Cyclotomic and Elliptic Polylogarithms and Motivic Extensions of $\mathbb{Q}$ by $\mathbb{Q}(m)$

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Dissertation submitted in partial fulfillment of the
requirements for the degree of Doctor of Philosophy
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ABSTRACT

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Abstract

Deligne and Goncharov [21] constructed a neutral tannakian category of mixed Tate motives MTM_N unramified over $\mathbb{Z}[\boldsymbol{\mu}_N, 1/N]$. Brown [10] and Hain–Matsumoto [38] computed the first order action of its motivic Galois group on the unipotent fundamental group of a smoothing of the once punctured Tate elliptic curve when $N = 1$. In this thesis, we extend their work to $N > 1$ by computing the first order action of the motivic Galois group on the unipotent fundamental group of a smoothing of the Tate elliptic curve with a cyclic subgroup of order N removed.

One of our main tools is the elliptic KZB connection for the level N congruence subgroup of $\mathrm{SL}_2(\mathbb{Z})$. We adapt it to the arithmetic setting and prove that it underlies an admissible variation of mixed Hodge structure over the corresponding universal elliptic curve with its N-torsion removed (Theorem 3.5.2). At the singular fiber above the distinguished cusp $q = 0$, we show that the KZB connection degenerates to the cyclotomic KZ connection and that the variation degenerates to a mixed Hodge structure that contains the Lie algebra of the motivic fundamental group $\pi_1^{\mathrm{mot}}(\mathbb{P}^1 - \{0, \boldsymbol{\mu}_N, \infty\}, \vec{\mathsf{v}})$ studied by Deligne and Goncharov. This observation allows us to construct a higher level analogue of the Hain map [37]. We further prove this map is a morphism in MTM_N by describing the Galois action on the motivic periods of the unipotent fundamental group of the punctured nodal cubic (Theorem 4.4.4).

The inclusion of $\pi_1^{\mathrm{mot}}(\mathbb{P}^1 - \{0, \boldsymbol{\mu}_N, \infty\}, \vec{\mathsf{v}})$ into the limit mixed Hodge structure of the KZB variation allows us to relate the periods of the N-cyclotomic polylog variation of MHS to the periods of the level N elliptic polylogarithm variation of MHS. This enables us to give a formula for the first order action of the Galois group of MTM_N on the elliptic polylogarithm. This is most explicit when N is a prime power (Theorem 5.5.7).

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Chapter 1

Introduction

In this thesis, we study periods of mixed Tate motives that are unramified over the ring $\mathcal{O}_N := \mathbb{Z}[\boldsymbol{\mu}_N, 1/N]$, where N is a positive integer and $\boldsymbol{\mu}_N$ denotes the Nth roots of unity in $\mathbb{C}$. Our main result is a computation of the "first order action" of the motivic Galois group on the unipotent fundamental group of a smoothing of the Tate elliptic curve with the cyclic subgroup corresponding to $\boldsymbol{\mu}_N$ removed. Before describing our results in detail, we recall some basic facts about periods and motives.

1.1 Periods

A *period* is a complex number which can be expressed as an integral of an algebraic differential form on a variety X defined over $\mathbb{Q}$ against a relative cycle in $X(\mathbb{C})$ with boundary in an algebraic subvariety $Y \subset X$ also defined over $\mathbb{Q}$. The periods form a ring and include many frequently occurring constants such as

$$2\pi i = \int_{|z|=1} \frac{dz}{z} \qquad \text{and} \qquad \zeta(3) = \iiint_{0\leq x\leq y\leq z\leq 1} \frac{dx}{1-x}\frac{dy}{y}\frac{dz}{z}.$$

More abstractly, periods are matrix entries of comparison isomorphisms

$$c : H^k_{\mathrm{DR}}(X, Y) \otimes \mathbb{C} \to H^k(X(\mathbb{C}), Y(\mathbb{C})) \otimes \mathbb{C}$$

between algebraic de Rham and Betti cohomology. Grothendieck conjectured all polynomial relations between periods come from maps between algebraic varieties [32]. An equivalent conjecture of Kontsevich and Zagier claims any such relation can be expressed using only addition, algebraic change of variables, and Stokes' theorem [45, 1].

1.2 Mixed Tate motives

If the pair (X, Y) is defined over a ring of S-integers $\mathcal{O}_{k,S}$ of a number field k and has good reduction at each prime, then the various incarnations (Betti, k-de Rham, ℓ-adic, etc.) of $H^\bullet(X, Y)$ are realizations of a motive over $\mathcal{O}_{k,S}$. Another version of Grothendieck's conjecture is that motives are determined by their periods. Using the work of Borel [6], Beilinson [2], and Levine [43], Deligne and Goncharov [21] proved the existence of a $\mathbb{Q}$-tannakian category of *mixed Tate motives* $\mathsf{MTM}(\mathcal{O}_{k,S})$ over $\mathcal{O}_{k,S}$. One may view these motives as those arising from cohomology groups of genus 0 curves and their moduli spaces [11].

The category $\mathsf{MTM}(\mathcal{O}_{k,S})$ is the category of finite dimensional representations of an affine group scheme $\mathcal{G}^{\mathrm{DR}}_{k,S}$ over $\mathbb{Q}$ that is an extension

$$1 \to \mathcal{K}^{\mathrm{DR}}_{k,S} \to \mathcal{G}^{\mathrm{DR}}_{k,S} \to \mathbb{G}_m \to 1$$

of the multiplicative group $\mathbb{G}_m$ by a prounipotent group. The Lie algebra of $\mathcal{K}^{\mathrm{DR}}_{k,S}$ is canonically isomorphic to the completion of a graded Lie algebra $\mathfrak{k}_{k,S}$, and the category $\mathsf{MTM}(\mathcal{O}_{k,S})$ is equivalent to the category of graded representations of $\mathfrak{k}_{k,S}$.

In the case $\mathcal{O}_N := \mathbb{Z}[\boldsymbol{\mu}_N, 1/N]$, Deligne and Goncharov proved the coordinate ring of the unipotent path torsor with suitable tangential base points of $\mathbb{P}^1 - \{0, \boldsymbol{\mu}_N, \infty\}$ is an ind-object of $\mathsf{MTM}(\mathcal{O}_N)$ and that it is a rich source of periods. The periods of this object are spanned by $2\pi i$ and the *multiple polylogarithms*

$$\mathrm{Li}_{n_1,\dots,n_m}(z_1, \dots, z_m) := \sum_{0<k_1<k_2<\cdots<k_m} \frac{z_1^{k_1} z_2^{k_2} \cdots z_m^{k_m}}{k_1^{n_1} k_2^{n_2} \cdots k_m^{n_m}}$$

evaluated at Nth roots of unity. We refer to these values as *N-cyclotomic multiple zeta values (MZVs)*. The classical MZV $\zeta(n_1, \dots, n_m)$ is the value of Li_m at $(1, \dots, 1)$. The integer m is called the "depth" of an MZV. Cyclotomic MZVs are filtered, but not graded,[1] by depth. We prove the following (Theorem 4.4.4)

[1]For example, $\zeta(4) = 4\zeta(1,3) = \zeta(1,1,2) = \frac{4}{3}\zeta(2,2)$.

Theorem 1. *The Lie algebra of the unipotent fundamental group of the Tate curve with $\boldsymbol{\mu}_N$ removed and suitable choice of tangential base point is a pro-object of MTM_N. Its periods are $\mathbb{Q}(2\pi i)$-linear combinations of N-cyclomotic MZVs.*

1.3 The depth filtration on $\mathfrak{k}_N$

A question of interest is whether $2\pi i$ and the N-cyclotomic MZVs generate all periods of all objects of $\mathsf{MTM}(\mathcal{O}_N)$. A tool for studying this problem is the "depth" filtration $D^\bullet$ on the Lie algebra $\mathfrak{k}_N$ of the prounipotent radical of the de Rham fundamental group of $\mathsf{MTM}(\mathcal{O}_N)$. This is a decreasing filtration of the Lie algebra $\mathfrak{k}_N$ corresponding to the degree in the generators $\mathbf{e}_\zeta$ under the Galois representation

$$\mathfrak{k}_N \longrightarrow \operatorname{Der}\operatorname{Lie}\pi_1^{\mathrm{un}}(\mathbb{P}^1 - \{0, \boldsymbol{\mu}_N, \infty\}, \vec{\mathrm{v}}_1)^{\mathrm{DR}} \cong \operatorname{Der}\mathbf{L}(\mathbf{e}_0, \mathbf{e}_\zeta \mid \zeta \in \boldsymbol{\mu}_N)^\wedge,$$

where $\mathbf{L}(\mathbf{e}_0, \mathbf{e}_\zeta \mid \zeta \in \boldsymbol{\mu}_N)^\wedge$ is the completed free Lie algebra on $\{\mathbf{e}_0, \mathbf{e}_\zeta \mid \zeta \in \boldsymbol{\mu}_N\}$. The filtration $D^\bullet$ is dual to an increasing filtration of the periods of $\mathsf{MTM}(\mathcal{O}_N)$ compatible with the depth filtration of the $\mathbb{Q}$-algebra generated by the N-cyclotomic MZVs.

Although the Lie algebra of $\mathfrak{k}_N$ is free, its associated depth graded Lie algebra $\operatorname{Gr}_D^\bullet \mathfrak{k}_N$ is not free, in general. Deligne [20] has shown $\operatorname{Gr}_D^\bullet$ is free when $N = 2$, 3, 4, 6, and 8. These are expected to be exceptional. The $N = 1$ case has well-known relations in $\operatorname{Gr}_D^2 \mathfrak{k}_1$ associated to cusp forms of $\mathrm{SL}_2(\mathbb{Z})$ [38, 46, 10, 26]. Meanwhile, Goncharov [27, 28, 29] has proven the existence of relations in $\operatorname{Gr}_D^\bullet \mathfrak{k}_N$ when N is prime and at least 5.

This thesis will study the graded quotient $\operatorname{Gr}_D^1 \mathfrak{k}_N$ for all $N \geq 1$. Our main task is to compute the motivic Galois representation

$$\phi_{\mathrm{ell}} : \operatorname{Gr}_D^1 \mathfrak{k}_N \longrightarrow \operatorname{Der}\left[\operatorname{Lie}\pi_1^{\mathrm{un}}(E_{\partial/\partial q} - \boldsymbol{\mu}_N, \vec{\mathrm{v}}_1)/D^2\right],$$

where $E_{\partial/\partial q}$ is the first order smoothing the Tate elliptic curve in the direction $\partial/\partial q$. The Lie algebra $\operatorname{Lie}\pi_1^{\mathrm{un}}(E_{\partial/\partial q} - \boldsymbol{\mu}_N, \vec{\mathrm{v}}_1)$ is isomorphic to the completed free Lie algebra

$$\mathbf{L}(\mathbf{X}, \mathbf{Y}, \mathbf{t}_\zeta \mid \zeta \in \boldsymbol{\mu}_N)^\wedge / \Big([\mathbf{X}, \mathbf{Y}] = \sum_{\zeta \in \boldsymbol{\mu}_N} \mathbf{t}_\zeta\Big),$$

and its depth filtration is given by the degree in the letters $\mathbf{t}_\zeta$:

$$D^m = \{\text{Lie words whose total degree in the } \mathbf{t}_\zeta \text{ is } \geq m\}.$$

Since depth is central, there are derivations

$$\epsilon^{\mathrm{op}}_{m+1,\zeta} \equiv \mathrm{ad}(\mathbf{Y}^{m-1} \cdot (\mathbf{t}_\zeta + (-1)^{m+1}\mathbf{t}_{\overline{\zeta}})) \mod D^2$$

(full formula in §3.2.1). Deligne and Goncharov [21, Thm. 6.8] showed that $\mathrm{Gr}^1_D \mathfrak{k}_N$ is isomorphic to $H_1(\mathfrak{k}_N)$. The following is a weak version of Theorem 5.5.7.

Theorem 2. *The action of $\mathfrak{k}_N$ on $\mathrm{Lie}\,\pi_1^{\mathrm{un}}(E_{\partial/\partial q} - \boldsymbol{\mu}_N, \vec{\mathrm{v}}_1)/D^2$ factors through the abelianization map $\mathfrak{k}_N \to H_1(\mathfrak{k}_N)$. The quotient $\mathrm{Gr}^1_D \mathfrak{k}_N$ has canonical generators $\boldsymbol{\sigma}_{m,\zeta}$ indexed by integers $m \geq 2$ and primitive Nth roots of unity $\zeta \in \boldsymbol{\mu}_N$, and complex conjugation induces the relations $\boldsymbol{\sigma}_{m,\zeta} = (-1)^{m+1}\boldsymbol{\sigma}_{m,\overline{\zeta}}$. The action of $\boldsymbol{\sigma}_{m,\zeta} \in \mathrm{Gr}^1_D \mathfrak{k}_N$ on $\mathrm{Lie}\,\pi_1^{\mathrm{un}}(E_{\partial/\partial q} - \boldsymbol{\mu}_N, \vec{\mathrm{v}}_1)/D^2$ is*

$$\phi_{\mathrm{ell}}(\boldsymbol{\sigma}_{m,\zeta}) \equiv \epsilon^{\mathrm{op}}_{m+1,\zeta} + \sum_{\substack{\xi \in \boldsymbol{\mu}_N \\ \textit{not primitive}}} c_\xi \epsilon^{\mathrm{op}}_{m+1,\xi} \mod D^2$$

where $c_\xi \in \mathbb{Q}$.

The derivations $\epsilon^{\mathrm{op}}_{m+1,\zeta}$ are cyclotomic generalizations of derivations ϵ_m first defined by Tsunogai [49, §3–4] and appear in the universal elliptic KZB connection [15, 42, 37]. We compute the constants c_ξ when N is a prime power.

1.4 Outline

Chapter 1 provides a brief review of modular curves, variations of mixed Hodge structure, Eisenstein series, unipotent completion of discrete groups, and iterated integration. All of these are necessary to construct the KZB connection in level N and prove its relevant properties.

Chapter 2 includes a description of the universal KZB connection over the modular curve $Y_\Gamma = \Gamma \backslash\backslash \mathfrak{h}$ of any level N congruence subgroup $\Gamma \subset \mathrm{SL}_2(\mathbb{Z})$. We compute the residues

of the connection at the cusps of Y_Γ and prove that the connection underlies an admissible variation of MHS.

Chapter 3 discusses the category of mixed Tate motives over $\mathbb{Z}[\boldsymbol{\mu}_N, 1/N]$ including its objects, extensions, fiber functors, and tannakian fundamental group. We introduce "motivic" periods and describe Deligne and Goncharov's unipotent path torsor of $\mathbb{P}^1 - \{0, \boldsymbol{\mu}_N, \infty\}$ as an example. Finally, we construct a level N version of the so-called "Hain map" and describe the action of the de Rham fundamental group of $\mathsf{MTM}(\mathcal{O}_N)$ on the motivic periods of $\operatorname{Lie} \pi_1^{\mathrm{un}}(E_{\partial/\partial q} - \boldsymbol{\mu}_N, \vec{\mathrm{v}}_1)$.

Chapter 4 first defines the cyclotomic and elliptic polylogarithm variations of MHS. We compute their limit MHS over an appropriate choices of tangential base points in $\mathbb{P}^1 - \{0, \boldsymbol{\mu}_N, \infty\}$ and $E_{\partial/\partial q} - \boldsymbol{\mu}_N$. We then find a basis of $\operatorname{Ext}^1_{\mathsf{MTM}_N}(\mathbb{Q}, \mathbb{Q}(n))$ and describe the action of $H_1(\mathfrak{t}_N)$ on each element. Since the limit MHS of the polylog variations are pull-backs of direct sums of simple extensions, this induces an action on $\operatorname{Lie} \pi_1^{\mathrm{un}}(E_{\partial/\partial q} - \boldsymbol{\mu}_N, \vec{\mathrm{v}}_1)$.

The appendices include a proof of the invariance of the level N KZB connection and a general fact about parallel transport of linearized connections. We refer to the latter to argue the Hain map is motivic. Appendix C is an index of notation.

1.5 Notation and conventions

In Chapters 1 and 2, we denote by γ the matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z})$, and a, b, c, and d will refer to its entries.

Throughout, we will use the topologist's convention for composition of paths. If X is a topological space, $\alpha, \beta : [0,1] \to X$, and $\alpha(1) = \beta(0)$, then $\alpha\beta$ denotes the path by first proceeding along α and then along β.

The one-dimensional pure $\mathbb{Q}$-Hodge structure of type $(-n, -n)$ will be denoted by $\mathbb{Q}(n)$. Its Betti and de Rham realizations are $\mathbb{Q}\mathbf{e}^B$ and $\mathbb{Q}\mathbf{e}^{\mathrm{DR}}$. The Betti to de Rham comparison isomorphism takes $\mathbf{e}^B$ to $(2\pi i)^n \mathbf{e}^{\mathrm{DR}}$.

We will denote by $\mathbf{L}(V)$ the free Lie algebra on a finite dimensional F-vector space

V. Recall that the universal enveloping algebra of $\mathbf{L}(V)$ is the tensor algebra $T(V)$. Let $\mathbf{L}(V)^\wedge$ be the completion of $\mathbf{L}(V)$ with respect to the lower central series, and let $T(V)^\wedge$ be the completion of $T(V)$ with respect to powers of the augmentation ideal $I = \ker \psi$ where $\psi : T(V) \to F$ and $\psi(v) = 0$ for all $v \in V$.

Denote the free associate algebra generated by $\{\mathbf{e}_j \mid j \in J\}$ by

$$F\langle \mathbf{e}_j \mid j \in J \rangle.$$

Its I-adic completion is the non-commutative power series ring

$$F\langle\langle \mathbf{e}_j \mid j \in J \rangle\rangle.$$

If $\{\mathbf{e}_j \mid j \in J\}$ is a basis of V, then there are canonical isomorphisms

$$F\langle \mathbf{e}_j : j \in J \rangle \cong T(V) \quad \text{and} \quad F\langle\langle \mathbf{e}_j : j \in J \rangle\rangle \cong T(V)^\wedge.$$

We have similar notation for Lie algebras. The free Lie algebra over F generated by $\{\mathbf{e}_j \mid j \in J\}$ is denoted by

$$\mathbf{L}(\mathbf{e}_j \mid j \in J).$$

The completion with respect to the lower central series is denoted

$$\mathbf{L}(\mathbf{e}_j \mid j \in J)^\wedge.$$

If $\{\mathbf{e}_j \mid j \in J\}$ is a basis of V, then there are canonical isomorphisms

$$\mathbf{L}(\mathbf{e}_j \mid j \in J) \cong \mathbf{L}(V) \quad \text{and} \quad \mathbf{L}(\mathbf{e}_j \mid j \in J)^\wedge \cong \mathbf{L}(V)^\wedge.$$

The adjoint action of an element $x \in T(V)$ on $y \in \mathbf{L}(V)$ will be denoted $x \cdot y$. This notation also applies to completions. If $a_n \in F$ and $x \in V$, then

$$\left(\sum_{n=0}^{\infty} a_n x^n \right) \cdot y := \sum_{n=0}^{\infty} a_n \operatorname{ad}_x^n(y).$$

For instance,

$$e^x \cdot y = \sum_{n=0}^{\infty} \frac{\operatorname{ad}_x^n(y)}{n!}.$$

Also, when it is clear we are working in the derivation algebra $\operatorname{Der} \mathfrak{g}$ of a Lie algebra $\mathfrak{g}$ with trivial center, such as a free Lie algebra of rank > 1, we will view $\mathfrak{g}$ as a subalgebra of $\operatorname{Der} \mathfrak{g}$ via the adjoint action $\operatorname{ad} : \mathfrak{g} \to \operatorname{Der} \mathfrak{g}$.

Chapter 2

Background

The following sections include topics needed to construct the KZB connection and prove its relevant properties. They also set notation for the rest of the thesis.

2.1 Modular curves

2.1.1 Level 1

A *framed elliptic curve* is a genus 1 curve E with marked point $P \in E$ together with a basis $\{\mathbf{a}, \mathbf{b}\}$ of $H_1(E, \mathbb{Z})$ such that $\mathbf{a} \cdot \mathbf{b} = 1$. The moduli space of framed elliptic curves is the upper half plane $\mathfrak{h}$. The point $\tau \in \mathfrak{h}$ corresponds to the elliptic curve $E_\tau := \mathbb{C}/\Lambda_\tau$ where $\Lambda_\tau := \mathbb{Z} \oplus \tau\mathbb{Z}$. The frames $\mathbf{a}$ and $\mathbf{b}$ are determined by the images of 1 and τ, respectively, under the natural isomorphism $H_1(E_\tau, \mathbb{Z}) \cong \Lambda_\tau$. Conversely, given a genus 1 curve X with marked point P and framing $\{\mathbf{a}, \mathbf{b}\}$ of $H_1(X, \mathbb{Z})$, let $\omega \in \Omega^1(X)$ be the unique 1-form such that $\int_{\mathbf{a}} \omega = 1$. Set $\tau = \int_{\mathbf{b}} \omega$. Since $\mathbf{a} \cdot \mathbf{b} = 1$, we must have $\operatorname{Im} \tau > 0$. There is then an isomorphism of Riemann surfaces $f : X \to E_\tau$ given by $f : x \mapsto \int_P^x \omega$.

The group $\mathrm{SL}_2(\mathbb{Z})$ acts on framings via

$$\gamma : \begin{pmatrix} \mathbf{b} \\ \mathbf{a} \end{pmatrix} \longmapsto \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} \mathbf{b} \\ \mathbf{a} \end{pmatrix} \tag{2.1.1}$$

and on the upper half plane via

$$\gamma : \tau \longmapsto \frac{a\tau + b}{c\tau + d}. \tag{2.1.2}$$

These actions are compatible with the identification of framed elliptic curves and $\mathfrak{h}$ above.

$$\begin{array}{ccc} \tau & \longleftrightarrow & (E_\tau, 0; 1, \tau) \\ \gamma \downarrow & & \gamma \downarrow \\ \frac{a\tau+b}{c\tau+d} & \longleftrightarrow & (E_\tau, 0; c\tau+d, a\tau+b) \end{array}$$

The orbifold quotient $\mathrm{SL}_2(\mathbb{Z})\backslash\backslash\mathfrak{h}$ is the moduli space of elliptic curves $\mathcal{M}_{1,1}$.

2.1.2 Higher level

Fix a natural number N. Denote the N-torsion points of an elliptic curve E_τ by $E_\tau[N]$. There is a natural isomorphism

$$E_\tau[N] = (N^{-1}\Lambda_\tau)/\Lambda_\tau \cong \Lambda_\tau/N\Lambda_\tau \cong H_1(E_\tau, \mathbb{Z}/N\mathbb{Z}).$$

A *level N structure* on E_τ is an isomorphism

$$H_1(E_\tau, \mathbb{Z}/N\mathbb{Z}) \to (\mathbb{Z}/N\mathbb{Z})^2$$

where the intersection pairing on $H_1(E_\tau, \mathbb{Z}/N\mathbb{Z})$ corresponds to the standard symplectic inner product on $(\mathbb{Z}/N\mathbb{Z})^2$. Under the action of $\mathrm{SL}_2(\mathbb{Z})$, such a structure is stabilized by the kernel of the mod N map $\mathrm{SL}_2(\mathbb{Z}) \to \mathrm{SL}_2(\mathbb{Z}/N)$. We call this group

$$\Gamma(N) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \equiv I \bmod N \right\} \subset \mathrm{SL}_2(\mathbb{Z}).$$

The orbifold quotient $Y(N) := \Gamma(N)\backslash\backslash\mathfrak{h}$ is the moduli space of elliptic curves with level N structure. The moduli space of elliptic curves $\mathcal{M}_{1,1}$ is the same as $Y(1)$.

A *level N congruence subgroup* is a subgroup of $\mathrm{SL}_2(\mathbb{Z})$ that contains $\Gamma(N)$. The orbifold quotient $\Gamma\backslash\backslash\mathfrak{h}$ of the upper half plane $\mathfrak{h}$ by any congruence subgroup Γ is called a *modular curve*. In addition to $\Gamma(N)$ itself, notable examples of congruence subgroups are

$$\Gamma_1(N) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \equiv \begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix} \bmod N \right\} \subset \mathrm{SL}_2(\mathbb{Z})$$

and

$$\Gamma_0(N) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \equiv \begin{pmatrix} * & * \\ 0 & * \end{pmatrix} \bmod N \right\} \subset \mathrm{SL}_2(\mathbb{Z}).$$

As with $\Gamma(N)$, both of these groups are stabilizers of the $\mathrm{SL}_2(\mathbb{Z})$-action on the set of elliptic curves with additional structure.

Given an elliptic curve E and a distinguished point P of order N, one may choose a framing $\{\mathbf{a}, \mathbf{b}\}$ of $H_1(E, \mathbb{Z})$ such that $P = \frac{1}{N}\mathbf{a} \in H_1(E, \mathbb{Z}/N\mathbb{Z})$. One can easily verify the stabilizer of P under the action of $\mathrm{SL}_2(\mathbb{Z})$ (2.1.1) is the subgroup $\Gamma_1(N)$. The orbifold quotient $Y_1(N) := \Gamma_1(N)\backslash\backslash\mathfrak{h}$ is the moduli space of elliptic curves with a distinguished point of order N.

Similarly, we observe $\Gamma_0(N)$ is the stabilizer of the order N cyclic subgroup $\langle \frac{1}{N}\mathbf{a} \rangle \subset H_1(E, \mathbb{Z}/N\mathbb{Z})$. The orbifold quotient $Y_0(N) := \Gamma_0(N)\backslash\backslash\mathfrak{h}$ is the moduli space of elliptic curves with distinguished cyclic subgroup of order N. The subgroup $\Gamma_1(N)$ is normal in $\Gamma_0(N)$ with quotient $(\mathbb{Z}/N)^\times$. Thus, $Y_1(N)$ is a Galois covering of $Y_0(N)$ with Galois group $(\mathbb{Z}/N\mathbb{Z})^\times$ where $k \in (\mathbb{Z}/N\mathbb{Z})^\times$ sends the point (E, P) in $Y_1(N)$ to (E, kP).

2.1.3 Compactification

Each modular curve can be compactified by carefully gluing in a finite number of disks. Let $\Gamma \subset \mathrm{SL}_2(\mathbb{Z})$ be a congruence subgroup. The quotient $\Gamma\backslash\mathbb{P}^1(\mathbb{Q})$ is finite and its elements are called *cusps* of the modular curve $Y_\Gamma := \Gamma\backslash\backslash\mathfrak{h}$. For each cusp $P \in \mathbb{P}^1(\mathbb{Q})$, there exists $\alpha \in \mathrm{SL}_2(\mathbb{Z})$ such that $\alpha\tau = i\infty$. The *width* of the cusp P is the smallest positive integer w_P such that

$$\alpha^{-1} \begin{pmatrix} 1 & w_P \\ 0 & 1 \end{pmatrix} \alpha \in \Gamma.$$

The stabilizer of P is given by

$$\Gamma_P = \Gamma \cap \left\{ \pm\alpha^{-1} \begin{pmatrix} 1 & w_P\mathbb{Z} \\ 0 & 1 \end{pmatrix} \alpha \right\}.$$

Let $C_\Gamma = \Gamma \cap \{\pm I\}$. For each cusp $P \in \Gamma\backslash\mathbb{P}^1(\mathbb{Q})$, let $\mathbb{D}_P$ be an open disk of radius $e^{-2\pi/w_P}$. The compactification $X_\Gamma := \overline{Y_\Gamma}$ is obtained by gluing the quotient $C_\Gamma\backslash\backslash\mathbb{D}_P$, where group C_Γ acts trivially on $\mathbb{D}_P$, to Y_Γ at each cusp along the maps

$$C_\Gamma\backslash\backslash\mathbb{D}_P \longleftarrow \Gamma_P\backslash\backslash\{\tau \in \mathfrak{h} \mid \mathrm{Im}(\tau) > 1\} \longrightarrow \Gamma\backslash\backslash\mathfrak{h}\ .$$

The left hand map sends $\tau \in \mathfrak{h}$ to $e^{2\pi i\tau/w_P} \in \mathbb{D}_P$ and includes C_Γ into Γ_P. We denote the compactifications of $Y_0(N)$, $Y_1(N)$, and $Y(N)$ by $X_0(N)$, $X_1(N)$, and $X(N)$, respectively.

2.2 Universal elliptic curves

The group $\mathrm{SL}_2(\mathbb{Z})$ acts on $\mathbb{Z}^2$ on the right by

$$\gamma : \begin{pmatrix} m & n \end{pmatrix} \mapsto \begin{pmatrix} m & n \end{pmatrix} \gamma.$$

The corresponding semidirect product $\mathrm{SL}_2(\mathbb{Z}) \ltimes \mathbb{Z}^2$ acts on $\mathbb{C} \times \mathfrak{h}$ by

$$\left(\gamma, \begin{pmatrix} m & n \end{pmatrix}\right) : (z, \tau) \mapsto ((c\tau + d)^{-1}(z + m\tau + n), \gamma\tau).$$

If Γ is a congruence subgroup of $\mathrm{SL}_2(\mathbb{Z})$, then the orbifold quotient

$$\mathcal{E}_\Gamma := (\Gamma \ltimes \mathbb{Z}^2)\backslash\backslash(\mathbb{C} \times \mathfrak{h})$$

is a fiber bundle over the modular curve $Y_\Gamma := \Gamma\backslash\backslash\mathfrak{h}$ whose fiber over $\tau \in \mathfrak{h}$ is the elliptic curve $\mathbb{C}/\Lambda_\tau$. The surface $\mathcal{E}_\Gamma$ is called the *universal elliptic curve* associated to Γ.

2.2.1 Compactification

The universal elliptic curve $\mathcal{E}_\Gamma$ over Y_Γ has a natural compactification $\overline{\mathcal{E}}_\Gamma$ over X_Γ obtained by gluing in stable nodal cubics above each cusp. If P is a cusp of Y_Γ with width w_P, then the fiber of $\overline{\mathcal{E}}_\Gamma$ above P is a w_P-gon of $\mathbb{P}^1$'s obtained by taking $\mathbb{P}^1 \times (\mathbb{Z}/w_P\mathbb{Z})$ and identifying (∞, n) with $(0, n+1)$ for each $n \in \mathbb{Z}/w_P\mathbb{Z}$. A full description is given in [23]. Above a cusp of width 1 (such as $\tau = i\infty$ in $X_1(N)$), the singular fiber is the nodal cubic E_0 (see [36, §5.2]).

Example 2.2.1. If N is an odd prime, then the set of cusps of the modular curve $Y_1(N)$ has two $\Gamma_0(N)$-orbits, each consisting of $\varphi(N)/2$ points. The cusps in the orbit that lies over $\tau = i\infty$ have width 1 and the orbit of cusps over $\tau = 0$ have width N. Thus, the singular fibers of $\overline{\mathcal{E}}_{\Gamma_1(N)}$ are each E_0 above the first orbit and N-gons of $\mathbb{P}^1$ above the second orbit.

2.2.2 Torsion sections

The universal curve $\overline{\mathcal{E}}_{\Gamma(N)}$ has N^2 sections cut out by the N-torsion of each fiber. The set of these N-torsion sections is isomorphic to $(N^{-1}\mathbb{Z}/\mathbb{Z})^2$. The element $\alpha = (\alpha_\tau, \alpha_1) \in (N^{-1}\mathbb{Z}/\mathbb{Z})^2$ corresponds to the torsion section

$$\tilde{\alpha} : \tau \to (\alpha_1 + \alpha_\tau \tau). \tag{2.2.1}$$

This defines a group isomorphism $(\mathbb{Z}/N\mathbb{Z})^2 \to \mathcal{E}_{\Gamma(N)}[N]$. The natural right action of $\mathrm{SL}_2(\mathbb{Z})$ on $(\mathbb{Z}/N\mathbb{Z})^2$ induces a right action on the N-torsion sections. This is the inverse monodromy action of $\mathrm{SL}_2(\mathbb{Z})$.

For general level N congruence subgroup Γ, the N-torsion sections of $\mathcal{E}_\Gamma$ may be multi-valued. The singled-valued sections are those corresponding to vectors in $(N^{-1}\mathbb{Z}/\mathbb{Z})^2$ that are fixed by the action of Γ. For example, the singled-valued sections of $\mathcal{E}_{\Gamma_1(N)}$ are those corresponding to $(0, k/N)$ for $k = 0, \ldots, (N-1)$. We will let $\mathcal{E}'_\Gamma$ denote $\mathcal{E}_\Gamma$ with its single-valued N-torsion sections removed.

The closures of the single-valued sections of $\mathcal{E}_{\Gamma_1(N)} \to Y_1(N)$ intersect the singular fiber over $\tau = i\infty$ at the roots of unity $\boldsymbol{\mu}_N \subset \mathbb{G}_m$, and they intersect the N-gon over $\tau = 0$ at the identity in each copy of $\mathbb{G}_m$. In the case of $\Gamma(N)$, the singular fibers of $\overline{\mathcal{E}}_{\Gamma(N)}$ are all N-gons of $\mathbb{P}^1$'s. The N-torsion sections of $\mathcal{E}_{\Gamma(N)}$ intersect these fibers at $\boldsymbol{\mu}_N \in \mathbb{G}_m$ in each component.

2.2.3 Moduli with tangent vectors

Let $\mathcal{L}_\Gamma \to X_\Gamma$ be the normal bundle of the identity section of the universal elliptic curve $\mathcal{E}_\Gamma$. The bundle $\mathcal{L}$ is the moduli of elliptic curves (E, P) with a Γ-structure and distinguished tangent vector $\vec{\mathrm{v}} \in T_P E$ [36, §5.4]. Equivalently, the fiber of $\mathcal{L}$ over an elliptic curve (E, P) is the tangent space $T_P E$.

Let $Y_{\Gamma,\vec{1}}(N)$ denote the $\mathbb{C}^\times$-bundle $\mathcal{L}^\times_{\Gamma(N)}$. It is the moduli space of elliptic curves with Γ-structure and choice of *nonzero* tangent vector.

2.3 Admissible variations of mixed Hodge structure

We assume familiarity with mixed Hodge structures (MHS). Here we give the definition of admissible variations of MHS and a standard example. The reader can find relevant background in [48, 44].

2.3.1 Definitions

Suppose X is a smooth projective variety over $\mathbb{C}$ and D is a divisor with normal crossings in X. Let $Y = X - D$. Relevant examples are:

- where Y is a modular curve, X is its natural compactification, and D is the set of cusps;
- where Y is the universal elliptic curve $\mathcal{E}'_\Gamma$ over the modular curve Y_Γ with single-valued N-torsion removed, X is the compactification $\overline{\mathcal{E}}_\Gamma$ over X_Γ defined in §2.2.1, and D is the union of the set of single-valued N-torsion sections and singular fibers over the cusps of Y_Γ.

Let $\mathbb{V}$ be a $\mathbb{Q}$-local system of finite rank over Y with unipotent local monodromy at every smooth point of D. Let $\mathcal{V} = \mathbb{V} \otimes_{\mathbb{Q}} \mathcal{O}_X$ be the associated flat vector bundle. Denote Deligne's canonical extension of $\mathcal{V}$ to X by $\overline{\mathcal{V}}$. Then $\overline{\mathcal{V}}$ has natural connection

$$\nabla : \overline{\mathcal{V}} \to \overline{\mathcal{V}} \otimes \Omega^1_X(\log D)$$

with logarithmic singularities along D. Since the local monodromy operators are unipotent, the residues of ∇ at each smooth point of D are nilpotent.

Definition 2.3.1. A *variation of MHS* $\mathbb{V}$ over Y consists of a local system $\mathbb{V}_{\mathbb{Q}}$ over Y of finite dimensional rational vector spaces, together with

(i) a finite increasing filtration $W_\bullet$ of $\mathbb{V}_{\mathbb{Q}}$ by $\mathbb{Q}$-local systems

$$0 \subseteq W_a\mathbb{V} \subseteq \cdots \subseteq W_{r-1}\mathbb{V} \subseteq W_r\mathbb{V} \subseteq \cdots \subseteq W_b\mathbb{V} = \mathbb{V},$$

and

(ii) a finite decreasing filtration $F^\bullet$ of $\mathcal{V}$ by holomorphic subbundles.

These are required to statisfy:

(i) *Griffiths' transversality*:

$$\nabla(F^p\mathcal{V}) \subseteq F^{p-1}\mathcal{V} \otimes \Omega^1_Y = F^p(\mathcal{V} \otimes \Omega^1_Y).$$

(ii) The fiber V_y above any point $y \in Y$ is a MHS with weight and Hodge filtrations cut out by $W_\bullet$ and $F^\bullet$, respectively.

Definition 2.3.2. Suppose that X is a curve. A variation of MHS $\mathbb{V}$ over Y is *admissible* if the following additional conditions hold.

(i) The subbundles $F^p\mathcal{V}$ extend to holomorphic subbundles of the canonical extension $\overline{\mathcal{V}}$.

(ii) For $P \in D$, let $L_P = -\operatorname{Res}_P \nabla$ and V_P be the fiber of $\overline{\mathcal{V}}$ above P. There exists an increasing *relative weight filtration* $M_\bullet$ such that

(a) $L_P(M_rV_P) \subseteq M_{r-2}V_P$ and $L_P(W_mV_P) \subseteq W_mV_P$ for all m and r, and

(b) L_P^r induces an isomorphism

$$L_P^r : \operatorname{Gr}^M_{m+r} \operatorname{Gr}^W_m V_P \to \operatorname{Gr}^M_{m-r} \operatorname{Gr}^W_m V_P$$

for all m and r.

If X is a curve and $\mathbb{V}$ is admissible, the fibers V_P over $P \in D$ have canonical *limit MHS* for each choice of tangent vector $\vec{\mathrm{v}} \in T_P X$. We typically denote this MHS by $V_{P,\vec{\mathrm{v}}}$ or $V_{\vec{\mathrm{v}}}$ if the choice of point is clear. The weight and Hodge filtrations of $V_{P,\vec{\mathrm{v}}}$ are $M_\bullet$ and the restriction of $F^\bullet$ to V_P, respectively. Meanwhile, the $\mathbb{Q}$-structure of $V_{P,\vec{\mathrm{v}}}$ is determined by the elements

$$\lim_{t\to 0} t^{-L_P} v(t) \in V_P, \tag{2.3.1}$$

where t is the local holomorphic coordinate of X centered at P such that $\vec{\mathrm{v}} = \partial/\partial t$ and $v(t)$ is a local flat section of $\mathbb{V}_{\mathbb{Q}}$.

Remark 2.3.3. $W_m V_{P,\vec{\mathrm{v}}}$ is a sub-MHS of $V_{P,\vec{\mathrm{v}}}$ for all m.

Remark 2.3.4. When $\dim X > 1$, a VMHS $\mathbb{V}$ is defined to be admissible if its restriction to any curve is admissible [44].

2.3.2 The local system $\mathbb{H}$

Fix a congruence subgroup Γ. Let $\mathbb{H}$ over Y_Γ be the local system whose fiber over τ is $H_1(\mathbb{C}/\Lambda_\tau, \mathbb{Q})$. It underlies a variation of polarized Hodge structures of weight -1 where the polarization is the intersection pairing. We will show $\mathbb{H}$ is admissible and compute its limit MHS at the distinguished cusp $\tau = i\infty$ at the tangent vector $\partial/\partial q$.

Let w be the width of the cusp $\tau = i\infty$. The local system $\mathbb{H}$ pulls back to both $\mathbb{D}^*$ and $\mathfrak{h}$

$$\begin{array}{ccccc} \mathbb{H}_{\mathfrak{h}} & \longrightarrow & \mathbb{H}_{\mathbb{D}^*} & \longrightarrow & \mathbb{H} \\ \downarrow & & \downarrow & & \downarrow \\ \mathfrak{h} & \xrightarrow[q]{} & \mathbb{D}^* & \longrightarrow & Y_\Gamma, \end{array}$$

where $q(\tau) = e^{2\pi i \tau/w}$. The pullback $\mathbb{H}_{\mathfrak{h}}$ has trivial framing given by the basis $\{\mathbf{a}, \mathbf{b}\}$ of $H_1(\mathbb{C}/\Lambda_\tau, \mathbb{Z})$ corresponding to the basis $\{1, \tau\}$ of Λ_τ via the natural isomorphism $\Lambda_\tau \cong H_1(\mathbb{C}/\Lambda_\tau, \mathbb{Z})$.

Let $\omega_\tau = dz \in H^0(\mathbb{C}/\Lambda_\tau, \Omega^1) \cong H^1(\mathbb{C}/\Lambda_\tau, \mathbb{C})$ be the unique holomorphic differential on

$\mathbb{C}/\Lambda_\tau$ taking the value 1 on $\mathbf{a}$. Then

$$\omega_\tau = \tau \mathbf{a}^{\mathrm{PD}} - \mathbf{b}^{\mathrm{PD}},$$

where $\mathbf{a}^{\mathrm{PD}}$ and $\mathbf{b}^{\mathrm{PD}}$ are the Poincaré duals of $\mathbf{a}$ and $\mathbf{b}$, respectively. Define $\mathbf{X} = \tau\mathbf{a} - \mathbf{b}$ to be the Poincaré dual of ω_τ.

