REAL HYPERSURFACES
IN
UNIMODULAR COMPLEX SURFACES

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ABSTRACT. A unimodular complex surface is a complex 2-manifold X endowed with a holomorphic volume form Υ. A strictly pseudoconvex real hypersurface in X inherits not only a CR-structure but a canonical coframing as well.

In this article, this canonical coframing is defined, its invariants are discussed and interpreted geometrically, and its basic properties are studied.

A natural evolution equation for strictly pseudoconvex real hypersurfaces in unimodular complex surfaces is defined, some of its properties are discussed, and several examples are computed.

It is shown that a real-analytic 3-manifold endowed with a real-analytic coframing satisfying the structure equations can be real-analytically embedded as a pseudoconvex hypersurface in a unimodular complex surface in such a way that the induced canonical coframing is the given one. Moreover, this embedding is essentially unique up to unimodular biholomorphism.

The locally homogeneous examples are determined and used to illustrate various features of the geometry of the induced structure on the hypersurface.

The invariants of the underlying CR-structure are expressed in terms of the invariants of the coframing.

1. INTRODUCTION

The purpose of this article is to define and study a flow for nondegenerate real hypersurfaces in a complex surface endowed with a holomorphic volume form. This flow is invariant under unimodular biholomorphisms and is weakly parabolic in an appropriate sense. Neither uniqueness nor short time existence for this flow has yet been proved and it should be an interesting problem to do so.

The remainder of this introduction will be a guide to the sections of the article.

A unimodular complex surface is a complex 2-manifold X endowed with a holomorphic volume form Υ. A strictly pseudoconvex real hypersurface in X inherits not only a CR-structure but a canonical coframing as well. This canonical coframing is defined in Section 2 by its structure equations (cf. Proposition 1).

It is worth emphasizing that the ambient holomorphic volume form is needed in order to define this coframing: There is no canonical coframing of strictly pseudoconvex real hypersurfaces in $\mathbb{C}^2$ that is invariant under the full pseudogroup of biholomorphisms of $\mathbb{C}^2$.

Specifically, it is shown that if $M \subset X$ is a strictly pseudoconvex real hypersurface in X, then the pullback of Υ to M can be written uniquely in the form $M^*\Upsilon = \theta \wedge \eta$ for 1-forms θ and η on M satisfying

$$(1.1) \qquad\qquad \theta = \bar\theta \qquad \text{and} \qquad \mathrm{d}\theta = \mathrm{i}\,\eta \wedge \overline{\eta}.$$

The 1-form θ depends on two derivatives of a local defining function for M while the 1-form η depends on three derivatives of such a local defining function.

It is then shown that there exist functions $a = \bar a$ and b on M such that

$$(1.2) \qquad\qquad \mathrm{d}\eta = 2\mathrm{i}\,\theta \wedge \left(a\,\eta + b\,\overline{\eta}\right).$$

The functions a and b are the *primary invariants* of M. They are the fourth order invariants of M as a hypersurface in (X, Υ): Given any point $p \in M$, there exist p-centered holomorphic coordinates (w, z) on a p-neighborhood $U \subset X$ such that $U^*\Upsilon = \mathrm{d}w \wedge \mathrm{d}z$ and so that $M \cap U$ is defined in U by an equation of the form

$$(1.3) \qquad \mathrm{Im}(w) = \tfrac{1}{2}\, z\bar z \left(1 + b(p)\, z^2 + \tfrac{3}{2}a(p)\, z\bar z + \overline{b(p)}\, \bar z^2\right) + R_5\left(z, \mathrm{Re}(w)\right)$$

where R_5 is a function on a neighborhood of $(0,0) \in \mathbb{C} \times \mathbb{R}$ that vanishes to order 5 at $(z, u) = (0, 0)$.

The invariants a and b are then discussed and interpreted geometrically, and their basic properties are studied. For example, a plays the role of 'mean curvature' in the sense that its vanishing is the Euler-Lagrange equation for the functional defined by integrating the canonical volume form $\tfrac{1}{2}\,\theta \wedge \mathrm{d}\theta$ over M. The condition for CR-flatness is expressed in terms of the invariants a and b and their derivatives. It is no surprise that there should be such an expression since a and b, together with the canonical coframing (θ, η), is a *complete* set of invariants (in Cartan's sense) for nondegenerate real hypersurfaces in unimodular surfaces.[1]

A nondegenerate hypersurface $M \subset X$ is locally homogeneous under the unimodular biholomorphism pseudogroup if and only if a and b are constant functions. It is shown that there is a 3-parameter family of examples of locally homogeneous hypersurfaces and these examples are constructed explicitly.

In Section 2.2, it is shown that a real-analytic 3-manifold endowed with a real-analytic coframing satisfying the structure equations of Proposition 1 can be real-analytically embedded as a pseudoconvex hypersurface in a unimodular complex surface in such a way that the induced canonical coframing is the given one. Moreover, this embedding is essentially unique up to unimodular biholomorphism. This shows that the coframing on the hypersurface captures all of the ambient local unimodular biholomorphism invariants.

In Section 2.3 a natural evolution equation for strictly pseudoconvex real hypersurfaces in unimodular complex surfaces is defined, some of its properties are discussed, and some homogeneous examples are computed. The basic idea is this: If T is the Reeb vector field on M associated to the contact form θ, then $\mathrm{i}T$ is a canonical normal vector field along $M \subset X$. The *unimodular normal flow* flow is then defined so that $\mathrm{i}T$ is the velocity field of the flow.

This flow is somewhat subtle. The normal vector field $\mathrm{i}T$ depends on third order information at each point of the hypersurface M, and the flow preserves the contact structure on M (i.e., θ changes only by a multiple). However, if one is allowed to reparametrize M during the flow (in a manner that is not invariant under

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[1]Roughly speaking, this means that any pointwise differential invariant for such hypersurfaces can be expressed in terms of a and b and their derivatives with respect to the coframing (θ, η).

the unimodular biholomorphism pseudogroup), then one can replace the flow by a second order flow, one whose linearization is subelliptic and that thus, presumably, has short-time existence and uniqueness for smooth initial data.

Explicitly, when M is expressed in local unimodular coordinates (w, z) as a graph of the form $\mathrm{Im}(w) = F_0\big(z, \mathrm{Re}(w)\big)$, the evolution equation for the function $F = F(t, z, u)$ that is induced by the unimodular normal flow is

$$(1.4) \quad F_t = \Big[2\big[(1+F_u{}^2)F_{z\bar{z}} - \mathrm{i}(1-\mathrm{i}\,F_u)F_{\bar{z}}F_{uz} + \mathrm{i}(1+\mathrm{i}\,F_u)F_z F_{u\bar{z}} + F_z F_{\bar{z}} F_{uu}\big]\Big]^{1/3},$$

with initial condition $F(0, z, u) = F_0(z, u)$. (The quantity inside the radical on the right hand side of this equation is always nonzero for a pseudoconvex hypersurface represented as a graph in this way.)

As a first step in understanding this flow, the evolution equations for the invariants are computed and examples of the flow are computed for homogeneous hypersurfaces. In most cases, the flow exists only for a finite time (either in the future or the past) and the nature of the singularity that develops is discussed. An example shows that this flow does not preserve CR-flatness, in general.

Section 3 is provided for the reader's convenience. It is an account (closely following Cartan's original article) of the equivalence method development of the Cartan connection associated to a pseudoconvex real hypersurface in a complex surface. This is used both as a contrast to the unimodular case and as a reference for the notions from CR-geometry that are used in the rest of the article.

Section 4 outlines the changes needed to generalize these constructions to the case of a pseudoconvex hypersurface M^{2n+1} in a higher dimensional unimodular complex manifold of dimension $n+1$ and to the case in which the ambient complex manifold is endowed with a real volume form (rather than a holomorphic volume form). It turns out, in either case, that one can still define a canonical $\mathrm{SU}(n)$-structure on M and a flow.

Finally, it is a pleasure to thank Richard Hamilton, whose questions about a different flow for CR-hypersurfaces in Kähler manifolds that was studied by Huisken and Klingenberg [7, 10] led me to search for a flow invariant under the unimodular biholomorphism group.

Contents

2. Real hypersurfaces in unimodular surfaces

Let X be a complex 2-manifold and let Υ be a nowhere-vanishing holomorphic 2-form on X. The pair (X, Υ) will be said to be a *unimodular surface.*

A coordinate chart $z = (z^1, z^2) : U \to \mathbb{C}^2$ on an open set $U \subset X$ will be said to be *unimodular* if the pullback of Υ to U is equal to $\mathrm{d}z^1 {\wedge} \mathrm{d}z^2$.

A unimodular surface has an atlas of unimodular coordinate charts. In fact, if $z^1 : U \to \mathbb{C}$ is a holomorphic function on $U \subset X$ and $\mathrm{d}z^1$ is nonvanishing at $p \in U$, then there is a p-neighborhood $V \subset U$ on which there exists another holomorphic function $z^2 : V \to \mathbb{C}$ such that $z = (z^1, z^2) : V \to \mathbb{C}^2$ is a unimodular coordinate chart. Note that, if the fibers of $z^1 : V \to \mathbb{C}$ are connected, then the function z^2 is determined up to the addition of a holomorphic function of z^1.

2.1. Induced geometry on hypersurfaces.
Now let $M \subset X$ be a real hypersurface. For simplicity, M will be assumed to be smooth, although weaker differentiability hypotheses would suffice for most of the constructions.

Let Φ be the pullback of Υ to M. Since $\mathrm{SL}(2, \mathbb{C})$ acts transitively on the set of real hyperplanes in $\mathbb{C}^2$, the algebraic type of Φ does not vary on M.

In particular, any point $p \in M$ will have an open neighborhood $V \subset M$ on which there will exist 1-forms $\theta = \overline{\theta}$ and η such that $V^* \Phi = \theta {\wedge} \eta$ and where $\theta {\wedge} \eta {\wedge} \overline{\eta} \neq 0$. These forms are not unique, but any other pair (θ^*, η^*) on V with these properties will satisfy

$$(2.1) \qquad \begin{pmatrix} \theta^* \\ \eta^* \end{pmatrix} = \begin{pmatrix} \lambda & 0 \\ \mu & \lambda^{-1} \end{pmatrix} \begin{pmatrix} \theta \\ \eta \end{pmatrix}$$

for some functions $\lambda = \bar{\lambda} \neq 0$ and μ on V.

In particular, the equation $\theta = 0$ defines a global 2-plane field $D \subset TM$ that is independent of the local choice of θ.[2] Since $\theta^* {\wedge} \mathrm{d}\theta^* = \lambda^2\, \theta {\wedge} \mathrm{d}\theta$, whether or not $\theta {\wedge} \mathrm{d}\theta$ vanishes on V is independent of the choice of coframing (θ, η).

Definition 1. A real hypersurface $M \subset X$ is said to be *nondegenerate* if its 2-plane field $D \subset TM$ is contact, i.e., if $\theta {\wedge} \mathrm{d}\theta$ is nowhere-vanishing for any locally defined 1-forms $\theta = \overline{\theta}$ and η satisfying $\Phi = \theta {\wedge} \eta$.

Remark 1 (Pseudoconvexity). The condition that is being called 'nondegenerate' here is also known as the condition of 'strict pseudoconvexity' in the CR-literature. This coincidence is special to hypersurfaces in complex 2-manifolds. In higher dimensions, 'nondegenerate' is implied by 'strict pseudoconvexity', but not *vice versa.*

Proposition 1 (The canonical coframing). *Let $M \subset X$ be a nondegenerate real hypersurface. Then there exist unique* 1*-forms $\theta = \overline{\theta}$ and η on M such that*

(1) $\Phi = \theta {\wedge} \eta$, *and*

(2) $\mathrm{d}\theta = \mathrm{i}\,\eta {\wedge} \overline{\eta}$.

Proof. Let $V \subset M$ be an open set on which there exist 1-forms $\theta = \overline{\theta}$ and η satisfying $V^* \Phi = \theta {\wedge} \eta$. Then there exist functions $L = \overline{L}$ and P on V such that

$$(2.2) \qquad \mathrm{d}\theta = \mathrm{i}L\,\eta \wedge \overline{\eta} + \mathrm{i}\left(P\,\eta - \bar{P}\,\overline{\eta}\right) \wedge \theta.$$

Since M is nondegenerate, $\theta {\wedge} \mathrm{d}\theta$ is nonvanishing on V. Thus, L is nowhere-vanishing on V.

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[2]The 2-plane field D is known as the *complex tangent space* of M in the CR literature.

Now let $\theta^* = \lambda\,\theta$ and $\eta^* = \mu\,\theta + \lambda^{-1}\,\eta$ for some functions $\lambda = \bar\lambda \neq 0$ and μ on V. Then there exist functions $L^* = \overline{L^*} \neq 0$ and P^* on V such that

$$(2.3) \qquad \mathrm{d}\theta^* = \mathrm{i} L^*\,\eta^* \wedge \overline{\eta^*} + \mathrm{i}\left(P^*\,\eta^* - \overline{P^*}\,\overline{\eta^*}\right) \wedge \theta^*.$$

Since θ^* is a multiple of θ, computation yields

$$(2.4) \qquad \left. \begin{aligned} \mathrm{d}\theta^* &\equiv \lambda\,\mathrm{d}\theta \equiv \lambda\,\mathrm{i} L\,\eta \wedge \overline{\eta} \\ &\equiv \lambda\,\mathrm{i} L\,(\lambda\eta^*) \wedge \overline{(\lambda\eta^*)} \equiv \mathrm{i}\,(\lambda^3 L)\,\eta^* \wedge \overline{\eta^*} \end{aligned} \right\} \quad \bmod \theta \;(\equiv \quad \bmod \theta^*).$$

Thus, $L^* = \lambda^3 L$. In particular, there is a unique choice of λ, namely $\lambda = L^{-1/3}$, such that $L^* = 1$.

