[B_\gamma \Pi\Big]v.$$

Since $b \geq 8\|P_{j,\gamma}|X-\eta|\|$ we have that

$$\|C_\gamma P_{j,\gamma}|X-\eta|\| \leq 4b^{-1}\|P_{j,\gamma}|X-\eta|\| \leq \frac{1}{2}.$$

Therefore,

$$\|(I-iC_\gamma P_{j,\gamma}|X-\eta|)^{-1}\| \leq 2,$$

so combining all of our bounds together gives:

$$\begin{aligned} \|B_{\gamma,(\eta,\eta')}v\| &\leq \Big[2\Big]\Big[4(1+K_2''\gamma)\Big]\Big[e^{\gamma\sqrt{1+2b^2}}\Big] \\ &\leq 16e^{\gamma\sqrt{1+2b^2}}, \end{aligned}$$

so long as

$$\gamma \leq (8K_2'')^{-1}$$
$$b \geq 8\Big(\|[P_{j,\gamma}, Y]\| + \|P_{j,\gamma}|X-\eta|\| + \||X-\eta|P_{j,\gamma}\| \Big).$$

This proves Lemma 4.4.

8. Extensions to three dimensions and higher

The proof of Theorem 4.1 generalizes to arbitrarily high dimensions under appropriate generalizations of the uniform spectral gaps assumption (Assumption 4.3) by an inductive procedure. We will explain in detail the necessary additional assumptions and adjustments of our argument for the proof in three dimensions, from which the necessary assumptions and adjustments in higher dimensions are obvious.

A sketch of the generalization of our main result to three dimensions is as follows. Assume regularity and spectral gap assumptions analogous to Assumptions 4.1 and 4.2, and consider position operators X, Y, and Z acting on $\mathcal{H} := L^2(\mathbb{R}^3)$ along directions corresponding to a three-dimensional basis. Let P be the Fermi projection, and consider the operator PXP. Assume PXP has uniform spectral gaps in the sense of Assumption 4.3, and let P_j denote spectral projections onto each of the separated components of the spectrum of PXP. Now assume the operators P_jYP_j *also* have uniform spectral gaps in the sense of Assumption 4.3, and let $P_{j,k}$ denote spectral projections onto each of the separated components of the spectrum of P_jYP_j. By analogous reasoning to the two dimensional case, functions in range$(P_{j,k})$ are quasi-one dimensional in the sense that they decay away from lines $x = c_1$, $y = c_2$ for constants c_1, c_2. We therefore claim that the set of eigenfunctions of the operator $P_{j,k}ZP_{j,k}$ will form an exponentially localized basis of range$(P_{j,k})$ for each j, k, and that the union of all of these eigenfunctions over j and k will form an exponentially-localized basis of range(P).

To make the above sketch rigorous, there are a few important steps in the proof which must be checked. We will discuss each step in turn.

Proving bounds on $P_{j,k}$. First, we must check that we can prove operator bounds on $P_{j,k}$ which are analogous to the operator bounds we prove in Appendices A and B on P and P_j (specifically Lemmas A.1 and B.1). To see that this is possible, note that when we proved Lemma B.1 on P_j, we only required Lemma A.1 on P. It follows that under a uniform gap assumption on P_jYP_j we can prove an analogous Lemma on $P_{j,k}$ by a similar calculation using only Lemma B.1 on P_j.

Proving $P_{j,k}ZP_{j,k}$ has compact resolvent. To prove $P_{j,k}ZP_{j,k}$ has compact resolvent, mimicking the calculations preceding (6.1), it is sufficient to prove that $P_{j,k}(Z + i\tilde{\mu})^{-1}P_{j,k}$ is compact for each j, k. We will first show how a naïve generalization of the proof that P_jYP_j is compact in two dimensions fails, and then present a correct generalization. Just as in equation (6.1), we can write

(8.1)
$$\begin{aligned} P_{j,k}(Z \pm i\tilde{\mu})^{-1}P_{j,k} = &[P_{j,k}(-\Delta+1)] \\ &\times \left[(-\Delta+1)^{-1}(Z \pm i\tilde{\mu})^{-1}(X-\eta_j+i)^{-1}(Y-\eta_k+i)^{-1}\right] \\ &\times [(Y-\eta_k+i)(X-\eta_j+i)P_{j,k}] \end{aligned}$$

where $\eta_j \in \sigma_j$, where σ_j is the jth separated component of $\sigma(PXP)$, and $\eta_k \in \sigma_{j,k}$, where σ_k is the kth separated component of $\sigma(P_jYP_j)$. To prove $P_{j,k}(Z + i\tilde{\mu})^{-1}P_{j,k}$ is compact, we must prove that the first and third terms in (8.1) are bounded, while the second is compact. That the second term is compact and the first term is bounded follow from essentially the same arguments as given in Section 6. Unfortunately, it is unclear if the last term

$$(Y - \eta_k + i)(X - \eta_j + i)P_{j,k}$$

is bounded. The trick is to write, instead of (8.1),

$$\begin{aligned} P_{j,k}(Z \pm i\tilde{\mu})^{-1}P_{j,k} = & [P_{j,k}(-\Delta + 1)] \\ & \times \left[(-\Delta + 1)^{-1}(Z \pm i\tilde{\mu})^{-1}(X - \eta_j + i)^{-1/2}(Y - \eta_k + i)^{-1/2}\right] \\ & \times \left[(Y - \eta_k + i)^{1/2}(X - \eta_j + i)^{1/2}P_{j,k}\right]. \end{aligned} \tag{8.2}$$

That the first term of (8.2) is bounded is clear. That the second term of (8.2) is compact follows from an almost identical argument as given in Section 6. We now show that the third term of (8.2) is bounded. Note that if $f \notin \mathrm{range}(P_{j,k})$ then the operator acting on f is clearly bounded. So let $f \in \mathrm{range}(P_{j,k})$. Then using the fact that the geometric mean is bounded by the arithmetic mean, we have that

$$\begin{aligned} & \left\|(Y - \eta_k + i)^{1/2}(X - \eta_j + i)^{1/2}P_{j,k}f\right\|^2 \\ & \leq \frac{1}{2}\left(\|(Y - \eta_k + i)P_{j,k}f\|^2 + \|(X - \eta_j + i)P_{j,k}f\|^2\right). \end{aligned}$$

The first term is bounded since $P_{j,k}$ is the projection onto a bounded subset of the spectrum of P_jYP_j (by the same proof as that of Lemma 6.1). The second term is bounded since $P_{j,k} = P_jP_{j,k}$ and $(X + \eta_j + i)P_j$ is bounded since P_j is the projection onto a bounded subset of the spectrum of PXP (Lemma 6.1).

Proving exponential localization of eigenfunctions of $P_{j,k}ZP_{j,k}$. The generalization of the proof of Lemma 4.4 to three dimensions is straightforward once we prove operator norm bounds on $P_{j,k}$ analogous to the operator norm bounds proved in Appendix B on P_j. The only modification necessary is that in three dimensions the choice of the operator O (7.6) must be changed to

$$O = b\Pi^X_{[\eta_j - b, \eta_j + b]}\Pi^Y_{[\eta_{j,k} - b, \eta_{j,k} + b]}\Pi^Z_{[\eta_{j,k,l} - b, \eta_{j,k,l} + b]} + |X - \eta_j| + |Y - \eta_{j,k}|,$$

where $\eta_j \in \sigma_j$, the jth component of $\sigma(PXP)$, $\eta_{j,k} \in \sigma_{j,k}$, the kth component of $\sigma(P_jYP_j)$, and $\eta_{j,k,l}$ denotes the lth eigenvalue of $P_{j,k}ZP_{j,k}$.

9. Generalization to Discrete Systems

While the statement of Theorem 4.1 only concerns continuum Hamiltonians acting on $L^2(\mathbb{R}^2)$, our proof can easily be adapted to discrete Hamiltonians (satisfying appropriate assumptions) acting on $l^2(\mathbb{Z}^2)$. The reason is that we only use properties of the ambient Hilbert space and Hamiltonian in certain specific points of the proof. For example, we use the precise form of the Hamiltonian to prove properties of the Fermi projector P (Lemma A.1), but the necessary properties of P_j are then proved from the properties of P without reference to the ambient Hilbert space or Hamiltonian.

The two places in the proof of Theorem 4.1 where we make use of specific properties of the Hamiltonian H are:

(1) We use the form of H to prove important estimates on P, specifically Lemma A.1.
(2) We use the fact that Δ is H-bounded (see Lemma 6.2) in the proof that $P_j Y P_j$ has compact resolvent (Lemma 4.3).

Regarding (1), in Appendix D we prove the following Lemma, which states that the required estimates on P hold for discrete Hamiltonians H with a spectral gap and entries which exponentially decay away from the diagonal:

Lemma 9.1. *For each $\lambda = (\lambda_x, \lambda_y) \in \mathbb{Z}^2$, let $e_\lambda \in l^2(\mathbb{Z}^2)$ denote a joint eigenvector of the position operators X and Y with eigenvalue λ_x and λ_y respectively:*

$$Xe_\lambda = \lambda_x e_\lambda \qquad Ye_\lambda = \lambda_y e_\lambda.$$

Furthermore, let $\|\cdot\|_2$ denote the Euclidean 2-norm on $\mathbb{Z}^2$. That is, $\|\lambda\|_2 := \sqrt{\lambda_x^2 + \lambda_y^2}$.

Next, let H be a self-adjoint operator on $l^2(\mathbb{Z}^2)$ with a spectral gap containing the Fermi level and P be the Fermi projection. Suppose further that for any $\lambda, \mu \in \mathbb{Z}^2$:

$$|\langle e_\lambda, He_\mu\rangle| \leq Ce^{-\gamma'\|\lambda-\mu\|_2},$$

where γ' and C are finite, positive constants. Under these assumptions, there exist finite, positive constants (γ', K_1', K_2') depending only on H so that for all $\gamma \leq \gamma'$:

(1) $\|P_\gamma - P\| \leq K_1'\gamma$
(2) $\|[P_\gamma, X]\| \leq K_2'$ and $\|[P_\gamma, Y]\| \leq K_2'$.

Proof. Given in Appendix D. □

Regarding (2), proving $P_j Y P_j$ has compact resolvent is easier on $l^2(\mathbb{Z}^2)$ as compared to $L^2(\mathbb{R}^2)$. In Section 6, we showed that to show that $P_j Y P_j$ has compact resolvent it is enough to show that the operator $P_j(Y \pm i\tilde{\mu})^{-1}P_j$ is compact for some $\tilde{\mu}$ sufficiently large. For any $\eta_j \in \sigma_j$ we have that:

$$\begin{aligned} P_j(Y \pm i\tilde{\mu})^{-1}P_j &= P_j(Y \pm i\tilde{\mu})^{-1}(X - \eta_j + i)^{-1}(X - \eta_j + i)P_j \\ &= P_j\Big[(Y \pm i\tilde{\mu})^{-1}(X - \eta_j + i)^{-1}\Big]\Big[(X - \eta_j + i)P_j\Big] \end{aligned}$$

Since the operators P_j and $(X - \eta_j + i)P_j$ are bounded by our estimates on P_j we only need to show that $(Y \pm i\tilde{\mu})^{-1}(X - \eta_j + i)^{-1}$ is compact. This is easy to see since if we define

$$\chi_N(x, y) = \begin{cases} 1 & |x| \leq N \text{ and } |y| \leq N \\ 0 & \text{otherwise} \end{cases}$$

then we have that

$$\begin{aligned} &(Y \pm i\tilde{\mu})^{-1}(X - \eta_j + i)^{-1} \\ &\qquad = (Y \pm i\tilde{\mu})^{-1}(X - \eta_j + i)^{-1}\chi_N + (Y \pm i\tilde{\mu})^{-1}(X - \eta_j + i)^{-1}(1 - \chi_N). \end{aligned}$$

Since χ_N projects onto a finite dimensional space the first term is compact for all N. Since additionally for all N sufficiently large

$$\|(Y \pm i\tilde{\mu})^{-1}(X - \eta_j + i)^{-1}(1 - \chi_N)\| = O(N^{-1})$$

we can conclude that in the operator norm topology:

$$(Y \pm i\tilde{\mu})^{-1}(X - \eta_j + i)^{-1}\chi_N \to (Y \pm i\tilde{\mu})^{-1}(X - \eta_j + i)^{-1}.$$

Hence $(Y \pm i\tilde{\mu})^{-1}(X - \eta_j + i)^{-1}$ is compact.