Monodromy about a simple positively-oriented loop around the puncture in $\mathbb{D}^*$ acts on the framing $\{\mathbf{a}, \mathbf{b}\}$ via the matrix $\begin{pmatrix} 1 & w \\ 0 & 1 \end{pmatrix} \in SL_2(\mathbb{Z})$. More precisely, by (2.1.2) and (2.1.1), this action maps $\tau \mapsto \tau + w$, $\mathbf{a} \mapsto \mathbf{a}$, and $\mathbf{b} \mapsto \mathbf{b} + w\mathbf{a}$. Thus, $\mathbf{a}$ and $\mathbf{X}$ trivialize the holomorphic vector bundle $\mathcal{H}_{\mathbb{D}^*} := \mathbb{H}_{\mathbb{D}_*} \otimes \mathcal{O}_{\mathbb{D}^*}$. The Hodge filtration on $\mathcal{H}_{\mathbb{D}^*}$ is $F^{-1} = \mathcal{H}_{\mathbb{D}^*}$, $F^0 = \mathcal{O}_{\mathbb{D}^*}\mathbf{X}$, and $F^1 = 0$. Since $\mathbf{a}$ and $\mathbf{X}$ trivialize $\mathcal{H}_{\mathbb{D}^*}$, we observe that $F^\bullet$ extends over the origin, which defines as extension $\mathcal{H}_{\mathbb{D}}$ of $\mathcal{H}_{\mathbb{D}^*}$ to $\mathbb{D}$.

The bundle $\mathcal{H}_{\mathbb{D}^*}$ has natural connection ∇ that vanishes on the flat sections $\mathbf{a}$ and $\mathbf{b}$. Thus, we have

$$\nabla \mathbf{a} = 0 \quad \text{and} \quad \nabla \mathbf{X} = \mathbf{a} d\tau = \frac{w}{2\pi i} \mathbf{a} \frac{dq}{q}.$$

We may then write the connection as

$$\nabla = d - L\frac{dq}{q}, \tag{2.3.2}$$

where L is the endomorphism

$$L = -\frac{w}{2\pi i} \mathbf{a} \frac{\partial}{\partial \mathbf{X}}.$$

Since $\mathrm{Res}_{q=0} \nabla = -L$ is nilpotent, the extension $\mathcal{H}_{\mathbb{D}}$ of $\mathcal{H}_{\mathbb{D}_*}$ constructed above is Deligne's canonical extension. The relative weight filtration $M_\bullet$ is given by $M_0 = \mathbb{C}\mathbf{X} \oplus \mathbb{C}\mathbf{a}$, $M_{-2} = \mathbb{C}\mathbf{a}$, and $M_{-3} = 0$.

Fix $\lambda \in \mathbb{C}^\times$ and set $\vec{\mathfrak{u}}$ to be the tangent vector $\lambda\partial/\partial q$ anchored at $q = 0$. Taking the limit (2.3.1) with flat sections $\mathbf{a}$ and $\mathbf{b}$ shows the variation $\mathbb{H}$ has a limit mixed Hodge structure $H_{\vec{\mathfrak{u}}}$ at $\vec{\mathfrak{u}}$ whose rational (or integral) structure is spanned by

$$\lim_{q \to 0} (q/\lambda)^{-L} \mathbf{a} = \mathbf{a} \quad \text{and} \quad \lim_{q \to 0} (q/\lambda)^{-L} \mathbf{b} = -\mathbf{X} - \frac{w \log \lambda}{2\pi i} \mathbf{a}.$$

Since we know $\mathbf{a} \in F^{-1}M_{-2}H_{\vec{\mathfrak{u}}}$ and $\mathbf{X} \in F^0M_0H_{\vec{\mathfrak{u}}}$, the MHS $H_{\vec{\mathfrak{u}}}$ is an extension of $\mathbb{Q}(0)$ by $\mathbb{Q}(1)$ with de Rham generators $\mathbf{Y} := \frac{1}{2\pi i}\mathbf{a}$ and $\mathbf{X}$. Note that $H_{\vec{\mathfrak{u}}}$ splits as $\mathbb{Q}(0) \oplus \mathbb{Q}(1)$ if and only if λ is a root of unity (and splits over $\mathbb{Z}$ if and only if $\lambda \in \boldsymbol{\mu}_w$). The projection of $\mathbf{X}$ and $\mathbf{Y}$ into $\mathcal{H}_{\mathbb{D}}$ are called the *de Rham framing* of $\mathcal{H}_{\mathbb{D}}$.[1]

The MHS $H_{\vec{\mathfrak{u}}}$ may be interpreted geometrically as $H_1(E_{\vec{\mathfrak{u}}})$ where $E_{\vec{\mathfrak{u}}}$ is the first order smoothing of the nodal cubic in the direction of $\vec{\mathfrak{u}}$ (see [37, Appendix C] for details). Since there is a retraction $E_{\vec{\mathfrak{u}}} \to E_0$ and inclusion $\mathbb{G}_m \to E_{\vec{\mathfrak{u}}}$ whose composition is the inclusion $\mathbb{G}_m \to E_0$, there is an exact sequence of MHS

$$0 \longrightarrow H_1(\mathbb{G}_m) \longrightarrow H_1(E_{\vec{\mathfrak{u}}}) \longrightarrow H_1(E_0) \longrightarrow 0.$$

As Hodge structures, $H_1(\mathbb{G}_m) \cong \mathbb{Q}(1)$ and $H_1(E_0) \cong \mathbb{Q}(0)$. The MHS $H_1(E_{\vec{\mathfrak{u}}})$ is the extension of $\mathbb{Q}(0)$ by $\mathbb{Q}(1)$ with period $w \log \lambda$.

The local system $\mathbb{H}$ may also be pulled back to $\mathcal{E}_\Gamma$, where the framing $\{\mathbf{X}, \mathbf{Y}\}$ is constant over each fiber. However, such a framing is not canonical as there will be an $\mathrm{SL}_2(\mathbb{Z})$-orbit of them.

2.4 Eisenstein series

Recall that a *modular form of weight k* for a congruence subgroup $\Gamma \subset \mathrm{SL}_2(\mathbb{Z})$ is a holomorphic function $f : \mathfrak{h} \to \mathbb{C}$ such that

- for all $\tau \in \mathfrak{h}$ and $\gamma \in \Gamma$,
$$f(\gamma\tau) = (c\tau + d)^k f(\tau);$$
- and for all $\gamma \in \mathrm{SL}_2(\mathbb{Z})$, the function $(c\tau + d)^{-k} f(\tau)$ is holomorphic at $\tau = i\infty$.

[1] The framing $\mathbf{X}$ and $\mathbf{Y}$ are related to Hain's T and A [37] by $\mathbf{X} = T$ and $\mathbf{Y} = A$ and Calaque and Gonzalez's x and y [31, 16] by $\mathbf{X} = 2\pi i x$ and $\mathbf{Y} = \frac{1}{2\pi i} y$.

2.4.1 Level 1

Suppose that k is an integer > 1. The normalized *Eisenstein series* of weight $2k$ is the analytic function on the upper half plane $\mathfrak{h}$ defined by

$$G_{2k}(\tau) = \sum_{\substack{\lambda \in \Lambda_\tau \\ \lambda \neq 0}} \frac{1}{\lambda^{2k}}.$$

The sum converges uniformly on $\mathfrak{h}$ and it is easily verified that it defines a modular form of weight $2k$ for the full modular group $\mathrm{SL}_2(\mathbb{Z})$. For later use, we also specify the normalization

$$\mathbb{G}_{2k}(\tau) = \frac{1}{2}\frac{(2k-1)!}{(2\pi i)^k} G_{2k}(\tau) = -\frac{B_{2k}}{4k} + \sum_{n=1}^{\infty} \sigma_{2k-1}(n) q^n,$$

where B_{2k} is the $2k$th Bernoulli number, $q = e^{2\pi i \tau}$, and $\sigma_k(n) = \sum_{d|n} d^k$. Note that all of the Fourier coefficients are rational and the coefficient of the q term is 1.

Some of these properties still hold when $k = 1$. The so-called "forbidden" Eisenstein series $\mathbb{G}_2(\tau)$ converges if summed in the correct order. It has the quasi-modularity property

$$\mathbb{G}_2(\gamma\tau) = (c\tau + d)^2 \mathbb{G}_2(\tau) + ic(c\tau + d)/4\pi$$

for $\gamma \in \mathrm{SL}_2(\mathbb{Z})$.

2.4.2 Higher level

There are also Eisenstein series $\mathbb{G}_{m,\alpha}(\tau)$ modular with respect to the congruence subgroup $\Gamma(N)$. These functions are indexed by weight and the N-torsion sections of $\mathcal{E}_{\Gamma(N)}$.

We define the Eisenstein series $G_{m,\alpha}(\tau)$ by

$$G_{m,\alpha}(\tau) = \sum_{(k,\ell) \neq (0,0)} \frac{e^{2\pi i (k\alpha_1 - \ell\alpha_\tau)}}{(k\tau + \ell)^m},$$

where α_1 and α_τ are the coordinates of the lift $\tilde{\alpha}$ (2.2.1). The definition is clearly invariant with respect to choice of lift $\tilde{\alpha}$. The series converge for $m \geq 2$ (although the case of $m = 2$ and α the zero section is exactly the "forbidden" level 1 series). They satisfy a twisted weight m modularity property

$$G_{m,\alpha}(\gamma\tau) = (c\tau + d)^m G_{m,\alpha\gamma}(\tau), \tag{2.4.1}$$

for all $\gamma \in \mathrm{SL}_2(\mathbb{Z})$. The right action of γ on α is as described in §2.2.2. Thus, $G_{m,\alpha}(\tau)$ is modular with respect to $\Gamma(N)$ for all N-torsion α and modular with respect to $\Gamma_1(N)$ if α is stabilized by $\Gamma_1(N)$. More generally, $G_{m,\alpha}(\tau)$ is modular with respect to a congruence subgroup Γ if and only α is a single-valued section of the universal curve $\mathcal{E}_\Gamma$.

The value of $G_{m,\alpha}(\tau)$ at the cusp $\tau = i\infty$ is

$$\lim_{\tau \to i\infty} G_{m,\alpha}(\tau) = \sum_{k \neq 0} \frac{e^{-2\pi i k \alpha_\tau}}{k^m} = -\frac{(-2\pi i)^m B_m([\alpha_\tau])}{m!},$$

where $B_m(x)$ is the mth Bernoulli polynomial and $[\alpha_\tau]$ is the unique rational number in $[0,1)$ such that $\alpha_\tau - [\alpha_\tau] \in \mathbb{Z}$.

2.4.3 With respect to $\Gamma_1(N)$

When $\alpha_\tau = 0$, the Eisenstein series $G_{m,\alpha}(\tau)$ is modular with respect to $G_1(N)$. In this case, we will sometimes write $G_{m,\alpha}(\tau)$ as $G_{m,\zeta}(\tau)$ where $\zeta = e^{2\pi i \alpha_1}$ is an Nth root of unity. Then we have series

$$G_{m,\zeta}(\tau) = \sum_{(k,\ell) \neq (0,0)} \frac{\zeta^k}{(k\tau + \ell)^m}.$$

If $q = e^{2\pi i \tau}$, then $G_{m,\zeta}(\tau)$ has Fourier series $\sum_{n=0}^{\infty} a_n q^n$, where

$$a_0 = (1 + (-1)^m)\zeta(m)$$

and

$$a_n = \frac{(-2\pi i)^m}{N^m (m-1)!} \left(\sum_{d|n} sgn(d) d^{m-1} + \sum_{\substack{(k,\ell) \in (\mathbb{Z}/N)^2 \\ (k,\ell) \neq (0,0)}} \zeta^k \sum_{\substack{d | nN \\ Nn/d \equiv k}} sgn(d) d^{m-1} e^{2\pi i \ell d/N} \right). \tag{2.4.2}$$

when $n \geq 1$. There are corresponding normalized Eisenstein series we denote by

$$\mathbb{G}_{m,\zeta}(\tau) = \frac{(m-1)!}{(2\pi i)^m} \frac{1}{\bar{\zeta} + (-1)^m \zeta} G_{m,\zeta}(\tau) = -\frac{B_m}{m(\bar{\zeta} + (-1)^m \zeta)} + q + \cdots.$$

Except when $m = 2$ and $\zeta = 1$, the functions $G_{m,\zeta}$ and $\mathbb{G}_{m,\zeta}$ satisfy the modularity properties

$$G_{m,\zeta}(\gamma\tau) = (c\tau + d)^m G_{m,\zeta^d}(\tau) \quad \text{if } \gamma \in \Gamma_0(N)$$

and

$$G_{m,\zeta}(\gamma\tau) = (c\tau + d)^m G_{m,\zeta}(\tau) \quad \text{if } \gamma \in \Gamma_1(N).$$

Complex conjugation induces the relations

$$G_{m,\bar{\zeta}}(\tau) = (-1)^m G_{m,\zeta}(\tau) \quad \text{and} \quad \mathbb{G}_{m,\bar{\zeta}}(\tau) = \mathbb{G}_{m,\zeta}(\tau).$$

The $\zeta = 1$ case reduces to the level 1 Eisenstein series

$$G_{m,1}(\tau) = G_m(\tau).$$

Values of $G_{m,\zeta}$ at the cusps are given by

$$G_{m,\zeta}(\tau)\big|_{q=0} = (1 + (-1)^m)\zeta(m)$$

and

$$(c\tau + d)^{-m} G_{m,\zeta}(\gamma\tau)\big|_{q=0} = \sum_{k\neq 0} \frac{\zeta^{-ck}}{k^m}.$$

2.5 Jacobi forms

Zagier [50] defines the Jacobi form $F_\tau^{\mathrm{Zag}}(u, v)$ by

$$F_\tau^{\mathrm{Zag}}(u, v) = \frac{\theta'(0|\tau)\theta(u + v|\tau)}{\theta(u|\tau)\theta(v|\tau)},$$

where $\theta(z|\tau)$ is the classical theta function

$$\theta(z|\tau) = \sum_{n\in\mathbb{Z}} (-1)^n e^{i\pi\tau(n+1/2)^2} e^{z(n+1/2)}.$$

Levin and Racinet [42] express the KZB connection in terms of the function $F^{\mathrm{LR}}(x, z, \tau)$, which is related to Zagier's function by

$$F^{\mathrm{LR}}(x, z, \tau) = 2\pi i F^{\mathrm{Zag}}(2\pi i x, 2\pi i z, \tau).$$

Calaque–Enriquez–Etinghof [15], Gonzalez [31], and Calaque–Gonzalez [16] implicitly use the same Jacobi form as Levin–Racinet.[2] We shall use the Jacobi form

$$F(x, z, \tau) := F^{\mathrm{LR}}(x/(2\pi i), z, \tau) = 2\pi i F^{\mathrm{Zag}}(x, 2\pi i z, \tau).$$

[2]The function $k(x, z|\tau)$ in [15, 31, 16] can be expressed as $F(x, z, \tau) - \frac{1}{x}$.

Zagier [50, §3] gives a number of properties of $F_\tau^{\mathrm{Zag}}(u,v)$, which imply the following for $F(x,z,\tau)$.

- (*Residues*) $F(x,z,\tau)$ has simple poles along $x=0$ and $z=0$ with residue $2\pi i$ and 1, respectively.
$$F(x,z,\tau) = \frac{2\pi i}{x} + \frac{1}{z} + \text{holomorphic terms}$$

- (*Symmetry property*)
$$F(x,z,\tau) = F(2\pi i z, x/(2\pi i), \tau) = -F(-x,-z,\tau)$$

- (*Elliptic property*)
$$F(x, z+m\tau+n, \tau) = e^{-mx}F(x,z,\tau) \tag{2.5.1}$$
for all $m,n \in \mathbb{Z}$.

- (*Modularity property*)
$$F\left(\frac{x}{c\tau+d}, \frac{z}{c\tau+d}, \gamma\tau\right) = (c\tau+d)\exp\left(\frac{czx}{c\tau+d}\right)F(x,z,\tau) \tag{2.5.2}$$
for all $\gamma \in \mathrm{SL}_2(\mathbb{Z})$.

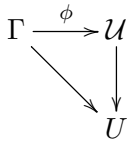
- (*Fourier expansion*) Set $q = e^{2\pi i\tau}$. Then
$$F(x,z,\tau) = \pi i\left(\coth(x/2) + \coth(\pi i z)\right) + 4\pi i \sum_{n=1}^{\infty}\sum_{d|n} \sinh\left(dx + \frac{2\pi i n z}{d}\right) q^n. \tag{2.5.3}$$

- (*Heat equation*)
$$\frac{\partial F}{\partial \tau}(x,z,\tau) = \frac{\partial^2 F}{\partial z \partial x}(x,z,\tau) \tag{2.5.4}$$

2.6 Unipotent completion

Let Γ be a discrete group. The unipotent completion of Γ over a field F of characteristic zero is a pro-unipotent group $\mathcal{U}$ together with a homomorphism $\phi : \Gamma \to \mathcal{U}$ with the following

universal property. If U is pro-unipotent and pro-algebraic over F, then any homomorphism $\Gamma \to U$ factors through ϕ.



The group $\mathcal{U}$ may be constructed concretely using I-adic completion. Let $F\Gamma$ denote the group algebra of Γ over F. Define the augmentation $\epsilon : F\Gamma \to F$ such that $\epsilon(\gamma) = 1$ for all $\gamma \in \Gamma$. Let $I = \ker \epsilon$ be the augmentation ideal, and define the I-adic completion

$$F\Gamma^\wedge = \varprojlim_n F\Gamma/I^n.$$

The group algebra $F\Gamma$ has a natural product as well as coproduct $\Delta : F\Gamma \to F\Gamma \otimes F\Gamma$ generated by $\Delta\gamma = \gamma \otimes \gamma$ for all $\gamma \in \Gamma$. This induces an augmentation preserving F-algebra homomorphism $\Delta : F\Gamma^\wedge \to F\Gamma^\wedge \hat{\otimes} F\Gamma^\wedge$.

The completed group ring $F\Gamma^\wedge$ has the structure of a Hopf algebra. We define the group-like elements

$$\mathcal{P}(F) = \{x \in F\Gamma^\wedge \mid \epsilon(x) = 1 \text{ and } \Delta x = x \otimes x\}$$

and the primitive elements

$$\mathfrak{p}(F) = \{x \in F\Gamma^\wedge \mid \Delta x = x \otimes 1 + 1 \otimes x\}.$$

The following proposition is easily verified [37].

Proposition 2.6.1. *(i) $\mathcal{P}(F)$ is a subgroup of $1 + I^\wedge$,*

(ii) $\mathfrak{p}(F)$ is a Lie algebra in $I^\wedge$,

(iii) the maps $\exp : I^\wedge \to (1+I^\wedge)$ and $\log : (1+I^\wedge) \to I^\wedge$ are inverse continuous bijections,

(iv) The restrictions $\exp : \mathfrak{p}(F) \to \mathcal{P}(F)$ and $\log : \mathcal{P}(F) \to \mathfrak{p}(F)$ are also inverse continuous bijections.

The last part of the proposition implies there is a group isomorphism

$$\exp : (\mathfrak{p}(F), \mathrm{BCH}) \to \mathcal{P}(F), \tag{2.6.1}$$

where the group structure of $\mathfrak{p}(F)$ is defined by the Baker–Campbell–Hausdorff formula [47].

The inclusion $\Gamma \to F\Gamma$ induces a natural homomorphism $\Gamma \to \mathcal{P}(F)$ since the image of $\Gamma \to F\Gamma^\wedge$ is contained in $1 + I^\wedge$ and $\Delta\gamma = \gamma \otimes \gamma$ for all $\gamma \in \Gamma$ by definition.

Definition 2.6.2. If $H_1(\Gamma; F)$ is finite-dimensional, the natural map $\Gamma \to \mathcal{P}(F)$ is called the *unipotent completion* of Γ over the field F.

By construction, the unipotent completion $\mathcal{P}$ is pro-unipotent. It is also pro-algebraic and the inclusion $\Gamma \to \mathcal{P}$ is Zariski dense. The Lie algebra $\mathfrak{p}$ of $\mathcal{P}$ is pro-nilpotent.

Example 2.6.3. Let $\Gamma = \langle x_1, \ldots, x_n \rangle$ be a free group on n generators. Then $\mathfrak{p}(F)$ is the completed free Lie algebra $\mathbf{L}(x_1, \ldots, x_n)^\wedge$ over F generated by $\{x_1, \cdots, x_n\}$.

Since we frequently work with fundamental groups, we will denote the unipotent completion of $\pi_1(X, x)$ by $\pi_1^{\mathrm{un}}(X, x)$ and its Lie algebra by $\mathfrak{p}(X, x)$.

Example 2.6.4. Let X be a torus with N points removed. Then for any base point $a \in X$

$$\pi_1(X, a) \cong \langle x, y, t_j \mid j = 0, \ldots, N-1 \rangle / \left(xyx^{-1}y^{-1} = \prod_{j=0}^{N-1} t_j \right),$$

where x and y are the usual generators of the fundamental group of the torus and each t_j is a homotopy class of small positively-oriented loop about the jth puncture. Define $X = \log x$, $Y = \log y$, and $T_j = \log t_j$ in $\mathfrak{p}(X, a)$. Then by the BCH formula,

$$\begin{aligned}
[X, Y] &= XY - YX \\
&= \log(e^X e^Y) - \log(e^Y e^X) \\
&= \log(xyx^{-1}y^{-1}) \\
&= \log \prod_{j=0}^{N-1} t_j \\
&= \sum_{j=0}^{N-1} T_j.
\end{aligned}$$

Thus, $\mathfrak{p}(X,a)$ is the completed Lie algebra generated by X, Y, and T_j with a single relation

$$\mathfrak{p}(X,a) = \mathbf{L}(X,Y,T_0,\dots,T_{N-1})^\wedge / \left([X,Y] = \textstyle\sum_j T_j\right).$$

This Lie algebra will appear when we define the KZB connection.

2.7 Iterated integrals

Suppose X is a manifold, $\omega_1,\dots,\omega_m$ are smooth 1-forms on X, and $\beta : [0,1] \to X$ is a piecewise smooth path. Define the *iterated integral*

$$\int_\beta \omega_1 \cdots \omega_m := \int_{0\le t_1 \le \cdots \le t_m \le 1} f_1(t_1)\cdots f_m(t_m)\, dt_1 \cdots dt_m,$$

where $\beta^*\omega_k = f_k(t)\,dt$. These integrals satisfy an inversion property

$$\int_{\beta^{-1}} \omega_1 \cdots \omega_m = (-1)^m \int_\beta \omega_m \cdots \omega_1 \tag{2.7.1}$$

and the *shuffle product*

$$\int_\beta \omega_1 \cdots \omega_m \int_\beta \omega_{m+1} \cdots \omega_{m+n} = \sum_\sigma \int_\beta \omega_{\sigma(1)} \cdots \omega_{\sigma(m+n)}, \tag{2.7.2}$$

where σ are permutations of $\{1,\dots,m+n\}$ that preserve the order of $\{1,\dots,m\}$ and $\{m+1,\dots,m+n\}$. By convention, the iterated integral with empty integrand is 1. See [17, 35] for more details.

We introduce two contexts in which iterated integrals regularly appear.

2.7.1 Cyclotomic multiple ζ-values

The *multiple polylogarithms* are analytic functions given by the power series

$$\mathrm{Li}_{n_1,\dots,n_m}(z_1,\dots,z_m) := \sum_{0<k_1<k_2<\cdots<k_m} \frac{z_1^{k_1} z_2^{k_2} \cdots z_m^{k_m}}{k_1^{n_1} k_2^{n_2} \cdots k_m^{n_m}}, \tag{2.7.3}$$

where $n_1,\dots,n_m$ are positive integers and $n_m \ge 2$. The series converges in the closed polydisk $|z_i| \le 1$ for all $i = 1,\dots,m$. Evaluated at $z_1 = \cdots = z_m = 1$, the multiple

polylogarithms yield the classical *multiple ζ-values* (MZVs)

$$\zeta(n_1,\dots,n_m) := \sum_{0<k_1<k_2<\cdots<k_m} \frac{1}{k_1^{n_1}k_2^{n_2}\cdots k_m^{n_m}},$$

where m is the *depth* and the sum $n_1+\cdots+n_m$ is the *weight* of the MZV. The values of the multiple polylogarithms at Nth roots of unity are called *N-cyclotomic MZVs* and have similar convention for length and depth.

Notably, the multiple polylogarithms may be written as iterated integrals.

Proposition 2.7.1.

$$\mathrm{Li}_{n_1,\dots,n_m}(z_1,\dots,z_m) = \int_0^{z_1\cdots z_m} \frac{dt}{1-t}\underbrace{\frac{dt}{t}\cdots\frac{dt}{t}}_{\substack{(n_1-1)\\ times}}\frac{dt}{z_1-t}\underbrace{\frac{dt}{t}\cdots\frac{dt}{t}}_{\substack{(n_2-1)\\ times}}\cdots\frac{dt}{z_1\cdots z_{m-1}-t}\underbrace{\frac{dt}{t}\cdots\frac{dt}{t}}_{\substack{(n_m-1)\\ times}}. \quad (2.7.4)$$

Proof. Express each term $\frac{1}{z-t}$ as the power series $\frac{1}{z}\sum_{r\geq 0}(t/z)^r$ and evaluate directly to recover the sum (2.7.3). □

This provides an alternative way of expressing the N-cyclotomic MZVs. The shuffle product of iterated integration (2.7.2) implies the $\mathbb{Q}$-span of the N-cyclotomic MZVs is closed under multiplication.

2.7.2 Chen's transport formula

Suppose $V\times X$ is a trivial vector bundle over a manifold X with connection $\nabla = d+\omega$, where ω is a smooth 1-form on X taking values in $\operatorname{End} V$.

Proposition 2.7.2. *Suppose $\beta:[0,1]\to X$ is a piecewise smooth path. Then the* inverse parallel transport *of $V\times X\to X$ with respect to ∇ along β is given by*

$$T(\beta)^{-1} = 1+\int_\beta \omega + \int_\beta \omega\omega + \int_\beta \omega\omega\omega + \cdots.$$

Proof. This is the inverse of Chen's formula [17, §3]. See [35, Lemma 2.5] and [37, Lemma 5.4] for a proof. □

If the connection ∇ is flat, then $T(\beta)^{-1}$ depends only on the homotopy class of β. As we use the topologist's convention for path multiplication, inverse transport is multiplicative

$$T(\alpha\beta)^{-1} = T(\alpha)^{-1}T(\beta)^{-1}.$$

This induces the following property of iterated integrals.

Proposition 2.7.3. *Suppose $\omega_1, \ldots, \omega_m \in E^1(X)$ and α and β are paths in X such that $\alpha(1) = \beta(0)$. Then*

$$\int_{\alpha\beta} \omega_1 \cdots \omega_m = \sum_{r=0}^{m} \int_{\alpha} \omega_1 \cdots \omega_r \int_{\beta} \omega_{r+1} \cdots \omega_m.$$

2.7.3 Regularized iterated integrals

Suppose X is a smooth curve. Fix a point $P \in X$ and suppose

$$\omega_1, \ldots, \omega_r \in H^0(X, \Omega^1_X(\log P))$$

are holomorphic 1-forms on $X - P$ with at worst logarithmic singularities at P. Fix a tangent vector $\vec{\mathrm{v}} \in T_P X$. We would like to define how to compute the iterated integrals

$$\int_{\vec{\mathrm{v}}}^{Q} \omega_1 \cdots \omega_r$$

where $Q \in X$. Integrals of this form are said to be *regularized at* $\vec{\mathrm{v}}$.

For simplicity, we will let $X = \mathbb{C}$ and P be the origin. This is the only case we need in the later sections of the thesis. Moreover, the definition is easily generalized to any smooth curve X by choosing a holomorphic coordinate centered at P. Consider a trivial bundle $V \times \mathbb{C}$ over $\mathbb{C}$ with flat connection $\nabla = d + \Omega$ where $\Omega \in H^0(\mathbb{C}, \Omega^1(\log 0)) \otimes \operatorname{End} V$ and $L = -\operatorname{Res}_0 \Omega$ is nilpotent (e.g. V underlies an admissible variation of MHS over $\mathbb{C}$). Set

$$\Omega = \omega_1 A_1 + \cdots + \omega_r A_r,$$

where each $\omega_j \in H^0(\mathbb{C}, \Omega^1(\log 0))$, $A_j \in \operatorname{End} V$, and

$$L = -\operatorname{Res}_0 \Omega = \sum_{j=1}^{r} \operatorname{Res}_0 \omega_j A_j$$

is nilpotent.

Set $\overline{\omega}_j = \mathrm{Res}_0\, \omega_j \frac{dz}{z}$ and

$$\overline{\Omega} = \overline{\omega}_1 A_1 + \cdots + \overline{\omega}_r A_r.$$

Observe that the transport with respect to the connection $d + \overline{\Omega}$ from $z \in \mathbb{C} - \{0\}$ to $\lambda \in \mathbb{C} - \{0\}$ is given by

$$\begin{aligned} T_{\overline{\Omega}}([z,\lambda]) &= T_{\overline{\Omega}}([\lambda, z])^{-1} \\ &= 1 + \int_\lambda^z \overline{\Omega} + \int_\lambda^z \overline{\Omega}\,\overline{\Omega} + \cdots \\ &= 1 + \sum_{m \geq 1} \frac{(\log z/\lambda)^m}{m!} (-L)^m \\ &= (z/\lambda)^{-L}. \end{aligned} \tag{2.7.5}$$

Fix a point $Q \in \mathbb{C} - \{0\}$. The *regularized iterated integrals* from $\lambda\partial/\partial z$ to Q are the coefficients of the transport (2.3.1) from $\lambda\partial/\partial z$ to Q with respect to the connection ∇

$$\lim_{z \to 0} (z/\lambda)^{-L} T_\Omega([Q, z/\lambda]).$$

Using the computation (2.7.5), the coefficient of the $A_{\ell_1} \cdots A_{\ell_n}$ term is given by

$$\int_{\lambda\partial/\partial z}^Q \omega_{\ell_1} \cdots \omega_{\ell_n} := \lim_{z \to 0} \sum_{j=1}^n \int_\lambda^z \overline{\omega}_{\ell_1} \cdots \overline{\omega}_{\ell_j} \int_z^Q \omega_{\ell_{j+1}} \cdots \omega_{\ell_n}.$$

This is precisely the same formula as Brown's "mortar board regularization" [10, §4].

Concretely, the regularization of an iterated integral at $\partial/\partial z \in T_0\mathbb{C}$ has the effect of setting $\lim_{z \to 0} \log z$ to zero. We illustrate this with the following example.

Example 2.7.4. The fibers of the logarithm variation $\mathbb{E}$ over $\mathbb{C} - \{0\}$ are extensions of $\mathbb{Z}$ by $\mathbb{Z}(1)$. The fiber over z has rational basis $\{2\pi i \mathbf{e}_1, \mathbf{e}_0 + (\log z)\mathbf{e}_1\}$ where $\mathbf{e}_0$ and $\mathbf{e}_1$ are the de Rham generators of $\mathbb{Z}$ and $\mathbb{Z}(1)$, respectively. The fiber at ± 1 splits over $\mathbb{Z}$ and the fibers at roots of unity split over $\mathbb{Q}$. The associated holomorphic vector bundle $\mathcal{V}$ has connection $\nabla = d - L\frac{dz}{z}$, where L is the nilpotent endomorphism

$$L : \begin{cases} \mathbf{e}_0 & \longmapsto & \mathbf{e}_1 \\ \mathbf{e}_1 & \longmapsto & 0. \end{cases}$$

Transport from $Q \in \mathbb{C} - \{0\}$ to $\lambda\partial/\partial z$ is given by

$$\begin{aligned} T([Q, \lambda\partial/\partial z]) &= 1 + \sum_{n \geq 1} L^n \int_{\lambda\partial/\partial z}^{Q} \underbrace{\frac{dz}{z} \cdots \frac{dz}{z}}_{n} \\ &= 1 + L \int_{\lambda\partial/\partial z}^{Q} \frac{dz}{z} \\ &= 1 + L \lim_{z \to 0} \left(\int_{\lambda}^{z} \frac{dz}{z} + \int_{z}^{Q} \frac{dz}{z} \right) \\ &= 1 + L(\log Q - \log \lambda). \end{aligned}$$

Observe that the $\log z$ terms canceled. Setting $Q = 1$, we also observe the limit MHS at $\lambda\partial/\partial z$ splits over $\mathbb{Z}$ if $\lambda = \pm 1$ and over Q if λ is a root of unity. In fact, the limit variation over $T_0\mathbb{C} - \{0\}$ is isomorphic to $\mathbb{E}$ via the natural identification $T_0\mathbb{C} \cong \mathbb{C}$.

Chapter 3

The elliptic KZB connection in level N

Calaque, Enriquez, and Etinghof [15] and Levin and Racinet [42] independently constructed the KZB connection for $\Gamma = \mathrm{SL}_2(\mathbb{Z})$. Hain [37] adapted their formulas for applications to mixed Hodge theory and mixed elliptic motives. Calaque and Gonzalez [16, 31] then generalized the connection to bilevel congruence subgroups and higher dimensional configuration spaces. In this chapter, we specialize Calaque and Gonzalez's formulas to $\Gamma(N)$ and other congruence subgroups. We choose generators and normalizations best suited for applications to Hodge theory and motives. In a sense, this chapter of the thesis serves as a generalization of many results in [37] to level $N > 1$.

3.1 The bundle $\boldsymbol{\mathcal{P}}_N$

The $\Gamma(N)$ KZB connection is defined on a trivial Lie algebra bundle $\mathfrak{p}_N \times \mathbb{C} \times \mathfrak{h}$ over $\mathbb{C} \times \mathfrak{h}$. The connection has logarithmic singularities along

$$\Lambda_N = \{(z + \lambda/N, \tau) \mid \lambda \in \Lambda_\tau\}$$

and is otherwise holomorphic. It is invariant with respect to $\Gamma(N) \ltimes \mathbb{Z}^2$ and thus descends to a connection on a bundle $\boldsymbol{\mathcal{P}}_N$ with fiber $\mathfrak{p}_N$ over the universal elliptic curve $\mathcal{E}_{\Gamma(N)}$ which is holomorphic over $\mathcal{E}'_{\Gamma(N)}$ and has logarithmic singularities along the N-torsion sections and the singular fibers above the cusps.

In this section, we first discuss the topological local system which KZB is describing. Then we define the action of $\Gamma(N) \ltimes \mathbb{Z}^2$ on the bundle $\boldsymbol{\mathcal{P}}_N$ on which KZB lives. The content of §3.1.1 and §3.1.2 are straightforward generalizations of [37].

3.1.1 The bundle $\boldsymbol{\mathcal{P}}_N^{\mathrm{top}}$

The bundle $\boldsymbol{\mathcal{P}}_N$ together with the KZB connection is the de Rham realization of a topological local system $\boldsymbol{\mathcal{P}}_N^{\mathrm{top}}$ over $\mathcal{E}_{\Gamma_1(N)}$. We describe this local system here.

Choose a base point $[E_o, \langle\,,\rangle_o, z_o]$ in $\mathcal{E}'_{\Gamma(N)}$, where E_o is an elliptic curve, $\langle\,,\rangle_o$ is a symplectic form on the N-torsion $E_o[N]$, and $z_o \in E'_o := E_o - E_o[N]$. Let y_o denote a lift of $[E_o, \langle\,,\rangle_o, z_o]$ to the universal cover U of $\mathcal{E}'_{\Gamma(N)}$. There is then a short exact sequence of fundamental groups

$$1 \to \pi_1(E'_o, z_o) \to \pi_1(\mathcal{E}'_{\Gamma(N)}, [E_o, \langle\,,\rangle_o, z_o]) \to \pi_1(Y(N), [E_o, \langle\,,\rangle_o]) \to 1. \tag{3.1.1}$$

Thus, $\pi_1(\mathcal{E}'_{\Gamma(N)}, [E_o, \langle\,,\rangle_o, z_o])$ acts on $\pi_1(E'_o, z_o)$ by conjugation. This induces a left action of $\pi_1(\mathcal{E}'_{\Gamma(N)}, [E_o, \langle\,,\rangle_o, z_o])$ on the unipotent completion $\mathcal{P}_o := \pi_1^{\mathrm{un}}(E'_o, z_o)$. Let $\boldsymbol{\mathcal{P}}_N^{\mathrm{top}}$ be the quotient

$$\pi_1(\mathcal{E}'_{\Gamma(N)}, [E_o, \langle\,,\rangle_o, z_o]) \backslash (\mathcal{P}_o \times U)$$

by the diagonal action of $\pi_1(\mathcal{E}'_{\Gamma(N)}, [E_o, \langle\,,\rangle_o, z_o])$ on $\mathcal{P}_o \times U$. Then $\boldsymbol{\mathcal{P}}_N^{\mathrm{top}}$ is a flat principal bundle whose fiber over $[E, \langle\,,\rangle, z]$ is canonically isomorphic to $\pi_1^{\mathrm{un}}(E', z)$. We will view the fibers as Lie algebras via the Baker-Campbell-Hausdorff formula (2.6.1).

In §3.3, we will prove $\boldsymbol{\mathcal{P}}_N$ together with the KZB connection is isomorphic to the flat holomorphic vector bundle associated with $\boldsymbol{\mathcal{P}}_N^{\mathrm{top}}$.

3.1.2 Factors of automorphy of $\boldsymbol{\mathcal{P}}_N$

We now construct the bundle $\boldsymbol{\mathcal{P}}_N$ on which the KZB connection is defined. Define the completed Lie algebra

$$\mathfrak{p}_N := \mathbf{L}\left(\mathbf{X}, \mathbf{Y}, \mathbf{t}_\alpha \mid \alpha \in (N^{-1}\mathbb{Z}/\mathbb{Z})^2\right)^\wedge \Big/ \left(\textstyle\sum_\alpha \mathbf{t}_\alpha = [\mathbf{X}, \mathbf{Y}]\right).$$

By Example 2.6.4, $\mathfrak{p}_N$ is (non-canonically) isomorphic to the Lie algebra of the unipotent completion of the fundamental group of an elliptic curve $E \cong \mathbb{C}/\Lambda$ punctured at its N-torsion points. We view the terms $\mathbf{X}$ and $\mathbf{Y}$ as the de Rham generators of $H_1(E, \mathbb{C})$ as in

§2.3.2. Meanwhile, each $\mathbf{t}_\alpha$ for $\alpha = \begin{pmatrix} k\mathbb{N} & \ell\mathbb{N} \end{pmatrix}$ corresponds to the homology class of a small positively-oriented loop around the N-torsion point $\frac{1}{N}(k\tau + \ell)$.

The semidirect product $\mathrm{SL}_2(\mathbb{Z}) \ltimes \mathbb{Z}^2$ acts on $\mathfrak{p}_N$ by the factor of automorphy

$$\tilde{M}_{\gamma,(m,n)}(z,\tau) = \begin{cases} M_\gamma(\tau) \circ \exp(cz\mathbf{X}/(c\tau+d)) & \gamma \in \mathrm{SL}_2(\mathbb{Z}) \\ \exp(-m\mathbf{X}) & (m,n) \in \mathbb{Z}^2, \end{cases} \tag{3.1.2}$$

where

$$M_\gamma(\tau) : \begin{cases} \mathbf{X} & \mapsto & (c\tau+d)\mathbf{X} \\ \mathbf{Y} & \mapsto & (c\tau+d)^{-1}\mathbf{Y} + \frac{c}{2\pi i}\mathbf{X} \\ \mathbf{t}_\alpha & \mapsto & \mathbf{t}_{\alpha\gamma^{-1}}. \end{cases}$$

The action on $\mathbf{X}$ and $\mathbf{Y}$ follows from (2.1.1) and the definitions $\mathbf{X} = \tau\mathbf{a} - \mathbf{b}$ and $\mathbf{Y} = \frac{1}{2\pi i}\mathbf{a}$. The action on the $\mathbf{t}_\alpha$ is induced by the isomorphism $\mathbb{C}/\Lambda_\tau \to \mathbb{C}/\Lambda_{\gamma\tau}$ where $z \mapsto (c\tau+d)^{-1}z$ and $\alpha \mapsto \alpha\gamma^{-1}$ (note this is a *left* action).

Proposition 3.1.1. *The factors of automorphy $\tilde{M}_{\gamma,(m,n)}(z,\tau)$ well-defined. That is,*

$$\tilde{M}_{\gamma_1\gamma_2,(m_1,n_1)\gamma_2+(m_2,n_2)}(z,\tau) = M_{\gamma_1,(m_1,n_1)}\left(\frac{z+m_2\tau+n_2}{c_2\tau+d_2}, \gamma_2\tau\right) M_{\gamma_2,(m_2,n_2)}(z,\tau)$$

for all $\gamma_k \in \mathrm{SL}_2(\mathbb{Z})$ and $(m_k,n_k) \in \mathbb{Z}^2$.

Proof. This is the same as Hain's proof in the level 1 case [37, §6]. □

Thus, for any level N congruence subgroup Γ, there is a well-defined left action of $\Gamma \ltimes \mathbb{Z}^2$ on the trivial bundle $\mathfrak{p}_N \times \mathbb{C} \times \mathfrak{h}$ over $\mathbb{C} \times \mathfrak{h}$

$$(\gamma,(m,n)) : (\mathbf{v},z,\tau) \mapsto \left(M_{\gamma,(m,n)}(\mathbf{v}), \frac{z+m\tau+n}{c\tau+d}, \gamma\tau\right).$$

We shall denote the quotient of $\mathfrak{p}_N \times \mathbb{C} \times \mathfrak{h}$ by this action by $\boldsymbol{\mathcal{P}}_\Gamma$. It is a bundle over the universal elliptic curve $\mathcal{E}_\Gamma$. In the next three sections, we consider the case of $\boldsymbol{\mathcal{P}}_{\Gamma(N)}$, abbreviated to $\boldsymbol{\mathcal{P}}_N$ for convenience. Once we define the bundle and its connection in this case, then only small modifications are necessary for other congruence subgroups.

3.2 The connection form

We now define the KZB connection. The formulas in §3.2.1 and §3.2.2 are adaptations of [16, 31].