It follows that the equation

$$(2.5) \qquad \mathrm{d}\theta = \mathrm{i}\,\eta \wedge \overline{\eta} + \mathrm{i}\left(P\,\eta - \bar{P}\,\overline{\eta}\right) \wedge \theta$$

uniquely defines θ on V and therefore globally defines θ on M. Consequently, η is defined on M up to the addition of a multiple of θ.

However, replacing η by $\eta + \bar P\,\theta$ in the coframing shows that one can arrange

$$(2.6) \qquad \mathrm{d}\theta = \mathrm{i}\,\eta \wedge \overline{\eta}.$$

Moreover, once θ is fixed, (2.6) uniquely determines η globally on M. $\qquad\square$

Remark 2 (The torsion tensor). As the construction in the proof demonstrates, two coframings related as in (2.1), satisfy

$$(2.7) \qquad L^*\left(\mathrm{i}\,\theta^* \wedge \eta^* \wedge \overline{\eta^*}\right)^3 = L\left(\mathrm{i}\,\theta \wedge \eta \wedge \overline{\eta}\right)^3,$$

so the local expression

$$(2.8) \qquad L\left(\mathrm{i}\,\theta \wedge \eta \wedge \overline{\eta}\right)^3 = (\theta \wedge \mathrm{d}\theta) \circ \left(\mathrm{i}\,\theta \wedge \eta \wedge \overline{\eta}\right)^2$$

is the restriction to the domain of the coframing of a well-defined global section C of the line bundle $\left(\Lambda^3(T^*M)\right)^{\otimes 3}$. The nondegeneracy condition is just that this section C be nowhere-vanishing.

Remark 3 (Coordinate expressions). It is useful to look at the tensor C in local unimodular coordinates. Suppose that p is a point of M and let $(w, z) : U \to \mathbb{C}^2$ be a unimodular coordinate system centered on p such that the hypersurface $\mathrm{Im}(w) = 0$ is tangent to M at p. Writing $w = u + \mathrm{i}v$, the hypersurface M can be described near p in these coordinates by an equation of the form

$$(2.9) \qquad v = F(u, z, \bar z)$$

where F is a real-valued function that vanishes at $(0,0,0)$ and has its first derivatives vanishing there.

One can then write $\Phi = M^*(\Upsilon) = \theta_0 \wedge \eta_0$ where

$$(2.10) \quad \theta_0 = \bar\theta_0 = \mathrm{d}u - \frac{\mathrm{i}\,F_z}{1 - \mathrm{i}\,F_u}\,\mathrm{d}z + \frac{\mathrm{i}\,F_{\bar z}}{1 + \mathrm{i}\,F_u}\,\mathrm{d}\bar z \qquad \text{and} \qquad \eta_0 = (1 + \mathrm{i}\,F_u)\,\mathrm{d}z.$$

Thus, $\mathrm{i}\,\theta_0 \wedge \eta_0 \wedge \overline{\eta_0} = \mathrm{i}\,(1 + F_u{}^2)\,\mathrm{d}u \wedge \mathrm{d}z \wedge \mathrm{d}\bar z$. Computation yields

$$(2.11) \qquad \mathsf{C} = L_0\left(\mathrm{i}\,\theta_0 \wedge \eta_0 \wedge \overline{\eta_0}\right)^3 = \left(1 + F_u{}^2\right)^3 L_0\left(\mathrm{i}\,\mathrm{d}u \wedge \mathrm{d}z \wedge \mathrm{d}\bar z\right)^3,$$

where

$$(2.12) \qquad L_0 = 2\,\frac{(1 + F_u{}^2)F_{z\bar z} + F_z F_{\bar z} F_{uu} - \mathrm{i}(1 - \mathrm{i}\,F_u)F_{\bar z}F_{uz} + \mathrm{i}(1 + \mathrm{i}\,F_u)F_z F_{u\bar z}}{\left(1 + F_u{}^2\right)^3}.$$

Thus, nondegeneracy at p is equivalent to $F_{z\bar{z}}(0,0,0) \neq 0$ (since the first derivatives of F vanish at $(0,0,0)$).

This computation shows that the normalized $\theta = (L_0)^{-1/3}\theta_0$ of Proposition 1 depends on two derivatives of the defining function F. Moreover, the normalized η depends on three derivatives of F.

2.1.1. *Invariants in the nondegenerate case.* Let $M \subset X$ be a nondegenerate hypersurface and let (θ, η) be the canonical coframing described in Proposition 1.

Primary invariants. Since $\mathrm{d}(\theta \wedge \eta) = \theta \wedge \mathrm{d}\eta = 0$, it follows that there exist functions a and b on M such that

(2.13) $$\mathrm{d}\eta = 2\mathrm{i}\,\theta \wedge (\,a\,\eta + b\,\overline{\eta}\,).$$

(The introduction of the coefficient 2i in this equation simplifies later formulae.) Since

(2.14) $$0 = \mathrm{d}(\mathrm{d}\theta) = \mathrm{d}(\mathrm{i}\,\eta \wedge \overline{\eta}) = 2(\bar{a} - a)\,\theta \wedge \eta \wedge \overline{\eta},$$

it follows that $a = \bar{a}$ is real-valued.

The functions a and b will be referred to as the *primary invariants* of M. Since the coframing (θ, η) depends on three derivatives of a defining function F as in (2.9), it follows that a and b are fourth-order expressions in F.

Remark 4 (Scaling effects). For any constant $\lambda \in \mathbb{C}^*$, the pair $(X, \lambda\Upsilon)$ defines a unimodular surface and this constant scaling will affect the canonical coframing.

Explicitly, replacing Υ by $\lambda\,\Upsilon$ replaces the canonical coframing (θ, η) on a nondegenerate hypersurface $M \subset X$ by the coframing

(2.15) $$\big(\,|\lambda|^{2/3}\theta,\; \lambda|\lambda|^{-2/3}\eta\,\big).$$

Hence, under this change, the primary invariants (a, b) are replaced by

(2.16) $$\big(\,|\lambda|^{-2/3}\,a,\; \lambda^{-2}|\lambda|^{4/3}\,b\,\big).$$

Finally, if one reverses the complex structure on X, effectively by replacing Υ by $\overline{\Upsilon}$, one finds that the canonical coframing (θ, η) is then replaced by $(-\theta, -\overline{\eta})$, so that the primary invariants (a, b) are replaced by $(a, \bar{b})$.

Secondary invariants. Since θ, η, and $\overline{\eta}$ are linearly independent and so form a basis for the 1-forms on M, there exist unique functions $a_\theta = \overline{a_\theta}$, a_η, and $a_{\overline{\eta}} = \overline{a_\eta}$ satisfying

(2.17) $$\mathrm{d}a = a_\theta\,\theta + a_\eta\,\eta + a_{\overline{\eta}}\,\overline{\eta}.$$

Similarly, there exist unique functions b_θ, b_η, and $b_{\overline{\eta}}$ satisfying

(2.18) $$\mathrm{d}b = b_\theta\,\theta + b_\eta\,\eta + b_{\overline{\eta}}\,\overline{\eta}.$$

Taking the exterior derivative of (2.13) yields

(2.19) $$b_\eta = a_{\overline{\eta}}\;\;(\,= \overline{a_\eta}\,).$$

The quantities a_θ, a_η, b_θ, and $b_{\overline{\eta}}$ are the *secondary invariants* of M.

Derivative commutants. In general, for a function f on M, set

(2.20) $$\mathrm{d}f = f_\theta\,\theta + f_\eta\,\eta + f_{\bar{\eta}}\,\overline{\eta}.$$

Of course, these differentiation operations are not independent. For example, taking the exterior derivative of (2.20) yields the relations

$$f_{\eta\bar\eta} - f_{\bar\eta\eta} = \mathrm{i} f_\theta\,,$$

(2.21)
$$f_{\theta\eta} - f_{\eta\theta} = 2\mathrm{i} a\, f_\eta - 2\mathrm{i}\bar b\, f_{\bar\eta}\,,$$

$$f_{\theta\bar\eta} - f_{\bar\eta\theta} = -2\mathrm{i} a\, f_{\bar\eta} + 2\mathrm{i} b\, f_\eta\,.$$

A subelliptic Laplacian. The operator $L : C^\infty(M) \to C^\infty(M)$ defined by

(2.22)
$$Lf = f_{\eta\bar\eta} + f_{\bar\eta\eta}$$

is a subelliptic real operator that plays the role of the Laplacian in this geometry.

2.1.2. *Some interpretations of the invariants.* Let (θ, η) be the canonical coframing of a nondegenerate hypersurface $M \subset X$ where (X, Υ) is a unimodular complex surface. The invariants found thus far have natural interpretations and lend themselves to various constructions.

For example, there is a natural volume form on M, namely $\theta \wedge \mathrm{d}\theta = \mathrm{i}\theta \wedge \eta \wedge \overline\eta$.

There is also a real vector field T on M defined by the conditions $\eta(T) = 0$ and $\theta(T) = 1$. The vector field T is the *Reeb vector field* of the contact form θ. The vector field $\mathrm{i}T$ is a canonical normal vector field along M. It points towards the 'pseudoconcave' side of the hypersurface, i.e., the side that contains the holomorphic discs whose boundaries lie in M.

This intrinsic normal can be used to describe variations of nondegenerate hypersurfaces: If $\iota : (-\epsilon, \epsilon) \times M \to X$ is a compactly supported 1-parameter family of immersions of M into X as nondegenerate hypersurfaces, then one can reparametrize this family so that

(2.23)
$$\iota_* \left(\frac{\partial}{\partial t}(t, p) \right) = f(t, p)\, J\big(T(t, p)\big),$$

for $(t, p) \in (-\epsilon, \epsilon) \times M$, i.e., so that the variation is 'normal'. The function $f : (-\epsilon, \epsilon) \times M \to \mathbb{R}$ then can be regarded as the 'normal displacement function'.

In this case, one has

(2.24)
$$\iota^*(\Upsilon) = (\theta + \mathrm{i} f\, \mathrm{d}t) \wedge \eta$$

where (θ, η) is the (t-dependent) induced canonical coframing on M. Computation yields

(2.25)
$$\mathrm{d}\theta = \mathrm{i}\,\eta \wedge \overline\eta + \left(\tfrac{1}{3}(4af + f_{\eta\bar\eta} + f_{\bar\eta\eta})\,\theta + \mathrm{i}(\,f_\eta\,\eta - f_{\bar\eta}\,\overline\eta)\right) \wedge \mathrm{d}t.$$

By this and integration by parts, if the variation is supported in a compact domain $K \subset M$, the volume function

(2.26)
$$V_K(t) = \int_{\{t\} \times K} \mathrm{i}\,\theta \wedge \eta \wedge \overline\eta$$

satisfies

(2.27)
$$V_K'(t) = -\frac{8}{3} \int_{\{t\} \times K} fa\, \mathrm{i}\,\theta \wedge \eta \wedge \overline\eta.$$

Since, for a given nondegenerate immersion $\iota_0 : M \to X$, one can construct a family $\iota : \mathbb{R} \times M \to X$ with any given compactly supported function $f_0 = f(0, \cdot) : M \to \mathbb{R}$, one sees that $\iota_0 : M \to X$ is volume critical for all compactly supported variations if and only if it satisfies $a_0 \equiv 0$. Thus, the invariant a can be regarded as an analog of 'mean curvature' of the nondegenerate immersion. (Keep in mind

that a is a fourth order invariant of the immersion. This is not surprising, though, since the volume functional is second order.)

One interpretation of the invariant b has to do with the geometry of the surface Σ that is the quotient of M by the flow of the Reeb vector field T. Since $\mathrm{i}\,\eta{\wedge}\overline\eta = \mathrm{d}\theta$ is closed and semi-basic for the projection $M \to \Sigma$, it follows that it is the pullback to M of a well-defined, nonvanishing 2-form on Σ. However, the quadratic form $\eta{\circ}\overline\eta$ satisfies

$$(2.28) \qquad\qquad \mathsf{L}_T\ \eta{\circ}\overline\eta = 2\mathrm{i}\left(b\,\overline\eta^2 - \overline b\,\eta^2\right).$$

Thus, the quadratic form $\eta{\circ}\overline\eta$ is not the pullback to M of a metric on Σ unless $b \equiv 0$.

However, the equation $b \equiv 0$ implies, via (2.19), that $a_{\overline\eta} = a_\eta = 0$. Then the exterior derivative of the equation $\mathrm{d}a = a_\theta\,\theta$ implies that $a_\theta = 0$. Thus, if b vanishes identically, then a is constant. As will be seen in §2.1.3, this implies that $M \subset X$ is locally homogeneous.

Invariants in normal coordinates. Another interpretation of the invariants a and b (and the canonical coframing) can be seen by considering a defining function for the hypersurface in so-called *normal coordinates*.

According to [2, §4.2], a holomorphic coordinate system (w, z) centered on $p \in M \subset X$ is said to be *normal* if there is an open p-neighborhood U in X such that $M \cap U$ is defined by an equation of the form

$$(2.29) \qquad\qquad \operatorname{Im} w = z\bar z\,\psi\big(z,\,\bar z,\,\operatorname{Re} w\big).$$

for some smooth function ψ of its arguments.

Acording to [2, Theorem 4.2.6], when M is real-analytic, normal coordinates centered on p exist and the function ψ can be regarded as an analytic function of its arguments. If M is merely smooth, then, for any integer k, there exist p-centered coordinates such that (2.29) holds up to order k.

There are two limitations of normal coordinates for our purposes: First, the proofs given in [2] do not show that one can choose the coordinates to be unimodular and, second, normal coordinates are not unique.