10. Conclusions and Future Work

In this work we have presented a new construction of exponentially-decaying generalized Wannier functions which generalizes the Kivelson-Nenciu-Nenciu scheme [27, 37] to higher spatial dimensions. We prove that in two spatial dimensions our construction is possible whenever the spectrum of the operator PXP has spectral gaps. In higher dimensions, our construction is possible whenever the operators PXP, P_jYP_j etc. each have spectral gaps. Since our proofs are operator theoretic, they go through for both continuum and discrete models with little modification required. When we implement our construction numerically in finite systems, with both periodic and Dirichlet boundary conditions, we see clearly that it yields generalized Wannier functions which decay exponentially away from their maxima.

Understanding the links between the gap assumption on PXP in two dimensions and the Chern number, and between the gap assumptions we must make in higher dimensions and higher dimensional topological invariants, is the subject of ongoing work. We hope that if this link can be established, it will open a new perspective on topological insulators which is potentially much more general than the current paradigm. For example, as we have shown in this paper, the PXP gap assumption makes sense for finite systems just as well as for infinite systems.

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Appendix A. Bounds for P

Recall that when H satisfies the regularity and spectral gap assumptions (Assumptions 4.1 and 4.2), we can define P by the contour integral

$$P = \frac{1}{2\pi i}\int_{\mathcal{C}} (\lambda - H)^{-1}\,\mathrm{d}\lambda \tag{A.1}$$

where $\mathcal{C}$ has finite length and $\sup_{\lambda\in\mathbb{C}} \|(\lambda - H)^{-1}\| < \infty$.

Recall also that we define the exponential growth operator $B_{\gamma,(a,b)}$, which depends on parameters $\gamma \geq 0$ and $(a, b) \in \mathbb{R}^2$ by:

$$B_{\gamma,(a,b)} = \exp\left(\gamma\sqrt{1 + (X - a)^2 + (Y - b)^2}\right).$$

Given any linear operator A, we define

$$A_{\gamma,(a,b)} := B_{\gamma,(a,b)} A B^{-1}_{\gamma,(a,b)}.$$

When the particular point (a, b) is irrelevant, for example if a result holds uniformly in $(a, b) \in \mathbb{R}^2$, we will omit it.

The goal of this section is to prove the following lemma:

Lemma A.1. *Let H satisfy the regularity and spectral gap assumptions (Assumptions 4.1 and 4.2). Let P be the Fermi projector* (A.1)*. Then there exists finite, positive constants (γ', K_1', K_2') depending only on H so that for all $\gamma \leq \gamma'$:*
(1) $\|P_\gamma - P\| \leq K_1'\gamma$
(2) $\|[P_\gamma, X]\| \leq K_2'$ *and* $\|[P_\gamma, Y]\| \leq K_2'$.

Note that Lemmas 5.1 and 5.2 are special cases of Lemma A.1.

We will prove Lemma A.1 by the following route. We will first prove Lemma A.1 under Assumption 4.2 whenever H satisfies certain operator-norm bounds. We will then prove that H satisfies these operator-norm bounds under Assumption 4.1. The operator-norm bounds we require on H are the subject of the following Lemma:

Lemma A.2. *Let H satisfy the regularity and spectral gap assumptions (Assumptions 4.1 and 4.2). Let $\mathcal{C}$ denote the contour appearing in* (A.1)*. Then there exist finite, positive constants (γ^*, C, C'), depending on $\mathcal{C}$, such that for all $\gamma \leq \gamma^*$ and all $\lambda \in \mathcal{C}$:*
(1) $\|(H_\gamma - H)(\lambda - H)^{-1}\| \leq C\gamma$
(2) $\|[H_\gamma, X](\lambda - H)^{-1}\| \leq C'$ *and* $\|[H_\gamma, Y](\lambda - H)^{-1}\| \leq C'$.

We will prove Lemma A.2 in Appendix A.2.

A.1. Proof of Lemma A.1 assuming Lemma A.2 on H.

For both of the bounds in Lemma A.1 it will be useful to first show that for γ sufficiently small we have that $\sup_{\lambda\in\mathcal{C}} \|(\lambda - H_\gamma)^{-1}\| < \infty$.

A.1.1. *Bounding* $(\lambda - H_\gamma)^{-1}$.

The proof follows the proof of Proposition 7 of [15]. We start with the algebraic manipulations:

$$\begin{aligned}(\lambda - H_\gamma)^{-1} &= (\lambda - H + H - H_\gamma)^{-1}\\ &= (\lambda - H)^{-1}(I - (H_\gamma - H)(\lambda - H)^{-1})^{-1}.\end{aligned}$$

Since $\|(H_\gamma - H)(\lambda - H)^{-1}\| \leq C\gamma$ by Lemma A.2, choosing $\gamma \leq \min\{(2C)^{-1}, \gamma^*\}$ gives

$$\|(I - (H_\gamma - H)(\lambda - H)^{-1})^{-1}\| \leq 2.$$

Hence,

$$\begin{aligned}\|(\lambda - H_\gamma)^{-1}\| &\leq \|(\lambda - H)^{-1}\| \|(I - (H_\gamma - H)(\lambda - H)^{-1})^{-1}\| \\ &\leq 2\|(\lambda - H)^{-1}\|.\end{aligned}$$

Since $\sup_{\lambda \in \mathcal{C}} \|(\lambda - H)^{-1}\| < \infty$ by assumption, we conclude that $\sup_{\lambda \in \mathcal{C}} \|(\lambda - H_\gamma)^{-1}\|$ is bounded for all $\gamma \leq \min\{(2C)^{-1}, \gamma^*\}$.

A.1.2. *Proof of Lemma A.1(1).* Let's begin by writing $P_\gamma - P$ in terms of contour integrals:

$$\begin{aligned}P_\gamma - P &= B_\gamma \left(\frac{1}{2\pi i} \int_{\mathcal{C}} (\lambda - H)^{-1} \mathrm{d}\lambda \right) B_\gamma^{-1} - \frac{1}{2\pi i} \int_{\mathcal{C}} (\lambda - H)^{-1} \mathrm{d}\lambda \\ &= \frac{1}{2\pi i} \int_{\mathcal{C}} (\lambda - H_\gamma)^{-1} - (\lambda - H)^{-1} \mathrm{d}\lambda.\end{aligned}$$

By the second resolvent identity we have

$$\begin{aligned}&(\lambda - H_\gamma)^{-1} - (\lambda - H)^{-1} \\ &\qquad = (\lambda - H_\gamma)^{-1}(H_\gamma - H)(\lambda - H)^{-1}.\end{aligned}$$

Therefore,

$$\begin{aligned}\|P_\gamma - P\| &\leq \frac{\ell(\mathcal{C})}{2\pi} \|(\lambda - H_\gamma)^{-1} - (\lambda - H)^{-1}\| \\ &= \frac{\ell(\mathcal{C})}{2\pi} \|(\lambda - H_\gamma)^{-1}(H_\gamma - H)(\lambda - H)^{-1}\| \\ &\leq \frac{\ell(\mathcal{C})}{2\pi} \|(\lambda - H_\gamma)^{-1}\| \|(H_\gamma - H)(\lambda - H)^{-1}\| \\ &\leq \frac{\ell(\mathcal{C})}{2\pi} \Big(\|(\lambda - H_\gamma)^{-1}\| \Big) \Big(C\gamma \Big),\end{aligned}$$

where the last inequality holds by Lemma A.2. Therefore, setting

$$K_1' := \frac{\ell(\mathcal{C})}{\pi} \|(\lambda - H_\gamma)^{-1}\| C,$$

we see that for all γ sufficiently small, $\|P_\gamma - P\| \leq K_1' \gamma$.

A.1.3. *Proof of Lemma A.1(2).* In this section, we will only prove that $\|[P_\gamma, X]\|$ is bounded; the fact that $\|[P_\gamma, Y]\|$ is bounded follows by an essentially identical argument. By the definition of P we have:

$$\begin{aligned}[P_\gamma, X] &= \frac{1}{2\pi i} \int_{\mathcal{C}} [(\lambda - H_\gamma)^{-1}, X] \mathrm{d}\lambda \\ &= \frac{1}{2\pi i} \int_{\mathcal{C}} (\lambda - H_\gamma)^{-1} [H_\gamma, X] (\lambda - H_\gamma)^{-1} \mathrm{d}\lambda \\ &= \frac{1}{2\pi i} \int_{\mathcal{C}} (\lambda - H_\gamma)^{-1} [H_\gamma, X] (\lambda - H + H - H_\gamma)^{-1} \mathrm{d}\lambda \\ &= \frac{1}{2\pi i} \int_{\mathcal{C}} (\lambda - H_\gamma)^{-1} [H_\gamma, X] (\lambda - H)^{-1} (I - (H_\gamma - H)(\lambda - H)^{-1})^{-1} \mathrm{d}\lambda\end{aligned}$$

Therefore,

$$\|[P_\gamma, X]\| \leq \frac{\ell(\mathcal{C})}{2\pi} \|(\lambda - H_\gamma)^{-1}\| \|[H_\gamma, X](\lambda - H)^{-1}\| \|(I - (H_\gamma - H)(\lambda - H)^{-1})^{-1}\|.$$

From Lemma A.2 we know that:

$$\|(H_\gamma - H)(\lambda - H)^{-1}\| \le C\gamma \text{ and } \|[H_\gamma, X](\lambda - H)^{-1}\| \le C'$$

Therefore, if we choose $\gamma \le \min\{(2C)^{-1}, \gamma^*\}$ then we have that

$$\|[P_\gamma, X]\| \le \frac{\ell(\mathcal{C})}{2\pi} 2C' \|(\lambda - H_\gamma)^{-1}\|.$$

But in Section A.1.1 we showed that $\|(\lambda - H_\gamma)^{-1}\|$ is bounded whenever $\gamma \le \min\{(2C)^{-1}, \gamma^*\}$. Therefore for all $\gamma \le \min\{(2C)^{-1}, \gamma^*\}$, $\|[P_\gamma, X]\| < \infty$.

A.2. **Proof of Lemma A.2, under Assumption 4.1.** We will now prove Lemma A.2, under Assumption 4.1. Throughout this section, for any $\vec{x} \in \mathbb{R}^2$, we will use $\|\vec{x}\|_2$ to denote the Euclidean 2-norm.

Recall the form of H:

$$\begin{aligned} H &= (-i\nabla + A)^2 + V \\ &= -\Delta - 2iA\cdot\nabla - (i\mathrm{div}(A) + A\cdot A + V), \end{aligned}$$

where A and V are such that $A \in L^\infty(\mathbb{R}^2; \mathbb{R}^2)$, $\mathrm{div}(A) \in L^\infty(\mathbb{R}^2; \mathbb{R})$, and $V \in L^\infty(\mathbb{R}^2; \mathbb{R})$.

