A HYBRID GLOBAL-LOCAL NUMERICAL METHOD FOR MULTISCALE PDES

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Abstract. We present a new hybrid numerical method for multiscale partial differential equations, which simultaneously captures both the global macroscopic information and resolves the local microscopic events. The convergence of the proposed method is proved for problems with bounded and measurable coefficient, while the rate of convergence is established for problems with rapidly oscillating periodic or almost-periodic coefficients. Numerical results are reported to show the efficiency and accuracy of the proposed method.

1. Introduction

Consider the elliptic problem with Dirichlet boundary condition

$$\begin{cases} -\operatorname{div}(a^{\varepsilon}(x)\nabla u^{\varepsilon}(x)) = f(x), & x\in D\subset\mathbb{R}^n,\\ u^{\varepsilon}(x)=0, & x\in\partial D,\end{cases} \tag{1.1}$$

where D is a bounded domain in $\mathbb{R}^n$ and ε is a small parameter that signifies explicitly the multiscale nature of the coefficient a^{ε}. We assume a^{ε} belongs to a set $M(\lambda,\Lambda,D)$ that is defined as

$$M(\lambda,\Lambda,D):=\Big\{B\in[L^{\infty}(D)]^{n\times n}\ \mid \xi\cdot B(x)\xi\geq\lambda\left|\xi\right|^2, \left|B(x)\xi\right|\leq\Lambda\left|\xi\right| \text{ for any } \xi\in\mathbb{R}^n \text{ and } a.e.\ x \text{ in } D\Big\},$$

where $|\cdot|$ denotes the Euclidean norm in $\mathbb{R}^n$. We will consider the general case that a^{ε} is not necessarily symmetric.

On the analytic side, the homogenization of (1.1) is well known. In the sense of H-convergence due to Murat and Tartar [31], for every $a^{\varepsilon}\in M(\lambda,\Lambda,D)$ and $f\in H^{-1}(D)$ the sequence $\{u^{\varepsilon}\}$ the solutions of (1.1) satisfies

$$\begin{aligned} u^{\varepsilon} &\rightharpoonup u_0 && \text{weakly in}\quad H_0^1(D), \text{ and}\\ a^{\varepsilon}\nabla u^{\varepsilon} &\rightharpoonup \mathcal{A}\nabla u_0 && \text{weakly in}\quad [L^2(D)]^n,\end{aligned}$$

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where u_0 is the solution of

$$\begin{cases} -\operatorname{div}\left(\mathcal{A}(x)\nabla u_0(x)\right) = f(x), & x \in D, \\ u_0(x) = 0, & x \in \partial D, \end{cases} \tag{1.2}$$

and $\mathcal{A} \in M(\lambda, \Lambda^2/\lambda, D)$. Here $H_0^1(D), L^2(D)$, and $H^{-1}(D)$ are standard Sobolev spaces [3].

For such multiscale PDEs, the quantities of major interest are usually the average behavior together with the local fluctuation of the solution [15]. Several numerical approaches have been developed in recent years for obtaining these information. Those methods can be roughly put into two categories.

The first class is the *global-local* approach firstly proposed in [34,35], and further developed in [5,6]. This is a two stage approach: One first computes the homogenized *global* solution u_0 over the whole domain, then finds the *local* fluctuation by solving an extra problem on a local part of the domain. The homogenized solution may be used as the boundary condition in solving the local problem. The convergence behavior of this approach has been investigated numerically in [30] for problems with many scales, within the heterogeneous multiscale method (HMM) framework [1,16–18].

Another class of methods is based on the idea of domain decomposition, which concerns handshaking multiple operators acting on different parts of the physical domain. Those operators may be either the restrictions of the same governing differential operators to the overlapping or non-overlapping sub-domains [14,19,20,26], or different differential operators that describe fundamentally different physical laws [2,13,23].

In this contribution, we propose a new hybrid method to capture both the average information and the local microscale information simultaneously, as we shall explain in more details below. The current approach is mainly inspired by the recent work [27,28] by two of the authors, in which a hybrid method that couples force balance equations from the atomistic model and the Cauchy-Born elasticity is proposed. Such method is proven to have sharp stability and optimal convergence rate. In the context of multiscale methods, our approach bears some similarity with the Arlequin method [7], which is a rather popular method in engineering literature for multi-modeling and multiscale coupling. The Arlequin method offers a framework to concurrently couple different models together. We refer to [8] for a review and numerical assessment of the Arlequin type methods. Compared with the global-local approach, the proposed hybrid method is a concurrent approach. Compared with the domain decomposition approach, the proposed method smoothly blends together the fine scale problem with the numerical homogenization problem, instead of the usual coupling in domain-decomposition approach via boundary conditions or volumetric matching.

More concretely, the hybrid method couples together the microscopic and the homogenized coefficients as follows. For a transition function ρ satisfying $0 \le \rho \le 1$, we define the hybrid coefficient as

$$b^{\varepsilon}(x) := \rho(x)a^{\varepsilon}(x) + (1-\rho(x))\mathcal{A}(x).$$

Recall that $\mathcal{A}$ is the homogenized coefficient of a^{ε}. We solve the following problem with the hybrid coefficient b^{ε}: find $v^{\varepsilon} \in H_0^1(D)$ such that

$$\langle b^{\varepsilon}\nabla v^{\varepsilon}, \nabla w\rangle = \langle f, w\rangle \qquad \text{for all} \quad w \in H_0^1(D), \tag{1.3}$$

where we denote the $L^2(D)$ inner product by $\langle\cdot,\cdot\rangle$, and the $L^2(\widetilde{D})$ inner product by $\langle\cdot,\cdot\rangle_{L^2(\widetilde{D})}$ for any measurable subset $\widetilde{D} \subset D$.

It is clear that $b^{\varepsilon} \in \mathcal{M}(\lambda, \Lambda^2/\lambda, D)$, and the existence and uniqueness of the solution of Problem (1.3) follows from the Lax-Milgram theorem.

Let $X_h \subset H_0^1(D)$ be a standard Lagrange finite element space consisting of piecewise polynomials of degree $r-1$, we find $v_h \in X_h$ such that

$$\langle b_h^{\varepsilon}\nabla v_h, \nabla w\rangle = \langle f, w\rangle \qquad \text{for all} \quad w \in X_h, \tag{1.4}$$

where

$$b_h^{\varepsilon} = \rho(x)a^{\varepsilon}(x) + (1-\rho(x))\mathcal{A}_h(x),$$

and $\mathcal{A}_h$ is an approximation of $\mathcal{A}$, which may be obtained by HMM type method or any other numerical homogenization / upscaling approaches. For practical concerns, we assume that the support of ρ is small, which means that we essentially solve the homogenized problem in the most part of the underlying domain, where $\rho \simeq 0$, while the original problem is solved wherever the microscale information is of particular interest, where $\rho \simeq 1$. The goal is to get the microscopic information together with the macroscopic behavior with computational cost comparable to solving the homogenized equation.

Note that $b^{\varepsilon}\nabla v^{\varepsilon} = \rho a^{\varepsilon}\nabla v^{\varepsilon} + (1-\rho)\mathcal{A}\nabla v^{\varepsilon}$ is a hybrid flux (i.e., a hybrid stress tensor for elasticity problem), which reads

$$b^{\varepsilon}\nabla v^{\varepsilon} = \begin{cases} a^{\varepsilon}\nabla v^{\varepsilon}, & \text{if} \quad \rho(x) = 1,\\ \mathcal{A}\nabla v^{\varepsilon}, & \text{if} \quad \rho(x) = 0,\\ \rho a^{\varepsilon}\nabla v^{\varepsilon} + (1-\rho)\mathcal{A}\nabla v^{\varepsilon}, & \text{otherwise.}\end{cases}$$

This implies that the proposed hybrid method actually mixes the flux/stress in a weak sense, which is in contrast to the approach in [27, 28] that mixes the forces in a strong sense. This is more appropriate because Problem (1.1) is in divergence form.

We emphasize that the working assumption is that the microscopic information is only desired on a region with relatively small size, which might lie in the interior or possibly near the boundary of the whole domain, or even abut the boundary of the domain. Outside the part where the oscillation is resolved, we could at best hope for capturing the macroscopic information of the solution. This motivates that we should only expect the convergence of the proposed method to the microscopic solution u^ε in a *local* energy norm instead of a global norm. Moreover, such local energy estimate should allow for highly refined grid that is quite often in practice, otherwise, the local events cannot be resolved properly.

The structure of the paper is as follows. In § 2, we study the H-limit of the hybrid method without taking into account the discretization. In § 3, the error estimate of the proposed method with discretization is proved, in particular, the local energy error estimate is established over a highly refined grid, which is the main theoretical result of this paper. One-dimensional example is constructed to show the size-dependence of the estimate over the measure of the support of the transition function ρ. In the last section, we report some numerical examples that validate the proposed method.

Throughout this paper, we shall use standard notations for Sobolev spaces, norms and seminorms, cf., [3], e.g.,

$$\|u\|_{H^1(D)} := \left(\int_D (u^2 + |\nabla u|^2)\,\mathrm{d}x\right)^{1/2}, \quad |u|_{W^{k,p}(D)} := \left(\sum_{|\alpha|=k} \|D^\alpha u\|^p_{L^p(D)}\right)^{1/p}.$$

We use C as a generic constant independent of ε and the mesh size h, which may change at different occurrence.

2. H-Convergence of the hybrid method

Before considering the convergence of the method, we first study the implication of the Arlequin strategy. To separate the influence of the discretization, we consider in this section the continuous Problem (1.3). By H-convergence theory, there exists a matrix $\mathcal{B} \in \mathcal{M}(\lambda, \Lambda^4/\lambda^3, D)$ that is the H-limit of b^ε. The following theorem shows that $\mathcal{B}$ is closely related to $\mathcal{A}$.

Theorem 2.1. *There holds*

$$\|\mathcal{A}(x) - \mathcal{B}(x)\| \le (\Lambda + \lambda)^2 \frac{\Lambda^3}{\lambda^4} \rho(x)\,(1 - \rho(x)) \quad a.e.\ x \in D. \tag{2.1}$$

Furthermore, if a^ε is symmetric, then the above estimate improves to

$$\|\mathcal{A}(x) - \mathcal{B}(x)\| \le 2(\sqrt{\Lambda} + \sqrt{\lambda}) \frac{\Lambda^{3/2}}{\lambda} \rho(x)(1 - \rho(x)) \quad a.e.\ x \in D. \tag{2.2}$$

It follows from the above result that $\mathcal{B} \equiv \mathcal{A}$ whenever $\rho(x) = 0$ or $\rho(x) = 1$, a.e., $x \in D$, which fits the intuition. When $0 < \rho < 1$, the above estimate gives a quantitative estimate about the distance between the effective matrices $\mathcal{B}$ and $\mathcal{A}$.

When a^{ε} is symmetric, the bound (2.2) indeed improves the general bound (2.1) because

$$\begin{aligned}(\Lambda+\lambda)^2\frac{\Lambda^3}{\lambda^4}-2(\sqrt{\Lambda}+\sqrt{\lambda})\frac{\Lambda^{3/2}}{\lambda} &= \frac{\Lambda^{3/2}}{\lambda^4}(\Lambda+\lambda)^2(\Lambda^{3/2}-\lambda^{3/2})\\ &\quad+\frac{\Lambda^{3/2}}{\lambda^{5/2}}\Big((\Lambda+\lambda)(\sqrt{\Lambda}+\sqrt{\lambda})+2\sqrt{\Lambda}\lambda\Big)(\sqrt{\Lambda}-\sqrt{\lambda})\\ &\ge 0.\end{aligned}$$

The proof is based on a perturbation result of H-limit, which can be stated as the following lemma in terms of our notation.

