$\mathcal{H}_2$-Conic Controller Synthesis

by

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Henry Pfister

A thesis submitted in partial fulfillment of the
requirements for the degree of Master of Science
in the Department of Mechanical Engineering and Materials Science
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ABSTRACT

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Abstract

Input-output stability theory is crucial in robust control. Since it does not necessarily involve investigations about properties of states within the system, but only examines the relationships between inputs and outputs, input-output theory simplifies analysis of stability for systems with complicated models or even no clear state-space expressions. As part of input-output theory, the Conic Sector Theorem can be used as a tool in controller synthesis. Compared to the commonly-used Passivity Theorem, the Conic Sector Theorem is applicable to more general cases. For example, the Passivity Theorem cannot be used to synthesize systems with passivity violations caused by factors such as noise, delays, and discretizations. This research investigates application of the Conic Sector Theorem in time-delay systems and develops a controller synthesis procedure that accounts for the optimal performance, robustness, and stability of the system.

Two key contributions are established in this work. First, a survey of theory and designs related to the Passivity Theorem and the conic sector theorem is given. Second, this research develops a method to synthesize conic, observer-based controllers by minimizing an upper-bound on the closed-loop $\mathcal{H}_2$-norm. The proposed method can be seen as the dual of an existing optimal synthesis method, but with an alternative initialization to expand the set of plants for which it is feasible. Moreover, the proposed method only involves solving convex optimization problems, thus making them readily solvable with existing software. Numerical simulations show that the new method leads to better performance in some examples and therefore provides a useful alternative tool for robust and optimal control.

Acknowledgements

Foremost, I would like to thank my supervisor, Leila Bridgeman, for her endless support and guidance. During my time in her group, she shared her knowledge, supported my research, sent me to conferences, and advised my career. Under her lead, I grew from a novice with little experience in control into a researcher in this area. Her patience and dedication to research influenced me profoundly and will stay as my fortune to carry on in my future career.

Next, I would like to thank our lab members. Ethan LoCicero helped to proofread my paper and provided some valuable feedback. Working with him and other lab colleagues inspired me with ideas in my research.

Lastly, I would like to thank my family for their love and support. I would not be able to go this far without them.

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Abbreviations

ARE Algebraic Riccati equation

BIBO Bounded-Input-Bounded-Output

LHS Left Hand Side

LMI Linear Matrix Inequality

LTI Linear Time-Invariant

RHS Right Hand Side

SISO Single-Input-Single-Output

Symbols

$\|\cdot\|$ A generic norm

$\langle\cdot,\cdot\rangle$ A generic inner product

$\mathcal{X}$ A normed inner-product space

$\mathcal{X}_e$ An extended normed inner-product space

$\mathcal{L}_2$ The Lebesgue space

$\mathcal{L}_{2e}$ The extended Lebesgue space

$\mathcal{H}_2$ $\mathcal{L}_2$ norm of the impulse response

$\mathbb{R}$ The set of real numbers

$\mathbb{S}^n$ The set of symmetric matrices of dimension $n \times n$

$\mathbb{R}^{>0}$ The set of positive real number

$\mathbb{R}^{<0}$ The set of negative real number

$\mathbf{I}$ The identity matrix

$\mathbf{0}$ The zero matrix

Chapter 1

Introduction

Control engineering has long been fundamental to all science. The means to model and manipulate systems so that they can perform as expected remains a core topic in all engineering. In the design of control systems, one vital criterion is to ensure system stability. That is, once an equilibrium of the system is broken by an external influence, the system should be able to return to the same or another equilibrium in finite time.

Stability analysis can be conducted based on numerous different theories. For systems with rational transfer functions, pole-zero plots and root locus plots convey information about the Bounded-Input, Bounded-Output (BIBO) stability [1]. Stability of Linear Time Invariant (LTI) systems can be determined from eigenvalues of the system matrix. The Nyquist stability criterion determines stability by examining the distributions of system roots on the complex plane[1]. Gain and phase margins can be indicated in a Bode plot to measure robustness of a closed-loop system [1]. Lyapunov stability theory is often applied to determine asymptotic stability for certain equilibrium states [2]. For closed-loop stability analysis, input-output stability theory is useful since it discards information about the internal states and only investigates the relationship between input and output. A system is input-output stable if it maps inputs in some inner-product space of reasonable inputs to outputs in an inner-product space of reasonable outputs [3]. Practically, this involves considering systems as mappings between function spaces. This especially simplifies the stability analysis for nonlinear systems [4]. Input-output stability theory has been widely used in control designs to ensure stability and robustness. For example, Strict Positive Real (SPR) controllers are frequently applied to passive systems for their ability to mitigate undesired oscillations caused by uncertainties.

Among numerous input-output stability results that have been developed, the Passivity Theorem [5] is commonly used for designs because it can be applied to a wide range of passive plants. The Conic Sector Theorem [6, 7] is also useful in controller synthesis, and its strength is the ability to control systems with passivity violations. This research seeks to enlarge the set of conic controllers by developing a new conic controller synthesis method.

The related work that inspired the creation of the new method can be traced back to the successful $\mathcal{H}_2$ control. In [8], J. C. Geromel and P. B. Gapski invented a controller synthesis method that optimizes the system performance by minimizing an upper-bound on the closed-loop $\mathcal{H}_2$-norm while ensures robust stability by retaining the controller strict positive realness. Later, T. Shimomura and S. Pullen suggested in [9] that this upper-bound can be further improved by an iterative scheme, and doing this can reduce the conservatism caused by over-bounding the closed-loop $\mathcal{H}_2$-norm and therefore result in better performance. Both of [8] and [9] inspired the work of [10], where L. J. Bridgeman replaced the SPR constraints in [9] with the ones to ensure stability based on the Conic Sector Theorem. In this way, although the performance is optimized by the similar iterative minimization of the upper-bound of the closed-loop $\mathcal{H}_2$ as in [9], the conic constraints allow the controller to possibly stabilize the systems that have passivity violations. Later, J. R. Forbes developed a dual approach to the method in [9]. This dual method makes use of the fact that the closed-loop $\mathcal{H}_2$-norm can be developed in two equivalent expressions [11, pp. 188-189], and imposes the bound of the closed-loop $\mathcal{H}_2$-norm in the dual expression to the one chosen in [8, 9].

Inspired by the idea of the dual SPR controller synthesis in [12], this research will propose a new method that solves a few dual problems to those of [10]. Unlike the way to derive closed-loop $\mathcal{H}_2$-norm in [10] as an expression of the observability Gramian, the new method calculates the closed-loop $\mathcal{H}_2$-norm with the controllability Gramian. Meanwhile, the design variable in the new method will be chosen as the observer matrix, rather than the feedback matrix in [10]. The new algorithm will be structured in parallel to the Algorithm 1 in [10]. Two essential optimization problems will be solved to develop the controller. The first problem will initialize a controller by minimizing an upper-bound on the closed-loop $\mathcal{H}_2$-norm. The second problem will be solved at each iteration to decrease this upper-bound. In each of the problems, conic constraints are imposed as LMIs to ensure stability. To highlight the advantage of conic controllers, in simulations, the new method will be applied to the control of a heat exchanger with passivity violations caused by time-delays. Ultimately, the numerical results of the performance and stability of the controller synthesized using the new method will be compared to the ones of some other conic controllers.

In this thesis, two key contributions are made. First, a survey about input-output stability theory and its application to controller synthesis will be provided in Chapters 3 and 4. Second, a new method for designing robust and optimal conic controllers will be introduced in Chapter 5. Specifically, after a brief review of convex optimization and $\mathcal{H}_2$ optimal control in Chapter 2, Chapter 3 will survey the existing input-output stability results surrounding the Passivity and Conic Sector Theorem. Next, Chapter 4 will demonstrate the design tools based on input-output theory by discussing synthesis methods of SPR controllers and conic controllers. In Chapter 5, a new synthesis method that minimizes an upper-bound on the closed-loop $\mathcal{H}_2$-norm of the system will be introduced. Ultimately, Chapter 6 concludes this work and specifies a few directions that might be carried on in the future.

Chapter 2

Preliminaries

To begin, some definitions have to be made. Since inner-product spaces will be investigated as the domain of signals within the system, definitions of vector spaces and inner-product spaces will be given first.

Definition 2.1 (Vector Space [13]). *A vector space consists of the following,*

1. *a field F of scalars*
2. *a set V of objects, called vectors*
3. *a defined vector addition, which for vectors $a, b, c \in V$, it follows that*
   (a) *addition is commutative:* $a + b = b + a$
   (b) *addition is associatice:* $a + (b + c) = (a + b) + c$
   (c) *there exists a unique vector* $\mathbf{0} \in V$ *such that* $a + \mathbf{0} = a$, $\forall a \in V$
   (d) *For each* $a \in V$ *there exists a unique vector* $-a \in V$ *such that* $a + (-a) = 0$
4. *a defined scalar multiplication, which for scalar $s \in F$ and vectors $a, b \in V$, it follows that*
   (a) $1a = a$, $\forall a \in V$
   (b) $(s_1 s_2)a = s_1(s_2 a)$
   (c) $s(a + b) = sa + sb$
   (d) $(s_1 + s_2)a = s_1 a + s_2 b$.

Definition 2.2 (Inner Product [13]). *Let F be a field of real numbers or of complex numbers, Let V be a vector space over F. An inner product on V is function that maps a pair of vectors $a, b \in V$ into a scalar $\langle a, b\rangle \in F$, so that for all $a, b, c \in V$ and* *any scalar* $s \in F$

1. $\langle a+b, c\rangle = \langle a, c\rangle \langle b, c\rangle$
2. $\langle sa, b\rangle = s\langle a, b\rangle$
3. $\langle a, b\rangle = \overline{\langle b, a\rangle}$ *where the bar above denotes complex conjugate*
4. $\langle a, a\rangle \geq 0$ *and* $\langle a, a\rangle = 0$ *iff* $a = \mathbf{0}$

Definition 2.3 (Inner-Product Space [13]). *A real or complex vector space equipped with an inner product is called an inner-product space.*

Definition 2.4 (Norm [13]). *A norm on vector space* V *is a function* $||\cdot|| :\to \mathbb{R}$ *that satisfies*

1. $||a|| \geq 0, \forall a \in V; \ ||a|| = 0$ *iff* $a = 0$
2. $||sa|| = |s|\ ||a||, \forall a \in V,\ s \in F$
3. $||a+b|| \leq ||a|| + ||b||, \forall a, b \in V.$

A real inner-product is also a normed vector space, since $\langle a, a\rangle = ||a||^2$, $\forall a \in V$.

Definition 2.5 ($\mathcal{L}_2$-Space and $\mathcal{L}_{2e}$-Space [14]). *An* $\mathcal{L}_2$*-space is defined as the set of all functions* $f(t) : \mathbb{R}^{\geq 0} \to \mathbb{R}$ *that satisfy* $\int_0^\infty |f(t)|^2 dt < \infty$. *An* $\mathcal{L}_{2e}$*-space is defined as all the functions that safisfy* $\int_0^\infty |f_T(t)|^2 dt < \infty$, $\forall T \in \mathbb{R}^{\geq 0}$, *where* $f_T(t)$ *is the truncation of* $f(t)$ *to the interval [0,T], that is,*

$$f_T(t) = \begin{cases} f(t),\ 0 \leq t < T \\ 0,\ t \geq T \end{cases} .$$

2.1 Convex Optimization and LMI's

The optimization problems that will be discussed in the following chapters involves solving optimization problems with a convex objective and LMI constraints. It would be wise to review the definitions and properties of convexity and LMI's here.

Definition 2.6 (Convex Set [13]). *Let V be a vector space over $\mathbb{R}$. Consider a subset $W \subseteq V$. If $\forall w_1, w_2 \in W$ and $\alpha \in (0,1)$, we have $\alpha w_1 + (1-\alpha)w_2 \in W$, then W is called a* ***convex set****.*

Definition 2.7 (Convex Function [13]). *Let V be a vector. Consider a convex set $W \subseteq V$ and a function $f : V \to \mathbb{R}$. If $\forall w_1, w_2 \in W$ and $\alpha \in (0,1)$, we have*

$$f(\alpha w_1 + (1-\alpha)w_2) \leq \alpha f(w_1) + (1-\alpha)f(w_2),$$

then the function f is called a ***convex function****. The function is* ***strictly convex*** *if equality happens only when $w_1 = w_2$.*

Theorem 2.1 (Uniqueness of a Convex Function [13]). *Suppose that $(V, ||\cdot||)$ is a normed vector space, $A \subseteq V$ is a convex set, and $f : V \to \mathbb{R}$ is a convex function on A. Then, any local minimum value of f on A is a global minimum value on A. If the function is strictly convex on A and achieves a local minimum value on A, then there exists a unique point $v_0 \in A$ that achieves the global minimum value on A.*

Theorem 2.1 indicates that if a function is convex, then its minimum point exists and is unique.

Definition 2.8 (Convex Optimization Problem [15]). *A convex optimization problem follows the form*

$$\begin{array}{ll} \text{Minimize} & f_0(x) \\ \text{Subject to} & f_i(x) \leq b_i, i = 1, ..., m, \end{array}$$

where the functions $f_0, ..., f_m : \mathbb{R}^n \to \mathbb{R}$ are convex.

Definition 2.9 (Feasibility [13]). *Consider the problem defined in Definition 2.8, if* *there exists a vector $v \in \mathbb{R}^n$ that satisfies*

$$f_i(v) \leq b_i, i = 1, \ldots, m,$$

*then the vector v is **feasible** to the problem. The set $F = \{x | f_i(x) \leq b_i, i = 1, \ldots, m\}$ is the **feasible set** to the problem. If $F = \emptyset$, then the problem is **infeasible**.*

Definition 2.10 (Matrix Definiteness)**.** *A matrix $A \in \mathbb{R}^{n \times n}$ is **positive-semidefinite** if $v^\mathsf{T} A v \geq 0, \forall v \in \mathbb{R}^n$. A matrix $A \in \mathbb{R}^{n \times n}$ is **positive-definite** if $v^\mathsf{T} A v > 0, \forall v \in \mathbb{R}^n, v \neq \mathbf{0}$. Definitions of matrix negative-semidefiniteness and negative-definiteness can be given respectively if all the equality signs are reversed.*

In this thesis, matrix definiteness are written as inequalities. For example, $\mathbf{A} < 0$ denotes that $\mathbf{A}$ is negative definite.

Definition 2.11 (Linear Matrix Inequality [16, 17])**.** *A linear matrix inequality $A : \mathbb{R}^m \rightarrow \mathbb{S}^n$ can be writen in the form*

$$A(\mathbf{w}) = A_0 + w_1 A_1 + \ldots + w_m A_m \geq 0$$

where $\mathbf{w} = [w_1, w_2, \ldots, w_m]^\mathsf{T}, A_i \in \mathbb{S}^n$, and $i = 0, \ldots, m$.

One can verify that solutions to an LMI forms a convex set by Definition 2.7. See that the following relationship holds for any $\mathbf{v}, \mathbf{w} \in \mathbb{R}^m$.

$$\begin{aligned}
A(\alpha\mathbf{v} + (1-\alpha)\mathbf{w}) &= A_0 + \sum_{i=1}^{m}(\alpha v_i + (1-\alpha)w_i)A_i \\
&= \alpha A_0 + \sum_{i=1}^{m}(\alpha v_i)A_i + (1-\alpha)A_0 + \sum_{i=1}^{m}((1-\alpha)w_i)A_i \\
&= \alpha A(\mathbf{v}) + (1-\alpha)A(\mathbf{w})
\end{aligned}$$

Based on the convexity property of LMI's, it is known that a convex objective function subject to LMI constraints is convex. Theorem 2.1 implies that a convex optimization problem with convex objective subject to LMI's has a unique minimizer.

LMI's can be concatenated together to form a new LMI. For example,

$$\mathbf{M}_{11} > 0, \mathbf{M}_{22} > 0 \iff \mathbf{M} = \begin{bmatrix} \mathbf{M}_{11} & \mathbf{0} \\ \mathbf{0} & \mathbf{M}_{22} \end{bmatrix} > 0.$$

Several software packages are powerful in solving optimization problems with LMI constraints. Packages such as `SeDuMi`[18], `MOSEK`[19], and `SDPT3`[20] were frequently used during the numerical simulations of this work. These packages can be conveniently imported and used using the MATLAB toolbox of `YALMIP`[21].

In this work, symmetric matrices will be abbreviated with asterisks as:

$$\begin{bmatrix} \mathbf{A} & \mathbf{B} \\ * & \mathbf{C} \end{bmatrix} = \begin{bmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{B}^{\mathsf{T}} & \mathbf{C} \end{bmatrix}.$$

This work will repeatedly use the Schur complement [16, p. 650], the principle of which is the fact that, if $\mathbf{A}$ is invertible, then

$$\mathbf{M}_1 = \begin{bmatrix} \mathbf{A} & \mathbf{B} \\ * & \mathbf{C} \end{bmatrix} < 0 \text{ and } \mathbf{M}_2 = \begin{bmatrix} \mathbf{C} & * \\ \mathbf{B} & \mathbf{A} \end{bmatrix} < 0$$

are equivalent to $\mathbf{M}_3 = \mathbf{C} - \mathbf{B}^\mathsf{T}\mathbf{A}^{-1}\mathbf{B} < 0$. The equivalence holds when all inequality signs are reversed.

2.2 $\mathcal{H}_2$-Optimal Control

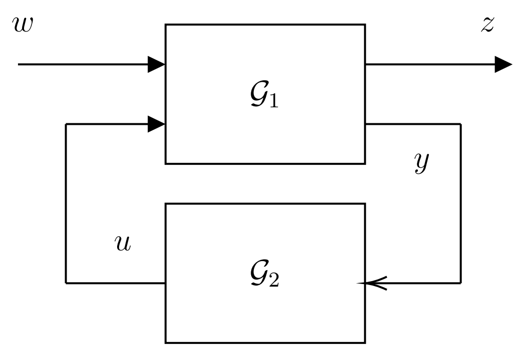


Figure 2.1: Block diagram of a common feedback interconnection for $\mathcal{H}_2$-optimal control

The controller synthesis methods that will be introduced in this work leverage the guaranteed performance brought $\mathcal{H}_2$-optimal control. As a popular control method, $\mathcal{H}_2$-optimal control seeks to minimize the $\mathcal{H}_2$-norm of the system to alleviate the effect of the disturbance w on the regulated output z. The goal of $\mathcal{H}_2$-optimal control is to find a controller $\mathcal{G}_2 : \mathcal{L}_{2e} \to \mathcal{L}_{2e}$ that minimizes the closed-loop $\mathcal{H}_2$-norm of its interconnection with the plant $\mathcal{G}_1 : \mathcal{L}_{2e} \to \mathcal{L}_{2e}$ represented in Figure 2.1. The regulated output z must be chosen as variables that need to be minimized.