A.3. **Proof of Lemma A.2(1).** Since $\mathrm{div}(A)$, $A\cdot A$, and V are all scalar functions we have

$$H_\gamma = -B_\gamma \Delta B_\gamma^{-1} - 2iB_\gamma\big(A\cdot\nabla\big)B_\gamma^{-1} + \big(-i\mathrm{div}(A) + A\cdot A + V\big),$$

and hence

$$(H_\gamma - H) = -B_\gamma \Delta B_\gamma^{-1} - 2iB_\gamma\big(A\cdot\nabla\big)B_\gamma^{-1}.$$

Suppose that B_γ is centered at the point (x_0, y_0) and define the notation $\vec{x} := (x - x_0, y - y_0)$. A straightforward calculation shows that

$$B_\gamma \Delta B_\gamma^{-1} = \Delta - \frac{2\gamma}{\sqrt{1 + \|\vec{x}\|_2^2}}\vec{x}\cdot\nabla + \frac{\gamma^2\|\vec{x}\|_2^2}{1 + \|\vec{x}\|_2^2} + \frac{\gamma\|\vec{x}\|_2^2}{(1 + \|\vec{x}\|_2^2)^{3/2}} - \frac{2\gamma}{\sqrt{1 + \|\vec{x}\|_2^2}}. \tag{A.2}$$

$$B_\gamma\big(A\cdot\nabla\big)B_\gamma^{-1} = \frac{-\gamma(A\cdot\vec{x})}{\sqrt{1 + \|\vec{x}\|_2^2}} + A\cdot\nabla. \tag{A.3}$$

Since

$$\left\| \frac{\gamma^2\|\vec{x}\|_2^2}{1 + \|\vec{x}\|_2^2} + \frac{\gamma\|\vec{x}\|_2^2}{(1 + \|\vec{x}\|_2^2)^{3/2}} - \frac{2\gamma}{\sqrt{1 + \|\vec{x}\|_2^2}} \right\|_{L^\infty} \le \gamma^2 + 3\gamma,$$

we have that

$$\begin{aligned} \|(B_\gamma H B_\gamma^{-1} - H)(\lambda - H)^{-1}\| &\le 2\gamma \left\| \frac{\vec{x}\cdot\nabla(\lambda - H)^{-1}}{\sqrt{1 + \|\vec{x}\|_2^2}} + i\frac{(A\cdot\vec{x})(\lambda - H)^{-1}}{\sqrt{1 + \|\vec{x}\|_2^2}} \right\| + (\gamma^2 + 3\gamma)\|(\lambda - H)^{-1}\| \\ &= 2\gamma \left\| \frac{\vec{x}\cdot(-i\nabla + A)(\lambda - H)^{-1}}{\sqrt{1 + \|\vec{x}\|_2^2}} \right\| + O(\gamma). \end{aligned} \tag{A.4}$$

Since for any $\vec{x} \in \mathbb{R}^2$ we have

$$\left\| \frac{\vec{x}}{\sqrt{1 + \|\vec{x}\|^2}} \right\|_2 \le 1,$$

we are done if we can show that $\vec{v}\cdot(-i\nabla+A)$ is H-bounded for any $\vec{v}\in\mathbb{R}^2$ with $\|\vec{v}\|_2\leq 1$. Specifically, we must prove

Lemma A.3. *Let $\vec{v}\in\mathbb{R}^2$, with $\|\vec{v}\|_2\leq 1$. Then there exist constants $a,b>0$ such that for any $\psi\in\mathcal{D}(H)$,*

$$\|\vec{v}\cdot(-i\nabla+A)\psi\|\leq a\|H\psi\|+b\|\psi\|.$$

Proof. The proof is a straightforward consequence of Lemma 6.2. First, note that using boundedness of $\vec{v}$ and A,

$$\|\vec{v}\cdot(-i\nabla+A)\psi\|\leq\|\vec{v}\cdot(-i\nabla)\psi\|+2\|\vec{v}\|_\infty\|A\|_\infty\|\psi\|. \tag{A.5}$$

On the other hand,

$$\|\vec{v}\cdot(-i\nabla)\psi\|\leq 2\|\vec{v}\|_\infty\left(\|\Delta\psi\|+\|\psi\|\right). \tag{A.6}$$

Substituting (A.6) into (A.5) gives

$$\|\vec{v}\cdot(-i\nabla+A)\psi\|\leq 2\|\vec{v}\|_\infty\|\Delta\psi\|+\left(2\|\vec{v}\|_\infty+2\|\vec{v}\|_\infty\|A\|_\infty\right)\|\psi\|. \tag{A.7}$$

The result now follows upon combining (A.7) with Lemma 6.2. □

A.4. **Proof of Lemma A.2(2).** We will begin by showing $\|[H_\gamma,X](\lambda-H)^{-1}\|$ is bounded; the corresponding bound for Y follows by analogous steps. We calculate

$$\begin{aligned}[H_\gamma,X]&=[-B_\gamma\Delta B_\gamma^{-1}+2iB_\gamma\big(A\cdot\nabla\big)B_\gamma^{-1}+\big(i\mathrm{div}(A)+A\cdot A+V\big),X]\\&=[-B_\gamma\Delta B_\gamma^{-1}+2iB_\gamma\big(A\cdot\nabla\big)B_\gamma^{-1},X].\end{aligned}$$

The calculations from Equations (A.2) and (A.3) therefore give us

$$\begin{aligned}&[H_\gamma,X]\\&\quad=\left[-\Delta+\frac{2\gamma}{\sqrt{1+\|\vec{x}\|_2^2}}\vec{x}\cdot\nabla+2iA\cdot\nabla,X\right]\\&\quad=-[\Delta,X]+2i[A\cdot\nabla,X]+2\gamma\left[\frac{\vec{x}}{\sqrt{1+\|\vec{x}\|_2^2}}\cdot\nabla,X\right]\\&\quad=-2\frac{\partial}{\partial x}+2iA_1+\frac{2\gamma(x-x_0)}{\sqrt{1+\|\vec{x}\|_2^2}}\\&\quad=2ie_1\cdot(i\nabla+A)+\frac{2\gamma(x-x_0)}{\sqrt{1+\|\vec{x}\|_2^2}}.\end{aligned}$$

The bound $\|[H_\gamma,X](\lambda-H)^{-1}\|\leq C'$ now follows immediately from Lemma A.3.

Appendix B. Bounds for P_j

Recall that when PXP satisfies the uniform gap assumption (Assumption 4.3), we can define band projectors

$$P_j:=\left(\frac{1}{2\pi i}\int_{\mathcal{C}_j}(\lambda-PXP)^{-1}\,\mathrm{d}\lambda\right)P \tag{B.1}$$

for each $j\in\mathcal{J}$, which project onto the spectral subspace corresponding to each separated component σ_j of the spectrum of PXP. Here $\mathcal{C}_j$ denotes a contour enclosing σ_j in the complex plane.

The goal of this section is to prove the following lemma:

Lemma B.1. *Let H satisfy the regularity and spectral gap assumptions (Assumptions 4.1 and 4.2), and PXP satisfy the uniform gap assumption (Assumption 4.3). Let P_j, $j \in I$ denote the band projectors defined by Definition 4.1 onto σ_j. Then there exist finite, positive constants $(\gamma'', K_1'', K_2'', K_3'')$, independent of j, such that for all $\gamma \leq \gamma''$*

(1) $\|P_{j,\gamma} - P_j\| \leq K_1''\gamma$
(2) $\|[P_{j,\gamma}, X]\| \leq K_2''$ *and* $\|[P_{j,\gamma}, Y]\| \leq K_2''$
(3) *For all* $\eta \in \sigma_j$*:*

$$\|(X - \eta)P_{j,\gamma}\| \leq K_3'' \text{ and } \|P_{j,\gamma}(X - \eta)\| \leq K_3''.$$

Note that Lemma B.1 is a re-statement of Lemma 7.1, and contains Lemma 6.1 as a special case.

The starting point of our proof will be Lemma A.1, which established operator-norm bounds on the Fermi projector P. In fact, the proof of Lemma B.1 makes no reference to the form of the original Hamiltonian, only requiring the following hypothesis on the projector P:

Assumption B.1. *Let P be an orthogonal projection and suppose that there exists finite, positive constants (γ', K_1', K_2') so that for all $\gamma \in [0, \gamma']$:*

(1) $\|P_\gamma - P\| \leq K_1'\gamma$
(2) $\|[P_\gamma, X]\| \leq K_2'$ *and* $\|[P_\gamma, Y]\| \leq K_2'$

Hence in this section we actually prove the more general result:

Lemma B.2. *Let P be an orthogonal projection satisfying Assumption B.1 with finite, positive constants (γ', K_1', K_2'). Now suppose that PXP has uniform spectral gaps in the sense of Assumption 4.3 with constants (d, D). If P_j is a band projection onto σ_j then there exists finite, positive constants $(\gamma'', K_1'', K_2'', K_3'')$, independent of j, such that for all $\gamma \leq \gamma''$*

(1) $\|P_{j,\gamma} - P_j\| \leq K_1''\gamma$
(2) $\|[P_{j,\gamma}, X]\| \leq K_2''$ *and* $\|[P_{j,\gamma}, Y]\| \leq K_2''$
(3) *For all* $\eta \in \sigma_j$*:*

$$\|(X - \eta)P_{j,\gamma}\| \leq K_3'' \text{ and } \|P_{j,\gamma}(X - \eta)\| \leq K_3''.$$

The constant γ'' only depends on (γ', K_1', K_2') and (d, D) and is independent of the system size.

The removal of H in Lemma B.2 makes this lemma a statement about orthogonal projectors in Hilbert spaces, not necessarily about differential operators. In particular, this allows us to generalize to discrete systems easily in Section 9.

B.1. **Proof of key estimate on $(\lambda - P_\gamma X P_\gamma)^{-1}$.** The most difficult step in the proof of Lemma B.2 is to prove the following proposition, which states that the operator $(\lambda - P_\gamma X P_\gamma)^{-1}$ is bounded uniformly for $\lambda \in \mathcal{C}_j$ for all $j \in \mathcal{J}$ by a constant which is independent of j.

Proposition B.1. *There constants $\gamma_0 > 0$ and $C > 0$ such that for all $\gamma \leq \gamma_0$*

$$\sup_{j \in \mathcal{J}} \sup_{\lambda \in \mathcal{C}_j} \|(\lambda - P_\gamma X P_\gamma)^{-1}\| \leq C. \tag{B.2}$$

To see why Proposition B.1 is relevant to Lemma B.2, recall the contour integral definition (B.1) of P_j and note that

$$\begin{aligned} P_{j,\gamma} &= B_\gamma\bigg(\frac{1}{2\pi i}\int_{\mathcal{C}_j}(\lambda - PXP)^{-1}\mathrm{d}\lambda\bigg)PB_\gamma^{-1} \\ &= B_\gamma\bigg(\frac{1}{2\pi i}\int_{\mathcal{C}_j}(\lambda - PXP)^{-1}\mathrm{d}\lambda\bigg)\Big(B_\gamma^{-1}B_\gamma\Big)PB_\gamma^{-1} \\ &= \bigg(\frac{1}{2\pi i}\int_{\mathcal{C}_j}B_\gamma(\lambda - PXP)^{-1}B_\gamma^{-1}\mathrm{d}\lambda\bigg)P_\gamma \\ &= \bigg(\frac{1}{2\pi i}\int_{\mathcal{C}_j}(\lambda - P_\gamma XP_\gamma)^{-1}\mathrm{d}\lambda\bigg)P_\gamma \end{aligned} \tag{B.3}$$

where in the last step we have used that $[B_\gamma, X] = 0$ and

$$B_\gamma PXPB_\gamma^{-1} = B_\gamma P\Big(B_\gamma^{-1}B_\gamma\Big)XPB_\gamma^{-1} = P_\gamma XP_\gamma.$$

Hence Proposition B.1 immediately implies, through (B.3), that for sufficiently small γ, there exists a constant $C > 0$ such that

$$\sup_{j\in\mathcal{J}}\|P_{j,\gamma}\| \le C.$$

With Proposition B.1 established, the other assertions of Lemma B.2 follow from relatively straightforward manipulations.