Lemma 2.1. *If $a^{\varepsilon}\in\mathcal{M}(\alpha,\beta,D)$ and $b^{\varepsilon}\in\mathcal{M}(\alpha',\beta',D)$ H-converges to $\mathcal{A}$ and $\mathcal{B}$, and $\|a^{\varepsilon}(x)-b^{\varepsilon}(x)\|\le\epsilon$ for a.e. $x\in D$, then*

$$\max_{x\in D}\|\mathcal{A}(x)-\mathcal{B}(x)\|\le\epsilon\frac{\beta\beta'}{\alpha\alpha'}. \tag{2.3}$$

If both a^{ε} and b^{ε} are symmetric, then

$$\max_{x\in D}\|\mathcal{A}(x)-\mathcal{B}(x)\|\le\epsilon\sqrt{\frac{\beta\beta'}{\alpha\alpha'}}. \tag{2.4}$$

Here $\|\cdot\|$ is the Euclidean norm of a matrix.

The above lemma is essentially contained in [39, Lemma 10.9]. We shall not directly use Lemma 2.1, while our proof largely follows the idea of the proof of this lemma.

Proof of Theorem 2.1 Firstly, we prove

$$\|\mathcal{A}(x)-\mathcal{B}(x)\|\le(\Lambda+\lambda)\frac{\Lambda^4}{\lambda^4}(1-\rho(x))\quad a.e.\ x\in D. \tag{2.5}$$

For any $f,g\in H^{-1}(D)$, we solve

$$\begin{cases}-\operatorname{div}(a^{\varepsilon}\nabla\phi^{\varepsilon})=f, & \text{in}\quad D,\\ \phi^{\varepsilon}=0, & \text{on}\quad\partial D,\end{cases}$$

and

$$\begin{cases}-\operatorname{div}\big((b^{\varepsilon})^t\nabla\psi^{\varepsilon}\big)=g, & \text{in}\quad D,\\ \psi^{\varepsilon}=0, & \text{on}\quad\partial D,\end{cases}$$

where $(b^{\varepsilon})^t$ is the transpose of the matrix b^{ε}. By H-limit theorem, there exist $\mathcal{A}\in\mathcal{M}(\lambda,\Lambda^2/\lambda,D)$ and $\mathcal{B}\in\mathcal{M}(\lambda,\Lambda^4/\lambda^3,D)$ such that

$$\begin{aligned}\phi^{\varepsilon}&\rightharpoonup\phi_0 &&\text{weakly in}\quad H_0^1(D),\\ a^{\varepsilon}\nabla\phi^{\varepsilon}&\rightharpoonup\mathcal{A}\nabla\phi_0 &&\text{weakly in}\quad[L^2(D)]^n,\end{aligned}$$

and

$$\begin{aligned}\psi^{\varepsilon}&\rightharpoonup\psi_0 &&\text{weakly in}\quad H_0^1(D),\\ (b^{\varepsilon})^t\nabla\psi^{\varepsilon}&\rightharpoonup(\mathcal{B})^t\nabla\psi_0 &&\text{weakly in}\quad[L^2(D)]^n,\end{aligned}$$

with

$$\begin{cases} -\operatorname{div}(\mathcal{A}\nabla\phi_0) = f, & \text{in} \quad D, \\ \phi_0 = 0, & \text{on} \quad \partial D, \end{cases}$$

and

$$\begin{cases} -\operatorname{div}\left(\mathcal{B}^t\nabla\psi_0\right) = g, & \text{in} \quad D, \\ \psi_0 = 0, & \text{on} \quad \partial D. \end{cases}$$

By the *Div-Curl Lemma* [38], we conclude

$$\begin{cases} \langle a^{\varepsilon}\nabla\phi^{\varepsilon}, \nabla\psi^{\varepsilon}\rangle \to \langle\mathcal{A}\nabla\phi_0, \nabla\psi_0\rangle & \text{in the sense of measure,} \\ \left\langle (b^{\varepsilon})^t\nabla\psi^{\varepsilon}, \nabla\phi^{\varepsilon}\right\rangle \to \left\langle(\mathcal{B})^t\nabla\psi_0, \nabla\phi_0\right\rangle & \text{in the sense of measure.} \end{cases} \tag{2.6}$$

Therefore, for any $\varphi \in C_0^\infty(D)$, we have

$$\lim_{\varepsilon\to 0} \langle\varphi(b^{\varepsilon} - a^{\varepsilon})\nabla\phi^{\varepsilon}, \nabla\psi^{\varepsilon}\rangle \to \langle\varphi(\mathcal{B} - \mathcal{A})\nabla\phi_0, \nabla\psi_0\rangle\,.$$

Let $\varphi \geq 0$, and we define

$$X{:} = \lim_{\varepsilon\to 0} \langle\varphi(b^{\varepsilon} - a^{\varepsilon})\nabla\phi^{\varepsilon}, \nabla\psi^{\varepsilon}\rangle\,.$$

It is clear that

$$\begin{aligned} X &\leq \limsup_{\varepsilon\to 0} \langle\varphi(1-\rho)\,|(\mathcal{A} - a^{\varepsilon})\nabla\phi^{\varepsilon}|\,, |\nabla\psi^{\varepsilon}|\rangle \\ &\leq \left(\Lambda + \Lambda^2/\lambda\right) \limsup_{\varepsilon\to 0} \langle\varphi(1-\rho)\,|\nabla\phi^{\varepsilon}|\,, |\nabla\psi^{\varepsilon}|\rangle\,. \end{aligned}$$

For any $\alpha > 0$, we bound X as

$$\begin{aligned} X &\leq \left(\frac{\Lambda}{\lambda} + \frac{\Lambda^2}{\lambda^2}\right)\left(\alpha\lambda \limsup_{\varepsilon\to 0}\left\langle\varphi(1-\rho), |\nabla\phi^{\varepsilon}|^2\right\rangle + \frac{\lambda}{4\alpha}\limsup_{\varepsilon\to 0}\left\langle\varphi(1-\rho), |\nabla\psi^{\varepsilon}|^2\right\rangle\right) \\ &\leq \left(\frac{\Lambda}{\lambda} + \frac{\Lambda^2}{\lambda^2}\right)\Big(\alpha \limsup_{\varepsilon\to 0}\langle\varphi(1-\rho)a^{\varepsilon}\nabla\phi^{\varepsilon}, \nabla\phi^{\varepsilon}\rangle \\ &\qquad\qquad + \frac{1}{4\alpha}\limsup_{\varepsilon\to 0}\langle\varphi(1-\rho)b^{\varepsilon}\nabla\psi^{\varepsilon}, \nabla\psi^{\varepsilon}\rangle\Big). \end{aligned}$$

Invoking the *Div-Curl Lemma* (2.6) once again, we obtain

$$\begin{aligned} X &\leq \left(\frac{\Lambda}{\lambda} + \frac{\Lambda^2}{\lambda^2}\right)\left(\alpha\langle\varphi(1-\rho)\mathcal{A}\nabla\phi_0, \nabla\phi_0\rangle + \frac{1}{4\alpha}\langle\varphi(1-\rho)\mathcal{B}\nabla\psi_0, \nabla\psi_0\rangle\right) \\ &\leq \left(\frac{\Lambda}{\lambda} + \frac{\Lambda^2}{\lambda^2}\right)\left(\alpha\frac{\Lambda^2}{\lambda}\left\langle\varphi(1-\rho), |\nabla\phi_0|^2\right\rangle + \frac{1}{4\alpha}\frac{\Lambda^4}{\lambda^3}\left\langle\varphi(1-\rho), |\nabla\psi_0|^2\right\rangle\right), \end{aligned}$$

which implies that for a.e. $x \in D$,

$$|(\mathcal{B} - \mathcal{A})\nabla\phi_0 \cdot \nabla\psi_0| \leq \left(\frac{\Lambda}{\lambda} + \frac{\Lambda^2}{\lambda^2}\right)\big(1 - \rho(x)\big)\left(\alpha\frac{\Lambda^2}{\lambda}\,|\nabla\phi_0|^2 + \frac{1}{4\alpha}\frac{\Lambda^4}{\lambda^3}\,|\nabla\psi_0|^2\right).$$

Optimizing α, we obtain

$$|(\mathcal{B} - \mathcal{A})\nabla\phi_0 \cdot \nabla\psi_0| \leq (\Lambda + \lambda)\,\frac{\Lambda^4}{\lambda^4}\big(1 - \rho(x)\big)\,|\nabla\phi_0|\,|\nabla\psi_0|\,,$$

from which we obtain (2.5).

Next, we prove

$$\|\mathcal{A}(x)-\mathcal{B}(x)\| \le (\Lambda+\lambda)\,\frac{\Lambda^3}{\lambda^3}\,\rho(x) \quad a.e.\ x\in D. \tag{2.7}$$

The proof of (2.7) is essentially the same with the one leads to (2.5) except that we define

$$X\colon=\lim_{\varepsilon\to 0}\langle\varphi(b^{\varepsilon}-\mathcal{A})\nabla\phi_0,\nabla\psi^{\varepsilon}\rangle \qquad \text{for}\quad \varphi\ge 0.$$

It is clear that

$$\begin{aligned} X &\le \limsup_{\varepsilon\to 0}\langle\varphi\rho\,|(\mathcal{A}-a^{\varepsilon})\nabla\phi_0|\,,|\nabla\psi^{\varepsilon}|\rangle \\ &\le \left(\Lambda+\Lambda^2/\lambda\right)\limsup_{\varepsilon\to 0}\langle\varphi\rho\,|\nabla\phi_0|\,,|\nabla\psi^{\varepsilon}|\rangle\,. \end{aligned}$$

For any $\alpha>0$, we bound X as

$$\begin{aligned} X &\le \left(\Lambda+\frac{\Lambda^2}{\lambda}\right)\left(\alpha\left\langle\varphi\rho,|\nabla\phi_0|^2\right\rangle+\frac{1}{4\alpha}\limsup_{\varepsilon\to 0}\left\langle\varphi\rho,|\nabla\psi^{\varepsilon}|^2\right\rangle\right) \\ &\le \left(\Lambda+\frac{\Lambda^2}{\lambda}\right)\left(\alpha\left\langle\varphi\rho\nabla\phi_0,\nabla\phi_0\right\rangle+\frac{1}{4\alpha\lambda}\limsup_{\varepsilon\to 0}\left\langle\varphi\rho b^{\varepsilon}\nabla\psi^{\varepsilon},\nabla\psi^{\varepsilon}\right\rangle\right). \end{aligned}$$

Applying the *Div-Curl Lemma* (2.6) to the second term in the right-hand side of the above equation, we obtain

$$\begin{aligned} X &\le \left(\Lambda+\frac{\Lambda^2}{\lambda}\right)\left(\alpha\left\langle\varphi\rho\nabla\phi_0,\nabla\phi_0\right\rangle+\frac{1}{4\alpha\lambda}\left\langle\varphi\rho\mathcal{B}\nabla\psi_0,\nabla\psi_0\right\rangle\right) \\ &\le \left(\Lambda+\frac{\Lambda^2}{\lambda}\right)\left(\alpha\left\langle\varphi\rho,|\nabla\phi_0|^2\right\rangle+\frac{1}{4\alpha}\frac{\Lambda^4}{\lambda^4}\left\langle\varphi\rho,|\nabla\psi_0|^2\right\rangle\right), \end{aligned}$$

which implies that for a.e. $x\in D$,

$$|(\mathcal{B}-\mathcal{A})\nabla\phi_0\cdot\nabla\psi_0| \le \left(\Lambda+\frac{\Lambda^2}{\lambda}\right)\rho\left(\alpha\,|\nabla\phi_0|^2+\frac{1}{4\alpha}\frac{\Lambda^4}{\lambda^4}\,|\nabla\psi_0|^2\right).$$

Optimizing α, we obtain

$$|(\mathcal{B}-\mathcal{A})\nabla\phi_0\cdot\nabla\psi_0| \le (\Lambda+\lambda)\,\frac{\Lambda^3}{\lambda^3}\,\rho(x)\,|\nabla\phi_0|\,|\nabla\psi_0|\,,$$

from which we obtain (2.7).