For an LTI system with state-space representation

$$\dot{x} = \mathbf{A}x + \mathbf{B}w$$
$$z = \mathbf{C}x,$$

suppose $\mathbf{A}$ is stable, then the impulse response of the system is

$$G(t) = \mathbf{C}e^{\mathbf{A}t}\mathbf{B}.$$

The $\mathcal{H}_2$-norm of the system is defined as

$$\|G\|_{\mathcal{H}_2} = ||G||_2 = \Big(\int_0^\infty \text{trace}(G^\mathsf{T}(t)G(t))dt \Big)^{\frac{1}{2}}.$$

It can also be defined using transfer function of the system $\hat{G}(s) = \mathbf{C}(sI - \mathbf{A})^{-1}\mathbf{B}$. The $\mathcal{H}_2$ norm is defined as [11]

$$\|G\|_{\mathcal{H}_2} = ||\hat{G}||_2 = \Big(\frac{1}{2\pi} \int_{-\infty}^\infty \text{trace}(\hat{G}^\mathsf{T}(jw)G(jw))dw \Big)^{\frac{1}{2}}.$$

Because $\hat{G}$ is the Laplace transform of G, based on Parseval's Theorem [22, Appendix B.2],

$$\|G\|_{\mathcal{H}_2} = ||\hat{G}||_2 = ||G||_2.$$

For the LTI system investigated above, the $\mathcal{H}_2$-norm can be computed in the following two ways.

1. Compute $\mathcal{H}_2$-norm using the observability Gramian,

$\mathbf{W}_o = \int_0^\infty e^{\mathbf{A}^\mathsf{T} t}\mathbf{C}^\mathsf{T}\mathbf{C}e^{\mathbf{A}t}dt$, that satisfies [11]

$$\mathbf{W}_o\mathbf{A} + \mathbf{A}^\mathsf{T}\mathbf{W}_o + \mathbf{C}^\mathsf{T}\mathbf{C} = 0. \tag{2.1}$$

The closed-loop $\mathcal{H}_2$-norm is

$$\begin{aligned}\|G(\cdot)\|_{\mathcal{H}_2} &= \int_0^\infty \text{trace}(G^\mathsf{T}(t)G(t))dt \\ &= \text{trace}(\int_0^\infty \mathbf{B}^\mathsf{T}e^{\mathbf{A}^\mathsf{T} t}\mathbf{C}^\mathsf{T}\mathbf{C}e^{\mathbf{A}t}\mathbf{B}dt) \\ &= \text{trace}(\mathbf{B}^\mathsf{T}\mathbf{W}_o\mathbf{B}).\end{aligned} \tag{2.2}$$

2. Compute $\mathcal{H}_2$-norm using the controllability Gramian, $\mathbf{W}_c = \int_0^\infty e^{\mathbf{A}t}\mathbf{B}\mathbf{B}^\mathsf{T}e^{\mathbf{A}^\mathsf{T} t}dt$, that satisfies [11]

$$\mathbf{A}\mathbf{W}_c + \mathbf{W}_c\mathbf{A}^\mathsf{T} + \mathbf{B}^\mathsf{T}\mathbf{B} = 0. \tag{2.3}$$

The closed-lood $\mathcal{H}_2$-norm is

$$\begin{aligned}\|G(\cdot)\|_{\mathcal{H}_2} &= \int_0^\infty \text{trace}(G(t)G^\mathsf{T}(t))dt \\ &= \text{trace}(\int_0^\infty \mathbf{C}e^{\mathbf{A}t}\mathbf{B}\mathbf{B}^\mathsf{T}e^{\mathbf{A}^\mathsf{T} t}\mathbf{C}^\mathsf{T}dt) \\ &= \text{trace}(\mathbf{C}\mathbf{W}_c\mathbf{C}^\mathsf{T}).\end{aligned} \tag{2.4}$$

To show that $\mathbf{W}_o$ satisfies Equation (2.1), consider replacing the $\mathbf{C}^\mathsf{T}\mathbf{C}$ in $\mathbf{W}_o =$

$\int_0^\infty e^{\mathbf{A}^\mathsf{T}t}\mathbf{C}^\mathsf{T}\mathbf{C}e^{\mathbf{A}t}dt$ with $-\mathbf{W}_o\mathbf{A} - \mathbf{A}^\mathsf{T}\mathbf{W}_o$, then

$$\begin{aligned}
\mathbf{W}_o &= -\int_0^\infty e^{\mathbf{A}^\mathsf{T}t}(\mathbf{W}_o\mathbf{A} + \mathbf{A}^\mathsf{T}\mathbf{W}_o)e^{\mathbf{A}t}dt \\
&= -\int_0^\infty \frac{d}{dt}(e^{\mathbf{A}^\mathsf{T}t})\mathbf{W}_o e^{\mathbf{A}t} + e^{\mathbf{A}^\mathsf{T}t}\mathbf{W}_o\frac{d}{dt}(e^{\mathbf{A}t})dt \\
&= -\int_0^\infty \frac{d}{dt}(e^{\mathbf{A}^\mathsf{T}t}\mathbf{W}_o e^{\mathbf{A}t})dt \\
&= -e^{\mathbf{A}^\mathsf{T}t}\mathbf{W}_o e^{\mathbf{A}t}|_0^\infty \\
&= \mathbf{W}_0.
\end{aligned}$$

The two different approaches of finding the $\mathcal{H}_2$-norm inspired the development of a new method that will be introduced in Chapter 5. Specifically, the idea is to develop an algorithm, where, comparing to an existing one, an objective function containing $\mathcal{H}_2$-norm will be reformulated as the dual expression.

Practically, an $\mathcal{H}_2$ controller can be designed following the procedure discussed below. Consider a plant with state-space realization [11]

$$\begin{aligned}
\dot{\mathbf{x}}_p =& \mathbf{A}\mathbf{x}_p + \mathbf{B}_1\mathbf{w} + \mathbf{B}_2\mathbf{u}_p, \\
\mathbf{z} =& \mathbf{C}_1\mathbf{x}_p + \mathbf{D}_{12}\mathbf{u}_p \\
\mathbf{y} =& \mathbf{C}_2\mathbf{x}_p + \mathbf{D}_{21}\mathbf{w},
\end{aligned} \tag{2.5}$$

where $\mathbf{x}_p \in \mathbb{R}^{n_x}$ is the state, $\mathbf{u}_p \in \mathbb{R}^{n_u}$ is the control input, $\mathbf{w} \in \mathbb{R}^{n_w}$ is the disturbance, $\mathbf{y} \in \mathbb{R}^{n_u}$ is the measured output, and $\mathbf{z} \in \mathbb{R}^{n_z}$ is the regulated output. The matrices are of appropriate dimension and follow the assumptions:

- $(\mathbf{A}, \mathbf{B}_1)$ is controllable and $(\mathbf{C}_1, \mathbf{A})$ is observable,
- $(\mathbf{A}, \mathbf{B}_2)$ is controllable and $(\mathbf{C}_2, \mathbf{A})$ is observable,
- $\mathbf{D}_{12}^\mathsf{T}\mathbf{C}_1 = \mathbf{0}$, $\mathbf{D}_{12}^\mathsf{T}\mathbf{D}_{12} > 0$, and

- $\mathbf{D}_{21}\mathbf{B}_1^\mathsf{T} = \mathbf{0}$, $\mathbf{D}_{21}\mathbf{D}_{21}^\mathsf{T} > 0$.

The state-space realization of the $\mathcal{H}_2$ controller is [11]

$$\boldsymbol{\mathcal{G}}_c : \{\mathbf{A} - \mathbf{B}_2\mathbf{K}_c - \mathbf{L}_c\mathbf{C}_2, \mathbf{L}_c, \mathbf{K}_c, \mathbf{0}\}, \tag{2.6}$$

where $\mathbf{A}_K = \mathbf{A} - \mathbf{B}_2\mathbf{K}_c$, and $\mathbf{K}_c \in \mathbb{R}^{n_u \times n_x} = \mathbf{R}_{12}^{-1}\mathbf{B}_2^\mathsf{T}\mathbf{\Pi}_1$ is the feedback matrix, while $\mathbf{L}_c \in \mathbb{R}^{n_x \times n_u} = \mathbf{\Pi}_2\mathbf{C}_2^\mathsf{T}\mathbf{R}_{21}^{-1}$ is the observer matrix, where $\mathbf{R}_{12} = \mathbf{D}_{12}^\mathsf{T}\mathbf{D}_{12}$, $\mathbf{R}_{21} = \mathbf{D}_{21}\mathbf{D}_{21}^\mathsf{T}$, and $\mathbf{\Pi}_1$ and $\mathbf{\Pi}_2 > 0$ are solutions to the algebraic Riccati equations (AREs)

$$\mathbf{A}\mathbf{\Pi}_1 + \mathbf{\Pi}_1\mathbf{A}^\mathsf{T} + \mathbf{C}_1\mathbf{C}_1^\mathsf{T} - \mathbf{\Pi}_1\mathbf{B}_2^\mathsf{T}\mathbf{R}_{12}^{-1}\mathbf{B}_2\mathbf{\Pi}_1 = \mathbf{0}; \tag{2.7}$$

$$\mathbf{A}\mathbf{\Pi}_2 + \mathbf{\Pi}_2\mathbf{A}^\mathsf{T} + \mathbf{B}_1\mathbf{B}_1^\mathsf{T} - \mathbf{\Pi}_2\mathbf{C}_2^\mathsf{T}\mathbf{R}_{21}^{-1}\mathbf{C}_2\mathbf{\Pi}_2 = \mathbf{0}. \tag{2.8}$$

AREs are easily solvable using software. The MATLAB function `care` can fulfil this task. To find a $\mathcal{H}_2$ controller, one can solve Equations (2.7) and (2.8), and then construct the controller using Equation (2.6). The procedure can be summarized in the following algorithm.

Algorithm 1 ($\mathcal{H}_2$-optimal controller). *Returns a set containing the controller state space realization.*

1: *Solve Equation* (2.7) *and Equation* (2.8) *and store* $(\mathbf{\Pi}_1, \mathbf{\Pi}_2)$.

2: *Calculate* $\mathbf{R}_{12} = \mathbf{D}_{12}^\mathsf{T}\mathbf{D}_{12}$ *and* $\mathbf{R}_{21} = \mathbf{D}_{21}\mathbf{D}_{21}^\mathsf{T}$, *then calculate* $\mathbf{L}_c = \mathbf{\Pi}_2\mathbf{C}_2^\mathsf{T}\mathbf{R}_{21}^{-1}$ *and* $\mathbf{K}_c = \mathbf{R}_{12}^{-1}\mathbf{B}_2^\mathsf{T}\mathbf{\Pi}_1$.

3: *Calculate* $(\mathbf{A}_c, \mathbf{B}_c, \mathbf{C}_c) = (\mathbf{A} - \mathbf{B}_2\mathbf{K}_c - \mathbf{L}_c\mathbf{C}_2, \mathbf{L}_c, \mathbf{K}_c)$.

4: ***return*** $\{\mathbf{A}_c, \mathbf{B}_c, \mathbf{C}_c, \mathbf{0}\}$.

2.3 Time Delay Systems

The numerical model for simulations that will be discussed in Section 5.6 is subject to a time delay. That is, it takes a period for changes in the control input to actually effect the states and outputs. To better introduce the model, it would be wise to first briefly discuss about time delay systems.

Since there is no information, energy, or objects that can transfer in an infinitely large speed, a period of time is consumed for such transfer to happen. Therefore, any physical system is a time-delay system and time delays exist for an action on the system to cause responses [23]. When controlling the system, sometimes it might be acceptable to ignore the time delays and control the system as there were no delays. This only happens when the delay is very small. If the delay is large, the control can be inaccurate or even fail if the delays are ignored.

Examples about time-delay systems are easy to find. For instance, the Moon is 384,400 km distant from the earth. When trying to remotely drive a rover on the Moon surface from the earth, it takes about 1.2 seconds for a control signal to travel before reaching the rover. This causes the real-time control to be a challenge, since the information from the Moon happened 1.2 seconds ago and the feedback going to the rover from the earth takes the same period of time to affect the rover.

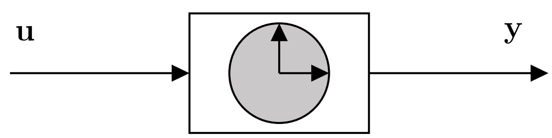


Figure 2.2: A Delay Element

For an open-loop time delay system (or delay element) subject to the delay h the input and output can be characterized by the following relationship:

$$\mathbf{y}(t) = \mathbf{u}(t-h).$$

The numerical example that will be discussed in Section 5.6 can be illustrated in Figure 2.3. The term h_i stands for input delay.

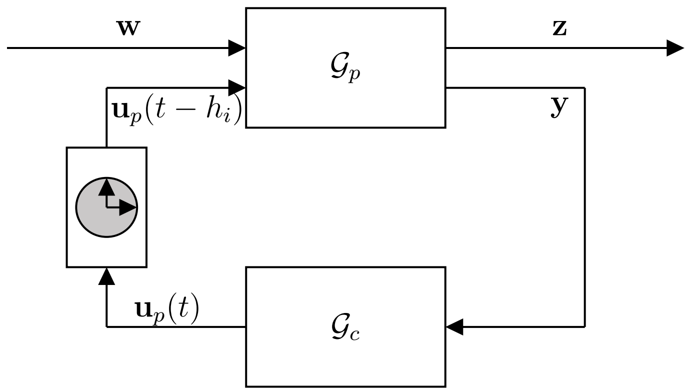


Figure 2.3: Block diagram of a closed-loop system with input time delay.

The state-space realization of the plant is

$$\begin{aligned}\dot{\mathbf{x}}_p(t) =& \mathbf{A}\mathbf{x}_p(t) + \mathbf{B}_1\mathbf{w}(t) + \mathbf{B}_2\mathbf{u}_p(t-h_i),\\ \mathbf{z}(t) =& \mathbf{C}_1\mathbf{x}_p(t) + \mathbf{D}_{12}\mathbf{u}_p(t)\\ \mathbf{y}(t) =& \mathbf{C}_2\mathbf{x}_p(t) + \mathbf{D}_{21}\mathbf{w}(t),\end{aligned}$$

while the state-space realization of the controller is

$$\begin{aligned}\dot{\mathbf{x}}_c(t) =& (\mathbf{A} - \mathbf{B}_2\mathbf{K}_c - \mathbf{L}_c\mathbf{C}_2)\mathbf{x}_c(t) + \mathbf{L}_c\mathbf{y}(t)\\ \mathbf{u}_p(t) =& -\mathbf{K}_c\mathbf{x}_c(t).\end{aligned}$$

Although there is a time delay between the controller output and the plant input, in the proposed method the $\mathbf{u}_p(t-h_i)$ in the state-space realization of the plant will be treated as $\mathbf{u}_p(t)$. However, to account for the time delay, the conic bounds of the system are found using the method in [24], which finds the conic bounds that ensure stability robust to delays lower than a limit H_i. By setting this limit when finding the conic bounds, the developed controller should be able to reduce the effects of delays on the performance and maintain stability when subject to delays $h_i \leq H_i$.

Chapter 3

Input-Output Stability Analysis

There exist multiple Input-Output stability results. Despite the widely used Small Gain Theorem [6] and the Passivity Theorem [5], γ-passivity [25], passivity indices [26], and large gain [27] can each be applied to their applicable situations. The newly-derived Extended Conic Sector Theorem [7] unifies these theorems by proving that these results are special cases of the new theorem [28].

Due to the pertinence to this research, in this chapter, detailed explanations about most of the above theorems will not be included. For a detailed review of input-output results and their relationships, [3, Ch. 2 and Ch. 10] and [14] can give detailed explanations. In this chapter, passivity and strict positive realness will be briefly discussed in Section 3.1. The criteria to determine system SPRness will also be discussed in Section 3.1. Next, the concept about the conic sector and the Conic Sector Theory will be reviewed in Section 3.2. Finally, since the conic sector theorem can be useful in controller synthesis only if conic bounds are predetermined, the means to select conic bounds will be introduced in Section 3.3.

3.1 Passivity and Strict Positive Realness

Figure 3.1 depicts a standard negative feedback connection. This interconnection is a typical structure of the problems that input-output stability theory investigates.

A good intuitive explanation about passivity is given in [29]. As it mentions, a passive system is a process that does not generate energy inside, but dissipates energy

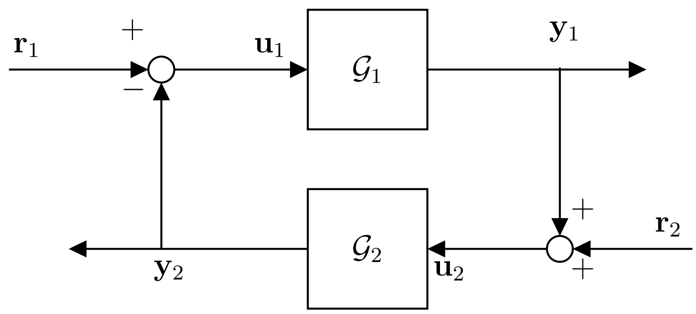


Figure 3.1: Block diagram of a negative feedback interconnection

from outside. If a strictly passive system is interconnected as the way in Figure 3.1 with a passive system. Then the closed loop is input-output stable. This can be understood as the energy dissipation in one system is repaid by the energy excess caused by the negative feedback with the other passive system. Thus a balance can be reached for the closed loop to stabilize.

Systems that inherently dissipate energy commonly exist, and this results in a wide range of cases where the Passivity Theorem can be applied to design stabilizing controllers. It has been determined that SPR controllers can be employed as a passive system [30]. Therefore the design of SPR controllers has received particular attentions. A few applications of SPR controller synthesis can be found in [8, 9, 12]. Theorem 3.1 provides a criterion to determine the SPRness of a system.

Theorem 3.1 (SPRness criterion [8, 30])**.** *Consider a system $\boldsymbol{\mathcal{G}} : \mathcal{L}_{2e} \mapsto \mathcal{L}_{2e}$ with state-space realization $\{\mathbf{A}, \mathbf{B}, \mathbf{C}, \mathbf{0}\}$, $\mathbf{A}$ is Hurwitz, and the system has a square transfer function $G(s) = \mathbf{C}(s\mathbf{I} - \mathbf{A})^{-1}\mathbf{B}$. $G(s)$ is **strict positive real** if there exists a matrix $\mathbf{P} = \mathbf{P}^{\mathsf{T}} > 0$ such that $\mathbf{PA} + \mathbf{A}^{\mathsf{T}}\mathbf{P} < 0$ and $\mathbf{C} = \mathbf{B}^{\mathsf{T}}\mathbf{P}$.*

3.2 The Conic Sector Theorem

Definition 3.1 (Conic System[6])**.** *For a square system, $\boldsymbol{\mathcal{G}} : \mathcal{L}_{2e} \to \mathcal{L}_{2e}$,*

- *if* $\langle b\mathbf{u} - \boldsymbol{\mathcal{G}}\mathbf{u}, \boldsymbol{\mathcal{G}}\mathbf{u} - a\mathbf{u}\rangle_T \geq \beta \;\forall\; \mathbf{u} \in \mathcal{L}_{2e}, T \in \mathbb{R}^{\geq 0}$, *where* β *depends only on the initial conditions, then* $\boldsymbol{\mathcal{G}}$ *is* **interior conic**, *denoted* $\boldsymbol{\mathcal{G}} \in \text{cone}[a,b]$, *in bounds* $a < b$, $b > 0$;
- *if* $\boldsymbol{\mathcal{G}} \in \text{cone}[a+\delta, b-\delta]$, *for some* $\delta > 0$, *then the system is* **strictly interior conic**, *denoted* $\boldsymbol{\mathcal{G}} \in \text{cone}(a,b)$;

The conic radius is defined as $r = \frac{b-a}{2}$ and the conic center is defined as $c = \frac{a+b}{2}$.