3.2.1 Derivations

The KZB connection on the bundle $\boldsymbol{\mathcal{P}}_N$ takes values in $\operatorname{Der}\mathfrak{p}_N$. The relevant derivations are indexed by integers $m \geq 0$ and N-torsion $\alpha \in (N^{-1}\mathbb{Z}/\mathbb{Z})^2$.

$$\begin{aligned}\delta_{m,\alpha}: \qquad & \mathbf{X} \longmapsto 0 \\ & \mathbf{Y} \longmapsto \sum_{j+k=m-1} \sum_{\beta} (-1)^j [\mathrm{ad}_{\mathbf{X}}^j(\mathbf{t}_\beta), \mathrm{ad}_{\mathbf{X}}^k(\mathbf{t}_{\alpha+\beta})] \\ & \mathbf{t}_\beta \longmapsto [\mathbf{t}_\beta, \mathrm{ad}_{\mathbf{X}}^m(\mathbf{t}_{\beta+\alpha}) + (-1)^m \mathrm{ad}_{\mathbf{X}}^m(\mathbf{t}_{\beta-\alpha})].\end{aligned}$$

One can check

$$\delta_{m,\alpha}([\mathbf{X}, \mathbf{Y}]) = \sum_{\beta} \delta_{m,\alpha}(\mathbf{t}_\beta),$$

and therefore $\delta_{m,\zeta}$ is well-defined on $\mathfrak{p}_N$. For $m \geq 0$, set

$$\epsilon_{m+2,\alpha} = \delta_{m,\alpha} + \mathrm{ad}_{\mathbf{X}^m}(\mathbf{t}_\alpha + (-1)^m \mathbf{t}_{-\alpha}). \tag{3.2.1}$$

The facts

$$\epsilon_{m+2,\alpha}(\mathbf{t}_0) = 0 \tag{3.2.2}$$

and

$$\epsilon_{m,\alpha} = (-1)^m \epsilon_{m,-\alpha} \tag{3.2.3}$$

are useful in calculations.

Remark 3.2.1. The derivations $\epsilon_{m,\zeta}$ are *special*. That is, they annihilate $\mathbf{t}_0$ and satisfy

$$\epsilon_{m,\alpha} : \mathbf{t}_\beta \mapsto [u_\beta, \mathbf{t}_\beta]$$

for some $u_\beta \in \mathfrak{p}_N$.

3.2.2 Coefficients

For each N-torsion section $\alpha \in (N^{-1}\mathbb{Z}/\mathbb{Z})^2 \cong (1/N)\Lambda_\tau/\Lambda_\tau$, choose a lift $\tilde{\alpha} = \alpha_1 + \alpha_\tau\tau \in (1/N)\Lambda_\tau$. Then define[1]

$$h_\alpha(x, z|\tau) := e^{-\alpha_\tau x}F(x, z - \tilde{\alpha}, \tau) - \frac{2\pi i}{x}.$$

The function h_α is independent of choice of lift $\tilde{\alpha}$ due to the elliptic property (2.5.1) of $F(x, z, \tau)$. Let $g_\alpha(x, z|\tau)$ denote the partial derivatives

$$g_\alpha(x, z|\tau) := \frac{\partial}{\partial x}h_\alpha(x, z|\tau).$$

A modularity property for h_α follows from (2.5.2)

$$h_\alpha\left(x, \frac{z}{c\tau + d}\middle|\gamma\tau\right) = (c\tau + d)e^{czx}h_{\alpha\gamma}((c\tau + d)x, z|\tau) + 2\pi i\frac{e^{czx} - 1}{x}. \tag{3.2.4}$$

Differentiating yields

$$\begin{aligned} g_\alpha\left(x, \frac{z}{c\tau + d}\middle|\gamma\tau\right) = {} & cz(c\tau + d)e^{czx}h_{\alpha\gamma}((c\tau + d)x, z|\tau) \\ & + (c\tau + d)^2e^{czx}g_\alpha((c\tau + d)x, z|\tau) \\ & + 2\pi i\frac{czxe^{czx} - e^{czx} + 1}{x^2}. \end{aligned} \tag{3.2.5}$$

Define $A_{m,\zeta}(\tau)$ to be the coefficients of the Taylor expansion of $g_\alpha(x, 0|\tau)$ with respect to x:

$$g_\alpha(x, 0|\tau) = \sum_{m=0}^{\infty} A_{m,\alpha}(\tau)x^m.$$

3.2.3 Formula

We can now write down an explicit formula for the $\Gamma(N)$ KZB connection. It is given by

$$\nabla_{\mathrm{KZB}_N} = d + \Omega_N,$$

[1]The function $k_\alpha(x, z|\tau)$ of Calaque and Gonzalez [31, 16] is equal to $2\pi i h_\alpha(2\pi i x, z|\tau)$ in our notation.

where Ω_N is a 1-form on $\mathbb{C} \times \mathfrak{h}$ with logarithmic singularities along Λ_N taking values in $\operatorname{Der} \mathfrak{p}_N$

$$\Omega_{\mathrm{KZB}_N} \in \Omega^1(\mathbb{C} \times \mathfrak{h}, \log \Lambda_N) \otimes \operatorname{Der} \mathfrak{p}_N.$$

Define the following 1-forms.

$$\psi = \frac{1}{2} \sum_{\substack{m \geq 0 \\ \alpha \in (N^{-1}\mathbb{Z}/\mathbb{Z})^2}} A_{m,\alpha}(\tau) \delta_{m,\alpha} \, d\tau$$

$$\nu_1 = \sum_{\alpha \in (N^{-1}\mathbb{Z}/\mathbb{Z})^2} g_\alpha \left(\mathbf{X}, z | \tau \right) \cdot \mathbf{t}_\alpha \, d\tau$$

$$\nu_2 = \left(2\pi i \mathbf{Y} + \sum_{\alpha \in (N^{-1}\mathbb{Z}/\mathbb{Z})^2} h_\alpha \left(\mathbf{X}, z | \tau \right) \cdot \mathbf{t}_\alpha \right) dz$$

Then the KZB connection form is given by

$$\Omega_N = 2\pi i \mathbf{Y} \frac{\partial}{\partial \mathbf{X}} \, d\tau + \psi + \nu_1 + \nu_2.$$

This connection satisfies each of the aforementioned properties, namely invariance with respect to $\Gamma(N) \ltimes \mathbb{Z}^2$, integrability, and equivalence to the local system $\boldsymbol{\mathcal{P}}_N^{\mathrm{top}}$.

3.2.4 Invariance

Suppose X is a topological space with a left G-action. A connection ∇ on a trivial bundle $V \times X$ over X with factors of automorphy $M_g : X \to \operatorname{Aut} V$ is G-invariant if

$$\nabla(M_g s) = M_g(\nabla s)$$

for all $g \in G$ and sections $s : X \to V$. If Ω is an endomorphism-valued 1-form on X such that $\nabla s = \Omega s$, the above equation requires

$$g^* \Omega = M_g \Omega M_g^{-1} - (dM_g) M_g^{-1}.$$

Proposition 3.2.2. *The $\Gamma(N)$ KZB connection is invariant with respect to the action of $\Gamma(N) \ltimes \mathbb{Z}^2$ on the trivial bundle $\mathfrak{p}_N \times \mathbb{C} \times \mathfrak{h}$. Thus, ∇_{KZB_N} is a well-defined connection on the bundle $\boldsymbol{\mathcal{P}}_N$ over $\mathcal{E}_{\Gamma(N)}$.*

Proof. See Appendix A. □

3.2.5 Integrability

A connection ∇ with form Ω is flat if and only if the curvature form $d\Omega + \Omega \wedge \Omega$ vanishes.

Lemma 3.2.3. *The KZB connection form Ω_N is closed.*

Proof. The 1-forms $\mathbf{Y}\partial/\partial\mathbf{X}$ and ψ do not depend on z and thus are closed. Meanwhile, we apply the heat equation (2.5.4) of the Jacobi form F and the multivariable chain rule to see

$$\begin{aligned}
d\nu_1 &= \sum_\alpha \frac{\partial}{\partial z} g_\alpha\left(\mathbf{X}, z|\tau\right) \cdot \mathbf{t}_\alpha \, dz \wedge d\tau \\
&= \sum_\alpha \left(-\alpha_\tau \frac{\partial h_\alpha}{\partial z}(\mathbf{X}, z|\tau) + e^{-\alpha_\tau} \frac{\partial^2 F}{\partial \mathbf{X} \partial z}(\mathbf{X}, z - \tilde{\alpha}, \tau) \right) \cdot \mathbf{t}_\alpha \, dz \wedge d\tau \\
&= \sum_\alpha \left(-\alpha_\tau \frac{\partial h_\alpha}{\partial z}(\mathbf{X}, z|\tau) + e^{-\alpha_\tau} \frac{\partial F}{\partial \tau}(\mathbf{X}, z - \tilde{\alpha}, \tau) \right) \cdot \mathbf{t}_\alpha \, dz \wedge d\tau \\
&= \sum_\alpha \frac{\partial}{\partial \tau} \left(h_\alpha(\mathbf{X}, z|\tau) \right) \cdot \mathbf{t}_\alpha \, dz \wedge d\tau \\
&= -d\nu_2.
\end{aligned}$$

Therefore, $d\Omega_{\mathrm{KZB}_N} = 0$. □

Lemma 3.2.4.

$$\Omega_N \wedge \Omega_N = 0$$

Proof. Hain [37, §9.5] proves this in level 1. Calaque and Gonzalez [16, Prop. 3.9] give a proof in higher level. Both calculations use a calculus developed by Levin and Racinet [42]. □

Corollary 3.2.5. *The level N KZB connection is flat.*

3.3 Rigidity

Here we prove the claim that $\boldsymbol{\mathcal{P}}_N$ together with KZB connection and the local system $\boldsymbol{\mathcal{P}}_N^{\mathrm{top}}$ are isomorphic as flat vector bundles. We argue this by identifying the fibers of $\boldsymbol{\mathcal{P}}_N$ and $\boldsymbol{\mathcal{P}}_N^{\mathrm{top}}$ over a single point then showing they have compatible monodromy representations.

3.3.1 Monodromy in general

Let $\nabla = d + \omega$ be a flat G-invariant connection on a trivial bundle $V \times X$ over X with factors of automorphy $M_g : X \to \operatorname{Aut} V$. Fix a base point $x_o \in G\backslash X$ and lift $\tilde{x}_o \in X$. This determines a surjection $\rho : \pi_1(G\backslash X, x_o) \to G$.

Proposition 3.3.1. *Inverse monodromy $\Theta_{x_o} : \pi_1(G\backslash X, x_o) \to \operatorname{Aut} V$ along ∇ is given by*

$$\Theta_{x_o}(\beta) = T(\tilde{\beta})^{-1} \circ M_{\rho(\beta)}(\tilde{x}_o),$$

where $\tilde{\beta}$ is a lift of β to X based at $\tilde{x}_o$ and $T(\tilde{\beta})^{-1}$ is Chen's transport formula (2.7.2).

Proof. Follow the identifications. See [37, §5.2] for detailed proof this is a group homomorphism. □

The choice to use inverse monodromy makes Θ_{x_o} multiplicative

$$\Theta_{x_o}(\alpha\beta) = \Theta_{x_o}(\alpha)\Theta_{x_o}(\beta).$$

3.3.2 Monodromy in a single fiber

Fix an elliptic curve $E = \mathbb{C}/\Lambda_\tau$ and a base point $x \in E'$. Parallel transport of the KZB connection induces a monodromy representation $\pi_1(\mathcal{E}'_{\Gamma(N)}, [E, \langle\,,\rangle, x]) \to \operatorname{Aut} \mathfrak{p}_N$. To only consider monodormy in the single fiber E', we restrict to

$$\Theta : \pi_1(E', x) \to \operatorname{Aut} \mathfrak{p}_N.$$

Any path $\beta = n\mathbf{a} + m\mathbf{b} + \sum_\alpha c_\alpha \mathbf{t}_\alpha \in \pi_1(E', x) \cong H_1(E', \mathbb{Z})$ lifts to a path $\tilde{\beta}$ in $\mathbb{C}$ from 0 to $m\tau + n$. Then by Proposition 3.3.1, we have

$$\Theta(\beta) = \left(1 + \int_{\tilde{\beta}} \Omega_N + \int_{\tilde{\beta}} \Omega_N \Omega_N + \cdots\right) \widetilde{M}_{I,(m,n)}(\tau).$$

Proposition 3.3.2. *For any $\beta \in \pi_1(E', x)$, we have*

$$\Theta(\beta) \equiv \exp(\beta) \bmod [\mathfrak{p}_N, \mathfrak{p}_N].$$

Proof. Since τ is fixed, we need only consider the dz component of Ω_N. We observe

$$\begin{aligned}\nu_2 &= 2\pi i\mathbf{Y} + \sum_{\alpha\in(N^{-1}\mathbb{Z}/\mathbb{Z})^2}\left(\frac{e^{-\alpha_\tau\mathbf{X}}\cdot\mathbf{t}_\alpha}{z-\tilde{\alpha}} + (\text{holomorphic in } z)\right)dz \\ &\equiv \mathbf{a} + \sum_{\alpha\in(N^{-1}\mathbb{Z}/\mathbb{Z})^2}\left(\frac{\mathbf{t}_\alpha}{z-\tilde{\alpha}} + (\text{holomorphic in } z)\right)dz \bmod [\mathfrak{p}_N,\mathfrak{p}_N].\end{aligned}$$

Thus,

$$\left(1+\int_{\tilde{\beta}}\Omega_N + \int_{\tilde{\beta}}\Omega_N\Omega_N + \cdots\right) \equiv 1 + (m\tau+n)\mathbf{a} + \sum_\alpha c_\alpha\mathbf{t}_\alpha \bmod [\mathfrak{p}_N,\mathfrak{p}_N],$$

and

$$\begin{aligned}\Theta(\beta) &\equiv \left(1+(m\tau+n)\mathbf{a}+\sum_\alpha c_\alpha\mathbf{t}_\alpha\right)e^{-m\mathbf{X}} \\ &\equiv 1+(m\tau+n)\mathbf{a}+\sum_\alpha c_\alpha\mathbf{t}_\alpha - m(\tau\mathbf{a}-\mathbf{b}) \\ &\equiv 1+n\mathbf{a}+m\mathbf{b}+\sum_\alpha c_\alpha\mathbf{t}_\alpha \\ &\equiv \exp(\beta) \bmod [\mathfrak{p}_N,\mathfrak{p}_N].\end{aligned}$$

□

Corollary 3.3.3. *The monodromy representation Θ induces an isomorphism*

$$\Theta : \pi_1^{\text{un}}(E',x) \to \exp\mathfrak{p}_N.$$

Proof. Let $\mathfrak{q}$ denote the Lie algebra of $\pi_1^{\text{un}}(E',x)$. The previous proposition implies that Θ induces an isomorphism $H_1(\mathfrak{q}) \to H_1(\mathfrak{p}_N)$. Since the Lie algebras $\mathfrak{p}_N$ and $\mathfrak{q}$ are free, this implies the induced map on Lie algebras $\mathfrak{q}\to\mathfrak{p}_N$ is also an isomorphism. Exponentiation gives the result. □

3.3.3 Rigidity

The isomorphism $\Theta : \pi_1^{\text{un}}(E',x) \to \exp\mathfrak{p}_N$ induces a group isomorphism

$$\log\Theta : \pi_1^{\text{un}}(E',x) \to \mathfrak{p}_N,$$

where the group operation of $\mathfrak{p}_N$ comes from the BCH formula. We identify the fibers of $\boldsymbol{\mathcal{P}}_N^{\mathrm{top}}$ and $\mathcal{P}_N$ above the base point $[E, \langle\,,\rangle, x]$ via this map. This also induces an identification of automorphism groups via

$$\mathrm{Ad}(\Theta^{-1} \circ \exp) : \mathrm{Aut}\,\mathfrak{p}_N \to \mathrm{Aut}\,\pi_1^{\mathrm{un}}(E', x).$$

Let G denote the fundamental group $\pi_1(\mathcal{E}_{\Gamma_1(N)}, [E, \langle\,,\rangle, x])$. There are monodromy representations

$$\rho^{\mathrm{KZB}} : G \to \mathrm{Aut}\,\mathfrak{p}_N \quad \text{and} \quad \rho^{\mathrm{top}} : G \to \mathrm{Aut}\,\pi_1^{\mathrm{un}}(E', x).$$

To show that $\boldsymbol{\mathcal{P}}_N$ and $\boldsymbol{\mathcal{P}}_N^{\mathrm{top}}$ are isomorphic, we must prove these monodromy representations agree, i.e. the following diagram commutes.

$$\begin{array}{ccc} & & \mathrm{Aut}\,\mathfrak{p}_N \\ & \overset{\rho^{\mathrm{KZB}}}{\nearrow} & \big\downarrow \\ G & & \mathrm{Ad}(\Theta^{-1}\circ\exp) \\ & \underset{\rho^{\mathrm{top}}}{\searrow} & \big\downarrow \\ & & \mathrm{Aut}\,\pi_1^{\mathrm{un}}(E', x) \end{array} \tag{3.3.1}$$

We apply a more general lemma of Hain. Let N be a normal subgroup of a discrete group Γ. Let $\mathcal{N}$ denote the unipotent completion of N. The conjugation homomorphism $\Gamma \to \mathrm{Aut}\,N$ induces a map $\phi : G \to \mathrm{Aut}\,\mathcal{N}$. Restriction of ϕ to N maps $n \in N$ to $\iota_{\theta(n)}$, where $\theta : N \to \mathcal{N}$ is inclusion and $\iota_u \in \mathrm{Aut}\,\mathcal{N}$ is conjugation by $u \in \mathcal{N}$.

Lemma 3.3.4 ([37, Lemma 14.1])**.** *If $\mathcal{N}$ has trivial center, then ϕ is the unique homomorphism $\Gamma \to \mathrm{Aut}\,\mathcal{N}$ whose restriction to N is $n \mapsto \iota_{\theta(n)}$.*

Corollary 3.3.5. *The diagram* (3.3.1) *commutes.*

Proof. We know from Example 2.6.4 that $\pi_1^{\mathrm{un}}(E', x)$ has trivial center and from (3.1.1) that ρ^{top} is induced by conjugation. Set Γ to be $G = \pi_1(\mathcal{E}_{\Gamma_1(N)}, [E, \langle\,,\rangle, z])$, $N = \pi_1(E', x)$, and $\phi = \rho^{\mathrm{top}}$. Applying the lemma, it suffices to show the diagram commutes when restricting to $N = \pi_1(E', x)$.

If $\beta \in \pi_1(E', x)$, then by Corollary 3.3.3

$$\rho^{\mathrm{KZB}}(\beta) = \exp(\operatorname{ad} \log \Theta(\beta)) = \operatorname{Ad}(\exp \log \Theta(\beta)) = \operatorname{Ad}(\Theta(\beta)).$$

Then

$$\begin{aligned}(\operatorname{Ad}(\Theta^{-1} \circ \exp) \cdot \rho^{\mathrm{KZB}})(\beta) &= (\Theta^{-1} \circ \exp) \circ \operatorname{Ad}(\Theta(\beta)) \circ \log \Theta \\ &= \Theta^{-1} \circ (\operatorname{Ad}(\Theta(\beta)) \circ \exp) \circ \log \Theta \\ &= \Theta^{-1} \circ \operatorname{Ad}(\Theta(\beta)) \circ \Theta \\ &= \operatorname{Ad}(\beta) \\ &= \rho^{\mathrm{top}}(\beta).\end{aligned}$$

□

Thus, $\boldsymbol{\mathcal{P}}_N^{\mathrm{top}}$ and $\boldsymbol{\mathcal{P}}_N$ with the KZB connection are isomorphic as flat vector bundles.

3.4 Restrictions and residues

In this section, we confirm the KZB connection has regular singularities along the N-torsion sections and singular fibers of $\overline{\mathcal{E}}_{\Gamma(N)}$. This is necessary to show $\boldsymbol{\mathcal{P}}_N$ is an admissible variation of MHS.

3.4.1 Restriction to torsion sections

We will first compute the restriction of the KZB connection to a first-order neighborhood of the identity section of $\mathcal{E}_{\Gamma(N)} \to Y(N)$ (see Appendix B). This is equivalent to restricting the connection to the moduli $Y_{\Gamma(N),\vec{1}}(N)$. Concretely, this restriction is given by

$$G(z,\tau)\, d\tau + H(z,\tau)\, \frac{dz}{z} \mapsto G(0,\tau)\, d\tau + H(0,\tau)\, \frac{dz}{z}.$$

Proposition 3.4.1. *The Taylor coefficients $A_{m,\alpha}(\tau)$ are level N Eisenstein series*

$$A_{m,\alpha}(\tau) = -\frac{(m+1)}{(2\pi i)^{m+1}} G_{m+2,\alpha}(\tau).$$

Proof. One can use (2.5.3) and (2.4.2) to show the functions above have identical Fourier expansions when $\alpha_1 = 1$ and $\alpha_\tau = 0$. It follows directly from the modularity equation (3.2.5) of g_α that

$$A_{m,\alpha}(\gamma\tau) = (c\tau + d)^{m+2} A_{m,\alpha\gamma}(\tau) \tag{3.4.1}$$

for $\gamma \in \mathrm{SL}_2(\mathbb{Z})$. This is analogous to the modularity (2.4.1) of Eisenstein series. □

Corollary 3.4.2. *The restricted $\Gamma(N)$ KZB connection form is*

$$\Omega'_N = \left(2\pi i \mathbf{Y} \frac{\partial}{\partial \mathbf{X}} - \frac{1}{2} \sum_{\substack{m \geq 2 \\ \alpha \in (N^{-1}\mathbb{Z}/\mathbb{Z})^2}} \frac{(m-1)}{(2\pi i)^{m-1}} G_{m,\alpha}(\tau) \epsilon_{m,\alpha} \right) d\tau + \mathbf{t}_0 \frac{dz}{z}.$$

In particular, the residue of Ω'_N along the zero section is $L_{z=0} = \mathbf{t}_0$. Similarly, along any lift of a torsion section $z = \tilde{\alpha}$, the residue of Ω'_N is

$$L_{z,\tilde{\alpha}} := \mathrm{Res}_{z=\tilde{\alpha}} \Omega'_N = e^{-\alpha_\tau \mathbf{X}} \cdot \mathbf{t}_\alpha.$$

3.4.2 Restriction to singular fibers

Let $q_N = e^{2\pi i \tau / N}$. Then $2\pi i \, d\tau = N \frac{dq_N}{q_N}$. Recall from §2.2, the singular fiber of $\overline{\mathcal{E}}_{\Gamma(N)}$ over $q_N = 0$ is an N-gon of $\mathbb{P}^1$'s. The restriction of Ω_N to a first order neighborhood of this singular fiber is given by

$$G(z,\tau)\frac{dq_N}{q_N} + H(z,\tau)\, dz \mapsto \left(G(z,\tau)\big|_{q_N=0} \right) \frac{dq_N}{q_N} + \left(H(z,\tau)\big|_{q_N=0} \right) dz.$$

We then compute

$$A_{m,\alpha}(\tau)\big|_{q_N=0} = -\frac{(m+1)}{(2\pi i)^{m+1}} G_{m+2,\alpha}(\tau)\big|_{q_N=0} = \frac{2\pi i (-1)^m B_{m+2}([\alpha_\tau])}{m!(m+2)}$$

and

$$\begin{aligned} g_\alpha(\mathbf{X}, z|\tau)\big|_{q_N=0} &= \frac{\partial}{\partial \mathbf{X}} \left(\pi i e^{-\alpha\tau \mathbf{X}} \left(\coth(\mathbf{X}/2) + \coth(\pi i (z - \tilde{\alpha})) \right) - \frac{2\pi i}{\mathbf{X}} \right) \\ &= \left(\pi i e^{-\alpha_\tau \mathbf{X}} \frac{\partial}{\partial \mathbf{X}} \coth(\mathbf{X}/2) \right) \\ &\qquad - \alpha_\tau \pi i e^{-\alpha\tau \mathbf{X}} \left(\coth(\mathbf{X}/2) + \coth(\pi i (z - \tilde{\alpha})) \right) + \frac{2\pi i}{\mathbf{X}^2} \\ &= \sum_{m=0}^{\infty} \frac{2\pi i (-1)^m B_{m+2}([\alpha_\tau])}{m!(m+2)} \mathbf{X}^m \end{aligned}$$

after computing the Taylor expansion. Let $w = e^{2\pi i z}$ and $\zeta = e^{2\pi i/N}$. Then

$$\begin{aligned} h_\alpha(\mathbf{X}, z|\tau) &= \pi i e^{-\alpha\tau\mathbf{X}} \left(\coth(\mathbf{X}/2) + \coth(\pi i(z-\tilde{\alpha}))\right) - \frac{2\pi i}{\mathbf{X}} \\ &= \pi i e^{-\alpha_\tau \mathbf{X}} \left(\frac{e^{\mathbf{X}}+1}{e^{\mathbf{X}}-1} + \frac{w q_N^{-\alpha_\tau}\zeta^{-\alpha_1}+1}{w q_N^{-\alpha_\tau}\zeta^{-\alpha_1}-1}\right) - \frac{2\pi i}{\mathbf{X}} \\ &= 2\pi i e^{-\alpha_\tau \mathbf{X}} \left(\frac{1}{e^{\mathbf{X}}-1} + \frac{w q_N^{-\alpha_\tau}\zeta^{-\alpha_1}}{w q_N^{-\alpha_\tau}\zeta^{-\alpha_1}-1}\right) - \frac{2\pi i}{\mathbf{X}}. \end{aligned}$$

Thus,

$$h_\alpha(\mathbf{X}, z|\tau)\big|_{q_N=0} = \begin{cases} 2\pi i \left(\frac{1}{e^{\mathbf{X}}-1} + \frac{w}{w-\zeta^{\alpha_1}}\right) - \frac{2\pi i}{\mathbf{X}} & \alpha_\tau = 0 \\ 2\pi i e^{-\alpha_\tau \mathbf{X}} \left(\frac{1}{e^{\mathbf{X}}-1} + 1\right) - \frac{2\pi i}{\mathbf{X}} & \alpha_\tau \neq 0. \end{cases}$$

Summing these gives the restriction to singular fiber

$$\begin{aligned} \Omega'_N = N \left(\mathbf{Y}\frac{\partial}{\partial \mathbf{X}} + \frac{1}{2} \sum_{\substack{\alpha \in (N^{-1}\mathbb{Z}/\mathbb{Z})^2 \\ m \geq 0}} \frac{(-1)^m B_{m+2}([\alpha_\tau])}{m!(m+2)} \epsilon_{m+2,\alpha} \right) \frac{dq_N}{q_N} \\ + \sum_\alpha \frac{e^{-\alpha_\tau \mathbf{X}}}{e^{\mathbf{X}}-1} \cdot \mathbf{t}_\alpha \frac{dw}{w} + \sum_{\alpha_\tau \in \mathbb{Z}} \mathbf{t}_\alpha \frac{dw}{w - e^{2\pi i \alpha_1}} + \sum_{\alpha_\tau \notin \mathbb{Z}} e^{-\alpha_\tau \mathbf{X}} \cdot \mathbf{t}_\alpha \frac{dw}{w}. \end{aligned}$$

Therefore, the residue of Ω_N along the singular fiber above $q_N = 0$ is constant

$$L_{q_N=0} := \mathrm{Res}_{q_N=0}\, \Omega_N = N \left(\mathbf{Y}\frac{\partial}{\partial \mathbf{X}} + \frac{1}{2} \sum_{\substack{\alpha \in (N^{-1}\mathbb{Z}/\mathbb{Z})^2 \\ m \geq 0}} \frac{(-1)^m B_{m+2}([\alpha_\tau])}{m!(m+2)} \epsilon_{m+2,\alpha} \right).$$

One can use modularity properties (3.2.4) and (3.2.5) to compute the restriction and residues along the other singular fibers. If P is a cusp such that $\gamma \in SL_2(\mathbb{Z})$ maps $q_N = 0$ to P, then the residue along the singular fiber above P is

$$L_{q_N,P} = N \left(\mathbf{Y}\frac{\partial}{\partial \mathbf{X}} + \frac{1}{2} \sum_{\substack{\alpha \in (N^{-1}\mathbb{Z}/\mathbb{Z})^2 \\ m \geq 0}} \frac{(-1)^m B_{m+2}([\alpha_\tau])}{m!(m+2)} \epsilon_{m+2,\alpha\gamma^{-1}} \right).$$

3.5 Admissibility of $\mathcal{P}_N$

Together with appropriate choices of Hodge, weight, and relative weight filtrations, the KZB connection underlies an admissible variation of MHS over $\mathcal{E}'_{\Gamma(N)}$.

Define Hodge and weight filtrations of the Lie algebra $\mathfrak{p}_N$ as follows.

$$F^{-p} = \{x \in \mathfrak{p}_N \mid \deg_{\mathbf{Y}}(x) + \sum \deg_{\mathbf{t}_\zeta}(x) \leq p\}$$
$$W_{-m} = \{x \in \mathfrak{p}_N \mid \deg_{\mathbf{Y}}(x) + \deg_{\mathbf{X}}(x) + 2\sum \deg_{\mathbf{t}_\zeta}(x) \geq m\}$$

There is also a relative weight filtration $M_\bullet$ along the singular fibers of $\mathcal{E}_{\Gamma(N)}$

$$M_{-m} = \{x \in \mathfrak{p}_N \mid 2\deg_{\mathbf{Y}}(x) + 2\sum \deg_{\mathbf{t}_\zeta}(x) \geq m\}$$

These are compatible with the filtrations on $\mathbb{H}$ in §2.3.2 and Lie bracket.

Lemma 3.5.1. *The Hodge and weight filtrations on $\mathfrak{p}_N$ descend to the local system $\boldsymbol{\mathcal{P}}_N \to \mathcal{E}'_{\Gamma(N)}$.*

Proof. The factors of automorphy $\widetilde{M}_{\gamma,(m,n)}$ are in $F_0W_0M_0 \operatorname{End} \mathfrak{p}_N$, and thus the filtrations are well-defined on $\mathcal{E}'_{\Gamma(N)}$. □

Theorem 3.5.2. *Together with these filtrations, $\boldsymbol{\mathcal{P}}_N \to \mathcal{E}'_{\Gamma(N)}$ is a pro-object of the category of admissible variations of MHS. The W-graded quotients of $\boldsymbol{\mathcal{P}}_N$ are direct sums of Tate twists of symmetric powers of $\mathbb{H}$. and the relative weight filtration of the limit MHS at $\partial/\partial q_N + \partial/\partial w$ is $M_\bullet$.*

Proof. We observed in the previous section that ∇_{KZB_N} has residues

$$L_{z,\tilde{\alpha}} = e^{-\alpha_\tau \mathbf{X}} \cdot \mathbf{t}_\alpha$$

and

$$L_{q_N,P} = N\left(\mathbf{Y}\frac{\partial}{\partial \mathbf{X}} + \frac{1}{2}\sum_{\substack{\alpha \in (N^{-1}\mathbb{Z}/\mathbb{Z})^2 \\ m \geq 0}} \frac{(-1)^m B_{m+2}([\alpha_\tau])}{m!(m+2)} \epsilon_{m+2,\alpha\gamma^{-1}}\right),$$

where $\gamma \in SL_2(\mathbb{Z})$ such that γP is the cusp $q_N = 0$. Thus, we have

$$L_{z,\tilde{\alpha}} \in F^{-1}M_{-2}W_{-2} \operatorname{Der} \mathfrak{p}_N.$$

and

$$L_{q,P} \in F^{-1}M_{-2}W_0 \operatorname{Der} \mathfrak{p}_N.$$

Therefore, ∇_{KZB_N} satisfies Griffiths transversality. Meanwhile, $L_{q,P}$ and $L_{q,P} + L_{w,\tilde{\alpha}}$ meet condition (ii)(a) of Definition 2.3.2.

The graded quotient $\mathrm{Gr}^W_{-m}\,\mathfrak{p}_N$ is a direct sum of variations of the form

$$S^{m-2k}\mathbb{H}\cdot f_k$$

where f_k is a Lie word in the terms $\mathbf{t}_\alpha$ of length k. Each $\mathbf{t}_\alpha$ cuts out a constant section of $\boldsymbol{\mathcal{P}}_N$ (trivial under monodromy action of $\Gamma(N)\ltimes\mathbb{Z}^2$). Thus each Lie word in the $\mathbf{t}_\alpha$ of length k spans a constant local system with fiber $\mathbb{Q}(k)$. Thus, $\mathrm{Gr}^W_{-m}\,\mathfrak{p}_N$ is a direct sum of variations of the form $S^{m-2k}\mathbb{H}(k)$. We then observe $\mathrm{Gr}^W_0\,L_{q,P} = N\mathbf{Y}\partial/\partial\mathbf{X}$. It follows from the representation theory of $\mathfrak{sl}_2$ that $L^r_{q,P}$ induces an isomorphism

$$L^r_{q,P} : \mathrm{Gr}^M_{-m+r}\,\mathrm{Gr}^W_{-m}\,\mathfrak{p}_N \to \mathrm{Gr}^M_{-m-r}\,\mathrm{Gr}^W_{-m}\,\mathfrak{p}_N.$$

Since $L_{w,\tilde{\alpha}} \in W_{-2}\operatorname{Der}\mathfrak{p}_N$, the sum $(L_{q,P}+L_{w,\tilde{\alpha}})^r$ induces the same isomorphism. Thus, $M_\bullet$ is a compatible relative weight filtration at every point along the singular fibers of $\mathcal{E}_{\Gamma(N)}$.

At points along the N-torsion sections of $\mathcal{E}_{\Gamma(N)}$ (and not at the singular fibers), since the residue $L_{z,\tilde{\alpha}}$ is in $W_{-2}\operatorname{Der}\mathfrak{p}_N$, the compatible relative weight filtration is the simply equal to $W_\bullet$. We conclude that $\boldsymbol{\mathcal{P}}_N$ is an admissible VMHS over $\mathcal{E}'_{\Gamma(N)}$. □

If $\vec{\mathfrak{w}} = \lambda\partial/\partial q + \mu\partial/\partial w$ is a tangent vector anchored at $(q=0, w=1)$, the limit MHS of $\boldsymbol{\mathcal{P}}_N$ at $\vec{\mathfrak{w}}$ is $\operatorname{Lie}\pi_1^{\mathrm{un}}(E_{\lambda\partial/\partial q}, \mu\partial/\partial w)$, where $E'_{\lambda\partial/\partial q}$ is the first order smoothing of the singular fiber above $q=0$ in the direction of $\lambda\partial/\partial q$ with its N-torsion removed and $\mu\partial/\partial w$ is a tangential base point anchored at the identity of $E_{\lambda\partial/\partial q}$.

Remark 3.5.3. Technically, the underlying local system of flat sections of $\boldsymbol{\mathcal{P}}_N$ is the admissible pro-VMHS. From §3.3, we know this local system is isomorphic to the local system over $\mathcal{E}_{\Gamma(N)}$ whose fiber over $[E, \langle\,,\rangle, x]$ is $\operatorname{Lie}\pi_1^{\mathrm{un}}(E', x)$.

3.6 Other congruence subgroups

The above formulas and proofs can be easily be adapted to any level N congruence subgroup Γ. This is done by modifying the generators $\mathbf{t}_\alpha$ of the Lie algebra $\mathfrak{p}_N$. Those $\mathbf{t}_\alpha$ such that

α is invariant under the right action of Γ on $(\mathbb{Z}/N\mathbb{Z})^2$ remain unchanged while all others are set to zero. The resulting connection form only has singularities along the single-valued torsion sections and is holomorphic over the punctured universal elliptic curve $\mathcal{E}'_\Gamma$.

Example 3.6.1. In the case of the full modular group $\Gamma(1) = \mathrm{SL}_2(\mathbb{Z})$, the only nontrivial $\mathbf{t}_\alpha$ is $\mathbf{t}_0$. Then the Lie algebra $\mathfrak{p}_1$ is free on the two generators $\mathbf{X}$ and $\mathbf{Y}$. The connection form Ω_1 reduces to Hain's formula[2] [37, §9.2].

The relevant congruence subgroup in our discussion of mixed Tate motives is $\Gamma_1(N)$. In this case, the KZB connection only includes $\mathbf{t}_\alpha$ where the first coordinate of α vanishes. These terms correspond to small loops around the real N-torsion of an elliptic curve $\mathbb{C}/\Lambda_\tau$. Since these torsion sections intersect the singular fiber above $\tau = i\infty$ at Nth roots of unity, we will reindex the connection by $\zeta \in \boldsymbol{\mu}_N$ instead of α. If $\alpha = \begin{pmatrix} 0 & \ell \end{pmatrix}$, then $\mathbf{t}_\alpha = \mathbf{t}_\zeta$ where $\zeta = e^{2\pi i \ell/N}$. The reason for this choice in notation will become clear when we discuss the Hain morphism in §4.3.2.

The $\Gamma_1(N)$ KZB connection is then $\nabla_{\Gamma_1(N)} = d + \Omega_{\Gamma_1(N)}$, where

$$\Omega_{\Gamma_1(N)} = \left(2\pi i \mathbf{Y} \frac{\partial}{\partial \mathbf{X}} + \frac{1}{2} \sum_{\substack{m \geq 0 \\ \zeta \in \boldsymbol{\mu}_N}} A_{m,\zeta}(\tau) \delta_{m,\zeta} + \sum_{\zeta \in \boldsymbol{\mu}_N} g_\zeta(\mathbf{X}, z|\tau) \cdot \mathbf{t}_\zeta \right) d\tau + \left(2\pi i \mathbf{Y} + \sum_{\zeta \in \boldsymbol{\mu}_N} h_\zeta(\mathbf{X}, z|\tau) \cdot \mathbf{t}_\zeta \right) dz.$$

The restriction of the $\Gamma_1(N)$ connection to the zero section is

$$\Omega'_{\Gamma_1(N)} = 2\pi i \left(\mathbf{Y} \frac{\partial}{\partial \mathbf{X}} - \frac{1}{2} \sum_{\substack{m \geq 2 \\ \zeta \in \boldsymbol{\mu}_N}} \frac{\overline{\zeta} + (-1)^m \zeta}{(m-2)!} \mathbb{G}_{m,\zeta}(\tau) \epsilon_{m,\zeta} \right) d\tau + \mathbf{t}_1 \frac{dz}{z}.$$

Letting $w = e^{2\pi i z}$, the resides along the torsion section $w = \zeta$ for $\zeta \in \boldsymbol{\mu}_N$ are simply

$$L_{w,\zeta} := \mathrm{Res}_{w=\zeta} \Omega_{\Gamma_1(N)} = \mathbf{t}_\zeta.$$

[2]The generators $\mathbf{X}$ and $\mathbf{Y}$ are equivalent to $2\pi i \mathbf{t} = T$ and $\frac{1}{2\pi i \mathbf{a}} = A$ in [37].

Similarly, the restriction of the $\Gamma_1(N)$ connection to a neighborhood of the singular fiber above $q_N = 0$ simplifies to

$$\Omega'_{\Gamma_1(N)} = \left(\mathbf{Y} \frac{\partial}{\partial \mathbf{X}} + \frac{1}{2} \sum_{\substack{\zeta \in \boldsymbol{\mu}_N \\ m \geq 0}} \frac{B_{2m+2}}{(2m)!(2m+2)} \epsilon_{2m+2,\zeta} \right) \frac{dq}{q} + \frac{\mathbf{X}}{e^{\mathbf{X}} - 1} \cdot \mathbf{Y} \frac{dw}{w} + \sum_{\zeta \in \boldsymbol{\mu}_N} \mathbf{t}_\zeta \frac{dw}{w - \zeta}. \quad (3.6.1)$$

We replace q_N with q since the cusp at $q = q_N = 0$ has width 1 in $Y_1(N)$ (it has width N in $Y(N)$). The resulting residue along the nodal cubic singular fiber above $q = 0$ is

$$L_{q=0} := \mathrm{Res}_{q=0} \, \Omega_{\Gamma_1(N)} = \mathbf{Y} \frac{\partial}{\partial \mathbf{X}} + \sum_{\substack{\zeta \in \boldsymbol{\mu}_N \\ m \geq 0}} \frac{B_{2m+2}}{(2m)!(2m+2)} \epsilon_{2m+2,\zeta}. \quad (3.6.2)$$

Remark 3.6.2. Since the KZB connection is flat and the sections $w = \zeta$ intersect the singular fiber above $q = 0$ transversely, we expect the residues $L_{w,\zeta}$ and $L_{q,0}$ to commute. This can be confirmed by a straightforward calculation of $[L_{q,0}, L_{w,\zeta}] = L_{q,0}(\mathbf{t}_\zeta) = 0$.

Chapter 4

Mixed Tate motives over $\mathbb{Z}[\boldsymbol{\mu}_N, 1/N]$

Deligne and Goncharov [21] constructed a neutral tannakian category of *mixed Tate motives* $\mathsf{MTM}(\mathcal{O}_{k,S})$ over a number field k and unramified outside a set S of primes of the ring of integers $\mathcal{O}_k$.

4.1 Review of mixed Tate motives

In this section we review some basic definitions and results from [21].

4.1.1 Tannakian categories

For a full discussion of tannakian categories including proof of Theorem 4.1.2, we refer the reader to [22]. We will give a very brief discussion of the main points.

Suppose F is a field of characteristic zero and G is an affine group scheme over F. Denote the category of F-linear representations of G by $\mathsf{Rep}_F(G)$ and the sub-category of finite-dimensional representations by $\mathsf{Rep}_F^{\mathrm{fte}}(G)$.

Definition 4.1.1. A *neutral tannakian category* $\mathcal{C}$ over F is a category equivalent to one of the form $\mathsf{Rep}_F^{\mathrm{fte}}(G)$ for some affine F-group G.