Finally we use the convex combination of (2.5) and (2.7) as

$$\begin{aligned} \|\mathcal{A}(x)-\mathcal{B}(x)\| &= \rho(x)\|\mathcal{A}(x)-\mathcal{B}(x)\|+(1-\rho(x))\,\|\mathcal{A}(x)-\mathcal{B}(x)\| \\ &\le (\Lambda+\lambda)^2\frac{\Lambda^3}{\lambda^4}\rho(x)\,(1-\rho(x)) \end{aligned}$$

for $a.e. x\in D$, this leads to (2.1).

If a^{ε} is symmetric, then $\mathcal{A},\mathcal{B}\in\mathcal{M}(\lambda,\Lambda,D)$. Repeating the above procedure, we obtain that the first bound (2.5) changes to

$$\|\mathcal{A}(x)-\mathcal{B}(x)\| \le \frac{2\Lambda^2}{\lambda}(1-\rho(x)) \quad a.e.\ x\in D,$$

and the second bound (2.7) changes to

$$\|\mathcal{A}(x)-\mathcal{B}(x)\| \le 2\Lambda\sqrt{\Lambda/\lambda}\,\rho(x) \quad a.e.\ x\in D.$$

A convex combination of the above two bounds yields (2.2) and finishes the proof. □

When a^ε is locally periodic, i.e., $a^\varepsilon = a(x, x/\varepsilon)$ with $a(x,\cdot)$ is Y-periodic with $Y = (-1/2, 1/2)^n$, we can characterize the effective matrix $\mathcal{B}$ more explicitly since b^ε is also locally periodic with the same period. By classical homogenization theory [9], the effective matrix $\mathcal{B}$ is given by

Theorem 2.2.

$$\mathcal{B}_{ij} = ⨍_Y \left(b_{ij} + b_{ik}\frac{\partial \chi_\rho^j}{\partial y_k}\right)(x,y)\,\mathrm{d}y, \tag{2.8}$$

where $\{\chi_\rho^j(x,y)\}_{j=1}^d$ *is periodic in* y *with period* Y *and it satisfies*

$$-\frac{\partial}{\partial y_i}\left(b_{ik}\frac{\partial \chi_\rho^j}{\partial y_k}\right)(x,y) = \frac{\partial b_{ij}}{\partial y_i}(x,y) \quad in \quad Y, \qquad \int_Y \chi_\rho^j(x,y)\,\mathrm{d}y = 0. \tag{2.9}$$

For $x\in D$ *with* $\rho(x)=0$ *or* $\rho(x)=1$*, we have* $\mathcal{B}=\mathcal{A}$.

In particular, for $n=1$, we have the following explicit formula for $\mathcal{B}$.

$$\mathcal{B}(x) = \left(\int_0^1 \frac{1}{\rho(x)a(x,y) + (1-\rho(x))\mathcal{A}(x)}\,\mathrm{d}y\right)^{-1},$$

where

$$\mathcal{A}(x) = \left(\int_0^1 \frac{1}{a(x,y)}\,\mathrm{d}y\right)^{-1}.$$

It is clear to see when $\rho(x)=0$ or $\rho(x)=1$, we always have $\mathcal{B}(x)=\mathcal{A}(x)$. This is of course expected.

3. Convergence Rate for the Discrete Problem

We now study the convergence rate of the discrete Problem (1.4). We assume that $\mathcal{A}_h \in \mathcal{M}(\lambda', \Lambda', D)$. This is true for any reasonable approximation of $\mathcal{A}$. For example, if we use HMM method [1, 17, 18] to compute the effective matrix, then $\mathcal{A}_h \in \mathcal{M}(\lambda, \Lambda^2/\lambda)$, so we may set $\lambda' = \lambda$ and $\Lambda' = \Lambda^2/\lambda$. By this assumption, we have

$$b_h^\varepsilon \in \mathcal{M}(\lambda\wedge\lambda', \Lambda\vee\Lambda', D).$$

If a^ε is symmetric, we further have $b_h^\varepsilon \in \mathcal{M}(\lambda, \Lambda, D)$.

To step further, let $\mathcal{T}_h$ be a triangulation of D with maximum mesh size h. Denote by h_τ the diameter of each element $\tau\in\mathcal{T}_h$. we assume that all elements in $\mathcal{T}_h$ is shape-regular in the sense of Ciarlet and Raviart [11], that is each $\tau\in\mathcal{T}_h$ contains a ball of radius $c_1 h_\tau$ and is contained in a ball of radius $C_1 h_\tau$ with fixed constants c_1 and C_1.

We assume that the solution of the homogenized problem (1.2) admits H^2-regularity, i.e., there exists C such that

$$\|u_0\|_{H^2(D)} \le C\|f\|_{L^2(D)}. \tag{3.1}$$

Denote $K = \text{supp}\,\rho$ and $|K| := \text{mes}K$, and define

$$\eta(K) = \begin{cases} \left|\ln|K|\right|^{1/2}, & \text{if} \quad n = 2, \\ 1, & \text{if} \quad n = 3. \end{cases}$$

We begin with the following inequality that will be frequently used later on.

Lemma 3.1. *For any $v \in H^1(D)$, and for any subset $\Omega \subset D$, we have*

$$\|v\|_{L^2(\Omega)} \le C\,|\Omega|^{1/n}\,\eta(\Omega)\|v\|_{H^1(D)}. \tag{3.2}$$

Proof. We distinguish the cases of $n = 2, 3$. For $n = 3$, by the Hölder's inequality, we obtain

$$\|v\|_{L^2(\Omega)} \le |\Omega|^{1/3}\,\|v\|_{L^6(\Omega)} \le |\Omega|^{1/3}\,\|v\|_{L^6(D)},$$

which together with the Sobolev immbeding inequality [3]

$$\|\phi\|_{L^6(D)} \le C\|\phi\|_{H^1(D)} \qquad \text{for all} \quad \phi \in H^1(D)$$

yields (3.2) with $n = 3$. The constant C in the above inequality depends only on the properties of D.

As to $n = 2$, for any $p > 2$, using the Hölder's inequality again we obtain

$$\|v\|_{L^2(\Omega)} \le |\Omega|^{1/2-1/p}\,\|v\|_{L^p(\Omega)} \le |\Omega|^{1/2-1/p}\,\|v\|_{L^p(D)},$$

which together with the Sobolev imbeding inequality

$$\|\phi\|_{L^p(D)} \le C\sqrt{p}\|\phi\|_{H^1(D)} \qquad \text{for all} \quad \phi \in H^1(D),$$

yields

$$\|v\|_{L^2(\Omega)} \le C\sqrt{p}\,|\Omega|^{1/2-1/p}\,\|v\|_{H^1(D)}.$$

Taking $p = \left|\ln|\Omega|\right|$ in the above inequality, we obtain (3.2) with $n = 2$. □

Remark. When $n = 1$, the prefactor in (3.2) is replaced by $|\Omega|^{1/2}$ by observing

$$\|v\|_{L^2(\Omega)} \le |\Omega|^{1/2}\,\|v\|_{L^\infty(\Omega)} \le C\,|\Omega|^{1/2}\,\|v\|_{L^\infty(D)} \le C\,|\Omega|^{1/2}\,\|v\|_{H^1(D)},$$

where we have used the imbeding $H^1(D) \hookrightarrow L^\infty(D)$ in the last step.

3.1. **Accuracy for retrieving the macroscopic information.** In this part, we estimate the approximation error between the hybrid solution and the homogenized solution. The following error estimate is based on the theorem of Berger, Scott and Strang [10].

Lemma 3.2. *Let* u_0 *and* v_h *be the solutions of Problem* (1.2) *and Problem* (1.4), *respectively. If we assume that the regularity estimate* (3.1) *holds true, then there exists* C *depends on* λ, Λ *and* D *such that*

$$\begin{aligned} \|\nabla(u_0 - v_h)\|_{L^2(D)} &\le C\left(h + |K|^{1/n}\,\eta(K) + e(HMM)\right)\|f\|_{L^2(D)}. \\ \|u_0 - v_h\|_{L^2(D)} &\le C\left(h^2 + |K|^{2/n}\,\eta^2(K) + e(HMM)\right)\|f\|_{L^2(D)}. \end{aligned} \tag{3.3}$$

where $e(HMM)\colon= \max_{x\in D\setminus K}\|(\mathcal{A}-\mathcal{A}_h)(x)\|$.

Proof. Let $\widetilde{u} \in X_h$ be the solution of

$$\langle \mathcal{A}\nabla\widetilde{u}, \nabla v\rangle = \langle f, v\rangle \qquad \text{for all} \quad v \in X_h.$$

By Cea's lemma, we obtain

$$\|\nabla(u_0 - \widetilde{u})\|_{L^2(D)} \le \frac{\Lambda'}{\lambda'} \inf_{\chi\in X_h} \|\nabla(u_0 - \chi)\|_{L^2(D)}.$$

Denote $w = \widetilde{u} - v_h$ and using the above equation, we obtain

$$\begin{aligned} \langle b_h^\varepsilon \nabla w, \nabla w\rangle &= \langle b_h^\varepsilon\nabla\widetilde{u}, \nabla w\rangle - \langle b_h^\varepsilon \nabla v_h, \nabla w\rangle = \langle b_h^\varepsilon\nabla\widetilde{u}, \nabla w\rangle - \langle f, w\rangle \\ &= \langle b_h^\varepsilon\nabla\widetilde{u}, \nabla w\rangle - \langle \mathcal{A}\nabla\widetilde{u}, \nabla w\rangle = \langle (b_h^\varepsilon - \mathcal{A})\nabla\widetilde{u}, \nabla w\rangle \\ &= \langle (b_h^\varepsilon - \mathcal{A})\nabla(\widetilde{u} - u_0), \nabla w\rangle + \langle (b_h^\varepsilon - \mathcal{A})\nabla u_0, \nabla w\rangle\,. \end{aligned}$$

Using the above estimate, we obtain

$$\begin{aligned} |\langle (b_h^\varepsilon - \mathcal{A})\nabla(\widetilde{u} - u_0), \nabla w\rangle| &\le (\Lambda + \Lambda')\,\|\nabla(u_0 - \widetilde{u})\|_{L^2(D)}\|\nabla w\|_{L^2(D)} \\ &\le (\Lambda + \Lambda')\,\frac{\Lambda'}{\lambda'} \inf_{\chi\in X_h} \|\nabla(u_0 - \chi)\|_{L^2(D)}\|\nabla w\|_{L^2(D)}. \end{aligned}$$

By definition, $b_h^\varepsilon - \mathcal{A} = \rho(a^\varepsilon - \mathcal{A}) + (1-\rho)(\mathcal{A}_h - \mathcal{A})$, using (3.2), we obtain

$$\begin{aligned} |\langle (b_h^\varepsilon - \mathcal{A})\nabla u_0, \nabla w\rangle| &\le \left(\Lambda + \Lambda^2/\lambda\right)\|\nabla u_0\|_{L^2(K)}\|\nabla w\|_{L^2(D)} \\ &\quad + e(\mathrm{HMM})\|\nabla u_0\|_{L^2(D)}\|\nabla w\|_{L^2(D)} \\ &\le C\,|K|^{1/n}\,\eta(K)\|\nabla u_0\|_{H^1(D)}\|\nabla w\|_{L^2(D)} \\ &\quad + e(\mathrm{HMM})\|\nabla u_0\|_{L^2(D)}\|\nabla w\|_{L^2(D)}, \end{aligned}$$

where C depends on Λ, λ and D.