To intuitively understand conic sectors, imagine a SISO, memoryless system, $\boldsymbol{\mathcal{G}} : \mathcal{L}_{2e} \to \mathcal{L}_{2e}$. The relationship between input and output can be visualized as a curve in graphs [6]. A clear description of this can be found in [3]. Suppose the output of the system is a function $f : \mathbb{R} \to \mathbb{R}$ of the input, that is, $(\boldsymbol{\mathcal{G}}u)(t) = f(u(t))$. If the trajectory of $f(u(t))$ with respect to $u(t)$ lies within the area bounded by lines with slopes a and b, then the system $\boldsymbol{\mathcal{G}} \in \text{cone}[a,b]$. This area is the conic sector of the system. This is clearly shown in Figure 3.2(a), where the plant is interior conic in bounds a and b, as the trajectory lies within the conic sector.

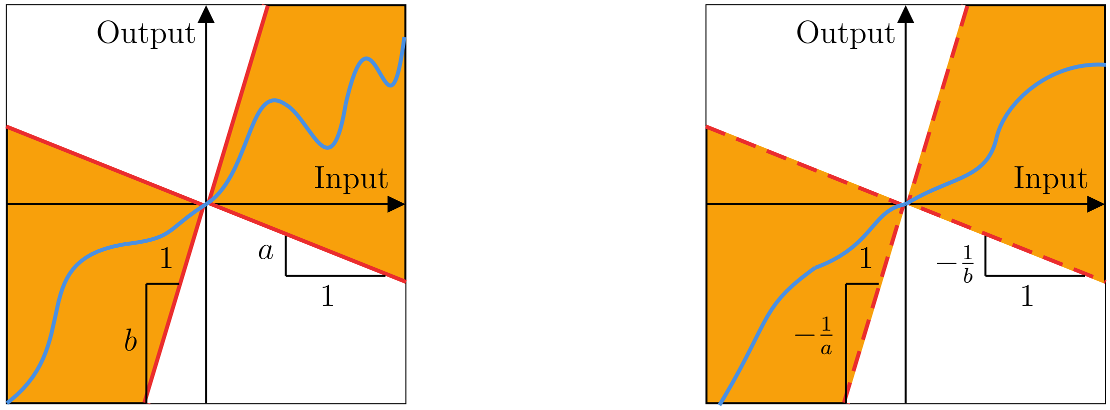
The Conic Sector Theorem determines how the conic properties of subsystems are related with input-output stability.

Theorem 3.2 (Conic Sector Theorem [6])**.** *Consider an interconnection shown in Figure 3.1, where* $\mathbf{r}^\mathsf{T} = [\mathbf{r_1}\ \mathbf{r_2}] \in \boldsymbol{\mathcal{L}_{2e}} \times \boldsymbol{\mathcal{L}_{2e}}$ *is the external signal, and* $\boldsymbol{\mathcal{G}}_1, \boldsymbol{\mathcal{G}}_2 : \mathcal{L}_{2e} \to \mathcal{L}_{2e}$. *The output and input matrices of the system satisfy the following relationship:*

$$\mathbf{y} = \begin{bmatrix} \mathbf{y}_1 \\ \mathbf{y}_2 \end{bmatrix} = \begin{bmatrix} \boldsymbol{\mathcal{G}}_1\mathbf{u}_1 \\ \boldsymbol{\mathcal{G}}_2\mathbf{u}_2 \end{bmatrix} \qquad \mathbf{u} = \begin{bmatrix} \mathbf{u}_1 \\ \mathbf{u}_2 \end{bmatrix} = \begin{bmatrix} \mathbf{r_1} - \mathbf{y_2} \\ \mathbf{r_2} + \mathbf{y_1} \end{bmatrix}$$

The closed-loop system $\boldsymbol{\mathcal{G}}$ *is* ***input-output stable*** *if* $\boldsymbol{\mathcal{G}}_1 \in \text{cone}[a,b]$ *and* $\boldsymbol{\mathcal{G}}_2 \in \text{cone}(-\frac{1}{b}, -\frac{1}{a})$, *for* $a \in (-\infty, 0], b \in (0, \infty)$.

Theorem 3.2 suggests that a closed-loop system is conic if its plant $\boldsymbol{\mathcal{G}}_1$ and con-



(a) Plant Cone (b) Controller Cone

Figure 3.2: Conic sectors of the plant, and the controller of a SISO, memoryless system [3]. Each of the trajectories is the output as function of the input of the system.

troller $\boldsymbol{\mathcal{G}}_2$ are each (strict) interior conic in certain conic bounds, that is, $\boldsymbol{\mathcal{G}}_1$cone$[a, b]$, and $\boldsymbol{\mathcal{G}}_2 \in \text{cone}(-\frac{1}{b}, -\frac{1}{a})$, for $a \leq 0 < b$.

Figures 3.2(a) and 3.2(b) illustrate the criteria for the closed-loop system to be conic when $a \leq 0$, and $b > 0$. Given that the system is memoryless, for the plant and the controller, each of their trajectory of the output as a function of the input has to lie in the sector colored in the figures so that the interconnection can be stable. Note that the trajectory can be plotted in graphs only if the system is memoryless. For a general system, Definition 3.1 and the Conic Sector Lemma 3.3 that will be discussed shortly certify if a system is conic.

For a Hurwitz, SISO, LTI system, $\boldsymbol{\mathcal{G}} : \mathcal{L}_{2e} \to \mathcal{L}_{2e}$, its conic sector can be shown in frequency domain [6]. For some $a < b$, define $r = \frac{b-a}{2}$ as the conic radius and $c = \frac{a+b}{2}$ as the conic center. If the Laplace transform of $\boldsymbol{\mathcal{G}}$, $G(s)$ satisfies

$$\Big| G(j\omega) - c \Big| \leq r, \forall \omega \in \mathbb{R}, \tag{3.1}$$

then $\boldsymbol{\mathcal{G}} \in \text{cone}[a, b]$ [6, Lemma 1, Part II].

The conic bounds can be described as a circle with radius r centers at $(c, 0)$ on the real axis. If the Nyquist trajectory of $G(s)$ is strictly bounded by the circle determined by the conic bounds, then $\boldsymbol{\mathcal{G}} \in \text{cone}[a, b]$. This is clearly shown in Figure 3.3. Although this property may not provide direct means to certify stability in controller synthesis, it can be used as a way to determine whether the system is bounded by conic constraints.

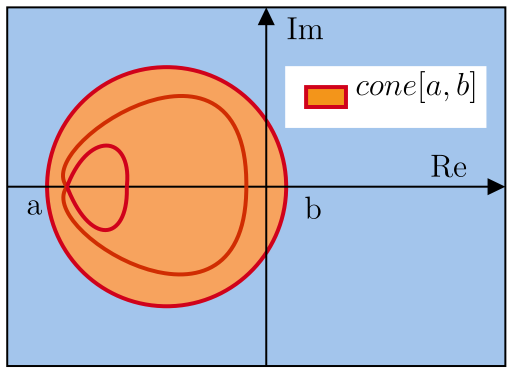


Figure 3.3: The Nyquist Plot of a system $\boldsymbol{\mathcal{G}} \in \text{cone}[a, b]$

3.3 Selection of Conic Bounds

In [31, 32], a way to determine if a system is interior conic is given. It is shown here as the Conic Sector Lemma.

Theorem 3.3 (Conic Sector Lemma [31, 32])**.** *Consider a square, stable, LTI system, $\boldsymbol{\mathcal{G}} : \mathcal{L}_{2e} \mapsto \mathcal{L}_{2e}$ with minimal state-space realization $\{\mathbf{A}, \mathbf{B}, \mathbf{C}, \mathbf{D}\}$. For $a, b \in \mathbb{R}$, $a < b$, $\boldsymbol{\mathcal{G}} \in \text{cone}[a, b]$ if and only if there exists $\mathbf{P} = \mathbf{P}^\mathsf{T} > 0$, such that*

$$\begin{bmatrix} \mathbf{PA}+\mathbf{A}^\mathsf{T}\mathbf{P}+\mathbf{C}^\mathsf{T}\mathbf{C} & * \\ \mathbf{B}^\mathsf{T}\mathbf{P}+(\mathbf{D}-c\mathbf{I})^\mathsf{T}\mathbf{C} & (\mathbf{D}-c\mathbf{I})^\mathsf{T}(\mathbf{D}-c\mathbf{I})-r^2\mathbf{I} \end{bmatrix} \leq 0, \tag{3.2}$$

where $r = \frac{b-a}{2}$ *and* $c = \frac{a+b}{2}$ *are the conic radius and center.*

Naturally, one wants to find the tightest conic bounds. As suggested in [32, 33], minimizing the conic radius r can fulfil this task. This involves solving the following optimization problem.

$$\begin{aligned} &\underset{\alpha,\beta,\mathbf{P}}{\text{Minimize}} && \alpha^2 - 2\beta \\ &\text{Subject to} && \begin{bmatrix} \mathbf{PA}+\mathbf{A}^{\mathsf{T}}\mathbf{P}+\mathbf{C}^{\mathsf{T}}\mathbf{C} & * \\ \mathbf{B}^{\mathsf{T}}\mathbf{P}+(\mathbf{D}-\alpha\mathbf{I})^{\mathsf{T}}\mathbf{C} & \mathbf{D}^{\mathsf{T}}\mathbf{D}-\alpha(\mathbf{D}+\mathbf{D}^{\mathsf{T}})+2\beta\mathbf{I} \end{bmatrix} \leq 0 \\ &&& \mathbf{P} > 0 \end{aligned} \tag{3.3}$$

The unknowns $\alpha = \frac{a+b}{2} = c$ and $\beta = -ab$ are chosen so that the constraint is linear. The objective is derived as $r^2 = \frac{(b-a)^2}{2} = \frac{(a+b)^2}{2} - 2ab = \alpha^2 - 2\beta$. Inequality (3.3) is derived by substituting α and β into Inequality (3.2). L. J. Bridgeman suggested in [34] that maximizing the lower conic bound can provide more design freedom. The reason for this is that although passive plants have conic bounds $a_p > 0$ and $b_p < \infty$, but due to small passivity violations, it can be usually expected that $|b_p|$ is larger and $a_p < 0$, $|a_p|$ is smaller than the ones in the ideal passive plant. Maximizing a_p will render b_p to increase. Thus gives actual freedom for a larger b_p. The reason of maximizing a_p instead of b_p is that, because $r_c = \frac{1}{2b_p} - \frac{1}{2a_p}$ and $|a_p| \approx 0, b_p >> 0$, minimizing $|a_p|$ can augment r_c more significantly [3, 34]. This approach involves solving the following optimization problem [34]

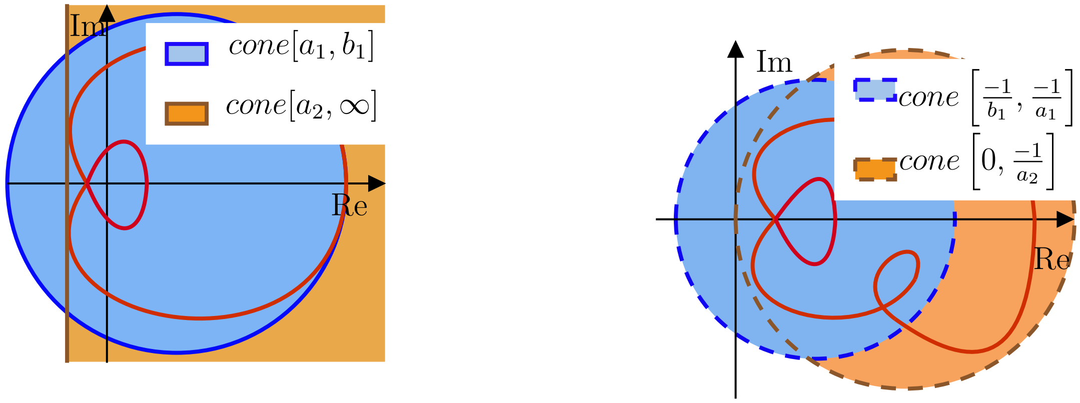
$$\begin{aligned} &\underset{a,\mathbf{P}}{\text{Minimize}} && -a \\ &\text{Subject to} && \begin{bmatrix} \mathbf{PA}+\mathbf{A}^\mathsf{T}\mathbf{P} & * \\ \mathbf{B}^\mathsf{T}\mathbf{P}-\frac{1}{2}\mathbf{C} & \frac{-\mathbf{D}-\mathbf{D}^\mathsf{T}}{2}+a\mathbf{I} \end{bmatrix} \leq 0. \\ & && \mathbf{P} > 0 \end{aligned} \tag{3.4}$$

After the lower bound a_p is found, the lowest b_p can be found by minimizing b_p subject to Inequality (3.2) with the known a_p. To see how Inequality (3.4) is derived, multiply both sides by $\frac{1}{b}$ in (3.2) and take the limit $b \to \infty$.

Figure 3.4 demonstrates the different outcomes of using the above two different ways of selecting conic bounds. The plant conic bounds $[a_1, b_1]$ are found by minimizing the conic radius while the bounds $[a_2, \infty]$ are determined by maximizing the lower conic bound. It can be understood in Figure 3.4(a) that maximizing the lower conic bound pushes the upper conic bound to ∞, thus the conic circle of $[a_1, b_1]$ becomes the half plane $[a_2, \infty)$, and allows a larger b_p due to small passivity violations. Maximizing a_p can also render a larger controller conic circle specified in Figure 3.4(b), thus improves the controller design freedom.

Although maximizing a_p theoretically provides more design freedom when confronted with small passivity violations, the two ways are both useful tools in finding conic bounds. The author has experienced situations where one of the approaches fails in finding conic bounds due to computational errors given by the solvers while the other way succeed in such attempts. It could be helpful to try the other way if one of them repeatedly fails in computations.

In [24], the LMI conditions implying conic bounds for systems subject to delays



(a) Nyquist plot of a plant

(b) Nyquist plot of a controller

Figure 3.4: Nyquist plots of a plant and a controller and conic bounds chosen using two different methods[3]

are given. The LMI conditions can be solved in optimization problems with objectives chosen as the mentioned r_p or $-a_p$ to find conic bounds that ensure stability robust to some maximum delays. And this method of selecting conic bounds can consider three types of delays: the state, input, and output delays, and can be implemented using existing software.

Chapter 4

Input-Output Controller Designs

Before introducing a new conic controller synthesis method, it is worth to talk about some existing controller design tools based on input-output stability theory. Because these methods are not only the former achievements that lead to the new method, but also good references to compare with. In this chapter, controller synthesis methods using the Passivity Theorem and Conic Sector Theorem will be discussed. First, in Section 4.1 some strict positive real controllers that inspire the new method in Chapter 5 will be discussed. Next, two nearly-optimal conic controllers, Conic B and Conic C controllers will be discussed in Section 4.2. Finally, in Section 4.3 discussions about the iterative $\mathcal{H}_2$-conic controller as dual to the method suggested in Chapter 5 will be provided.

4.1 SPR Controller Synthesis

Strictly Positive Real controllers are very useful in overcoming the oscillations brought by system uncertainties. It is known that if an SPR controller is connected with a passive plant shown in Figure 2.1, then the closed-loop system is stable.

In 1997, J. C. Geromel and P. B. Gapski [8] developed a procedure for constructing SPR observer-based controllers. To optimize system performance, they provided a way to minimize an upper-bound on the system closed-loop $\mathcal{H}_2$-norm.

The method considers an LTI system with a plant modeled as

$$\begin{aligned}\dot{\mathbf{x}}_p =& \mathbf{A}\mathbf{x}_p + \mathbf{B}_1\mathbf{w} + \mathbf{B}_2\mathbf{u}_p,\\ \mathbf{z} =& \mathbf{C}_1\mathbf{x}_p + \mathbf{D}_{12}\mathbf{u}_p\\ \mathbf{y} =& \mathbf{C}_2\mathbf{x}_p + \mathbf{D}_{21}\mathbf{w}.\end{aligned} \tag{4.1}$$

The controller has the state-space realization

$$\boldsymbol{\mathcal{G}}_c : \{\mathbf{A} - \mathbf{B}_2\mathbf{K}_c - \mathbf{L}_c\mathbf{C}_2, \mathbf{L}_c, \mathbf{K}_c, \mathbf{0}\}, \tag{4.2}$$

where $\mathbf{A}_K = \mathbf{A} - \mathbf{B}_2\mathbf{K}_c$, and $\mathbf{K}_c \in \mathbb{R}^{n_u \times n_x}$ is the feedback matrix, while $\mathbf{L}_c \in \mathbb{R}^{n_x \times n_u} = \mathbf{\Pi}\mathbf{C}_2^{\mathsf{T}}\mathbf{R}_{21}^{-1}$ is the observer matrix, where $\mathbf{R}_{21} = \mathbf{D}_{21}\mathbf{D}_{21}^{\mathsf{T}}$, and $\mathbf{\Pi} > 0$ is the solution to the Riccati equation

$$\mathbf{A}\mathbf{\Pi} + \mathbf{\Pi}\mathbf{A}^{\mathsf{T}} + \mathbf{B}_1\mathbf{B}_1^{\mathsf{T}} - \mathbf{\Pi}\mathbf{C}_2^{\mathsf{T}}\mathbf{R}_{12}^{-1}\mathbf{C}_2\mathbf{\Pi} = \mathbf{0}. \tag{4.3}$$

The crux of this method can be summarized as the following optimization problem:

$$\underset{\mathbf{Z},\mathbf{W}}{\text{Minimize}} \quad \text{trace}(\mathbf{Z}) \tag{4.4}$$

$$\text{Subject to} \quad \begin{bmatrix} \mathbf{W} & \mathbf{L}_c\mathbf{D}_{21} \\ * & \mathbf{Z} \end{bmatrix} \geq 0 \tag{4.5}$$

$$\begin{bmatrix} \mathbf{A}\mathbf{W} + \mathbf{W}\mathbf{A}^{\mathsf{T}} - \mathbf{Q} & \mathbf{W}\mathbf{C}_1^{\mathsf{T}} - \mathbf{L}_c\mathbf{D}_{21}^{\mathsf{T}} \\ \mathbf{C}_1\mathbf{W} - \mathbf{D}_{12}\mathbf{L}_c^{\mathsf{T}} & -\mathbf{I} \end{bmatrix} \leq 0 \tag{4.6}$$

$$\mathbf{W}(\mathbf{A} - \mathbf{L}_c\mathbf{C}_2)^{\mathsf{T}} + (\mathbf{A} - \mathbf{L}_c\mathbf{C}_2)\mathbf{W} - \mathbf{Q} \leq -\epsilon\mathbf{I}, \tag{4.7}$$

where $\epsilon > 0$ is a small tolerance, $\mathbf{Q} = \mathbf{L}_c\mathbf{B}_2^\mathsf{T} + \mathbf{B}_2\mathbf{L}_c^\mathsf{T}$, $\mathbf{W} = \mathbf{W}_o^{-1} = \mathbf{W}^\mathsf{T} > 0$, and $\mathbf{Z} = \mathbf{Z}^\mathsf{T} \geq 0$. The controller feedback matrix $\mathbf{K}_c = \mathbf{L}_c^\mathsf{T}\mathbf{W}_o^{-1}$ can be calculated at last.