Suppose $(\mathcal{C}, \otimes)$ is a rigid abelian F-linear tensor category with identity object $\mathbb{1}$ such that $\mathrm{End}_{\mathcal{C}}(\mathbb{1}) = F$. Suppose $\omega : \mathcal{C} \to \mathsf{Vec}_F$ is a faithful F-linear exact tensor functor, which we call a *fiber functor*. Given fiber functors ω_1 and ω_2, a *natural isomorphism* η from ω_1 to ω_2 is a family of isomorphisms $\eta_V : \omega_1(V) \to \omega_2(V)$ indexed by $V \in \mathrm{Ob}\,\mathcal{C}$ such that $\eta_{V_2} \circ \omega_1(f) = \omega_2(f) \circ \eta_{V_1}$ for all morphisms $f \in \mathrm{Hom}_{\mathcal{C}}(V_1, V_2)$. A natural isomorphism from a fiber functor ω to itself is called a *natural automorphism*. Let $\mathrm{Aut}^{\otimes}(\omega)$ denote the set of natural automorphisms of ω compatible with the tensor product.

Theorem 4.1.2 ([22, Thm. 3.2]). *If ω is a fiber functor of a neutral tannakian category, then* $\mathrm{Aut}^{\otimes}(\omega)$ *is represented by an affine group scheme G over F and there is an equivalence of categories $\mathcal{C} \to \mathsf{Rep}_F^{\mathrm{fte}}(G)$ corresponding to ω.*

We refer to affine group scheme $\mathrm{Aut}^{\otimes}(\omega)$ as the *fundamental group of $\mathcal{C}$ with respect to* ω, denoted $\pi_1(\mathcal{C}, \omega)$. When the choice of fiber functor is clear, we sometimes omit ω from the notation.

Given two fiber functors ω_1 and ω_2, we define $\mathrm{Isom}^{\otimes}(\omega_1, \omega_2)$ as the set of natural isomorphisms from ω_1 to ω_2. It has the structure of a right $\pi_1(\mathcal{C}, \omega_1)$-torsor and left $\pi_1(\mathcal{C}, \omega_2)$-torsor.

4.1.2 Realizations of objects of $\mathsf{MTM}(\mathcal{O}_{k,S})$

Betti and de Rham realizations

The category $\mathsf{MTM}(\mathcal{O}_{k,S})$ of mixed Tate motives over the ring $\mathcal{O}_{k,S}$ of S-integers of a number field k is neutral tannakian with respect to the natural fiber functors

$$\omega^{\mathrm{DR}} : \mathsf{MTM}(\mathcal{O}_{k,S}) \to \mathsf{Vec}_k \quad \text{and} \quad \omega^{B} : \mathsf{MTM}(\mathcal{O}_{k,S}) \to \mathsf{Vec}_{\mathbb{Q}}.$$

The images $\omega^{\mathrm{DR}}(V)$ and $\omega^{B}(V)$ of an object V of $\mathsf{MTM}(\mathcal{O}_{k,S})$ are called the de Rham and Betti realizations of V, respectively. We will often use the shorthand V_k^{DR} for $\omega^{\mathrm{DR}}(V)$ and V^B for $\omega^{\mathrm{DR}}(V)$. Each object V of $\mathsf{MTM}(\mathcal{O}_{k,S})$ has a weight filtration $M_\bullet$ by subobjects of V in $\mathsf{MTM}(\mathcal{O}_{k,S})$. It induces filtrations on V_k^{DR} and V^B. The de Rham realization has an additional filtration $F^\bullet$ called the Hodge filtration. For each embedding $\sigma : k \hookrightarrow \mathbb{C}$, there is a comparison isomorphism of filtered vector spaces

$$c_{V,\sigma}^{\mathrm{DR},B} : (V_k^{\mathrm{DR}}, M_\bullet) \otimes_\sigma \mathbb{C} \xrightarrow{\sim} (V^{\mathrm{B}}, M_\bullet) \otimes_{\mathbb{Q}} \mathbb{C}. \tag{4.1.1}$$

Simple objects of $\mathsf{MTM}(\mathcal{O}_{k,S})$

The simple object $\mathbb{Q}(-1)$ is defined to be the motive of $H^1(\mathbb{G}_m)$. The de Rham realization $\mathbb{Q}(-1)_k^{\mathrm{DR}}$ is the algebraic de Rham cohomology $H^1_{\mathrm{DR}}(\mathbb{G}_m) \cong k[\frac{dz}{z}]$ with coefficients in k. The

Betti realization $\mathbb{Q}(-1)^B$ is the singular cohomology with rational coefficients $H^1_{\text{sing}}(\mathbb{G}_m, \mathbb{Q})$. The comparison isomorphism

$$c^{\mathrm{DR},B}_{\mathbb{Q}(-1),\sigma} : H^1_{\mathrm{DR}}(\mathbb{G}_m) \otimes \mathbb{C} \to H^1_B(\mathbb{G}_m) \otimes \mathbb{C}$$

maps $[\frac{dz}{z}] \mapsto 2\pi i \gamma^\vee$, where γ is the positive generator of $H_1^{\text{sing}}(\mathbb{G}_m, \mathbb{Z})$ and $\gamma^\vee$ denotes its Poincaré dual.

We then set $\mathbb{Q}(-n) := \mathbb{Q}(-1)^{\otimes n}$ for any $n \in \mathbb{Z}$. The Hodge and weight filtrations of $V = \mathbb{Q}(n)$ are given by

$$V^{\mathrm{DR}} = F^{-n}V^{\mathrm{DR}} \supset F^{-n+1}V^{\mathrm{DR}} = \{0\}$$

and

$$\{0\} = M_{-2n-1}V \subset M_{-2n}V = V.$$

The de Rham and Hodge realizations of $V = \mathbb{Q}(n)$ are given by $V^{\mathrm{DR}} = k$ and $V^B = \mathbb{Q}$. The comparison isomorphism $c^{\mathrm{DR},B}_{V,\sigma} : V^{\mathrm{DR}} \otimes \mathbb{C} \to V^{\mathrm{B}} \otimes \mathbb{C}$ is multiplication by $(2\pi i)^{-n}$.

Canonical $\mathbb{Q}$-structure of V^{DR}

Define a functor $\omega : \mathsf{MTM}(\mathcal{O}_{k,S}) \to \mathsf{Vec}_{\mathbb{Q}}$ by

$$\omega(V) := \bigoplus_{m \in \mathbb{Z}} \mathrm{Hom}_{\mathsf{MTM}(\mathcal{O}_{k,S})}(\mathbb{Q}(m), \mathrm{Gr}^M_{-2m} V).$$

It is a fiber functor and $\omega(V)$ has Hodge and weight filtrations defined by

$$F^{-m}W_{-2m}\omega(V) = \mathrm{Hom}_{\mathsf{MTM}(\mathcal{O}_{k,S})}(\mathbb{Q}(m), \mathrm{Gr}^M_{-2m} V).$$

The de Rham realization V^{DR}_k of every object V of $\mathsf{MTM}(\mathcal{O}_{k,S})$ has a canonical splitting

$$V^{\mathrm{DR}}_k \cong \bigoplus F^{-m}V^{\mathrm{DR}}_k \cap M_{-2m}V^{\mathrm{DR}}_k \cong \bigoplus \mathrm{Gr}^M_{-2m} V^{\mathrm{DR}}_k \tag{4.1.2}$$

as k-vector spaces. By [21, Prop. 2.10], for each V in $\mathsf{MTM}(\mathcal{O}_{k,S})$, there is a canonical isomorphism

$$V^{\mathrm{DR}}_k \cong \omega(V) \otimes_{\mathbb{Q}} k. \tag{4.1.3}$$

that preserves the Hodge and weight filtrations. Thus, the $\mathbb{Q}$-vector space $\omega(V)$ is a rational form of V_k^{DR}. We will denote the $\mathbb{Q}$ vector space $\omega(V)$ by V^{DR}. With this notation $V_k^{\mathrm{DR}} \cong \mathbb{V}^{\mathrm{DR}} \otimes_{\mathbb{Q}} k$. The comparison isomorphism (4.1.1) becomes

$$c_V^{\mathrm{DR},B} : V^{\mathrm{DR}} \otimes_{\mathbb{Q}} \mathbb{C} \xrightarrow{\sim} V^B \otimes_{\mathbb{Q}} \mathbb{C}.$$

We show by example that this isomorphism still depends on the choice of embedding $\sigma : k \hookrightarrow \mathbb{C}$.

Example 4.1.3. Let $U = \mathbb{A}^1 - D$ where D is a finite set of points defined over k. For each $s \in S$, the map $z \mapsto z - s$ induces an isomorphism $H^1(U) \to H^1(\mathbb{A}^1 - \{s\})$ between objects of $\mathsf{MTM}(\mathcal{O}_{k,S})$. The inclusion $U \hookrightarrow \mathbb{A}^1 - \{s\}$ induces a morphism $H^1(\mathbb{A}^1 - \{s\}) \to H^1(U)$ between objects of $\mathsf{MTM}(\mathcal{O}_{k,S})$. We then have the commutative diagram below where the horizontal maps are each the canonical isomorphism (4.1.3).

$$\begin{array}{ccc}
H^1_{\mathrm{DR}}(\mathbb{G}_m) & \xrightarrow{\cong} & \omega(H^1(\mathbb{G}_m)) \otimes k \\
\downarrow & & \downarrow \\
H^1_{\mathrm{DR}}(\mathbb{A}^1 - \{\zeta\}) & \xrightarrow{\cong} & \omega(H^1(\mathbb{A}^1 - \{\zeta\})) \otimes k \\
\uparrow & & \uparrow \\
H^1_{\mathrm{DR}}(U) & \xrightarrow{\cong} & \omega(H^1(U)) \otimes k
\end{array}$$

Since the class $\frac{dw}{w} \in H^1_{\mathrm{DR}}$ is rational in $\omega(H^1(\mathbb{G}_m)) \otimes k$, it follows that $\frac{dw}{w-s}$ is rational (in the sense of (4.1.3)) in both $H^1_{\mathrm{DR}}(\mathbb{A}^1 - \{s\})$ and $H^1_{\mathrm{DR}}(U)$.

For each embedding $\sigma : k \hookrightarrow \mathbb{C}$ we have the commutative diagram

$$\begin{array}{ccc}
H^1_{\mathrm{DR}}(U) \otimes_\sigma \mathbb{C} & \xrightarrow{c_{V,\sigma}^{\mathrm{DR},B}} & H^1_B(U) \otimes \mathbb{C} \\
\cong \uparrow & & \| \\
\omega(H^1(U)) \otimes_\sigma \mathbb{C} & \xrightarrow{c_V^{\mathrm{DR},B}} & H^1_B(U) \otimes \mathbb{C}.
\end{array}$$

The comparison isomorphism $c_{V,\sigma}^{\mathrm{DR},B}$ maps the rational generator $\frac{dw}{w-s}$ of $H^1_{\mathrm{DR}}(U)$ to $2\pi i \gamma^\vee_{\sigma(s)}$, where $\gamma_{\sigma(s)}$ is the homology class of a small positive loop around $\sigma(s)$.

Taken together, the de Rham realization V^{DR} over $\mathbb{Q}$, the Betti realization V^B, the filtrations $M_\bullet$ and $F^\bullet$, and the comparison isomorphism $c_V^{\mathrm{DR},B}$ form a mixed Hodge structure

that we call the "Hodge realization" of V. More precisely, the $\mathbb{Q}$-vector space $c_V^{B,\mathrm{DR}}(V^B)$ underlies a MHS whose complexification is $V^{\mathrm{DR}} \otimes \mathbb{C}$. This MHS has weight graded quotients of Tate type, and its periods are the matrix coefficients of $c_V^{\mathrm{DR},B}$. These numbers are also called the *periods* of the mixed Tate motive V.

The ℓ-adic realization

There are also fiber functors $\omega_\ell : \mathsf{MTM}(\mathcal{O}_{k,S}) \to \mathsf{Vec}_{\mathbb{Q}_\ell}$ for each prime ℓ that does not divide N. The $\mathbb{Q}_\ell$-vector space $V_\ell := \omega_\ell(V)$ is a $\mathrm{Gal}(\overline{\mathbb{Q}}/K)$-representation filtered by $M_\bullet$, unramified outside ℓ, and crystalline at ℓ. It is called the ℓ-adic étale realization of V. There is also a comparison isomorphism of filtered $\mathbb{Q}_\ell$-vector spaces

$$c_V^{B,\ell} : (V^B, M_\bullet) \otimes \mathbb{Q}_\ell \xrightarrow{\sim} (V_\ell, M_\bullet).$$

The Hodge realization of $\mathsf{MTM}(\mathcal{O}_{k,S})$ is fully faithful [21]. Thus, the ℓ-adic realizations are completely determined by the de Rham and Betti realizations together with their filtrations and comparison isomorphism $c_V^{\mathrm{DR},B}$. In this thesis, we focus on the de Rham and Betti fiber functors.

4.1.3 Extensions

Deligne and Goncharov [21] showed the extensions of simple objects in $\mathsf{MTM}(\mathcal{O}_{k,S})$ have the property

$$\mathrm{Ext}^j_{\mathsf{MTM}(\mathcal{O}_{k,S})}(\mathbb{Q}, \mathbb{Q}(n)) \cong \begin{cases} \mathbb{Q} & j = n = 0 \\ K_{2n-1}(\mathcal{O}_{k,S}) \otimes \mathbb{Q} & j = 1, n \geq 0 \\ 0 & \text{otherwise,} \end{cases} \tag{4.1.4}$$

where $K_\bullet(\mathcal{O}_N)$ are the algebraic K-groups of $\mathcal{O}_N$.

In the later sections, we will focus on the case where k is the cyclotomic field $\mathbb{Q}(\boldsymbol{\mu}_N)$ and S is the set of primes dividing N. In this setting, $\mathcal{O}_{k,S} = \mathbb{Z}[\boldsymbol{\mu}_N, 1/N]$, which we shall

denote by $\mathcal{O}_N$. We further abbreviate $\mathsf{MTM}(\mathcal{O}_N)$ to MTM_N. Restricting to this case, it follows from Borel [7] that

$$\dim_{\mathbb{Q}} \operatorname{Ext}^1_{\mathsf{MTM}(\mathcal{O}_N)}(\mathbb{Q},\mathbb{Q}(n)) = \begin{cases} 1 & N=1, n \text{ odd} \geq 3 \\ 0 & N=1, n \text{ otherwise} \\ 1 & N=2, n \text{ odd} \\ 0 & N=2, n \text{ even} \\ \varphi(N)/2 - 1 + \omega(N) & N \geq 3, n = 1 \\ \varphi(N)/2 & N \geq 3, n \geq 2, \end{cases} \tag{4.1.5}$$

where $\omega(N) = \sum_{p|N} 1$ is the prime omega function and $\varphi(N)$ is the Euler phi function. In §5.4, we will specify a natural basis of $\operatorname{Ext}^1_{\mathsf{MTM}(\mathcal{O}_N)}(\mathbb{Q},\mathbb{Q}(n))$.

4.1.4 Tannakian fundamental group of $\mathsf{MTM}(\mathcal{O}_{k,S})$

Let $\mathsf{MTM}(\mathcal{O}_{k,S})^{\mathrm{ss}}$ denote the category of semi-simple mixed Tate motives over $\mathcal{O}_N$. This category is tannakian with respect to the de Rham and Betti fiber functors. For $\bullet \in \{\mathrm{DR}, B\}$, the action of $\pi_1(\mathsf{MTM}(\mathcal{O}_{k,S})^{\mathrm{ss}}, \omega^\bullet)$ on $\omega^\bullet(\mathbb{Q}(-1))$ determines the action on any object of $\mathsf{MTM}(\mathcal{O}_{k,S})^{\mathrm{ss}}$. The complex points of $\pi_1(\mathsf{MTM}(\mathcal{O}_{k,S})^{\mathrm{ss}}, \omega^\bullet)$ are given by

$$\begin{aligned} \pi_1(\mathsf{MTM}(\mathcal{O}_{k,S})^{\mathrm{ss}}, \omega^\bullet)(\mathbb{C}) &= \operatorname{Aut}^{\otimes}(\omega^\bullet|_{\mathsf{MTM}(\mathcal{O}_{k,S})^{\mathrm{ss}}})(\mathbb{C}) \\ &= \operatorname{Aut}^{\otimes}(\omega^\bullet|_{\mathbb{Q}(-1)})(\mathbb{C}) \\ &= \operatorname{Aut}_{\mathbb{C}}(\mathbb{Q} \otimes \mathbb{C}) \\ &= \mathbb{G}_m. \end{aligned}$$

The simple object $\mathbb{Q}(m)$ corresponds to the mth power of the standard character of $\mathbb{G}_m$.

Let $\mathcal{G}^\bullet_{k,S}$ denote the fundamental group of $\mathsf{MTM}(\mathcal{O}_{k,S})$ with respect to the fiber functor $\omega^\bullet$ for $\bullet \in \{\mathrm{DR}, B\}$. The inclusion $\mathsf{MTM}(\mathcal{O}_{k,S})^{\mathrm{ss}} \to \mathsf{MTM}(\mathcal{O}_{k,S})$ induces a surjection $\chi^\bullet : \mathcal{G}^\bullet_{k,S} \to \mathbb{G}_m$. The kernel $\mathcal{K}^\bullet_{k,S}$ acts trivially on objects of $\mathsf{MTM}(\mathcal{O}_{k,S})^{\mathrm{ss}}$ and thus is prounipotent. Thus, for each fiber functor, we have the short exact sequence

$$1 \longrightarrow \mathcal{K}^\bullet_{k,S} \longrightarrow \mathcal{G}^\bullet_{k,S} \xrightarrow{\chi^\bullet} \mathbb{G}_m \longrightarrow 1. \tag{4.1.6}$$

In light of §4.1.2, the de Rham realization V^{DR} of every object V of $\mathsf{MTM}(\mathcal{O}_{k,S})$ has a canonical splitting

$$V^{\mathrm{DR}} \cong \bigoplus F^m V^{\mathrm{DR}} \cap M_{2m} V^{\mathrm{DR}}$$

as $\mathbb{Q}$-vector spaces. Thus, the functor $\mathrm{Gr}^M_\bullet : \mathsf{MTM}(\mathcal{O}_{k,S}) \to \mathsf{MTM}(\mathcal{O}_{k,S})^{\mathrm{ss}}$ induces a splitting of $\mathcal{G}^{\mathrm{DR}}_{k,S} \to \mathbb{G}_m$ and defines an isomorphism

$$\mathcal{G}^{\mathrm{DR}}_{k,S} \cong \mathcal{K}^{\mathrm{DR}}_{k,S} \rtimes \mathbb{G}_m.$$

The Lie algebra of $\mathcal{K}^{\mathrm{DR}}_{k,S}$ is the de Rham realization of a pro-object $\mathfrak{k}^{\wedge}_{k,S}$ of $\mathsf{MTM}(\mathcal{O}_{k,S})$. It is therefore the degree completion of its de Rham realization $\mathfrak{k}^{\mathrm{DR}}_{k,S} := \omega(\mathfrak{k}^{\wedge}_{k,S})$. It is referred to as the *motivic Lie algebra* of $\mathsf{MTM}(\mathcal{O}_{k,S})$. It is free. The de Rham realization V^{DR} of each object V of $\mathsf{MTM}(\mathcal{O}_{k,S})$ is a graded representation of $\mathfrak{k}^{\mathrm{DR}}_{k,S}$, and $\mathsf{MTM}(\mathcal{O}_{k,S})$ is equivalent to the category of graded representations of $\mathfrak{k}^{\mathrm{DR}}_{k,S}$ on finite dimensional rational vector spaces. The focus of Chapter 5 is computing this action in the case $\mathcal{O}_{k,S} = \mathbb{Z}[\boldsymbol{\mu}_N, 1/N]$ to "first order" on the Lie algebra of the unipotent completion of the fundamental group of the first order smoothing of the Tate curve minus $\boldsymbol{\mu}_N$.

4.1.5 Motivic periods and their Galois action

We define the rings of *de Rham* and *Betti periods* of $\mathsf{MTM}(\mathcal{O}_{k,S})$ to be the coordinate rings of $\mathcal{G}^{\mathrm{DR}}_{k,S}$ and $\mathcal{G}^{B}_{k,S}$, respectively

$$\mathcal{P}^{\bullet}_{k,S} := \mathcal{O}(\mathcal{G}^{\bullet}_{k,S}) \quad \text{for } \bullet \in \{\mathrm{DR}, B\}.$$

We also define the ring of *motivic periods* of $\mathsf{MTM}(\mathcal{O}_{k,S})$ by

$$\mathcal{P}^{\mathfrak{m}}_{k,S} := \mathcal{O}(\mathrm{Isom}^{\otimes}(\omega, \omega^B)).$$

The left and right actions of $\mathcal{G}^{\mathrm{DR}}_{k,S}$ and $\mathcal{G}^{B}_{k,S}$ on $\mathrm{Isom}^{\otimes}(\omega, \omega^B)$ induce right and left coactions

$$\Delta^{\mathrm{DR}} : \mathcal{P}^{\mathfrak{m}}_{k,S} \to \mathcal{P}^{\mathfrak{m}}_{k,S} \otimes \mathcal{P}^{\mathrm{DR}}_{k,S} \quad \text{and} \quad \Delta^{B} : \mathcal{P}^{\mathfrak{m}}_{k,S} \to \mathcal{P}^{B}_{k,S} \otimes \mathcal{P}^{\mathfrak{m}}_{k,S}. \tag{4.1.7}$$

The action of $\mathcal{G}^{\mathrm{DR}}_{k,S}$ is well-defined because the isomorphism (4.1.3) is canonical.

Let R be a subring of $\mathbb{C}$. Then $\mathcal{G}^{\bullet}_{k,S}(R) = \mathrm{Hom}(\mathcal{P}^{\bullet}_{k,S}, R)$ and the coactions $\Delta^{\bullet}$ induce *Galois actions* of $\mathcal{G}^{\bullet}_{k,S}(R)$ on $\mathcal{P}^{\mathfrak{m}}_{k,S}$. The de Rham action is given by

$$\begin{aligned} \mathcal{G}^{\mathrm{DR}}(R) \times \mathcal{P}^{\mathfrak{m}}_{k,S} &\longrightarrow \mathcal{P}^{\mathfrak{m}}_{k,S} \otimes R \\ (g, p^{\mathfrak{m}}) &\longmapsto ((\mathrm{id} \otimes g) \circ \Delta^{\mathrm{DR}})(p^{\mathfrak{m}}). \end{aligned}$$

The Betti Galois action is defined analogously.

The motivic periods of $\mathcal{P}^{\mathfrak{m}}_{k,S}$ may be written more concretely as symbols $[V, \eta, \gamma]^{\mathfrak{m}}$, where $V \in \mathrm{Ob}(\mathsf{MTM}(\mathcal{O}_{k,S}))$, $\eta \in V^{\mathrm{DR}}$, and $\gamma \in (V^B)^{\vee}$. In many situations, they may be viewed as formal unevaluated integrals. They satisfy bilinearity relations and respect tensor products and morphisms in $\mathsf{MTM}(\mathcal{O}_{k,S})$ [12, §2]. For each object V, there is a motivic comparison $c^{\mathfrak{m}}_V$

$$\begin{aligned} c^{\mathfrak{m}}_V : V^{\mathrm{DR}} &\longrightarrow V^B \otimes \mathcal{P}^{\mathfrak{m}}_{k,S} \\ \eta &\longmapsto \sum_j [V, \eta, e_j^{\vee}] e_j, \end{aligned} \tag{4.1.8}$$

where $\{e_j\}$ is a basis of V^B. This comparison may be viewed as a $\mathcal{P}^{\mathfrak{m}}_{k,S}$-rational point of $\mathrm{Isom}^{\otimes}(\omega, \omega^B)$ and thus is natural with respect to the morphisms in $\mathsf{MTM}(\mathcal{O}_{k,S})$.

The *period map* $\mathrm{per} : \mathcal{P}^{\mathfrak{m}}_{k,S} \to \mathbb{C}$ sends

$$\mathrm{per} : [V, \gamma, \eta] \mapsto \gamma(c_V^{\mathrm{DR},B}(\eta)),$$

where $c_V^{\mathrm{DR},B} : V^{\mathrm{DR}} \otimes \mathbb{C} \to V^B \otimes B$ is the comparison isomorphism of V. This map may be interpreted as the evaluation of the integral of the "form" η against the "cycle" γ. It is straightforward to confirm that the diagram

$$\begin{array}{ccc} V^{\mathrm{DR}} & \xrightarrow{c^{\mathfrak{m}}_V} & V^B \otimes \mathcal{P}^{\mathfrak{m}}_{k,S} \\ & \searrow^{c_V^{\mathrm{DR},B}} & \downarrow{\scriptstyle \mathrm{id} \otimes \mathrm{per}} \\ & & V^B \otimes \mathbb{C} \end{array}$$

commutes. In the case V^{DR} and V^B are the de Rham and Betti cohomology of a variety defined over $\mathbb{Q}$, the image of per contains periods in the sense of Kontsevich and Zagier [45]. Restricted to mixed Tate motives, Grothendieck's *period conjecture* claims the map $\mathrm{per} : \mathcal{P}^{\mathfrak{m}}_{k,S} \to \mathbb{C}$ is injective [32]. An equivalent claim is that all polynomial relations between

periods of objects of $\mathsf{MTM}(\mathcal{O}_{k,S})$ are of "geometric origin" (i.e. arise from algebraic change of variables and Stokes' theorem) [1].

Example 4.1.4. The *Lefshetz period* is the simplest non-trivial example of a motivic period. Recall that $\mathbb{Q}(-1) = H^1(\mathbb{G}_m)$. Let $\gamma \in H^1_B(\mathbb{G}_m, \mathbb{Q})^\vee$ be dual to a simple closed positively-oriented loop about the origin. Then the Lefshetz period, denoted $\mathbb{L}$, is given by

$$\mathbb{L} = [H^1(\mathbb{G}_m), [dz/z], \sigma]^{\mathfrak{m}}.$$

Evaluating gives $\operatorname{per} \mathbb{L} = \sigma(c^{\mathrm{DR},B}_{\mathbb{Q}(-1)}([dz/z])) = 2\pi i$.

The Galois actions of $\mathcal{G}^\bullet_{k,S}(R)$ on $\mathcal{P}^{\mathfrak{m}}_{k,S}$ may be described in terms of matrix entries. Every $g \in \mathcal{G}^\bullet_{k,S}(R)$ is a natural isomorphism of $\omega^\bullet \otimes R$. Thus, for every realization $\omega^\bullet(V)$, the element g defines an automorphism of $\omega^\bullet(V) \otimes R$. The left action of $\mathcal{G}^{\mathrm{DR}}_{k,S}$ is given by

$$g \cdot [V, \eta, \gamma]^{\mathfrak{m}} = [V, g \cdot \eta, \gamma]^{\mathfrak{m}} \in \mathcal{P}^{\mathfrak{m}}_{k,S} \otimes R$$

for $g \in \mathcal{G}^{\mathrm{DR}}_{k,S}$. The right action of $\mathcal{G}^N_B$ is analogous:

$$[V, \eta, \gamma]^{\mathfrak{m}} \cdot g = [V, \eta, \gamma \cdot g]^{\mathfrak{m}} \in \mathcal{P}^{\mathfrak{m}}_{k,S} \otimes R$$

for $g \in \mathcal{G}^B_{k,S}$. This action is equivalent to the coaction (4.1.7) [12, §2.3].

Example 4.1.5. We can compute the action of $\mathcal{G}^{\mathrm{DR}}_{k,S}$ and $\mathcal{G}^B_{k,S}$ on the Lefschetz period $\mathbb{L}$.

Since $\mathbb{Q}(-1)$ is semi-simple, the action of $\mathcal{G}^\bullet_{k,S}(\mathbb{C})$ factors through $\chi^\bullet : \mathcal{G}^\bullet_{k,S} \to \mathbb{G}_m$. Thus, if $g \in \mathcal{G}^{\mathrm{DR}}_{k,S}(\mathbb{C})$, we have

$$g \cdot \mathbb{L} = [H^1(\mathbb{G}_m), g \cdot [dz/z], \gamma] = [H^1(\mathbb{G}_m), \chi^{\mathrm{DR}}(g)^{-1}[dz/z], \gamma] = \chi^{\mathrm{DR}}(g)^{-1}\mathbb{L}.$$

Similarly, if $g \in \mathcal{G}^B_{k,S}$, then

$$\mathbb{L} \cdot g = [H^1(\mathbb{G}_m), [dz/z], \gamma \cdot g] = [H^1(\mathbb{G}_m), [dz/z], \chi^B(g)^{-1}\gamma] = \chi^B(g)^{-1}\mathbb{L}.$$

4.2 The motivic path torsor of $\mathbb{P}^1 - \{0, \boldsymbol{\mu}_N, \infty\}$

Deligne and Goncharov [21] showed the coordinate ring of the motivic path torsor of $\mathbb{P}^1 - \{0, \boldsymbol{\mu}_N, \infty\}$ with appropriate choice of base points is an object of $\mathsf{MTM}(\mathcal{O}_N)$ where $\mathcal{O}_N =$

$\mathbb{Z}[\boldsymbol{\mu}_N, 1/N]$. This object is a rich source of motivic periods. These periods will appear as coefficients of the motivic version of the cyclotomic Drinfeld associator.

4.2.1 Brown's category of generalized Hodge realizations

Brown [12, §3] defines a category $\mathcal{H}$ of generalized Hodge realizations. An object V of $\mathcal{H}$ is a triple $(V^B, V^{\mathrm{DR}}, c)$ consisting of

(1) A finite-dimensional $\mathbb{Q}$-vector space V^B with a finite increasing filtration $M_\bullet$,

(2) A finite-dimensional $\mathbb{Q}$-vector space V^{DR} with finite increasing filtration $M_\bullet$ and finite decreasing filtration $F^\bullet$,

(3) An isomorphism
$$c : V^{\mathrm{DR}} \otimes \mathbb{C} \xrightarrow{\sim} V^B \otimes \mathbb{C},$$

(4) An involution $F_\infty : V^B \xrightarrow{\sim} V^B$ called the real Frobenius.

These are also required to satisfy the following two conditions.

- If c_{DR} (resp. c_B) is the $\mathbb{C}$-antilinear involution on $V^{\mathrm{DR}} \otimes \mathbb{C}$ (resp. $V^B \otimes \mathbb{C}$), then $c \circ c_{\mathrm{DR}} = (F_\infty \otimes c_B) \circ c$.
- The filtrations $M_\bullet V^B$ and $cF^\bullet(V^{\mathrm{DR}} \otimes \mathbb{C})$ equip V^B with a graded-polarizable $\mathbb{Q}$-MHS.

There are fiber functors $\omega_{\mathcal{H}}^B$ and $\omega_{\mathcal{H}}^{\mathrm{DR}}$ from $\mathcal{H}$ to $\mathsf{Vec}_{\mathbb{Q}}$ taking V to its corresponding component. Deligne [19, §1] showed $\mathcal{H}$ is Tannakian with respect to both $\omega_{\mathcal{H}}^B$ and $\omega_{\mathcal{H}}^{\mathrm{DR}}$.

The category $\mathsf{MTM}(\mathcal{O}_{k,S})$ of mixed Tate motives over $\mathcal{O}_{k,S}$ embeds as a full subcategory of $\mathcal{H}$ via the functor
$$\omega^{\mathcal{H}} : V \longmapsto (V^B, V^{\mathrm{DR}}, c_V^{\mathrm{DR},B})$$
[21, §1]. Also, the discussion of motivic periods in §4.1.5 extends analogously to the category $\mathcal{H}$, where the ring of periods $\mathcal{P}_{\mathcal{H}}^{\mathfrak{m}}$ is the coordinate ring $\mathcal{O}(\mathrm{Isom}^{\otimes}(\omega_{\mathcal{H}}^{\mathrm{DR}}, \omega_{\mathcal{H}}^B))$. Thus, since $\omega^{\mathcal{H}}$ is fully faithful, it induces an inclusion
$$\mathcal{P}_{k,S}^{\mathfrak{m}} \hookrightarrow \mathcal{P}_{\mathcal{H}}^{\mathfrak{m}}.$$

In the next section, we review the construction of the canonical MHS on fundamental groups. These MHS will be viewed as objects of $\mathcal{H}$.

4.2.2 The MHS of fundamental groups

There is a canonical MHS on the fundamental group of any smooth pointed complex algebraic variety X [35, 34]. We recall the construction in the case $U = \mathbb{P}^1 - (D \cup \{\infty\})$, where $D \subset \mathbb{A}^1(k)$ is a finite set of points defined over a number field k.

Let I be the augmentation ideal of the group algebra $\mathbb{Q}\pi_1(U, x)$. Denote the I-adic completion by $\mathbb{Q}\pi_1(U, x)^\wedge$ as in §2.6. We also define the I-adic completion of the path torsor $\pi(U; x, y)$ by

$$\mathbb{Q}\pi(U; x, y)^\wedge := \varprojlim_n \mathbb{Q}\pi(U; x, y)/I^n\pi(U; x, y). \tag{4.2.1}$$

Denote the 1-form $\frac{dw}{w-s}$ by ω_s. Let V be the $\mathbb{Q}$-vector space

$$V = \omega(H^1(U)) = \bigoplus_{s \in D} \mathbb{Q}\omega_s$$

(see Example 4.1.3). The dual space $V^\vee = \mathrm{Hom}_{\mathbb{Q}}(V, \mathbb{Q})$ has basis $\{\mathbf{e}_s \mid s \in D\}$. Define the tensor algebra $A := T(V^\vee)$. It has the structure of a Hopf algebra with coproduct generated by $\Delta\mathbf{e}_s = \mathbf{e}_s \otimes 1 + 1 \otimes \mathbf{e}_s$ for each $s \in D$. There is also an augmentation ideal J of A generated by $V^\vee$. Denote the J-adic completion of A by $A^\wedge$. There are natural Hodge and weight filtrations on $A^\wedge$ since each $\mathbf{e}_s$ has type $(-1, -1)$.

Define the universal connection form

$$\Omega = \sum_{s \in D} \omega_s \mathbf{e}_s \in V \otimes V^\vee.$$

Chen's transport of Ω (2.7.2) defines a map

$$\Theta_{x,y} : \mathbb{C}\pi(U; x, y)^\wedge \to A^\wedge \otimes \mathbb{C}. \tag{4.2.2}$$

This is an isomorphism of complete Hopf algebras [17, (3.5)]. It induces a MHS on $\mathbb{Q}\pi(U; x, y)^\wedge$. The periods are iterated integrals of the 1-forms ω_s over paths from x to

y. The triple

$$(\mathbb{Q}\pi(U;x,y)^\wedge, A^\wedge, \Theta_{x,y})$$

is a pro-object of Brown's category $\mathcal{H}$ [35, §5].

Remark 4.2.1. In the case $x = y$, the completion $\mathbb{Q}\pi_1(U,x)$ is the universal enveloping algebra of Lie algebra $\mathfrak{p}(U,x)$ over $\mathbb{Q}$. The natural inclusion $\mathfrak{p}(U,x) \hookrightarrow \mathbb{Q}\pi_1(U,x)^\wedge$ induces a MHS on $\mathfrak{p}(U,x)$. The comparison isomorphism $\mathfrak{p}(U,x) \to \mathbf{L}(V^\vee)^\wedge$ is also Chen's transport of Ω but each $\mathbf{e}_s$ acts by left adjoint.

The coordinate ring $\mathcal{O}(A^\wedge \otimes \mathbb{C})$ is the $\mathbb{C}$-vector space spanned by the iterated integrals

$$\int \omega_{s_1} \cdots \omega_{s_r}.$$

This is the tensor coalgebra on $H^0(\mathbb{P}^1, \Omega^1_{\mathbb{P}^1}(\log D))$ with the shuffle product (2.7.2). The coordinate ring is endowed with a $\mathbb{Q}$-mixed Hodge structure. Since each ω_s is of type $(1,1)$, the Hodge and weight filtrations are

$$F^p = \operatorname{span}\left\{ \int \omega_{s_1} \cdots \omega_{s_r} \middle| r \geq p \right\}$$
$$W_{2m+1} = W_{2m} = \operatorname{span}\left\{ \int \omega_{s_1} \cdots \omega_{s_r} \middle| r \leq m \right\}.$$

The $\mathbb{Q}$-structure is spanned by the integrals which take values in $\mathbb{Q}$ over the loops in $\pi_1(U,x)$.

By the construction above, the path torsors define a variation of MHS over $U \times U$

$$\{\mathbb{Q}\pi(U;x,y)^\wedge\}_{(x,y)} \to U \times U.$$

The connection form Ω is in $F^{-1}W_{-2}$ and has pro-nilpotent residues. Monodromy $\Theta_{x,x}$ is the identity on the weight graded quotients and thus is unipotent. Thus, the relative weight filtration at each point in $(D \times U) \cup (U \times D)$ is equal to the weight filtration. Hence, the variation is admissible [39].

Restricting the variation to the diagonal, the fibers are each completed fundamental groups $\mathbb{Q}\pi_1(U,x)^\wedge$. Transport from x to y is conjugation by $\Theta_{y,x}$. The inclusion $\mathfrak{p}(U,x) \hookrightarrow \mathbb{Q}\pi_1(U,x)^\wedge$ generates a sub-variation of MHS over U

$$\{\mathfrak{p}(U,x)\}_x \to U. \tag{4.2.3}$$

Transport from x to y is left adjoint by $\Theta_{y,x}$.

4.2.3 Tangential base points

Let X be a Riemann surface and D a finite set of points on X. The *real oriented blow-up* $\mathrm{Bl}^o_D X$ of X at D is a Riemann surface with boundary circle replacing each point of D. There exists a continuous projection $\mathrm{Bl}^o_D X \to X$, where the preimage of a point $P \in D$ is a circle viewed as the quotient of $T_P X - \{0\}$ by the positive real numbers. The restriction of the projection $(\mathrm{Bl}^o_D X - \partial\, \mathrm{Bl}^o_D X) \to X - D$ is a biholomorphism.

Each tangent vector $\vec{v} \in T_P X - \{0\}$ determines a point $[\vec{v}] \in \mathrm{Bl}^o_D X$. Deligne [19] defines the fundamental group $\pi_1(X - D, \vec{v})$ of $X - D$ with tangential base point $\vec{v}$ to be $\pi_1(\mathrm{Bl}^o_D X, [\vec{v}])$. When r is a positive real number, the biholomorphism $(\mathrm{Bl}^o_D X - \partial\, \mathrm{Bl}^o_D X) \to X - D$ induces a canonical isomorphism between $\pi_1(X-D, \vec{v})$ and $\pi_1(X-D, r\vec{v})$. However, the MHS on the unipotent completions $\pi_1^{\mathrm{un}}(X-D, \vec{v})$ and $\pi_1^{\mathrm{un}}(X-D, r\vec{v})$ are *not* necessarily isomorphic unless the local monodromy at P is trivial (see §2.3.1).

4.2.4 Realizations of Π^{mot}

We can now describe the Hodge realization and motivic periods of Deligne and Goncharov's object in the category MTM_N of mixed Tate motives over $\mathcal{O}_N = \mathbb{Z}[\boldsymbol{\mu}_N, 1/N]$. These are discussed in greater detail in [21, §5] and [9, §2].

For convenience, we set

$$U_N = \mathbb{P}^1 - \{0, \boldsymbol{\mu}_N, \infty\} = \mathbb{G}_m - \boldsymbol{\mu}_N.$$

Observe that U_N is a scheme over $\mathbb{Q}$

$$U_N = \mathrm{Spec}\, \mathbb{Q}[Z]/(Z^N - 1).$$

Fix tangential base points $\vec{\mathsf{w}}_\zeta = \zeta\partial/\partial w \in T_0 U_N$ and $\vec{\mathsf{v}}_\zeta = \zeta\partial/\partial w \in T_\zeta U_N$ for all $\zeta \in \boldsymbol{\mu}_N$ (see Figure 4.1). Set ${}_x\Pi^B_y := \mathbb{Q}\pi(U_N; x, y)^\wedge$, the I-adic completion of the path torsor (4.2.1).

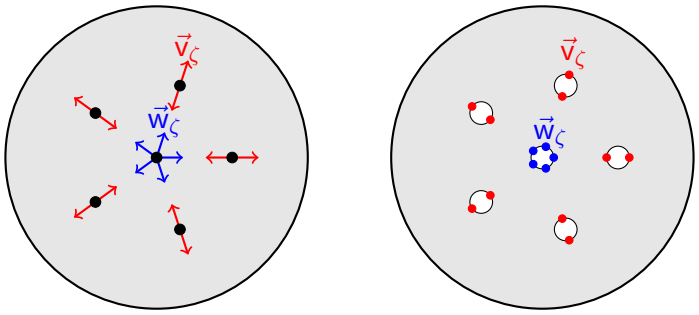


Figure 4.1: The tangential base points $\pm\vec{v}_\zeta$ and $\vec{w}_\zeta$ in U_N and $\mathrm{Bl}^o_{\{0,\boldsymbol{\mu}_N\}} U_N$

Likewise, set ${}_x\Pi_y^{\mathrm{DR}}$ to be the tensor coalgebra $A^\wedge$ as defined in §4.2.2 with $U = U_N$ and $D = \{0\} \cup \boldsymbol{\mu}_N$. There is a comparison isomorphism

$$c^{B,\mathrm{DR}} : {}_x\Pi_y^B \otimes_{\mathbb{Q}} \mathbb{C} \to {}_x\Pi_y^{\mathrm{DR}} \otimes_{\mathbb{Q}} \mathbb{C} \tag{4.2.4}$$

as defined in (4.2.2).