It is not difficult to show, however, that, for any point $p \in M$, there exist p-centered unimodular holomorphic coordinates (w, z) with $w = u + \mathrm{i}\,v$ such that M is defined in these coordinates by an equation of the form (2.9) where

$$(2.30) \qquad F = \tfrac12\,z\bar z\left(1 + b(p)\,z^2 + \tfrac32 a(p)\,z\bar z + \overline{b(p)}\,\bar z^2\right) + \mathrm{O}(5)$$

and where the unwritten terms vanishing to order 5 and higher may contain u as well as z and $\bar z$.[3]

CR-invariants. Of course, the usual (i.e., nonunimodular) CR-invariants of M can be expressed in terms of unimodular CR-invariants.[4]

Proposition 2 (Criterion for CR-flatness)**.** *The* CR-*structure on M is* CR-*flat if and only if*

$$(2.31) \qquad\qquad a_{\overline\eta\overline\eta} + 2\mathrm{i}\,b_\theta + 6ab = 0.$$

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[3]In the real-analytic case, it is possible to canonically define unique p-centered unimodular coordinates by requiring that they satisfy certain natural geometric conditions. However, these coordinates do not seem to be particularly useful and their description is less than satisfactory, so their definition has not been included here.

[4]For the reader's convenience, the definition of the Cartan connection of a real hypersurface in a complex surface and its invariants are reviewed in Section 3.

Proof. By the analysis conducted in Section 3, one knows that there exist 1-forms α, β, $\sigma = \bar{\sigma}$ and a function s on M such that the following equations hold

$$
\begin{aligned}
\mathrm{d}\theta &= -(\alpha + \bar{\alpha}) \wedge \theta + \mathrm{i}\,\eta \wedge \bar{\eta}\,, \\
\mathrm{d}\eta &= -\beta \wedge \theta - \alpha \wedge \eta\,, \\
\mathrm{d}\alpha &= -\sigma \wedge \theta - \mathrm{i}\,\beta \wedge \bar{\eta} - 2\mathrm{i}\,\bar{\beta} \wedge \eta\,, \\
\mathrm{d}\beta &= -\sigma \wedge \eta + \bar{\alpha} \wedge \beta - s\,\bar{\eta} \wedge \theta\,.
\end{aligned}
$$
(2.32)

Moreover, the underlying CR-structure is CR-flat if and only if the function s vanishes identically.

Thus, to verify the proposition, it suffices to determine a triple of forms α, β, and $\sigma = \bar{\sigma}$ and a function s satisfying (2.32) in terms of the primary invariants and their derivatives.

Now, (2.32) does not uniquely determine the unknown forms α, β and $\sigma = \bar{\sigma}$, but, following the calculations in Section 3, any solution is seen to be of the form

$$
\begin{aligned}
\alpha &= (u - \tfrac{3}{2}\mathrm{i}\,a)\,\theta\,, \\
\beta &= (u + \tfrac{1}{2}\mathrm{i}a)\,\eta + 2\mathrm{i}b\,\bar{\eta} - a_{\bar{\eta}}\,\theta\,, \\
\sigma &= -\mathrm{d}u - \tfrac{1}{2}\mathrm{i}\big(a_\eta\,\eta - a_{\bar{\eta}}\,\bar{\eta}\big) + (u^2 + \tfrac{1}{4}\,a^2 - 4b\bar{b} - \tfrac{1}{2}a_{\eta\bar{\eta}} - \tfrac{1}{2}a_{\bar{\eta}\eta})\,\theta\,, \\
s &= a_{\bar{\eta}\bar{\eta}} + 2\mathrm{i}\,b_\theta + 6ab\,.
\end{aligned}
$$
(2.33)

for some real-valued function u on M. (In particular, taking $u = 0$ gives the only solution that abides $\alpha = -\bar{\alpha}$.) Consequently, the underlying CR-structure is CR-flat if and only if $a_{\bar{\eta}\bar{\eta}} + 2\mathrm{i}\,b_\theta + 6ab = 0$, as claimed. $\qquad\square$

2.1.3. *Locally homogeneous examples.* A real hypersurface $M \subset X$ is said to be locally homogeneous, if, for any two points $p, q \in M$, there exists an open p-neighborhood $U \subset X$ and a unimodular biholomorphism $\phi : U \to \phi(U) \subset X$ such that $\phi(p) = q$ and $\phi(M \cap U) = M \cap \phi(U)$.

If M is locally homogeneous, then the canonical coframing (θ, η) must also be locally homogeneous. Consequently, the primary invariants a and b must be constant functions on M.

Conversely, for any constants $a = \bar{a}$ and b, there exists a connected, simply connected 3-dimensional Lie group $G_{a,b}$ on which there exist left-invariant 1-forms $\theta = \bar{\theta}$ and η satisfying $\mathrm{i}\theta \wedge \eta \wedge \bar{\eta} \neq 0$ and

$$
\mathrm{d}\theta = \mathrm{i}\eta \wedge \bar{\eta} \qquad \text{and} \qquad \mathrm{d}\eta = 2\mathrm{i}\,\theta \wedge \big(a\,\eta + b\,\bar{\eta}\big).
$$
(2.34)

The group $G_{a,b}$ and the coframing (θ, η) are unique up to isomorphism.

Thus, there is a 3-parameter family of homogeneous abstract unimodular CR-structures. Note that, by Proposition 2, the underlying CR-structure is CR-flat if and only if either $a = 0$ or $b = 0$.

When $a > |b|$, the group $G_{a,b}$ is isomorphic to SU(2). When $a = \pm|b|$ but $(a, b) \neq (0,0)$, the group $G_{a,b}$ is isomorphic to a semi-direct product of $\mathbb{R}$ with $\mathbb{R}^2$. The group $G_{0,0}$ is isomorphic to the Heisenberg group. In all other cases, the group $G_{a,b}$ is isomorphic to the simply-connected cover of SL(2, $\mathbb{R}$).

Example 1 (The case $(a, b) = (0, 0)$). Let (w, z) be standard coordinates on $\mathbb{C}^2$, let $\Upsilon = \mathrm{d}w \wedge \mathrm{d}z$, and let $w = u + \mathrm{i}v$. Then, for the hypersurface $M \subset \mathbb{C}^2$ defined by $v = \mathrm{Im}\,w = \frac{1}{2}\,z\bar{z}$, one computes

$$
\theta = \mathrm{d}u + \tfrac{\mathrm{i}}{2}\,(z\,\mathrm{d}\bar{z} - \bar{z}\,\mathrm{d}z) \qquad \text{and} \qquad \eta = \mathrm{d}z,
$$
(2.35)

so that the structure equations are

$$(2.36) \qquad d\theta = i\,\eta \wedge \overline{\eta} \qquad \text{and} \qquad d\eta = 0.$$

Thus, the primary invariants a and b both vanish.

This example is seen to be homogeneous by inspection. The 3-parameter group of unimodular biholomorphic transformations defined by

$$(2.37) \qquad f(w,z) = \left(w + i\,\overline{z_0}\,z + u_0 + \tfrac{i}{2}\,z_0\overline{z_0}\,,\ z + z_0\right)$$

for arbitrary $z_0 \in \mathbb{C}$ and $u_0 \in \mathbb{R}$ preserves M and acts transitively on it.

Example 2 (Homogeneous examples with $a > |b|$). To construct these homogeneous examples, whose symmetry groups are isomorphic to either $SU(2)$ or $SO(3)$, explicitly as hypersurfaces in unimodular complex surfaces, one can exploit the geometry of $SO(3)$-actions.

Let $SO(3)$ act on $\mathbb{C}^3$ as the complexification of its standard action on $\mathbb{R}^3$. This action leaves invariant the holomorphic quadratic function $Q(z) = z \cdot z$ and the positive definite Hermitian form $H(z) = z \cdot \bar{z} = |z|^2$. These satisfy the obvious inequalities

$$(2.38) \qquad |z|^2 \geq |z \cdot z| \geq 0.$$

The $SO(3)$-orbit of a nonzero z_0 has dimension 3 if and only if $|z_0|^2 > |z_0 \cdot z_0|$ since $|z_0|^2 = |z_0 \cdot z_0| > 0$ if and only if $z_0 = \lambda x_0$ for some $\lambda \in \mathbb{C}^*$ and some nonzero $x_0 \in \mathbb{R}^3$. In fact, $SO(3)$ acts freely on the open set where $|z|^2 > |z \cdot z|$.

Thus, for constants $h \in \mathbb{R}^+$ and $q \in \mathbb{C}$ with $h > |q|$, the submanifold $M_{h,q}$ defined by

$$(2.39) \qquad M_{h,q} = \left\{ z \in \mathbb{C}^3 \mid z \cdot z = q,\ |z|^2 = h \right\}$$

is an $SO(3)$-invariant, 3-dimensional hypersurface in the $SO(3)$-invariant quadric surface $X_q \subset \mathbb{C}^3$ defined by $z \cdot z = q$. It is not difficult to see that $M_{h,q}$ is CR-nondegenerate and is an $SO(3)$-orbit.

On the other hand, the locus $M_{|q|,q} \subset X_q$ defined by setting $h = |q|$ in (2.39) is either a point (in the case $q = 0$) or a smoothly embedded, totally real 2-sphere (in the case $q \neq 0$). These singular orbits will become significant when the behavior of the homogeneous hypersurfaces under the unimodular normal flow (yet to be defined) is studied.

Now, consider the $SO(3)$-invariant 2-form

$$(2.40) \qquad \Upsilon = \frac{z^1\,dz^2 \wedge dz^3 + z^2\,dz^3 \wedge dz^1 + z^3\,dz^1 \wedge dz^2}{(z^1)^2 + (z^2)^2 + (z^3)^2},$$

which is well-defined away from the singular quadric $z \cdot z = 0$. It is not difficult to see that Υ pulls back to each quadric surface X_q with $q \neq 0$ to be a nowhere-vanishing 2-form Υ_q, one that is invariant under the action of $SO(3)$.

Since $M_{h,q}$ is homogeneous under the unimodular action of $SO(3)$ on (X_q, Υ_q), its primary invariants are constant. Direct computation yields

$$(2.41) \qquad a = \frac{h}{2\left(h^2 - |q|^2\right)^{2/3}} \qquad \text{and} \qquad b = \frac{-q}{2\left(h^2 - |q|^2\right)^{2/3}}.$$

These formulae can be solved for h and q in the form

$$(2.42) \qquad h = \frac{a}{8\left(a^2 - |b|^2\right)^2} \qquad \text{and} \qquad q = \frac{-b}{8\left(a^2 - |b|^2\right)^2}.$$

Consequently, any $(a, b) \in \mathbb{R} \times \mathbb{C}$ with $a > |b| > 0$ is represented by a unique $M_{h,q}$ with $h > |q| > 0$.

The reader may wonder what happens in the case $q = 0$, which is, apparently, not covered by the above formulae. It is an interesting fact that, even though Υ is not well-defined on the locus $z \cdot z = 0$, there is a well-defined Υ_0 on X_0 that is, in a natural sense, the limit of Υ_q on X_q as $q \to 0$ and that is a nonvanishing 2-form on the smooth part of X_0, i.e., away from $z = 0$. One sees this by noticing that, on the open set where $z^1 \neq 0$, the 2-form

$$(2.43) \qquad \Psi_1 = \frac{\mathrm{d}z^2 {\wedge} \mathrm{d}z^3}{z^1}$$

pulls back to each level set X_q with $q \neq 0$ to agree with Υ_q. Moreover, Ψ_1 pulls back to be well-defined and nonvanishing in the open set in X_0 on which $z^1 \neq 0$ and thus can naturally be regarded as the limit Υ_0 of Υ_q as $q \to 0$ on this open set. By cyclically permuting the indices, one sees that Υ_0 can be defined in this manner on all of X_0 away from the singular point $z = 0$. Taking this Υ_0 as the volume form on X_0, the formulae (2.41) are seen to be valid even for $q = 0$.

One more comment about this construction: For $q \neq 0$, the surface X_q is smooth, connected, and simply-connected. When $h > |q|$, the hypersurface $M_{h,q} \subset X_q$ is compact and bounds the compact, simply-connected domain defined by $|z|^2 \leq h$, even though $M_{h,q}$, being diffeomorphic to $SO(3)$, is not, itself, simply-connected. As will be seen below, it is very unlikely that the simply-connected cover of $M_{h,q}$ can be embedded as a hypersurface in a unimodular surface in such a way that it bounds a compact domain.

On the other hand, the surface X_0 is singular at $z = 0$ and this singularity is what is known as a 'double point' in algebraic geometry. In fact, one can realize (X_0, Υ_0) as the quotient of $(\mathbb{C}^2, \mathrm{d}w {\wedge} \mathrm{d}z)$ by the unimodular involution $(w, z) \to (-w, -z)$. Under this identification, the $SO(3)$-action on X_0 lifts to the standard $SU(2)$-action on $\mathbb{C}^2$ (which is, of course, unimodular). The level sets $M_{h,0}$ are the quotients of the hyperspheres centered on $(0, 0) \in \mathbb{C}^2$. Thus, the simply-connected models for the case of constant invariants $a > 0$ and $b = 0$ can be embedded the standard hyperspheres in $\mathbb{C}^2$. For more detail on this case, see Example 5.

Example 3 (Homogeneous examples with $a < 0$ and $b = 0$). Let $\mathbb{D}$ be the unit disk in $\mathbb{C}$ endowed with the Poincaré metric of constant curvature -1, whose oriented isometry group is $\mathrm{PSL}(2, \mathbb{R})$. Let $X = T^*\mathbb{D} \to \mathbb{D}$ be the canonical bundle endowed with the standard (nowhere-vanishing) holomorphic 2-form Υ that it inherits as a (holomorphic) cotangent bundle. Let $|\cdot| : X \to \mathbb{R}$ denote the hermitian norm on X that it inherits from the Poincaré metric.

For $r > 0$, let $M_r = \{\zeta \in X \mid |\zeta|^2 = \frac{1}{2} r^2\}$. Then M_r is a hypersurface in X that is easily seen to be nondegenerate and one computes without difficulty that the invariants satisfy $a = -\frac{1}{2} r^{-2/3}$ and $b = 0$.

Of course, M_r is not simply-connected but is diffeomorphic to $S^1 \times \mathbb{D}$. One can get the simply-connected example by considering the inverse image of M_r under the universal covering $\tilde{X}^* \to X^*$ where $X^* \subset X$ is the complement of the zero section of $X \to \mathbb{D}$.