The objective function (4.4) and the constraint (4.5) impose the bound

$$\text{trace}(\mathbf{C}_1\mathbf{\Pi}\mathbf{C}_1^\mathsf{T})+\text{trace}(\mathbf{Z}) \geq \text{trace}(\mathbf{C}_1\mathbf{\Pi}\mathbf{C}_1^\mathsf{T}) + \text{trace}(\mathbf{D}_{21}^\mathsf{T}\mathbf{L}_c^\mathsf{T}\mathbf{W}_o\mathbf{L}_c\mathbf{D}_{21}),$$

over the approximated squared $\mathcal{H}_2$-norm

$$\|H_{zw}(s)\|_{\mathcal{H}_2}^2 = \text{trace}(\mathbf{C}_1\mathbf{\Pi}\mathbf{C}_1^\mathsf{T}) + \|H_K(s)\|_{\mathcal{H}_2}^2 .$$

The term $\|H_K(s)\|_{\mathcal{H}_2}^2$ comes from [35, 36], where $H_K(s) = (\mathbf{C}_1 - \mathbf{D}_{12}\mathbf{K}_c)(s\mathbf{I} - \mathbf{A} + \mathbf{B}_2\mathbf{K}_c)^{-1}(\mathbf{L}_c\mathbf{D}_{21})$. Using Equation (2.2), $\|H_K(s)\|_{\mathcal{H}_2}^2$ can be calculated as

$$\|H_K(s)\|_{\mathcal{H}_2}^2 = \text{trace}(\mathbf{D}_{21}^\mathsf{T}\mathbf{L}_c^\mathsf{T}\mathbf{W}_O\mathbf{L}_c\mathbf{D}_{21}),$$
$$\text{where } \mathbf{W}_O(\mathbf{A} - \mathbf{B}_2\mathbf{K}_c) + (\mathbf{A} - \mathbf{B}_2\mathbf{K}_c)^\mathsf{T}\mathbf{W}_O + (\mathbf{C}_1 - \mathbf{D}_{12}\mathbf{K}_c)^\mathsf{T}(\mathbf{C}_1 - \mathbf{D}_{12}\mathbf{K}_c) = 0.$$

The detailed proof of the bounding of $\|H_{zw}(s)\|_{\mathcal{H}_2}^2$ will be given in Section 5.3. To show that the resulting controller is SPR, recall that $\mathbf{K}_c = \mathbf{L}_c^\mathsf{T}\mathbf{W}^{-1}$, the constraint (4.7) can be written as

$$\mathbf{W}(\mathbf{A} - \mathbf{B}_2\mathbf{K}_c - \mathbf{L}_c\mathbf{C}_2)^\mathsf{T} + (\mathbf{A} - \mathbf{B}_2\mathbf{K}_c - \mathbf{L}_c\mathbf{C}_2)\mathbf{W} \leq -\epsilon\mathbf{I} < 0. \qquad (4.8)$$

According to Theorem 3.1, (4.8) implies that the controller $\boldsymbol{\mathcal{G}}_c$ is SPR.

Later, T. Shimomura and S. Pullen [9] claimed that the upper-bound on $\mathcal{H}_2$-norm in [8] can result in excessive conservatism. They suggested a new approach that further minimizes the cost through an iterative scheme.

Although the detailed iterative procedure and proofs will be saved for Chapter 5, it would be worthwhile to give some introduction here about some basic ideas about this scheme.

The iterative scheme makes use of a 'dilation trick' that arises from the following fact. For any $\mathbf{R} > 0$ and $\mathbf{J}, \mathbf{N}$ that are real matrices of appropriate dimensions, $(\mathbf{RN} - \mathbf{J})^{\mathsf{T}}\mathbf{R}^{-1}(\mathbf{RN} - \mathbf{J}) \geq 0$. For a particular $\mathbf{N}$,

$$-\mathbf{J}^{\mathsf{T}}\mathbf{R}^{-1}\mathbf{J} \leq -\mathbf{J}^{\mathsf{T}}\mathbf{N} - \mathbf{N}^{\mathsf{T}}\mathbf{J} + \mathbf{N}^{\mathsf{T}}\mathbf{RN} \tag{4.9}$$

'dilates' a quadratic expression in $\mathbf{J}$ to an LMI in $\mathbf{J}$. In each iteration, an optimization problem with convex, LMI constraints is posed. Different $\mathbf{N}$ matrices are chosen in different iterations so that the RHS of Inequality (4.9) with the $\mathbf{N}$ selection is no less than the LHS representing a value in the first iteration. The selection of different $\mathbf{N}$ matrices dictates how tightly the objective bounds $\|\boldsymbol{\mathcal{G}}_{cl}\|^2_{\mathcal{H}_2}$ above, so the iteration can be thought of as a search for better choices of auxiliary matrices.

Following Shimomura and Pullen's work, J. R. Forbes [12] suggested a dual SPR controller synthesis method that iteratively minimizes an upper-bound on the closed-loop $\mathcal{H}_2$-norm. This method makes uses the dual approaches to derive the $\mathcal{H}_2$-norm. Similar to the two ways discussed in Section 2.2, the two expressions of $\|\boldsymbol{\mathcal{G}}_{cl}\|^2_{\mathcal{H}_2}$ are shown in Table 4.1. Previously, [8] and [9] developed their algorithms using Approach 1. However, [12] constructed an algorithm similar to [9], but chose the expression from Approach 2 when bounding $\|\boldsymbol{\mathcal{G}}_{cl}\|^2_{\mathcal{H}_2}$. This resulted in a different computational process, where more iterations were taken in the numerical example they implemented. Still, the dual approach provides an alternative tool to synthesize $\mathcal{H}_2$-optimal SPR controllers.

Approach 1	Approach 2
Derive the closed-loop $\mathcal{H}_2$ norm using the observability Gramian $\mathbf{W}_o$, leaving the feedback matrix $\mathbf{K}_c$ as the design variable.	Derive the closed-loop $\mathcal{H}_2$ norm using the controllability Gramian $\mathbf{W}_c$, leaving the observer matrix $\mathbf{L}_c$ as the design variable.
$\Vert\boldsymbol{\mathcal{G}}_{cl}\Vert^2_{\mathcal{H}_2} = \mathrm{trace}(\mathbf{C}_1^\mathsf{T}\boldsymbol{\Pi}\mathbf{C}_1) + \mathrm{trace}(\mathbf{L}_c\mathbf{D}_{12}\mathbf{W}_o\mathbf{D}_{12}^\mathsf{T}\mathbf{L}_c^\mathsf{T})$	$\Vert\boldsymbol{\mathcal{G}}_{cl}\Vert^2_{\mathcal{H}_2} = \mathrm{trace}(\mathbf{B}_1^\mathsf{T}\boldsymbol{\Pi}\mathbf{B}_1) + \mathrm{trace}(\mathbf{D}_{12}\mathbf{K}_c\mathbf{W}_c\mathbf{K}_c^\mathsf{T}\mathbf{D}_{12}^\mathsf{T})$

Table 4.1: Two approaches to construct closed-loop $\mathcal{H}_2$-norms for the system specified in Equations (4.1) and (4.2)

Inspired by the success of SPR controller synthesis in [8, 9], L. J. Bridgeman suggested that a conic controller synthesis method [10] could be developed using a similar iterative algorithm that minimizes the upper-bound on closed-loop $\mathcal{H}_2$-norm. Instead of imposing SPR constraints, the new method imposes conic constraints to ensure stability. More talks about this method shall be given in Section 4.3. It is worth to mention that in [10], $\Vert\boldsymbol{\mathcal{G}}_{cl}\Vert^2_{\mathcal{H}_2}$ was constructed using Approach 1 in Table 4.1. Combining the idea of [10, 12], a dual approach to the method in [10] can be developed. This will be the main topic in Chapter 5.

4.2 Conic B controller and Conic C controller

Two controller synthesis methods, Conic B and Conic C, were developed in [34]. These two approaches share similar procedures and find controllers that are nearly-optimal. To construct a controller that has $\mathcal{H}_2$ properties close to that of an $\mathcal{H}_2$-optimal controller, these two approaches seek to minimize the difference between the proposed controller and an $\mathcal{H}_2$ optimal controller.

The controller structure is defined as.

$$\mathcal{G}_c : \{\mathbf{A}_c, \mathbf{B}_c, \mathbf{C}_c, \mathbf{0}\}, \tag{4.10}$$

where $\mathbf{A}_c = \mathbf{A} - \mathbf{B}_2\mathbf{K}_c - \mathbf{L}_c\mathbf{C}_2$, $\mathbf{B}_c = \mathbf{L}_c$, and $\mathbf{C}_c = \mathbf{K}_c$. Definitions of these matrices are defined in Section 2.2.

The Conic C controller will be discussed first. To minimize the difference to a $\mathcal{H}_2$ controller, the following objective function is minimized.

$$\mathcal{J} = \text{trace}((\mathbf{C}_c - \mathbf{K}_c)\mathbf{W}_c(\mathbf{C}_c - \mathbf{K}_c)), \tag{4.11}$$

where $\mathbf{C}_c$ is the design variable and the controllability Gramian $\mathbf{W}_c$ satisfies

$$\mathbf{A}_c\mathbf{W}_c + \mathbf{W}_c\mathbf{A}_c^{\mathsf{T}} + \mathbf{B}_c^{\mathsf{T}}\mathbf{B}_c = 0. \tag{4.12}$$

To ensure stability, the conic constraint is developed by applying the state-space matrices and predetermined conic constraints into Theorem 3.3. The resulted conic constraint is

$$\begin{bmatrix} \mathbf{P}\mathbf{A}_c + \mathbf{A}_c^{\mathsf{T}}\mathbf{P} & \mathbf{P}\mathbf{B}_c & \mathbf{C}_c^{\mathsf{T}} \\ * & -\frac{(a_c-b_c)^2}{4b_c}\mathbf{I} & \frac{-(a_c+b_c)}{2}\mathbf{I} \\ * & * & -b_c\mathbf{I} \end{bmatrix} \leq 0 \tag{4.13}$$

To sum up, the conic C controller is constructed by solving an optimization problem with Equation (4.11) subject to Equation (4.13).

A conic B controller can be found in a similar procedure. The difference from conic C method is that in conic B the design variable is chosen as $\mathbf{B}_c$. As a consequence, the cost function changes to

$$\mathcal{J} = \text{trace}((\mathbf{B}_c - \mathbf{L}_c)\mathbf{W}_o(\mathbf{B}_c - \mathbf{L}_c)), \tag{4.14}$$

and the observability Gramian $\mathbf{W}_o$ satisfies

$$\mathbf{W}_o\mathbf{A}_c + \mathbf{A}_c^{\mathsf{T}}\mathbf{W}_o + \mathbf{C}_c^{\mathsf{T}}\mathbf{C}_c = 0. \tag{4.15}$$

The conic constraint derived from Theorem 3.3 is

$$\begin{bmatrix} \mathbf{A}_c\mathbf{P}^{-1} + \mathbf{P}^{-1}\mathbf{A}_c^{\mathsf{T}} & \mathbf{B}_c & \mathbf{P}^{-1}\mathbf{C}_c^{\mathsf{T}} \\ * & -\frac{(a_c-b_c)^2}{4b_c}\mathbf{I} & \frac{-(a_c+b_c)}{2}\mathbf{I} \\ * & * & -b_c\mathbf{I} \end{bmatrix} \leq 0 \tag{4.16}$$

A conic B controller can be developed by solving an optimization problem that minimizes the cost function in Equation (4.14) subject to Equation (4.16).

4.3 Iterative $\mathcal{H}_2$-Conic Controller

The method of iterative $\mathcal{H}_2$-conic controller synthesis developed by L. J. Bridgeman in [10] is inspired by the iterative algorithm for designs of SPR controllers in [8] and [9]. The algorithm consists of two parts: initialization and iterative improvement. First a conic controller is initialized by over-bounding the closed-loop $\mathcal{H}_2$-norm, $\|\boldsymbol{\mathcal{G}}_{cl}\|_{\mathcal{H}_2}$, as in [8], and then minimize this bound through iterations that take ideas from [9].

To bound $\|\boldsymbol{\mathcal{G}}_{cl}\|_{\mathcal{H}_2}$, the algorithm makes use of properties of the observability Gramian, $\mathbf{W}_O$. For the system defined in Equation (4.1), it is known that $\mathbf{W}_O > 0$

satisfies the following equation

$$\mathbf{M}_1(\mathbf{W}_O, \mathbf{K}_c) =$$
$$\mathbf{W}_O(\mathbf{A} - \mathbf{B}_2\mathbf{K}_c) + (\mathbf{A} - \mathbf{B}_2\mathbf{K}_c)^\top \mathbf{W}_O + (\mathbf{C}_1 - \mathbf{D}_{12}\mathbf{K}_c)^\top (\mathbf{C}_1 - \mathbf{D}_{12}\mathbf{K}_c) = 0. \quad (4.17)$$

The expression from [8] is

$$\|\boldsymbol{\mathcal{G}}_{cl}\|^2_{\mathcal{H}_2} = \text{trace}(\mathbf{C}_1 \boldsymbol{\Pi} \mathbf{C}_1^\top) + \text{trace}(\mathbf{L}_c \mathbf{D}_{12} \mathbf{W}_O \mathbf{D}_{12}^\top \mathbf{L}_c^\top)$$

If there exists $\mathbf{W}_o > 0$, and $\mathbf{M}_1(\mathbf{W}_o, \mathbf{K}_c) \leq 0$, then

$$\text{trace}(\mathbf{C}_1 \boldsymbol{\Pi} \mathbf{C}_1^\top) + \text{trace}(\mathbf{L}_c \mathbf{D}_{12} \mathbf{W}_o \mathbf{D}_{12}^\top \mathbf{L}_c^\top) \geq \text{trace}(\mathbf{C}_1 \boldsymbol{\Pi} \mathbf{C}_1^\top) + \text{trace}(\mathbf{L}_c \mathbf{D}_{12} \mathbf{W}_O \mathbf{D}_{12}^\top \mathbf{L}_c^\top).$$

The upper-bound is defined as

$$\mathcal{J}(\mathbf{W}_o) = \text{trace}(\mathbf{C}_1 \boldsymbol{\Pi} \mathbf{C}_1^\top) + \text{trace}(\mathbf{L}_c \mathbf{D}_{12} \mathbf{W}_o \mathbf{D}_{12}^\top \mathbf{L}_c^\top) \quad (4.18)$$

The idea to iteratively improve the initial controller is as discussed in Section 4.1. It utilize Inequality (4.9), and chooses different auxiliary matrices $\mathbf{N}$ to mitigate the conservatism brought by the initial bounding of $\|\boldsymbol{\mathcal{G}}_{cl}\|_{\mathcal{H}_2}$.

Ultimately, the decreasing process of the upper-bound and $\|\boldsymbol{\mathcal{G}}_{cl}\|_{\mathcal{H}_2}$ applied to the same example in [10], with maximum input delay of the plant set to be $1s$ is shown in Figure 4.1. Through the iterative scheme, the upper-bound on the closed-loop $\mathcal{H}_2$-norm is reduced, pushing the closed-loop $\mathcal{H}_2$-norm to converge at a value that is above the cost of a $\mathcal{H}_2$ optimal controller. This difference results from the fact that the performance is sacrificed as stability constraints are imposed.

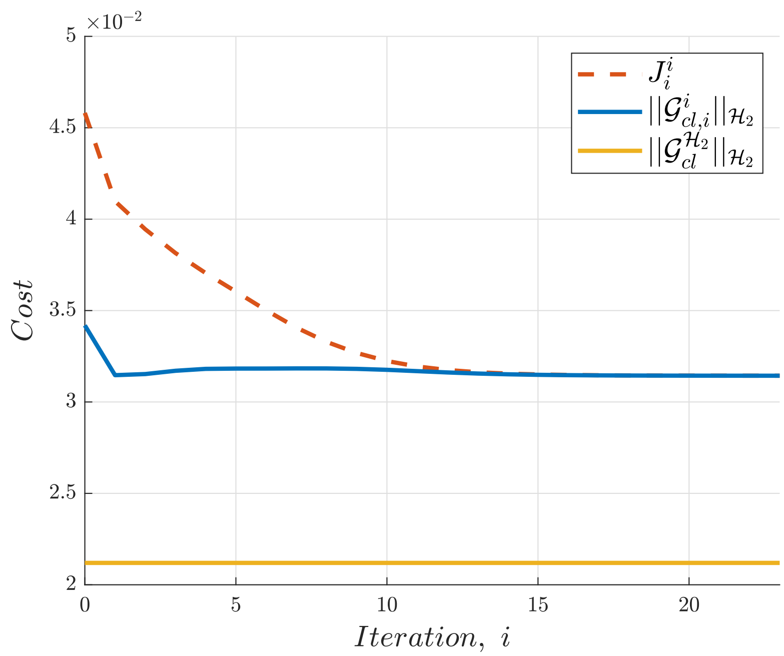


Figure 4.1: The decreases in the upper-bound $\mathcal{J}$ and closed-loop $\mathcal{H}_2$-norm $\left\|\boldsymbol{\mathcal{G}}_{cl,i}^{i}\right\|_{\mathcal{H}_2}$ through iterations in the implementation of the method and numerical example in [10]. The conic bound is found for input delays less than 1 second. $\left\|\boldsymbol{\mathcal{G}}_{cl}^{\mathcal{H}_2}\right\|_{\mathcal{H}_2}$ stands for the closed-loop $\mathcal{H}_2$-norm of an $\mathcal{H}_2$ optimal controller.