Theorem 4.2.2 ([21]). *Suppose $x, y, z \in \{\pm\vec{v}_\zeta, \vec{w}_\zeta \mid \zeta \in \boldsymbol{\mu}_N\}$. The coordinate rings $\mathcal{O}({}_x\Pi_y^B)$ and $\mathcal{O}({}_x\Pi_y^{\mathrm{DR}})$ are the Betti and de Rham realizations, respectively, of an ind-object $\mathcal{O}({}_x\Pi_y^{\mathrm{mot}})$ in the category MTM_N. The comparison isomorphism is induced by* (4.2.4). *The coproducts $\mathcal{O}({}_x\Pi_z^{\mathrm{mot}}) \to \mathcal{O}({}_x\Pi_y^{\mathrm{mot}}) \otimes \mathcal{O}({}_y\Pi_z^{\mathrm{mot}})$ are morphisms in MTM_N.*

4.2.5 Periods of Π^{mot}

The universal 1-form ω is given by

$$\Omega = \omega_0 \mathbf{e}_0 + \sum_{\zeta \in \boldsymbol{\mu}_N} \omega_\zeta \mathbf{e}_\zeta.$$

By the theorem, the transport $c^{B,\mathrm{DR}}$ of Ω is the transpose of the comparison isomorphism of $\mathcal{O}({}_x\Pi_y^{\mathrm{mot}})$. Let $\mathcal{P}_N^{\mathfrak{m}}$ denote the motivic periods of MTM_N. The matrix coefficients of $c^{B,\mathrm{DR}}$ are in $\mathcal{P}_N^{\mathfrak{m}}$, and thus $c^{B,\mathrm{DR}}$ restricts to a homomorphism

$$c^{B,\mathrm{DR}} : \mathbb{Q}\pi_1(U_N; x, y)^\wedge \to \mathcal{P}_N^{\mathfrak{m}} \langle\langle \mathbf{e}_0, \mathbf{e}_\zeta \mid \zeta \in \boldsymbol{\mu}_N \rangle\rangle$$

of cocommutative Hopf algebras. The image is generated as an algebra by two distinct types of elements: exponentials and Drinfeld associators.

Exponentials

In this case, suppose $x, y \in \{\pm\vec{\mathsf{v}}_\zeta, \vec{\mathsf{w}}_\zeta \mid \zeta \in \boldsymbol{\mu}_N\}$ are anchored at the same point $r \in \{0\} \cup \boldsymbol{\mu}_N$. Let $\gamma_r^{x,y}$ denote the counterclockwise path about r from x to y. If $x = y$, then let $\gamma_r^{x,x}$ be a simple closed loop about r based at x. Then

$$\begin{aligned} c^{B,\mathrm{DR}}(\gamma_r^{x,y}) &= \sum_{[\omega_{r_1}|\cdots|\omega_{r_n}] \in \mathcal{O}({}_x\Pi_y^{\mathrm{DR}})} [\mathcal{O}({}_x\Pi_y^{\mathrm{mot}}), [\omega_{r_1}|\cdots|\omega_{r_n}], (\gamma_r^{x,y})^\vee]^{\mathfrak{m}} \mathbf{e}_{r_1} \cdots \mathbf{e}_{r_n} \\ &= \sum_{n \geq 0} [\mathcal{O}({}_x\Pi_y^{\mathrm{mot}}), [\underbrace{\omega_r|\cdots|\omega_r}_{n \text{ times}}], (\gamma_r^{x,y})^\vee]^{\mathfrak{m}} \mathbf{e}_r^n \\ &= \sum_{n \geq 0} \frac{(k\mathbb{L}/N)^n \mathbf{e}_r^n}{n!} \\ &= \exp(k\mathbb{L}\mathbf{e}_r/N), \end{aligned}$$

where $k \in \{1, \ldots, N\}$ such that $y = e^{2\pi i k/N} x$.

Cyclotomic Drinfeld associators

Fix $x = \vec{\mathsf{w}}_1$ and $y = -\vec{\mathsf{v}}_1$. Let $\mathrm{dch} \in \pi_1(U_N; \vec{\mathsf{w}}_1, -\vec{\mathsf{v}}_1)$ denote the straight line path, or "droit chemin," from $\vec{\mathsf{w}}_1$ and $-\vec{\mathsf{v}}_1$. Then

$$c^{B,\mathrm{DR}}(\mathrm{dch}) = \sum_{[\omega_{r_1}|\cdots|\omega_{r_n}] \in \mathcal{O}({}_x\Pi_y^{\mathrm{DR}})} [\mathcal{O}({}_x\Pi_y^{\mathrm{mot}}), [\omega_{r_1}|\cdots|\omega_{r_n}], \mathrm{dch}^\vee]^{\mathfrak{m}} \mathbf{e}_{r_1} \cdots \mathbf{e}_{r_n}, \tag{4.2.5}$$

where

$$\mathrm{per}[\mathcal{O}({}_x\Pi_y^{\mathrm{mot}}), [\omega_{r_1}|\cdots|\omega_{r_n}], \mathrm{dch}^\vee]^{\mathfrak{m}} = \int_{\mathrm{dch}} \omega_{r_1} \cdots \omega_{r_n}.$$

To account for the limit MHS at $\vec{\mathsf{v}}_0$ and $-\vec{\mathsf{v}}_1$, the integrals above are regularized at $\vec{\mathsf{v}}_0$ if $r_1 = 0$ and regularized at $-\vec{\mathsf{v}}_1$ if $r_n \neq 0$ (see §2.7.3). By (2.7.4), each integral evaluates to an N-cyclotomic MZV (see §2.7.1) [25]. Thus, we shall refer to these unevaluated motivic periods as *motivic N-cyclotomic MZVs*. The right hand side of (4.2.5) is called the *motivic Drinfeld associator* and denoted $\Phi_{01}^{\mathfrak{m}}$. It is group-like in $\mathcal{P}_N^{\mathfrak{m}}\langle\langle \mathbf{e}_0, \mathbf{e}_\zeta \rangle\rangle$ [24].

The image of the straight line path from $\vec{\mathsf{w}}_\eta$ to $-\vec{\mathsf{v}}_\eta$ under $c^{B,\mathrm{DR}}$ is obtained by the change of variables $w \mapsto w\eta$. This is simply the image of $\Phi_{01}^{\mathfrak{m}}$ under the action $\mathbf{e}_r \mapsto \mathbf{e}_{r\eta}$

for all $r \in \{0\} \cup \boldsymbol{\mu}_N$. We denote this associator by $\Phi^{\mathfrak{m}}_{0\eta}$. Its periods are also the motivic N-cyclotomic MZVs.

Remark 4.2.3. The periods of $\mathcal{O}({}_x\Pi^{\mathrm{mot}}_y)$ are generated by $\mathbb{L}$ and the motivic N-cyclotomic MZVs. Deligne [20] showed these periods in fact generate *all* of the periods of objects of the category MTM_N when $N = 2$, 3, 4, 6, or 8. Brown [8] proved the same is true for $N = 1$. This last problem was more difficult due to relations in the "associated depth graded" of the motivic Lie algebra $\mathfrak{k}_1$ (we describe this Lie algebra and its depth filtration in Chapter 5). These small values of N are thought to be exceptional cases. Goncharov [27] showed that when N is a prime at least 5, there are relations in the associated depth graded of $\mathfrak{k}_N$ and that this implies the periods of objects of MTM_N are not generated by the N-cyclotomic MZVs. However, no explicit examples of such exotic periods are currently known.

4.3 The KZ local system

As in §2.6 we will denote the $\mathbb{Q}$-Lie algebra of $\pi_1^{\mathrm{un}}(U_N, a)$ by $\mathfrak{p}(U_N, a)$. Denote by $\boldsymbol{\mathcal{P}}_N^{\mathrm{KZ,top}}$ the $\mathbb{Q}$-local system over U_N whose fiber over $w \in U_N$ is $\mathfrak{p}(U_N, w)$.

4.3.1 The local system

The completed group ring $\mathbb{Q}\pi_1(U_N, w)^\wedge$ is canonically isomorphic to the universal enveloping algebra of the completed free Lie algebra $\mathfrak{p}(U_N, w)$. Thus, the discussion in §4.2.2 gives $\boldsymbol{\mathcal{P}}_N^{\mathrm{KZ,top}}$ the structure of an admissible variation of MHS over U_N. After tensoring with $\mathbb{C}$, each fiber is isomorphic to the completed free Lie algebra

$$\begin{aligned}\mathfrak{p}_N^{\mathrm{KZ}} :&= \mathbf{L}(\mathbf{e}_0, \mathbf{e}_\infty, \mathbf{e}_\zeta \mid \zeta \in \boldsymbol{\mu}_N)^\wedge \Big/ \left(\mathbf{e}_0 + \mathbf{e}_\infty + \sum_\zeta \mathbf{e}_\zeta = 0\right) \\ &\cong \mathbf{L}(\mathbf{e}_0, \mathbf{e}_\zeta \mid \zeta \in \boldsymbol{\mu}_N)^\wedge,\end{aligned}$$

Define $\boldsymbol{\mathcal{P}}_N^{\mathrm{KZ}}$ to be the trivial bundle $\mathfrak{p}_N^{\mathrm{KZ}} \times U_N$ over U_N with flat connection

$$\nabla_{\mathrm{KZ}_N} = d + \Omega_{\mathrm{KZ}_N} = d + \mathbf{e}_0 \frac{dw}{w} + \sum_{\zeta \in \boldsymbol{\mu}_N} \mathbf{e}_\zeta \frac{dw}{w - \zeta}, \tag{4.3.1}$$

where $\mathbf{e}_0$ and $\mathbf{e}_\zeta$ act on fibers $\mathfrak{p}_N^{\mathrm{KZ}}$ via left adjoint. This is a well-known cyclotomic generalization of the Knizhnik–Zamolodchikov (KZ) connection [41, 30]. The variation $\boldsymbol{\mathcal{P}}_N^{\mathrm{KZ}}$ is a special case of (4.2.3). In particular, it is a unipotent admissible variation of MHS and is isomorphic as a flat vector bundle to $\boldsymbol{\mathcal{P}}_N^{\mathrm{KZ,top}}$.

Proposition 4.3.1. *At each tangential base point $\vec{v} \in \{\vec{\mathrm{w}}_\zeta, \pm\vec{\mathrm{v}}_\zeta \mid \zeta \in \boldsymbol{\mu}_N\}$, the limit MHS of $\boldsymbol{\mathcal{P}}_N^{\mathrm{KZ}}$ is the Hodge realization of a pro-object of MTM_N.*

Proof. The Lie algebra $\mathfrak{p}(U_N, \vec{v})$ is canonically isomorphic to the Zariski tangent space of ${}_{\vec{v}}\Pi_{\vec{v}}^{\mathrm{DR}}$ and thus dual to a subquotient of $\mathcal{O}({}_{\vec{v}}\Pi_{\vec{v}}^{\mathrm{DR}})$. The result follows from Theorem 4.2.2. □

Corollary 4.3.2. *The* inverse *transport of the KZ connection along the straight line path* dch *from $\mathfrak{p}(U_N, \vec{\mathrm{w}}_1)$ to $\mathfrak{p}(U_N, -\vec{\mathrm{v}}_1)$ is the* left *adjoint of the motivic Drinfeld associator $\Phi_{01}^{\mathfrak{m}}$.*

Proof. By the discussion at the end of §4.2.2, transport of $\boldsymbol{\mathcal{P}}_N^{\mathrm{KZ}}$ from -$\vec{\mathrm{v}}_1$ to $\vec{\mathrm{w}}_1$ along the inverse of dch is left adjoint by

$$\Theta^{\mathrm{KZ}}_{\vec{\mathrm{w}}_1,-\vec{\mathrm{v}}_1} = T(\mathrm{dch})^{-1} = c^{B,\mathrm{DR}}(\mathrm{dch}) = \Phi_{01}^{\mathfrak{m}}.$$

□

Remark 4.3.3. As dual elements of the algebraic de Rham cohomology $H^1_{\mathrm{DR}}(U_N)$, the basis elements $\mathbf{e}_\zeta$ are defined over $\mathbb{Q}(\boldsymbol{\mu}_N)$. However, by the argument in Example 4.1.3, they form a *rational* basis of $\omega(\mathfrak{p}(U_N, \vec{\mathrm{v}}_1)) = \mathfrak{p}(U_N, \vec{\mathrm{v}}_1)^{\mathrm{DR}}$.

4.3.2 The Hain map

Any elliptic curve $E_\tau = \mathbb{C}/\Lambda_\tau$ may be written as $E_q := \mathbb{C}^*/q^{\mathbb{Z}}$ where $q = e^{2\pi i\tau}$. Following the notation of [37], set

$$\mathcal{A}_{|q|}; = \{w \in \mathbb{C}^* \mod |q|^{1/2} \leq |w| \leq |q|^{-1/2}\}.$$

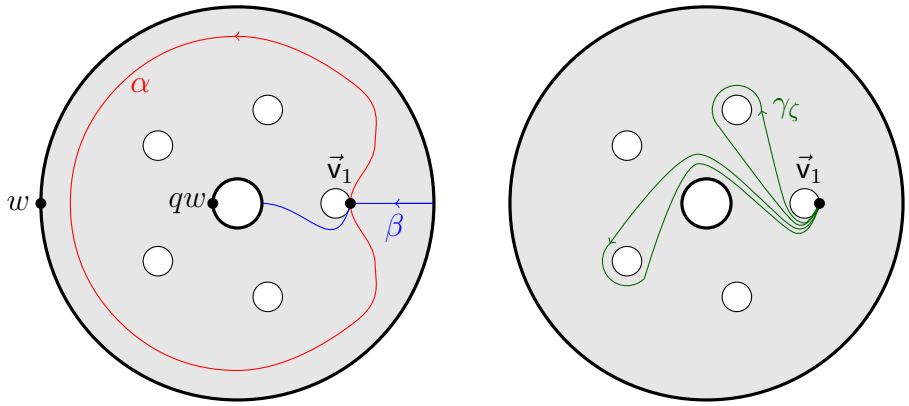


Figure 4.2: $\mathrm{Bl}^o_{\boldsymbol{\mu}_N} E_q$ seen as quotient of $\mathrm{Bl}^o_{\boldsymbol{\mu}_N} \mathcal{A}_{|q|}$ with generators of $\pi_1(E_q - \boldsymbol{\mu}_N, \vec{v}_1)$

Denote the inner and outer boundaries of this annulus by $\partial_- \mathcal{A}_{|q|}$ and $\partial_+ \mathcal{A}_{|q|}$, respectively. Then E_q is the quotient of $\mathcal{A}_{|q|}$ with $w \in \partial_+ \mathcal{A}_{|q|}$ identified with $qw \in \partial_- \mathcal{A}_{|q|}$. The real oriented blow-up $\mathrm{Bl}^o_{\boldsymbol{\mu}_N} E_q$ is a similar quotient of $\mathrm{Bl}^o_{\boldsymbol{\mu}_N} \mathcal{A}_{|q|}$.

Roughly speaking, as $q \to 0$, the space $\mathcal{A}_{|q|}$ converges to $\mathrm{Bl}^o_{\{0,\infty\}} \mathbb{P}^1$ and $\mathrm{Bl}^o_{\boldsymbol{\mu}_N} \mathcal{A}_{|q|}$ converges to $\mathrm{Bl}^o_{\{0,\boldsymbol{\mu}_N,\infty\}} \mathbb{P}^1$. Thus, as $R \to 0$, the elliptic curve $E_{Re^{i\theta}}$ converges to the quotient of $\mathrm{Bl}^o_{\{0,\infty\}} \mathbb{P}^1$ with the boundary circles at 0 and ∞ identified by multiplication by $e^{i\theta}$. This is the "first order smoothing" $E_{e^{i\theta}\partial/\partial q}$ of the nodal elliptic curve in the direction of $e^{i\theta}\partial/\partial q$. The scheme U_N includes into $E_{e^{i\theta}\partial/\partial q}$ in the obvious way.

Fix $\vec{\mathrm{u}} = \partial/\partial q$ at the origin of the q-disk. We observe that the inclusion $U_N \hookrightarrow E_{\vec{\mathrm{u}}}$ is compatible with the KZ_N and $\Gamma_1(N)$ KZB connections. That is, upon a change of basis, the linearization of the KZB bundle (3.6.1) over $E_{\vec{\mathrm{u}}}$ "pulls back" to the KZ_N bundle $\boldsymbol{\mathcal{P}}_N^{\mathrm{KZ}}$ over U_N.

Proposition 4.3.4. *Consistent with notation from Chapter 3, let $E'_{\vec{\mathrm{u}}} := E_{\vec{\mathrm{u}}} - \boldsymbol{\mu}_N$. The inclusion $U_N \hookrightarrow E'_{\vec{\mathrm{u}}}$ induces the commutative diagram*

$$\begin{array}{ccc} \mathfrak{p}(U_N, \vec{v}_1) & \longrightarrow & \mathfrak{p}(E'_{\vec{\mathrm{u}}}, \vec{v}_1) \\ \downarrow{\scriptstyle \mathrm{KZ}_N} & & \downarrow{\scriptstyle \mathrm{KZB}_{\Gamma_1(N)}} \\ \mathfrak{p}_N^{\mathrm{KZ}} & \xrightarrow{\Psi_N} & \mathfrak{p}_N. \end{array}$$

The vertical maps are the comparison isomorphisms of the canonical MHS on $\mathfrak{p}(U_N, \vec{v}_1)$ and $\mathfrak{p}(E'_{\vec{\mathrm{u}}}, \vec{v}_1)$ induced by transport of the the KZ_N and $\Gamma_1(N)$ KZB connections, respectively.

The lower horizontal map Ψ_N is the level N "Hain map"

$$\Psi_N : \begin{cases} \mathbf{e}_0 & \longmapsto & \frac{\mathbf{X}}{e^{\mathbf{X}}-1} \cdot \mathbf{Y} \\ \mathbf{e}_\infty & \longmapsto & \frac{\mathbf{X}}{e^{-\mathbf{X}}-1} \cdot \mathbf{Y} \\ \mathbf{e}_\zeta & \longmapsto & \mathbf{t}_\zeta. \end{cases} \tag{4.3.2}$$

Furthermore, Ψ_N is a morphism of MHS that preserves relative weight filtrations.

Proof. Recall the restriction of $\Gamma_1(N)$ KZB to a first order neighborhood of the nodal cubic (3.6.1). Pulling back along the inclusion $U_N \hookrightarrow E'_{\vec{\mathbf{u}}}$ yields the connection

$$d + \frac{\mathbf{X}}{e^{\mathbf{X}}-1} \cdot \mathbf{Y} \frac{dw}{w} + \sum_{\zeta \in \boldsymbol{\mu}_N} \mathbf{t}_\zeta \frac{dw}{w-\zeta}.$$

This is precisely the image of the cyclotomic KZ connection (4.3.1) under the given formula (4.3.2) for Ψ_N.

For the last claim, recall each $\mathbf{e}_r$ is of type $(-1,-1)$ and $M_\bullet = W_\bullet$ on $\mathfrak{p}_N^{\mathrm{KZ}}$. It follows that Ψ_N is compatible with the filtrations on $\mathfrak{p}_N$ defined in §3.5. □

4.4 The motivic limit MHS of $\boldsymbol{\mathcal{P}}_N$

The Lie algebra $\mathfrak{p}(E'_{\vec{\mathbf{u}}}, \vec{\mathbf{v}}_1)$ is endowed with the limit MHS of the elliptic KZB variation on $\boldsymbol{\mathcal{P}}_{\Gamma_1(N)}$ at $\partial/\partial q + \partial/\partial w$ above the cusp $q = 0$. We now argue this MHS is in fact the Hodge realization of a pro-object of MTM_N. This will build on the results about the cyclotomic KZ connection from the previous section.

Define loops α and β in $\pi_1(E_q - \boldsymbol{\mu}_N, \vec{\mathbf{v}}_1)$ appearing in Figure 4.2. Their homology classes $\mathbf{a}$ and $\mathbf{b}$ are those defined in §2.3.2. They have intersection number 1. It is also clear that $\mathbf{a}$ is the class of the vanishing cycle as $q \to 0$ and that $\mathbf{X} = \frac{1}{2\pi i}(\log q)\mathbf{a} - \mathbf{b}$ is invariant as q travels around $\mathbb{D}^*$. For each $\zeta \in \boldsymbol{\mu}_N$, there is a generator $\gamma_\zeta \in \pi_1(E_q - \boldsymbol{\mu}_N, \vec{\mathbf{v}}_1)$ chosen to be the homotopy class of the loop traveling clockwise half way around the boundary circle at 1, counter clockwise around the inner boundary circle $|w| = |q|^{1/2}$ (as necessary) and then counterclockwise around the boundary circle at ζ (see green loops in Figure 4.2). Dividing

β into two paths, one can easily verify the expected relation

$$\alpha\beta\alpha^{-1}\beta^{-1} = \gamma_1\gamma_\zeta\cdots\gamma_{\zeta^{N-1}},$$

where here $\zeta = e^{2\pi i/N}$.

Letting $q \to 0$ in the direction of $\vec{\mathfrak{u}} = \partial/\partial q$, the homotopy classes of α, β, and γ_ζ for $\zeta \in \boldsymbol{\mu}_N$ converge to generators of the topological fundamental group $\pi_1(E'_{\vec{\mathfrak{u}}}, \vec{\mathfrak{v}}_1)$. We will construct a MHS on $\mathfrak{p}(E'_{\vec{\mathfrak{u}}}, \vec{\mathfrak{v}}_1)$ by transporting the KZ_N connection form along these generators of $\pi_1(E_{\vec{\mathfrak{u}}}, \vec{\mathfrak{v}}_1)$ as in §4.2.2 and then applying the Hain map.

Recall from §4.2.5 the regularized transport of KZ_N from $\vec{\mathfrak{w}}_\eta \in T_0U_N$ to $-\vec{\mathfrak{v}}_\eta \in T_\eta U_N$ is the motivic Drinfeld associator $\Phi^{\mathfrak{m}}_{0\eta} \in \mathcal{P}^{\mathfrak{m}}_N\langle\langle \mathbf{e}_0, \mathbf{e}_\zeta \mid \zeta \in \boldsymbol{\mu}_N\rangle\rangle$. The involution $w \mapsto 1/w$ allows us to define a motivic transport from $\vec{\mathfrak{v}}_1$ to ∞, denoted $\Phi^{\mathfrak{m}}_{1\infty}$ by exchanging $\mathbf{e}_0$ for $\mathbf{e}_1$, $\mathbf{e}_1$ for $\mathbf{e}_\infty = -\mathbf{e}_0 - \mathbf{e}_1$, and $\mathbf{e}_\zeta$ for $\mathbf{e}_{\bar{\zeta}}$ in the formula of $\Phi^{\mathfrak{m}}_{01}$ (4.2.5).

For the remainder of the section, denote by $A_{E'_{\vec{\mathfrak{u}}}}$ the group algebra

$$A_{E'_{\vec{\mathfrak{u}}}} := \mathbb{Q}\langle\langle \mathbf{X}, \mathbf{Y}, \mathbf{t}_\zeta\rangle\rangle \Big/ \left(\sum_{\zeta\in\boldsymbol{\mu}_N} \mathbf{t}_\zeta = [\mathbf{X}, \mathbf{Y}] \right).$$

Following the paths in Figure 4.2, *inverse* monodromy

$$\Theta : \pi_1(E'_{\vec{\mathfrak{u}}}, \vec{\mathfrak{v}}_1) \to \mathcal{P}^{\mathfrak{m}}_N \otimes A_{E'_{\vec{\mathfrak{u}}}} \tag{4.4.1}$$

is given by the formulas

$$\begin{aligned}
\alpha &\longmapsto \Phi^{\mathfrak{m}}_{\infty 1} e^{\mathbb{L}\mathbf{e}_\infty} \Phi^{\mathfrak{m}}_{1\infty} \\
\beta &\longmapsto e^{\mathbb{L}\mathbf{e}_1/2} \Phi^{\mathfrak{m}}_{01} e^{-\mathbf{X}} \Phi^{\mathfrak{m}}_{1\infty} \\
\gamma_\zeta &\longmapsto e^{\mathbb{L}\mathbf{e}_1/2} \Phi^{\mathfrak{m}}_{01} e^{k\mathbb{L}\mathbf{e}_0/N} \Phi^{\mathfrak{m}}_{\zeta 0} e^{-\mathbb{L}\mathbf{e}_\zeta} \Phi^{\mathfrak{m}}_{0\zeta} e^{-k\mathbb{L}\mathbf{e}_0/N} \Phi^{\mathfrak{m}}_{10} e^{-\mathbb{L}\mathbf{e}_1/2},
\end{aligned}$$

where $\zeta = e^{2\pi i k/N}$ and the motivic Drinfeld associators $\Phi^{\mathfrak{m}}$ actually indicate their image under the Hain map. The $e^{-\mathbf{X}}$ in the formula for β corresponds to the factor of automorphy $\widetilde{M}_{I,(1,0)}$ associated with identifying the inner and outer boundaries of the annulus (see §3.1.2 and Figure 4.3). Note that $\Theta(\gamma_1)$ simplifies to $e^{-\mathbb{L}\mathbf{e}_1}$.

The associators $\Phi^{\mathfrak{m}}$ satisfy relations consistent with the topology of $E_{\vec{\mathfrak{u}}}$. The loop α in Figure 4.2 is homotopy equivalent to the loop α' based at $\vec{\mathfrak{v}}_1$ in Figure 4.4.

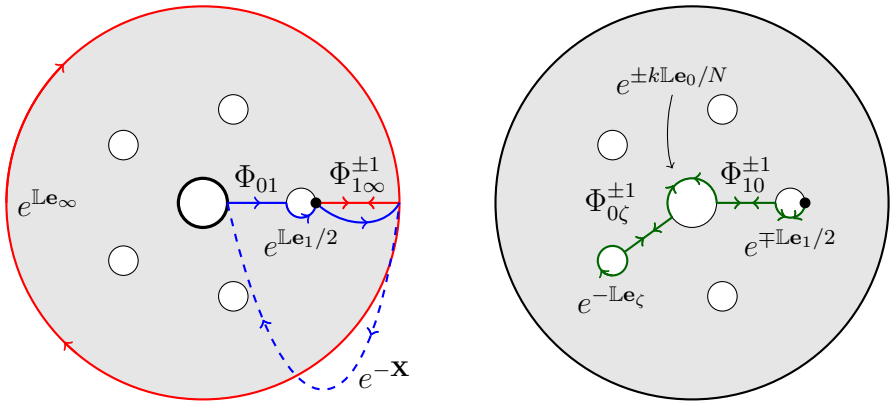


Figure 4.3: Inverse transport operators for α, β, and γ_ζ

Set $\xi = e^{2\pi i/N}$. Then

$$\Theta(\alpha') = e^{-\mathbb{L}\mathbf{e}_1/2}\Phi_{01}^{\mathfrak{m}} e^{-\mathbb{L}\mathbf{e}_0/N}(\Phi_{0\xi}^{\mathfrak{m}})^{-1} e^{-\mathbb{L}\mathbf{e}_\xi}\Phi_{0\xi}^{\mathfrak{m}} \cdots \\ \cdots e^{-\mathbb{L}\mathbf{e}_{\xi^{N-1}}}\Phi_{0\xi^{N-1}}^{\mathfrak{m}} e^{-\mathbb{L}\mathbf{e}_0/N}(\Phi_{01}^{\mathfrak{m}})^{-1} e^{-\mathbb{L}\mathbf{e}_1/2}.$$

Since the KZ_N connection is flat, we know $\operatorname{per}\Theta(\alpha) = \operatorname{per}\Theta(\alpha') \in \mathbb{C} \otimes A_{E'_{\vec{u}}}$. We shall call this the "star relation." It is a cyclotomic version of Drinfeld's "hexagon relation" [24].

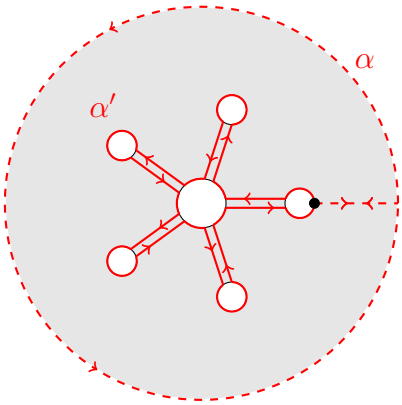


Figure 4.4: The star relation

Lemma 4.4.1. *The star relation is motivic, that is* $\Theta(\alpha) = \Theta(\alpha')$ *in* $\mathcal{P}_N^{\mathfrak{m}} \otimes A_{E'_{\vec{u}}}$.

Proof. The motivic comparison isomorphism $c_{\mathfrak{m}}$ (4.1.8) is natural with respect to morphisms

in MTM_N. Thus, by Theorem 4.2.2, we have the commutative diagram

$$\begin{array}{ccc} \pi_1(U_N,\vec{\mathrm{v}}_1)\times\pi_1(U_N,\vec{\mathrm{v}}_1) & \longrightarrow & \pi_1(U_N,\vec{\mathrm{v}}_1) \\ \Big\downarrow{\scriptstyle\Theta} & & \Big\downarrow{\scriptstyle\Theta} \\ \mathcal{P}_N^{\mathfrak{m}}\langle\langle \mathbf{e}_0,\mathbf{e}_\zeta\rangle\rangle\otimes\mathcal{P}_N^{\mathfrak{m}}\langle\langle \mathbf{e}_0,\mathbf{e}_\zeta\rangle\rangle & \longrightarrow & \mathcal{P}_N^{\mathfrak{m}}\langle\langle \mathbf{e}_0,\mathbf{e}_\zeta\rangle\rangle, \end{array}$$

where the top horizontal map is composition of loops and the bottom horizontal map is multiplication. Since $\alpha^{-1}\alpha'$ is homotopic to the constant path, the image of $\alpha^{-1}\alpha'$ in the bottom right must be 1. Hence, $\Theta(\alpha)=\Theta(\alpha')$. □

We also have a motivic "cylinder relation" involving the factor of automorphy $e^{-\mathbf{X}}$.

Lemma 4.4.2. *We have*

$$e^{\mathbf{X}}e^{\mathbb{L}\mathbf{e}_0}e^{-\mathbf{X}}e^{\mathbb{L}\mathbf{e}_\infty}=1$$

Proof. Apply the factor of automorphy $\widetilde{M}_{I,(1,0)}=e^{-\mathbf{X}}$ to identify the circle at ∞ to the circle at zero to observe

$$e^{\mathbf{X}}\mathbf{e}_0e^{-\mathbf{X}}=-\mathbf{e}_\infty.$$

Hain gives a computational proof of this in [37, Lemma 18.1]. Then multiply both sides by $\mathbb{L}$ and exponentiate. □

Corollary 4.4.3. *Inverse monodromy* $\Theta:\mathfrak{p}(E'_{\vec{\mathrm{u}}},\vec{\mathrm{v}}_1)\to\mathcal{P}_N^{\mathfrak{m}}\otimes A_{E'_{\vec{\mathrm{u}}}}$ *is well-defined.*

Proof. Apply the previous lemma to confirm

$$\Theta(\alpha\beta\alpha^{-1}\beta^{-1})=\Theta(\gamma_1\gamma_\xi\gamma_{\xi^2}\cdots\gamma_{\xi^{N-1}}).$$

□

Theorem 4.4.4. *The generalized Hodge realization*

$$\Big(\mathfrak{p}(E'_{\vec{\mathrm{u}}},\vec{\mathrm{v}}_1),A_{E'_{\vec{\mathrm{u}}}},\Theta\Big)$$

is the Hodge realization of a pro-object of MTM_N.

Proof. By Proposition 4.3.1, $\mathfrak{p}(U_N, \vec{\mathsf{v}}_1$ is an object of MTM_N. The corresponding object of Brown's category $\mathcal{H}$

$$\left(\mathfrak{p}(U_N, \vec{\mathsf{v}}_1), \mathbb{Q}\langle\langle \mathbf{e}_0, \mathbf{e}_\zeta \mid \zeta \in \boldsymbol{\mu}_N\rangle\rangle, \Theta^{\mathrm{KZ}}_{\vec{\mathsf{v}}_1,\vec{\mathsf{v}}_1}\right)$$

is its Hodge realization. The action of $\pi_1(\mathcal{H}, \omega^B_{\mathcal{H}})$ on $\mathfrak{p}(U_N, \vec{\mathsf{v}}_1)$ factors through $\mathcal{G}^B_N = \pi_1(\mathsf{MTM}_N, \omega^B)$. The Hain map

$$\Psi_N : \left(\mathfrak{p}(U_N, \vec{\mathsf{v}}_1), \mathbb{Q}\langle\langle \mathbf{e}_0, \mathbf{e}_\zeta \mid \zeta \in \boldsymbol{\mu}_N\rangle\rangle, \Theta^{\mathrm{KZ}}_{\vec{\mathsf{v}}_1,\vec{\mathsf{v}}_1}\right) \longmapsto \left(\mathfrak{p}(E'_{\vec{\mathsf{u}}}, \vec{\mathsf{v}}_1), A_{E'_{\vec{\mathsf{u}}}}, \Theta\right)$$

is a morphism in $\mathcal{H}$ and thus equivariant with respect to the action of $\pi_1(\mathcal{H}, \omega^B_{\mathcal{H}})$. Thus, the action of $\pi_1(\mathcal{H}, \omega^B_{\mathcal{H}})$ on the restriction of the image of Ψ_N in $\mathfrak{p}(E'_{\vec{\mathsf{u}}}, \vec{\mathsf{v}}_1)$ also factors through $\mathcal{G}^B_N$, the Betti fundamental group of MTM_N.

To see the action of $\pi_1(\mathcal{H}, \omega^B_{\mathcal{H}})$ on the entirety of $\mathfrak{p}(E'_{\vec{\mathsf{u}}}, \vec{\mathsf{v}}_1)$ factors through $\mathcal{G}^B_N$, it remains to show $\mathcal{G}^B_N$ respects the cylinder relation. However, we know this is the case since $\mathbf{e}_0$ and $\mathbf{e}_\infty$ each span copies of $\mathbb{Q}(1)$ and $\mathbf{X}$ spans a copy of $\mathbb{Q}(0)$ in $\mathfrak{p}(E'_{\vec{\mathsf{u}}}, \vec{\mathsf{v}}_1)$. Since $\mathcal{G}^B_N$ acts trivially on $\mathbb{Q}(0)$, the action respects the cylinder relation. Thus, there is a well-defined $\mathcal{G}^B_N$-action on $\mathfrak{p}(E'_{\vec{\mathsf{u}}}, \vec{\mathsf{w}}_1)$. Hence, $\mathfrak{p}(E'_{\vec{\mathsf{u}}}, \vec{\mathsf{w}}_1)$ is the Betti realization of an object of MTM_N. □

A priori, it is not obvious the Hain map should be $\pi_1(\mathcal{H}, \omega^B_{\mathcal{H}})$-equivariant. While Ψ_N certainly preserves the Hodge and weight filtrations, we have not shown it preserves rational structures. Since the Hain map is induced by the relationship between KZ_N and $\Gamma_1(N)$ KZB, this problem is equivalent to showing the MHS on $\mathfrak{p}(E'_{\vec{\mathsf{u}}}, \vec{\mathsf{v}}_1)$ in the previous theorem is the same as the canonical MHS induced by the $\Gamma_1(N)$ KZB connection. This follows from a more general fact about parallel transport with respect to linearized connections. See discussion in Appendix B.

Chapter 5

Representation of the motivic Lie algebra $\mathfrak{k}_N$

The results of the previous chapter prove the existence of representations ϕ_{cyc} and ϕ_{ell} of $\mathcal{G}_N^{\mathrm{DR}}$ that fit into a commutative diagram

$$\begin{array}{ccc} \mathcal{G}_N^{\mathrm{DR}} & \xrightarrow{\phi_{\mathrm{cyc}}} & \mathrm{Aut}\,\mathfrak{p}(U_N,\vec{\mathrm{v}}_1)^{\mathrm{DR}} \\ & \searrow^{\phi_{\mathrm{ell}}} & \downarrow \Psi_N \\ & & \mathrm{Aut}\,\mathfrak{p}(E'_{\vec{\mathrm{u}}},\vec{\mathrm{v}}_1)^{\mathrm{DR}}, \end{array}$$

where the vertical map is induced by the Hain map. Restricting to the unipotent radical $\mathcal{K}_N^{\mathrm{DR}}$ of $\mathcal{G}_N^{\mathrm{DR}}$ (4.1.6) induces representations of the motivic Lie algebra $\mathfrak{k}_N^{\mathrm{DR}}$.

$$\begin{array}{ccc} \mathfrak{k}_N^{\mathrm{DR}} & \xrightarrow{\phi_{\mathrm{cyc}}} & \mathrm{Der}\,\mathfrak{p}(U_N,\vec{\mathrm{v}}_1)^{\mathrm{DR}} \\ & \searrow^{\phi_{\mathrm{ell}}} & \downarrow \Psi_N \\ & & \mathrm{Der}\,\mathfrak{p}_N(E'_{\vec{\mathrm{u}}},\vec{\mathrm{v}}_1)^{\mathrm{DR}}. \end{array} \tag{5.0.1}$$

As mentioned in Remark 4.2.3, relations between motivic periods of MTM_N are closely related to relations in the associated depth graded of the Lie algebra $\mathfrak{k}_N^{\mathrm{DR}}$. In this chapter, we define this depth filtration and give an explicit formula up to "first order" for the representation ϕ_{ell} when N is a prime power.

5.1 Depth filtrations and polylogarithm quotients

5.1.1 Filtrations and compatibility

Fix a (possibly tangential) base point x of U_N. The inclusion $U_N \hookrightarrow \mathbb{G}_m$ induces a map $\mathfrak{r}_{\mathrm{cyc}} : \mathfrak{p}(U_N, x)^\bullet \to \mathfrak{p}(\mathbb{G}_m, x)^\bullet$ for $\bullet \in \{B, \mathrm{DR}\}$. Define the *depth filtration* on $\mathfrak{p}(U_N, x)^\bullet$ by

$$D^d\mathfrak{p}(U_N, x)^\bullet = \begin{cases} \mathfrak{p}(U_N, x)^\bullet & d = 0 \\ \ker \mathfrak{r}_{\mathrm{cyc}} & d = 1 \\ [D^1, D^{d-1}] & d > 1. \end{cases}$$

We may identify $\mathfrak{p}(U_N, x)^{\mathrm{DR}}$ with $\mathbf{L}(\mathbf{e}_0, \mathbf{e}_\zeta \mid \zeta \in \boldsymbol{\mu}_N)^\wedge$ and $\mathfrak{p}(\mathbb{G}_m, x)^{\mathrm{DR}}$ with $\mathbf{L}(\mathbf{e}_0)^\wedge$ via the KZ connection (4.3.1) and (4.2.2). Then $\mathfrak{r}(\mathbf{e}_0) = \mathbf{e}_0$ and $\mathfrak{r}(\mathbf{e}_\zeta) = 0$. Thus, $D^d\mathfrak{p}(U_N, w)^{\mathrm{DR}}$ is spanned by the Lie words with degree at least d in the $\mathbf{e}_\zeta$ terms.

There is an analogous filtration in the elliptic case. Let $E_q = \mathbb{G}_m/q^{\mathbb{Z}}$ be a smooth elliptic curve and let $E'_q = E_q - \boldsymbol{\mu}_N$. Fix a (possibly tangential) base point $x \in E'_q$. The inclusion $E'_q \hookrightarrow E_q$ induces a map $\mathfrak{r}_{\mathrm{ell}} : \mathfrak{p}(E'_q, x)^\bullet \to \mathfrak{p}(E_q, x)^\bullet$. The depth filtration on $\mathfrak{p}(E'_q, x)$ is then

$$D^d\mathfrak{p}(E'_q, x)^\bullet = \begin{cases} \mathfrak{p}(E'_q, x)^\bullet & d = 0 \\ \ker \mathfrak{r}_{\mathrm{ell}} & d = 1 \\ [D^1, D^{d-1}] & d > 1. \end{cases}$$

Use the KZB connection to identify $\mathfrak{p}(E'_q, x)^{\mathrm{DR}}$ with

$$\mathbf{L}(\mathbf{x}, \mathbf{y}, \mathbf{t}_\zeta)^\wedge / \left(\sum \mathbf{t}_\zeta = [\mathbf{x}, \mathbf{y}]\right)$$

and $\mathfrak{p}(E_q, x)^{\mathrm{DR}}$ with $\mathbf{L}(\mathbf{x}, \mathbf{y})^\wedge$. Then $\mathfrak{r}_{\mathrm{ell}}(\mathbf{x}) = \mathbf{x}$, $\mathfrak{r}_{\mathrm{ell}}(\mathbf{y}) = \mathbf{y}$, and $\mathfrak{r}_{\mathrm{ell}}(\mathbf{t}_\zeta) = 0$. Thus, $D^d\mathfrak{p}(E'_q, x)^{\mathrm{DR}}$ is spanned by the Lie words with degree at least d in the $\mathbf{t}_\zeta$ terms.