In the other direction, for any group $\Gamma \subset \mathrm{PSL}(2, \mathbb{R})$ acting freely on $\mathbb{D}$ with compact quotient $\Gamma \backslash \mathbb{D}$, the action extends unimodularly and freely to (X, Υ). The quotient $\Gamma \backslash M_r$ is then a compact example of a nondegenerate hypersurface in $(\Gamma \backslash X, \Upsilon)$ that has constant invariants and that bounds a compact domain.

For more discussion about this example, see Example 7.

Example 4 (Homogeneous examples with $a < -|b|$ or $|a| < |b|$). Examples of these two types have symmetry group isomorphic to some covering of $\mathrm{PSL}(2,\mathbb{R}) = \mathrm{SL}(2,\mathbb{R})/\{\pm \mathrm{I}_2\}$. To construct them up to a covering, one can exploit the geometry of the irreducible 3-dimensional representation of $\mathrm{PSL}(2,\mathbb{R})$.

Let $\mathrm{SL}(2,\mathbb{R})$ act on the space $S_2(\mathbb{R})$ of 2-by-2 symmetric matrices with real entries in the usual way: $A \cdot s = A s \, {}^t\!A$ for $A \in \mathrm{SL}(2,\mathbb{R})$ and $s \in S_2(\mathbb{R})$. This action is almost faithful, with $\{\pm \mathrm{I}_2\} \simeq \mathbb{Z}_2$ acting trivially, so that this is actually a representation of $\mathrm{PSL}(2,\mathbb{R})$. This action preserves the quadratic form $s \mapsto \det(s)$, which has maximal negative definite subspaces of dimension 2. Let $r \cdot s$ denote the inner product on $S_2(\mathbb{R})$ associated to this quadratic form, i.e., $s \cdot s = \det(s)$. Because $\mathrm{PSL}(2,\mathbb{R})$ is simply connected, it also preserves an orientation of $S_2(\mathbb{R})$ as well as the 'positive time-like cone' $S_2^+(\mathbb{R})$ that consists of the positive definite matrices.

Let $\mathrm{PSL}(2,\mathbb{R})$ act on $S_2(\mathbb{C}) = \mathbb{C} \otimes S_2(\mathbb{R})$ by extension of scalars and extend the inner product on $S_2(\mathbb{R})$ complex linearly to $S_2(\mathbb{C})$. Then $\mathrm{PSL}(2,\mathbb{R})$ preserves both the complex quadratic form $Q(z) = z \cdot z$ and the Hermitian form $H(z) = z \cdot \bar{z}$. For $q \in \mathbb{C}$, define

$$(2.44) \qquad X_q = Q^{-1}(q) = \{\, z \in S_2(\mathbb{C}) \mid z \cdot z = q \,\},$$

so that X_q is an $\mathrm{SL}(2,\mathbb{R})$-invariant quadric surface, nonsingular if $q \neq 0$ while X_0 is a (singular) cone.

Just as in the $\mathrm{SO}(3)$-invariant case that has already been treated, one can define a canonical, $\mathrm{PSL}(2,\mathbb{R})$-invariant holomorphic 2-form Υ_q on X_q so that, when $q \neq 0$, the inclusion mapping $z : X_q \to S_2(\mathbb{C}) \simeq \mathbb{C}^3$ satisfies

$$(2.45) \qquad \tfrac{1}{2}\, z \wedge \mathrm{d}z \wedge \mathrm{d}z = q \Upsilon_q \, \mathbf{v}$$

where $\mathbf{v} \in \Lambda^3\big(S_2(\mathbb{R})\big)$ is the volume 3-vector associated to an orthnormal, oriented basis of $S_2(\mathbb{R}) \simeq \mathbb{R}^3$ and so that, when $q = 0$, the 2-form Υ_0, well-defined on the smooth part of X_0, is the limit of Υ_q as q approaches 0 in a suitable sense. (The details, which are entirely analogous to those in the $\mathrm{SO}(3)$ case, will be left to the reader). In this way, (X_q, Υ_q) becomes a $\mathrm{PSL}(2,\mathbb{R})$-invariant[5] unimodular complex surface.

Now, the $\mathrm{PSL}(2,\mathbb{R})$ action must preserve the level sets of H: For $h \in \mathbb{R}$, set

$$(2.46) \qquad M_{h,q} = Q^{-1}(q) \cap H^{-1}(h) = \{\, z \in X_q \mid z \cdot \bar{z} = h \,\}.$$

The triangle inequality for the indefinite quadratic form $\det$ on $S_2(\mathbb{R})$ implies that $H(z) \leq |Q(z)|$. In other words, $M_{h,q}$ is empty unless $h \leq |q|$. It is sometimes useful to note the identity

$$(2.47) \qquad \lambda \cdot M_{h,q} = M_{|\lambda|^2 h, \, \lambda^2 q},$$

valid for $\lambda \in \mathbb{C}^*$, and the fact that scalar multiplication by λ induces a mapping from X_q to $X_{\lambda^2 q}$ that pulls back $\Upsilon_{\lambda^2 q}$ to $\lambda \Upsilon_q$.

Computation shows that each $M_{h,q}$ with $h < |q|$ and $h \neq -|q|$ is a (nonempty) smooth, nondegenerate real hypersurface in (X_q, Υ_q) that has constant invariants

$$(2.48) \qquad a = \frac{h}{2(h^2 - |q|^2)^{2/3}} \qquad \text{and} \qquad b = \frac{-q}{2(h^2 - |q|^2)^{2/3}}\,.$$

---

[5]In fact, $\mathrm{PSL}(2,\mathbb{C})$-invariant.

Referring to the inversion formulae (2.42), one sees that, when $a < -|b|$, equations (2.48) can be solved for h and q uniquely with $h < -|q|$. On the other hand, when $|a| < |b|$, the equations (2.48) can be solved for h and q uniquely with $|h| < |q|$. Thus, these provide homogeneous (though not simply connected) models of the desired kinds.

Now, when $h^2 \neq |q|^2$, the hypersurface $M_{h,q}$ is not connected.

When $h < -|q|$, any element of $M_{h,q}$ is of the form $x - iy$ where x and y in $S_2(\mathbb{R})$ are linearly independent and span a negative definite 2-dimensional subspace of $S_2(\mathbb{R})$. Say that $x - iy$ lies in $M_{h,q}^+$ if the basis (I_2, x, y) is a positively oriented basis of $S_2(\mathbb{R})$ and that $x - iy$ lies in $M_{h,q}^-$ if the basis $(-I_2, x, y)$ is a positively oriented basis of $S_2(\mathbb{R})$. Then $M_{h,q}$ is the disjoint union of the connected sets $M_{h,q}^\pm$. In fact, $\mathrm{PSL}(2, \mathbb{R})$ acts simply transitively on each of $M_{h,q}^+$ and $M_{h,q}^-$, as is not difficult to see.

On the other hand, when $|h| < |q|$, the situation is a bit more subtle: The hypersurface $M_{h,q}$ has two components, but there is no natural way to label one as positive and the other as negative. This is most easily seen as follows: Consider the case $q = 1$. Each element of $M_{h,1}$ is of the form $x - iy$ where $x, y \in S_2(\mathbb{R})$ satisfy

$$(2.49) \qquad x \cdot x = \tfrac{1}{2}(h+1) > 0, \quad y \cdot y = \tfrac{1}{2}(h-1) < 0, \quad \text{and} \quad x \cdot y = 0.$$

Now, one can define $M_{h,1}^+$ to be those elements $x - iy \in M_{h,1}$ such that x lies in $S_2^+(\mathbb{R})$ and $M_{h,1}^-$ to be those elements $x - iy \in M_{h,1}$ such that $-x$ lies in $S_2^+(\mathbb{R})$. Then $M_{h,1}$ is the disjoint union of the two connected components $M_{h,1}^+$ and $M_{h,q}^-$. Moreover, $\mathrm{PSL}(2, \mathbb{R})$ acts simply transitively on each of $M_{h,1}^+$ and $M_{h,1}^-$. Now, using the scaling property (2.47), one might be tempted to try to 'define' $M_{h,q}^\pm$ as the image under scalar multiplication by $\sqrt{q} \in \mathbb{C}^*$ of $M_{h/|q|,1}^\pm$. However, the choice of the square root affects which component is mapped to which component, and, of course, there is no continuous way to choose this square root.

For use in the study of the effect of unimodular normal flow on these examples, it is also worthwhile to look at the sets $M_{h,q}$ when $h = \pm|q|$.

The case $M_{0,0}$ is special. It is not difficult to see that

$$(2.50) \qquad M_{0,0} = \{ \, \lambda x \mid \lambda \in \mathbb{C}, \ x \in S_2(\mathbb{R}) \text{ and } \det(x) = 0 \, \}.$$

Thus, $M_{0,0}$ is a union of a real 1-parameter family of complex lines. In particular, $M_{0,0}$ is a totally degenerate real hypersurface in X_0. It is not homogeneous under $\mathrm{PSL}(2, \mathbb{R})$, as the orbits are 2-dimensional, except for the origin itself, which is 0-dimensional.

The subset $M_{1,1}$ is described as

$$(2.51) \qquad M_{1,1} = \{ \, x \in S_2(\mathbb{R}) \mid \det(x) = 1 \, \},$$

and hence is a 2-dimensional surface, the (real) unit hyperboloid of 2-sheets, sometimes denoted by H_1. Its two sheets are $M_{1,1}^+ = H_1^+$, consisting of the positive definite $s \in S_2(\mathbb{R})$ with determinant 1, and $M_{1,1}^- = -M_{1,1}^+ = -H_1^+ = H_1^-$.

The subset $M_{-1,1}$ is somewhat more complicated: It is 3-dimensional and singular. It can be described in equations in the form

$$(2.52) \qquad M_{-1,1} = \{ \, x - \mathrm{i}\,y \mid x, y \in S_2(\mathbb{R}), \ x \cdot x = x \cdot y = 0, \ y \cdot y = -1 \, \}.$$

Thus, $M_{-1,1}$ is the union of two smooth, closed, connected 3-dimensional hypersurfaces in X_1, say $M_{-1,1}^\pm$, each of which is seen to be an oriented line bundle over

the hyperboloid of one sheet

$$(2.53) \qquad H_{-1} = \{ \, y \in S_2(\mathbb{R}) \mid \det(y) = -1 \, \}$$

via the mapping $x - \mathrm{i} \, y \mapsto y$. (Each fiber of the map $M_{-1,1} \to H_{-1}$ is a union of two distinct null lines in $S_2(\mathbb{R})$ and these lines are oriented by taking the positive ray to lie in the closure of the positive cone $S_2^+(\mathbb{R})$.)

The (CR-degenerate) hypersurfaces $M_{-1,1}^+$ and $M_{-1,1}^-$ intersect transversely along the singular locus of $M_{-1,1}$,

$$(2.54) \qquad \mathrm{sing}\big(M_{-1,1}\big) = \{ \, \mathrm{i} \, y \in S_2(\mathbb{R}) \mid \det(y) = -1 \, \} = \mathrm{i} \, H_{-1}.$$

The smooth part $M_{-1,1}^*$ of $M_{-1,1}$ is the disjoint union of 4 components, which can be labeled as $M_{-1,1}^{\pm;\pm}$. Each of these components is acted upon simply transitively by $\mathrm{PSL}(2, \mathbb{R})$.

2.1.4. *Cohomogeneity one examples.* The homogeneous examples display many interesting phenomena, but they are necessarily of somewhat limited use in understanding the general case. A somewhat more interesting case is that of examples that have cohomogeneity one, i.e., the group of symmetries acts with principal orbits of dimension 2. It turns out that, in this case, the structure equations can be analyzed fully and these examples characterized.

If the pseudogroup of local symmetries of (M, Φ) has 2-dimensional orbits on M, then, using (2.13), the invariants of the canonical coframing satisfy

$$(2.55) \qquad \begin{pmatrix} \mathrm{d}a \\ \mathrm{d}b \\ \mathrm{d}\bar{b} \end{pmatrix} = \begin{pmatrix} a_\theta & a_\eta & \overline{a_\eta} \\ b_\theta & \overline{a_\eta} & b_{\bar\eta} \\ \overline{b_\theta} & b_{\bar\eta} & a_\eta \end{pmatrix} \begin{pmatrix} \theta \\ \eta \\ \bar\eta \end{pmatrix},$$

where the 3-by-3 matrix of covariant derivatives in (2.55) must have rank at most 1 since $(a, b) : M \to \mathbb{R} \times \mathbb{C}$ must be constant on the 2-dimensional orbits of the pseudogroup of symmetries.

Now, the rank assumption implies, in particular, that $|a_\eta|^2 = |b_{\bar\eta}|^2$. Thus, if the locus $a_\eta = 0$ has any interior, it will follow that $\mathrm{d}a$ and $\mathrm{d}b$ are multiples of θ on that interior and hence, since θ is a contact form, that $\mathrm{d}a$ and $\mathrm{d}b$ must vanish on the interior of the locus $a_\eta = 0$. In particular, a and b must be constant on each component of the interior of the locus $a_\eta = 0$. Consequently, the unimodular CR-structure will be locally homogeneous on each such component. For simplicity, this case will be set aside and it will be assumed for the rest of this subsubsection that the locus $a_\eta = 0$ has no interior.