Chapter 5

Dual, Iterative $\mathcal{H}_2$-Conic Controller Synthesis

This chapter will introduce a new method that can be considered as a dual approach to the method in [10], discussed in Section 4.3. The dual approach is inspired by [12], where a dual approach to SPR controller synthesis was developed. In [10], the observability Gramian was used to calculate the closed-loop $\mathcal{H}_2$-norm, leaving the feedback matrix as a design variable. Conversely, in the proposed method, the closed-loop $\mathcal{H}_2$-norm is computed with the controllability Gramian, and the observer matrix will be a design variable. This new approach provides an alternative method for [10], that in some cases may decrease iterations in minimizing the cost and generate controllers with smaller closed-loop $\mathcal{H}_2$-norms. The structure of the algorithm in this paper will parallel that of [10]. Optimization problems will be constructed sequentially to maintain convexity and feasibility. First, similar to [8, 10], an initial conic controller will be selected by posing an optimization problem where an upper-bound of the closed-loop $\mathcal{H}_2$ is minimized with LMI constraints that impose the conic property. Next, the chosen controller will be improved iteratively based on convex relaxation paralleling [9]. In each iteration, relevant LMIs will be posed to satisfy the $\mathcal{H}_2$ performance and conic requirements. The numerical example considers a heat exchanger subject to a passivity violation introduced in [24]. Finally, the simulation results, including stability and performance of the system using the proposed method, will be compared to those obtained from the previous methods.

5.1 Problem Statement

Consider a linear plant $\boldsymbol{\mathcal{G}}_p : \mathcal{L}_{2e} \mapsto \mathcal{L}_{2e}$ that has the following state-space realization:

$$
\begin{aligned}
\dot{\mathbf{x}}_p =& \mathbf{A}\mathbf{x}_p + \mathbf{B}_1\mathbf{w} + \mathbf{B}_2\mathbf{u}_p, \\
\mathbf{z} =& \mathbf{C}_1\mathbf{x}_p + \mathbf{D}_{12}\mathbf{u}_p \\
\mathbf{y} =& \mathbf{C}_2\mathbf{x}_p + \mathbf{D}_{21}\mathbf{w},
\end{aligned}
\tag{5.1}
$$

where $\mathbf{x}_p \in \mathbb{R}^{n_x}$ is the state, $\mathbf{u}_p \in \mathbb{R}^{n_u}$ is the control input, $\mathbf{w} \in \mathbb{R}^{n_w}$ is the disturbance, $\mathbf{y} \in \mathbb{R}^{n_u}$ is the measured output, and $\mathbf{z} \in \mathbb{R}^{n_z}$ is the regulated output. The matrices are of appropriate dimension and follow the assumptions:

- $(\mathbf{A}, \mathbf{B}_1)$ is controllable and $(\mathbf{C}_1, \mathbf{A})$ is observable,
- $(\mathbf{A}, \mathbf{B}_2)$ is controllable and $(\mathbf{C}_2, \mathbf{A})$ is observable,
- $\mathbf{D}_{12}^{\mathsf{T}}\mathbf{C}_1 = \mathbf{0}$, $\mathbf{D}_{12}^{\mathsf{T}}\mathbf{D}_{12} > 0$, and
- $\mathbf{D}_{21}\mathbf{B}_1^{\mathsf{T}} = \mathbf{0}$, $\mathbf{D}_{21}\mathbf{D}_{21}^{\mathsf{T}} > 0$.

Ideally, the goal of this work would be to design a conic, observer-based controller that directly minimizes the closed-loop $\mathcal{H}_2$-norm. An expression for closed-loop $\mathcal{H}_2$-norm for time-delay systems is established in [37]. However, it is nonlinear in the design variables when applied to controller synthesis here. Therefore, in this work an upper bound on $\|\boldsymbol{\mathcal{G}}_{cl}\|_{\mathcal{H}_2}$ is minimized by solving a convex optimization problem. The idea of taking this upper bound is inspired by the synthesis method for positive real controllers from [8]. Incorporated as an optimization problem, the upper bound of $\|\boldsymbol{\mathcal{G}}_{cl}\|_{\mathcal{H}_2}$ can be minimized efficiently using applicable solvers.

The paper will consider the closed-loop $\mathcal{H}_2$-norm of the system $\boldsymbol{\mathcal{G}}_{cl} : \mathcal{L}_{2e} \mapsto \mathcal{L}_{2e}$

defined by

$$\mathbf{z} = \boldsymbol{\mathcal{G}}_{cl}\mathbf{w}, \begin{bmatrix} \mathbf{y} \\ \mathbf{z} \end{bmatrix} = \boldsymbol{\mathcal{G}}_p \begin{bmatrix} \mathbf{u}_p \\ \mathbf{w} \end{bmatrix}, \ \mathbf{u}_p = -\boldsymbol{\mathcal{G}}_c\mathbf{y}. \tag{5.2}$$

The controller structure was inspired by applying the dual of the SPR synthesis methods of [8, 9], and provides means to calculate and over-bound of $\|\boldsymbol{\mathcal{G}}_{cl}\|_{\mathcal{H}_2}$. The state-space realization of the controller will be

$$\boldsymbol{\mathcal{G}}_c : \{\mathbf{A}_K - \mathbf{L}_c\mathbf{C}_2, \mathbf{L}_c, \mathbf{K}_c, \mathbf{0}\}, \tag{5.3}$$

where $\mathbf{A}_K = \mathbf{A} - \mathbf{B}_2\mathbf{K}_c$, and $\mathbf{K}_c \in \mathbb{R}^{n_u \times n_x}$ is the feedback matrix, while $\mathbf{L}_c \in \mathbb{R}^{n_x \times n_u}$ is the observer matrix. We choose $\mathbf{K}_c = \mathbf{R}_{12}^{-1}\mathbf{B}_2^{\mathsf{T}}\mathbf{\Pi}$, where $\mathbf{R}_{12} = \mathbf{D}_{12}^{\mathsf{T}}\mathbf{D}_{12}$, and $\mathbf{\Pi} > 0$ is the solution to the Riccati equation

$$\mathbf{\Pi}\mathbf{A} + \mathbf{A}^{\mathsf{T}}\mathbf{\Pi} + \mathbf{C}_1^{\mathsf{T}}\mathbf{C}_1 - \mathbf{\Pi}\mathbf{B}_2\mathbf{R}_{12}^{-1}\mathbf{B}_2^{\mathsf{T}}\mathbf{\Pi} = \mathbf{0}. \tag{5.4}$$

Unlike the synthesis method in [10], where $\mathbf{K}_c$ is a design variable, in this work the observer matrix, $\mathbf{L}_c$, will be the design variable. This allows the conic constraint, $\boldsymbol{\mathcal{G}}_c \in \text{cone}[a_c, b_c]$ for some $a_c < 0 < b_c$, to be imposed.

Assuming a particular structure for $\boldsymbol{\mathcal{G}}_c$ and selecting an LTI plant, $\boldsymbol{\mathcal{G}}_p$, is still not sufficient to make minimizing $\|\boldsymbol{\mathcal{G}}_{cl}\|_{\mathcal{H}_2}$ subject to a controller conic constraint a well-posed optimization problem with guaranteed uniqueness of solutions. Thus, this work presents an iterative synthesis method that repeatedly selects conic controllers that minimize upper-bounds on $\|\boldsymbol{\mathcal{G}}_{cl}\|_{\mathcal{H}_2}$ and relies only on solving minimization problems with convex objectives subject to LMI constraints that can be solved efficiently with existing software.

5.2 Algorithm Overview

The method developed in this paper centers on the following algorithm. The problems inside will be defined later in Sections 5.3 to 5.5.

Algorithm 2 (Dual, iterative conic controllers). *Returns a sequence of observer matrices and associated costs.*

1: *Set the allowable maximum iterations, $M > 1$, and minimum improvement per iteration, $\epsilon > 0$.*

2: *Design an initial, non-optimal conic controller by solving Problem 1a) and store the results $(\gamma, \mathbf{W}, \mathbf{W}_L)$.*

3: *Find an upper-bound on the closed-loop $\mathcal{H}_2$-norm with the initial conic controller by solving Problem 1b) with $(\gamma, \mathbf{W}, \mathbf{W}_L)$ and store the solutions as $(\mathbf{W}_C^0, \mathbf{P}_c^0, \mathbf{L}_c^0)$.*

4: *Store the initial cost as $J^0 = \mathrm{trace}(\mathbf{B}_1^\mathsf{T}\mathbf{\Pi}\mathbf{B}_1) + \mathrm{trace}(\mathbf{D}_{21}\mathbf{K}_c\mathbf{W}_C^0\mathbf{K}_c^\mathsf{T}\mathbf{D}_{21}^\mathsf{T})$.*

5: *Initialize $i = 0$, $\tilde{\mathbf{P}} = (\mathbf{P}_c^0)^{-1}$.*

6: ***repeat***

7: $\quad i = i + 1$

8: $\quad$ *Calculate $\mathbf{N}_c = (a_c b_c \mathbf{C}_2 - c\mathbf{K}_c)\tilde{\mathbf{P}}$ and $\mathbf{N}_{bd} = (\mathbf{D}_{21}\mathbf{D}_{21}^\mathsf{T})^{-1}\mathbf{C}_2\mathbf{W}_C$ to ensure that the previous conic controller respectively satisfies dilated conic constraint* (5.5) *and overbounding constraint* (5.6).

9: $\quad$ *Find a new conic controller with lower cost by solving Problem 2 with $\mathbf{N}_c$ and $\mathbf{N}_{bd}$ and store the results as $(\tilde{\mathbf{P}}^i, \mathbf{W}_C^i, \mathbf{L}_c^i)$.*

10: $\quad$ *Store the cost as $J^i = \mathrm{trace}(\mathbf{B}_1^\mathsf{T}\mathbf{\Pi}\mathbf{B}_1) + \mathrm{trace}(\mathbf{D}_{21}\mathbf{K}_c\mathbf{W}_C^i\mathbf{K}_c^\mathsf{T}\mathbf{D}_{21}^\mathsf{T})$*

11: ***until*** $i > M$ ***or*** $J^i - J^{i-1} < \epsilon$

12: ***return*** $\{\mathbf{L}_c^j\}_{j=0}^i$ *and* $\{J^j\}_{j=0}^i$.

The crux of this paper derives convex objectives subject to LMI constraints that bound $\|\boldsymbol{\mathcal{G}}_{cl}\|_{\mathcal{H}_2}^2$ while imposing conic bounds. Problems 1 and 2 fulfil this task, and

Algorithm 2 iteratively modifies them while maintaining feasibility.

5.3 Initial Optimization

The method to bound $\|\boldsymbol{\mathcal{G}}_{cl}\|^2_{\mathcal{H}_2}$ is presented first. By applying the dual of the work in [10], the upper-bound of $\|\boldsymbol{\mathcal{G}}_{cl}\|^2_{\mathcal{H}_2}$ can be constructed. From [8, 36], the $\mathcal{H}_2$-norm can be calculated as $\text{trace}(\mathbf{C}_1^\mathsf{T}\boldsymbol{\Pi}\mathbf{C}_1) + \|\boldsymbol{\mathcal{G}}_{bd}\|^2_{\mathcal{H}_2}$. Applying the dual of that yields

$$\|\boldsymbol{\mathcal{G}}_{cl}\|^2_{\mathcal{H}_2} = \text{trace}(\mathbf{B}_1^\mathsf{T}\boldsymbol{\Pi}\mathbf{B}_1) + \|\boldsymbol{\mathcal{G}}_{bd}\|^2_{\mathcal{H}_2},$$

where $\boldsymbol{\mathcal{G}}_{bd}$ has the state-space realization

$$\{(\mathbf{A} - \mathbf{L}_c\mathbf{C}_2), \mathbf{B}_1 - \mathbf{L}_c\mathbf{D}_{21}, \mathbf{D}_{12}\mathbf{K}_c, \mathbf{0}\}.$$

From [11, pp. 188-189], $\|\boldsymbol{\mathcal{G}}_{bd}\|^2_{\mathcal{H}_2}$ can be calculated as

$$\|\boldsymbol{\mathcal{G}}_{bd}\|^2_{\mathcal{H}_2} = \text{trace}(\mathbf{D}_{12}\mathbf{K}_c\mathbf{W}_c\mathbf{K}_c^\mathsf{T}\mathbf{D}_{12}^\mathsf{T}).$$

The controlability Gramian of $\boldsymbol{\mathcal{G}}_{bd}$ is $\mathbf{W}_c > 0$, which satisfies $\mathbf{M}_{b1}(\mathbf{W}_c, \mathbf{L}_c) = \mathbf{0}$ for

$$\begin{aligned}\mathbf{M}_{b1}(\mathbf{W}, \mathbf{L}) =& \mathbf{W}(\mathbf{A} - \mathbf{L}\mathbf{C}_2)^\mathsf{T} + (\mathbf{A} - \mathbf{L}\mathbf{C}_2)\mathbf{W} \\ &+ (\mathbf{B}_1 - \mathbf{L}\mathbf{D}_{21})(\mathbf{B}_1 - \mathbf{L}\mathbf{D}_{21})^\mathsf{T}.\end{aligned}$$

Following [8], the cost is defined to bound $\|\boldsymbol{\mathcal{G}}_{cl}\|^2_{\mathcal{H}_2}$ as

$$\begin{aligned}\mathcal{J}_1(\mathbf{W}_C) &:= \text{trace}(\mathbf{B}_1^\mathsf{T}\boldsymbol{\Pi}\mathbf{B}_1) + \text{trace}(\mathbf{D}_{12}\mathbf{K}_c\mathbf{W}_C\mathbf{K}_c^\mathsf{T}\mathbf{D}_{12}^\mathsf{T}) \\ &\geq \mathcal{J}_1(\mathbf{W}_c) = \|\boldsymbol{\mathcal{G}}_{cl}\|^2_{\mathcal{H}_2},\end{aligned}$$

where $\mathbf{W}_C > 0$ and $\mathbf{M}_{b1}(\mathbf{W}_C, \mathbf{L}_c) \leq \mathbf{0}$.

The method to impose the conic property is demonstrated next. Applying Theroem 3.3 to the state-space realization from Equation (5.3) shows that $\boldsymbol{\mathcal{G}}_c \in \text{cone}[a_c, b_c]$ if there exists $\mathbf{P}_c > 0$ such that $\mathbf{M}_{c1}(\mathbf{P}_c, \mathbf{L}_c) \leq 0$ for

$$\mathbf{M}_{c1}(\mathbf{P}, \mathbf{L}) =$$
$$\begin{bmatrix} \mathbf{P}(\mathbf{A}_K - \mathbf{L}\mathbf{C}_2) + (\mathbf{A}_K - \mathbf{L}\mathbf{C}_2)^\mathsf{T}\mathbf{P} + \mathbf{K}_c^\mathsf{T}\mathbf{K}_c & * \\ \mathbf{L}^\mathsf{T}\mathbf{P} - c_c\mathbf{K}_c & a_c b_c \mathbf{I} \end{bmatrix},$$

where $c_c = (a_c + b_c)/2$. To maintain convexity and linearity in the design variables, the LMI constraints associated with bounding $\|\boldsymbol{\mathcal{G}}_{cl}\|_{\mathcal{H}_2}^2$ and satisfying conic bounds are developed separately for initialization and iterative improvements.

The optimization problem solved in controller initialization consists of two steps, as follows.

Problem 1 (Initial Controller). *Solve the following two sub-problems.*

1a) *Find matrices* $\mathbf{P}_c$ *and* $\mathbf{W}_L$ *that satisfy* $\mathbf{M}_{c2}(\mathbf{P}_c, \mathbf{W}_L) \leq 0$ *and* $\mathbf{P}_c \geq 0$, *where*

$$\mathbf{M}_{c2}(\mathbf{P}_c, \mathbf{W}_L) =$$
$$\begin{bmatrix} \mathbf{P}_c\mathbf{A}_K + \mathbf{A}_K^\mathsf{T}\mathbf{P}_c - \mathbf{W}_L\mathbf{C}_2 - \mathbf{C}_2^\mathsf{T}\mathbf{W}_L^\mathsf{T} + \mathbf{K}_c^\mathsf{T}\mathbf{K}_c & * \\ \mathbf{W}_L^\mathsf{T} - c_c\mathbf{K}_c & a_c b_c \mathbf{I} \end{bmatrix};$$

1b) *Calculate* $\mathbf{L}_c = \mathbf{P}_c^{-1}\mathbf{W}_L$, *then solve the following optimization problem with the known* $\mathbf{L}_c$

$$\begin{aligned} &\underset{\mathbf{W}_C}{\text{Minimize}} && \text{trace}(\mathbf{D}_{12}\mathbf{K}_c\mathbf{W}_C\mathbf{K}_c^\mathsf{T}\mathbf{D}_{12}^\mathsf{T}) \\ &\text{Subject to} && \mathbf{M}_{b1}(\mathbf{W}_C; \mathbf{L}_c) \leq 0, \mathbf{W}_C \geq 0. \end{aligned}$$

The LMI $\mathbf{M}_{c2}(\mathbf{P}_c, \mathbf{W}_L) \leq 0$ is derived by replacing the terms of $\mathbf{P}_c\mathbf{L}_c$ in $\mathbf{M}_{c1}(\mathbf{P}_c, \mathbf{L}_c) \leq 0$ with $\mathbf{W}_L$. Notice that the two-step structure is not quite the dual of the counterpart in [9] or [10] where the controller is initialized by solving one optimization problem. The duals of [9] and [10] would be equivalent to solving Problems 1a) and 1b) simultaneously while imposing that $\mathbf{P}_c = \gamma \mathbf{W}_c$ for some $\gamma > 0$. This approach could be used here, but the additional assumption was observed to render the problem infeasible in certain numerical examples. While solving 1a) and 1b) separately was observed to mitigate these issues, it engenders another potential problem: even if a conic controller exists for which 1b) can be solved, 1a) might not select it.

5.4 Stepwise Optimization

The initial optimization provides a conic controller and its closed-loop $\mathcal{H}_2$-norm. However, the controller can still be improved iteratively by solving optimization problems that make use of Equation (4.9). At each iteration in Algorithm 2, the following problem is solved.