Lemma 5.1.1. *Suppose $\varphi : \mathfrak{f} \to \mathfrak{f}'$ is a homomorphism of free Lie algebras. If the induced map $\varphi_* : H_1(\mathfrak{f}) \to H_1(\mathfrak{f}')$ is injective, then φ is injective and strict with respect to the lower central series. That is,*

$$\varphi(L^j\mathfrak{f}') = \varphi(\mathfrak{f}) \cap L^j\mathfrak{f}'.$$

To prove the result, it suffices to show that $\mathrm{Gr}_L^\bullet \mathfrak{f} \to \mathrm{Gr}_L^\bullet \mathfrak{f}'$ *is injective.*

Proof. Since $\mathfrak{f}$ and $\mathfrak{f}'$ are free, there are a canonical isomorphisms $\mathrm{Gr}_L^\bullet \mathfrak{f} \cong \mathbf{L}(H_1(\mathfrak{f}))$ and $\mathrm{Gr}_L^\bullet \mathfrak{f}' \cong \mathbf{L}(H_1(\mathfrak{f}'))$ [47]. We then have the commutative diagram

$$\begin{array}{ccc} \mathrm{Gr}_L^\bullet \mathfrak{f} & \xrightarrow{\mathrm{Gr}_L^\bullet \varphi} & \mathrm{Gr}_L^\bullet \mathfrak{f}' \\ \downarrow & & \downarrow \\ \mathbf{L}(H_1(\mathfrak{f})) & \xrightarrow{\varphi_*} & \mathbf{L}(H_1(\mathfrak{f}')). \end{array}$$

The result follows immediately. □

Proposition 5.1.2. *The depth filtration on* $\mathfrak{p}(U_N, x)$ *(resp.* $\mathfrak{p}(E'_q, x)$*) induces filtrations* $D^\bullet$ *on the Betti local system* $\boldsymbol{\mathcal{P}}_N^{\mathrm{KZ,top}}$ *(resp.* $\boldsymbol{\mathcal{P}}_{\Gamma_1(N)}^{\mathrm{top}}$*) and de Rham holomorphic vector bundle* $\boldsymbol{\mathcal{P}}_N^{\mathrm{KZ}}$ *(resp.* $\boldsymbol{\mathcal{P}}_{\Gamma_1(N)}$*) with flat connection* ∇_{KZ_N} *(resp.* $\nabla_{\mathrm{KZB}_{\Gamma_1(N)}}$*). The subbundles* $D^d\boldsymbol{\mathcal{P}}_N^{\mathrm{KZ,top}}$ *and* $D^d\boldsymbol{\mathcal{P}}_N^{\mathrm{KZ}}$ *(resp.* $D^d\boldsymbol{\mathcal{P}}_{\Gamma_1(N)}^{\mathrm{top}}$ *and* $D^d\boldsymbol{\mathcal{P}}_{\Gamma_1(N)}$*) are isomorphic as flat vector bundles.*

Proof. The Lie algebras $\mathrm{Gr}_\bullet^W \mathfrak{p}_N$ and $\mathrm{Gr}_\bullet^W \mathfrak{p}_N^{\mathrm{KZ}}$ are free and thus by the Shirishov–Witt theorem so are $D^1 \mathrm{Gr}_\bullet^W \mathfrak{p}_N$ and $D^1 \mathrm{Gr}_\bullet^W \mathfrak{p}_N^{\mathrm{KZ}}$. Both the KZ and KZB connections take values in D^0F^{-1} and thus transport along any path is injective on $\mathrm{Gr}_D^1 \mathrm{Gr}_\bullet^W$. Since $D^d = L^d(D^1)$, the result follows from the lemma and taking inverse limits with respect to the weight filtration. □

The choice of notation $D^\bullet$ is intentionally ambiguous since the filtrations are compatible with the respect to the Hain map.

Lemma 5.1.3. *The Hain map induces an injection*

$$H_1(\mathrm{Gr}_\bullet^W D^1\mathfrak{p}(U_N, \vec{\mathrm{v}}_1)^{\mathrm{DR}}) \to H_1(\mathrm{Gr}_\bullet^W D^1\mathfrak{p}(E'_{\vec{\mathrm{u}}}, \vec{\mathrm{v}}_1)^{\mathrm{DR}}).$$

Proof. Observe

$$H_1(\mathrm{Gr}_\bullet^W D^1\mathfrak{p}(U_N, \vec{\mathrm{v}}_1)^{\mathrm{DR}}) = \bigoplus_{\substack{n\geq 0 \\ \zeta\in\boldsymbol{\mu}_N}} \mathbb{Q}\mathbf{e}_0^n \cdot \mathbf{e}_\zeta$$

and

$$H_1(\mathrm{Gr}^W_\bullet\, D^1\mathfrak{p}(E'_{\vec{\mathrm{u}}},\vec{\mathrm{v}}_1)^{\mathrm{DR}}) = \bigoplus_{\substack{n\geq 0\\ \zeta\in\boldsymbol{\mu}_N}} S^m H\cdot \mathbf{t}_\zeta.$$

The Hain map sends $\Psi_N : \mathbf{e}_0^n\cdot\mathbf{e}_\zeta \mapsto \mathbf{Y}^n\cdot\mathbf{t}_\zeta \bmod D^2\mathfrak{p}(E'_{\vec{\mathrm{u}}},\vec{\mathrm{v}}_1)^{\mathrm{DR}}$. This is clearly injective. □

Corollary 5.1.4. *The Hain map is strict with respect to the depth filtrations.*

Proof. It follows from the Lemmas 5.1.1 and 5.1.3 that

$$\mathrm{Gr}^W_\bullet\,\Psi_N : \mathrm{Gr}^W_\bullet\,\mathfrak{p}(U_N,\vec{\mathrm{v}}_1)^{\mathrm{DR}} \to \mathrm{Gr}^W_\bullet\,\mathfrak{p}(E'_{\vec{\mathrm{u}}},\vec{\mathrm{v}}_1)^{\mathrm{DR}}$$

is strict with respect to $D^\bullet$. Apply inverse limits with respect to the weight filtration for the result. □

The depth filtration $D^\bullet$ pulls back along ϕ_{cyc} and ϕ_{ell} in (5.0.1) to induce a depth filtration on $\mathfrak{k}_N$.

5.1.2 Polylog variations

Here we define the elliptic and cyclotomic polylog VMHS over $\mathcal{E}'_{\Gamma_1(N)}$ and U_N, respectively. In the next section, we will use the Hain map to relate their limit MHS at $\vec{\mathrm{u}}+\vec{\mathrm{v}}_1$.

Elliptic case

The *elliptic polylog variation* $\mathbf{Pol}_{\mathrm{ell},N}$ over $\mathcal{E}'_{\Gamma_1(N)}$ is the quotient $\boldsymbol{\mathcal{P}}_{\Gamma_1(N)}/D^2\boldsymbol{\mathcal{P}}_{\Gamma_1(N)}$. We have the obvious short exact sequence of VMHS

$$0 \longrightarrow \mathrm{Gr}^1_D\,\boldsymbol{\mathcal{P}}_{\Gamma_1(N)} \longrightarrow \mathbf{Pol}_{\mathrm{ell},N} \longrightarrow \mathrm{Gr}^0_D\,\boldsymbol{\mathcal{P}}_{\Gamma_1(N)} \longrightarrow 0.$$

The KZB connection modulo D^1 is simply the connection (2.3.2) on $\mathbb{H}$. Thus,

$$\mathrm{Gr}^0_D\,\boldsymbol{\mathcal{P}}_{\Gamma_1(N)} = \mathrm{Gr}^W_{-1}\,\boldsymbol{\mathcal{P}}_{\Gamma_1(N)} = \mathbb{H}$$

and

$$\mathrm{Gr}^W_\bullet\,\mathrm{Gr}^1_D\,\boldsymbol{\mathcal{P}}_{\Gamma_1(N)} = \mathrm{Gr}^W_\bullet \bigoplus_{\zeta\in\boldsymbol{\mu}_N} \mathrm{Sym}\,\mathbb{H}\cdot\mathbf{t}_\zeta \cong \mathrm{Gr}^W_\bullet \bigoplus_{\zeta\in\boldsymbol{\mu}_N} (\mathrm{Sym}\,\mathbb{H})(1)$$

since each $\mathbf{t}_\zeta$ generates a constant variation MHS $\mathbb{Q}(1)$ modulo D^2. Thus, the polylog variation $\mathbf{Pol}_{\mathrm{ell},N}$ is a direct sum of extensions of $\mathbb{H}$ by symmetric powers $S^m\mathbb{H}(1)$. This is a level N or cyclotomic generalization of Beilinson and Levin's elliptic polylog variation [4].

Cyclotomic case

The *cyclotomic polylog variation* is defined analogously except with the KZ_N variation of MHS. Define $\mathbf{Pol}_{\mathrm{cyc}_N}$ to be the quotient $\boldsymbol{\mathcal{P}}_N^{\mathrm{KZ}}/D^2\boldsymbol{\mathcal{P}}_N^{\mathrm{KZ}}$. As in the elliptic case, there is short exact sequence

$$0 \longrightarrow \mathrm{Gr}_D^1 \boldsymbol{\mathcal{P}}_N^{\mathrm{KZ}} \longrightarrow \mathbf{Pol}_{\mathrm{cyc}_N} \longrightarrow \mathrm{Gr}_D^0 \boldsymbol{\mathcal{P}}_N^{\mathrm{KZ}} \longrightarrow 0.$$

We know the holomorphic vector bundle associated to the kernel $\mathrm{Gr}_D^1 \boldsymbol{\mathcal{P}}_N^{\mathrm{KZ}} \otimes \mathcal{O}_{U_N}$ is spanned by $\mathbf{e}_0^m \cdot \mathbf{e}_\zeta$. In the next section, we will show $\mathrm{Gr}_D^1 \boldsymbol{\mathcal{P}}_N^{\mathrm{KZ}}$ in fact splits as a MHS

$$\mathrm{Gr}_\bullet^M \mathrm{Gr}_D^1 \boldsymbol{\mathcal{P}}_N^{\mathrm{KZ}} \cong \mathrm{Gr}_\bullet^M \bigoplus_{\substack{m\geq 0 \\ \zeta \in \boldsymbol{\mu}_N}} \mathbb{Q}(m+1)^{\mathrm{DR}},$$

where $\mathbf{e}_0^m \cdot \mathbf{e}_\zeta$ each span a copy of $\mathbb{Q}(m+1)^{\mathrm{DR}}$. Meanwhile, the quotient $\mathrm{Gr}_D^0 \boldsymbol{\mathcal{P}}_N^{\mathrm{KZ}}$ is spanned by the class of $\mathbf{e}_0$, hence $\mathrm{Gr}_D^0 \boldsymbol{\mathcal{P}}_N^{\mathrm{KZ}}$ is isomorphic to the constant VMHS $\mathbb{Q}(1)$ over U_N.

Remark 5.1.5. One can also obtain the variation $\mathbf{Pol}_{\mathrm{cyc}_N}$ by restricting $\mathbf{Pol}_{\mathrm{ell}_N}$ to a neighborhood of the cusp $q = 0$ and taking monodromy invariants. We know from (3.6.1) that the residue of the connection on $\mathbf{Pol}_{\mathrm{ell}_N}$ is $\mathbf{Y}\partial/\partial\mathbf{X}$, and thus monodromy around the cusp is given by $\exp(-\mathbf{Y}\partial/\partial\mathbf{X})$. The invariants are generated by $\mathbf{Y}$ and $\mathbf{t}_\zeta$, which is precisely the image of $\mathbf{Pol}_{\mathrm{cyc}_N}$ under the Hain map.

5.2 Limit MHS of polylogarithm variations

To compute the action (5.0.1) of $\mathfrak{k}_N$, we must know the limit MHS of the cyclotomic and elliptic polylogarithm variations at $\vec{\mathsf{v}}_1$. We proceed by first computing the limit MHS of the cyclotomic polylog at $\vec{\mathsf{w}}_1$ and then use parallel transport of the KZ_N connection along dch to compute the limit MHS of the cyclotomic polylog at $\vec{\mathsf{v}}_1$.

5.2.1 Preliminaries

If X is a topological space, denote the free abelian group on the set of homotopy classes of paths in X from x to y by $H_0(P_{x,y}X)$. These groups form a local system

$$\{H_0(P_{x,y}X)\}_{(x,y)} \to X \times X. \tag{5.2.1}$$

If $x = y$, then $H_0(P_{x,y}X)$ is canonically isomorphic to $\mathbb{Q}\pi_1(X, x)$. Powers of the augmentation ideal I of $\mathbb{Q}\pi_1(X, x)$ define a filtration on $H_0(P_{x,y}X)$ which extends to a flat filtration of the local system (5.2.1). Each fiber has a completion $H_0(P_{x,y}X, \mathbb{Q})^\wedge$ in the I-adic topology. The corresponding variation of completed groups

$$\{H_0(P_{x,y}X, \mathbb{Q})^\wedge\}_{(x,y)} \to X \times X$$

is an admissible variation of MHS whose fiber over (x, x) is canonically isomorphic to the MHS on $\mathbb{Q}\pi_1(X, x)^\wedge$ [40].

5.2.2 The limit MHS of $\mathbf{Pol}_{\mathrm{cyc}_N}$ at $\vec{\mathsf{w}}_1$

The variation (5.2.1) can be described explicitly when $X = U_N$. Restrict the variation $\{H_0(P_{x,y}U_N, \mathbb{Q})^\wedge\}_{(x,y)}$ to $U_N \times \{\vec{\mathsf{w}}_1\}$

$$\{H_0(P_{w,\vec{\mathsf{w}}_1}U_N, \mathbb{Q})^\wedge\}_{w \in U_N} \to U_N.$$

By (4.2.2), the restricted variation is isomorphic, as a flat vector bundle, to

$$U_N \times \mathbb{C}\langle\langle \mathbf{e}_0, \mathbf{e}_\zeta \mid \zeta \in \boldsymbol{\mu}_N \rangle\rangle \to U_N$$

with connection $\nabla = d + \mathbf{e}_0\omega_0 + \sum \mathbf{e}_\zeta\omega_\zeta$, where each $\mathbf{e}_r$ acts via left multiplication. This is a cyclotomic version of the classical polylogarithm variation [33].

Recall the augmentation ideal J of $\mathbb{C}\langle\langle \mathbf{e}_0, \mathbf{e}_\zeta \mid \zeta \in \boldsymbol{\mu}_N \rangle\rangle$. Define the left ideal $\mathcal{J}$ of $H_0(P_{\vec{\mathsf{w}}_1,\vec{\mathsf{w}}_1}U_N)^\wedge$ generated by $\{J\mathbf{e}_0, \mathbf{e}_\zeta J \mid \zeta \in \boldsymbol{\mu}_N\}$. Set $\mathcal{J}_m = \mathcal{J} + J^m$. Then the quotient $J/\mathcal{J}_m$ has basis

$$\big\{\mathbf{e}_0, \mathbf{e}_\zeta, \mathbf{e}_0\mathbf{e}_\zeta, \ldots, \mathbf{e}_0^{m-1}\mathbf{e}_\zeta \mid \zeta \in \boldsymbol{\mu}_N\big\}.$$

Let $\mathbb{J}_{N,m}$ be the variation over U_N given by

$$U_N \times J/\mathcal{J}_m \to U_N.$$

Denote the inverse limit of the $\mathbb{J}_{N,m}$ by $\mathbb{J}_N$. It is simply the variation

$$U_N \times J/\mathcal{J} \to U_N.$$

(As written, $\mathbb{J}_N$ and $\mathbb{J}_{N,m}$ are only holomorphic vectors bundles with flat connection and Hodge and weight filtrations. In order to view them as variations, we consider the rational structure induced by Chen's transport formula as in §4.2.2.)

Proposition 5.2.1. *There is an isomorphism of variations of MHS* $\mathbf{Pol}_{\mathrm{cyc}_N} \to \mathbb{J}_N$ *given by the expected formula*

$$\mathbf{e}_0 \longmapsto \mathbf{e}_0$$
$$\mathbf{e}_0^n \cdot \mathbf{e}_\zeta \longmapsto \mathbf{e}_0^n \mathbf{e}_\zeta.$$

Proof. The variations $\mathbf{Pol}_{\mathrm{cyc}_N}$ and $\mathbb{J}_N$ are both sub-quotients of $\{H_0(P_{w,\vec{\mathbf{w}}_1} U_N, \mathbb{Q})^\wedge\}_{w \in U_N}$. Moreover, we know

$$\mathbf{e}_0^n \cdot \mathbf{e}_\zeta \equiv \mathbf{e}_0 \mathbf{e}_\zeta \bmod \mathcal{J}.$$

Thus, the map preserves the connection and the rational structure. □

Next, for each $\zeta \in \boldsymbol{\mu}_N$, let $U_\zeta := \mathbb{G}_m - \{\zeta\}$. Define J_ζ to be the augmentation ideal of $\mathbb{C}\langle\langle \mathbf{e}_0, \mathbf{e}_\zeta \rangle\rangle$ viewed as a subalgebra of $\mathbb{C}\langle\langle \mathbf{e}_0, \mathbf{e}_\zeta : \zeta \in \boldsymbol{\mu}_N \rangle\rangle$. Define left ideals

$$\mathcal{J}_\zeta = \mathcal{J} \cap \mathbb{C}\langle\langle \mathbf{e}_0, \mathbf{e}_\zeta \rangle\rangle \quad \text{and} \quad \mathcal{J}_{m,\zeta} = \mathcal{J}_m \cap \mathbb{C}\langle\langle \mathbf{e}_0, \mathbf{e}_\zeta \rangle\rangle.$$

Let $\mathbb{J}^\zeta_{N,m}$ be the variation over U_ζ given by

$$U_\zeta \times (J_\zeta/\mathcal{J}_{m,\zeta}) \to U_\zeta.$$

The rational structure of $\mathbb{J}^\zeta_{N,m}$ is pulled back from $\mathbb{J}_{1,m} = \mathbb{J}^1_{N,m}$ along the map $z \mapsto z/\zeta$. Thus, $\mathbb{J}^\zeta_{N,m}$ is isomorphic to the trivial $\mathbb{C}\langle\langle \mathbf{e}_0, \mathbf{e}_\zeta \rangle\rangle$-bundle over U_ζ with connection $\nabla = d + \mathbf{e}_0\omega_0 + \mathbf{e}_\zeta\omega_\zeta$.

The inclusion $U_\zeta \hookrightarrow \mathbb{G}_m$ induces a surjection at $\vec{\mathsf{w}}_1$

$$\mathbb{Q}\pi_1(U_\zeta, \vec{\mathsf{w}}_1)^\wedge \to \mathbb{Q}\pi_1(\mathbb{G}_m, \vec{\mathsf{w}}_1)^\wedge.$$

This induces a surjection on the quotients

$$\Upsilon^\zeta_{N,m} : \mathbb{J}^\zeta_{N,m}\big|_{\vec{\mathsf{w}}_1} \to \frac{J \cap \mathbb{C}\langle\langle \mathbf{e}_0 \rangle\rangle}{\mathcal{J}_m \cap \mathbb{C}\langle\langle \mathbf{e}_0 \rangle\rangle} \cong \mathbb{C}\mathbf{e}_0.$$

The inclusion $\mathbb{D}^* \hookrightarrow U_\zeta$ induces a splitting of map $\Upsilon^\zeta_{N,m}$ above and thus the limit MHS at $\vec{\mathsf{w}}_1$ splits as

$$\mathbb{J}^\zeta_{N,m}\big|_{\vec{\mathsf{w}}_1} \cong \mathbb{Q}(1) \oplus \ker \Upsilon^\zeta_{N,m}\big|_{\vec{\mathsf{w}}_1}, \tag{5.2.2}$$

where the de Rham realization of the $\mathbb{Q}(1)$ summand is spanned by $\mathbf{e}_0$. The other summand $\ker \Upsilon^\zeta_{N,m}$ is spanned by $\{\mathbf{e}_\zeta, \mathbf{e}_0\mathbf{e}_\zeta, \dots, \mathbf{e}^{m-1}\mathbf{e}_\zeta\}$. The connection on $\ker \Upsilon^\zeta_{N,m}$ is $\nabla = d + \mathbf{e}_0\omega_0$ since $\mathbf{e}_\zeta$ acts trivially.

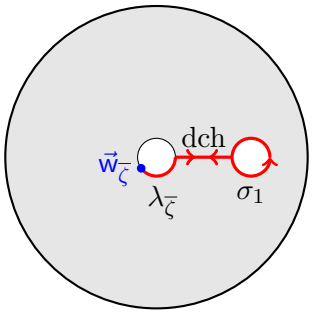
Lemma 5.2.2. *The limit MHS on* $\ker \Upsilon^\zeta_{N,m}$ *splits completely at* $\vec{\mathsf{w}}_1$ *and* $\vec{\mathsf{v}}_1$ *as* $\mathbb{Q}(1) \oplus \mathbb{Q}(2) \oplus \cdots \oplus \mathbb{Q}(m-1)$*, where the de Rham generator of the copy of* $\mathbb{Q}(n)$ *summand is* $\mathbf{e}_0^{n-1}\mathbf{e}_\zeta$*.*

Proof. The kernel of $\Upsilon^\zeta_{N,m}$ and the mth symmetric power of the logarithm variation are isomorphic as holomorphic vector bundles with flat connection. Thus, if the limit MHS of $\ker \Upsilon^\zeta_{N,m}$ splits at $\vec{\mathsf{w}}_1$, it does at $\vec{\mathsf{v}}_1$ as well (see Example 2.7.4).

To show the limit MHS splits at $\vec{\mathsf{w}}_1$, first observe

$$H_0(P_{w,\vec{\mathsf{w}}_1} U_\zeta, \mathbb{Q})^\wedge \cong H_0(P_{w\bar{\zeta},\vec{\mathsf{w}}_{\bar{\zeta}}} U_1, \mathbb{Q})^\wedge$$

via the change of variables $w \mapsto w/\zeta$ and $\mathbf{e}_\zeta \mapsto \mathbf{e}_1$. Let γ be the loop in U_1 based at $\vec{\mathsf{w}}_\zeta$ which travels counterclockwise to $\vec{\mathsf{w}}_1$, via dch to $-\vec{\mathsf{v}}_1$, around 1 counterclockwise, and back to to $\vec{\mathsf{w}}_{\bar{\zeta}}$.



For convenience, denote the path from $\vec{\mathsf{w}}_{\overline{\zeta}}$ to $\vec{\mathsf{w}}_1$ by $\lambda_{\overline{\zeta}}$ and the loop about 1 by σ_1. We will use Chen's transport formula (4.2.2) along γ to compute an element of the rational structure of $H_0(P_{\vec{\mathsf{w}}_{\overline{\zeta}},\vec{\mathsf{w}}_{\overline{\zeta}}}U_1,\mathbb{Q})^\wedge$ which descends to a rational vector in $J/\mathcal{J}^1_{N,m}$. We first compute some *regularized* iterated integrals needed in the proof. These are

1. $\displaystyle\int_{\lambda_{\overline{\zeta}}} \underbrace{\omega_0\cdots\omega_0}_{n} = \frac{1}{n!}\frac{(2\pi i)^n}{k^n}$ where $0\le k<N$ and $\zeta = e^{2\pi i k/N}$,
2. $\displaystyle\int_{\mathrm{dch}} \underbrace{\omega_0\cdots\omega_0}_{n} = \frac{1}{n!}\left(\int_{\mathrm{dch}}\omega_0\right)^n = 0$,
3. $\displaystyle\int_{\sigma_1}\omega_{r_1}\cdots\omega_{r_n} = \frac{1}{n!}\prod_{k=1}^{n}\int_{\sigma_1}\omega_{r_k} = 0$ unless $r_k=1$ for all k,
4. $\displaystyle\int_{\sigma_1}\omega_1 = 2\pi i$.

Chen's transport along the loop based at $\vec{\mathsf{w}}_1$ yields

$$\begin{aligned}
&\Theta_{\vec{\mathsf{w}}_1}(\mathrm{dch}\cdot\sigma_1\cdot\mathrm{dch}^{-1})\\
&\quad\equiv 1+\sum_{\substack{n\ge1\\ r_s\in\{0,1\}}}\mathbf{e}_{r_1}\cdots\mathbf{e}_{r_n}\int_\gamma\omega_{r_1}\cdots\omega_{r_n}\\
&\quad\equiv 1+\sum_{\substack{n\ge1\\ j+k+\ell=n\\ r_s\in\{0,1\}}}\mathbf{e}_{r_1}\cdots\mathbf{e}_{r_n}\int_{\mathrm{dch}}\omega_1\cdots\omega_{r_j}\int_{\sigma_1}\omega_{r_{j+1}}\cdots\omega_{r_k}\int_{\mathrm{dch}^{-1}}\omega_{r_{k+1}}\cdots\omega_{r_\ell}\\
&\quad\equiv 1+2\pi i\mathbf{e}_1 \bmod \mathcal{J}^1_{N,m}
\end{aligned}$$

since every product vanishes except when $j=\ell=0$, $k=1$, and $r_1=1$. It follows that

$$\begin{aligned}
\Theta_{\vec{\mathsf{w}}_{\overline{\zeta}}}(\gamma) &\equiv \left(1+\sum_{n\ge1}\int_{\lambda_{\overline{\zeta}}}\underbrace{\omega_0\cdots\omega_0}_{n}\right)(1+2\pi i\mathbf{e}_1)\left(1+\sum_{n\ge1}\int_{\lambda_{\overline{\zeta}}^{-1}}\underbrace{\omega_0\cdots\omega_0}_{n}\right)\\
&\equiv 1+\sum_{n=0}^{m-2}\frac{1}{n!}\frac{(2\pi i)^{n+1}}{k^n}\mathbf{e}_0^n\mathbf{e}_1 \bmod \mathcal{J}^1_{N,m}.
\end{aligned}$$

Let $1_{\vec{\mathsf{w}}_{\overline{\zeta}}}$ be the constant path at $\vec{\mathsf{w}}_{\overline{\zeta}}$. Then

$$\Theta_{\vec{\mathsf{w}}_{\overline{\zeta}}}(\gamma-1_{\vec{\mathsf{w}}_{\overline{\zeta}}}) \equiv \sum_{n=0}^{m-2}\frac{1}{n!}\frac{(2\pi i)^{n+1}}{k^n}\mathbf{e}_0^n\mathbf{e}_1 \bmod \mathcal{J}^1_{N,m}$$

is a $\mathbb{Q}$-Betti element of $\ker \Upsilon^{\zeta}_{N,m}$. We also know from (5.2.2) that $2\pi i\mathbf{e}_0$ is a $\mathbb{Q}$-Betti element. Thus, all products of $2\pi i\mathbf{e}_0$ and $\Theta_{\vec{\mathsf{w}}_{\overline{\zeta}}}(\gamma)$ are also $\mathbb{Q}$-Betti. In particular, $(2\pi i)^{n+1}\mathbf{e}_0^n\mathbf{e}_1$ is a $\mathbb{Q}$-Betti element of $\ker \Upsilon^{\zeta}_{N,m}$ for $n = 0, \ldots, (m-2)$. □

Corollary 5.2.3. *The limit MHS of* $\mathbf{Pol}_{\mathrm{cyc}_N}$ *at* $\vec{\mathsf{w}}_1$ *splits completely as*

$$\mathbf{Pol}_{\mathrm{cyc}_N}\big|_{\vec{\mathsf{w}}_1} \cong \mathbb{Q}(0) \oplus \bigoplus_{\substack{n\geq 0\\ \zeta\in\boldsymbol{\mu}_N}} \mathbb{Q}(n+1),$$

where $\mathbf{e}_0$ *spans* $\mathbb{Q}(0)^{\mathrm{DR}}$ *and each* $\mathbf{e}_0^n \cdot \mathbf{e}_\zeta$ *spans a copy of* $\mathbb{Q}(n+1)^{\mathrm{DR}}$.

Proof. The inclusions $U_N \hookrightarrow U_\zeta$ induce a morphism of variations of MHS $\mathbb{J}_{N,m} \to \mathbb{J}^{\zeta}_{N,m}$ for each $\zeta \in \boldsymbol{\mu}_N$. The inclusion $U_N \hookrightarrow \mathbb{G}_m$ induces a split surjection

$$\Upsilon_{N,m} : \mathbb{J}_{N,m}\big|_{\vec{\mathsf{w}}_1} \to \mathbb{Q}(1).$$

The variation $\mathbb{J}_{N,m}$ is the pullback

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \displaystyle\bigoplus_{\zeta\in\boldsymbol{\mu}_N} \ker \Upsilon^{\zeta}_{N,m} & \longrightarrow & \displaystyle\bigoplus_{\zeta\in\boldsymbol{\mu}_N} \mathbb{J}^{\zeta}_{N,m} & \longrightarrow & \displaystyle\bigoplus_{\zeta\in\boldsymbol{\mu}_N} \mathbb{Q}(1) & \longrightarrow & 0\\
 & & \| & & \uparrow & & \uparrow{\scriptstyle \Delta} & & \\
0 & \longrightarrow & \ker \Upsilon_{N,m} & \longrightarrow & \mathbb{J}_{N,m} & \longrightarrow & \mathbb{Q}(1) & \longrightarrow & 0,
\end{array}$$

where Δ is the diagonal map. After taking inverse limits, it follows from (5.2.2) and the previous lemma that the top row (and hence the bottom row) splits. The result follows from Proposition 5.2.1. □

5.2.3 The limit MHS of $\mathbf{Pol}_{\mathrm{cyc}_N}$ at $\vec{\mathsf{v}}_1$

The results of the previous section imply that the fiber $\mathfrak{p}(U_N, \vec{\mathsf{w}}_1)/D^2$ of the polylog variation of MHS at $\vec{\mathsf{w}}_1$ splits as a MHS:

$$\mathfrak{p}(U_N, \vec{\mathsf{w}}_1)/D^2 \cong \mathbb{Q}(1) \oplus \bigoplus_{\substack{m\geq 0\\ \zeta\in\boldsymbol{\mu}_N}} \mathbb{Q}(m+1)$$

with de Rham basis $\mathbf{e}_0$ and $\mathbf{e}_0^m \cdot \mathbf{e}_\zeta$ and Betti basis $2\pi i\mathbf{e}_0$ and $(2\pi i)^{m+1}\mathbf{e}_0 \cdot \mathbf{e}_\zeta$. We may now compute the limit MHS of $\mathbf{Pol}_{\mathrm{cyc}_N}$ at $\vec{\mathsf{v}}_1$ by transporting the $\mathbb{Q}$-Betti structure of the

limit MHS at $\vec{\mathsf{w}}_1$ to $\vec{\mathsf{v}}_1$. We first transport along the straight line path dch to $-\vec{\mathsf{v}}_1$ and then around the puncture at $1 \in \mathbb{G}_m$ to $\vec{\mathsf{v}}_1$.

At $\vec{\mathsf{v}}_1$, the limit MHS has de Rham basis $\mathbf{e}_0$ and $\mathbf{e}_0^m \cdot \mathbf{e}_\zeta$. The parallel transport from $\vec{\mathsf{v}}_1$ to $\vec{\mathsf{w}}_1$ is given by left multiplication of the regularized transport $T(\mathrm{dch})$ of the KZ_N connection (4.3.1) along dch. We have:

$$\begin{aligned} T(\mathrm{dch}) &\equiv T(\mathrm{dch}^{-1})^{-1} \\ &\equiv 1 + \sum_{\substack{m\geq 1 \\ \zeta\in\boldsymbol{\mu}_N}} \mathbf{e}_0^m \cdot \mathbf{e}_\zeta \int_{\mathrm{dch}^{-1}} \underbrace{\omega_0 \cdots \omega_0}_{m} \omega_\zeta \\ &\equiv 1 + \sum_{\substack{m\geq 1 \\ \zeta\in\boldsymbol{\mu}_N}} (-1)^{m+1} \mathbf{e}_0^m \cdot \mathbf{e}_\zeta \int_{\mathrm{dch}} \omega_\zeta \underbrace{\omega_0 \cdots \omega_0}_{m} \\ &\equiv 1 + \sum_{\substack{m\geq 1 \\ \zeta\in\boldsymbol{\mu}_N}} (-1)^{m} \mathrm{Li}_{m+1}(\overline{\zeta}) \mathbf{e}_0^m \cdot \mathbf{e}_\zeta \end{aligned}$$

which acts via the left adjoint action. The third and fourth lines follow from (2.7.1) and (2.7.4), respectively. Applying $T(\mathrm{dch})$ to the $\mathbb{Q}$-Betti basis of the limit MHS at $\vec{\mathsf{w}}_1$ yields the $\mathbb{Q}$-Betti basis of the limit MHS at $-\vec{\mathsf{v}}_1$:

$$T(\mathrm{dch}) : \begin{cases} (2\pi i)^{n+1} \mathbf{e}_0^n \cdot \mathbf{e}_\zeta & \longmapsto & (2\pi i)^{n+1} \mathbf{e}_0^n \cdot \mathbf{e}_\zeta \\ \\ 2\pi i \mathbf{e}_0 & \longmapsto & 2\pi i \mathbf{e}_0 + 2\pi i \displaystyle\sum_{\substack{m\geq 2 \\ \zeta\in\boldsymbol{\mu}_N}} (-1)^m \mathrm{Li}_m(\overline{\zeta}) \mathbf{e}_0^m \cdot \mathbf{e}_\zeta \end{cases}$$

Finally, transport from $-\vec{\mathsf{v}}_1$ to $\vec{\mathsf{v}}_1$ is simply $\exp(2\pi i \mathbf{e}_1/2) \equiv 1 + \pi i \mathbf{e}_1 \bmod D^2$. Applying this to the image of $T(\mathrm{dch})$ yields a $\mathbb{Q}$-Betti basis of $\mathfrak{p}(U_N, \vec{\mathsf{v}}_1)$ given in the following table.

Weight	$\mathbb{Q}$-Betti generators
-2	$2\pi i \mathbf{e}_0 - \dfrac{(2\pi i)^2}{2} \mathbf{e}_0 \cdot \mathbf{e}_1 + 2\pi i \displaystyle\sum_{\substack{m\geq 2 \\ \zeta\in\boldsymbol{\mu}_N}} (-1)^m \mathrm{Li}_m(\overline{\zeta})\, \mathbf{e}_0^m \cdot \mathbf{e}_\zeta$
$-2m-2$	$(2\pi i)^{m+1} \mathbf{e}_0^m \cdot \mathbf{e}_\zeta$

For later use, we define a slightly different basis. We first need a lemma.

Lemma 5.2.4. *If $m \geq 2$ and $\zeta \in \boldsymbol{\mu}_N$, then*

$$\mathrm{Li}_m(\zeta) + (-1)^m \mathrm{Li}_m(\overline{\zeta}) \in (2\pi i)^m \mathbb{Q}$$

Proof. The mth Bernoulli polynomial $B_m(x)$ has Fourier expansion

$$B_m(x) = -\frac{m!}{(2\pi i)^m} \sum_{k \neq 0} \frac{e^{2\pi i k x}}{k^m}.$$

Thus,

$$\begin{aligned} -\frac{(2\pi i)^m}{m!} B_m(x) &= \sum_{k<0} \frac{(e^{2\pi i x})^k}{k^m} + \sum_{k>0} \frac{(e^{2\pi i x})^k}{k^m} \\ &= \sum_{k>0} \frac{(e^{-2\pi i x})^k}{(-k)^m} + \sum_{k>0} \frac{(e^{2\pi i x})^k}{k^m} \\ &= (-1)^m \mathrm{Li}_m(e^{-2\pi i k x}) + \mathrm{Li}_m(e^{2\pi i k x}). \end{aligned}$$

Set $x = \frac{1}{2\pi i} \log \zeta$ with choice of logarithm such that $x \in [0, 1)$. The result follows from the fact both x and the coefficients of the polynomial $B_m(x)$ are in $\mathbb{Q}$. □

Applying the lemma, we can write down another a simpler $\mathbb{Q}$-Betti basis of $\mathfrak{p}(U_N, \vec{\mathsf{w}}_1)$. It will streamline computations in the later sections.

Weight	$\mathbb{Q}$-Betti generators
-2	$2\pi i \mathbf{e}_0 - 2\pi i \sum_{\substack{m \geq 2 \\ \zeta \in \boldsymbol{\mu}_N}} \mathrm{Li}_m(\zeta)\, \mathbf{e}_0^m \cdot \mathbf{e}_\zeta$
$-2m-2$	$(2\pi i)^{m+1} \mathbf{e}_0^m \cdot \mathbf{e}_\zeta$

Let $E_{m,\zeta} \in \mathrm{Ext}^1_{\mathsf{MHS}}(\mathbb{Q}, \mathbb{Q}(m))$ denote the extension of $\mathbb{Q}$ by $\mathbb{Q}(m)$ with period $\mathrm{Li}_m(\zeta)$. The quotient of $\mathfrak{p}(U_N, \vec{\mathsf{v}}_1)/D^2$ generated by $\mathbf{e}_0$ and $\mathbf{e}_0^m \cdot \mathbf{e}_\zeta$ is an extension of $\mathbb{Q}(m+1)$ by $\mathbb{Q}(1)$ with period $-\mathrm{Li}_m(\zeta)$, hence isomorphic to $-E_{m,\zeta}(1)$ as a MHS. Thus, $\mathfrak{p}(U_N, \vec{\mathsf{v}}_1)/D^2$ is the pull-back of the direct product of all extensions $-E_{m,\zeta}(1)$ via the diagonal map

$\Delta : \mathbb{Q}(1) \to \prod \mathbb{Q}(1)$.

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \prod\limits_{\substack{m\geq 0\\ \zeta\in\boldsymbol{\mu}_N}} \mathbb{Q}(m+1) & \longrightarrow & \mathfrak{p}(U_N,\vec{\mathrm{v}}_1)/D^2 & \longrightarrow & \mathbb{Q}(1) & \longrightarrow & 0 \\
 & & \| & & \downarrow & & \downarrow{\scriptstyle \Delta} & & \\
0 & \longrightarrow & \prod\limits_{\substack{m\geq 0\\ \zeta\in\boldsymbol{\mu}_N}} \mathbb{Q}(m+1) & \longrightarrow & \mathbf{E} & \longrightarrow & \prod\limits_{\substack{m\geq 2\\ \zeta\in\boldsymbol{\mu}_N}} \mathbb{Q}(1) & \longrightarrow & 0
\end{array}$$

The extension $\mathbf{E}$ in the diagram is

$$\mathbf{E} = \bigoplus_{\zeta\in\boldsymbol{\mu}_N} (\mathbb{Q}(1)\oplus\mathbb{Q}(1)) \oplus \bigoplus_{\zeta\in\boldsymbol{\mu}_N} (\mathbb{Q}(2)\oplus\mathbb{Q}(1)) \oplus \prod_{\substack{m\geq 2\\ \zeta\in\boldsymbol{\mu}_N}} -E_{m,\zeta}(1),$$

where each copy of $\mathbb{Q}(1)\oplus\mathbb{Q}(1)$ is a trivial extension spanned by $2\pi i\mathbf{e}_0$ and $2\pi i\mathbf{e}_\zeta$ each copy of $\mathbb{Q}(2)\oplus\mathbb{Q}(1)$ is a trivial extension spanned by $(2\pi i)^2\mathbf{e}_0\cdot\mathbf{e}_\zeta$ and $2\pi i\mathbf{e}_0$.

5.2.4 The limit MHS of $\mathbf{Pol}_{\mathrm{ell}_N}$ at $\vec{\mathrm{v}}_1$

Recall $\mathbf{Pol}_{\mathrm{ell}_N}$ is an extension

$$0 \longrightarrow \bigoplus_{\substack{m\geq 0\\ \zeta\in\boldsymbol{\mu}}} S^m\mathbb{H}(1) \longrightarrow \mathbf{Pol}_{\mathrm{ell}_N} \longrightarrow \mathbb{H} \longrightarrow 0,$$

of variations of MHS, where the copy of $S^m\mathbb{H}(1)$ corresponding to $\zeta\in\boldsymbol{\mu}_N$ is $S^m\mathbb{H}\cdot\mathbf{t}_\zeta$. The limit of this variation of MHS at $\vec{\mathrm{u}}+\vec{\mathrm{v}}_1$ is isomorphic to $\mathfrak{p}(E'_{\vec{\mathrm{u}}},\vec{\mathrm{v}}_1)/D^2$. It is an extension

$$0 \longrightarrow \prod_{\substack{m\geq 0\\ \zeta\in\boldsymbol{\mu}_N}} S^m H_{\vec{\mathrm{u}}}\cdot\mathbf{t}_\zeta \longrightarrow \mathfrak{p}(E'_{\vec{\mathrm{u}}},\vec{\mathrm{v}}_1)/D^2 \longrightarrow H_{\vec{\mathrm{u}}} \longrightarrow 0,$$

of MHS, where $H_{\vec{\mathrm{u}}}$ is the limit MHS of $\mathbb{H}$ at $\vec{\mathrm{u}}$ (see §2.3.2). The MHS $H_{\vec{\mathrm{u}}}$ is isomorphic to $\mathbb{Q}(0)\oplus\mathbb{Q}(1)$ and has $\mathbb{Q}$-DR realization $\mathbb{Q}\mathbf{X}\oplus\mathbb{Q}\mathbf{Y}$ where $\mathbb{Q}(0)$ has de Rham generator $\mathbf{X}$ and $\mathbb{Q}(1)$ has de Rham generator $\mathbf{Y}$.

Let $V_{m,\zeta}\in \mathrm{Ext}^1_{\mathsf{MHS}}(H_{\vec{\mathrm{u}}}, S^m H_{\vec{\mathrm{u}}}(1))$ be the sub-quotient of $\mathfrak{p}(E'_{\vec{\mathrm{u}}},\vec{\mathrm{v}}_1)/D^2$ generated by $H_{\vec{\mathrm{u}}}$ and $S^m H_{\vec{\mathrm{u}}}\cdot\mathbf{t}_\zeta$. As with the cyclotomic polylog, we observe that $\mathfrak{p}(E'_{\vec{\mathrm{u}}},\vec{\mathrm{v}}_1)/D^2$ is the

pull-back of the direct product of the extensions $V_{m,\zeta}$ via the diagonal map $\Delta : H_{\vec{\mathfrak{u}}} \to \prod H_{\vec{\mathfrak{u}}}$ (see diagram):

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \prod\limits_{\substack{m\geq 0\\ \zeta\in\boldsymbol{\mu}_N}} S^m H_{\vec{\mathfrak{u}}}\cdot \mathbf{t}_\zeta & \longrightarrow & \mathfrak{p}(E'_{\vec{\mathfrak{u}}},\vec{\mathfrak{v}}_1)/D^2 & \longrightarrow & H_{\vec{\mathfrak{u}}} & \longrightarrow & 0\\
 & & \| & & \downarrow & & \downarrow{\scriptstyle \Delta} & & \\
0 & \longrightarrow & \prod\limits_{\substack{m\geq 0\\ \zeta\in\boldsymbol{\mu}_N}} S^m H_{\vec{\mathfrak{u}}}\cdot \mathbf{t}_\zeta & \longrightarrow & \prod\limits_{\substack{m\geq 0\\ \zeta\in\boldsymbol{\mu}_N}} V_{m,\zeta} & \longrightarrow & \prod\limits_{\substack{m\geq 0\\ \zeta\in\boldsymbol{\mu}_N}} H_{\vec{\mathfrak{u}}} & \longrightarrow & 0
\end{array}$$

Since the Hain map preserves depth filtrations, it induces a map on the polylog quotients. On the fiber over $\vec{\mathfrak{u}}+\vec{\mathfrak{v}}_1$, we have the commutative diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \prod\limits_{\substack{m\geq 0\\ \zeta\in\boldsymbol{\mu}_N}} \mathbb{C}\mathbf{e}_0^m\cdot \mathbf{e}_\zeta & \longrightarrow & \mathfrak{p}(U_N,\vec{\mathfrak{v}}_1)/D^2 & \longrightarrow & \mathbb{C}\mathbf{e}_0 & \longrightarrow & 0\\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \prod\limits_{\substack{m\geq 0\\ \zeta\in\boldsymbol{\mu}_N}} S^m H_{\vec{\mathfrak{u}}}\cdot \mathbf{t}_\zeta & \longrightarrow & \mathfrak{p}(E'_{\vec{\mathfrak{u}}},\vec{\mathfrak{v}}_1)/D^2 & \longrightarrow & H_{\vec{\mathfrak{u}}} & \longrightarrow & 0,
\end{array} \tag{5.2.3}$$

of de Rham realizations, where each vertical arrow is induced by the Hain map. Our task is to use the knowledge of the periods of the upper extension to compute the periods of the lower extension. To do this, it is best to decompose the extension.