Let $M^* \subset M$ be the open set on which $a_\eta \neq 0$ and write

$$(2.56) \qquad \mathrm{d}a = r \left(2s \, \theta + \mathrm{e}^{\mathrm{i}\phi} \, \eta + \mathrm{e}^{-\mathrm{i}\phi} \, \bar\eta \right)$$

for some real functions $r = |a_\eta| > 0$, s, and some function ϕ defined up to an integral multiple of 2π on M^*. Since r, s, and ϕ are also invariant under the action of the symmetry pseudogroup, their differentials must be multiples of $\mathrm{d}a$. In particular, the 1-form

$$(2.57) \qquad \alpha = 2s \, \theta + \mathrm{e}^{\mathrm{i}\phi} \, \eta + \mathrm{e}^{-\mathrm{i}\phi} \, \bar\eta$$

must be closed. Moreover, since $\mathrm{d}b$ must be a multiple of $\mathrm{d}a$, and, in particular, $a_{\bar\eta} \, \mathrm{d}a - a_\eta \, \mathrm{d}b = 0$, it follows that

$$(2.58) \qquad \mathrm{d}b = \mathrm{e}^{-2\mathrm{i}\phi} \, \mathrm{d}a = r \mathrm{e}^{-2\mathrm{i}\phi} \, \alpha.$$

Now, computing the exterior derivatives of the structure equations and the identity $\mathrm{d}(\mathrm{d}\alpha) = 0$ yields that, first

$$(2.59) \qquad\qquad\qquad\qquad \mathrm{d}\phi = s\,\alpha,$$

and then, that

$$(2.60) \qquad\qquad\qquad\qquad b = \mathrm{e}^{-2\mathrm{i}\phi}\left(a + s^2 + \mathrm{i}\,v\right)$$

for some real-valued function v. Moreover, one finds that

$$(2.61) \qquad\qquad \mathrm{d}s = -v\,\alpha \qquad \text{and} \qquad \mathrm{d}v = 2s(a + s^2)\,\alpha.$$

In conclusion, one has structure equations of the form

$$\mathrm{d}\theta = \mathrm{i}\,\eta \wedge \overline{\eta},$$

$$\mathrm{d}\eta = 2\mathrm{i}\,\theta \wedge \left(a\,\eta + \mathrm{e}^{-2\mathrm{i}\phi}\left(a + s^2 + \mathrm{i}\,v\right)\overline{\eta}\right),$$

$$(2.62) \qquad \begin{pmatrix} \mathrm{d}a \\ \mathrm{d}v \\ \mathrm{d}s \\ \mathrm{d}\phi \end{pmatrix} = \begin{pmatrix} r \\ 2s(a+s^2) \\ -v \\ s \end{pmatrix} \left(2s\,\theta + \mathrm{e}^{\mathrm{i}\phi}\,\eta + \mathrm{e}^{-\mathrm{i}\phi}\,\overline{\eta}\right).$$

Note that the exterior derivatives of these structure equations imply

$$(2.63) \qquad \mathrm{d}\left(2s\,\theta + \mathrm{e}^{\mathrm{i}\phi}\,\eta + \mathrm{e}^{-\mathrm{i}\phi}\,\overline{\eta}\right) = \mathrm{d}r \wedge \left(2s\,\theta + \mathrm{e}^{\mathrm{i}\phi}\,\eta + \mathrm{e}^{-\mathrm{i}\phi}\,\overline{\eta}\right) = 0,$$

but no other identities on the derivatives of the forms θ and η or the functions a, v, s, or ϕ. According to Cartan's generalization of the third fundamental theorem of Lie, the solutions of (2.62) depend on one arbitrary function of one variable up to diffeomorphism and any coframing (θ, η) satisfying these structure equations has its pseudogroup acting transitively on the leaves of the integrable 1-form α as defined in (2.57).

2.2. **Embedding results.** It is natural to ask how much of the local geometry of $M \subset X$ is captured by the coframing (θ, η). In this section, some of these questions will be answered, at least in the real-analytic category. First, it will be useful to have an abstract notion of unimodular CR-structure:

Definition 2 (Unimodular CR-structures). A closed, complex-valued 2-form Ψ on a 3-manifold M will be said to be a *unimodular CR-structure* on M if the real and imaginary parts of Ψ are linearly independent at each point of M. Given such a Ψ, the unique 2-plane field $D \subset TM$ that consists of the 2-planes $D_x \subset T_x M$ to which Ψ pulls back to zero will be called the *complex tangent space* of Ψ. If D is a contact plane field, then Ψ will be said to be *nondegenerate*.

Remark 5 (Canonical coframing for unimodular CR-structures). It is clear that, if Ψ is a nondegenerate unimoduclar CR-structure, then the method employed in the proof of Proposition 1 will prove that there exists a unique coframing (θ, η) with $\theta = \bar{\theta}$ such that $\Psi = \theta \wedge \eta$ and $\mathrm{d}\theta = \mathrm{i}\,\eta \wedge \overline{\eta}$. This coframing will, of course, be referred to as the canonical coframing of (M, Ψ).

The first result provides local uniqueness for realizing a real-analytic unimodular CR-structure as a hypersurface in a unimodular complex surface:

Proposition 3 (Ambient uniqueness)**.** *Let M^3 be a real-analytic 3-manifold endowed with a real-analytic unimodular CR-structure Ψ.*

Suppose that (X_1, Υ_1) and (X_2, Υ_2) are two unimodular complex surfaces such that there exist real-analytic embeddings $\phi_i : M \to X_i$ satisfying

$$(2.64) \qquad\qquad \phi_1{}^*(\Upsilon_1) = \phi_2{}^*(\Upsilon_2) = \Psi.$$

Then there exist open sets $U_i \subset X_i$, with U_i containing $\phi_i(M)$ and a unimodular biholomorphism $\Phi : (U_1, \Upsilon_1) \to (U_2, \Upsilon_2)$ such that $\phi_2 = \Phi \circ \phi_1$.

Moreover, (Φ, U_1, U_2) is locally unique in the sense that, if $(\tilde{\Phi}, \tilde{U}_1, \tilde{U}_2)$ is another triple with these properties, then there exists an open subset $V_1 \subset U_1 \cap \tilde{U}_1$ containing $\phi_1(M)$ such that $\tilde{\Phi} = \Phi$ on V_1.

Proof. The proof is an application of the Cartan-Kähler Theorem: Consider the product manifold $X_1 \times X_2$ with projections $\pi_i : X_1 \times X_2 \to X_i$ and the ideal $\mathcal{I}$ on $X_1 \times X_2$ generated by the real and imaginary components of the closed 2-form $\pi_2^* \Upsilon_2 - \pi_1^* \Upsilon_1$. The ideal $\mathcal{I}$ with independence condition $\Upsilon_1 \wedge \overline{\Upsilon_1} \neq 0$ is evidently involutive, with Cartan characters

$$(2.65) \qquad\qquad (s_0, s_1, s_2, s_3, s_4) = (0, 2, 2, 0, 0).$$

The graph $\phi = (\phi_1, \phi_2) : M \to X_1 \times X_2$ is, by assumption, an integral manifold of $\mathcal{I}$ of dimension 3. Inspection shows that it is regular and that, moreover, its index of indeterminacy is zero. By the Cartan-Kähler theorem, it lies in a (locally unique) 4-dimensional integral manifold $Y \subset X_1 \times X_2$ of $\mathcal{I}$ to which the 4-form $\Upsilon_1 \wedge \overline{\Upsilon_1}$ pulls back to be a volume form. It follows that, near $\phi(M)$, the submanifold Y is the graph of a real-analytic invertible mapping $\Phi : U_1 \to U_2$ (where $U_i \subset X_i$ is an open neighborhood of $\phi_i(M)$) that satisfies $\Phi^* \Upsilon_2 = \Upsilon_1$. Thus Φ is a unimodular biholomorphism.

The stated local uniqueness follows from the local uniqueness of Y. $\qquad\square$

Proposition 4 (Ambient existence)**.** *Let M^3 be a real-analytic 3-manifold endowed with a real-analytic unimodular CR-structure Ψ.*

Then there exists a unimodular complex surface (X, Υ) and a real-analytic embedding $\phi : M \to X$ such that $\phi^* \Upsilon = \Psi$.

Proof. First, an application of the Cartan-Kähler Theorem will show local existence: On the product manifold $M \times \mathbb{C}^2$, consider the ideal $\mathcal{I}$ generated by the real and imaginary parts of the closed 2-form $\pi_2^*(\mathrm{d}z^1 \wedge \mathrm{d}z^2) - \pi_1^*(\Psi)$. If Ω is any volume form on M, then the ideal $\mathcal{I}$ with independence condition $\pi_1^*(\Omega) \neq 0$ is involutive, with characters

$$(2.66) \qquad\qquad (s_0, s_1, s_2, s_3) = (0, 2, 2, 0).$$

It follows that every point $p \in M$ has an open neighborhood U on which there exists a real-analytic embedding $z : U \to \mathbb{C}^2$ such that $z^*(\mathrm{d}z^1 \wedge \mathrm{d}z^2) = \Psi$.

Second, a patching argument finishes the proof: Consider the collection $\mathcal{U}$ consisting of those open sets $U \subset M$ for which there exists a unimodular complex surface (X, Υ) and a real-analytic embedding $\phi : U \to X$ satisfying $\phi^* \Upsilon = \Psi$. By the first part of the argument, $\mathcal{U}$ is an open covering of M. Let $\mathcal{U}'$ be a countable, locally finite refinement of $\mathcal{U}$ and use the uniqueness guaranteed by Proposition 3 to show that the union of the elements of $\mathcal{U}'$ belongs to $\mathcal{U}$. $\qquad\square$

Remark 6 (Cohomogeneity 0 and 1 examples). Suppose now that (M, Ψ) is a real-analytic, nondegenerate unimodular CR-structure that is either of cohomogeneity 0 (i.e., symmetry group G of dimension 3) or 1 (symmetry group G dimension 2). It then follows from Proposition 3 that the symmetry group action extends to any unimodular surface thickening (X, Υ) of (M, Ψ), at least to a neighborhood of $M \subset X$. Since M is a nondegenerate CR-structure, the orbits of G cannot be complex curves and hence it follows that the complexification $G^{\mathbb{C}}$ of G extends to an infinitesimally homogeneous unimodular action on (X, Υ). Thus, in the usual way, (X, Υ) can be regarded (up to a covering) as a coadjoint orbit of $G^{\mathbb{C}}$.

In particular, the examples of cohomogeneity 0 and 1 can be studied in a standard way by considering a 2-dimensional coadjoint orbit (X, Υ) of the complexification of a 2- or 3-dimensional Lie group G and looking at the G-invariant hypersurfaces in these coadjoint orbits.

This gives a uniform way of interpreting the formulae found in the previous subsection for the homogeneous examples.

Also, in the case that G is a 2-dimensional Lie group acting freely on (X, Υ), the quotient $G \backslash X$ will be a 2-dimensional surface and the image of a G-invariant hypersurface in X in this quotient will be a curve in this surface. This gives a geometric interpretation to the result, found by Cartan-Kähler analysis in §2.1.4, that the cohomogeneity 1 examples depend on one function of one variable.

This point of view also simplifies the study of evolution of such cohomogeneity 1 structures, as defined in the next section.

2.3. Hypersurface flows in unimodular surfaces.
In this section the behavior of the canonical coframing and its invariants under deformation will be investigated. A canonical flow for nondegenerate hypersurfaces will be defined and some examples computed.

2.3.1. *Deformation formulae.*
Consider a 1-parameter family $\iota : (-\epsilon, \epsilon) \times M \to X$ of immersions of M into X as nondegenerate hypersurfaces, parametrized so that

$$(2.67) \qquad \iota_* \left(\frac{\partial}{\partial t}(t, p) \right) = f(t, p) \, J\big(T(t, p)\big),$$

for $(t, p) \in (-\epsilon, \epsilon) \times M$, i.e., so that the variation is 'normal'. The function $f : (-\epsilon, \epsilon) \times M \to \mathbb{R}$ is the *normal displacement function*.

By assumption,

$$(2.68) \qquad \iota^*(\Upsilon) = (\theta + \mathrm{i} f \, \mathrm{d}t) \wedge \eta$$

where (θ, η) is the (t-dependent) induced canonical coframing on M. Expansion of the identities $\mathrm{d}\big((\theta + \mathrm{i} f \, \mathrm{d}t) \wedge \eta\big) = \mathrm{d}(\mathrm{d}\theta) = 0$ yields equations of the form

$$(2.69) \quad \begin{aligned} \mathrm{d}\theta &= \mathrm{i}\, \eta \wedge \overline{\eta} + \left(\tfrac{1}{3}(4af + f_{\eta\bar\eta} + f_{\bar\eta\eta})\, \theta + \mathrm{i}(\, f_\eta \, \eta - f_{\bar\eta} \, \overline{\eta})\right) \wedge \mathrm{d}t \\ \mathrm{d}\eta &= 2\mathrm{i}\,(\theta + \mathrm{i}\, f \, \mathrm{d}t) \wedge (a\, \eta + b\, \overline{\eta}) + \big(\mathrm{i}u\, \theta - (\tfrac{1}{3}(4af + f_{\eta\bar\eta} + f_{\bar\eta\eta}) + \mathrm{i}f_\theta)\eta\big) \wedge \mathrm{d}t \end{aligned}$$

where

$$(2.70) \qquad u = \tfrac{1}{3}(4af + f_{\eta\bar\eta} + f_{\bar\eta\eta})_{\overline{\eta}} + \mathrm{i}\ f_{\theta\overline{\eta}} + 4f_\eta \overline{b}\,.$$

The evolution of the invariants a and b can be computed by taking the exterior derivatives of the equations (2.69).

Note, in particular, that the normal deformation preserves the contact form θ up to a multiple if and only if $f_\eta = f_{\bar\eta} = 0$. By (2.21), this implies that $f_\theta = 0$ as

well, i.e., that f is a function of t alone. In such a case, assuming that $f(0) \neq 0$, one can reparametrize in t so as to arrange that $f(t) \equiv 1$ for t small.

2.3.2. *Unimodular normal flow.* The above formulae motivate the study of the case $f \equiv 1$ for a deformation.

Definition 3. A 1-parameter family $\iota : (-\epsilon, \epsilon) \times M \to X$ of immersions of M into X as nondegenerate hypersurfaces will be said to be a *unimodular normal flow* if its normal displacement function f satisfies $f \equiv 1$.