Problem 2 (Improved Controller).

$$\begin{aligned}
&\underset{\tilde{\mathbf{P}}, \mathbf{W}_C, \mathbf{L}_c}{\text{Minimize}} && \operatorname{trace}(\mathbf{B}_1^\mathsf{T} \mathbf{\Pi} \mathbf{B}_1) + \operatorname{trace}(\mathbf{D}_{12}\mathbf{K}_c\mathbf{W}_C\mathbf{K}_c^\mathsf{T}\mathbf{D}_{12}^\mathsf{T}) \\
&\text{Subject to} && \tilde{\mathbf{P}} > 0,\ \mathbf{W}_C > 0, \mathbf{M}_{bN}(\mathbf{W}_C, \mathbf{L}_c; \mathbf{N}_{bd}) \leq 0, \\
& && \textit{and } \mathbf{M}_{cN}(\tilde{\mathbf{P}}, \mathbf{L}_c; \mathbf{N}_c) \leq 0.
\end{aligned}$$

In the above optimization problem, $\tilde{\mathbf{P}} = \mathbf{P}_c^{-1}$, $\mathbf{W}_C$, and $\mathbf{L}_c$ are design variables. The inequalities $\mathbf{M}_{bN}(\mathbf{W}_C, \mathbf{L}_c; \mathbf{N}_{bd}) \leq 0$ and $\mathbf{M}_{cN}(\tilde{\mathbf{P}}, \mathbf{L}_c; \mathbf{N}_c) \leq 0$, to be derived in the coming paragraphs, result from applying Equation (4.9) to matrix inequalities that respectively impose the upper bound and conic bounds while introducing auxiliary matrices $\mathbf{N}_{bd}$ and $\mathbf{N}_c$. This is done to impose these qualities through convex, LMI constraints. Their selection dictates how tightly the objective bounds $\|\boldsymbol{\mathcal{G}}_{cl}\|_{\mathcal{H}_2}^2$ above, so the iteration can be thought of as a search for better choices of auxiliary matrix.

To derive $\mathbf{M}_{cN}(\tilde{\mathbf{P}}, \mathbf{L}_c; \mathbf{N}_c) \leq 0$ from $\mathbf{M}_{c1}(\mathbf{P}_c, \mathbf{L}_c) \leq 0$, take the Schur complement to see that $\mathbf{M}_{c1}(\mathbf{P}_c, \mathbf{L}_c) \leq 0$ is equivalent to

$$\begin{aligned}
&\mathbf{P}_c(\mathbf{A}_K - \mathbf{L}_c\mathbf{C}_2) + (\mathbf{A}_K - \mathbf{L}_c\mathbf{C}_2)^\mathsf{T}\mathbf{P}_c + \mathbf{K}_c^\mathsf{T}\mathbf{K}_c \\
&\quad - \frac{1}{a_cb_c}\left(\mathbf{L}_c^\mathsf{T}\mathbf{P}_c - c_c\mathbf{K}_c\right)^\mathsf{T}\left(\mathbf{L}_c^\mathsf{T}\mathbf{P}_c - c_c\mathbf{K}_c\right) \\
&= \mathbf{P}_c\mathbf{A}_K + \mathbf{A}_K^\mathsf{T}\mathbf{P}_c - \frac{1}{a_cb_c}\left(\mathbf{P}_c\mathbf{L}_c + (a_cb_c\mathbf{C}_2^\mathsf{T} - c_c\mathbf{K}_c^\mathsf{T})\right) \\
&\quad \left(\mathbf{L}_c^\mathsf{T}\mathbf{P}_c + (a_cb_c\mathbf{C}_2 - c_c\mathbf{K}_c)\right) - \frac{r_c^2}{a_cb_c}\mathbf{K}_c^\mathsf{T}\mathbf{K}_c \\
&\quad + a_cb_c\left(\frac{c_c}{a_cb_c}\mathbf{K}_c - \mathbf{C}_2\right)^\mathsf{T}\left(\frac{c_c}{a_cb_c}\mathbf{K}_c - \mathbf{C}_2\right) \leq 0.
\end{aligned}$$

Let $\mathbf{R} = -a_cb_c\mathbf{I}$ and $\mathbf{J} = (c_c\mathbf{K}_c - a_cb_c\mathbf{C}_2)\tilde{\mathbf{P}}$, multiplying left and right by $\tilde{\mathbf{P}} = \mathbf{P}_c^{-1} > 0$ reveals that

$$\begin{aligned}
&\mathbf{A}_K\tilde{\mathbf{P}} + \tilde{\mathbf{P}}\mathbf{A}_K^\mathsf{T} - \frac{1}{a_cb_c}(\mathbf{L}_c - \mathbf{J}^\mathsf{T})(\mathbf{L}_c^\mathsf{T} - \mathbf{J}) - \frac{r_c^2}{a_cb_c}\tilde{\mathbf{P}}\mathbf{K}_c^\mathsf{T}\mathbf{K}_c\tilde{\mathbf{P}} \\
&+ a_cb_c\tilde{\mathbf{P}}\left(\frac{c_c}{a_cb_c}\mathbf{K}_c - \mathbf{C}_2\right)^\mathsf{T}\left(\frac{c_c}{a_cb_c}\mathbf{K}_c - \mathbf{C}_2\right)\tilde{\mathbf{P}} \leq 0.
\end{aligned}$$

Taking the Schur complement twice, this is equivalent to

$$\begin{bmatrix} \mathbf{A}_K\tilde{\mathbf{P}} + \tilde{\mathbf{P}}\mathbf{A}_K^\mathsf{T} - \mathbf{J}^\mathsf{T}\mathbf{R}^{-1}\mathbf{J} & * & * \\ \mathbf{L}_c^\mathsf{T} + \mathbf{J} & -\mathbf{R} & * \\ \mathbf{K}_c\tilde{\mathbf{P}} & \mathbf{0} & -r_c^{-2}\mathbf{R} \end{bmatrix} \leq 0,$$

By applying Equation (4.9), the quadratic term in $\mathbf{J}$ can be ‘dilated’ into an LMI in

$\mathbf{J}$. Consequently, $\mathbf{M}_{c1}(\mathbf{P}_c, \mathbf{L}_c) \leq 0$ if there exists $\mathbf{N}_c$ such that $\mathbf{M}_{cN}(\mathbf{P}_c^{-1}, \mathbf{L}_c; \mathbf{N}_c) \leq 0$, where

$$\mathbf{M}_{cN}(\tilde{\mathbf{P}}, \mathbf{L}_c; \mathbf{N}) = \begin{bmatrix} \mathbf{M}_{tl1} & * & * \\ \mathbf{L}_c^\mathsf{T} + \mathbf{J} & -\mathbf{R} & * \\ \mathbf{K}_c\tilde{\mathbf{P}} & \mathbf{0} & -r_c^{-2}\mathbf{R} \end{bmatrix} \leq 0 \tag{5.5}$$

where $\mathbf{M}_{tl1} = \mathbf{A}_K\tilde{\mathbf{P}} + \tilde{\mathbf{P}}\mathbf{A}_K^\mathsf{T} - \mathbf{J}^\mathsf{T}\mathbf{N} - \mathbf{N}^\mathsf{T}\mathbf{J} + \mathbf{N}^\mathsf{T}\mathbf{R}\mathbf{N}$.

Moreover, if $\mathbf{M}_{c1}(\mathbf{P}_c, \mathbf{L}_c) \leq 0$ then there exists $\mathbf{N}_c$ such that $\mathbf{M}_{cN}(\mathbf{P}_c^{-1}, \mathbf{L}_c; \mathbf{N}_c) \leq 0$. Namely, $\mathbf{N}_c = \mathbf{R}^{-1}\mathbf{J} = (a_c b_c \mathbf{C}_2 - c_c\mathbf{K}_c)\tilde{\mathbf{P}}$, which achieves equality in Equation (4.9).

Next, $\mathbf{M}_{b1}(\mathbf{W}_C, \mathbf{L}_c) \leq 0$ must be replaced with an LMI in the design variables. To achieve this, recall that $\mathbf{D}_{21}\mathbf{B}_1^\mathsf{T} = \mathbf{0}$ and re-write the constraint as

$$\begin{aligned} \mathbf{M}_{b1}(\mathbf{W}_C, \mathbf{L}_c) =& \mathbf{W}_C\mathbf{A}^\mathsf{T} - \mathbf{W}_C\mathbf{C}_2^\mathsf{T}\mathbf{L}_c^\mathsf{T} + \mathbf{A}\mathbf{W}_C \\ & - \mathbf{L}_c\mathbf{C}_2\mathbf{W}_C + \mathbf{B}_1\mathbf{B}_1^\mathsf{T} + \mathbf{L}_c\mathbf{D}_{21}\mathbf{D}_{21}^\mathsf{T}\mathbf{L}_c^\mathsf{T} \\ =& \mathbf{W}_C\mathbf{A}^\mathsf{T} + \mathbf{A}\mathbf{W}_C + \mathbf{B}_1\mathbf{B}_1^\mathsf{T} \\ & + (\mathbf{L}_c\mathbf{D}_{21}\mathbf{D}_{21}^\mathsf{T} - \mathbf{W}_C\mathbf{C}_2^\mathsf{T})(\mathbf{D}_{21}\mathbf{D}_{21}^\mathsf{T})^{-1} \\ & (\mathbf{D}_{21}\mathbf{D}_{21}^\mathsf{T}\mathbf{L}_c^\mathsf{T} - \mathbf{C}_2\mathbf{W}_C) \\ & - \mathbf{W}_C\mathbf{C}_2^\mathsf{T}(\mathbf{D}_{21}\mathbf{D}_{21}^\mathsf{T})^{-1}\mathbf{C}_2\mathbf{W}_C. \end{aligned}$$

Selecting $(\mathbf{R}, \mathbf{J}) = (\mathbf{D}_{21}\mathbf{D}_{21}^\mathsf{T}, \mathbf{C}_2\mathbf{W}_C)$ in Equation (4.9) implies that

$$\begin{aligned} &-\mathbf{W}_C\mathbf{C}_2^\mathsf{T}(\mathbf{D}_{21}\mathbf{D}_{21}^\mathsf{T})^{-1}\mathbf{C}_2\mathbf{W}_C \\ &\quad \leq -\mathbf{N}^\mathsf{T}\mathbf{C}_2\mathbf{W}_C - \mathbf{W}_C\mathbf{C}_2^\mathsf{T}\mathbf{N} + \mathbf{N}^\mathsf{T}(\mathbf{D}_{21}\mathbf{D}_{21}^\mathsf{T})^{-1}\mathbf{N}. \end{aligned}$$

Taking the Schur complement reveals that $\mathbf{M}_{b1}(\mathbf{W}_C, \mathbf{L}_c) \leq 0$ if there exists $\mathbf{N}_{bd}$ such that $\mathbf{M}_{bN}(\mathbf{W}_C, \mathbf{L}_c; \mathbf{N}_{bd}) \leq 0$ where

$$\mathbf{M}_{bN}(\mathbf{W}_C, \mathbf{L}_c; \mathbf{N}_{bd}) = \begin{bmatrix} \mathbf{M}_{tl2} & \mathbf{L}_c\mathbf{D}_{21}\mathbf{D}_{21}^\mathsf{T} - \mathbf{W}_C\mathbf{C}_2^\mathsf{T} \\ \mathbf{D}_{21}\mathbf{D}_{21}^\mathsf{T}\mathbf{L}_c^\mathsf{T} - \mathbf{C}_2\mathbf{W}_C & -\mathbf{D}_{21}\mathbf{D}_{21}^\mathsf{T} \end{bmatrix} \leq 0. \tag{5.6}$$

where $\mathbf{M}_{tl2} = \mathbf{W}_C(\mathbf{A}^\mathsf{T} - \mathbf{C}_2^\mathsf{T}\mathbf{N}_{bd}) + (\mathbf{A} - \mathbf{N}_{bd}^\mathsf{T}\mathbf{C}_2)\mathbf{W}_C + \mathbf{B}_1\mathbf{B}_1^\mathsf{T} + \mathbf{N}_{bd}^\mathsf{T}(\mathbf{D}_{21}\mathbf{D}_{21}^\mathsf{T})^{-1}\mathbf{N}_{bd}$. Note that $\mathbf{N}_{bd} = \mathbf{R}^{-1}\mathbf{J} = (\mathbf{D}_{21}\mathbf{D}_{21}^\mathsf{T})^{-1}\mathbf{C}_2\mathbf{W}_C$ implies that equality is achieved in Equation (4.9).

5.5 Iteration

Convergence may not happen in finite iterations or when Problem 1 is infeasible. Otherwise, convergence follows from the monotonic decrease of the cost and the lower-bound set by the closed-loop $\mathcal{H}_2$-norm of an $\mathcal{H}_2$ optimal controller. Similar to discussions in [9, Section II], if the conic constraints in Problems 1 and 2 are absent, then the cost $\mathcal{J}_1$ reduces to the closed-loop $\mathcal{H}_2$-norm of a $\mathcal{H}_2$-optimal controller, therefore $\mathcal{J}_1$ is lower-bounded.

Each of Problems Problem 1 and Problem 2 minimizes a conservative upper-bound on $\|\boldsymbol{\mathcal{G}}_{cl}\|_{\mathcal{H}_2}^2$. The proposed Algorithm 2 begins by solving Problem 1 and then successively changes the auxiliary matrices $\mathbf{N}_{bd}$ and $\mathbf{N}_c$ in Problem 2 to decrease this conservatism.

To move from the initialization step to the first iteration, observe that Section 5.4 showed that $\mathbf{M}_{b1}(\mathbf{W}_C, \mathbf{L}_c) \leq 0$ implies $\mathbf{M}_{bN}(\mathbf{W}_C, \mathbf{L}_c; \mathbf{N}_{bd}) \leq 0$ for $\mathbf{N}_{bd} = (\mathbf{D}_{21}\mathbf{D}_{21}^\mathsf{T})^{-1}\mathbf{C}_2\mathbf{W}_C$. Section 5.4 also showed that $\mathbf{M}_{c1}(\mathbf{P}_c, \mathbf{L}_c) \leq 0$ implies $\mathbf{M}_{cN}(\tilde{\mathbf{P}}, \mathbf{L}_c; \mathbf{N}_c) \leq$

0 for $\mathbf{N}_c = (a_c b_c \mathbf{C}_2 - c_c \mathbf{K}_c)\tilde{\mathbf{P}}$. As a result, the solutions of Problem 1 $(\mathbf{W}_C^0, \mathbf{P}_c^0, \mathbf{L}_c^0)$ satisfy the constraints of Problem 1 with $\mathbf{N}_{bd} = (\mathbf{D}_{21}\mathbf{D}_{21}^\mathsf{T})^{-1}\mathbf{C}_2(\mathbf{W}_C^0)^{-1}$ and $\mathbf{N}_c = (a_c b_c \mathbf{C}_2 - c_c \mathbf{K}_c)\tilde{\mathbf{P}}^0$. As a consequence, the controller of Problem 1 is feasible for Problem 2. Problem 2 will only select a different controller if it decreases the cost, so the cost will either decrease or remain constant.

Similar to moving from the initialization, note that if $(\mathbf{P}_c^i, \mathbf{W}_C^i, \mathbf{L}_c^i)$ is a solution to the i^{th} iteration of Problem 1, $\mathbf{M}_{bN}(\mathbf{W}_C^i, \mathbf{L}_c^i; \mathbf{N}_{bd}^i) \leq 0$ implies that $\mathbf{M}_{b1}(\mathbf{W}_C^i, \mathbf{L}_c^i) \leq 0$, and subsequently $\mathbf{M}_{bN}(\mathbf{W}_C^i, \mathbf{L}_c^i; \mathbf{N}_{bd}^{i+1}) \leq 0$ for $\mathbf{N}_{bd}^{i+1} = (\mathbf{D}_{21}\mathbf{D}_{21}^\mathsf{T})^{-1}\mathbf{C}_2\mathbf{W}_C^i$. The similar derivation from $\mathbf{M}_{cN}(\tilde{\mathbf{P}}^i, {\mathbf{L}_c}^i; \mathbf{N}_c^i) \leq 0$ to
$\mathbf{M}_{cN}(\tilde{\mathbf{P}}^i, {\mathbf{L}_c}^i; \mathbf{N}_c^{i+1}) \leq 0$ also holds. Hence, $(\mathbf{P}_c^i, \mathbf{W}_C^i, \mathbf{L}_c^i)$ satisfies the constraints of the $(i+1)^{\text{st}}$ iteration of Problem 2 and $\mathcal{J}_1(\mathbf{W}_C^{i+1}) \leq \mathcal{J}_1(\mathbf{W}_C^i)$.

5.6 Numerical Example

To directly compare different conic controller synthesis methods, this section revisits the heat exchanger considered in [24] and [10]. The plant is subject to input delays that cause passivity violations [29]. The parameters of the plant from [38, pp. 133-140] are listed in Table 5.1.

Table 5.1: System Parameters

Parameters	Aniline	Benzine	Units
A	-	7.57	m^2
U	-	391	$\frac{J}{s \cdot m^2 \cdot {}^\circ C}$
v	12	7.17	$\frac{m^3}{s} \times 10^{-4}$
ρ	1022	879	$\frac{kg}{m^3}$
c_p	2.18	1.76	$\frac{J}{kg \cdot {}^\circ C} \times 10^{-3}$
V	9.41	3.75	$m^3 \times 10^{-2}$

After derivations based on the heat exchanger model discussed in [39, pp. 55-60], the states are defined in [40] as:

$$\mathbf{x}_p = \begin{bmatrix} T_c^o(t) - T_{c,d}^o \\ T_h^o(t) - [T_{c,d}^o + \frac{v_c c_{pc} \rho_c}{UA}(T_{c,d}^o - t_c^i)] \end{bmatrix},$$

where $T_c^o(t)$ and $T_h^o(t)$ are the outlet temperature of the cold and hot streams, t_c^i is the inlet temperature of the cold stream, and $T_{c,d}$ is the desired temperature at the outlet of the cold stream. With an input delay, the system state-space realization can be indicated as:

$$\begin{aligned} \dot{\mathbf{x}}_p(t) &= \mathbf{A}\mathbf{x}_p(t) + \mathbf{B}_2 \mathbf{u}_p(t - h_i) \\ \mathbf{y}(t) &= \mathbf{C}_2 \mathbf{x}_p(t), \end{aligned}$$

where the state-space matrices are defined as:

$$\mathbf{A} = \begin{bmatrix} -\frac{v_c}{V_c} - \frac{UA}{c_{pc}\rho_c V_c} & \frac{UA}{c_{pc}\rho_c V_c} \\ \frac{UA}{c_{ph}\rho_h V_h} & -\frac{v_h}{V_h} - \frac{UA}{c_{ph}\rho_h V_h} \end{bmatrix}, \ \mathbf{C}_2^\mathsf{T} = \begin{bmatrix} 1 \\ 0 \end{bmatrix},$$

$$\mathbf{B}_1 = \begin{bmatrix} \sqrt{0.1} & 0 & 0 \\ 0 & \sqrt{30} & 0 \end{bmatrix}, \ \mathbf{C}_1^\mathsf{T} = \begin{bmatrix} \sqrt{30} & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix},$$

$$\mathbf{B}_2 = \begin{bmatrix} 0 \\ \frac{v_h}{V_h} \end{bmatrix}, \ \mathbf{D}_{12} = \begin{bmatrix} 0 \\ \sqrt{0.1} \end{bmatrix}, \text{ and } \mathbf{D}_{21} = \begin{bmatrix} 0 & 1 \end{bmatrix}.$$

To compare with the proposed method, an $\mathcal{H}_2$-optimal controller, $\boldsymbol{\mathcal{G}}_2^{\mathcal{H}}$, is designed. This controller assumes no delays in the nominal plant and is therefore expected to have good performance and stability only for small delays. A conic controller, $\boldsymbol{\mathcal{G}}^B$, as discussed in [34], is also developed for comparison by setting the input matrix as the design variable and minimizing the distance to the $\mathcal{H}_2$-optimal controller.