Summing up all the estimates, using the standard interpolate estimate and the regularity estimate (3.1), we conclude $(3.3)_1$.

We exploit *Nitsche's trick* [32] to prove the L^2 estimate. For any $g \in L^2(D)$, we find $\phi \in H_0^1(D)$ such that

$$\langle \nabla\phi, \mathcal{A}\nabla\psi\rangle = \langle g, \psi\rangle \qquad \text{for all} \quad \psi \in H_0^1(D). \tag{3.4}$$

A direct calculation gives

$$
\begin{aligned}
\langle g, u_0 - v_h\rangle &= \langle \mathcal{A}\nabla(u_0 - v_h), \nabla\phi\rangle \\
&= \left\langle \mathcal{A}\nabla(u_0 - v_h), \nabla(\phi - \widetilde{\phi})\right\rangle + \left\langle \mathcal{A}\nabla(u_0 - v_h), \nabla\widetilde{\phi}\right\rangle,
\end{aligned} \tag{3.5}
$$

where $\widetilde{\phi} \in X_h$ is the Lagrange interpolant of ϕ.

The first term can be bounded by

$$
\begin{aligned}
\left|\left\langle \mathcal{A}\nabla(u_0 - v_h), \nabla(\phi - \widetilde{\phi})\right\rangle\right| &\le \frac{\Lambda^2}{\lambda}\|\nabla(u_0 - v_h)\|_{L^2(D)}\|\nabla(\phi - \widetilde{\phi})\|_{L^2(D)} \\
&\le Ch\|\nabla(u_0 - v_h)\|_{L^2(D)}\|\nabla\phi\|_{H^1(D)}.
\end{aligned}
$$

The second term in the right-hand side of (3.5) can be rewritten into

$$
\begin{aligned}
\left\langle \mathcal{A}\nabla(u_0 - v_h), \nabla\widetilde{\phi}\right\rangle &= \left\langle f, \widetilde{\phi}\right\rangle - \left\langle \mathcal{A}\nabla v_h, \nabla\widetilde{\phi}\right\rangle \\
&= \left\langle (b_h^\varepsilon - \mathcal{A})\nabla v_h, \nabla\widetilde{\phi}\right\rangle.
\end{aligned}
$$

By $b_h^\varepsilon - \mathcal{A} = \rho(a^\varepsilon - \mathcal{A}) + (1-\rho)(\mathcal{A}_h - \mathcal{A})$, we obtain

$$
\begin{aligned}
\left|\left\langle \mathcal{A}\nabla(u_0 - v_h), \nabla\widetilde{\phi}\right\rangle\right| \le \Big(&(\Lambda + \Lambda^2/\lambda)\|\nabla v_h\|_{L^2(K)}\|\nabla\widetilde{\phi}\|_{L^2(K)} \\
&+ e(\text{HMM})\|\nabla v_h\|_{L^2(D)}\|\nabla\widetilde{\phi}\|_{L^2(D)}\Big).
\end{aligned}
$$

Using (3.2), we obtain

$$
\begin{aligned}
\|\nabla v_h\|_{L^2(K)} &\le \|\nabla(u_0 - v_h)\|_{L^2(K)} + \|\nabla u_0\|_{L^2(K)} \\
&\le \|\nabla(u_0 - v_h)\|_{L^2(D)} + |K|^{1/n}\,\eta(K)\|\nabla u_0\|_{H^1(D)},
\end{aligned}
$$

and

$$
\begin{aligned}
\|\nabla\widetilde{\phi}\|_{L^2(K)} &\le \|\nabla(\phi - \widetilde{\phi})\|_{L^2(K)} + \|\nabla\widetilde{\phi}\|_{L^2(K)} \\
&\le C\left(h + |K|^{1/n}\,\eta(K)\right)\|\nabla\phi\|_{H^1(D)}.
\end{aligned}
$$

Summing up all the above estimates, using the triangle inequality and $(3.3)_1$, we obtain $(3.3)_2$. □

As in Remark 3, the factors $|K|^{1/n}\,\eta(K)$ in (3.3) should be replaced by $|K|^{1/2}$ in one-dimension. The above estimates show that the solution of the hybrid problem is a good approximation of the solution of the homogenized problem provided that $|K|$ is small. This is expected since in this case we essentially solve the homogenized problem over the main part of the domain D. The dependence on $|K|$ in the estimate $(3.3)_1$ is also essential and sharp, as will be shown by an explicit one-dimensional example below in § 3.3. Similar constructions can be also done for higher dimensions, though it becomes much more tedious.

3.2. **Accuracy for retrieving the local microscopic information.** Parallel to the above results, we have the following energy error estimate for $u^\varepsilon - v_h$. Our proof relies on the *Meyers' regularity* result [29] for Problem (1.1) in an essential way. We state Meyers' results as follows. There exists $p_0 > 2$ that depends on $D, \varLambda$ and λ, such that for all $p \le p_0$,

$$\|\nabla u^\varepsilon\|_{L^p(D)} \le C\|f\|_{L^p(D)} \tag{3.6}$$

with C depends on $D, \varLambda$ and λ.

As a consequence of the higher integrability results for gradients of quasiconformal mappings due to Astala [4], the threshold p_0 established by Astala, Leonetti and Nesi [24] equals to $2\sqrt{\varLambda}/(\sqrt{\varLambda} - \sqrt{\lambda})$, i.e., which depends on the contrast of the media.

Lemma 3.3. *Let u^ε and v_h be the solutions of Problem* (1.1) *and Problem* (1.4)*, respectively. There holds, for any $2 < p < p_0$,*

$$\|\nabla(u^\varepsilon - v_h)\|_{L^2(D)} \le C\left(\inf_{\chi\in X_h}\|\nabla(u^\varepsilon - \chi)\|_{L^2(D)} + |D\setminus K|^{1/2-1/p}\|f\|_{L^p(D)}\right),$$

where C depends on $\lambda, \lambda', \varLambda, \varLambda'$ and D.

Proof. Let $\widetilde{u} \in X_h$ be the solution of

$$\langle a^\varepsilon\nabla\widetilde{u}, \nabla v\rangle = \langle f, v\rangle \qquad \text{for all} \quad v \in X_h.$$

By Cea's lemma, we obtain

$$\|\nabla(u^\varepsilon - \widetilde{u})\|_{L^2(D)} \le \frac{\varLambda}{\lambda}\inf_{\chi\in X_h}\|\nabla(u^\varepsilon - \chi)\|_{L^2(D)}.$$

Denote $w = \widetilde{u} - v_h$ and using the above equation, we obtain

$$\begin{aligned}\langle b_h^\varepsilon\nabla w, \nabla w\rangle &= \langle b_h^\varepsilon\nabla\widetilde{u}, \nabla w\rangle - \langle b_h^\varepsilon\nabla v_h, \nabla w\rangle = \langle b_h^\varepsilon\nabla\widetilde{u}, \nabla w\rangle - \langle f, w\rangle\\ &= \langle b_h^\varepsilon\nabla\widetilde{u}, \nabla w\rangle - \langle a^\varepsilon\nabla\widetilde{u}, \nabla w\rangle = \langle (b_h^\varepsilon - a^\varepsilon)\nabla\widetilde{u}, \nabla w\rangle\\ &= \langle (b_h^\varepsilon - a^\varepsilon)\nabla(\widetilde{u} - u^\varepsilon), \nabla w\rangle + \langle (b_h^\varepsilon - a^\varepsilon)\nabla u^\varepsilon, \nabla w\rangle\,.\end{aligned}$$

By definition, $b_h^\varepsilon - a^\varepsilon = (1-\rho)(\mathcal{A}_h - a^\varepsilon)$, we obtain

$$\begin{aligned}|\langle (b_h^\varepsilon - a^\varepsilon)\nabla(\widetilde{u} - u^\varepsilon), \nabla w\rangle| &\le (\varLambda + \varLambda')\|\nabla(u^\varepsilon - \widetilde{u})\|_{L^2(D)}\|\nabla w\|_{L^2(D)}\\ &\le (\varLambda + \varLambda')\frac{\varLambda}{\lambda}\inf_{\chi\in X_h}\|\nabla(u^\varepsilon - \chi)\|_{L^2(D)}\|\nabla w\|_{L^2(D)},\end{aligned}$$

and for $2 < p < p_0$, using (3.6), we obtain

$$\begin{aligned}|\langle (b_h^\varepsilon - a^\varepsilon)\nabla u^\varepsilon, \nabla w\rangle| &\le (\varLambda + \varLambda')\|\nabla u^\varepsilon\|_{L^2(D\setminus K)}\|\nabla w\|_{L^2(D)}\\ &\le C\,|D\setminus K|^{1/2-1/p}\|f\|_{L^p(D)}\|\nabla w\|_{L^2(D)}.\end{aligned}$$

Combining the above three inequalities, we obtain the desired estimate and finish the proof. ∎

The above estimate indicates that the global microscopic information can be retrieved provided that $|K|$ is big, namely we solve (1.1) almost everywhere, which seems to contradict with our motivation. In fact, our projective is less ambitious, since what we need is the local microscopic information. Therefore, the most relevant error notion is often related to the local norm instead of the global energy error. The following local energy estimate is the main result of this part.

We assume that $\rho \equiv 1$ on $K_0 \subset\subset K$, and $\text{dist}(K_0, \partial K \setminus \partial D) = d \geq \kappa h$ for a sufficiently large $\kappa > 0$, moreover $|\nabla \rho| \leq C/d$ for certain constant C. For a subset $B \in D$, we define

$$H^1_<(B) := \left\{ u \in H^1(D) \mid u|_{D \setminus B} = 0 \right\}.$$

In order to prove the localized energy error estimate, we state some properties of X_h confined to K following those of [12]. More detailed discussion on these properties may be found in [33]. Let G_1 and G be subsets of K with $G_1 \subset G$ and $\text{dist}(G_1, \partial G \setminus \partial D) = \widetilde{d} > 0$. The following assumptions are assumed to hold:

A1: *Local interpolant.* There exists a local interpolant such that for any $u \in H^1_<(G_1) \cap C(G_1)$, $Iu \in X_h \cap H^1_<(G)$.

A2: *Inverse properties.* For each $\chi \in X_h$ and $\tau \in \mathcal{T}_h \cap K, 1 \leq p \leq q \leq \infty$, and $0 \leq \nu \leq s \leq r$,

$$\|\chi\|_{W^{s,q}(\tau)} \leq C h_\tau^{\nu - s + n/p - n/q} \|\chi\|_{W^{\nu,p}(\tau)}. \tag{3.7}$$

A3. *Superapproximation.* Let $\omega \in C^\infty(K) \cap H^1_<(G_1)$ with $|\omega|_{W^{j,\infty}(K)} \leq C \widetilde{d}^{-j}$ for integers $0 \leq j \leq r$ for each $\chi \in X_h \cap H^1_<(G)$ and for each $\tau \in \mathcal{T}_h \cap K$ satisfying $h_\tau \leq d$,

$$\|\omega^2 \chi - I(\omega^2 \chi)\|_{H^1(\tau)} \leq C \left(\frac{h_\tau}{\widetilde{d}} \|\nabla(\omega\chi)\|_{L^2(\tau)} + \frac{h_\tau}{\widetilde{d}^2} \|\chi\|_{L^2(\tau)} \right), \tag{3.8}$$

where the interpolant I is defined in A1.