We will prove Proposition B.1 across the next few subsections, with the proof of one lemma postponed until Appendix C. We start by showing how a naïve approach at bounding (B.2) yields an estimate which is not uniform in j.

B.1.1. *Failure of a naïve approach to yield an estimate uniform in j.* We start with the formal manipulations

$$\begin{aligned} (\lambda - P_\gamma XP_\gamma)^{-1} &= (\lambda - PXP_\gamma + PXP_\gamma - P_\gamma XP_\gamma)^{-1} \\ &= (\lambda - PXP_\gamma - (P_\gamma - P)XP_\gamma)^{-1} \\ &= (\lambda - PXP_\gamma)^{-1}\left(I - (P_\gamma - P)XP_\gamma(\lambda - PXP_\gamma)^{-1}\right)^{-1}. \end{aligned} \tag{B.4}$$

Recall that $\|P_\gamma - P\| = O(\gamma)$ by Assumption B.1. It follows that if

$$(\lambda - PXP_\gamma)^{-1} \text{ and } XP_\gamma(\lambda - PXP_\gamma)^{-1}$$

can be bounded for all $\lambda \in \mathcal{C}_j$ independently of j we are done by taking γ sufficiently small. Unfortunately, a direct attempt to bound $(\lambda - PXP_\gamma)^{-1}$ fails in this respect. More formal manipulations yield:

$$\begin{aligned} (\lambda - PXP_\gamma)^{-1} &= (\lambda - PXP - PX(P_\gamma - P))^{-1} \\ &= (\lambda - PXP - PX(P_\gamma - P))^{-1}(\lambda - PXP)(\lambda - PXP)^{-1} \\ &= \left(I - (\lambda - PXP)^{-1}PX(P_\gamma - P)\right)^{-1}(\lambda - PXP)^{-1}. \end{aligned}$$

Again, since $\|P_\gamma - P\| = O(\gamma)$ by Assumption B.1, we obtain a bound on $(\lambda - PXP_\gamma)^{-1}$ if we can bound

$$(\lambda - PXP)^{-1}PX$$

for all $\lambda \in \mathcal{C}_j$ independently of j. Let

$$Q := I - P$$

denote the orthogonal projector onto the orthogonal complement of range P. Since $P+Q=I$,

$$\begin{aligned}\left\|(\lambda-PXP)^{-1}PX\right\| &= \left\|(\lambda-PXP)^{-1}PX(P+Q)\right\| \\ &= \left\|(\lambda-PXP)^{-1}(PXP-\lambda+\lambda+PXQ)\right\| \\ &\leq 1+|\lambda|\left\|(\lambda-PXP)^{-1}\right\|+\left\|(\lambda-PXP)^{-1}PXQ\right\|.\end{aligned} \tag{B.5}$$

It is easy to see that PXQ is bounded, since

$$PXQ = PX(I-P) = PX - PXP = -P[X,P],$$

which is bounded by Assumption B.1, and $(\lambda-PXP)^{-1}$ is bounded for all $\lambda\in\mathcal{C}_j$ independently of j by assumption. Substituting these observations into (B.5) yields a bound on $(\lambda-PXP)^{-1}PX$ with the form

$$\left\|(\lambda-PXP)^{-1}PX\right\| \leq C + C|\lambda|,$$

which can be bounded uniformly for $\lambda\in\mathcal{C}_j$ for any fixed value of $j\in\mathcal{J}$, but not independently of j.

Remark B.1. *The inserting of $PXP_\gamma - PXP_\gamma$ in Equation* (B.4) *is non-obvious. Initially, one might be tempted to insert $PXP-PXP$ instead. To make this step work, one would need to show that*

$$\|(P_\gamma XP_\gamma - PXP)(\lambda-PXP)^{-1}\| = O(\gamma).$$

Unfortunately, some numerical tests for finite systems suggest that:

$$\|(P_\gamma XP_\gamma - PXP)(\lambda-PXP)^{-1}\| \sim |\textit{system size}|\gamma.$$

Since it does not seem possible to control $\|(P_\gamma XP_\gamma - PXP)(\lambda-PXP)^{-1}\|$ in general, we have to resort to the more complicated argument we give above.

B.1.2. *Strategy for proving an estimate which is uniform in j.* We now explain how to improve on the j-dependent bound proved in Section B.1.1. First, for each $j\in\mathcal{J}$, let $\eta_j\in\sigma_j$ be arbitrary, and define a j-dependent "shift" of $\lambda - P_\gamma XP_\gamma$ by

$$\begin{aligned}&(\lambda_{\eta_j} - P_\gamma X_{\eta_j} P_\gamma)^{-1}, \\ &\lambda_{\eta_j} := \lambda-\eta_j, \quad X_{\eta_j} := X-\eta_j.\end{aligned} \tag{B.6}$$

We will proceed by the following steps:

(1) Using the argument sketched in Section B.1.1, we will bound each shifted operator (B.6) uniformly in λ over the contour $\lambda\in\mathcal{C}_j$, and uniformly over $\eta\in\sigma_j$, by a constant which is independent of $j\in\mathcal{J}$.
(2) We will deduce the bound (B.2) from the set of bounds proved in part (1).

Before embarking on the proof of estimate (B.2), we pause to introduce some notation and prove some elementary bounds which follow from Assumption B.1.

B.1.3. *Notation and Some Easy Estimates.* Recall that we defined $Q := I-P$ to be the orthogonal projection onto the orthogonal complement of P. We now make the following further definitions

$$\begin{aligned}Q_\gamma &:= B_\gamma Q B_\gamma = I - P_\gamma \\ E &:= P_\gamma - P.\end{aligned}$$

We now note some easy estimates which follow from Assumption B.1. We will prove the following proposition:

Proposition B.2. *For any P satisfying Assumption B.1 and for all γ sufficiently small:*

(1) $\|P_\gamma - P\| = \|Q - Q_\gamma\| = \|E\| \leq K'\gamma$
(2) $\|P_\gamma\| \leq 1 + K'\gamma$, *and* $\|Q_\gamma\| \leq 1 + K'\gamma$.

For any $\eta \in \mathbb{C}$, let $X_\eta = X - \eta$ and $\lambda_\eta = \lambda - \eta$. Then

(3) $\|P_\gamma X_\eta Q_\gamma\| < \infty$ *and* $\|Q_\gamma X_\eta P_\gamma\| < \infty$.

Proof. (1) An elementary manipulation implies that

$$Q - Q_\gamma = P_\gamma - P = E,$$

from which the claim immediately follows.
(2) The triangle inequality implies that

$$\begin{aligned} \|P_\gamma\| &\leq \|P_\gamma - P\| + \|P\| \\ &\leq 1 + K_1'\gamma. \end{aligned} \tag{B.7}$$

An identical calculation implies $\|Q_\gamma\| \leq 1 + K_1'\gamma$.
(3) Since P is a projection, we have that

$$P_\gamma^2 = B_\gamma P B_\gamma^{-1} B_\gamma P B_\gamma^{-1} = B_\gamma P^2 B_\gamma^{-1} = P_\gamma.$$

Therefore, P_γ is also a projection. An immediate consequence of the fact that P_γ is a projection is $P_\gamma Q_\gamma = P_\gamma - P_\gamma^2 = 0$. Since $P_\gamma Q_\gamma = 0$ we also have that

$$P_\gamma X_\eta Q_\gamma = P_\gamma (X - \eta) Q_\gamma = P_\gamma X Q_\gamma$$

and

$$P_\gamma X Q_\gamma = P_\gamma Q_\gamma X + P_\gamma [X, Q_\gamma] = -P_\gamma [X, P_\gamma],$$

where the last equality follows from $Q_\gamma = Q + P - P_\gamma = I - P_\gamma$. Therefore

$$\|P_\gamma X_\eta Q_\gamma\| = \|P_\gamma [X, P_\gamma]\| \leq \|P_\gamma\| \|[X, P_\gamma]\|.$$

Using equation (B.7) and Assumption B.1, $\|P_\gamma\| \|[X, P_\gamma]\| < \infty$, and hence $\|P_\gamma X_\eta Q_\gamma\| < \infty$. Similar calculations show that $\|Q_\gamma X_\eta P_\gamma\| < \infty$. □

B.1.4. *Estimating the j-dependent shifted operators* (B.6). We now move on to bounding each of the j-dependent shifted operators (B.6) uniformly in λ over the contour $\lambda \in \mathcal{C}_j$, and uniformly in $\eta \in \sigma_j$, by a constant independent of $j \in \mathcal{J}$.

For simplicity of notation, we will drop the subscript j from η_j in this section. Assuming for now that $(\lambda_\eta - P X_\eta P_\gamma)^{-1}$ is well defined, then an identical calculation to (B.4) gives

$$(\lambda_\eta - P_\gamma X_\eta P_\gamma)^{-1} = (\lambda_\eta - P X_\eta P_\gamma)^{-1} \Big(I - E X_\eta P_\gamma (\lambda_\eta - P X_\eta P_\gamma)^{-1} \Big)^{-1},$$

where $E := P_\gamma - P$. If we could show that

$$(\lambda_\eta - P X_\eta P_\gamma)^{-1} \text{ and } X_\eta P_\gamma (\lambda_\eta - P X_\eta P_\gamma)^{-1} \tag{B.8}$$

are both bounded, then since $\|E\| = O(\gamma)$ we can choose γ sufficiently small so that $(\lambda_\eta - P_\gamma X_\eta P_\gamma)^{-1}$ can be written as a product of bounded operators and we are done.

We will prove that $\|(\lambda_\eta - P X_\eta P_\gamma)^{-1}\|$ is bounded in Section B.1.5 we will then use that result to prove that $\|X_\eta P_\gamma (\lambda_\eta - P X_\eta P_\gamma)^{-1}\|$ is bounded in Section B.1.6.