For a unimodular normal flow, after normal reparametrization, one has structure equations of the form

(2.71)
$$d\theta = i\,\eta \wedge \overline{\eta} + \tfrac{4}{3} a\,\theta \wedge dt$$
$$d\eta = 2i\,(\theta + i\,dt) \wedge (a\,\eta + b\,\overline{\eta}) + \tfrac{4}{3}\big(ia_{\overline{\eta}}\,\theta - a\,\eta\big) \wedge dt,$$

while the primary invariants satisfy evolution equations of the form

(2.72)
$$a_t = \tfrac{1}{3}(a_{\eta\overline{\eta}} + a_{\overline{\eta}\eta}) + \tfrac{4}{3}\,a^2 + 4\,b\overline{b}$$
$$b_t = \tfrac{2}{3}\,a_{\overline{\eta}\overline{\eta}} + i\,b_\theta + \tfrac{16}{3}\,ab.$$

Remark 7 (real-analytic uniqueness). By (2.71) and (2.72), for a unimodular normal flow, the first derivatives of the coframing and primary invariants with respect to t are expressed in terms of the covariant derivatives of the invariants. By induction, it follows that all higher time derivatives of the coframing and primary invariants can be expressed in terms of the coframing and iterated covariant derivatives of the invariants. In particular, a real-analytic 'initial' hypersurface can be embedded into a real-analytic unimodular normal flow in at most one way. Presumably, this is a special case of a uniqueness result for the flow in the smooth case.

Remark 8 (In unimodular coordinates). Since the Reeb vector field T depends on three derivatives of the embedding, the normal vector field iT also depends on three derivatives of the embedding. In particular, the equation for unimodular normal flow is, strictly speaking, a third order flow.

However, a closer examination reveals that, although iT is indeed a third order expression and is the lowest order normal vector field invariant under unimodular biholomorphism, one can express iT (noninvariantly) as the sum of a second order vector field and a vector field tangential to M. In other words, modulo reparametrization of M (in a possibly non-normal fashion), one can write the flow as a second order flow.

For example, suppose that the variation can be written locally in unimodular coordinates (w, z) in the graphical form

(2.73)
$$v = F(z, \overline{z}, u, t)$$

where $w = u + i\,v$. Then one finds that this family represents a unimodular normal flow (after normal reparametrization) if and only if F satisfies the nonlinear equation

(2.74) $\quad F_t = \Big[2\big[(1+F_u{}^2)F_{z\overline{z}} + F_z F_{\overline{z}} F_{uu} - i(1-i\,F_u)F_{\overline{z}}F_{uz} + i(1+i\,F_u)F_z F_{u\overline{z}}\big]\Big]^{1/3}.$

Note that the expression inside the cube root is nonvanishing as long as the family consists of nondegenerate hypersurfaces.

The linearization of (2.74) at a nondegenerate hypersurface $v = F(z, \bar{z}, u)$ is an equation of the form

$$(2.75) \qquad G_t = S_F(G)$$

where S_F is a second-order subelliptic operator of the same symbol type as the $\bar{\partial}_b$-Laplacian. Thus, the linearization is only weakly parabolic, so questions of short-time existence or uniqueness do not appear to be easy to resolve.

Example 5 (Concentric spheres). Consider the hypersurface foliation on $\mathbb{C}^2$ minus the origin defined by the level sets of the function $r = |z|^2 + |w|^2$, i.e., the spheres of positive radius centered on the origin.

Set

$$(2.76) \qquad \theta + \mathrm{i}\,\mathrm{d}t = -\mathrm{i}\,\frac{\bar{z}\,\mathrm{d}z + \bar{w}\,\mathrm{d}w}{\left(|z|^2 + |w|^2\right)^{1/3}} \qquad \text{and} \qquad \eta = \mathrm{i}\,\frac{z\,\mathrm{d}w - w\,\mathrm{d}z}{\left(|z|^2 + |w|^2\right)^{2/3}}.$$

Then $(\theta + \mathrm{i}\,\mathrm{d}t) \wedge \eta = \mathrm{d}z \wedge \mathrm{d}w$ and

$$(2.77) \qquad \mathrm{d}t = -\frac{\bar{z}\,\mathrm{d}z + \bar{w}\,\mathrm{d}w + z\,\mathrm{d}\bar{z} + w\,\mathrm{d}\bar{w}}{2\left(|z|^2 + |w|^2\right)^{1/3}} = -\tfrac{1}{2}r^{-1/3}\,\mathrm{d}r = \mathrm{d}\!\left(-\tfrac{3}{4}\,r^{2/3}\right).$$

Thus, the level sets of $r > 0$ are the integral surfaces of $\mathrm{d}t$ and one sees that this is a unimodular normal flow when one sets $t - t_0 = \tfrac{3}{4}r_0{}^{2/3} - \tfrac{3}{4}r^{2/3}$.

Computation yields

$$(2.78) \qquad \mathrm{d}\theta = \mathrm{i}\,\eta \wedge \bar{\eta} + \tfrac{4}{3}r^{-2/3}\,\theta \wedge \tau,$$

so (θ, η) pulls back to each level set of r to be the canonical coframing on that level set. The structure equation for η turns out to be

$$(2.79) \qquad \mathrm{d}\eta = 2\mathrm{i}\,\theta \wedge \left(r^{-2/3}\,\eta\right) + \tfrac{2}{3}r^{-2/3}\,\eta \wedge \tau.$$

Note that the time t flow of the level set $r = r_0 > 0$ is the level set $r = R(t)$, where

$$(2.80) \qquad R(t) = \left(r_0{}^{2/3} - \tfrac{4}{3}\,t\right)^{3/2},$$

and hence that this flow contracts to the center of the sphere in finite time.

Example 6 (Abstract Homogeneous Flows). Suppose now that M^3 is endowed with an abstract homogeneous unimodular CR-structure, i.e., a coframing (θ_0, η_0) satisfying

$$(2.81) \qquad \begin{aligned} \mathrm{d}\theta_0 &= \mathrm{i}\,\eta_0 \wedge \overline{\eta_0}, \\ \mathrm{d}\eta_0 &= 2\mathrm{i}\,\theta_0 \wedge \left(a_0\,\eta_0 + b_0\,\overline{\eta_0}\right) \end{aligned}$$

for some constants $a_0 = \overline{a_0}$ and b_0.

Of course, this coframing is real-analytic in appropriate coordinates and hence the realization of M^3 as a nondegenerate hypersurface in a unimodular complex surface is locally unique. In this particular case, one can explicitly define the complex structure on a neighborhood of $M \times \{0\} \subset M \times \mathbb{R}$ that realizes this with a holomorphic volume form

$$(2.82) \qquad \Upsilon = (\theta + \mathrm{i}\,\mathrm{d}t) \wedge \eta$$

where θ and η are t-dependent 1-forms on M.

Computation yields that, when regarded as 1-forms on an open set in $M \times \mathbb{R}$, the 1-forms θ and η must satisfy

$$(2.83) \qquad \begin{aligned} \mathrm{d}\theta &= \mathrm{i}\,\eta \wedge \overline{\eta} + \tfrac{4}{3}\,a\,\theta \wedge \mathrm{d}t \,, \\ \mathrm{d}\eta &= 2\mathrm{i}\,(\theta + \mathrm{i}\,\mathrm{d}t) \wedge \left(a\,\eta + b\,\overline{\eta}\right) - \tfrac{4}{3}\,a\,\eta \wedge \mathrm{d}t \end{aligned}$$

where $a = \overline{a}$ and b are functions of t that satisfy the differential equations

$$(2.84) \qquad \begin{aligned} a'(t) &= \tfrac{4}{3}\,a(t)^2 + 4\,b(t)\,\overline{b(t)} \\ b'(t) &= \tfrac{16}{3}\,a(t)b(t) \end{aligned}$$

and initial conditions $a(0) = a_0$ and $b(0) = b_0$.

Note that (2.84) does not have any periodic solutions other than the fixed point $(a,b) \equiv (0,0)$ since, except in this case, a' is strictly positive.

Thus, if $I_0 \subset \mathbb{R}$ is the maximal interval containing $0 \in \mathbb{R}$ on which a and b can be defined satisfying (2.84) and the given initial conditions, then Cartan's standard existence theorem implies that, for any 3-dimensional M_0 endowed with a coframing (θ_0, η_0) satisfying (2.81), there will exist a time-dependent coframing (θ, η) on $M_0 \times I_0$ satisfying (2.83) where a and b satisfy (2.84) and where (θ, η) pull back to $M_0 \times \{0\}$ to become (θ_0, η_0).

In fact, though, one does not need to invoke Cartan's theorem, since one can explicitly find (θ, η) in the form

$$(2.85) \qquad \theta = \left(\det F(t)\right)\theta_0 \qquad \text{and} \qquad \begin{pmatrix} \eta \\ \overline{\eta} \end{pmatrix} = F(t)\begin{pmatrix} \eta_0 \\ \overline{\eta_0} \end{pmatrix}$$

where $F(t)$, defined for $t \in I_0$, satisfies the initial condition $F(0) = I_2$ and the differential equation

$$(2.86) \qquad F'(t) = -2\begin{pmatrix} \tfrac{1}{3}a(t) & b(t) \\ \overline{b(t)} & \tfrac{1}{3}a(t) \end{pmatrix} F(t).$$

where $\left(a(t), b(t)\right)$ is the solution on I_0 of (2.84) that satisfies $a(0) = a_0$ and $b(0) = b_0$.

Now, the general solution of (2.84) is expressed in elliptic functions. However, the problem can be simplified and a partial explicit integration effected: For any solution of (2.84), either $b(t)$ vanishes identically or it never vanishes.

When $b(t) \equiv 0$, the remaining equation for a is integrated as

$$(2.87) \qquad a(t) = \frac{a_0}{1 - \tfrac{4}{3}a_0 t}.$$

It then follows that

$$(2.88) \qquad \theta = \left(1 - \tfrac{4}{3}a_0 t\right)\theta_0 \qquad \text{and} \qquad \eta = \left(1 - \tfrac{4}{3}a_0 t\right)^{\frac{1}{2}}\eta_0 \,.$$

When $a_0 > 0$, this gives the evolution of the sphere of radius $a_0^{-3/4}$ in $\mathbb{C}^2$ (endowed with its standard volume form).

Example 7 (Circle bundles over $K = -1$ surfaces). This is a continuation of the discusion begun in Example 3. Recall that the hypersurfaces $M_r \subset T^*\mathbb{D}$ constructed there have $a = -\tfrac{1}{2}r^{-2/3}$. It follows from (2.87) that the unimodular normal flow with M_{r_0} as initial hypersurface is described by $M_{r(t)}$ where

$$(2.89) \qquad r(t) = (r_0^{2/3} + \tfrac{2}{3}\,t)^{3/2}.$$

Of course, this flow exists for all positive time, but exists only for a finite time in the past. Note that as t decreases towards the singularity at $t = -\frac{3}{2} r_0^{2/3}$, the hypersurface $M_{r(t)}$ converges to the zero section of $X \to \mathbb{D}$, i.e., the hypersurface collapses onto a complex curve.

Assume, henceforth, that $b(t)$ is nowhere-vanishing. Then one finds that the ratio $\bar{b}/b$ is constant and that, after applying scaling as discussed in Remark 4, all solutions can be deduced from solutions in which $b(t)$ is positive and real, so assume this from now on. Then the obvious homogeneity of (2.84) implies that there must be a homogeneous first integral and, indeed, one finds that the function $(a^2 - b^2)^2/b$ is constant. In other words, each integral curve of (2.84) with b positive and real lies in a quartic plane curve of the form

$$(2.90) \qquad\qquad (a^2 - b^2)^2 - \lambda^3 b = 0$$

for some constant $\lambda \geq 0$.

When $\lambda = 0$, one has the solutions

$$(2.91) \qquad a(t) = \frac{a_0}{1 - \frac{16}{3} a_0 t} \qquad \text{and} \qquad b(t) = \frac{b_0}{1 - \frac{16}{3} a_0 t}$$

with $a_0 = \pm b_0 \neq 0$.

When $\lambda > 0$, the curve (2.90) is irreducible over $\mathbb{C}$ and of genus 1. It is smooth in the finite part of the plane and has two ordinary double points on the line at infinity. The functions a and b each have four simple poles on the normalized curve, representing the four points on the line at infinity. The function b has a quadruple zero at $(a, b) = (0, 0)$ and no other zeros, while the function a has four simple zeros, only two of which, those at $(a, b) = (0, 0)$ and $(a, b) = (0, \lambda)$, are real. The curve (2.90) has two real branches: One that passes through $(a, b) = (0, 0)$ and that, except for this point, lies in the union of the sectors defined by $0 < b < -a$ and $0 < b < a$ and one that passes through $(a, b) = (0, \lambda)$ and lies in the sector $b > |a|$. Away from the point $(a, b) = 0$, one has

$$(2.92) \qquad\qquad dt = \frac{3\, db}{16ab} = \frac{3\, da}{4(a^2 + 3b^2)},$$

implying that dt is a meromorphic differential with a double pole at $(a, b) = 0$ and no other zeros or poles on the real branches (note that db vanishes at $(a, b) = (0, \lambda)$), even on the line at infinity.

In particular, the integral of dt over the branch in the sector $b > |a|$ is finite, so that these solutions exist for only a finite time both forward and backwards.

Similarly, any solution of (2.90) that lies in the sector $0 < b < -a$ exists for only a finite time in the past but an infinite time in the future, while any solution of (2.90) that lies in the sector $0 < b < a$ exists for an infinite time in the past but only a finite time in the future.

Note that, since $\big(\log\big(\det F(t)\big)\big)' = -\frac{4}{3} a(t) = -\frac{1}{4}\big(\log b(t)\big)'$, it follows that

$$(2.93) \qquad\qquad \det F(t) = \big(b(t)/b_0\big)^{-\frac{1}{4}},$$

so

$$(2.94) \qquad\qquad \theta = \big(b(t)/b_0\big)^{-\frac{1}{4}} \theta_0.$$

Example 8 (Explicit homogeneous hypersurface flows). Finally, consider the flow for the homogenous nondegenerate hypersurfaces $M_{h,q} \subset X_q$ in either Example 2 (the SO(3)-homogeneous examples) or Example 4 (the PSL$(2,\mathbb{R})$-homogeneous examples).