The assumptions A1,A2 and A3 are satisfied by standard Lagrange finite element defined on shape-regular grids. In particular, the *Superapproximation* property (3.8) was recently proved in [12, Theorem 2.1], which is the key for the validity of the local energy estimate over shape-regular grids.

Theorem 3.1. *Let $K_0 \subset K \subset D$ be given, and let dist$(K_0, \partial K \setminus \partial D) = d$. Let assumptions A1, A2 and A3 hold with $\widetilde{d} = d/16$, in addition, let $\max_{\tau \cap K \neq \emptyset} h_\tau / d \leq 1/16$. Then*

$$\|\nabla(u^\varepsilon - v_h)\|_{L^2(K_0)} \leq C \left(\inf_{\chi \in X_h} \|\nabla(u^\varepsilon - \chi)\|_{L^2(K)} + d^{-1} \|u^\varepsilon - v_h\|_{L^2(K)} \right), \tag{3.9}$$

where C depends only on $\Lambda, \lambda, \Lambda', \lambda'$ and D.

In the above local energy estimate, the first contribution comes from local approximation, the local events may be resolved by the adaptive method that may require highly refined mesh, which is allowed by the above theorem. All the other contributions are encapsulated in the second term $\|u^\varepsilon - v_h\|_{L^2(K)}$, which is a direct consequence of the L^2 estimate $(3.3)_2$ and the triangle inequality, and is included in the following

$$\begin{aligned}\|u^\varepsilon - v_h\|_{L^2(K)} \le C\left(h^2 + |K|^{2/n}\,\eta^2(K) + \max_{x\in D\setminus K}\|(\mathcal{A}-\mathcal{A}_h)(x)\|\right)\|f\|_{L^2(D)}\\ + \|u^\varepsilon - u_0\|_{L^2(D)}.\end{aligned} \tag{3.10}$$

The convergence rate of the approximated solution in L^2 consists of two parts, the first one is how the solution approximates the homogenized solution, which relies on the smoothness of the homogenized solution, the size of the support of the transition function ρ, and the error committed by the approximation of the effective matrix. The second source of the error comes from the convergence rate in L^2 for the homoginization problem. For any bounded and measurable a^ε, $\|u^\varepsilon - u_0\|_{L^2(D)}$ converges to zero as ε tends to zero by H-convergence theory. More structures have to be assumed on a^ε if one were to seek for convergence rate. There are a lot of results for estimating $\|u^\varepsilon - u_0\|_{L^2(D)}$ under various conditions on a^ε. Roughly speaking, $\|u^\varepsilon - u_0\|_{L^2(D)} \simeq \mathcal{O}(\varepsilon^\gamma)$, where γ depends on the properties of the coefficient a^ε and the domain D. We refer to [22] for a careful study of this problem for elliptic system with periodic coefficients. For elliptic systems with almost-periodic coefficients, we refer to [37] and references therein for related discussions.

Remark. It is worth pointing that the above estimate (3.9) is also valid even if the subdomain K_0 abuts the original domain D, which makes practical implementation convenient.

The proof of this theorem is in the same spirit of [33] by combining the ideas of [12] and [36]. In particular, the following Caccioppoli-type estimate for *discrete harmonic function* is a natural adaption of [12, Lemma 3.3], which is crucial for the local energy error estimate.

Lemma 3.4. *Let $K_0 \subset K \subset D$ be given, and let dist$(K_0, \partial K\setminus\partial D) = d$. Let assumptions A1, A2 and A3 hold with $\widetilde{d} = d/4$, and assume that $u_h \in X_h$ satisfies*

$$\langle b_h^\varepsilon \nabla u_h, \nabla v\rangle = 0 \qquad \forall v \in X_h \cap H^1_<(K).$$

In addition, let $\max_{\tau\cap K\neq\emptyset} h_\tau/d \le 1/4$. Then, there exists C such that

$$\|u_h\|_{H^1(K_0)} \le \frac{C}{d}\|u_h\|_{L^2(K)}, \tag{3.11}$$

where C depends only on the constants in (3.7), (3.8), Λ' and λ'.

Proof of Theorem 3.1. Without loss of generality, we may assume that K_0 is the intersection of a ball $B_{d/2}$ with D, the general case may be proved by a covering argument as in [33, Theorem 5.1 and Theorem 5.2]. Let $\widetilde{K}$ be the intersection of a ball $B_{3d/4}$ with D and K be the intersection of a ball B_d with D. Therefore, we have $K_0 \subset\subset \widetilde{K} \subset\subset K$, and $\mathrm{dist}(K_0, \partial\widetilde{K} \setminus \partial D) = d/4$. Let $\widehat{u} = \rho u^\varepsilon$ and define $\widehat{u}_h \in X_h \cap H^1_<(K)$ as the local Galerkin projection of $\widehat{u}$, i.e., $\widehat{u}_h \in X_h \cap H^1_<(K)$ satisfying

$$\langle b_h^\varepsilon \nabla(\widehat{u} - \widehat{u}_h), \nabla v\rangle = F(v) \qquad \forall v \in X_h \cap H^1_<(K),$$

where $F(v) := \langle (b_h^\varepsilon - a^\varepsilon)\nabla u^\varepsilon, \nabla v\rangle$. By coercivity of b_h^ε, we immediately have the stability estimate

$$\|\nabla \widehat{u}_h\|_{L^2(K)} \le C\left(\|\nabla \widehat{u}\|_{L^2(K)} + \|\nabla u^\varepsilon\|_{L^2(K\cap K_1)}\right) \tag{3.12}$$

for certain C that depends only on $\Lambda, \lambda, \Lambda'$ and λ'.

By definition and recalling that $\rho \equiv 1$ on $\widetilde{K}$, we may verify that $\widehat{u}_h - v_h$ is *discrete harmonic* in the sense that for any $v \in X_h \cap H^1_<(\widetilde{K})$, there holds

$$\begin{aligned}
\langle b_h^\varepsilon \nabla(\widehat{u}_h - v_h), \nabla v\rangle &= \langle b_h^\varepsilon \nabla \widehat{u}_h, \nabla v\rangle - \langle f, v\rangle \\
&= \langle b_h^\varepsilon \nabla \widehat{u}, \nabla v\rangle - F(v) - \langle f, v\rangle \\
&= \langle b_h^\varepsilon \nabla u^\varepsilon, \nabla v\rangle - \langle a^\varepsilon \nabla u^\varepsilon, \nabla v\rangle - F(v) \\
&= 0.
\end{aligned}$$

Using (3.11) and recalling that $\rho \equiv 1$ on $\widetilde{K}$, we obtain

$$\begin{aligned}
\|\nabla(u^\varepsilon - v_h)\|_{L^2(K_0)} &\le \|\nabla(\widehat{u} - \widehat{u}_h)\|_{L^2(K_0)} + \|\nabla(\widehat{u}_h - v_h)\|_{L^2(K_0)} \\
&\le \|\nabla(\widehat{u} - \widehat{u}_h)\|_{L^2(K)} + \frac{C}{d}\|\widehat{u}_h - v_h\|_{L^2(\widetilde{K})} \\
&\le \|\nabla(\widehat{u} - \widehat{u}_h)\|_{H^1(K)} + \frac{C}{d}\left(\|\widehat{u}_h - \widehat{u}\|_{L^2(\widetilde{K})} + \|u^\varepsilon - v_h\|_{L^2(\widetilde{K})}\right).
\end{aligned}$$

Using Poincare's inequality, we obtain

$$\|\widehat{u}_h - \widehat{u}\|_{L^2(\widetilde{K})} \le Cd\|\nabla(\widehat{u}_h - \widehat{u})\|_{L^2(\widetilde{K})}.$$

Combining the above two inequalities, we obtain

$$\|\nabla(u^\varepsilon - v_h)\|_{L^2(K_0)} \le C\|\nabla(\widehat{u} - \widehat{u}_h)\|_{L^2(K)} + \frac{C}{d}\|u^\varepsilon - v_h\|_{L^2(\widetilde{K})}. \tag{3.13}$$

Next we use the triangle inequality and (3.12), we obtain,

$$\begin{aligned}
\|\nabla(\widehat{u} - \widehat{u}_h)\|_{L^2(K)} &\le C\left(\|\nabla \widehat{u}\|_{L^2(K)} + \|\nabla u^\varepsilon\|_{L^2(K\cap K_1)}\right) \\
&\le C\left(\|\nabla u^\varepsilon\|_{L^2(K)} + d^{-1}\|u^\varepsilon\|_{L^2(K)} + \|\nabla u^\varepsilon\|_{L^2(K\cap K_1)}\right) \\
&\le C\left(\|\nabla u^\varepsilon\|_{L^2(K)} + d^{-1}\|u^\varepsilon\|_{L^2(K)}\right).
\end{aligned}$$

Substituting the above inequality into (3.13), we obtain

$$\|\nabla(u^\varepsilon - v_h)\|_{L^2(K_0)} \le C\left(\|\nabla u^\varepsilon\|_{L^2(K)} + d^{-1}\|u^\varepsilon\|_{L^2(K)} + \frac{C}{d}\|u^\varepsilon - v_h\|_{L^2(\widetilde{K})}\right).$$

For any $\chi\in X_h$, we write $u^\varepsilon-v_h=(u^\varepsilon-\chi)-(v_h-\chi)$, and we employ the above inequality with u^ε taken to be $u^\varepsilon-\chi$ and v_h taken to be $v_h-\chi$. This implies

$$\begin{aligned}\|\nabla(u^\varepsilon-v_h)\|_{L^2(K_0)}\le C\inf_{\chi\in X_h}&\bigl(\|\nabla(u^\varepsilon-\chi)\|_{L^2(K)}+d^{-1}\|u^\varepsilon-\chi\|_{L^2(K)}\bigr)\\&+\frac{C}{d}\|u^\varepsilon-v_h\|_{L^2(\widetilde{K})}.\end{aligned}\tag{3.14}$$

Let $\chi^*=\arg\inf_{\chi\in X_h}\|\nabla(u^\varepsilon-\chi)\|_{L^2(K)}$, we let

$$\chi=\chi^*+\fint_K(u^\varepsilon-\chi^*)\,\mathrm{d}x.$$

Using Poincare's inequality, there exists C independent the size of K such that

$$\|u^\varepsilon-\chi\|_{L^2(K)}=\|u^\varepsilon-\chi^*-\fint_K(u^\varepsilon-\chi^*)\|_{L^2(K)}\le Cd\|\nabla(u^\varepsilon-\chi^*)\|_{L^2(K)}.$$

Substituting the above inequality into (3.14), we obtain (3.9) and complete the proof. □

3.3. **Example.** To better appreciate the estimates established in (3.3), which was crucial in our analysis, let us consider a one-dimensional problem

$$\begin{cases}-\dfrac{\mathrm{d}}{\mathrm{d}x}\left(a^\varepsilon(x)\dfrac{\mathrm{d}u}{\mathrm{d}x}\right)=0,\quad x\in(0,1),\\[2mm] u(0)=0,\quad a^\varepsilon(1)\dfrac{\mathrm{d}u}{\mathrm{d}x}(1)=1,\end{cases}$$

where $a^\varepsilon(x)=2+\sin(x/\varepsilon)$. A direct calculation gives that the effective coefficient $\mathcal{A}=\sqrt{3}$ and the solution of the homogenized problem is $u_0(x)=x/\mathcal{A}$.