B.1.5. *Bounding* $(\lambda_\eta - PX_\eta P_\gamma)^{-1}$. Essentially the same calculations as in (B.1.1) give the following

$$(\lambda_\eta - PX_\eta P_\gamma)^{-1} = \Big(I - (\lambda_\eta - PX_\eta P)^{-1} PX_\eta E\Big)^{-1} (\lambda_\eta - PX_\eta P)^{-1}.$$

Hence

$$\|(\lambda_\eta - PX_\eta P_\gamma)^{-1}\| \le \|\Big(I - (\lambda_\eta - PX_\eta P)^{-1} PX_\eta E\Big)^{-1}\| \|(\lambda_\eta - PX_\eta P)^{-1}\|. \tag{B.9}$$

To show that the first term is bounded it is enough to show that for γ sufficiently small

$$\|(\lambda_\eta - PX_\eta P)^{-1} PX_\eta E\| \le \frac{1}{2}.$$

Inserting $P + Q = I$, we have that

$$\begin{aligned}\|(\lambda_\eta - PX_\eta P)^{-1} PX_\eta\| &= \|(\lambda_\eta - PX_\eta P)^{-1} PX_\eta (P+Q)\| \\ &\le \|(\lambda_\eta - PX_\eta P)^{-1} PX_\eta P\| + \|(\lambda_\eta - PX_\eta P)^{-1}\| \|PX_\eta Q\| \\ &\le 1 + |\lambda_\eta| \|(\lambda_\eta - PX_\eta P)^{-1}\| + \|(\lambda_\eta - PX_\eta P)^{-1}\| \|PX_\eta Q\|.\end{aligned}$$

By the definition of the contour $\mathcal{C}_j$ (recall Assumption 4.3 and Definition 4.1), we know that both $|\lambda_\eta|$ and $\|(\lambda_\eta - PX_\eta P)^{-1}\|$ are both bounded by constants which are independent of $\eta \in \sigma_j$ and j. Since by Proposition B.2, we also know that $\|PX_\eta Q\|$ is bounded, we can conclude that $\|(\lambda_\eta - PX_\eta P)^{-1} PX_\eta\|$ is bounded by a constant independent of λ and j.

Since $\|E\| \le K_1' \gamma$ if we pick γ so that

$$\gamma \le (2K_1')^{-1} \Big(1 + \Big(|\lambda_\eta| + \|PX_\eta Q\|\Big) \|(\lambda_\eta - PX_\eta P)^{-1}\|\Big)^{-1}$$

we see that

$$\|(\lambda_\eta - PX_\eta P)^{-1} PX_\eta E\| \le \frac{1}{2}$$

and therefore by Equation B.9, $\|(\lambda_\eta - PX_\eta P_\gamma)^{-1}\|$ is bounded uniformly for $\lambda \in \mathcal{C}_j$ by a constant independent of j.

B.1.6. *Bounding* $X_\eta P_\gamma(\lambda_\eta - PX_\eta P_\gamma)^{-1}$. The key trick proving this bound is noticing that $E = Q - Q_\gamma$ and therefore $Q = Q_\gamma + E$. Inserting a copy of $(P+Q)$ into the quantity we want to bound gives

$$\begin{aligned}X_\eta P_\gamma(\lambda_\eta - PX_\eta P_\gamma)^{-1} &= (P+Q) X_\eta P_\gamma (\lambda_\eta - PX_\eta P_\gamma)^{-1} \\ &= (P + Q_\gamma + E) X_\eta P_\gamma (\lambda_\eta - PX_\eta P_\gamma)^{-1}\end{aligned} \tag{B.10}$$

Now we can move the term containing E on the right hand side of Equation B.10 to get:

$$(I - E) X_\eta P_\gamma (\lambda_\eta - PX_\eta P_\gamma)^{-1} = (P + Q_\gamma) X_\eta P_\gamma (\lambda_\eta - PX_\eta P_\gamma)^{-1}$$

Since $\|E\| \le K_1'\gamma$ so long as we pick $\gamma \le (2K_1')^{-1}$ we can invert $(I - E)$ to get:

$$X_\eta P_\gamma (\lambda_\eta - PX_\eta P_\gamma)^{-1} = (I-E)^{-1} (P + Q_\gamma) X_\eta P_\gamma (\lambda_\eta - PX_\eta P_\gamma)^{-1}$$

Therefore,

$$\begin{aligned}&\|X_\eta P_\gamma(\lambda_\eta - PX_\eta P_\gamma)^{-1}\| \\ &\quad \le \|(I-E)^{-1}\| \Big(\|PX_\eta P_\gamma(\lambda_\eta - PX_\eta P_\gamma)^{-1}\| + \|Q_\gamma X_\eta P_\gamma (\lambda_\eta - PX_\eta P_\gamma)^{-1}\|\Big).\end{aligned}$$

Now to finish the proof we only need to show that both

$$PX_\eta P_\gamma(\lambda_\eta - PX_\eta P_\gamma)^{-1} \text{ and } Q_\gamma X_\eta P_\gamma(\lambda_\eta - PX_\eta P_\gamma)^{-1}$$

are bounded independent of λ and j. However, with the estimates we have now both of these bounds are fairly easy:

$$\|PX_\eta P_\gamma(\lambda_\eta - PX_\eta P_\gamma)^{-1}\| \leq 1 + |\lambda_\eta|\|(\lambda_\eta - PX_\eta P_\gamma)^{-1}\|$$
$$\|Q_\gamma X_\eta P_\gamma(\lambda_\eta - PX_\eta P_\gamma)^{-1}\| \leq \|Q_\gamma X_\eta P_\gamma\|\|(\lambda_\eta - PX_\eta P_\gamma)^{-1}\|.$$

By the proof in Section B.1.5, we know that $\|(\lambda_\eta - PX_\eta P_\gamma)^{-1}\|$ is bounded. Since $|\lambda_\eta|$ is bounded by the construction of $\mathcal{C}_j$ and from Proposition B.2 we know that $\|Q_\gamma X_\eta P_\gamma\|$ is also bounded, by the above logic we conclude that $X_\eta P_\gamma(\lambda_\eta - PX_\eta P_\gamma)^{-1}$ is bounded. Therefore, by the above logic, for all γ sufficiently small the operator $\|(\lambda_\eta - P_\gamma X_\eta P_\gamma)^{-1}\|$ is bounded uniformly for $\lambda \in \mathcal{C}_j$ by a constant which is independent of $j \in \mathcal{J}$. Since our arguments make no reference to any particular $\eta_j \in \sigma_j$, this estimate is also uniform in $\eta \in \sigma_j$.

B.1.7. *Deducing the key estimate* (B.2) *from the uniform bound proved on the shifted operators* (B.6). The key estimate (B.2) can be deduced from the uniform bound proved on the shifted operators (B.6) in the previous section via the following Lemma, whose proof we postpone until Appendix C.

Lemma B.3. *Suppose P satisfies Assumption B.1 with constants (γ', K_1', K_2') and suppose that PXP has uniform spectral gaps with decomposition $\{\sigma_j\}_{j\in\mathcal{J}}$ and corresponding contours $\{\mathcal{C}_j\}_{j\in\mathcal{J}}$. Then the following are equivalent for all $0 \leq \gamma < \gamma'$:*

(1) There exists a $C > 0$, independent of j, such that

$$\sup_{\lambda\in\mathcal{C}_j} \|(\lambda - P_\gamma XP_\gamma)^{-1}\| \leq C.$$

There exists a $C' > 0$, independent of j, such that for each $j \in \mathcal{J}$:

$$\sup_{\lambda\in\mathcal{C}_j} \sup_{\eta_j\in\sigma_j} \|(\lambda_{\eta_j} - P_\gamma X_{\eta_j} P_\gamma)^{-1}\| \leq C'$$

Furthermore, for any $0 \leq \gamma < \gamma'$ if $\|(\lambda - P_\gamma XP_\gamma)^{-1}\|$ is bounded we have for any $j \in \mathcal{J}$, $\lambda \in \mathcal{C}_j$, and $\eta_j \in \sigma_j$:

$$(\lambda - P_\gamma XP_\gamma)^{-1}P_\gamma = (\lambda_{\eta_j} - P_\gamma X_{\eta_j} P_\gamma)^{-1}P_\gamma.$$

Proof. Given in Appendix C. □

With Proposition B.1 proved, we can now proceed to prove the assertions of Lemma B.2.

B.2. **Proof of Lemma B.2(1).** Let $\eta \in \sigma_j$ be arbitrary. Writing $P_{j,\gamma}$ and P_j in terms of their contour integrals gives:

$$P_{j,\gamma} - P_j = \frac{1}{2\pi i}\int_{\mathcal{C}_j} (\lambda - P_\gamma XP_\gamma)^{-1}P_\gamma - (\lambda - PXP)^{-1}P\mathrm{d}\lambda.$$

Similar to the proof in Section B.1, we will want to work with the shifted versions of $(\lambda - P_\gamma XP_\gamma)^{-1}$ and $(\lambda - PXP)^{-1}$. Due to Lemma B.3 we know that:

$$(\lambda - P_\gamma XP_\gamma)^{-1}P_\gamma = (\lambda_\eta - P_\gamma X_\eta P_\gamma)^{-1}P_\gamma$$
$$(\lambda - PXP)^{-1}P = (\lambda_\eta - PX_\eta P)^{-1}P.$$

Therefore, we have that

$$\begin{aligned}
&P_{j,\gamma} - P_j \\
&= \frac{1}{2\pi i}\int_{\mathcal{C}_j} (\lambda - P_\gamma X P_\gamma)^{-1}P_\gamma - (\lambda - PXP)^{-1}P\mathrm{d}\lambda \\
&= \frac{1}{2\pi i}\int_{\mathcal{C}_j} (\lambda_\eta - P_\gamma X_\eta P_\gamma)^{-1}P_\gamma - (\lambda_\eta - PX_\eta P)^{-1}P\mathrm{d}\lambda \\
&= \frac{1}{2\pi i}\int_{\mathcal{C}_j} \Big((\lambda_\eta - P_\gamma X_\eta P_\gamma)^{-1} - (\lambda_\eta - PX_\eta P)^{-1}\Big)P + (\lambda_\eta - P_\gamma X_\eta P_\gamma)^{-1}(P_\gamma - P)\mathrm{d}\lambda \\
&= \frac{1}{2\pi i}\int_{\mathcal{C}_j} \Big((\lambda_\eta - P_\gamma X_\eta P_\gamma)^{-1} - (\lambda_\eta - PX_\eta P)^{-1}\Big)P + (\lambda_\eta - P_\gamma X_\eta P_\gamma)^{-1}E\mathrm{d}\lambda.
\end{aligned}$$

Therefore

$$\begin{aligned}
&\|P_{j,\gamma} - P_j\| \\
&\qquad \leq \frac{\ell(\mathcal{C}_j)}{2\pi}\Big(\|(\lambda_\eta - P_\gamma X_\eta P_\gamma)^{-1} - (\lambda_\eta - PX_\eta P)^{-1}\|\|P\| + \|(\lambda_\eta - P_\gamma X_\eta P_\gamma)^{-1}\|\|E\|\Big).
\end{aligned}$$

The second term is $O(\gamma)$ since $\|(\lambda_\eta - P_\gamma X_\eta P_\gamma)^{-1}\|$ is bounded and $\|E\| = O(\gamma)$. Therefore to finish the proof we need only need to show the first term is $O(\gamma)$.