To apply the proposed method, conic bounds for the plant that hold true for delays up to a limit are determined by implementing the method of [24]: the plant conic radius $r_p = \frac{a_p+b_p}{2}$ was minimized subject to the constraints listed in [24, Theorem 3.1]. Notably, [10] selects conic bounds that maximize a_p. However, for this example, this selection sometimes renders infeasible controller initialization problems when the input matrix is a design variable. Instead, the plant conic radius is minimized here. It is worth mentioning that, for $h_i < 1s$, minimizing r_p in finding the conic bounds results in a converged closed-loop $\mathcal{H}_2$-norm of 0.031 after 23 iterations for $\boldsymbol{\mathcal{G}}^{[10]}$, which is slightly less than that of 0.033 after 22 iterations by initially maximizing a_p as in [10]. For input delays $h_i < 1s$, the plant conic bounds are $a_p = -0.075$ and $b_p = 0.37$. According to Theorem 3.2, any controllers within the cone$[a_c, b_c]$, where $a_c = -2.69$ and $b_c = 13.26$ will ensure closed-loop input-output robust stability for input delays $h_i < 1s$.

Next, Algorithm 2 is implemented to obtain a sequence of controllers and their associated costs. For $h_i < 1s$, it takes 11 iterations to achieve the defined minimum improvement of 10^{-6} in the cost. Figure 5.1 displays the resulting costs and the closed-loop $\mathcal{H}_2$-norms in comparison with those obtained following [10]. At each iteration, $i \in \mathbb{N}$, the cost is denoted as J_i, and the closed-loop $\mathcal{H}_2$-norm is denoted as $||\mathcal{G}_{cl,i}||_{\mathcal{H}_2}$. Notations *dual* and [10] respectively indicate the methods from this paper and [10]. It can be observed in Figure 5.1 that the objective bounds the closed-loop $\mathcal{H}_2$-norm and decreases until achieving the minimum improvement. The proposed synthesis method results in a faster convergence and a smaller final closed-loop $\mathcal{H}_2$-norm.

The Nyquist plots of the controllers designed for $h_i \leq 2(s)$ is shown in Figure 5.2. Recall from [6] that if the Nyquist plot of a system lies in the circle with radius $(b-a)/2$ centered at $((a+b)/2, 0)$ on the real axis, then the system is in cone$[a, b]$. Therefore, Figure 5.2 confirms that the conic controllers satisfy the conic constraints, while the $\mathcal{H}_2$ optimal controller does not. Figure 5.2 also depicts that the iterations increase the gain of the controller while respecting the conic bounds. As shown in the Bode diagrams of Figure 5.3, the gain of $\boldsymbol{\mathcal{G}}^{dual}$ is improved by Algorithm 2 to approach that of $\boldsymbol{\mathcal{G}}^{\mathcal{H}_2}$. In both Figure 5.2 and Figure 5.3, the curves of $\boldsymbol{\mathcal{G}}^{[10]}$ and $\boldsymbol{\mathcal{G}}^{dual}$ are similar, revealing the closeness in their gains. Figure 5.4 illustrates the calculated closed-loop $\mathcal{H}_2$-norm following [37]. It is shown that $\boldsymbol{\mathcal{G}}^{\mathcal{H}_2}$ yields better performance for low delays but immediately deteriorates as delays increase. The proposed iterative conic controller has performance that is close to, but better than, the performance of existing conic controllers for all delays. Finally, Figure 5.5 compares the closed-loop responses of the controllers when tracking the desired output described in [24] at input delays of 1 and 2 seconds. It reveals that $\boldsymbol{\mathcal{G}}^{\mathcal{H}_2}$ results in faster convergence than conic controllers at low delays, but its performance degrades and is eventually unable to stabilize the system. Despite converging more slowly at lower delays due to maintaining unnecessary robustness, the conic controllers manage to stabilize the systems as delays increase. In this example, the proposed method has a very close performance and stability features in comparison to that of [10], but requires fewer iterations during design.

Some advice can be useful for implementation. Minimizing r_p or a_p are two alternative approaches to find the conic bounds. Extending M allows convergence if needed. Adding a small tolerance to matrix definiteness releases computational flexibility to the solvers when solving LMI's.

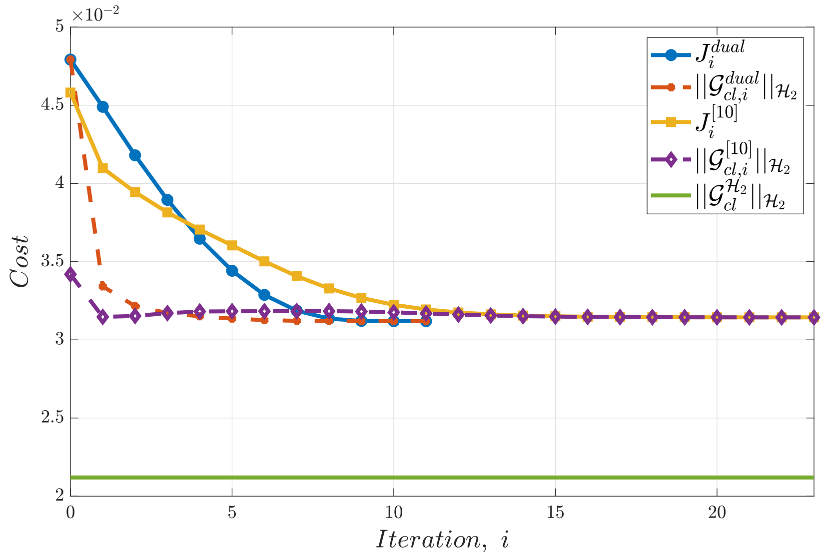


Figure 5.1: With conic controllers designed for $h_i \leq 1(s)$, the decreases in costs and closed-loop $\mathcal{H}_2$-norms through iterations from the controller synthesis methods proposed in this paper and [10].

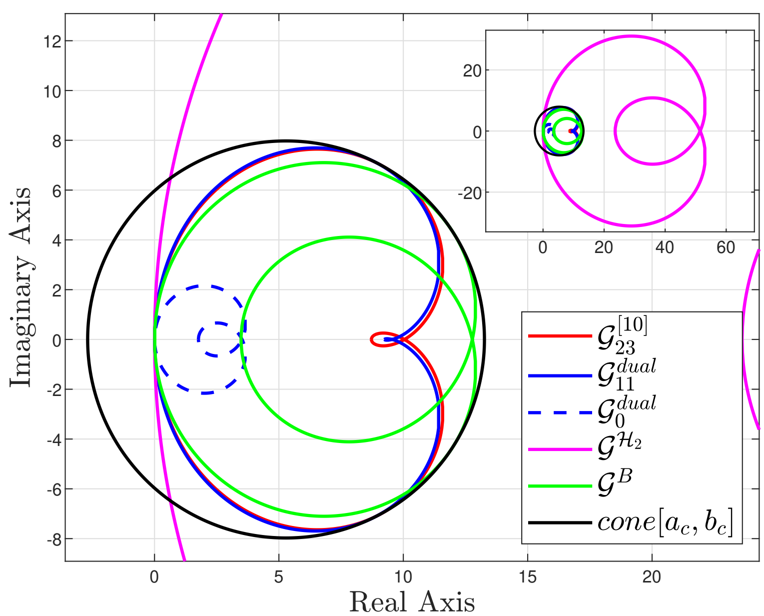


Figure 5.2: Nyquist plots of controllers ensuring stability for $h_i \leq 2(s)$.

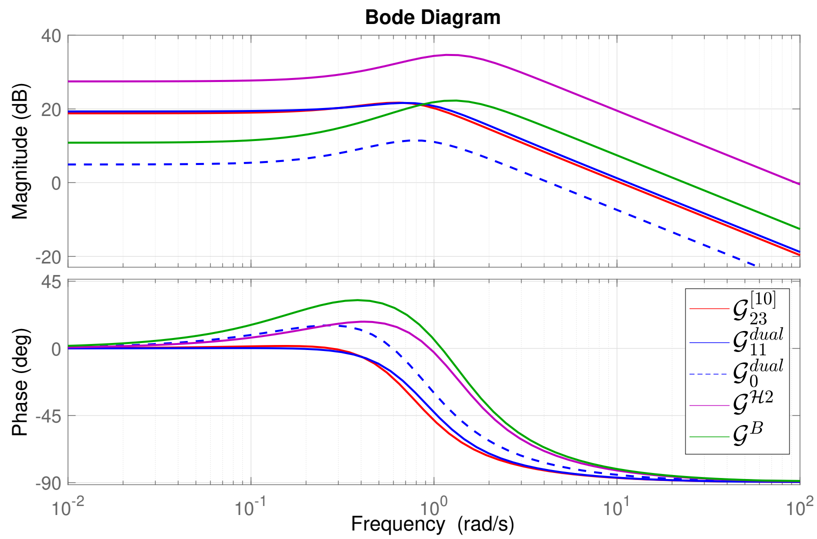


Figure 5.3: Bode diagrams of the designed controllers ensuring stability for $h_i \leq 2(s)$.

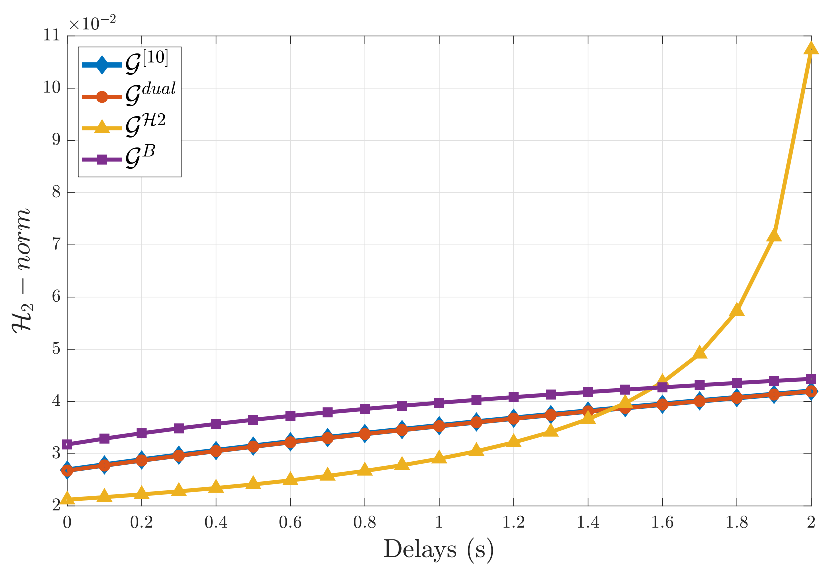


Figure 5.4: Closed-loop $\mathcal{H}_2$-norms of the designed controllers with different delays.

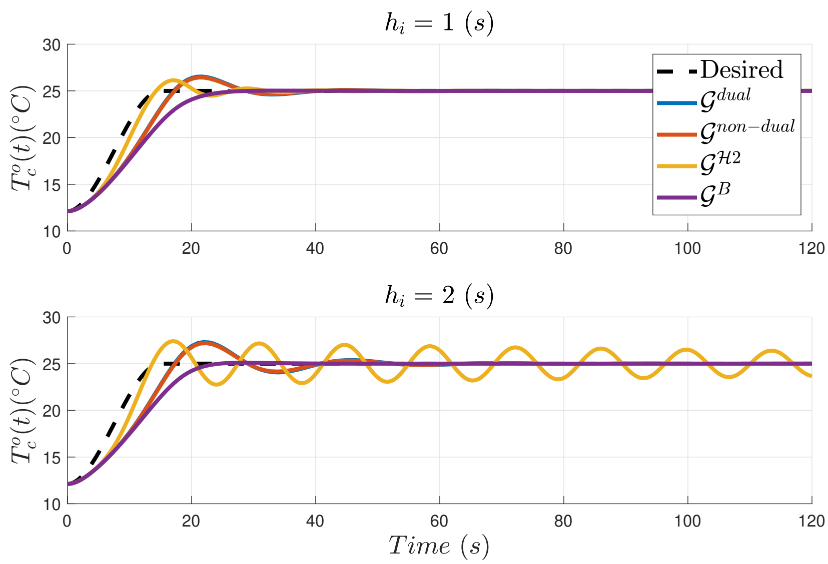


Figure 5.5: Closed-loop responses when tracking the desired output.

5.7 An Alternative Approach

An alternative approach to initialize the controller will be presented in this section. This procedure can replace the steps 2 and 3 in Algorithm 2. In Algorithm 2 the controller is constructed by minimizing the upper-bound on $\mathcal{H}_2$-norm to make it become closer to an $\mathcal{H}_2$-optimal controller in the aspect of performance. In contrast, the idea of this new approach is to first develop an $\mathcal{H}_2$ optimal controller and then gradually make it satisfy the conic constraint. Although the implementation of this algorithm has not been performed, this new method is desirable since it does not seek to over-bound the closed-loop $\mathcal{H}_2$-norm but tries to generate a controller with this closed-loop norm directly. This can result in better performance if the implementation is sucessful.

Algorithm 3 (Conic A controllers). *Returns a sequence of observer matrices.*

Set the allowable maximum iterations, $M > 1$, *and a small tolerance* $\epsilon > 0$.

Initialize $i = 0$.

Solve Equation (2.8) *to get* $\mathbf{\Pi}_2$. *Then calculate the observer matrix of an* $\mathcal{H}_2$-*optimal controller* $\mathbf{L}_0 = \mathbf{\Pi}_2 \mathbf{C}_2^{\mathsf{T}} (\mathbf{D}_{21}\mathbf{D}_{21})^{-1}$.

Initialize a controller by solving Problem 3, and store $(\mathbf{W}_i, \mathbf{P}_i, \alpha_i)$.

repeat

    $i = i + 1$.

    Find a controller with an improved conic constraint by solving Problem 4 and get $(\Delta\mathbf{W}, \Delta\mathbf{P}, \alpha_i)$.

    Update $(\mathbf{W}_i, \mathbf{P}_i, \mathbf{L}_i)$ *as* $(\mathbf{W}_{i-1} + \Delta\mathbf{W}, \mathbf{P}_{i-1} + \Delta\mathbf{P}, \mathbf{L}_{i-1} + \Delta\mathbf{L})$.

until $i > M$ ***or*** $\alpha < \epsilon$

return $\{\mathbf{L}_c^j\}_{j=0}^i$

Problems 3 and 4 are defined below.

Problem 3 ($\mathcal{H}_2$-Optimal Controller). *Solve the following problem*

$$\begin{aligned} \underset{\mathbf{W}_i,\mathbf{P}_i,\alpha_i}{\text{Minimize}} \quad & \alpha_i \\ \text{Subject to} \quad & \mathbf{M}_{c1}(\mathbf{P}_i,\mathbf{L}_i) \leq \alpha_i, \mathbf{M}_{b1}(\mathbf{W}_i;\mathbf{L}_i) \leq 0, \mathbf{W}_i \geq 0, \mathbf{P}_i \geq 0. \end{aligned}$$

Problem 4 (Improved $\mathcal{H}_2$-Optimal Controller). *Solve the following problem*

$$\begin{aligned} \underset{\Delta\mathbf{W},\Delta\mathbf{P},\Delta\mathbf{L},\alpha_i}{\text{Minimize}} \quad & \alpha_i \\ \text{Subject to} \quad & \mathbf{M}_{c\alpha}(\Delta\mathbf{P},\Delta\mathbf{L},\alpha_i) \leq 0, \mathbf{M}_{b\alpha}(\Delta\mathbf{W},\Delta\mathbf{L}) \leq 0, \\ & \Delta\mathbf{W}+\mathbf{W}_{i-1} \geq 0, \Delta\mathbf{P}+\mathbf{P}_{i-1} \geq 0. \end{aligned}$$

Problem 4 is solved at each iteration with the objective as α, and the conic constraint is set to be the inequality $\mathbf{M}_{c\alpha}(\Delta\mathbf{P},\Delta\mathbf{L},\alpha_i) \leq 0$ that implies $\mathbf{M}_{c1}(\mathbf{P}_i,\mathbf{L}_i) \leq \alpha_i$. The constraint $\mathbf{M}_{b\alpha}(\Delta\mathbf{W},\Delta\mathbf{L}) \leq 0$ implies $\mathbf{M}_{b1}(\mathbf{W}_i,\mathbf{L}_i) \leq 0$. As Problem 4 is solved iteratively, only the variables that renders α to decrease will be chosen and finally pushes α to be negative, thus forcing $\mathbf{M}_{c1}(\mathbf{P}_i,\mathbf{L}_i) \leq 0$. To derive the constraints in Problem 4, recall from Section 5.3,

$$\mathbf{M}_{c1}(\mathbf{P},\mathbf{L}) = \begin{bmatrix} \mathbf{P}(\mathbf{A}_K-\mathbf{L}\mathbf{C}_2)+(\mathbf{A}_K-\mathbf{L}\mathbf{C}_2)^{\mathsf{T}}\mathbf{P}+\mathbf{K}_c^{\mathsf{T}}\mathbf{K}_c & * \\ \mathbf{L}^{\mathsf{T}}\mathbf{P}-c_c\mathbf{K}_c & a_cb_c\mathbf{I} \end{bmatrix}$$

$$\mathbf{M}_{b1}(\mathbf{W},\mathbf{L}) = \mathbf{W}(\mathbf{A}-\mathbf{L}\mathbf{C}_2)^{\mathsf{T}}+(\mathbf{A}-\mathbf{L}\mathbf{C}_2)\mathbf{W}+(\mathbf{B}_1-\mathbf{L}\mathbf{D}_{21})(\mathbf{B}_1-\mathbf{L}\mathbf{D}_{21})^{\mathsf{T}}.$$