We consider a uniform mesh given by

$$x_0=0<x_1=h<\cdots<x_i=ih<\cdots<x_{2N}=1,$$

where $h=1/(2N)$. The finite element space X_h is simply the piecewise linear element associated with the above mesh with zero boundary condition at $x=0$.

Case $h\gg\varepsilon$. We first consider the case that $h\gg\varepsilon$, while the precise relation between h and ε will be made clear below. Denote $v_h(x_j)=v_j$ and the interval $I_j=(x_{j-1},x_j)$, the mean of the coefficients b^ε over each I_j is denoted by $b_j=\fint_{I_j}b^\varepsilon(x)\,\mathrm{d}x$.

We define the transition function ρ as a piecewise linear function that is supported in $(-2L,2L)$, where L is a fixed number with $0<L<1/4$. Without loss of generality, we assume that $L=Mh$ with M an integer. In particular,

$$\rho=\begin{cases}0 & 0\le x\le x_{N-2M},\\[1mm] \dfrac{x-x_{N-2M}}{L} & x_{N-2K}\le x\le x_{N-M},\\[1mm] 1 & x_{N-M}\le x\le x_{N+M},\\[1mm] \dfrac{x_{N+2M}-x}{L} & x_{N+M}\le x\le x_{N+2M},\\[1mm] 0 & x_{N+2M}\le x\le x_{2N}=1.\end{cases}$$

By construction, we get the size of the support of the transition function ρ is $|K| = 4L$.

We easily obtain the linear system for $\{v_j\}_{j=1}^{2N}$ as

$$\begin{cases} -b_j v_{j-1} + (b_j + b_{j+1}) v_j - b_{j+1} v_{j+1} = 0, \quad j = 1, \cdots, 2N-1, \\ -b_{2N} v_{2N-1} + b_{2N} v_{2N} = h. \end{cases}$$

Define $c_j \colon = (v_j - v_{j-1}) b_j / h$, we rewrite the above equation as

$$c_j - c_{j-1} = 0, \quad j = 1, \cdots, 2N-1, \qquad c_{2N} = 1.$$

Hence $c_j = 1$ for $j = 1, \cdots, 2N$, and the above linear system reduces to

$$(v_j - v_{j-1}) b_j = h.$$

Using $v_0 = 0$, we obtain

$$v_j = h \sum_{i=1}^{j} \frac{1}{b_i}. \tag{3.15}$$

Observing that $v_h(x) = u_0(x)$ for $x \in [0, x_{N-2M}]$ because they are linear functions that coincide at all the nodal points x_i for $i = 0, \cdots, N-2M$.

For $x \in I_{N-2M+j+1}$, we obtain

$$u_0(x) - v_h(x) = h \sum_{i=1}^{j} \left(\frac{1}{\mathcal{A}} - \frac{1}{b_{N-2M+i}} \right) + (x - x_{N-2M+j}) \left(\frac{1}{\mathcal{A}} - \frac{1}{b_{N-2M+j+1}} \right).$$

Define $S_j \colon = h \sum_{i=1}^{j} \left(\frac{1}{\mathcal{A}} - \frac{1}{b_{N-2M+i}} \right)$, we rewrite the above equation as

$$u_0(x) - v_h(x) = \frac{x_{N-2M+j+1} - x}{h} S_j + \frac{x - x_{N-2M+j+1}}{h} S_{j+1}, \tag{3.16}$$

which immediately yields

$$\begin{aligned} \int_{x_{N-2M}}^{x_{N-M}} \left| u_0'(x) - v_h'(x) \right|^2 \mathrm{d}x &= h \sum_{j=1}^{M} \left| \frac{1}{\mathcal{A}} - \frac{1}{b_{N-2M+j}} \right|^2 \\ &\geq \frac{h}{27} \sum_{j=1}^{M} \left| \mathcal{A} - b_{N-2M+j} \right|^2. \end{aligned} \tag{3.17}$$

This is the starting point of later derivation. A direct calculation gives

$$\begin{aligned} b_{N-2M+j} - \mathcal{A} &= ⨍_{I_{N-2M+j}} \rho(x) (a^\varepsilon(x) - \mathcal{A}) \, \mathrm{d}x \\ &= \frac{2 - \mathcal{A}}{2} \big(\rho(x_{N-2M+j-1}) + \rho(x_{N-2M+j}) \big) + ⨍_{I_{N-2M+j}} \sin \frac{x}{\varepsilon} \, \mathrm{d}x \\ &= \frac{(2 - \mathcal{A}) h}{2L} (2j - 1) + ⨍_{I_{N-2M+j}} \sin \frac{x}{\varepsilon} \, \mathrm{d}x, \end{aligned}$$

and an integration by parts yields

$$\fint_{I_{N-2M+j}}\sin\frac{x}{\varepsilon}\,\mathrm{d}x=\frac{2j\varepsilon}{L}\sin\frac{h}{2\varepsilon}\sin\frac{x_{N-2M+j-1/2}}{\varepsilon}$$
$$-\frac{\varepsilon}{L}\cos\frac{x_{N-2M+j-1}}{\varepsilon}+\frac{\varepsilon^2}{Lh}\Bigl(\cos\frac{x_{N-2M+j-1}}{\varepsilon}-\cos\frac{x_{N-2M+j}}{\varepsilon}\Bigr).$$

Combining the above two equations, we obtain

$$b_{N-2M+j}-\mathcal{A}=\frac{(2-\mathcal{A})h}{2L}(2j-1)+\frac{2j\varepsilon}{L}\sin\frac{h}{2\varepsilon}\sin\frac{x_{N-2M+j-1/2}}{\varepsilon}+\mathrm{REM},\tag{3.18}$$

where the remainder term

$$\mathrm{REM}:=-\frac{\varepsilon}{L}\cos\frac{x_{N-2M+j-1}}{\varepsilon}+\frac{\varepsilon^2}{Lh}\Bigl(\cos\frac{x_{N-2M+j-1}}{\varepsilon}-\cos\frac{x_{N-2M+j}}{\varepsilon}\Bigr),$$

which can be bounded as

$$\begin{aligned}|\mathrm{REM}|&\le\frac{\varepsilon}{L}+\frac{2\varepsilon^2}{Lh}\Bigl|\sin\frac{h}{2\varepsilon}\Bigr|\Bigl|\cos\frac{x_{N-2M+j-1/2}}{\varepsilon}\Bigr|\\&\le\frac{\varepsilon}{L}+\frac{2\varepsilon^2}{Lh}\frac{h}{2\varepsilon}=\frac{2\varepsilon}{L}.\end{aligned}$$

Note that $\sum_{j=1}^M(2j-1)^2=M(4M^2-1)/3$, and

$$\sum_{j=1}^M j^2\sin^2\frac{x_{N-2M+j-1/2}}{\varepsilon}\le\sum_{j=1}^M j^2=\frac{1}{6}M(M+1)(2M+1).$$

Summing up all the above estimates and using the elementary inequality

$$(a+b+c)^2+b^2+c^2\ge\frac{a^2}{3}\quad\text{for any}\quad a,b,c\in\mathbb{R},$$

we have, for $M\ge3$,

$$\begin{aligned}\sum_{j=1}^M\left|\mathcal{A}-b_{N-2M+j}\right|^2&\ge\frac{1}{3}\frac{(2-\mathcal{A})^2h^2}{4L^2}\sum_{j=1}^M(2j-1)^2-\frac{4\varepsilon^2}{L^2}\sin^2\frac{h}{2\varepsilon}\sum_{j=1}^M j^2\sin^2\frac{x_{N-2M+j-1/2}}{\varepsilon}\\&\quad-\frac{4Mh\varepsilon^2}{L^2}\\&\ge\frac{(2-\mathcal{A})^2h^2}{36L^2}M(4M^2-1)-\frac{2\varepsilon^2}{3L^2}M(M+1)(2M+1)-\frac{4M\varepsilon^2}{L^2}\\&\ge\frac{(2-\mathcal{A})^2h^2}{36L^2}M(4M^2-1)-\frac{2\varepsilon^2}{3L^2}M(4M^2-1)\\&\ge\frac{(2-\mathcal{A})^2h^2}{72L^2}M(4M^2-1)\end{aligned}$$

provided that $\varepsilon/h\le(2-\mathcal{A})/(4\sqrt{3})$. Substituting the above estimate into (3.17), we obtain

$$\begin{aligned}\int_{x_{N-2M}}^{x_{N-M}}\left|u_0'(x)-v_h'(x)\right|^2\,\mathrm{d}x&\ge\frac{(2-\mathcal{A})^2h^3}{1944L^2}M(4M^2-1)\\&\ge\frac{(2-\mathcal{A})^2h^3}{648L^2}M^3=\frac{(2-\mathcal{A})^2}{648}L.\end{aligned}$$

This implies

$$\|u_0' - v_h'\|_{L^2(1/2-2L,1/2-L)} \ge \frac{2-\mathcal{A}}{18\sqrt{2}} L^{1/2} = \frac{2-\mathcal{A}}{36\sqrt{2}} |K|^{1/2}.$$

This shows that the factor $|K|^{1/2}$ in $(3.3)_1$ is sharp.

Using (3.16), and proceeding along the same line that leads to the above lower bound, we may find that there exists C depending on $\mathcal{A}$ such that

$$\|u_0 - v_h\|_{L^2(1/2-2L,1/2-L)} \ge CL^{3/2} = \frac{C}{8} |K|^{3/2},$$

which explains the size-dependence of $|K|$ in $(3.3)_2$.