In either case, the evolution equations (2.84) for the invariants (a,b) coupled with the equations (2.41) or (2.48) imply that the hypersurface $M_{h_0,q}$ will evolve as the family $M_{h(t),q}$ where $h(t)$ satisfies the ordinary differential equation initial value problem

$$(2.95) \qquad h'(t) = 2\big(|q|^2 - h(t)^2\big)^{1/3}, \qquad h(0) = h_0.$$

If $h_0 > |q|$, which happens in the SO(3)-invariant examples (and only in those examples), it follows that the flow exists for an infinite time into the past (and $h(t) \to \infty$ as $t \to -\infty$) but only a finite time into the future, as $h(t)$ will go down to $|q|$ in time

$$(2.96) \qquad T(h_0, q) = \int_{|q|}^{h_0} \frac{\mathrm{d}h}{2\big(h^2 - |q|^2\big)^{1/3}} < \infty.$$

In particular $M_{h(t),q}$ will converge in finite time to $M_{|q|,q} \subset X_q$, which is a point if $q = 0$[6] and an embedded, totally real 2-sphere in X_q if $|q| > 0$.

If $h_0 < -|q|$, which happens in the PSL$(2,\mathbb{R})$-invariant examples (and only in those examples), it follows that the flow exists for all future time (and $h(t) \to -\infty$ as $t \to \infty$) but extends only a finite time into the past, as $h(t)$ will have come down from a starting value of $-|q|$ at the (past, i.e., negative) time

$$(2.97) \qquad T(h_0, q) = \int_{h_0}^{-|q|} \frac{\mathrm{d}h}{2\big(|q|^2 - h^2\big)^{1/3}} > -\infty.$$

Recall that $M_{h,q}$ is the disjoint union of $M_{h,q}^+$ and $M_{h,q}^-$ (as defined in Example 4).

It is interesting to consider what happens to each of these smooth hypersurfaces as h approaches $|q|$ from below. When $q = 0$, the hypersurfaces $M_{h,0}^{\pm}$ as $h \to 0^-$ converge to $M_{0,0}$ in the sense that every point of $M_{0,0}$ is a limit of a sequence $z_k \in M_{h_k,0}^+$ with $h_k \to 0^-$, with a similar statement for the hypersurfaces $M_{h,0}^-$. This is so in spite of the fact that, under the actual unimodular normal flow, all of the points of $M_{h,0}$ with $h < 0$ flow to the origin itself. Thus, there is a significant difference between the limit of the pointwise flow and the 'hypersurface flow'. When $q \neq 0$, the situation is even more interesting. As $h \to -|q|^-$, the hypersurface $M_{h,q}^+$ converges to a proper subset of $M_{-|q|,q}$ that consists of the singular locus $\mathrm{i}\,H_{-1}$ (a surface) and two of the four components of the smooth locus of $M_{-|q|,q}$, say, $M_{-|q|,q}^{+,+}$ and $M_{-|q|,q}^{-,-}$. In particular, the union of these three PSL$(2,\mathbb{R})$-orbits is a singular, degenerate hypersurface, with singular locus equal to the 2-dimensional orbit $\mathrm{i}\,H_{-1}$. On the other hand, as $h \to -|q|^-$, the hypersurface $M_{h,q}^-$ converges to a proper subset of $M_{-|q|,q}$ that consists of the singular locus $\mathrm{i}\,H_{-1}$ (a surface) and two of the four components of the smooth locus of $M_{-|q|,q}$, say, $M_{-|q|,q}^{+,-}$ and $M_{-|q|,q}^{-,+}$. In particular, the union of these three PSL$(2,\mathbb{R})$-orbits is a singular, degenerate hypersurface, with singular locus equal to the 2-dimensional orbit $\mathrm{i}\,H_{-1}$. In each case, under

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[6]The reader may wonder what would happen if one were to replace the singular unimodular quadric (X_0, Υ_0) by its crepant resolution $(\hat{X}_0, \hat{\Upsilon}_0)$. The answer, not surprisingly, is that $M_{h(t),0}$, regarded as a hypersurface in $\hat{X}_0$, would collapse in finite time to the exceptional curve in $\hat{X}_0$.

the actual unimodular normal flow, the points of $M_{h,q}$ with $h < -|q|$ converge to the 2-dimensional singular locus $\mathrm{i}\, H_{-1}$, while the hypersurface itself converges to a (singular) 3-dimensional hypersurface.

Finally, consider what happens to the hypersurfaces $M_{h_0,q} \subset X_q$ with $|h_0| < |q|$. Again, these are $\mathrm{PSL}(2,\mathbb{R})$-invariant hypersurfaces. The unimodular normal flow with this initial hypersurface $M_{h(t),q}$ exists for $T^-(h_0,q) < t < T^+(h_0,q)$, where

$$(2.98) \qquad T^+(h_0,q) = \int_{h_0}^{|q|} \frac{\mathrm{d}h}{2\big(|q|^2 - h^2\big)^{1/3}} < +\infty$$

and

$$(2.99) \qquad T^-(h_0,q) = \int_{h_0}^{-|q|} \frac{\mathrm{d}h}{2\big(|q|^2 - h^2\big)^{1/3}} > -\infty.$$

Several things are interesting about these flows: First, the hypersurface $M_{0,q}$ is CR-flat (since these have $ab = 0$) but $M_{h,q}$ is not CR-flat when $hq \neq 0$. Thus, this furnishes an example of a CR-flat hypersurface for which the unimodular flow does not preserve CR-flatness. Second, as h increases to the value $|q|$, the hypersurfaces $M_{h,q}^+$ converge to the totally real surface $M_{|q|,q}^+ \subset X_q$, i.e., the upper nappe of the (real) unit hyperboloid of two sheets while the hypersurfaces $M_{h,q}^-$ converge to the totally real surface $M_{|q|,q}^- \subset X_q$, i.e., the lower nappe of the (real) unit hyperboloid of two sheets. Thus, the 'singularity' that develops in forward time from $M_{h,q}$ under the unimodular normal flow is collapse onto a totally real surface. Third, as h decreases to the value $-|q|$, the hypersurfaces $M_{h,q}^+$ converge to the singular real hypersurface in X_q that is the union of the three $\mathrm{PSL}(2,\mathbb{R})$-orbits $M_{-|q|,q}^{+,+}$, $M_{-|q|,q}^{-,+}$, and $\mathrm{i}\, H_{-1}$ while the hypersurfaces $M_{h,q}^-$ converge to the singular real hypersurface in X_q that is the union of the three $\mathrm{PSL}(2,\mathbb{R})$-orbits $M_{-|q|,q}^{+,-}$, $M_{-|q|,q}^{-,-}$, and $\mathrm{i}\, H_{-1}$. (Nevertheless, under this flow, each point of $M_{h,q}$ converges to a point of $\mathrm{i}\, H_{-1}$.)

3. Invariants of 3-dimensional CR-manifolds

This section contains an exposition of Cartan's solution [4, 5] in 1932 of the equivalence problem for nondegenerate hypersurfaces in complex 2-manifolds (with no volume specified). The main reason for including it here is that it will be used to compute the invariants of the underlying CR-structure associated to the canonical coframing of a pseudoconvex real hypersurface in a unimodular complex surface.

This equivalence problem is much more subtle than the case of hypersurfaces in a unimodular surface. For another exposition of Cartan's solution in more 'modern' language, the reader might compare Jacobowitz' [8, Chapters 5-7], though the notation is different from that of this article.

Of course, the theory has been extensively developed since Cartan's work, with the general solution for a nondegenerate hypersurface in a complex n-manifold being the subject of a famous paper by Chern and Moser [6], and an earlier paper by Tanaka [12].

3.1. **The geometric problem and its G-structure.** Suppose that $M^3 \subset X$ is a smooth real hypersurface in a complex 2-manifold X, which, since all the considerations are local, can be taken to be $\mathbb{C}^2$ if desired.

3.1.1. *The notion of a CR-structure.* For each $x \in M$, the tangent plane T_xM cannot be a complex subspace of T_xX, but contains a unique complex subspace $D_x \subset T_xM$ of complex dimension 1. Thus, M inherits a geometric structure from being immersed as a hypersurface in a complex 2-manifold.

Definition 4. A (smooth) *CR-structure* on a 3-manifold M is a choice of a (smooth) rank 2 subbundle $D \subset TM$ together with a choice of complex structure on D, i.e., a smooth bundle map $J : D \to D$ satisfying $J^2 = -\mathrm{Id}_D$.

In the real-analytic category, every CR-structure on a 3-manifold is locally induced by an immersion into $\mathbb{C}^2$.

Proposition 5 (Local realization of analytic CR-structures). *Let (D, J) be a real-analytic CR-structure on M^3. Then for each point $x \in M$ there exists an x-neighborhood U and a real-analytic embedding $Z : U \to \mathbb{C}^2$ so that (D, J) is the CR-structure on U induced by the embedding Z.*

Proof. On a neighborhood U of x choose a real-analytic, nonvanishing real 1-form ρ that annihilates D and a real-analytic, complex valued 1-form η linearly independent from ρ that satisfies $\eta(Jv) = \mathrm{i}\,\eta(v)$ for all $v \in D$. Then any complex-valued 1-form ζ on U that satisfies $\zeta(Jv) = \mathrm{i}\,\zeta(v)$ is a linear combination of ρ and η.

To construct the desired Z, it suffices to find two complex functions z^1 and z^2 in a neighborhood of x whose differentials are linearly independent and that satisfy $\mathrm{d}z^k(Jv) = \mathrm{i}\,\mathrm{d}z^k(v)$, i.e., so that $\mathrm{d}z^k \wedge \rho \wedge \omega = 0$.

Now, on $N = U \times \mathbb{C}$ with second projection $z : N \to \mathbb{C}$, let $\mathcal{I}$ be the ideal generated by the two 3-forms that are the real and imaginary parts of $\mathrm{d}z \wedge \rho \wedge \omega$. The characters of $\mathcal{I}$ are $s_i = 0$ for $i \neq 2$ and $s_2 = 2$. Meanwhile, the space of 3-dimensional integral elements of $\mathcal{I}$ that satisfy the independence condition $\rho \wedge \omega \wedge \bar{\omega} \neq 0$ is of dimension 4. Thus, the system $\mathcal{I}$ is in involution.

Choose two integral manifolds Σ_i, $i = 1, 2$ of $\mathcal{I}$ that pass through $(x, 0) \in N$ but that are not tangent there. Each is then the graph of a function z^i that satisfies $\mathrm{d}z^k \wedge \rho \wedge \omega = 0$ and the condition that the two integral manifolds not be tangent is equivalent to $\mathrm{d}z^1 \wedge \mathrm{d}z^2 \neq 0$.   □

Remark 9 (The need for the analyticity hypothesis). The famous Levi-Nirenberg example [9, 11] shows that the assumption of real-analyticity is necessary here.

3.1.2. *A G-structure associated to a CR-structure.* Suppose now that M^3 is endowed with a *CR*-structure (D, J). Let $V = \mathbb{R} \oplus \mathbb{C}$ and think of V as the space of columns of height 2 whose first entry is real and whose second entry is complex.

A coframe $u : T_xM \to V$ will be said to be 0-*adapted* to (D, J) if $u(D_x) = \mathbb{C} \subset V$ and, moreover, $u(Jv) = \mathrm{i}\,u(v)$ for all $v \in D_x$. Let $B_0 \subset F^*(M, V)$ denote the space of 0-adapted V-valued coframes on M.

If u and u^* lie in B_0 and share the same basepoint, then

$$(3.1) \qquad\qquad u^* = \begin{pmatrix} r & 0 \\ b & a \end{pmatrix} u,$$

where r is a real number and a and b are complex, with $a \neq 0$. Thus, B_0 is a G_0-structure on M where

$$(3.2) \qquad G_0 = \left\{ \begin{pmatrix} r & 0 \\ b & a \end{pmatrix} \,\middle|\, r \in \mathbb{R}^*,\ a \in \mathbb{C}^*,\ \text{and } b \in \mathbb{C} \right\}.$$

Conversely, given a G_0-structure B_0 on M, there is canonically associated to it a unique CR-structure (D, J) that gives rise to it via this construction. Thus, the two sorts of structures are equivalent.

3.2. The first analysis. Now let B_0 be a G_0-structure on M^3. Write the canonical V-valued 1-form ω on B_0 in the form

$$(3.3) \qquad \omega = \begin{pmatrix} \theta \\ \eta \end{pmatrix}$$

where θ is a real-valued 1-form and η is a complex-valued 1-form.

3.2.1. *The first structure equation.* The first structure equation can be written in the form

$$(3.4) \qquad d \begin{pmatrix} \theta \\ \eta \end{pmatrix} = - \begin{pmatrix} \rho_0 & 0 \\ \beta_0 & \alpha_0 \end{pmatrix} \wedge \begin{pmatrix} \theta \\ \eta \end{pmatrix} + \begin{pmatrix} \theta \wedge (b\,\eta + \bar{b}\,\bar{\eta}) + \mathrm{i}L\,\eta \wedge \bar{\eta} \\ \theta \wedge (c\,\eta + e\,\bar{\eta}) + T\,\eta \wedge \bar{\eta} \end{pmatrix}$$

where L is a real function on B_0 but the other coefficients are allowed to be complex. Clearly, by adding multiples of θ, η and $\bar{\eta}$ to the pseudo-connection forms ρ_0, α_0, and β_0, it can be arranged that $b = c = e = T = 0$, but L cannot be affected by such changes.