From $\mathbf{M}_{c1}(\mathbf{P},\mathbf{L}) \leq \alpha^0$, substitute in $\mathbf{P}_0 + \Delta\mathbf{P}$ and $\mathbf{L}_0 + \Delta\mathbf{L}$ to get

$$
\begin{aligned}
&\mathbf{M}_{c1}(\mathbf{P}_0 + \Delta\mathbf{P}, \mathbf{L}_0 + \Delta\mathbf{L}) = \\
&\begin{bmatrix} (\mathbf{P}_0 + \Delta\mathbf{P})(\mathbf{A}_K - (\mathbf{L}_0 + \Delta\mathbf{L})\mathbf{C}_2) & * \\ +(\mathbf{A}_K - (\mathbf{L}_0 + \Delta\mathbf{L})\mathbf{C}_2)^\mathsf{T}(\mathbf{P}_0 + \Delta\mathbf{P}) + \mathbf{K}_c^\mathsf{T}\mathbf{K}_c & \\ (\mathbf{L}_0 + \Delta\mathbf{L})^\mathsf{T}(\mathbf{P}_0 + \Delta\mathbf{P}) - c_c\mathbf{K}_c & a_c b_c \mathbf{I} \end{bmatrix} - \alpha^0\mathbf{I} \\
&= \mathbf{M}_{c0} - \alpha^0\mathbf{I} + \begin{bmatrix} -\Delta\mathbf{P}\Delta\mathbf{L}\mathbf{C}_2 - \mathbf{C}_2^\mathsf{T}\Delta\mathbf{L}^\mathsf{T}\Delta\mathbf{P} & * \\ \Delta\mathbf{L}^\mathsf{T}\Delta\mathbf{P} & \mathbf{0} \end{bmatrix} \leq 0,
\end{aligned}
\tag{5.7}
$$

where

$$
\mathbf{M}_{c0} = \begin{bmatrix} \mathbf{P}_0\mathbf{A}_K - \mathbf{P}_0\mathbf{L}_0\mathbf{C}_2 - \mathbf{P}_0\Delta\mathbf{L}\mathbf{C}_2 + \Delta\mathbf{P}\mathbf{A}_K - \Delta\mathbf{P}\mathbf{L}_0\mathbf{C}_2 + \mathbf{A}_K^\mathsf{T}\mathbf{P}_0 & * \\ +\mathbf{A}_K^\mathsf{T}\Delta\mathbf{P} - \mathbf{C}_2^\mathsf{T}\mathbf{L}_0^\mathsf{T}\mathbf{P}_0 - \mathbf{C}_2^\mathsf{T}\Delta\mathbf{L}\mathbf{P}_0 - \mathbf{C}_2^\mathsf{T}\mathbf{L}_0^\mathsf{T}\Delta\mathbf{P} + \mathbf{K}_c^\mathsf{T}\mathbf{K}_c & \\ \mathbf{L}_0^\mathsf{T}\mathbf{P}_0 + \mathbf{L}_0^\mathsf{T}\Delta\mathbf{P} + \Delta\mathbf{L}^\mathsf{T}\mathbf{P}_0 - c_c\mathbf{K}_c & a_c b_c\mathbf{I} \end{bmatrix}.
$$

Using Inequality (4.9), (5.7) can be dilated into

$$
\mathbf{M}_{c0} - \alpha^0\mathbf{I} + \begin{bmatrix} -\Delta\mathbf{P}^2 + \mathbf{C}_2^\mathsf{T}\Delta\mathbf{L}^\mathsf{T}\Delta\mathbf{P} & * \\ \Delta\mathbf{L}^\mathsf{T}\Delta\mathbf{P} & \mathbf{0} \end{bmatrix} \leq 0
$$

Applying Schur Complement to get

$$
\left[\begin{array}{cc|c|c} \multicolumn{2}{c|}{\mathbf{M}_{c0} - \alpha^0 + \begin{bmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \Delta\mathbf{L}^\mathsf{T}\Delta\mathbf{L} \end{bmatrix}} & \begin{matrix} \Delta\mathbf{P} \\ \Delta\mathbf{L} \end{matrix} & \begin{matrix} \mathbf{C}_2^\mathsf{T}\Delta\mathbf{L}^\mathsf{T} \\ \mathbf{0} \end{matrix} \\ \hline \Delta\mathbf{P} & \Delta\mathbf{L} & -\mathbf{I} & \mathbf{0} \\ \Delta\mathbf{L}\mathbf{C}_2 & \mathbf{0} & -\mathbf{0} & -\mathbf{I} \end{array}\right] \leq 0,
$$

which is implied by

$$\mathbf{M}_{c\alpha}(\Delta\mathbf{P},\Delta\mathbf{L},\alpha)=\left[\begin{array}{cc|c|c} \mathbf{M}_{c0}-\alpha^0 & & \Delta\mathbf{P} & \mathbf{C}_2^\mathsf{T}\Delta\mathbf{L}^\mathsf{T} \\ & & \Delta\mathbf{L} & \mathbf{0} \\ \hline \Delta\mathbf{P} & \Delta\mathbf{L} & -\mathbf{I} & \mathbf{0} \\ \Delta\mathbf{L}\mathbf{C}_2 & \mathbf{0} & -\mathbf{0} & -\mathbf{I} \end{array}\right]\leq 0. \tag{5.8}$$

From $\mathbf{M}_{b1}(\mathbf{W},\mathbf{L})\leq 0$, substitute in $\mathbf{W}_0+\Delta\mathbf{W}$ and $\mathbf{L}_0+\Delta\mathbf{L}$ to get

$$\begin{aligned} &\mathbf{M}_{b1}(\mathbf{W}_0+\Delta\mathbf{W},\mathbf{L}_0,\Delta\mathbf{L}) \\ &=(\mathbf{W}_0+\Delta\mathbf{W})(\mathbf{A}-(\mathbf{L}_0+\Delta\mathbf{L})\mathbf{C}_2)^\mathsf{T}+(\mathbf{A}-(\mathbf{L}_0+\Delta\mathbf{L})\mathbf{C}_2)(\mathbf{W}_0+\Delta\mathbf{W})+ \\ &(\mathbf{B}_1-(\mathbf{L}_0+\Delta\mathbf{L})\mathbf{D}_{21})(\mathbf{B}_1-(\mathbf{L}_0+\Delta\mathbf{L})\mathbf{D}_{21})^\mathsf{T} \\ &=\mathbf{M}_{b0}(\Delta\mathbf{W},\Delta\mathbf{L};\mathbf{W}_0,\mathbf{L}_0)-\Delta\mathbf{W}\mathbf{C}_2^\mathsf{T}\Delta\mathbf{L}^\mathsf{T}-\Delta\mathbf{L}\mathbf{C}_2\Delta\mathbf{W}+\Delta\mathbf{L}\mathbf{D}_{21}\mathbf{D}_{21}^\mathsf{T}\Delta\mathbf{L}^\mathsf{T}\leq 0 \end{aligned} \tag{5.9}$$

where

$$\begin{aligned} &\mathbf{M}_{b0}(\Delta\mathbf{W},\Delta\mathbf{L};\mathbf{W}_0,\mathbf{L}_0) \\ &=\mathbf{W}_0\mathbf{A}^\mathsf{T}-\mathbf{W}_0\mathbf{C}_2^\mathsf{T}(\mathbf{L}_0+\Delta\mathbf{L})^\mathsf{T}+\Delta\mathbf{W}\mathbf{A}^\mathsf{T}+\mathbf{A}\mathbf{W}_0+\mathbf{A}\Delta\mathbf{W}-(\mathbf{L}_0+\Delta\mathbf{L})\mathbf{C}_2\mathbf{W}_0 \\ &+\mathbf{B}_1\mathbf{B}_1^\mathsf{T}-\mathbf{B}_1\mathbf{D}_{21}^\mathsf{T}(\mathbf{L}_0+\Delta\mathbf{L})^\mathsf{T}-(\mathbf{L}_0+\Delta\mathbf{L})\mathbf{D}_{21}\mathbf{B}_1^\mathsf{T}-\Delta\mathbf{W}\mathbf{C}_2^\mathsf{T}\mathbf{L}_0^\mathsf{T}-\mathbf{L}_0\mathbf{C}_2\Delta\mathbf{W} \\ &+\mathbf{L}_0\mathbf{D}_{21}\mathbf{D}_{21}^\mathsf{T}\mathbf{L}_0^\mathsf{T}+\mathbf{L}_0\mathbf{D}_{21}\mathbf{D}_{21}^\mathsf{T}\Delta\mathbf{L}^\mathsf{T}+\Delta\mathbf{L}\mathbf{D}_{21}\mathbf{D}_{21}^\mathsf{T}\mathbf{L}_0^\mathsf{T}. \end{aligned}$$

Inequality (5.9) is equivalent to

$$\mathbf{M}_{b\alpha}(\Delta\mathbf{W}, \Delta\mathbf{L}) = \begin{bmatrix} \mathbf{M}_{b0}(\Delta\mathbf{W}, \Delta\mathbf{L}; \mathbf{W}_0, \mathbf{L}_0) & \Delta\mathbf{L}\mathbf{D}_{21} & \Delta\mathbf{L}\mathbf{C}_2 & \Delta\mathbf{W} \\ * & -\mathbf{I} & \mathbf{0} & \mathbf{0} \\ * & * & -\mathbf{I} & \mathbf{0} \\ * & * & * & -\mathbf{I} \end{bmatrix} \leq 0 \quad (5.10)$$

Chapter 6

Conclusions

The 1990s and 2000s witnessed the accelerated improvements in the computational power of computers [41]. This promoted the development of software capable of rapidly solving optimization problems [15]. As a result, control engineers are equipped with tools to confront systems that used to be considered hard to control. And this motivates the explorations of advanced control tools. It is known that the Conic Sector Theorem can be used to resolve problems that some other theorems cannot. For instance, systems with passivity violations and the open-loop unstable systems that the Passivity Theorem does not cover. This research seeks to further discover the potential value of the Conic Sector Theorem and to enrich the toolbox of robust control.

The various results within input-output stability theory are sometimes overwhelming for people who first step into this area. The survey included in this work serves as a guide to the basic theory and designs of passivity and conic sector theorem.

The novelty of this work resides in the introduction of a new observer-based controller synthesis method that ensures robust stability via the Conic Sector Theorem. An algorithm is proposed, that centers on initializing a conic controller and then iteratively decreasing an upper-bound over the closed-loop $\mathcal{H}_2$-norm. Despite core optimization problems developed as duals to the existing iterative $\mathcal{H}_2$-conic controller developed in [10], the proposed method terminated faster and achieved a smaller closed-loop $\mathcal{H}_2$-norm in some examples. This work developed a new controller synthesis method achieving good performance and input-output stability, adding an additional valuable tool in the family of conic controllers.

Three on-going improvements are carried on in this research. First, a subsystem that uses a Padé approximation to estimate the response of the control input $\mathbf{u}$ is added to better model the time-delay system. This subsystem can replace the delayed element between the controller and the plant in Figure 2.3. In this way the time delay may be better approximated. Second, inspired by the close convergences in the results of the proposed method and the one in [10], a proof is under derivation to seek equivalence of the convergences. Third, an expression of the closed-loop $\mathcal{H}_2$-norm for time-delay systems is given in [37] and can possibly be used as the objective function to replace the upper-bound. If this can be done, the conservatism brought by bounding the closed-loop $\mathcal{H}_2$-norm can be decreased and the performance can be further optimized.

Appendix A

Resulting Publications

Liangting Wu and Leila Jasmine Bridgeman. *Dual, Iterative $\mathcal{H}_2$-Conic Controller Synthesis.* In *American Control Conference*, Denver, CO, USA, July 2020. (In Proceeding).

Liangting Wu. *$\mathcal{H}_2$/Conic Controller Synthesis.* In *Southeast Controls Conference*, Atlanta, GA, USA, November 2019. (Published Abstract).

Bibliography

[1] G. F. Franklin, J. D. Powell, and A. Emami-Naeini. *Feedback control of dynamic systems*, volume 3. Addison-Wesley Reading, 1994.

[2] M. Corless. *Introduction to Dynamic Systems*. School of Aeronautics & Astronautics, Purdue University, West Lafayette, USA, 2011. URL `http://www.hamilton.ie/ollie/Downloads/Dyn.pdf`.

[3] L. J. Bridgeman. *Methods Exploiting and Extending the Conic Sector Theorem*. The Department of Mechanical Engineering, McGill University, Montreal, Canada, 2016.

[4] R. Ortega. Applications of input-output techniques to control problems. In *European Ctrl Conf.*, pages 1307–1315, Gernoble, FRA, 2 - 5 July 1991.

[5] M. Vidyasagar. $\mathcal{L}_2$-stability of interconnected systems using a reformulation of the Passivity Theorem. *IEEE Trans. Circuits Systems*, 24(11):637–645, 1977.

[6] G. Zames. On the input-output stability of time-varying nonlinear feedback systems parts I & II. *IEEE Trans. Aut. Ctrl.*, 11(2):229–231, 1966.

[7] L. J. Bridgeman and J. R. Forbes. The extended conic sector theorem. *IEEE Trans. Aut. Ctrl.*, 61(7):1931–1937, July 2016.

[8] J. C. Geromel and P. B. Gapski. Synthesis of positive real $\mathcal{H}_2$ controllers. *IEEE Trans. Aut. Ctrl.*, 42(7):988–992, 1997.

[9] T. Shimomura and S.P. Pullen. Strictly positive real H_2 controller synthesis via iterative algorithms for convex optimization. *J. Guid., Ctrl. Dyn.*, 25(6): 1003–1011, 2002.

[10] Leila Jasmine Bridgeman and James Richard Forbes. Iterative $\mathcal{H}_2$-conic controller synthesis. *Int. J. Robust Nonlin. Ctrl.*, 29(11):3701–3714, 2019. doi: 10.1002/rnc.4581.

[11] G.E. Dullerud and F.G. Paganini. *A Course in Robust Control Theory: A Convex Approach*. Springer, New York, USA, 2005.

[12] J.R. Forbes. Dual approaches to strictly positive real controller synthesis with a $\mathcal{H}_2$ performance using linear matrix inequalities. *Int. J. Robust Nonlin. Ctrl.*, 23(8):903–918, 2013.

[13] J. Chamberland and H. D. Pfister. *Engineering Fundamentals*. The Department of Electrical Engineering, Duke University, Durham, USA, 2011. URL `http://pfister.ee.duke.edu/courses/ece586/notes.pdf`.

[14] A. Schaft. *L2-Gain and Passivity in Nonlinear Control*. Springer-Verlag, Berlin, Heidelberg, 2nd edition, 1999. ISBN 1852330732.

[15] S. Boyd and L. vandenberghe. *Convex optimization*. Cambridge university press, Cambridge, UK, 2004.

[16] S. Boyd, L. El Ghaoui, E. Feron, and V. Balakrishnan. *Linear Matrix Inequalities in System an Control Theory*. SIAM, Philadelphia, USA, 1994.

[17] R. J. Caverly and J. R. Forbes. LMI properties and applications in systems, stability, and control theory, 2019.

[18] J. Strum. Using SeDuMi 1.02, a MATLAB toolbox for optimization over symmetric cones. *Optimization Methods and Software*, 11(12):625–653, 1999. Special Issue on Interior Point Methods.

[19] MOSEK ApS. *The MOSEK optimization toolbox for MATLAB manual. Version 7.1 (Revision 28).*, 2015. URL `http://docs.mosek.com/7.1/toolbox/index.html`.

[20] K.C. Toh, M.J. Todd, and RH Tutuncu. SDPT3 - a MATLAB software package for semidefinite programming. *Optimization Methods and Software*, 11:545–581, 1999.

[21] J. Löftberg. YALMIP: A toolbox for modeling and optimization in MATLAB. In *Computer Aided Ctrl Systems Design*, Taipei, TWN, 4 Sept. 2004.

[22] C.A. Desoer and M. Vidyasagar. *Feedback Systems: Input-Output Properties*. SIAM, New York, USA, 2009.

[23] Q.C. Zhong. *Robust Control of Time-delay Systems*. Springer London, 2006.

[24] L. J. Bridgeman and J. R. Forbes. Conic bounds for systems subject to delays. *IEEE Trans. Aut. Ctrl.*, 62(4):2006–2013, 2017. doi: 10.1109/TAC.2016.2596680.

[25] N. Sakamoto and M. Suzuki. γ-passive system and its phase property and synthesis. *IEEE Trans. Aut. Ctrl.*, 41(6):859–865, 1996.

[26] M. Xia, P.J. Antsaklis, and V. Gupta. Passivity indices and passivation of systems with application to systems with input/output delay. In *IEEE Conf. Decision and Ctrl.*, pages 783–788, Los Angeles, USA, 15-17 Dec. 2014. IEEE.

[27] V. Zahedzadeh, H.J. Marquez, and T Chen. On the input-output stability of nonlinear systems: Large Gain Theorem. In *Amer. Ctrl. Conf.*, pages 3440–3445, Seattle, USA, 11-13 June 2008. IEEE.

[28] L.J. Bridgeman and J.R. Forbes. A comparative study of input–output stability results. *IEEE Trans. Aut. Ctrl.*, 2018.

[29] J. Bao and P.L. Lee. *Process Control: The Passive Systems Approach*. Springer-Verlag, London, GBR, 2007.

[30] W. Sun, P. P. Khargonekar, and D. Shim. Solution to the positive real control problem for linear time-invariant systems. *IEEE Transactions on Automatic Control*, 39(10):2034–2046, 1994.

[31] S. Gupta and S.M. Joshi. Some properties and stability results for sector-bounded LTI systems. In *IEEE Conf. Decision and Ctrl.*, volume 3, pages 2973–2978, Lake Buena Vista, USA, 14-16 Dec. 1994. IEEE.

[32] S. Gupta. Robust stabilization of uncertain systems based on energy dissipation concepts. Contractor Report 4713, NASA, 1996. URL `http://ntrs.nasa.gov/archive/nasa/casi.ntrs.nasa.gov/19960016570.pdf`.

[33] S.M. Joshi and A.G. Kelkar. Design of norm-bounded and sector-bounded LQG controllers for uncertain systems. *J. Optimization Theory and Applications*, 113 (2):269–282, 2002.

[34] L.J. Bridgeman and J.R. Forbes. Conic-sector-based control to circumvent passivity violations. *Int. J. Ctrl*, 87:1467–1477, 2014.

[35] J. Doyle, K. Glover, P. Khargonekar, and B. Francis. State-space solutions to standard $\mathcal{H}_2$ and $\mathcal{H}_\infty$ control problems. 1988.

[36] P.P. Khargonekar and M.A. Rotea. Mixed H_2/H_∞ control: A convex optimization approach. *IEEE Trans. Aut. Ctrl.*, 36:824–837, 1991.

[37] E. Jarlebring, J. Vanbiervliet, and W. Michiels. Characterizing and computing the $\mathcal{H}_2$ norm of time-delay systems by solving the delay lyapunov equation. *IEEE Trans. Aut. Ctrl.*, 56(4):814–825, 2011. doi: 10.1109/TAC.2010.2067510.

[38] R.W. Serth. *Process Heat Transfer*. Academic Press, Oxford, GBR, 2007.

[39] K.M. Hangos and J. Bokor. *Analysis and Control of Nonlinear Process Systems*. Springer, London, GBR, 2004.

[40] L.J. Bridgeman and J.R. Forbes. Conic controller synthesis that minimizes an upper bound on the closed-loop $\mathcal{H}_2$-norm. In *Amer. Ctrl. Conf.*, pages 2405–2409, Seattle, USA, 24-26 May 2017. IEEE.

[41] C. A. Mack. Fifty years of moore's law. *IEEE Transactions on Semiconductor Manufacturing*, 2011.