Using the second resolvent identity we get

$$\begin{aligned}
&(\lambda_\eta - P_\gamma X_\eta P_\gamma)^{-1} - (\lambda_\eta - PX_\eta P)^{-1} \\
&\quad = (\lambda_\eta - P_\gamma X_\eta P_\gamma)^{-1}\Big(P_\gamma X_\eta P_\gamma - PX_\eta P\Big)(\lambda_\eta - PX_\eta P)^{-1} \\
&\quad = (\lambda_\eta - P_\gamma X_\eta P_\gamma)^{-1}\Big(P_\gamma X_\eta P_\gamma - P_\gamma X_\eta P + P_\gamma X_\eta P - PX_\eta P\Big)(\lambda_\eta - PX_\eta P)^{-1} \\
&\quad = (\lambda_\eta - P_\gamma X_\eta P_\gamma)^{-1}\Big(P_\gamma X_\eta E + EX_\eta P\Big)(\lambda_\eta - PX_\eta P)^{-1} \\
&\quad = (\lambda_\eta - P_\gamma X_\eta P_\gamma)^{-1}P_\gamma X_\eta E(\lambda_\eta - PX_\eta P)^{-1} + (\lambda_\eta - P_\gamma X_\eta P_\gamma)^{-1}EX_\eta P(\lambda_\eta - PX_\eta P)^{-1}.
\end{aligned}$$

Therefore,

$$\begin{aligned}
&\|(\lambda_\eta - P_\gamma X_\eta P_\gamma)^{-1} - (\lambda_\eta - PX_\eta P)^{-1}\| \\
&\quad \leq \|(\lambda_\eta - P_\gamma X_\eta P_\gamma)^{-1}P_\gamma X_\eta\|\|E\|\|(\lambda_\eta - PX_\eta P)^{-1}\| \\
&\qquad + \|(\lambda_\eta - P_\gamma X_\eta P_\gamma)^{-1}\|\|E\|\|X_\eta P(\lambda_\eta - PX_\eta P)^{-1}\|
\end{aligned}$$

Since we already know that $\|(\lambda_\eta - PX_\eta P)^{-1}\|$ and $\|(\lambda_\eta - P_\gamma X_\eta P_\gamma)^{-1}\|$ are bounded and $\|E\| = O(\gamma)$ we only need to show that $\|(\lambda_\eta - P_\gamma X_\eta P_\gamma)^{-1}P_\gamma X_\eta\|$ and $\|X_\eta P(\lambda_\eta - PX_\eta P)^{-1}\|$ are bounded.

This is straightforward since

$$\begin{aligned}
&\|(\lambda_\eta - P_\gamma X_\eta P_\gamma)^{-1}P_\gamma X_\eta\| \\
&\quad = \|(\lambda_\eta - P_\gamma X_\eta P_\gamma)^{-1}P_\gamma X_\eta(P_\gamma + Q_\gamma)\| \\
&\quad \leq \|(\lambda_\eta - P_\gamma X_\eta P_\gamma)^{-1}P_\gamma X_\eta P_\gamma\| + \|(\lambda_\eta - P_\gamma X_\eta P_\gamma)^{-1}\|\|P_\gamma X_\eta Q_\gamma\| \\
&\quad \leq 1 + |\lambda_\eta|\|(\lambda_\eta - P_\gamma X_\eta P_\gamma)^{-1}\| + \|(\lambda_\eta - P_\gamma X_\eta P_\gamma)^{-1}\|\|P_\gamma X_\eta Q_\gamma\|,
\end{aligned}$$

which is bounded by our previous estimates.

Repeating similar steps for $X_\eta P(\lambda_\eta - PX_\eta P)^{-1}$ (where one inserts $P+Q$ instead of $P_\gamma + Q_\gamma$) shows that $X_\eta P(\lambda_\eta - PX_\eta P)^{-1}$ is bounded. Therefore, by the previous logic, we conclude that $\|P_{j,\gamma} - P_j\| = O(\gamma)$ as we wanted to show.

B.3. **Proof of Lemma B.2(2).** We will show that $\|[P_{j,\gamma}, Y]\|$ is bounded, that $\|[P_{j,\gamma}, X]\|$ is also bounded follows by similar calculations. By definition we have:

$$[P_{j,\gamma}, Y] = \frac{1}{2\pi i}\int_{\mathcal{C}_j} [(\lambda - P_\gamma X P_\gamma)^{-1} P_\gamma, Y] \mathrm{d}\lambda.$$

Hence

$$\|[P_{j,\gamma}, Y]\| \leq \frac{\ell(\mathcal{C}_j)}{2\pi} \sup_{\lambda \in \mathcal{C}_j} \|[(\lambda - P_\gamma X P_\gamma)^{-1} P_\gamma, Y]\|.$$

Noticing that

$$\begin{aligned}&(\lambda - P_\gamma X P_\gamma)^{-1} P_\gamma - P_\gamma (\lambda - P_\gamma X P_\gamma)^{-1}\\ &\quad = (\lambda - P_\gamma X P_\gamma)^{-1} \left[P_\gamma(\lambda - P_\gamma X P_\gamma) - (\lambda - P_\gamma X P_\gamma) P_\gamma\right] (\lambda - P_\gamma X P_\gamma)^{-1},\end{aligned}$$

and hence $[(\lambda - P_\gamma X P_\gamma)^{-1}, P_\gamma] = 0$, we have:

$$\begin{aligned}[(\lambda - P_\gamma X P_\gamma)^{-1} P_\gamma, Y] =& [(\lambda - P_\gamma X P_\gamma)^{-1} P_\gamma, Y]\\ =& (\lambda - P_\gamma X P_\gamma)^{-1} P_\gamma Y - Y P_\gamma (\lambda - P_\gamma X P_\gamma)^{-1}\\ =& (\lambda - P_\gamma X P_\gamma)^{-1} P_\gamma Y (P_\gamma + Q_\gamma) - (P_\gamma + Q_\gamma) Y P_\gamma (\lambda - P_\gamma X P_\gamma)^{-1}\\ =& \ [(\lambda - P_\gamma X P_\gamma)^{-1}, P_\gamma Y P_\gamma]\\ &+ (\lambda - P_\gamma X P_\gamma)^{-1} P_\gamma Y Q_\gamma - Q_\gamma Y P_\gamma (\lambda - P_\gamma X P_\gamma)^{-1}.\end{aligned} \tag{B.11}$$

The last two terms in Equation (B.11) are bounded since $\|Q_\gamma Y P_\gamma\|$ and $\|P_\gamma Y Q_\gamma\|$ are bounded (see Proposition B.2) and we showed that $\|(\lambda - P_\gamma X P_\gamma)^{-1}\|$ is bounded in Section B.1. Therefore, to show that $\|[P_{j,\gamma}, Y]\|$ is bounded it suffices to show that $\|[(\lambda - P_\gamma X P_\gamma)^{-1}, P_\gamma Y P_\gamma]\|$ is bounded.

An elementary commutator identity gives that

$$[(\lambda - P_\gamma X P_\gamma)^{-1}, P_\gamma Y P_\gamma] = (\lambda - P_\gamma X P_\gamma)^{-1} [P_\gamma Y P_\gamma, P_\gamma X P_\gamma] (\lambda - P_\gamma X P_\gamma)^{-1}.$$

Since P_γ is idempotent, an easy calculation shows that (proven below in Lemma B.4):

$$[P_\gamma Y P_\gamma, P_\gamma X P_\gamma] = -P_\gamma [[X, P_\gamma], [Y, P_\gamma]].$$

Hence

$$\begin{aligned}\|[(\lambda - P_\gamma X P_\gamma)^{-1}, P_\gamma Y P_\gamma]\| &\leq \|(\lambda - P_\gamma X P_\gamma)^{-1}\| \|[P_\gamma Y P_\gamma, P_\gamma X P_\gamma]\| \|(\lambda - P_\gamma X P_\gamma)^{-1}\|\\ &= \|P_\gamma [[X, P_\gamma], [Y, P_\gamma]]\| \|(\lambda - P_\gamma X P_\gamma)^{-1}\|^2\\ &\leq \|P_\gamma\| \|[[X, P_\gamma], [Y, P_\gamma]]\| \|(\lambda - P_\gamma X P_\gamma)^{-1}\|^2\\ &\leq 2\|P_\gamma\| \|[X, P_\gamma]\| \|[Y, P_\gamma]\| \|(\lambda - P_\gamma X P_\gamma)^{-1}\|^2\end{aligned}$$

and so from the bounds in Proposition B.2 and above logic, $\|[P_{j,\gamma}, Y]\|$ is bounded.

Lemma B.4. *Suppose that P is a idempotent (i.e. $P^2 = P$) and $[X, Y] = 0$ then*

$$[PXP, PYP] = P[[X, P], [Y, P]]$$

Proof. This is just a straightforward calculation

$$
\begin{aligned}
P[[X,P],[Y,P]] &= P[XP-PX, YP-PY] \\
&= P\Big((XP-PX)(YP-PY)-(YP-PY)(XP-PX)\Big) \\
&= P\Big(\Big(XPYP - XPY - PXYP + PXPY\Big) - \Big(YPXP - YPX - PYXP + PYPX\Big)\Big) \\
&= \Big(PXPYP - PXPY - PXYP + PXPY\Big) - \Big(PYPXP - PYPX - PYXP + PYPX\Big) \\
&= \Big(PXPYP - PXYP\Big) - \Big(PYPXP - PYXP\Big) \\
&= PXPYP - PYPXP \\
&= [PXP, PYP].
\end{aligned}
$$

□

B.4. **Proof of Lemma B.2(3).** For this section, let us fix some $\eta \in \sigma_j$, we will prove that $\|(X-\eta)P_{j,\gamma}\|$ is bounded. The fact that $\|P_{j,\gamma}(X-\eta)\|$ is bounded follows by essentially the same steps. Recalling we define $\lambda_\eta := \lambda - \eta$ and $X_\eta := X - \eta$ and using Lemma B.3 we have

$$
\begin{aligned}
(X-\eta)P_{j,\gamma} &= \frac{1}{2\pi i}\int_{\mathcal{C}_j}(X-\eta)(\lambda - P_\gamma X P_\gamma)^{-1}P_\gamma \mathrm{d}\lambda \\
&= \frac{1}{2\pi i}\int_{\mathcal{C}_j} X_\eta(\lambda_\eta - P_\gamma X_\eta P_\gamma)^{-1}P_\gamma \mathrm{d}\lambda \\
&= \frac{1}{2\pi i}\int_{\mathcal{C}_j} X_\eta P_\gamma(\lambda_\eta - P_\gamma X_\eta P_\gamma)^{-1} \mathrm{d}\lambda \\
&= \frac{1}{2\pi i}\int_{\mathcal{C}_j} (P_\gamma + Q_\gamma) X_\eta P_\gamma(\lambda_\eta - P_\gamma X_\eta P_\gamma)^{-1} \mathrm{d}\lambda.
\end{aligned}
$$

where we have used that P_γ commutes with $(\lambda_\eta - P_\gamma X_\eta P_\gamma)^{-1}$. Therefore,

$$
\|(X-\eta)P_{j,\gamma}\| \leq \frac{\ell(\mathcal{C}_j)}{2\pi}\Big(\|P_\gamma X_\eta P_\gamma(\lambda_\eta - P_\gamma X_\eta P_\gamma)^{-1}\| + \|Q_\gamma X_\eta P_\gamma\|\|(\lambda_\eta - P_\gamma X_\eta P_\gamma)^{-1}\|\Big).
$$

Note that

$$
\|P_\gamma X_\eta P_\gamma(\lambda_\eta - P_\gamma X_\eta P_\gamma)^{-1}\| \leq 1 + |\lambda_\eta|\|(\lambda_\eta - P_\gamma X_\eta P_\gamma)^{-1}\|.
$$

Therefore, since $\|(\lambda_\eta - P_\gamma X_\eta P_\gamma)^{-1}\|$ and $\|Q_\gamma X_\eta P_\gamma\|$ are both bounded, we can conclude that $\|(X-\eta)P_{j,\gamma}\|$ is bounded as we wanted to show.