Lemma 5.2.5. *The mth symmetric power of the limit MHS $H_{\vec{\mathfrak{u}}}$ splits as*

$$S^m H_{\vec{\mathfrak{u}}} = \mathbb{Q}(0)\oplus\cdots\oplus\mathbb{Q}(m).$$

The de Rham generator of $\mathbb{Q}(r)$ is $\mathbf{X}^{m-r}\mathbf{Y}^r$.

Recall $-E_{m,\zeta}(1)$ is the extension of $\mathbb{Q}(1)$ by $\mathbb{Q}(m+1)$ with period $-\mathrm{Li}_m(\zeta)$. For $m\geq 2$, the diagram (5.2.3) decomposes into a direct product of maps $-E_{m,\zeta}(1)\to V_{m,\zeta}$.

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathbb{C}\mathbf{e}_0^m\cdot\mathbf{e}_\zeta & \longrightarrow & -E_{m,\zeta}(1) & \longrightarrow & \mathbb{C}\mathbf{e}_0 & \longrightarrow & 0\\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \bigoplus\limits_{r=0}^{m}\mathbb{C}\mathbf{X}^{m-r}\mathbf{Y}^r\cdot\mathbf{t}_\zeta & \longrightarrow & V_{m,\zeta} & \longrightarrow & \mathbb{C}\mathbf{X}\oplus\mathbb{C}\mathbf{Y} & \longrightarrow & 0.
\end{array}$$

The Betti $\mathbb{Q}$-basis of the top extension $-E_{m,\zeta}(1)$ maps to a Betti $\mathbb{Q}$-basis of the bottom extension $V_{m,\zeta}$.

M-weight	$\mathbb{Q}$-Betti generator of $V_{m,\zeta}$
-2	$2\pi i\mathbf{Y} - 2\pi i\,\mathrm{Li}_m(\zeta)\mathbf{Y}^m \cdot \mathbf{t}_\zeta$
$-2r-2$	$(2\pi i)^{r+1}\mathbf{X}^{m-r}\mathbf{Y}^r \cdot \mathbf{t}_\zeta$

There must also be a $\mathbb{Q}$-Betti vector

$$\mathbf{X}^B = \mathbf{X} - \sum_{r=0}^{m} c_r \mathbf{X}^{m-r}\mathbf{Y}^r \cdot \mathbf{t}_\zeta$$

in $V_{m,\zeta}$ that projects onto $\mathbf{X}$ in $H_{\vec{\mathfrak{u}}}$. Monodromy preserves the rational structure. Recall from (3.6.2) the nilpotent residue at $q=0$ given by $L_{q,0} \equiv \mathbf{Y}\partial/\partial\mathbf{X} \bmod D^2$. Thus,

$$(\exp(2\pi i L_{q,0}) - 1)\mathbf{X}^B = 2\pi i\mathbf{Y} - \exp(2\pi i L_{q,0}) \sum_{r=0}^{m} c_r \mathbf{X}^{m-r}\mathbf{Y}^r \cdot \mathbf{t}_\zeta$$

is also rational. Given the rational vectors in the table above, we may choose $c_{m-1} = \mathrm{Li}_m(\zeta)$ and $c_r = 0$ for $r \neq m-1$. We have proven the following.

Proposition 5.2.6. *When $m \geq 2$, the extension $V_{m,\zeta}$ has a de Rham splitting given by*

$$\mathbf{X} \mapsto \mathbf{X} - \mathrm{Li}_m(\zeta)\mathbf{X}\mathbf{Y}^{m-1} \cdot \mathbf{t}_\zeta$$
$$\mathbf{Y} \mapsto \mathbf{Y} - \mathrm{Li}_m(\zeta)\mathbf{Y}^m \cdot \mathbf{t}_\zeta.$$

Corollary 5.2.7. *When $m \geq 2$ and $\zeta \neq 1$, the pull-back of $V_{m,\zeta} \oplus V_{m,\bar{\zeta}}$ along the diagonal map $H_{\vec{\mathfrak{u}}} \to H_{\vec{\mathfrak{u}}} \oplus H_{\vec{\mathfrak{u}}}$ has de Rham splitting*

$$\mathbf{X} \mapsto \mathbf{X} - \mathrm{Li}_m(\zeta)\mathbf{X}\mathbf{Y}^{m-1} \cdot (\mathbf{t}_\zeta + (-1)^{m+1}\mathbf{t}_{\bar{\zeta}})$$
$$\mathbf{Y} \mapsto \mathbf{Y} - \mathrm{Li}_m(\zeta)\mathbf{Y}^m \cdot (\mathbf{t}_\zeta + (-1)^{m+1}\mathbf{t}_{\bar{\zeta}}).$$

Proof. Use the splitting in the previous statement for ζ and $\bar{\zeta}$. Then recall $\mathrm{Li}_m(\zeta) + (-1)^m\mathrm{Li}_m(\bar{\zeta}) \in (2\pi i)^m\mathbb{Q}$. □

This splitting may be described in terms of the derivations which appear in the KZB connection. For $m \geq 2$, define

$$\epsilon^{\mathrm{op}}_{m+1,\zeta} := \frac{1}{(m-1)!}\left(\mathbf{Y}\frac{\partial}{\partial \mathbf{X}}\right)^{m-1} \cdot \epsilon_{m+1,\zeta},$$

where $\epsilon_{m+1,\zeta}$ is the derivation defined in (3.2.1).

Corollary 5.2.8. *The de Rham splitting of the pull-back of $V_{m,\zeta} \oplus V_{m,\bar{\zeta}}$ along the diagonal map $H_{\vec{\mathsf{u}}} \to H_{\vec{\mathsf{u}}} \oplus H_{\vec{\mathsf{u}}}$ is*

$$1 + \mathrm{Li}_m(\zeta)\epsilon^{\mathrm{op}}_{m+1}.$$

Proof. Use (3.2.1) to observe

$$\epsilon_{m+1,\zeta} \equiv \mathrm{ad}\left(\mathbf{X}^{m-1} \cdot (\mathbf{t}_\zeta + (-1)^{m+1}\mathbf{t}_{\bar{\zeta}})\right) \bmod D^2\mathfrak{p}(E'_{\vec{\mathsf{u}}}, \vec{\mathsf{v}}_1).$$

Apply $\frac{1}{(m-1)!}\left(\mathbf{Y}\frac{\partial}{\partial \mathbf{X}}\right)^{m-1}$ to get the splittings in Corollary 5.2.7. □

Note that $\mathrm{Li}_m(\zeta)\epsilon^{\mathrm{op}}_{m,\zeta}$ is invariant under complex conjugation since by (3.2.3)

$$\epsilon^{\mathrm{op}}_{m+1,\zeta} = (-1)^{m+1}\epsilon^{\mathrm{op}}_{m+1,\bar{\zeta}}$$

and by Lemma 5.2.4

$$\mathrm{Li}_m(\zeta) \equiv (-1)^{m+1}\mathrm{Li}_m(\bar{\zeta}) \in \mathbb{C}/(2\pi i)^m \cong \mathrm{Ext}^1_{\mathsf{MHS}}(\mathbb{Q}, \mathbb{Q}(m)).$$

Remark 5.2.9. In the case $\zeta = 1$, we see $V_{m,1}$ splits when m is even and has de Rham splitting $\frac{1}{2}\zeta(m)\epsilon^{\mathrm{op}}_{m+1,1}$ when m is odd.

Remark 5.2.10. Parallel transport via regularized iterated integrals of KZB_N of the extensions $V_{m,\zeta}$ gives extensions of $\mathbb{H}$ by $S^m\mathbb{H}(1)$ over $\mathcal{E}'_{\Gamma_1(N)}$ (and $Y_1(N)$ and $Y_{\Gamma_1(N),\vec{1}}(N)$ by restriction). Thus, each $V_{m,\zeta}$ is the fiber over $\vec{\mathsf{v}} + \vec{\mathsf{w}}$ of a variation of MHS over $\mathcal{E}'_N$ that represents a class of

$$\mathrm{Ext}^1_{\mathsf{MHS}(\mathcal{E}'_{\Gamma(N)})}(\mathbb{H}, S^m\mathbb{H}(1))$$

whose monodromy invariants at $\vec{\mathsf{u}}$ are an element of $\mathrm{Ext}^1_{\mathsf{MHS}(U_N)}(\mathbb{Q}(1), \mathbb{Q}(m+1))$.

5.3 Hodge realizations of $\mathrm{Ext}^1_{\mathsf{MTM}_N}(\mathbb{Q},\mathbb{Q}(m))$

We observed in §5.2.3 that, up to Tate twists, the Hodge realization of $\mathfrak{p}(U_N,\vec{\mathrm{v}}_1)/D^2$ is the sum (as extensions) of extensions $-E_{m,\zeta}(1)$ in $\mathrm{Ext}^1_{\mathsf{MHS}}(\mathbb{Q}(1),\mathbb{Q}(m+1))$. Since the extensions $E_{m,\zeta}$ are subquotients of the mixed Tate motive $\mathfrak{p}(U_N,\vec{\mathrm{v}}_1)$, they are realizations of rational classes in $\mathrm{Ext}^1_{\mathsf{MTM}_N}(\mathbb{Q},\mathbb{Q}(m))$ [21, §5]. In this section, we use these extensions to construct an explicit basis of $\mathrm{Ext}^1_{\mathsf{MTM}_N}(\mathbb{Q},\mathbb{Q}(m))$.

When $m\geq 1$, the period map

$$\mathrm{Ext}^1_{\mathsf{MHS}}(\mathbb{Q},\mathbb{Q}(m))\to \mathbb{C}/(2\pi i)^m\mathbb{Q}$$

on 1-extensions of MHS is an isomorphism [14]. Thus, we have the composition

$$\begin{aligned}\mathrm{span}_{\mathbb{Q}}\{E_{m,\zeta}\mid \zeta\in\boldsymbol{\mu}_N\}\hookrightarrow \mathrm{Ext}^1_{\mathsf{MTM}_N}(\mathbb{Q},\mathbb{Q}(m))&\\ \to \mathrm{Ext}^1_{\mathsf{MHS}}(\mathbb{Q},\mathbb{Q}(m))\xrightarrow{\sim}\mathbb{C}/(2\pi i)^m\mathbb{Q}.&\quad (5.3.1)\end{aligned}$$

The image of each extension is $E_{m,\zeta}$ in $\mathbb{C}/(2\pi i)^m\mathbb{Q}$ is the period $\mathrm{Li}_m(\zeta)$. When $m\geq 2$, Goncharov [30] proved there is an isomorphism

$$\mathrm{span}_{\mathbb{Q}}\{\mathrm{Li}_m\zeta\mid\zeta\in\boldsymbol{\mu}_N\}\cong K_{2m-1}(\mathcal{O}_N)\otimes\mathbb{Q}.$$

Thus, by (4.1.4) the inclusion in (5.3.1) is an isomorphism. Therefore, $\mathbb{Q}$-linear relations in $\mathrm{Ext}^1_{\mathsf{MTM}_N}(\mathbb{Q},\mathbb{Q}(m))$ correspond exactly to $\mathbb{Q}$-linear relations between the periods $\mathrm{Li}_m(\zeta)$ in $\mathbb{C}/(2\pi i)^m\mathbb{Q}$.

Remark 5.3.1. When $m=1$, the extensions $E_{1,\zeta}$ do *not* span $\mathrm{Ext}^1_{\mathsf{MTM}_N}(\mathbb{Q},\mathbb{Q}(1))$. One must add extensions with period $\log p$ for all primes p that divide N. This is essentially equivalent to Dirichlet's unit theorem.

We recall a useful fact about polylogarithms.

Lemma 5.3.2. *Suppose ℓ is a positive integer. Then the* distribution relation *of polylogarithms is*

$$\ell^{m-1}\sum_{w^\ell=z}\mathrm{Li}_m(w)=\mathrm{Li}_m(z).\qquad (5.3.2)$$

The proof follows directly from the power series expansion of the polylogarithm function. Details can be found in [30, §2].

Corollary 5.3.3. *Suppose ζ is a primitive dth root of unity. If p is prime, then $\mathrm{Li}_m(\zeta)$ is a $\mathbb{Q}$-linear combination of values of Li_m at primitive (dp)th roots of unity.*

Proof. There exists a primitive (dp)th root of unity ξ such that $\xi^p = \zeta$. Then by the distribution relation,

$$\mathrm{Li}_m(\zeta) = p^{m-1} \sum_{w^p=\zeta} \mathrm{Li}_m(w) = p^{m-1} \sum_{k=0}^{p-1} \mathrm{Li}_m(\xi^{kd+1}).$$

If $p \mid d$, then $\gcd(kd+1, p) = 1$ for all k, hence $\gcd(kd+1, dp) = 1$ and each ξ^{kd+1} is primitive, completing the proof. On the other hand, if $p \nmid d$, then there is exactly one $\ell \in \{0, 1, \ldots, p-1\}$ such that $p \mid \ell d + 1$. Then $\xi^{\ell d+1}$ is a dth primitive root of unity. Every root ξ^{kd+1} for $k \neq \ell$ is a primitive (dp)th root of unity. One can then use the distribution relation again to rewrite $\mathrm{Li}_m(\xi^{\ell d+1})$ as a sum of values of Li_m. After no more than p iterations, the original primitive dth root of unity ζ will appear in the sum. One can then solve for $\mathrm{Li}_m(\zeta)$ in terms of $\mathrm{Li}_m(\eta)$ with $\eta \in \boldsymbol{\mu}_{dp}$ primitive. □

Proposition 5.3.4. *When $m \geq 2$, the extensions $E_{m,\zeta}$ with $\zeta \in \boldsymbol{\mu}_N$ primitive and $\operatorname{Im} \zeta > 0$ form a basis of $\mathrm{Ext}^1_{\mathsf{MTM}_N}(\mathbb{Q}, \mathbb{Q}(m))$.*

Proof. Suppose $d \mid N$ and η is a primitive dth root of unity. There are primes $p_1, \ldots, p_n$ (not necessarily distinct) such that $N = dp_1 \cdots p_n$. One can apply the result of the previous lemma n times to write $\mathrm{Li}_m(\eta)$ as $\mathbb{Q}$-linear combination of $\mathrm{Li}_m(\zeta)$ with $\zeta \in \boldsymbol{\mu}_N$ primitive. Thus, the extensions $E_{m,\zeta}$ with $\zeta \in \boldsymbol{\mu}_N$ primitive generate $\mathrm{Ext}^1_{\mathsf{MTM}_N}(\mathbb{Q}, \mathbb{Q}(m))$. It follows from Lemma 5.2.4 that $E_{m,\overline{\zeta}} \equiv (-1)^{m+1} E_{m,\zeta} \bmod \mathbb{Q}(m)$. Thus, we need only consider those $E_{m,\zeta}$ with positive imaginary part. The result follows from counting dimension and the fact that the inclusion in (5.3.1) is an isomorphism. □

Example 5.3.5. Suppose $N = p^n$ is a prime power. Let $\zeta \in \boldsymbol{\mu}_N$ be a primitive (p^k)th root

of unity for $0 < k \leq n$. It follows from the distribution relation (5.3.2)

$$\mathrm{Li}_m(\zeta) = (p^{n-k})^{m-1} \sum_{\substack{\eta \in \boldsymbol{\mu}_N \\ \eta^{p^{n-k}} = \zeta}} \mathrm{Li}_m(\eta). \tag{5.3.3}$$

Note that each of the roots η in the sum are primitive. Then

$$\begin{aligned}
\zeta(m) = \mathrm{Li}_m(1) &= N^{m-1} \sum_{\zeta \in \boldsymbol{\mu}_N} \mathrm{Li}_m(\zeta) \\
&= N^{m-1}\zeta(m) + N^{m-1} \sum_{k=1}^{n} \sum_{\substack{\zeta \in \boldsymbol{\mu}_N \\ p^k\text{-primitive}}} \left((p^{n-k})^{m-1} \sum_{\substack{\eta \in \boldsymbol{\mu}_N \\ \eta^{p^{n-k}} = \zeta}} \mathrm{Li}_m(\eta) \right) \\
&= N^{m-1}\zeta(m) + N^{m-1} \sum_{\substack{\zeta \in \boldsymbol{\mu}_N \\ \text{primitive}}} \sum_{k=1}^{n} (p^{n-k})^{m-1} \mathrm{Li}_m(\zeta) \\
&= N^{m-1}\zeta(m) + N^{m-1} \frac{1 - N^{m-1}}{1 - p^{m-1}} \sum_{\substack{\zeta \in \boldsymbol{\mu}_N \\ \text{primitive}}} \mathrm{Li}_m(\zeta).
\end{aligned}$$

Rearranging,

$$\zeta(m) = \frac{N^{m-1}}{1 - p^{m-1}} \sum_{\substack{\zeta \in \boldsymbol{\mu}_N \\ \text{primitive}}} \mathrm{Li}_m(\zeta).$$

Thus, as extensions in $\mathrm{Ext}^1_{\mathsf{MTM}_N}(\mathbb{Q}, \mathbb{Q}(m))$,

$$E_{m,1} = \frac{N^{m-1}}{1 - p^{m-1}} \sum_{\substack{\zeta \in \boldsymbol{\mu}_N \\ \text{primitive}}} E_{m,\zeta}, \tag{5.3.4}$$

where summation is the Baer sum of extensions. Note that by Lemma 5.2.4 this last statement is trivial when m is even.

5.4 The action of $H_1(\mathfrak{k}_N)$

For the remainder of the thesis, we are concerned only with the de Rham fiber functor of MTM_N. Thus, we will omit the mention of DR from $\mathcal{G}_N^{\mathrm{DR}}$, $\mathcal{K}_N^{\mathrm{DR}}$, and $\mathfrak{k}_N^{\mathrm{DR}}$.

The Hochschild–Serre spectral sequence associated to the short exact sequence (4.1.6) converges

$$E_2^{s,t} = H^s(\mathbb{G}_m, H^t(\mathcal{K}_N, \mathbb{Q}(m))) \Rightarrow H^{s+t}(\mathcal{G}_N, \mathbb{Q}(m)).$$

The group $\mathbb{G}_m$ is reductive, and thus $H^s(\mathbb{G}_m, -) = 0$ for $s > 0$. It follows that

$$\begin{aligned} H^t(\mathcal{K}_N, \mathbb{Q}(m))^{\mathbb{G}_m} &= H^0(\mathbb{G}_m, H^t(\mathcal{K}_N, \mathbb{Q}(m))) \\ &= H^t(\mathcal{G}_N, \mathbb{Q}(m)) \\ &= \mathrm{Ext}^t_{\mathsf{MTM}_N}(\mathbb{Q}, \mathbb{Q}(m)). \end{aligned}$$

Therefore,

$$H_1(\mathfrak{k}_N, \mathbb{Q}) = \prod_{m>0} \mathrm{Ext}^1_{\mathsf{MTM}_N}(\mathbb{Q}, \mathbb{Q}(m))^\vee \otimes \mathbb{Q}(m).$$

When $m \geq 2$, we have a basis of $\mathrm{Ext}^1_{\mathsf{MTM}_N}(\mathbb{Q}, \mathbb{Q}(m))$ from Proposition 5.3.4. One can then specify a dual basis

$$\{\overline{\boldsymbol{\sigma}}_{m,\zeta} \mid \zeta \in \boldsymbol{\mu}_N \text{ primitive}, \operatorname{Im}\zeta > 0\}$$

of $H_1(\mathfrak{k}_N)$. We can also define $\overline{\boldsymbol{\sigma}}_{m,\zeta}$ for ζ primitive and $\operatorname{Im}\zeta < 0$ by the relations induced by Lemma 5.2.4

$$\overline{\boldsymbol{\sigma}}_{m,\bar{\zeta}} = (-1)^{m+1}\overline{\boldsymbol{\sigma}}_{m,\zeta}.$$

We may then choose non-canonical lifts $\boldsymbol{\sigma}_{m,\zeta} \in \mathfrak{k}_N$ of each $\overline{\boldsymbol{\sigma}}_{m,\zeta} \in H_1(\mathfrak{k}_N)$ satisfying the same relation

$$\boldsymbol{\sigma}_{m,\bar{\zeta}} = (-1)^{m+1}\boldsymbol{\sigma}_{m,\zeta}. \tag{5.4.1}$$

Suppose $\{\mathbf{v}_0^{\mathrm{DR}}, \mathbf{v}_m^{\mathrm{DR}}\}$ is a $\mathbb{Q}$-de Rham basis of $E_{m,\zeta}$ with $\mathbf{v}_m \in M_{-2m}E_{m,\zeta}$. The action of $\exp\boldsymbol{\sigma}_{m,\zeta} \in \mathcal{K}_N$ on $E_{m,\zeta}$ stabilizes the associated graded $\mathrm{Gr}^M_\bullet E_{m,\zeta}$. Thus,

$$\exp\boldsymbol{\sigma}_{m,\zeta} : \begin{cases} \mathbf{v}_0^{\mathrm{DR}} \longmapsto \mathbf{v}_0^{\mathrm{DR}} + c\mathbf{v}_m^{\mathrm{DR}} \\ \mathbf{v}_m^{\mathrm{DR}} \longmapsto \mathbf{v}_m^{\mathrm{DR}} \end{cases}$$

for some $c \in \mathbb{Q}$. We may scale $\boldsymbol{\sigma}_{m,\zeta}$ such that $c = 1$ and set

$$\boldsymbol{\sigma}_{m,\zeta} : \begin{cases} \mathbf{v}_0^{\mathrm{DR}} \longmapsto \mathbf{v}_m^{\mathrm{DR}} \\ \mathbf{v}_m^{\mathrm{DR}} \longmapsto 0. \end{cases}$$

The action of $\boldsymbol{\sigma}_{m,\zeta}$ on basis elements $E_{m,\xi}$ with $\operatorname{Im}\xi > 0$ and $\xi \neq \zeta$ is trivial.

Proposition 5.4.1. *Suppose $V_1, \dots, V_n \in \operatorname{Ext}^1_{\mathsf{MTM}_N}(\mathbb{Q}, \mathbb{Q}(m))$. The action of $\boldsymbol{\sigma}_{m,\zeta}$ on the pull-back of $\bigoplus_k V_k$ along the diagonal map $\Delta : \mathbb{Q} \to \mathbb{Q}^n$ is the sum of the actions of $\boldsymbol{\sigma}_{m,\zeta}$ on each V_k individually.*

Proof. Consider the diagram of de Rham realizations below.

$$\begin{array}{ccccccccc}
0 & \longrightarrow & (\mathbb{Q}(m)^{\mathrm{DR}})^n & \longrightarrow & \bigoplus_k V_k^{\mathrm{DR}} & \longrightarrow & (\mathbb{Q}^{\mathrm{DR}})^n & \longrightarrow & 0 \\
 & & \| & & \uparrow & & \uparrow{\scriptstyle \Delta} & & \\
0 & \longrightarrow & (\mathbb{Q}(m)^{\mathrm{DR}})^n & \longrightarrow & \Delta^*(\bigoplus_k V_k^{\mathrm{DR}}) & \longrightarrow & \mathbb{Q}^{\mathrm{DR}} & \longrightarrow & 0
\end{array}$$

The vertical maps are reaizations of morphisms in MTM_N and thus $\mathfrak{k}_N$-equivariant. Thus, the direct sum of the actions of $\boldsymbol{\sigma}_{m,\zeta}$ on each V_k pulls back to the their sum. □

Corollary 5.4.2. *The action of $\boldsymbol{\sigma}_{m,\zeta}$ on $E_{m,\bar{\zeta}}$ is $(-1)^{m+1}$ times the action of the $\boldsymbol{\sigma}_{m,\zeta}$ on $E_{m,\zeta}$.*

Proof. Consider the Baer sum $E_{m,\zeta} + E_{m,\bar{\zeta}}$ in the diagram below.

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathbb{Q}(m)^{\mathrm{DR}} \oplus \mathbb{Q}(m)^{\mathrm{DR}} & \longrightarrow & \Delta^*(E_{m,\zeta} \oplus (-1)^m E_{m,\bar{\zeta}}) & \longrightarrow & \mathbb{Q}^{\mathrm{DR}} & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \Sigma} & & \downarrow & & \| & & \\
0 & \longrightarrow & \mathbb{Q}(m)^{\mathrm{DR}} & \longrightarrow & E_{m,\zeta} + (-1)^m E_{m,\bar{\zeta}} & \longrightarrow & \mathbb{Q}(m)^{\mathrm{DR}} & \longrightarrow & 0
\end{array}$$

Since $E_{m,\zeta} + (-1)^m E_{m,\bar{\zeta}}$ splits as a MHS and thus also in MTM_N, the action of $\mathfrak{k}_N$ is trivial. The results follows from $\mathfrak{k}_N$-equivariance of the vertical maps. □

5.5 Galois representation of $\mathfrak{k}_N$

Recall action of $\mathfrak{k}_N$ on $\mathfrak{p}(U_N, \vec{\mathrm{v}}_1)$ and $\mathfrak{p}(E'_{\vec{\mathrm{u}}}, \vec{\mathrm{v}}_1)$

$$\begin{array}{ccc}
\mathfrak{k}_N & \xrightarrow{\phi_{\mathrm{cyc}}} & \operatorname{Der}\mathfrak{p}(U_N, \vec{\mathrm{v}}_1) \\
 & \searrow^{\phi_{\mathrm{ell}}} & \downarrow{\scriptstyle \Psi_N} \\
 & & \operatorname{Der}\mathfrak{p}_N(E'_{\vec{\mathrm{u}}}, \vec{\mathrm{v}}_1),
\end{array}$$

where the Hain map (4.3.2) is $\mathfrak{k}_N$-equivariant. If $\delta \in \operatorname{Gr}^M_{-2m} \operatorname{Der}\mathfrak{p}(E'_{\vec{\mathrm{u}}}, \vec{\mathrm{v}}_1)$ is a derivation, define its *head* to be the projection of δ on the highest W-graded quotient for which it is

nonzero. For each generator $\boldsymbol{\sigma}_{m,\zeta} \in \mathfrak{k}_N$, our goal is to compute the head of $\phi_{\mathrm{ell}}(\boldsymbol{\sigma}_{m,\zeta}) \in \mathrm{Gr}^M_{-2m} \operatorname{Der} \mathfrak{p}(E'_{\vec{\mathrm{u}}}, \vec{\mathrm{v}}_1)$.

Lemma 5.5.1. *The image of ϕ_{ell} commutes with the residue $L_{q=0}$ of the $\Gamma_1(N)$ KZB connection* (3.6.2)*, that is,*

$$[\phi_{\mathrm{ell}}(\boldsymbol{\sigma}_{m,\zeta}), L_{q=0}] = 0$$

for all $\boldsymbol{\sigma}_{m,\zeta} \in \mathfrak{k}_N$.

Proof. The residue $L_{q=0}$ spans a copy of $\mathbb{Q}(1)$ in $\operatorname{Der} \mathfrak{p}(E'_{\vec{\mathrm{u}}}, \vec{\mathrm{v}}_1)$. The action of $\mathfrak{k}_N$ on semisimple objects of MTM_N is trivial. □

Define the *special derivations* $\operatorname{SDer} \mathfrak{p}_N$ to be the subalgebra

$$\{\delta \in \operatorname{Der} \mathfrak{p}_N \mid \delta(\mathbf{t}_1) = 0, \delta(\mathbf{t}_\zeta) = [u_\zeta, \mathbf{t}_\zeta] \text{ for some } u_\zeta \in \mathfrak{p}_N\}.$$

This is the Lie algebra of the *special automorphisms* of $\mathfrak{p}_N$ given by

$$\{\sigma \in \operatorname{Aut} \mathfrak{p}_N \mid \sigma(\mathbf{t}_1) = \mathbf{t}_1, \sigma(\mathbf{t}_\zeta) = e^{u_\zeta} \mathbf{t}_\zeta e^{-u_\zeta} \text{ for some } u_\zeta \in \mathfrak{p}_N\}.$$

Lemma 5.5.2. *The sub-variation of MHS of $\boldsymbol{\mathcal{P}}_{\Gamma_1(N)}$ restricted to $Y_{\Gamma_1(N),\vec{1}}(N)$ with fiber $\operatorname{SDer} \mathfrak{p}_N$ is admissible.*

Proof. Each of the derivations $\epsilon_{m,\zeta}$ is special (see Remark 3.2.1). Thus, the residues $L_{q,P}$ at each cusp are also special. The residue along the zero section $L_{z=0} = \operatorname{ad} \mathbf{t}_1$ is clearly special as well. Thus, the sub-VMHS generated by special derivations is admissible. □

Lemma 5.5.3. *The image of ϕ_{ell} is contained in $\operatorname{SDer} \mathfrak{p}(E'_{\vec{\mathrm{u}}}, \vec{\mathrm{v}}_1)$.*

Proof. We will show the image of the Galois representation $\mathcal{K}_N \to \operatorname{Aut} \mathfrak{p}(E'_{\vec{\mathrm{u}}}, \vec{\mathrm{v}}_1)$ lies in the special automorphisms of $\mathfrak{p}(E'_{\vec{\mathrm{u}}}, \vec{\mathrm{v}}_1)$.

Fix an Nth root of unity $\zeta \in \boldsymbol{\mu}_N \subset \mathbb{G}_m \subset E_{\vec{\mathrm{u}}}$. Let ℓ denote a path from $\vec{\mathrm{v}}_1$ to $\vec{\mathrm{v}}_\zeta$, and let κ denote a small positively oriented loop based at $\vec{\mathrm{v}}_\zeta$ traveling around ζ once. Then the term $\exp \mathbf{t}_\zeta \in \exp \mathfrak{p}(E'_{\vec{\mathrm{u}}}, \vec{\mathrm{v}}_1)$ corresponds to the homotopy class of $\ell \kappa \ell^{-1}$.

Let D be a contractible neighborhood of ζ containing no other roots of unity. Then $\mathfrak{p}(D - \{\zeta\}, \vec{\mathsf{v}}_\zeta) \cong \mathbb{L}(\log \kappa)^\wedge$, which is a pro-object of $\mathsf{MTM}_N^{\mathrm{ss}}$. Thus, the action of $\mathcal{K}_N$ fixes $\log \kappa$ and also κ. Therefore, if $\exp \boldsymbol{\sigma} \in \mathcal{K}_N$, we have

$$\begin{aligned}(\exp \boldsymbol{\sigma})(\exp \mathbf{t}_\zeta) &= \ell^\sigma \kappa (\ell^\sigma)^{-1} \\ &= \ell^\sigma (\ell^{-1}\ell) \kappa (\ell^{-1}\ell)(\ell^\sigma)^{-1} \\ &= (\ell^\sigma \ell^{-1})(\exp \mathbf{t}_\zeta)(\ell^\sigma \ell^{-1})^{-1}.\end{aligned}$$

Note that $\ell^\sigma \ell^{-1}$ is a loop based at $\vec{\mathsf{v}}_1$. Thus, by Proposition 2.6.1, $\ell^\sigma \ell^{-1} = e^u$ for some $u \in \mathfrak{p}(E'_{\vec{\mathsf{u}}}, \vec{\mathsf{v}}_1)$. Then $\sigma(\exp \mathbf{t}_\zeta) = e^u e^{\mathbf{t}_\zeta} e^{-u}$. Applying a logarithm gives

$$\begin{aligned}(\exp \boldsymbol{\sigma})(\mathbf{t}_\zeta) &= \log(e^u e^{\mathbf{t}_\zeta} e^{-u}) \\ &= \log(\mathrm{Ad}(e^u) \exp \mathbf{t}_\zeta) \\ &= \log(\exp \mathrm{Ad}(e^u) \mathbf{t}_\zeta) \\ &= \mathrm{Ad}(e^u)\mathbf{t}_\zeta.\end{aligned}$$

If $\zeta = 1$ and ℓ is the trivial path, then $(\exp \boldsymbol{\sigma})(\mathbf{t}_1) = \mathbf{t}_1$. Thus, $\mathcal{K}_N$ acts on $\mathfrak{p}(E'_{\vec{\mathsf{u}}}, \vec{\mathsf{v}}_1)$ via special automorphisms. Again applying logarithm shows

$$\boldsymbol{\sigma}(\mathbf{t}_\zeta) = \log(\mathrm{Ad}(e^u))(\mathbf{t}_\zeta) = \log(\exp(\operatorname{ad} u)) \cdot \mathbf{t}_\zeta = [u, \mathbf{t}_\zeta].$$

Thus, $\phi_{\mathrm{ell}}(\boldsymbol{\sigma})$ is a special derivation of $\mathfrak{p}(E'_{\vec{\mathsf{u}}}, \vec{\mathsf{v}}_1)$. □

Lemma 5.5.4. *$W_{-2} \operatorname{SDer} \mathfrak{p}_N = M_{-2} W_{-2} \operatorname{SDer} \mathfrak{p}_N$.*

Proof. Suppose $\delta \in W_{-2} \operatorname{SDer} \mathfrak{p}_N$. Every nontrivial component of $\delta(\mathbf{X})$ are Lie words with degree at least one in $\mathbf{Y}$ or $\mathbf{t}_\zeta$, and thus $\delta(\mathbf{X}) \in M_{-2}\mathfrak{p}_N$. We also have $\delta(\mathbf{t}_\zeta) = [u_\zeta, \mathbf{t}_\zeta]$ since δ is special. If $u_\zeta \in W_{-2}$, then $u_\zeta \in M_{-2}$ as well (see filtrations in §3.5).

It remains to show $\delta(\mathbf{Y}) \in M_{-4}\mathfrak{p}_N$. We know

$$[\mathbf{X}, \delta(\mathbf{Y})] = \delta([\mathbf{X}, \mathbf{Y}]) - [\delta(\mathbf{X}), \mathbf{Y})].$$

Since $[\mathbf{X}, \mathbf{Y}] = \sum \mathbf{t}_\zeta$, we have $\delta([\mathbf{X}, \mathbf{Y}]) \in M_{-4}\mathfrak{p}_N$. Since $\delta(\mathbf{X}) \in M_{-2}\mathfrak{p}_N$, we also know $[\delta(\mathbf{X}), \mathbf{Y}] \in M_{-4}\mathfrak{p}_N$. It follows that $\delta(\mathbf{Y}) \in M_{-4}\mathfrak{p}_N$, hence $\delta \in M_{-2} \operatorname{SDer} \mathfrak{p}_N$. □

Proposition 5.5.5. *For $m \geq 2$, $\phi_{\mathrm{ell}}(\boldsymbol{\sigma}_{m,\zeta}) \in W_{-m-1}\operatorname{SDer}\mathfrak{p}(E'_{\vec{\mathrm{u}}}, \vec{\mathrm{v}}_1)$.*

Proof. Suppose $\phi_{\mathrm{ell}}(\boldsymbol{\sigma}_{m,\zeta}) \in W_{-k}\operatorname{SDer}\mathfrak{p}(E'_{\vec{\mathrm{u}}}, \vec{\mathrm{v}}_1)$ and $\phi_{\mathrm{ell}}(\boldsymbol{\sigma}_{m,\zeta}) \notin W_{-k-1}\operatorname{SDer}\mathfrak{p}(E'_{\vec{\mathrm{u}}}, \vec{\mathrm{v}}_1)$. Since the sub-variation of MHS of special derivations is admissible, it follows from Lemma 5.5.1 that $\phi_{\mathrm{ell}}(\boldsymbol{\sigma}_{m,\zeta})$ is a lowest weight vector in the $\mathfrak{sl}_2$ representation $\operatorname{Gr}^W_{-k}\operatorname{SDer}\mathfrak{p}(E'_{\vec{\mathrm{u}}}, \vec{\mathrm{v}}_1)$. Since $\phi_{\mathrm{ell}}(\boldsymbol{\sigma}_{m,\zeta}) \in M_{-2m}\operatorname{SDer}\mathfrak{p}(E'_{\vec{\mathrm{u}}}, \vec{\mathrm{v}}_1)$, the isomorphism in Definition 2.3.2 implies the corresponding highest weight vector is in

$$M_{2m-2k}\operatorname{Gr}^W_{-k}\operatorname{SDer}\mathfrak{p}(E'_{\vec{\mathrm{u}}}, \vec{\mathrm{v}}_1).$$

By assumption $m \geq 2$ and thus $k \geq 2$ since $\operatorname{Gr}^M_{-1}\mathfrak{p}_N = 0$. Since the highest weight vector is by definition nontrivial, it follows from the previous lemma that $k \geq m+1$. □

Theorem 5.5.6. *When $N \geq 2$ and $\zeta \in \boldsymbol{\mu}_N$ is primitive, the head of $\phi_{\mathrm{ell}}(\boldsymbol{\sigma}_{m,\zeta})$ is congruent to*

$$\epsilon^{\mathrm{op}}_{m+1,\zeta} + \sum_{\eta \text{ not primitive}} c_\eta \epsilon^{\mathrm{op}}_{m+1,\xi}$$

in $\operatorname{Gr}^{-m-2}_W \operatorname{Der}(\mathfrak{p}(E'_{\vec{\mathrm{u}}}, \vec{\mathrm{v}}_1)/D^2)$ for some constants $c_\eta \in \mathbb{Q}$.

Proof. Recall from §5.2.3 the limit MHS of the cyclotomic polylog $\mathfrak{p}(U_N, \vec{\mathrm{v}}_1)/D^2$ is the pull-back of the direct product of extensions $-E_{m,\zeta}(1)$. For $\zeta \in \boldsymbol{\mu}_N$ primitive, the action of $\boldsymbol{\sigma}_{m,\zeta}$ (see §5.4) on $-E_{m,\zeta}(1)$ is given by

$$\boldsymbol{\sigma}_{m,\zeta} : \mathbf{e}_0 \longmapsto -\mathbf{e}_0^m \cdot \mathbf{e}_\zeta.$$

And since $-E_{m,\bar{\zeta}} \equiv (-1)^m E_{m,\zeta}$ as extensions of MHS, the action of $\boldsymbol{\sigma}_{m,\zeta}$ on $-E_{m,\bar{\zeta}}(1)$ is

$$\boldsymbol{\sigma}_{m,\zeta} : \mathbf{e}_0 \longmapsto (-1)^m \mathbf{e}_0^m \cdot \mathbf{e}_{\bar{\zeta}}.$$

Applying the Hain map, the action of $\boldsymbol{\sigma}_{m,\zeta}$ on the pull-back of the direct sum $V_{m,\zeta} \oplus V_{m,\zeta}$ is equivalent to $\epsilon^{\mathrm{op}}_{m+1,\zeta}$ (this is essentially the same argument as in Corollary 5.2.7).

Moreover, $\boldsymbol{\sigma}_{m,\zeta}$ acts trivially on $-E_{m,\xi}(1)$ for all primitive $\xi \notin \{\zeta, \bar{\zeta}\}$. Thus, $\phi_{\mathrm{ell}}(\boldsymbol{\sigma}_{m,\zeta})$ has a trivial $\epsilon^{\mathrm{op}}_{m+1,\xi}$ component. The constants c_η are determined by the decomposition of

$E_{m,\eta}$ for η primitive into a sum of extensions $E_{m,\xi}$ with ξ primitive. By Lemma 5.2.4, we know such a decomposition exists and that the coefficients are rational.

Since each derivation $\epsilon^{\mathrm{op}}_{m+1,\zeta}$ has W-weight $-m-1$, it follows from the previous result that this is the head of $\phi_{\mathrm{ell}}(\boldsymbol{\sigma}_{m,\zeta})$. □

When N is a prime power, we can use Example 5.3.5 to compute the constants c_η.

Theorem 5.5.7. *When $N = p^n$ is a prime power and ζ is a primitive Nth root of unity, the head of $\phi_{\mathrm{ell}}(\boldsymbol{\sigma}_{m,\zeta})$ is congruent to*

$$\epsilon^{\mathrm{op}}_{m+1,\zeta} + \frac{N^{m-1}}{1-p^{m-1}}\epsilon^{\mathrm{op}}_{m+1,1} + \sum_{k=1}^{n-1} p^{k(m-1)}\epsilon^{\mathrm{op}}_{m+1,\zeta^{p^k}}$$

in $\mathrm{Gr}_W^{-m-2}\,\mathrm{Der}(\mathfrak{p}(E'_{\vec{\mathrm{u}}},\vec{\mathrm{v}}_1)/D^2)$.

Proof. By the previous statement, $\epsilon^{\mathrm{op}}_{m+1,\zeta}$ is the only derivation indexed by a primitive root of unity to appear in the head.