Case $h \ll \varepsilon$. We next consider the case when $h \ll \varepsilon$. In fact, we may employ coarser mesh with mesh size H outside the defect region with $H \gg h$, while a finer mesh with mesh size h inside the defect region. The above derivation remains true and we still have $v_h(x) = u_0(x)$ for $x \in [0, 1/2 - 2L]$. We start from the inequality (3.17). Notice that the dominant term in the expression of $b_{N-2M+j} - \mathcal{A}$ is the oscillatory one in (3.18). Denote $\phi = 2h/\varepsilon$. A direct calculation gives

$$\begin{aligned}
&\sum_{j=1}^M j^2 \sin^2 \frac{x_{N-2M+j-1/2}}{\varepsilon} = \frac{1}{2}\sum_{j=1}^M j^2 - \frac{1}{2}\sum_{j=1}^M \cos\frac{x_{2N-4M+2j-1}}{\varepsilon}\\
&= \frac{1}{12} M(M+1)(2M+1)\\
&\quad - \Big\{ \frac{M(M+1)}{4\sin(\phi/2)} \sin[(N-M)\phi] + \frac{M+1}{4\sin^2\phi/2} \cos[(N-M)\phi]\cos\frac{\phi}{2}\\
&\qquad - \frac{\cos[(N-3M/2-1)\phi]\cos\frac{\phi}{2}\sin\frac{M+1}{2}\phi}{4\sin^3(\phi/2)} \Big\}.
\end{aligned}$$

We assume that

$$\sin\frac{\phi}{2} \ge \frac{5}{M}. \tag{3.19}$$

Denote the terms in the curled bracket by I. Given (3.19), using the elementary inequalities $\sin x \le x$ and $\sin x \ge 2x/\pi$ for $0 \le x \le \pi/2$, we bound I as

$$\begin{aligned}
|I| &\le \frac{(M+1)M}{4\sin(\phi/2)} + \frac{M+1}{4\sin^2(\phi/2)} + \frac{(M+1)\phi/2}{4\sin^3(\phi/2)}\\
&\le \frac{(M+1)M}{4\sin(\phi/2)} + \frac{M+1}{4\sin^2(\phi/2)} + \frac{(M+1)\pi}{8\sin^2(\phi/2)}\\
&\le \frac{M^2(M+1)}{12},
\end{aligned}$$

which immediately yields

$$\sum_{j=1}^M j^2 \sin^2 \frac{x_{N-2M+j-1/2}}{\varepsilon} \ge \frac{M^3}{12}.$$

This implies

$$\frac{4\varepsilon^2}{L^2}\sin^2\frac{h}{2\varepsilon}\sum_{j=1}^{M} j^2\sin^2\frac{x_{N-2M+j-1/2}}{\varepsilon} \geq \frac{4\varepsilon^2}{L^2}\left(\frac{2}{\pi}\frac{h}{2\varepsilon}\right)^2\frac{M^3}{12} = \frac{M}{3\pi^2}.$$

Note also

$$\frac{(2-\mathcal{A})^2h^2}{4L^2}\sum_{j=1}^{M}(2j-1)^2 \leq \frac{(2-\mathcal{A})^2}{3}M.$$

Combining the above two estimates, we obtain

$$\begin{aligned}\sum_{j=1}^{M}\left|\mathcal{A}-b_{N-2M+j}\right|^2 &\geq \frac{1}{2}\sum_{j=1}^{M}\left(\frac{2\varepsilon}{L}\sin\frac{h}{2\varepsilon}j\sin\frac{x_{N-2M+j-1/2}}{\varepsilon}+\frac{(2-\mathcal{A})h}{2L}(2j-1)\right)^2\\ &\quad -\frac{4M\varepsilon^2}{L^2}\\ &\geq \left(\frac{1}{6}\left(1/\pi+\mathcal{A}-2\right)^2-\frac{4\varepsilon^2}{L^2}\right)M > 0\end{aligned}$$

provided that

$$h > \frac{\sqrt{6}\varepsilon}{(1/\pi+\mathcal{A}-2)M}.$$

This condition suffices for the validity of (3.19), which is satisfied under a weaker condition $h > 5\pi\varepsilon/(2M)$.

Substituting the above estimate into (3.17), we may find that there exists C depending only on $\mathcal{A}$ such that

$$\|u_0'-v_h'\|_{L^2(1/2-2L,1/2-L)} \geq CL^{1/2} = \frac{C}{2}\left|K\right|^{1/2}.$$

This proves that the factor $|K|^{1/2}$ is sharp for $(3.3)_1$. The same argument shows the size-dependence of $|K|$ in the estimate $(3.3)_2$.

4. Numerical Examples

In this section, we present some numerical examples to demonstrate the accuracy and efficiency of the proposed hybrid method. The domain K_0 whose microstructure is of interesting is assumed to be a square for simplicity, i.e., $K_0 = (-L, L)^2$. The first step in implementation is to construct the transition function ρ. For any parameter $\delta > 0$, we let $\gamma_\delta : [-L-\delta, L+\delta] \to [0,1]$ be a first order differentiable function such that $\gamma_\delta(t) \equiv 1$ for $0 \leq t \leq L$, and $\gamma_\delta'(L) = 0, \gamma_\delta(L+\delta) = \gamma_\delta'(L+\delta) = 0$. Moreover, γ_δ is an even function with respect to the origin. We may extend γ_δ to a function defined on $\mathbb{R}$ by taking $\gamma_\delta(t) = 0$ for $|t| \geq L+\delta$. Finally we define the transition function $\rho(x) := \gamma_\delta(x_1)\gamma_\delta(x_2)$. In particular, we denote $K := (-L-\delta, L+\delta)^2$.

For both examples, we compute the following two quantities

$$\|u^\varepsilon - v_h\|_{H^1(K_0)} \qquad \text{and} \qquad \|u_0 - v_h\|_{H^1(D\setminus K)},$$

which are the two quantities of major interest, and the domain $D = (0,1)^2$. The reference solution u^ε and u_0 are not available analytically, we compute both u^ε and u_0 over very refined mesh, and the details will be given below.

4.1. **An example with two scales.** In this example, we take

$$a^\varepsilon(x) = \frac{(R_1 + R_2\sin(2\pi x_1))(R_1 + R_2\cos(2\pi x_2))}{(R_1 + R_2\sin(2\pi x_1/\varepsilon))(R_1 + R_2\sin(2\pi x_2/\varepsilon))} I,$$

where I is a two by two identity matrix. The effective matrix is given by

$$\mathcal{A}(x) = \frac{(R_1 + R_2\sin(2\pi x_1))(R_1 + R_2\cos(2\pi x_2))}{R_1\sqrt{R_1^2 - R_2^2}} I.$$

In the simulation, we let $R_1 = 2.5, R_2 = 1.5$ and $\varepsilon = 0.01$. The forcing term $f \equiv 1$ and the homogeneous Dirichlet boundary condition is imposed. The subdomain around the defect is $K_0 = (0.5, 0.5) + (-L, L)^2$ with $L = 0.05$.

We compute u^ε with standard linear finite element over a very refined uniform mesh with mesh size $3.33e-4$ which amounts to putting 30 points inside each wave length, i.e., $h \simeq \varepsilon/30$. The homogenized solution u_0 is computed by directly solving the homogenized problem (1.2) with the standard linear finite element over a uniform mesh with mesh size $3.33e-4$, and the above effective matrix $\mathcal{A}$ is employed in the simulation. We take these numerical solutions as the reference solutions, which are still denoted by u^ε and u_0, respectively. Note that the mesh is a body-fitted grid with respective to the defect domain K_0.

We solve the hybrid problem (1.4) over a non-uniform mesh as in Figure 1, which is also a body-fitted mesh.

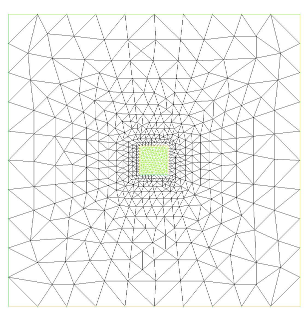


FIGURE 1. Mesh for the hybrid method

We plot the solutions u^ε, u_0 and v_h in Figure 2. It does not seem there is difference among these three solutions.

We also plot the zoomed solutions inside the defect domain K_0 in Figure 3, it seems the hybrid solution approximates the original solution very well because it captures the oscillation of the microstructures inside K_0.

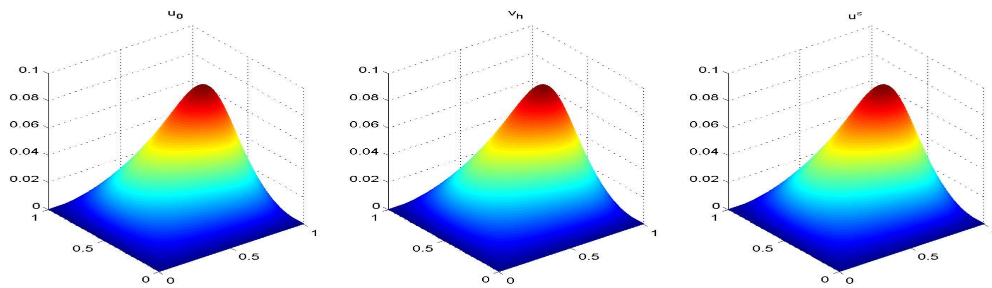


FIGURE 2. Solutions in D, $\delta = 0.05$. Left: the homogenized solution u_0; Middle: the solution of the hybrid method; Right: the solution of the original problem 1.1.

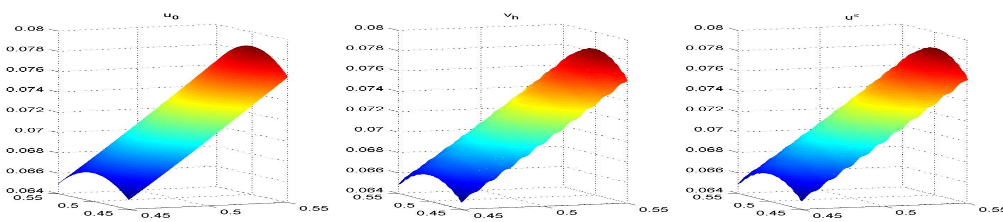


FIGURE 3. Solutions in the defect domain K_0 and $\delta = 0.05$. Left: the homogenized solution u_0; Middle: the solution of the hybrid method; Right: the solution of the original problem 1.1.

We plot the zoomed solutions in $D\backslash K$ in Figure 4, i.e., outsie the defect domain K_0, it seems that the hybrid solution approximates the homogenized solution very well because it is as smooth as the homogenized solution, while there is oscillation in u^ε.

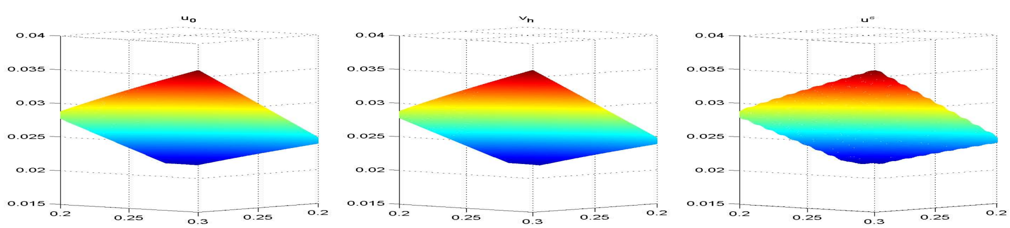


FIGURE 4. Solution in a subdomain of D/K, and $\delta = 0.05$. Left: the homogenized solution u_0; Middle: the solution of the hybrid method; Right: the solution of the original problem 1.1.

Next we plot the localized error $\|u^\varepsilon - v_h\|_{H^1(K_0)}$ in Figure 5, which suggests that the parameter δ has little influence on the local energy error. Similar phenomenon has been observed in [2].

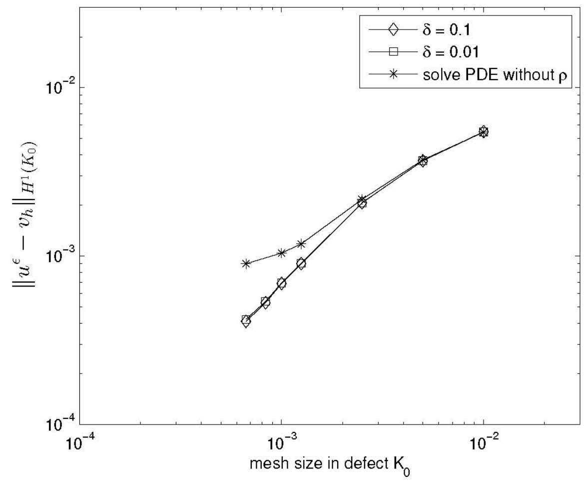


FIGURE 5. Localized H^1 error with respective to the mesh size in defect; different lines represent different parameter δ.

Finally, we plot the error $\|u_0 - v_h\|_{H^1(D\setminus K)}$ in Figure 6. For fixed L, this quantity decreases as the mesh outside K is refined.