Appendix C. Shifting Lemma

Let's recall the result we would like to prove:

Lemma C.1. *Suppose P satisfies Assumption B.1 with constants (γ', K_1', K_2') and suppose that PXP has uniform spectral gaps with decomposition $\{\sigma_j\}_{j\in\mathcal{J}}$ and corresponding contours $\{\mathcal{C}_j\}_{j\in\mathcal{J}}$. For arbitrary $\eta \in \mathbb{C}$, define $\lambda_\eta := \lambda - \eta$ and $X_\eta := X - \eta$. Then the following are equivalent for all $0 \le \gamma < \gamma'$:*

1. *There exists a $C > 0$, independent of j, such that*
$$\sup_{\lambda\in\mathcal{C}_j} \|(\lambda - P_\gamma X P_\gamma)^{-1}\| \le C$$
2. *There exists a $C' > 0$, independent of j, such that*
$$\sup_{\lambda\in\mathcal{C}_j} \sup_{\eta\in\sigma_j} \|(\lambda_\eta - P_\gamma X_\eta P_\gamma)^{-1}\| \le C'$$

Furthermore, for any $\gamma < \gamma'$ if $\|(\lambda - P_\gamma X P_\gamma)^{-1}\|$ is bounded we have for any $j \in \mathcal{J}$ and $\eta_j \in \mathcal{C}_j$:
$$(\lambda - P_\gamma X P_\gamma)^{-1} P_\gamma = (\lambda_{\eta_j} - P_\gamma X_{\eta_j} P_\gamma)^{-1} P_\gamma.$$

The basic steps to prove this lemma are the following:
$$\begin{aligned}(\lambda_\eta - P_\gamma X_\eta P_\gamma)^{-1} &= (\lambda - \eta - P_\gamma(X - \eta)P_\gamma)^{-1}\\ &= (\lambda - \eta - P_\gamma X P_\gamma + \eta P_\gamma)^{-1}\\ &= (\lambda - P_\gamma X P_\gamma - \eta Q_\gamma)^{-1}.\end{aligned} \tag{C.1}$$

Since $P_\gamma + Q_\gamma = I$, because of this calculation we know that
$$\begin{aligned}&\|(\lambda_\eta - P_\gamma X_\eta P_\gamma)^{-1}\| = \|(\lambda - P_\gamma X P_\gamma - \eta Q_\gamma)^{-1}\|\\ &\qquad \le \|(\lambda - P_\gamma X P_\gamma - \eta Q_\gamma)^{-1} P_\gamma\| + \|(\lambda - P_\gamma X P_\gamma - \eta Q_\gamma)^{-1} Q_\gamma\|.\end{aligned}$$

Since $P_\gamma Q_\gamma = Q_\gamma P_\gamma = 0$, we should expect that shifting by ηQ_γ should not change what happens on $\mathrm{range}\,(P_\gamma)$. Similarly, the action of $P_\gamma X P_\gamma$ should not change what happens on $\mathrm{range}\,(Q_\gamma)$. This observation leads us to expect that:
$$(\lambda - P_\gamma X P_\gamma - \eta Q_\gamma)^{-1} P_\gamma = (\lambda - P_\gamma X P_\gamma)^{-1} P_\gamma \tag{C.2}$$
$$(\lambda - P_\gamma X P_\gamma - \eta Q_\gamma)^{-1} Q_\gamma = (\lambda - \eta Q_\gamma)^{-1} Q_\gamma. \tag{C.3}$$

By similar reasoning:
$$(\lambda - \eta Q_\gamma)^{-1} Q_\gamma = (\lambda - \eta + \eta P_\gamma)^{-1} Q_\gamma = (\lambda - \eta)^{-1} Q_\gamma. \tag{C.4}$$

Assuming Equations (C.2), (C.3), (C.4) are true, we conclude that:
$$\|(\lambda_\eta - P_\gamma X_\eta P_\gamma)^{-1}\| \le \|(\lambda - P_\gamma X P_\gamma)^{-1}\|\|P_\gamma\| + |\lambda - \eta|^{-1}\|Q_\gamma\|. \tag{C.5}$$

Since P satisfies Assumption B.1 we know that $\|P_\gamma\|$ and $\|Q_\gamma\|$ are bounded. Because of the uniform spectral gaps assumption on PXP, since we have chosen $\eta \in \sigma_j$ and $\lambda \in \mathcal{C}_j$ we also know that $|\lambda - \eta|^{-1}$ is bounded by a constant independent of j and η. Therefore, Equation (C.5) shows that
$$\|(\lambda - P_\gamma X P_\gamma)^{-1}\| < \infty \Longrightarrow \|(\lambda_\eta - P_\gamma X_\eta P_\gamma)^{-1}\| < \infty.$$

We can prove the reverse implication by instead starting with the calculation
$$\begin{aligned}(\lambda - P_\gamma X P_\gamma)^{-1} &= (\lambda - \eta + \eta - P_\gamma(X - \eta + \eta)P_\gamma)^{-1}\\ &= (\lambda_\eta - P_\gamma X_\eta P_\gamma + \eta Q_\gamma)^{-1},\end{aligned}$$

and proceeding along similar steps.

What remains to finish the proof of Lemma B.3 is to prove that Equations (C.2), (C.3), (C.4) are all true. For this, we have the following technical lemma:

Lemma C.2. *Let $\tilde{P}, \tilde{Q}$ be any pair of bounded operators such that $\tilde{P}\tilde{Q} = \tilde{Q}\tilde{P} = 0$. Next, let A, B be possibly unbounded operators densely defined on a common domain $\mathcal{D}$. Suppose further that $\|[\tilde{P}, A]\|$ both $\|[\tilde{Q}, B]\|$ are bounded.*

If $\tilde{\lambda} \in \mathbb{C}$ is any scalar such that $\|(\tilde{\lambda} + \tilde{P}A\tilde{P})^{-1}\|$ is bounded then $\|(\tilde{\lambda} + \tilde{P}A\tilde{P} + \tilde{Q}B\tilde{Q})^{-1}\tilde{P}\|$ is also bounded and

$$(\tilde{\lambda} + \tilde{P}A\tilde{P})^{-1}\tilde{P} = (\tilde{\lambda} + \tilde{P}A\tilde{P} + \tilde{Q}B\tilde{Q})^{-1}\tilde{P}.$$

Note that applying Lemma C.2 three times proves that Equations (C.2), (C.3), (C.4) are all true.

The assumption that $\|[\tilde{P}, A]\|$ and $\|[\tilde{Q}, B]\|$ are bounded is purely a technical assumption which ensures that $\tilde{P}A\tilde{P}$ and $\tilde{Q}B\tilde{Q}$ are a well defined operators on $\mathcal{D}$. To see, why observe that

$$\tilde{P}A\tilde{P} = \tilde{P}[A, \tilde{P}] + \tilde{P}\tilde{P}A \text{ and } \tilde{Q}B\tilde{Q} = \tilde{Q}[B, \tilde{Q}] + \tilde{Q}\tilde{Q}B.$$

For our purposes, the only unbounded operator we will need to be careful with is the operator X. Since P satisfies Assumption B.1 we know that $\|[P_\gamma, X]\| = \|[Q_\gamma, X]\| < \infty$, therefore we may apply Lemma C.2 without worry.

Proof of Lemma C.2. First, note that $\tilde{\lambda} + \tilde{P}A\tilde{P} + \tilde{P}B\tilde{P}$ is injective on range(P) since for arbitrary non-zero $v \in \text{range}(P) \cap \mathcal{D}$,

$$\begin{aligned}\left\|\left(\tilde{\lambda} + \tilde{P}A\tilde{P} + \tilde{Q}B\tilde{Q}\right) v\right\| &= \left\|\left(\tilde{\lambda} + \tilde{P}A\tilde{P} + \tilde{Q}B\tilde{Q}\right) \tilde{P}v\right\| \\ &= \left\|\left(\tilde{\lambda} + \tilde{P}A\tilde{P}\right) v\right\| \geq \|(\lambda + PAP)^{-1}\|^{-1}\|v\|.\end{aligned}$$

Now observe that since $\tilde{P}\tilde{Q} = \tilde{Q}\tilde{P} = 0$

$$[(\tilde{\lambda} + \tilde{P}A\tilde{P} + \tilde{Q}B\tilde{Q}), (\tilde{\lambda} + \tilde{P}A\tilde{P})] = 0.$$

Since $(\tilde{\lambda} + \tilde{P}A\tilde{P})^{-1}$ is well defined, this implies that

$$[(\tilde{\lambda} + \tilde{P}A\tilde{P} + \tilde{Q}B\tilde{Q}), (\tilde{\lambda} + \tilde{P}A\tilde{P})^{-1}] = 0$$

Since $\tilde{Q}\tilde{P} = 0$ we also have that

$$\begin{aligned}&(\tilde{\lambda} + \tilde{P}A\tilde{P} + \tilde{Q}B\tilde{Q})\tilde{P} = (\tilde{\lambda} + \tilde{P}A\tilde{P})\tilde{P} \\ &\iff (\tilde{\lambda} + \tilde{P}A\tilde{P})^{-1}(\tilde{\lambda} + \tilde{P}A\tilde{P} + \tilde{Q}B\tilde{Q})\tilde{P} = \tilde{P} \\ &\iff (\tilde{\lambda} + \tilde{P}A\tilde{P} + \tilde{Q}B\tilde{Q})(\tilde{\lambda} + \tilde{P}A\tilde{P})^{-1}\tilde{P} = \tilde{P}.\end{aligned}$$

The final equality implies that range$(\tilde{P}) \subseteq$ range$(\tilde{\lambda} + \tilde{P}A\tilde{P} + \tilde{Q}B\tilde{Q})$. Since $(\tilde{\lambda} - \tilde{P}A\tilde{P})^{-1}$ is bounded we can conclude that $(\tilde{\lambda} + \tilde{P}A\tilde{P} + \tilde{Q}B\tilde{Q})$ is invertible on range$(\tilde{P})$ and so

$$(\tilde{\lambda} + \tilde{P}A\tilde{P})^{-1}\tilde{P} = (\tilde{\lambda} + \tilde{P}A\tilde{P} + \tilde{Q}B\tilde{Q})^{-1}\tilde{P}.$$

□

Appendix D. Discrete Hamiltonian Estimates

In this section, we will prove the following lemma:

Lemma D.1. *For each $\lambda = (\lambda_x, \lambda_y) \in \mathbb{Z}^2$, let $e_\lambda \in l^2(\mathbb{Z}^2)$ denote a joint eigenvector of the position operators X and Y with eigenvalue λ_x and λ_y respectively:*

$$Xe_\lambda = \lambda_x e_\lambda \qquad Ye_\lambda = \lambda_y e_\lambda.$$

Furthermore, let $\|\cdot\|_2$ denote the Euclidean 2-norm on $\mathbb{Z}^2$. That is, $\|\lambda\|_2 := \sqrt{\lambda_x^2 + \lambda_y^2}$.

Next, let H be a self-adjoint operator on $l^2(\mathbb{Z}^2)$ with a spectral gap containing the Fermi level and P is the Fermi projection. Suppose further that for any $\lambda, \mu \in \mathbb{Z}^2$:

$$|\langle e_\lambda, He_\mu \rangle| \leq Ce^{-\gamma' \|\lambda - \mu\|_2}.$$

where γ' and C are finite, positive constants. Under these assumptions, there exist finite, positive constants (γ', K_1', K_2') depending only on H so that for all $\gamma \leq \gamma'$:

(1) $\|P_\gamma - P\| \leq K_1' \gamma$
(2) $\|[P_\gamma, X]\| \leq K_2'$ and $\|[P_\gamma, Y]\| \leq K_2'$.