For $1 \leq k < n$, one deduces from (5.3.3) that $\boldsymbol{\sigma}_{m,\zeta}$ acts on $-E_{m,\zeta^{p^k}}(1)$ by

$$\boldsymbol{\sigma}_{m,\zeta} : \mathbf{e}_0 \longmapsto -p^{k(m-1)}\mathbf{e}_0^m \cdot \mathbf{e}_{\zeta^{p^k}}.$$

Similarly, the action on $-E_{m,\overline{\zeta}^{p^k}}(1)$ is

$$\boldsymbol{\sigma}_{m,\zeta} : \mathbf{e}_0 \longmapsto (-1)^m p^{k(m-1)}\mathbf{e}_0^m \cdot \mathbf{e}_{\overline{\zeta}^{p^k}}.$$

The extension $E_{m,1}$ decomposes as a $\mathbb{Q}$-linear combination of $E_{m,\zeta}$ with ζ primitive (see Example 5.3.5). It follows that $\boldsymbol{\sigma}_{m,\zeta}$ acts on $-E_{m,1}(1)$ by

$$\boldsymbol{\sigma}_{m,\zeta} : \mathbf{e}_0 \longmapsto -\frac{N^{m-1}}{1-p^{m-1}}\left(1+(-1)^m\right)\mathbf{e}_0^m \cdot \mathbf{e}_1.$$

Applying the Hain map yields the result

$$\phi_{\mathrm{ell}}(\boldsymbol{\sigma}_{m,\zeta}) \equiv \epsilon^{\mathrm{op}}_{m+1,\zeta} + \frac{N^{m-1}}{1-p^{m-1}}\epsilon^{\mathrm{op}}_{m+1,1} + \sum_{k=1}^{n-1} p^{k(m-1)}\epsilon^{\mathrm{op}}_{m+1,\zeta^{p^k}}.$$

□

This formula respects the relations (5.4.1) thanks to the relation between derivations (3.2.3).

Corollary 5.5.8. *When N is prime, the head of $\phi_{\mathrm{ell}}(\boldsymbol{\sigma}_{m,\zeta})$ reduces to*

$$\phi_{\mathrm{ell}}(\boldsymbol{\sigma}_{m,\zeta}) \equiv \epsilon^{\mathrm{op}}_{m+1,\zeta} + \frac{1}{N^{1-m}-1}\epsilon^{\mathrm{op}}_{m+1,1}.$$

Remark 5.5.9. When $N = 1$, one simply has $\phi_{\mathrm{ell}}(\boldsymbol{\sigma}_{m,1}) \equiv \frac{1}{2}\epsilon_{m,1}$. This is consistent with [38, Prop. 29.4].

To determine the head of each $\phi_{\mathrm{ell}}(\boldsymbol{\sigma}_{m,\zeta})$ when N is not a prime power, one must decompose each $E_{m,\zeta}$ in terms of a basis of $\mathrm{Ext}^1_{\mathsf{MTM}_N}(\mathbb{Q}, \mathbb{Q}(m))$. An algorithm to do so in outlined in the proof of Lemma 5.2.4, but the computation is arduous for N large. We give an example by computing the heads of ϕ_{ell} when $N = 6$. This is the only one of Deligne's exceptional values of N [20] that is not a prime power.

Example 5.5.10. Fix a primitive sixth root of unity ζ. One can use the distribution relation (5.3.2) to verify

$$\mathrm{Li}_m(-1) = (2^{1-m} - 1)\mathrm{Li}_m(1)$$

and

$$\mathrm{Li}_m(\zeta^2) \equiv \frac{\mathrm{Li}_m(\zeta)}{2^{1-m} + (-1)^m} \bmod (2\pi i)^m\mathbb{Q}.$$

It follows from Lemma 2.7.4 that

$$\mathrm{Li}_m(\zeta^5) \equiv (-1)^{m+1}\mathrm{Li}_m(\zeta) \quad \text{and} \quad \mathrm{Li}_m(\zeta^4) \equiv (-1)^{m+1}\mathrm{Li}_m(\zeta^2)$$

mod $(2\pi i)^m\mathbb{Q}$. If m is odd, then

$$\begin{aligned}\zeta(m) &= 6^{m-1}\sum_{\eta\in\boldsymbol{\mu}_N}\mathrm{Li}_m(\eta)\\ &\equiv 6^{m-1}\left(2^{1-m}\zeta(m) + \frac{2}{1-2^{m-1}}\mathrm{Li}_m(\zeta)\right) \bmod (2\pi i)^m\mathbb{Q}.\end{aligned}$$

When m is even, we of course have $\zeta(m) \equiv 0$. Rearranging yields

$$\zeta(m) \equiv \frac{1+(-1)^m}{(6^{1-m}-2^{1-m})(1-2^{m-1})}\mathrm{Li}_m(\zeta) \bmod (2\pi i)^m\mathbb{Q}.$$

Thus, the action of $\phi_{\mathrm{cyc}}(\boldsymbol{\sigma}_{m,\zeta})$ on $\mathfrak{p}(U_6, \vec{\mathrm{v}}_1)$ is

$$\begin{aligned}\phi_{\mathrm{cyc}}(\boldsymbol{\sigma}_{m,\zeta}) : \mathbf{e}_0 \longmapsto & -\mathbf{e}_0^m \cdot (\mathbf{e}_\zeta + (-1)^{m+1}\mathbf{e}_{\zeta^5}) \\ & - \frac{1}{2^{1-m} + (-1)^m}\mathbf{e}_0^m \cdot (\mathbf{e}_{\zeta^2} + (-1)^{m+1}\mathbf{e}_{\zeta^4}) \\ & - \frac{1 + (-1)^{m+1}}{(6^{1-m} - 2^{1-m})(1 - 2^{m-1})}\mathbf{e}_0^m \cdot \mathbf{e}_1 \\ & - \frac{1 + (-1)^{m+1}}{(3^{1-m} - 1)}\mathbf{e}_0^m \cdot \mathbf{e}_{-1}.\end{aligned}$$

Finally, apply the Hain map to see

$$\begin{aligned}\phi_{\mathrm{ell}}(\boldsymbol{\sigma}_{m,\zeta}) \equiv \epsilon^{\mathrm{op}}_{m+1,\zeta} & + \frac{1}{2^{1-m} + (-1)^m}\epsilon^{\mathrm{op}}_{m+1,\zeta^2} \\ & + \frac{1}{(6^{1-m} - 2^{1-m})(1 - 2^{m-1})}\epsilon^{\mathrm{op}}_{m+1,1} \\ & + \frac{1}{(3^{1-m} - 1)}\epsilon^{\mathrm{op}}_{m+1,-1}\end{aligned}$$

mod $W_{-m-2}\operatorname{Der}(\mathfrak{p}(E'_{\vec{\mathrm{u}}}, \vec{\mathrm{v}}_1)/D^2)$.

Chapter 6

Conclusions

In §5.5 we computed an explicit action in depth 1 of the motivic Lie algebra $\mathfrak{t}_N^{\mathrm{DR}}$ on the Lie algebra of the unipotent fundamental group of a first order smoothing of the nodal cubic with $\boldsymbol{\mu}_N$ removed. This was done by proving the KZB connection underlies an admissible variation of MHS in §3.5, constructing a cyclotomic analogue of the Hain morphism in §4.3.2, proving $\mathfrak{p}(E_{\vec{\mathrm{u}}}, \vec{\mathrm{v}}_1)$ is a pro-object of the category of mixed Tate motives over $\mathbb{Z}[\boldsymbol{\mu}_N, 1/N]$ in §4.4, and relating the cyclotomic and elliptic polylogarithm quotients above the nodal cubic in §5.1 and §5.2.

Via the representation computed in §5.5, the relations between the derivations $\epsilon_{m+1,\zeta}$ appearing the KZB connection are the same as relations in the depth graded quotient $\mathrm{Gr}_D^2 \mathfrak{t}_N^{\mathrm{DR}}$. These are expected to be related to cusp forms of $\Gamma_1(N)$. Pollack [46] computed all such relations when $N = 1$, and his relations are exactly dual to the Gangl–Kaneko–Zagier relations between depth 2 MZVs [26]. We hope to use the results of §5.5 to compute a presentation of $\mathrm{Gr}_D^2 \mathfrak{t}_N^{\mathrm{DR}}$ and gain an analogous understanding of the relationship between N-cyclotomic MZVs and cusp forms of level $N > 1$. Such a presentation may also provide explanations using modular forms for results of Deligne [20] and Goncharov [27, 28, 29] concerning the depth graded quotients of $\mathfrak{t}_N^{\mathrm{DR}}$ and the periods of MTM_N.

Appendix A

Invariance of the KZB connection

The following is a proof of Proposition 3.2.2. The calculation is similar to the level 1 proof in [37, §9].

A.1 Review of Lie theory

First recall some elementary Lie theory. Let V be a vector space and $\varphi, \phi \in \operatorname{End} V$. Then

$$\exp(\operatorname{ad}\phi)(\varphi) = e^{\phi} \circ \varphi \circ e^{-\phi}. \tag{A.1.1}$$

Thus, in our notation, if $\delta \in \operatorname{Der}\mathfrak{p}_N$ and $\phi \in \mathbb{C}\langle\langle \mathbf{X}, \mathbf{Y}, \mathbf{t}_\alpha \rangle\rangle$, then

$$\exp(\phi) \cdot \delta = e^{\phi} \circ \delta \circ e^{-\phi}.$$

Lemma A.1.1. *If $f \in \mathbb{C}\langle\langle \mathbf{X}, \mathbf{Y}, \mathbf{t}_\alpha \rangle\rangle$ and δ is a continuous derivation of $\mathbb{C}\langle\langle \mathbf{X}, \mathbf{Y}, \mathbf{t}_\alpha \rangle\rangle$, then*

$$e^{-f}\delta(e^f) = \frac{1 - \exp(-\operatorname{ad}_f)}{\operatorname{ad}_f}\delta(f). \tag{A.1.2}$$

Proof. Both $e^{-sf}\delta(e^{sf})$ and $\frac{1-\exp(-s\operatorname{ad}_f)}{\operatorname{ad}_f}\delta(f)$ satisfy the differential equation $X'(s) = \delta(f) - \operatorname{ad}_u(X)$ and vanish at $s = 0$. Therefore, they are equal when $s = 1$. See [37, §4]. □

Lemma A.1.2. *If $\delta \in \operatorname{Der}\mathfrak{p}_N$ and $m \in \mathbb{C}$, then*

$$e^{-m\mathbf{X}} \cdot \delta = \delta + \frac{1 - e^{-m\mathbf{X}}}{\mathbf{X}}\delta(\mathbf{X}). \tag{A.1.3}$$

Proof. This is a straightforward application of (A.1.1) and (A.1.2). Suppose $\mathbf{v} \in \mathfrak{p}_N$. Then

$$\begin{aligned}(e^{-m\mathbf{X}} \cdot \delta)(\mathbf{v}) &= (e^{-m\mathbf{X}} \circ \delta \circ e^{m\mathbf{X}})(\mathbf{v}) \\ &= e^{-m\mathbf{X}} \cdot (\delta(e^{m\mathbf{X}} \cdot \mathbf{v})) \\ &= e^{-m\mathbf{X}} \cdot (e^{m\mathbf{X}} \cdot (\delta(\mathbf{v}))) + e^{-m\mathbf{X}} \cdot (\delta(e^{m\mathbf{X}}) \cdot \mathbf{v}) \\ &= \delta(\mathbf{v}) + \frac{1 - \exp(-m\mathbf{X})}{m\mathbf{X}} \delta(m\mathbf{X}) \cdot \mathbf{v} \\ &= \delta(\mathbf{v}) + \frac{1 - \exp(-m\mathbf{X})}{\mathbf{X}} \delta(\mathbf{X}) \cdot \mathbf{v}.\end{aligned}$$

□

Corollary A.1.3. *For all $\mathbf{v} \in \mathfrak{p}_N$ and $m \in \mathbb{C}$,*

$$e^{-m\mathbf{X}} \cdot (\operatorname{ad} \mathbf{v}) = \operatorname{ad}(e^{-m\mathbf{X}} \cdot \mathbf{v}).$$

Proof. It follows from the previous lemma

$$\begin{aligned}e^{-m\mathbf{X}} \cdot (\operatorname{ad} \mathbf{v}) &= (\operatorname{ad} \mathbf{v}) + \frac{1 - e^{-m\mathbf{X}}}{\mathbf{X}} [\mathbf{v}, \mathbf{X}] \\ &= (\operatorname{ad} \mathbf{v}) - (1 - e^{-m\mathbf{X}}) \cdot \mathbf{v} \\ &= e^{-m\mathbf{X}} \cdot \mathbf{v}.\end{aligned}$$

□

A.2 Invariance of KZB_N

The following statements prove Proposition 3.2.2. We proceed by first showing elliptic invariance with respect to $\mathbb{Z}^2$ and then modular invariance with respect to $\Gamma(N)$.

Lemma A.2.1. *For all $(m, n) \in \mathbb{Z}^2$,*

$$(m, n)^* \Omega_{\mathrm{KZB}_N} = e^{-m\mathbf{X}} \cdot \Omega_{\mathrm{KZB}_N}.$$

Proof. The forms $\mathbf{Y}\partial/\partial\mathbf{X}\, d\tau$ and ψ are invariant with respect to $\mathbb{Z}^2$. It follows from Lemma

A.1.2 that

$$(m,n)^*\left(2\pi i\mathbf{Y}\frac{\partial}{\partial\mathbf{X}}\,d\tau\right) = 2\pi i\mathbf{Y}\frac{\partial}{\partial\mathbf{X}}\,d\tau$$
$$= e^{-m\mathbf{X}}\cdot\left(2\pi i\mathbf{Y}\frac{\partial}{\partial\mathbf{X}}\right)\,d\tau - 2\pi i\frac{1-e^{-m\mathbf{X}}}{\mathbf{X}}\mathbf{Y} \qquad \text{(A.2.1)}$$

and

$$(m,n)^*\psi = \psi = e^{-m\mathbf{X}}\psi. \qquad \text{(A.2.2)}$$

Meanwhile,

$$(m,n)^*\left(g_\alpha(\mathbf{x},\ z|\tau)\cdot\mathbf{t}_\zeta\,d\tau\right)$$
$$= g_\alpha(\mathbf{x}, z+m\tau+n|\tau)\cdot\mathbf{t}_\alpha\,d\tau$$
$$= \frac{\partial}{\partial\mathbf{X}}\left(e^{-\alpha_\tau\mathbf{X}}F(\mathbf{X}, z+m\tau+n-\tilde{\alpha},\tau)-\frac{2\pi i}{\mathbf{X}}\right)\cdot\mathbf{t}_\alpha\,d\tau$$
$$= \frac{\partial}{\partial\mathbf{X}}\left(e^{-(\alpha_\tau+m)\mathbf{X}}F(\mathbf{X}, z-\tilde{\alpha},\tau)-\frac{2\pi i}{\mathbf{X}}\right)\cdot\mathbf{t}_\alpha\,d\tau$$
$$= \Big(-(\alpha_\tau+m)e^{-(\alpha_\tau+m)\mathbf{X}}F(\mathbf{X}, z-\tilde{\alpha},\tau)$$
$$+e^{-(\alpha_\tau+m)\mathbf{X}}\frac{\partial F}{\partial\mathbf{X}}(\mathbf{X}, z-\tilde{\alpha},\tau)+\frac{2\pi i}{\mathbf{X}^2}\Big)\cdot\mathbf{t}_\alpha\,d\tau$$
$$= \left(e^{-m\mathbf{X}}\frac{\partial g_\alpha}{\partial\mathbf{X}}(\mathbf{X}, z|\tau)-me^{-(\alpha_\tau+m)\mathbf{X}}F(\mathbf{X}, z-\tilde{\alpha},\tau)+\frac{2\pi i}{\mathbf{X}^2}\right)\cdot\mathbf{t}_\alpha\,d\tau$$

and

$$(m,n)^*\left(h_\alpha(\mathbf{X}, z|\tau)\ \cdot\mathbf{t}_\alpha\,dz\right)$$
$$= \left(e^{-\alpha_\tau\mathbf{X}}F(\mathbf{X}, z+m\tau+n-\tilde{\alpha},\tau)-\frac{2\pi i}{\mathbf{X}}\right)\cdot\mathbf{t}_\alpha\,d(z+m\tau+n)$$
$$= \left(e^{-(\alpha_\tau+m)\mathbf{X}}F(\mathbf{X}, z-\tilde{\alpha},\tau)-\frac{2\pi i}{\mathbf{X}}\right)\cdot\mathbf{t}_\alpha\,(dz+m\,d\tau)$$

Then

$$(m,n)^*(\nu_1+\nu_2) = 2\pi i\mathbf{Y}+\sum_{\alpha\in(N^{-1}\mathbb{Z}/\mathbb{Z})^2}\left(e^{-m\mathbf{X}}\frac{\partial g_\alpha}{\partial\mathbf{X}}(\mathbf{X}, z|\tau)+\frac{2\pi i}{\mathbf{X}^2}\right)\cdot\mathbf{t}_\alpha\,d\tau$$
$$+\sum_{\alpha\in(N^{-1}\mathbb{Z}/\mathbb{Z})^2}\left(e^{-m\mathbf{X}}h_\alpha(\mathbf{X}, z|\tau)-\frac{2\pi i}{\mathbf{X}}\right)\cdot\mathbf{t}_\alpha\,dz$$
$$= e^{-m\mathbf{X}}\cdot\nu_1+2\pi i\frac{1-e^{-m\mathbf{X}}}{\mathbf{X}}\mathbf{Y}+e^{-m\mathbf{X}}\cdot\nu_2. \qquad \text{(A.2.3)}$$

Summing (A.2.1), (A.2.2), and (A.2.3) proves the lemma. □

Corollary A.2.2. *For $(m,n) \in \mathbb{Z}^2$ and $I \in SL_2(\mathbb{Z})$ the identity matrix,*

$$(m,n)^*\Omega_{\mathrm{KZB}_N} = \mathrm{Ad}(\widetilde{M}_{I,(m,n)})\Omega_{\mathrm{KZB}_N} - d\widetilde{M}_{I,(m,n)}\widetilde{M}^{-1}_{I,(m,n)}.$$

Proof. For all $(m,n) \in \mathbb{Z}^2$, the factor of automorphy is $\widetilde{M}_{I,(m,n)} = \exp(-m\mathbf{X})$. Thus,

$$\begin{aligned}\mathrm{Ad}(\widetilde{M}_{I,(m,n)})\Omega_{\mathrm{KZB}_N} &= \exp(-m\mathbf{X}) \circ \Omega_{\mathrm{KZB}_N} \circ \exp(m\mathbf{X}) \\ &= \exp(-m\mathbf{X}) \cdot \Omega_{\mathrm{KZB}_N} \\ &= (m,n)^*\Omega_{\mathrm{KZB}_N}.\end{aligned}$$

The factor of automorphy does not depend on z or τ and thus $d\widetilde{M}_{I,(m,n)} = 0$. □

We next show ∇_{KZB_N} is invariant with respect to $\Gamma(N)$. We shall abbreviate $\widetilde{M}_{\gamma,(0,0)}$ to simply $\widetilde{M}_\gamma$.

Lemma A.2.3. *For all $\gamma \in \Gamma(N)$,*

$$\gamma^*\psi = \mathrm{Ad}(\widetilde{M}_\gamma)\psi. \tag{A.2.4}$$

Proof. By (3.4.1), we have

$$\begin{aligned}\gamma^*\left(A_{m,\alpha}(\tau)\delta_{s,\alpha}\, d\tau\right) &= A_{m,\alpha}(\gamma\tau)\delta_{s,\alpha}\, d(\gamma\tau) \\ &= (c\tau+d)^{s+2}A_{m,\alpha}(\tau)\delta_{s,\alpha}\,\frac{d\tau}{(c\tau+d)^2} \\ &= (c\tau+d)^{s}A_{m,\alpha}(\tau)\delta_{s,\alpha}\, d\tau.\end{aligned}$$

Meanwhile, we know $e^{cz\mathbf{X}/(c\tau+d)} \cdot \delta_{s,\alpha}$ since $\delta_{s,\alpha}$ annihilates $\mathbf{X}$. Thus, we have

$$\begin{aligned}\mathrm{Ad}(\widetilde{M}_\gamma(\tau))\delta_{s,\alpha} &= \mathrm{Ad}(M_\gamma(\tau))\delta_{s,\alpha} \\ &= (c\tau+d)^s\delta_{s,\alpha}.\end{aligned}$$

□

Lemma A.2.4. *For all $\gamma \in \Gamma(N)$,*

$$\begin{aligned}\gamma^*\nu_1 &= \sum_{\alpha\in(N^{-1}\mathbb{Z}/\mathbb{Z})^2} g_\alpha\left(\mathbf{X}, \frac{z}{c\tau+d}\middle|\gamma\tau\right)\cdot \mathbf{t}_\alpha\, d(\gamma\tau)\\ &= cz(c\tau+d)\sum_\alpha e^{cz\mathbf{X}}h_\alpha((c\tau+d)\mathbf{X}, z|\tau)\cdot\mathbf{t}_\alpha\,\frac{d\tau}{(c\tau+d)^2}\\ &\quad + (c\tau+d)^2\sum_\alpha e^{cz\mathbf{X}}g_\alpha((c\tau+d)\mathbf{X}, z|\tau)\cdot\mathbf{t}_\alpha\,\frac{d\tau}{(c\tau+d)^2}\\ &\quad + 2\pi i\frac{cz\mathbf{X}e^{cz\mathbf{X}}-e^{cz\mathbf{X}}+1}{\mathbf{X}}\cdot\mathbf{Y}\frac{d\tau}{(c\tau+d)^2}\end{aligned}$$

Proof. Follows directly from (3.2.5). □

Lemma A.2.5. *For all $\gamma \in \Gamma(N)$,*

$$\mathrm{Ad}(\widetilde{M}_\gamma(\tau))(\nu_1) = \sum_\alpha e^{cz\mathbf{X}}g_\alpha((c\tau+d)\mathbf{X}, z|\tau)\cdot\mathbf{t}_\alpha\, d\tau.$$

Proof. Follows from the formula for the factor of automorphy and Corollary A.1.3. □

Lemma A.2.6. *For all $\gamma \in \Gamma(N)$,*

$$\begin{aligned}\gamma^*\nu_2 &= 2\pi i\mathbf{Y} + \sum_\alpha h_\alpha\left(\mathbf{X}, \frac{z}{c\tau+d}\middle|\gamma\tau\right)\cdot\mathbf{t}_\alpha\, d\left(\frac{z}{c\tau+d}\right)\\ &= 2\pi i\mathbf{Y} + \sum_\alpha\left((c\tau+d)h_\alpha((c\tau+d)\mathbf{X}, z|\tau) + 2\pi i\frac{e^{cz\mathbf{X}}-1}{\mathbf{X}}\right)\cdot\mathbf{t}_\alpha\, d\left(\frac{z}{c\tau+d}\right)\\ &= 2\pi i e^{cz\mathbf{X}}\cdot\mathbf{Y} + (c\tau+d)\sum_\alpha h_\alpha((c\tau+d)\mathbf{X}, z|\tau)\cdot\mathbf{t}_\alpha\left(\frac{dz}{c\tau+d} - \frac{cz\,d\tau}{(c\tau+d)^2}\right).\end{aligned}$$

Proof. Follows directly from (3.2.4). □

Lemma A.2.7. *For all $\gamma \in \Gamma(N)$,*

$$\mathrm{Ad}(\widetilde{M}_\gamma(\tau))(\nu_2) = e^{cz\mathbf{X}}\cdot\left(\frac{2\pi i\mathbf{Y}}{c\tau+d} + c\mathbf{X}\right)dz + \sum_\alpha e^{cz\mathbf{X}}h_\alpha((c\tau+d)\mathbf{X}, z|\tau)\cdot\mathbf{t}_\alpha\, dz.$$

Proof. Follows from the formula for the factor of automorphy and Corollary A.1.3. □

Corollary A.2.8. *For all $\gamma \in \Gamma(N)$,*

$$(\gamma^* - \mathrm{Ad}(\widetilde{M}_\gamma(\tau)))(\nu_1+\nu_2) = -c\mathbf{X}\,dz - 2\pi i\frac{e^{cz\mathbf{X}}-1}{\mathbf{X}}\frac{d\tau}{(c\tau+d)^2}. \tag{A.2.5}$$

Proof. Sum the results of the four previous lemmas. □

Lemma A.2.9. *For all* $\gamma \in \Gamma(N)$,

$$\begin{aligned}
&\mathrm{Ad}(\widetilde{M}_\gamma(\tau)) \left(2\pi i \mathbf{Y} \frac{\partial}{\partial \mathbf{X}}\, d\tau\right) \\
&\qquad = \mathrm{Ad}(M_\gamma(\tau)) \left(2\pi i \mathbf{Y} \frac{\partial}{\partial \mathbf{X}}\, d\tau + 2\pi i \frac{1 - e^{cz\mathbf{X}/(c\tau+d)}}{\mathbf{X}} \cdot \mathbf{Y}\, d\tau\right) \\
&\qquad = \mathrm{Ad}(M_\gamma(\tau)) \left(2\pi i \mathbf{Y} \frac{\partial}{\partial \mathbf{X}}\, d\tau\right) \\
&\qquad\qquad + 2\pi i \frac{1 - e^{cz\mathbf{X}}}{\mathbf{X}} \cdot \left(\mathbf{Y} + \frac{c(c\tau+d)}{2\pi i}\mathbf{X}\right) \frac{d\tau}{(c\tau+d)^2} \\
&\qquad = \mathrm{Ad}(M_\gamma(\tau)) \left(\frac{1}{2\pi i}\mathbf{y} \frac{\partial}{\partial \mathbf{x}}\, d\tau\right) \\
&\qquad\qquad + \left(2\pi i \frac{1 - e^{cz\mathbf{X}}}{\mathbf{X}} \cdot \mathbf{Y} - c^2 z (c\tau+d)\mathbf{X}\right) \frac{d\tau}{(c\tau+d)^2} \qquad \text{(A.2.6)}
\end{aligned}$$

Lemma A.2.10. *For all* $\gamma \in \Gamma(N)$,

$$\begin{aligned}
d\widetilde{M}_\gamma(\tau)\widetilde{M}_\gamma(\tau)^{-1} &= d(M_\gamma(\tau)e^{cz\mathbf{X}/(c\tau+d)})e^{-cz\mathbf{X}/(c\tau+d)}M_\gamma(\tau)^{-1} \\
&= \Big(dM_\gamma(\tau)e^{cz\mathbf{X}/(c\tau+d)} + c\mathbf{X}M_\gamma(\tau)e^{cz\mathbf{X}/(c\tau+d)}\, dz \\
&\qquad - \frac{c^2 z M_\gamma(\tau)}{(c\tau+d)^2}\mathbf{X}e^{cz\mathbf{X}/(c\tau+d)}\, d\tau\Big) e^{-cz\mathbf{X}/(c\tau+d)}M_\gamma(\tau)^{-1} \\
&= dM_\gamma(\tau)M_\gamma(\tau)^{-1} + c\mathbf{X}\, dz - c^2 z\mathbf{X}\, \frac{d\tau}{(c\tau+d)} \qquad \text{(A.2.7)}
\end{aligned}$$

Proposition A.2.11. *The* KZB_N *connection is invariant with respect to* $\Gamma(N)$.

Proof. Sum the equations (A.2.4), (A.2.5), (A.2.6), and (A.2.7) to get

$$\begin{aligned}
&\gamma^*\Omega_{\mathrm{KZB}_N} - \mathrm{Ad}(\widetilde{M}_\gamma(\tau))\Omega_{\mathrm{KZB}_N} + d\widetilde{M}_\gamma(\tau)\widetilde{M}_\gamma(\tau)^{-1} \\
&\qquad = \gamma^* \left(2\pi i \mathbf{Y} \frac{\partial}{\partial \mathbf{X}}\, d\tau\right) - \mathrm{Ad}(M_\gamma(\tau)) \left(2\pi i \mathbf{Y} \frac{\partial}{\partial \mathbf{X}}\, d\tau\right) + dM_\gamma(\tau)M_\gamma(\tau)^{-1}.
\end{aligned}$$

The right hand side is zero by a simple calculation [37, Lemma 9.16]. □

Remark A.2.12. The above proof holds for any congruence subgroup Γ of level N so long as we only include N-torsion α invariant under the action of Γ.

Appendix B

Regularized transport and linearized connections

Let $\mathcal{V}$ be a trivial vector bundle $V \times \mathbb{C}^2 \to \mathbb{C}^2$ with flat connection

$$\nabla = d + \Omega = d + A(x,y)\frac{dx}{x} + B(x,y)\frac{dy}{y}.$$

The functions A and B are holomorphic and take values in $\operatorname{End} V$. Further, suppose the residues $A(0,y)$ and $B(x,0)$ are nilpotent. Let ∇_0 denote the linearized connection along the divisor $y = 0$

$$\nabla_0 = d + \Omega_0 = d + A(x,0)\frac{dx}{x} + B(x,0)\frac{d\lambda}{\lambda}.$$

This defines a connection on the normal bundle of $y = 0$, where the point (x,λ) is the vector $\lambda\partial/\partial y \in T_{(x,0)}\mathbb{C}^2$.

Then $\mathcal{V}$ is Deligne's canonical extension of the local system of flat sections of $\mathcal{V}$ over $\mathbb{C}^\times \times \mathbb{C}^\times$ [18]. Thus, there is holomorphic change of gauge $h : \mathbb{C}^2 \to \operatorname{Aut} V$ with $h(0,0) = \mathrm{id}$ such that

$$\widetilde{\Omega} = h\Omega h^{-1} - (dh)h^{-1} = A(0,0)\frac{dx}{x} + B(0,0)\frac{dy}{y}.$$

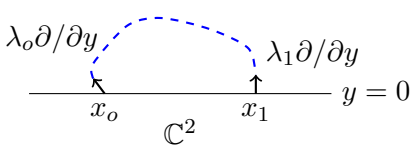
For convenience, set $A = A(0,0)$ and $B(0,0)$. Since the connection is flat, we have $[A,B] = 0$. Computing residues yields

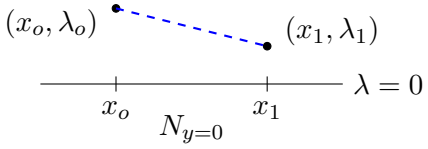
$$h(x,0)B(x,0)h(x,0)^{-1} = B(0,0).$$

Thus, we also have

$$\widetilde{\Omega} = h(x,0)\Omega_0 h(x,0)^{-1} - (dh(x,0))h(x,0)^{-1}.$$

Proposition B.1. *Regularized transport with respect to ∇ from $\mathbf{v}_o = \lambda_o\partial/\partial y$ anchored at $(x_o,0)$ to $\mathbf{v}_1 = \lambda_1\partial/\partial y$ anchored at $(x_1,0)$ is equal to transport with respect to ∇_0 from (x_o,λ_o) to (x_1,λ_1).*





Proof. Apply the definition of regularized transport and use the facts above.

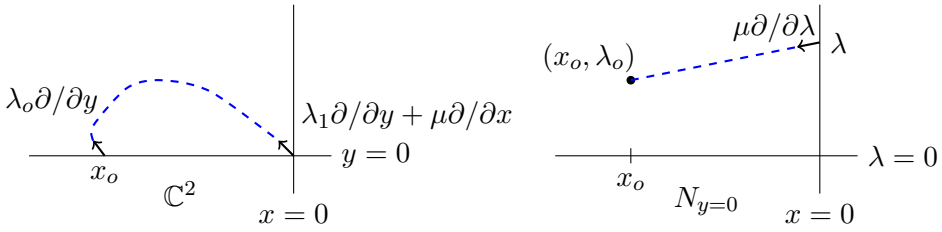
$$\begin{aligned}
T_\Omega(\mathbf{v}_o,\mathbf{v}_1) &= \lim_{y\to 0}(y/\lambda_1)^{B(x_1,0)}T_\Omega((x_o,y),(x_1,y))(y/\lambda_o)^{-B(x_o,0)}\\
&= \lim_{y\to 0}(y/\lambda_1)^{B(x_1,0)}h^{-1}(x_1,y)T_{\widetilde{\Omega}}((x_o,y),(x_1,y))h(x_o,y)(y/\lambda_o)^{-B(x_o,0)}\\
&= \lim_{y\to 0}(y/\lambda_1)^{B(x_1,0)}h(x_1,0)^{-1}(x_o/x_1)^{-A}h(x_o,0)(y/\lambda_o)^{-B(x_o,0)}\\
&= h(x_1,0)^{-1}\left(\lim_{y\to 0}(y/\lambda_1)^{B}(x_1/x_o)^{-A}(y/\lambda_o)^{-B}\right)h(x_o,0)\\
&= h(x_1,0)^{-1}T_{\widetilde{\Omega}}((x_o,\lambda_o),(x_1,\lambda_1))h(x_o,0)\\
&= T_{\Omega_0}((x_o,\lambda_o),(x_1,\lambda_1))
\end{aligned}$$

□

Corollary B.1. *Regularized transport from* $\lambda_o\partial/\partial y \in T_{(x_o,0)}\mathbb{C}^2$ *to*

$$\lambda_1\partial/\partial y + \mu\partial/\partial x \in T_{(0,0)}\mathbb{C}^2$$

with respect to ∇ *is equal to regularized transport from* (x_o,λ_o) *to* $\mu\partial/\partial\lambda \in T_{(0,\lambda_1)}\mathbb{C}$ *with respect to* ∇_0.



Proof. From the previous statement, we know

$$T_{\nabla_0}((x_o,\lambda_o),(x_1,\lambda_1)) = \lim_{y\to 0}(y/\lambda_1)^{B(x_1,0)}T_\nabla((x_o,y),(x_1,y))(y/\lambda_o)^{-B(x_o,0)}.$$

Right multiplying by $(x_1/\mu)^{A(0,0)}$ and taking limits as $x_1 \to 0$ yields the standard regularized transport operators

$$\lim_{x_1 \to 0} (x_1/\mu)^{A(0,0)} T_{\nabla_0}((x_o, \lambda_o), (x_1, \lambda_1)) \\ = \lim_{\substack{x_1 \to 0 \\ y \to 0}} (x_1/\mu)^{A(0,y)} (y/\lambda_1)^{-L_{x_1}} T_{\nabla}((x_o, y), (x_1, y))(y/\lambda_o)^{L_{x_o}}.$$

Note that $\exp(-2\pi i A(0, y))$ is the monodromy operator about $x = 0$ at (x_1, y). □

Corollary B.2. *The Hain map sends the canonical limit MHS of* $\mathfrak{p}(U_N, \vec{\mathrm{v}}_1)$ *to the canonical limit MHS of* $\mathfrak{p}(E'_{\vec{\mathrm{u}}}, \vec{\mathrm{v}}_1)$.

Proof. By the previous statement, regularized transport in $\boldsymbol{\mathcal{P}}_N$ to $\mathfrak{p}(E'_{\vec{\mathrm{u}}}, \vec{\mathrm{v}}_1)$ is equal first transporting to a smooth point of the nodal cubic E_0 and then transporting with respect to the KZ connection to $\vec{\mathrm{v}}_1$. □

Appendix C

Index of notation

$\mathbb{G}_m$	the multiplicative group of a field (usually $\mathbb{C}$)	p. 2
$\boldsymbol{\mu}_N$	the set of Nth roots of unity	p. 1
$\mathfrak{h}$	the upper half plane	p. 7
Y_Γ	the modular curve $\Gamma\backslash\backslash\mathfrak{h}$	p. 9
X_Γ	the compactification of Y_Γ	p. 10
$\mathcal{E}_\Gamma$	the universal elliptic curve over Y_Γ	p. 10
$\overline{\mathcal{E}}_\Gamma$	the compactification of $\mathcal{E}_\Gamma$	p. 10
$\mathcal{E}'_\Gamma$	$\mathcal{E}_\Gamma$ minus single-valued N-torsion	p. 11
E_0	the nodal cubic	p. 10
$\mathbb{H}$	the local system $R_1 f_* \mathbb{Q}$ associated to $f : \mathcal{E}_\Gamma \to Y_\Gamma$	p. 14
$\mathbf{a}, \mathbf{b}$	the Betti $\mathbb{Q}$-basis of $\mathbb{H}$	p. 14
$\mathbf{X}, \mathbf{Y}$	the de Rham basis of $\mathbb{H}$	p. 16
$E_{\vec{\mathsf{u}}}$	the first order smoothing of E_0 in the direction of $\vec{\mathsf{u}}$	p. 16
$G_{m,\alpha}$	$\Gamma(N)$ Eisenstein series of weight m	p. 17
$G_{m,\zeta}$	$\Gamma_1(N)$ Eisenstein series of weight m	p. 18
$\mathbf{L}(S)$	the free Lie algebra on the set S	p. 22
$\pi_1^{\mathrm{un}}(X, x)$	the unipotent completion of $\pi_1(X, x)$	p. 22
$\mathfrak{p}(X, x)$	the Lie algebra of $\pi_1^{\mathrm{un}}(X, x)$	p. 22
$\mathrm{Li}_m(z)$	the mth polylogarithm function	p. 23
$\mathfrak{p}_N$	non-canonical fiber of KZB local system	p. 29
$\epsilon_{m,\alpha}$	a derivation of $\mathfrak{p}_N$ indexed by an N-torsion section α	p. 31
$\mathsf{MTM}(\mathcal{O}_{k,S})$	the category of mixed Tate motives over $\mathcal{O}_{k,S}$	p. 45
$\mathcal{O}_{k,S}$	the ring of S-integers of a number field k	p. 46
$\omega^\bullet$	fiber functor of MTM_N for $\bullet \in \{B, \mathrm{DR}\}$	p. 46

V_k^{DR}	the de Rham realization of an object V of $\mathsf{MTM}(\mathcal{O}_{k,S})$	p. 46
V^B	the Betti realization of an object V of $\mathsf{MTM}(\mathcal{O}_{k,S})$	p. 46
ω	rational fiber functor of $\mathsf{MTM}(\mathcal{O}_{k,S})$	p. 47
V^{DR}	the canonical rational form of V_k^{DR}	p. 48
$\mathcal{G}_{k,S}^{\bullet}$	Fundamental group of $\mathsf{MTM}(\mathcal{O}_{k,S})$ with respect to $\omega^{\bullet}$	p. 50
$\mathcal{K}_{k,S}^{\bullet}$	the prounipotent radical of $\mathcal{G}_{k,S}^{\bullet}$	p. 50
$\mathcal{O}_N$	the ring of S-integers of $\mathbb{Q}(\boldsymbol{\mu}_N)$ with $S = \{p \mid p\|N\}$	p. 50
MTM_N	the category of mixed Tate motives over $\mathcal{O}_N$	p. 50
$\mathfrak{k}_{k,S}^{\mathrm{DR}}$	the Lie algebra of $\mathcal{K}_{k,S}^{\mathrm{DR}}$	p. 51
$\mathcal{P}_{k,S}^{\mathfrak{m}}$	the ring of motivic periods of $\mathsf{MTM}(\mathcal{O}_{k,S})$	p. 51
$\mathbb{L}$	the Lefschetz period of $\mathbb{Q}(-1)$	p. 53
$\mathcal{H}$	Brown's category of Hodge realizations	p. 54
$\omega_{\mathcal{H}}^{\bullet}$	fiber functors of $\mathcal{H}$	p. 54
ω_s	the 1-form $\frac{dw}{w-s}$	p. 55
U_N	$\mathbb{G}_m - \boldsymbol{\mu}_N$	p. 57
$\vec{\mathsf{w}}_\zeta$	the tangent vector $\zeta\partial/\partial w \in T_0\mathbb{C}$	p. 57
$\vec{\mathsf{v}}_\zeta$	the tangent vector $\zeta\partial/\partial w \in T_\zeta\mathbb{C}$	p. 57
$\mathcal{P}_N^{\mathfrak{m}}$	the ring of motivic periods of MTM_N	p. 58
dch	the straight line path from $\vec{\mathsf{w}}_1$ to $-\vec{\mathsf{v}}_1$	p. 59
$\Phi_{0\eta}^{\mathfrak{m}}$	the motivic Drinfeld associator from $\vec{\mathsf{w}}_\zeta$ to $-\vec{\mathsf{v}}_\zeta$	p. 59
$\mathfrak{p}_N^{\mathrm{KZ}}$	non-canonical fiber of KZ local system	p. 60
$\vec{\mathsf{u}}$	the tangent vector $\partial/\partial q$ at $q = 0$ in Y_Γ	p. 62
$E'_{\vec{\mathsf{u}}}$	$E_{\vec{\mathsf{u}}}$ punctured at $\boldsymbol{\mu}_N \hookrightarrow E_{\vec{\mathsf{u}}}$	p. 62
$\mathcal{G}_N^{\bullet}$	Fundamental group of MTM_N with respect to $\omega^{\bullet}$	p. 67
$\mathcal{K}_N^{\bullet}$	the prounipotent radical of $\mathcal{G}_N^{\bullet}$	p. 68
$\mathfrak{k}_N^{\mathrm{DR}}$	the Lie algebra of $\mathcal{K}_N^{\mathrm{DR}}$	p. 68
$E_{m,\zeta}$	the extension of $\mathbb{Q}$ by $\mathbb{Q}$ with period $\mathrm{Li}_m(\zeta)$	p. 79
$\overline{\boldsymbol{\sigma}}_{m,\zeta}$	generator of $H_1(\mathfrak{k}_N)$ dual to $E_{m,\zeta}$	p. 87

$\boldsymbol{\sigma}_{m,\zeta}$	non-canonical lift of $\overline{\boldsymbol{\sigma}}_{m,\zeta}$ to $\mathfrak{k}_N$	p. 87
SDer	special derviations	p. 89

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