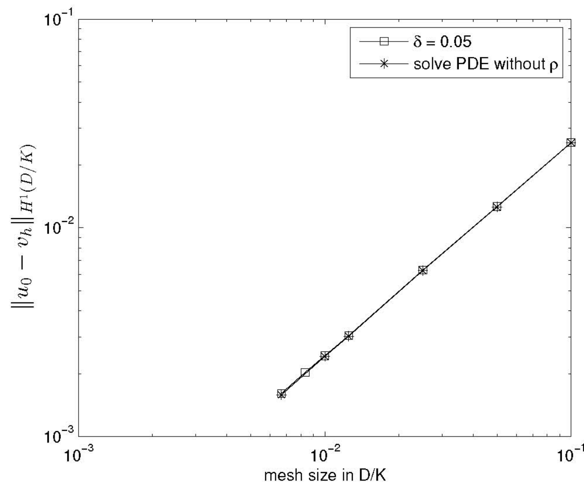


FIGURE 6. H^1 error in D/K and $\delta = 0.05$

The results in Table 1 show the convergence rate of the hybrid solution to the homogenized solution. It is optimal in the sense that the solution of the hybrid problem converges to the homogenized solution with first order in the energy norm, and it converges with second order in the L^2 norm. This seems consistent with the theoretical estimates (3.3), because

$$\begin{aligned} \|\nabla(u_0 - v_h)\|_{L^2(D)} &\le C\left(h + L\left|\ln L\right|^{1/2}\right), \\ \|u_0 - v_h\|_{L^2(D)} &\le C\left(h^2 + L^2\left|\ln L\right|\right). \end{aligned}$$

When $L \simeq h$, the convergence rate is of first order with respect to the energy norm, while the mesh size is smaller than L, the dominant term in the error bound is $L\left|\ln L\right|^{1/2}$, the convergence rate deteriorates a little bit, which is clear from the last line of Table 1. The same scenario applies to the L^2 error estimate.

TABLE 1. Error between the hybrid solution and the homogenized solution outside K.

h	$\|u_0 - v_h\|_{L^2(D\backslash K)}$	order	$\|u_0 - v_h\|_{H^1(D\backslash K)}$	order
1/10	9.04E-04		2.56E-02	
1/20	2.47E-04	1.87	1.26E-02	1.02
1/40	7.96E-05	1.64	6.28E-03	1.01
1/80	3.41E-05	1.22	3.04E-03	1.04
1/160	2.47E-05	0.47	1.61E-03	0.92

4.2. **An example without scale separation in the defect domain.** The setup for the second example is the same with the first one except that the coefficient is replaced by

$$a^\varepsilon = \chi_{K_0}\tilde{a} + (1 - \chi_{K_0})\tilde{a}^\varepsilon,$$

where

$$\tilde{a} = 3 + \frac{1}{7}\sum_{j=0}^{4}\sum_{i=0}^{j}\frac{1}{j+1}\cos\left(\left\lfloor 8(ix_2 - \frac{x_1}{i+1})\right\rfloor + \lfloor 150ix_1\rfloor + \lfloor 150x_2\rfloor\right),$$

and

$$\tilde{a}^\varepsilon = \left(2.1 + \cos(2\pi x_1/\varepsilon)\cos(2\pi x_2/\varepsilon) + \sin(4x_1^2x_2^2)\right) I.$$

The above coefficient is taken from [2], which has no clear scale inside K_0; while it is locally periodic outside K_0. We plot the coefficient a^ε in Figure 7 with $\varepsilon = 0.1$.

We let $\varepsilon = 0.0063$ for the sake of comparison with those in [2]. We still compute u^ε over a very refined uniform mesh with mesh size $3.33e{-4}$. The effective matrix $\mathcal{A}$ is computed through a fast solver based on the discrete least-squares reconstruction in the framework of HMM. We reconstruct $\mathcal{A}_h$ with high accuracy so that the reconstruction error is negligible. We refer to [25] and [21] for details of such fast algorithm. The homogenized solution u_0 is computed by solving Problem (1.2) with this reconstructed effective matrix $\mathcal{A}_h$, which has also been used in solving the hybrid problem (1.4).

We solve Problem (1.4) over a non-uniform mesh as in Figure 1, and the solutions u^ε, v_h and u_0 and the zoomed solution are plot in Figure 8. The difference between

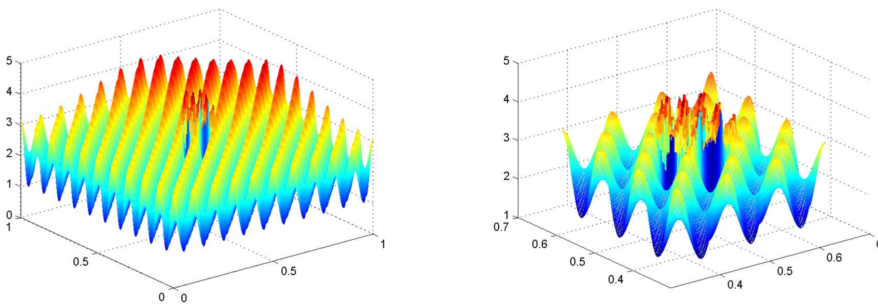
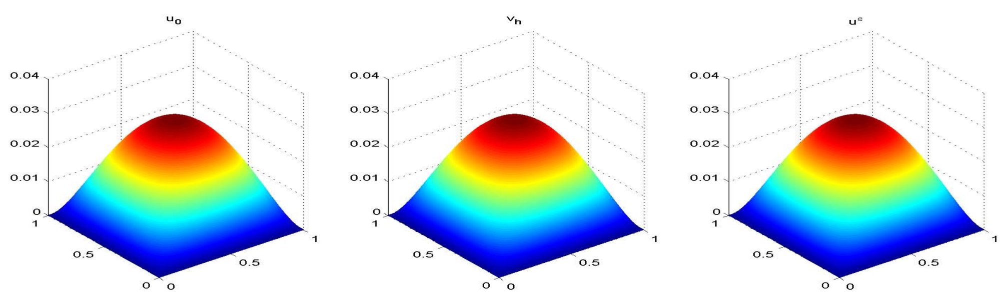
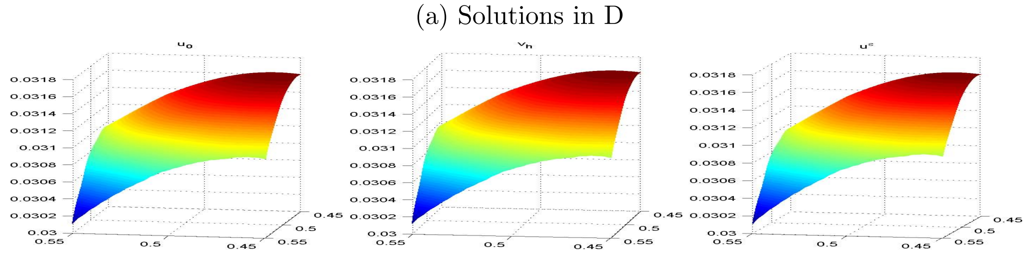


FIGURE 7. Coefficient a^ε with $\varepsilon = 0.1$. The right one is the zoomed plot near the defect.

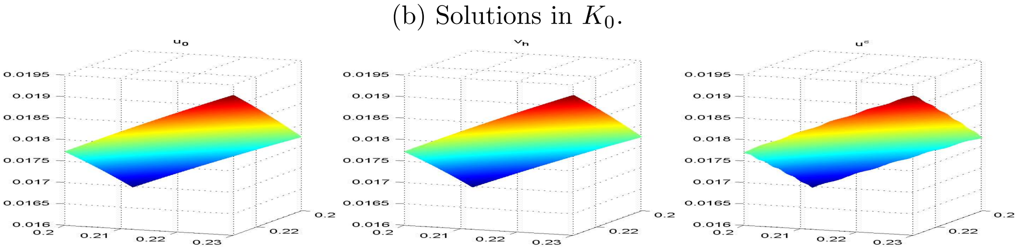
the solutions are small because there is no explicit scale inside the defect domain K_0.



(a) Solutions in D



(b) Solutions in K_0.



(c) Solutions in a subdomain of D/K.

FIGURE 8. The solution of hybrid method with $\delta = 0.05$ in the simulation.

We report the localized H^1 error in Figure 9. It seems the hybrid method converges slightly faster than the direct method, and the parameter δ has no significant effect on the results.

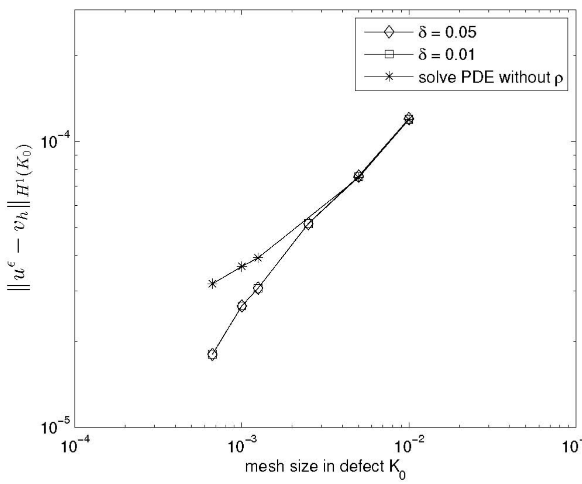


FIGURE 9. Localized H^1 error with respective to mesh size in K_0.

Next we plot the error outside K in Figure 10. The results in Table 2 shows that the convergence rate of the hybrid solution to the homogenized solution is optimal with respect to both the energy norm and the L^2 norm.

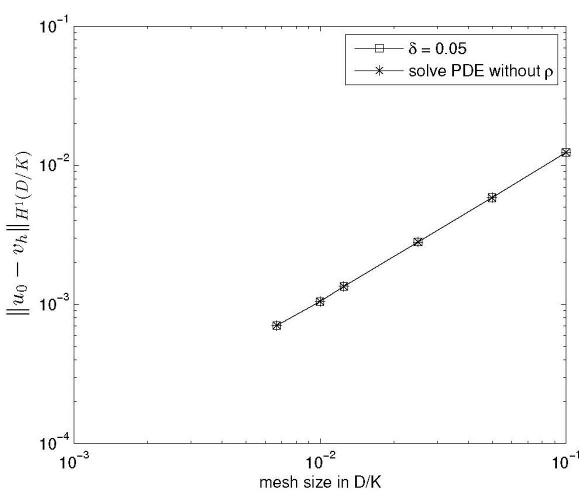


FIGURE 10. Error in D/K.

TABLE 2. Error between the hybrid solution and the homogenized solution outside K.

h	$\|u_0 - v_h\|_{L^2(D\setminus K)}$	order	$\|u_0 - v_h\|_{H^1(D\setminus K)}$	order
1/10	3.93E-04		1.24E-02	
1/20	8.92E-05	2.14	5.86E-03	1.08
1/40	2.06E-05	2.11	2.81E-03	1.06
1/80	4.92E-06	2.07	1.35E-03	1.06
1/160	1.45E-06	1.76	7.05E-04	0.94

5. Conclusion

We propose a new hybrid method that retrieves the global macroscopic information and resolves the local events simultaneously. The efficiency and accuracy of the proposed method have been demonstrated for problems with or without scale separation. The rate of convergence has been established when the coefficient is either periodic or almost-periodic.

For possible future directions, the formulation of the method can be naturally extended to deal with problems with random coefficients and also time-dependent problems. For the discretization of the hybrid equation with coefficient b^ε, it is also interesting to study the case when the local mesh inside the defect domain is not body-fitted. We shall leave these for further exploration.

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