Recall the contour integral definition of P:

$$P = \int_{\mathcal{C}} (\lambda - H)^{-1} \mathrm{d}\lambda.$$

The proof of Lemma A.1 in Appendix A is easily generalized to prove Lemma D.1 as long as we can show that there exist finite, positive constants $(\gamma,' K_1', K_2')$ such that for all $\lambda \in \mathcal{C}$

(1) $\|(H_\gamma - H)(\lambda - H)^{-1}\| \leq K_1' \gamma$
(2) $\|[H, X](\lambda - H)^{-1}\| \leq K_2'$

It turns out that the assumptions on H imply that $\|H\|$ is bounded (to see this, replace $H_\gamma - H$ by H in the proof of assertion (1) of Lemma D.2 below). As a consequence of this, we do not need the resolvent $(\lambda - H)^{-1}$ to control $(H_\gamma - H)$ and $[H, X]$. Therefore, we can show the stronger result:

Lemma D.2. *Suppose that H is an operator on $l^2(\mathbb{Z}^2)$ such that there exist finite positive constants (C, γ') such that:*

$$|\langle e_\lambda, He_\mu \rangle| \leq Ce^{-\gamma' \|\lambda - \mu\|_2} \tag{D.1}$$

Then for all $\gamma \leq \frac{1}{2}\gamma'$, there exist constants (C', C'') depending only on C and γ' such that:

(1) $\|H_\gamma - H\| \leq C' \gamma$
(2) $\|[H, X]\| \leq C''$

This lemma implies the two estimates needed for the proof from Appendix A since $\sup_{\lambda \in \mathcal{C}} \|(\lambda - H)^{-1}\| < \infty$.

Proof. By definition of the spectral norm

$$\|H_\gamma - H\| = \sup_{\|v\|=1} \|(H_\gamma - H)v\|.$$

Since the collection $\{e_\lambda\}_{\lambda\in\mathbb{Z}^2}$ forms a complete orthogonal basis for $l^2(\mathbb{Z}^2)$ we have for any v:

$$\begin{aligned}
\|(H_\gamma - H)v\|^2 &= \sum_\lambda |\langle e_\lambda, (H_\gamma - H)v\rangle|^2 \\
&= \sum_\lambda |\langle e_\lambda, (H_\gamma - H)\left(\sum_\mu \alpha_\mu e_\mu\right)\rangle|^2 \\
&= \sum_\lambda |\sum_\mu \alpha_\mu \langle e_\lambda, (H_\gamma - H)e_\mu\rangle|^2 \\
&\leq \sum_\lambda \left(\sum_\mu |\alpha_\mu|\,|\langle e_\lambda, (H_\gamma - H)e_\mu\rangle|\right)^2. \qquad \text{(D.2)}
\end{aligned}$$

Now recall the definition of H_γ

$$H_\gamma = e^{\gamma\sqrt{1+(X-a)^2+(Y-b)^2}} H e^{-\gamma\sqrt{1+(X-a)^2+(Y-b)^2}}.$$

Therefore since e_λ is a simultaneous eigenvector of X and Y we have

$$\langle e_\lambda, H_\gamma e_\mu\rangle = e^{\gamma\sqrt{1+(\lambda_x-a)^2+(\lambda_y-b)^2}} e^{-\gamma\sqrt{1+(\mu_x-a)^2+(\mu_y-b)^2}} \langle e_\lambda, H_\gamma e_\mu\rangle.$$

Now notice that

$$\sqrt{1+(\lambda_x-a)^2+(\lambda_y-b)^2} = \left\| \begin{bmatrix} 1 \\ \lambda_x - a \\ \lambda_y - b \end{bmatrix} \right\|_2$$

and similarly for μ. Therefore, by the reverse triangle inequality we know that

$$\begin{aligned}
&\left|\sqrt{1+(\lambda_x-a)^2+(\lambda_y-b)^2} - \sqrt{1+(\mu_x-a)^2+(\mu_y-b)^2}\,\right| \\
&\leq \sqrt{(1-1)^2 + ((\lambda_x-a)-(\mu_x-a))^2 + ((\lambda_y-b)-(\mu_y-b))^2} \\
&= \|\lambda-\mu\|_2.
\end{aligned}$$

Therefore,

$$|\langle e_\lambda, H_\gamma e_\mu\rangle| \leq e^{\gamma\|\lambda-\mu\|_2} |\langle e_\lambda, He_\mu\rangle|.$$

Returning to Equation (D.2), this calculation shows that

$$\begin{aligned}
\|(H_\gamma - H)v\|^2 &\leq \sum_\lambda \left(\sum_\mu |\alpha_\mu|\,|\langle e_\lambda, (H_\gamma - H)e_\mu\rangle|\right)^2 \\
&\leq \sum_\lambda \left(\sum_\mu |\alpha_\mu|\, \left(e^{\gamma\|\lambda-\mu\|_2} - 1\right) |\langle e_\lambda, He_\mu\rangle|\right)^2.
\end{aligned}$$

The mean value theorem combined with the fact that e^x is strictly convex gives us that

$$\begin{aligned}
e^{\gamma\|\lambda-\mu\|_2} - 1 &= e^{\gamma\|\lambda-\mu\|_2} - e^0 \\
&\leq \gamma\|\lambda-\mu\|_2 e^{\gamma\|\lambda-\mu\|_2}.
\end{aligned}$$

From our assumption on H (D.1) we have:

$$\|(H_\gamma - H)v\|^2 \leq \gamma^2 \sum_\lambda \left(\sum_\mu |\alpha_\mu|\,\|\lambda-\mu\|_2 e^{\gamma\|\lambda-\mu\|_2} |\langle e_\lambda, He_\mu\rangle|\right)^2 \qquad \text{(D.3)}$$

$$\leq C^2\gamma^2 \sum_\lambda \left(\sum_\mu |\alpha_\mu| \, \|\lambda - \mu\|_2 e^{-(\gamma'-\gamma)\|\lambda-\mu\|_2} \right)^2 .$$

Therefore, after all of these steps we have found that for all $v \in l^2(\mathbb{Z}^2)$:

$$\|(H_\gamma - H)v\| \leq C\gamma \left(\sum_\lambda \left(\sum_\mu |\alpha_\mu| \, \|\lambda - \mu\|_2 e^{-(\gamma'-\gamma)\|\lambda-\mu\|_2} \right)^2 \right)^{1/2} .$$

Now notice that the right hand side of this equation in simply the l^2-norm of a discrete convolution. Therefore, by Young's convolution inequality we have:

$$\|(H_\gamma - H)v\| \leq C\gamma \left(\sum_\mu |\alpha_\mu|^2 \right)^{1/2} \left(\sum_\mu \|\mu\|_2 e^{-(\gamma'-\gamma)\|\mu\|_2} \right).$$

When $\|v\| = 1$, we know that $\sum_\mu |\alpha_\mu|^2 = 1$ so

$$\sup_{\|v\|=1} \|(H_\gamma - H)v\| \leq C\gamma \left(\sum_\mu \|\mu\|_2 e^{-(\gamma'-\gamma)\|\mu\|_2} \right).$$

Since for any $\mu \in \mathbb{Z}^2$, we know that $\|\mu\|_2^2 \in \mathbb{Z}^2$, we can partition $\mathbb{Z}^2$ as follows:

$$\mathbb{Z}^2 = \bigcup_{n=0}^{\infty} \{\mu \in \mathbb{Z}^2 : \|\mu\|_2 = \sqrt{n}\}.$$

Therefore, we can rewrite the above sum as:

$$\sum_\mu \|\mu\|_2 e^{-(\gamma'-\gamma)\|\mu\|_2} = \sum_{n=0}^{\infty} \sum_{\{\mu\in\mathbb{Z}^2:\|\mu\|_2=\sqrt{n}\}} \sqrt{n} e^{-(\gamma'-\gamma)\sqrt{n}}.$$

Now the set $\{x \in \mathbb{R}^2 : \|x\|_2 = \sqrt{n}\}$ defines a circle of radius $\sqrt{n}$ and hence has circumference $2\pi\sqrt{n}$. Since the minimum spacing between points on the integer lattice is 1, we know that for any n:

$$\#\{\mu \in \mathbb{Z}^2 : \|\mu\|_2 = \sqrt{n}\} \leq 2\pi\sqrt{n}.$$

Therefore,

$$\begin{aligned} \sum_{n=0}^{\infty} \sum_{\{\mu\in\mathbb{Z}^2:\|\mu\|_2=\sqrt{n}\}} \sqrt{n} e^{-(\gamma'-\gamma)\sqrt{n}} &\leq \sum_{n=0}^{\infty} 2\pi n e^{-(\gamma'-\gamma)\sqrt{n}} \\ &= 2\pi \sum_{n=1}^{\infty} n e^{-(\gamma'-\gamma)\sqrt{n}} \\ &\leq 2\pi \int_0^\infty n e^{-(\gamma'-\gamma)\sqrt{n}} \mathrm{d}n \\ &= \frac{24\pi}{(\gamma'-\gamma)^4}. \end{aligned}$$

Combining this estimate with the previous steps, we have shown that

$$\|H_\gamma - H\| \leq \frac{24\pi C\gamma}{(\gamma'-\gamma)^4}$$

If $\gamma \leq \frac{1}{2}\gamma'$ then $(\gamma' - \gamma)^4 \geq 16\gamma'^4$ so

$$\|H_\gamma - H\| \leq \frac{3\pi C\gamma}{4\gamma'^4}$$

which finishes the first bound.

The proof that $\|[H_\gamma, X]\|$ is bounded follows by essentially the same calculation. Following the same steps as before for any $v \in l^2(\mathbb{Z}^2)$ (cf. Equation (D.2))

$$\|[H_\gamma, X]v\|^2 \leq \sum_\lambda \left(\sum_\mu |\alpha_\mu| \, |\langle e_\lambda, [H_\gamma, X] e_\mu \rangle| \right)^2 .$$

Now

$$\begin{aligned} |\langle e_\lambda, [H_\gamma, X] e_\mu \rangle| &\leq |\langle e_\lambda, (B_\gamma H B_\gamma^{-1} X - X B_\gamma H B_\gamma^{-1}) e_\mu \rangle| \\ &\leq |\mu_x - \lambda_x| e^{\gamma \sqrt{1 + (\lambda_x - a)^2 + (\lambda_y - b)^2}} e^{-\gamma \sqrt{1 + (\mu_x - a)^2 + (\mu_y - b)^2}} |\langle e_\lambda, H e_\mu \rangle| \\ &\leq \|\lambda - \mu\|_2 e^{\gamma \sqrt{1 + (\lambda_x - a)^2 + (\lambda_y - b)^2}} e^{-\gamma \sqrt{1 + (\mu_x - a)^2 + (\mu_y - b)^2}} |\langle e_\lambda, H e_\mu \rangle|. \end{aligned}$$

Following the same argument used previously:

$$e^{\gamma \sqrt{1 + (\lambda_x - a)^2 + (\lambda_y - b)^2}} e^{-\gamma \sqrt{1 + (\mu_x - a)^2 + (\mu_y - b)^2}} \leq e^{\gamma \|\lambda - \mu\|_2},$$

so

$$|\langle e_\lambda, [H_\gamma, X] e_\mu \rangle| \leq \|\lambda - \mu\|_2 e^{\gamma \|\lambda - \mu\|_2} |\langle e_\lambda, H e_\mu \rangle|.$$

Hence,

$$\|[H_\gamma, X]v\|^2 \leq \sum_\lambda \left(\sum_\mu |\alpha_\mu| \, \|\lambda - \mu\|_2 e^{\gamma \|\lambda - \mu\|_2} |\langle e_\lambda, H e_\mu \rangle| \right)^2 .$$

But the right hand side is the same quantity we controlled in the proof that $\|H_\gamma - H\| \leq C'\gamma$ (see Equation (D.3)). Therefore, we can use those calculations to immediately conclude that for all $\gamma \leq \frac{1}{2}\gamma'$:

$$\|[H_\gamma, X]\| \leq \frac{3C\pi}{4\gamma'^4}.$$

□