Thus, the structure equations can be assumed to have the form

$$(3.5) \qquad d \begin{pmatrix} \theta \\ \eta \end{pmatrix} = - \begin{pmatrix} \rho_0 & 0 \\ \beta_0 & \alpha_0 \end{pmatrix} \wedge \begin{pmatrix} \theta \\ \eta \end{pmatrix} + \begin{pmatrix} \mathrm{i}L\,\eta \wedge \bar{\eta} \\ 0 \end{pmatrix}.$$

Differentiating the first equation $d\theta = -\rho_0 \wedge \theta + \mathrm{i}L\,\eta \wedge \bar{\eta}$ and reducing modulo θ gives the relation

$$(3.6) \qquad dL \equiv L(\alpha_0 + \bar{\alpha}_0 - \rho_0) \mod \theta, \eta, \bar{\eta},$$

so, on a given fiber of B_0, either L vanishes identically or is nowhere zero there.

3.2.2. *The case $L \equiv 0$.* The case where L vanishes identically, i.e., the intrinsic torsion of the G_0-structure vanishes, turns out not to be very interesting. In this case, one can calculate that the characters of the Lie algebra $\mathfrak{g}_0$ are $s_1 = 3$, $s_2 = 1$, and $s_3 = 0$. Moreover the variability of the pseudo-connection is of dimension $5 = s_1 + 2s_2$, so G_0 is semi-involutive and all of the real-analytic G_0-structures with vanishing torsion are equivalent. Thus, it makes sense to concentrate on the (generic) case where L is nowhere-vanishing.[7]

3.2.3. *The case $L \neq 0$.* Now, there is a direct geometric interpretation of L. Since θ is a nonzero multiple of $\pi^*(\sigma)$, where σ is any nonvanishing 1-form with $D = \ker \sigma$, it follows that $\theta \wedge d\theta = \mathrm{i}L\,\theta \wedge \eta \wedge \bar{\eta}$ is nonzero if and only if $\sigma \wedge d\sigma$ is nonzero, i.e., if and only if D is a contact plane field on M^3.

Definition 5. A CR-structure (D, J) on M^3 is *nondegenerate* if D is nowhere-integrable, i.e., is a contact structure on M.

Thus, the condition that L be nowhere-vanishing is the condition that the CR-structure be nondegenerate. From now on, this will be assumed to be the case.

---

[7]The 'intermediate' case, in which L vanishes on some proper closed subset has also been intensively studied, this body of work will play no role in this article.

3.2.4. *The first structure reduction.* This assumption leads directly to the first re-duction: Set

$$(3.7) \qquad\qquad B_1 = \{ \, u \in B_0 \,\big|\, L(u) = 1 \, \}.$$

Then B_1 is a G_1-structure on M where

$$(3.8) \qquad\qquad G_1 = \left\{ \, \begin{pmatrix} a\bar{a} & 0 \\ b & a \end{pmatrix} \,\bigg|\, a \in \mathbb{C}^* \text{ and } b \in \mathbb{C} \, \right\}.$$

Pulling all of the forms on B_0 back to B_1 and giving them the same names, the structure equations on B_1 now read

$$(3.9) \qquad \mathrm{d}\begin{pmatrix} \theta \\ \eta \end{pmatrix} = - \begin{pmatrix} \alpha_0 + \bar{\alpha}_0 & 0 \\ \beta_0 & \alpha_0 \end{pmatrix} \wedge \begin{pmatrix} \theta \\ \eta \end{pmatrix} + \begin{pmatrix} (\alpha_0 + \bar{\alpha}_0 - \rho_0)\wedge\theta + \mathrm{i}\,\eta\wedge\bar{\eta} \\ 0 \end{pmatrix}.$$

where $\alpha_0 + \bar{\alpha}_0 - \rho_0 = a\theta + b\eta + \bar{b}\bar{\eta}$ for some functions a and b on B_1. Subtracting $b\eta$ from α_0 reduces the function b to zero and the structure equations become

$$(3.10) \qquad\qquad \mathrm{d}\begin{pmatrix} \theta \\ \eta \end{pmatrix} = - \begin{pmatrix} \alpha_0 + \bar{\alpha}_0 & 0 \\ \beta_0 & \alpha_0 \end{pmatrix} \wedge \begin{pmatrix} \theta \\ \eta \end{pmatrix} + \begin{pmatrix} \mathrm{i}\,\eta\wedge\bar{\eta} \\ 0 \end{pmatrix}.$$

Now the torsion is constant.

If the algebra $\mathfrak{g}_1$ were involutive, then reaching this point would imply that any two nondegenerate G_0-structures were locally equivalent. However, one easily computes that the characters of this algebra are $s_1 = 3$, $s_2 = 1$, and $s_3 = 0$ while the pseudo-connections with this torsion are determined up to a replacement of the form $(\alpha_0, \beta_0) \mapsto (\alpha_0^*, \beta_0^*)$ where

$$(3.11) \qquad\qquad \begin{pmatrix} \alpha_0^* \\ \beta_0^* \end{pmatrix} = \begin{pmatrix} \alpha_0 \\ \beta_0 \end{pmatrix} + \begin{pmatrix} s^1 & 0 \\ s^2 & s^1 \end{pmatrix} \begin{pmatrix} \theta \\ \eta \end{pmatrix},$$

and s^1 and s^2 are arbitrary complex-valued functions on B_1. Thus, $\dim \mathfrak{g}_1^{(1)} = 4 < s_1 + 2s_2 + 3s_3 = 5$, so $\mathfrak{g}_1$ is not involutive. Hence, there remains the possibility that there will be differential invariants at some higher order.

3.3. **Prolongation and further reductions.** According to the prescription of the method of equivalence, I now construct a $\mathfrak{g}_1^{(1)}$-bundle $B_1^{(1)}$ over B_1 that consists of the coframes on B_1 with values in $V \oplus \mathfrak{g}_1$ that satisfy the structure equations of B_1.

For simplicity, I will identify $V \oplus \mathfrak{g}_1$ with $\mathbb{R} \oplus \mathbb{C}^3$, thought of as the columns of height 4 with the first entry real and the remaining three complex.

In the trivialization $B_1^{(1)} = B_1 \times \mathfrak{g}_1^{(1)}$ induced by the section $B_1 \to B_1^{(1)}$ represented by a choice of α_0 and β_0 on B_1 satisfying (3.10), the canonical 1-form $\omega^{(1)}$ has the form

$$(3.12) \qquad \omega^{(1)} = \begin{pmatrix} \theta \\ \eta \\ \alpha \\ \beta \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ s^1 & 0 & 0 & 0 \\ s^2 & s^1 & 0 & 0 \end{pmatrix}^{-1} \begin{pmatrix} \theta \\ \eta \\ \alpha_0 \\ \beta_0 \end{pmatrix} = \begin{pmatrix} \theta \\ \eta \\ \alpha_0 - s^1\theta \\ \beta_0 - s^2\theta - s^1\eta \end{pmatrix}$$

where, of course, the functions s^1 and s^2 now represent coordinates on $\mathfrak{g}_1^{(1)}$ and so are independent from the functions on B_1. The structure equations on $B_1^{(1)}$ have

the form:

$$(3.13) \qquad d \begin{pmatrix} \theta \\ \eta \\ \alpha \\ \beta \end{pmatrix} = - \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \sigma_0^1 & 0 & 0 & 0 \\ \sigma_0^2 & \sigma_0^1 & 0 & 0 \end{pmatrix} \begin{pmatrix} \theta \\ \eta \\ \alpha \\ \beta \end{pmatrix} + \begin{pmatrix} -(\alpha + \bar\alpha) \wedge \theta + i\, \eta \wedge \bar\eta \\ -\beta \wedge \theta - \alpha \wedge \eta \\ T_\alpha \\ T_\beta \end{pmatrix}$$

where T_α and T_β represent the torsion terms associated to those components of the canonical 1-form while σ_0^1 and σ_0^2 are 1-forms that satisfy $\sigma_0^i \equiv ds^i$ modulo forms semi-basic for the projection $B_1^{(1)} \to B_1$ but that are otherwise arbitrary.

Computing the exterior derivatives of the first two equations of (3.13) yields

$$(3.14) \qquad \begin{aligned} 0 = d(d\theta) &= -\left(T_\alpha + \overline{T_\alpha} - i\,\beta \wedge \bar\eta + i\,\bar\beta \wedge \eta \right) \wedge \theta, \\ 0 = d(d\eta) &= -\left(T_\beta + \beta \wedge \bar\alpha\right) \wedge \theta - \left(T_\alpha + i\,\beta \wedge \bar\eta\right) \wedge \eta, \end{aligned}$$

Setting $T_\alpha^* = T_\alpha + i\,\beta \wedge \bar\eta + 2i\,\bar\beta \wedge \eta$ and $T_\beta^* = T_\beta + \beta \wedge \bar\alpha$, these equations can be written in the form

$$(3.15) \qquad \left(T_\alpha^* + \overline{T_\alpha^*}\right) \wedge \theta = \left(T_\beta^*\right) \wedge \theta + \left(T_\alpha^*\right) \wedge \eta = 0$$

and the second of these equations implies, via Cartan's Lemma, that there exist 1-forms ψ_1, ψ_2, and ψ_3 so that

$$(3.16) \qquad \begin{aligned} T_\alpha^* &= \psi_2 \wedge \theta + \psi_1 \wedge \eta \\ T_\beta^* &= \psi_3 \wedge \theta + \psi_2 \wedge \eta \end{aligned}$$

Since T_α^* and T_β^* are semi-basic, the ψ_i must be also. Thus, by subtracting ψ_2 from σ_0^1 and ψ_3 from σ_0^2, it can be arranged that $\psi_2 = \psi_3 = 0$. Then the remaining equation on T_α^* becomes

$$(3.17) \qquad \left(\psi_1 \wedge \eta + \overline{\psi_1 \wedge \eta}\right) \wedge \theta = 0,$$

which implies that $\psi_1 \wedge \eta = b\, \eta \wedge \theta + R\, \eta \wedge \bar\eta$, where b is a complex function and R is a real function. By adding $b\,\eta$ to σ_0^1, it can be arranged that $b = 0$, so that the structure equations now take the form

$$(3.18) \qquad d \begin{pmatrix} \theta \\ \eta \\ \alpha \\ \beta \end{pmatrix} = - \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \sigma_0^1 & 0 & 0 & 0 \\ \sigma_0^2 & \sigma_0^1 & 0 & 0 \end{pmatrix} \begin{pmatrix} \theta \\ \eta \\ \alpha \\ \beta \end{pmatrix} + \begin{pmatrix} -(\alpha + \bar\alpha) \wedge \theta + i\, \eta \wedge \bar\eta \\ -\beta \wedge \theta - \alpha \wedge \eta \\ -i\,\beta \wedge \bar\eta - 2i\,\bar\beta \wedge \eta + R\, \eta \wedge \bar\eta \\ -\beta \wedge \bar\alpha \end{pmatrix}.$$

Now, computing the exterior derivative of the $d\alpha$ equation modulo θ yields

$$(3.19) \qquad 0 = d(d\alpha) \equiv \left(dR - (\alpha + \bar\alpha)R - 2i(\overline{\sigma_0^1} - \sigma_0^1)\right) \wedge \eta \wedge \bar\eta \quad \mod \theta,$$

which implies

$$(3.20) \qquad dR \equiv (\alpha + \bar\alpha)R + 2i(\overline{\sigma_0^1} - \sigma_0^1) \quad \mod \theta, \eta, \bar\eta.$$

In particular, on each fiber of $B_1^{(1)} \to B_1$, the relation $dR = 2i\, d(\overline{s^1} - s^1)$ holds.

It follows that the equation $R = 0$ defines a G_2-structure $B_2 \subset B_1^{(1)}$ on B_1, where G_2 is the subgroup consisting of those matrices in $\mathfrak{g}_1^{(1)}$ for which s^1 is real.

Now pull back all of the forms and functions on $B_1^{(1)}$ to B_2, write $\sigma_0^1 = \sigma_0 + i\,\tau$ where σ_0 and τ are real 1-forms, and write the structure equations on B_2 in the

form

$$
(3.21) \qquad \mathrm{d}\begin{pmatrix}\theta\\\eta\\\alpha\\\beta\end{pmatrix} = -\begin{pmatrix}0&0&0&0\\0&0&0&0\\\sigma_0&0&0&0\\\sigma_0^2&\sigma_0&0&0\end{pmatrix}\begin{pmatrix}\theta\\\eta\\\alpha\\\beta\end{pmatrix} + \begin{pmatrix}-(\alpha+\bar\alpha)\wedge\theta + \mathrm{i}\,\eta\wedge\bar\eta\\-\beta\wedge\theta-\alpha\wedge\eta\\-\mathrm{i}\,\beta\wedge\bar\eta-2\mathrm{i}\,\bar\beta\wedge\eta-\mathrm{i}\,\tau\wedge\theta\\-\beta\wedge\bar\alpha-\mathrm{i}\,\tau\wedge\eta\end{pmatrix}.
$$

The congruence (3.20) now implies that $\tau = a\,\theta + b\,\eta + \bar b\,\bar\eta$ for some real function a and complex function b on B_2. By adding $\mathrm{i}a\,\eta$ to σ_0^2, it can be arranged that $a = 0$, but b cannot be absorbed.

The last two structure equations now read

$$
(3.22) \qquad \begin{aligned}
\mathrm{d}\alpha &= -\sigma_0\wedge\theta - \mathrm{i}\,\beta\wedge\bar\eta - 2\mathrm{i}\,\bar\beta\wedge\eta - \mathrm{i}\,(b\,\eta + \bar b\,\bar\eta)\wedge\theta,\\
\mathrm{d}\beta &= -\sigma_0^2\wedge\theta - \sigma_0\wedge\eta - \beta\wedge\bar\alpha + \mathrm{i}\,\bar b\,\eta\wedge\bar\eta,
\end{aligned}
$$

and it remains to determine how b varies on the fibers of $B_2 \to B_1$.

To do this, compute the exterior derivative of the first of these equations and write it in the form

$$
(3.23) \qquad \begin{aligned}
0 = \mathrm{d}(\mathrm{d}\alpha) = &-\left(\mathrm{d}\sigma_0 - (\alpha+\bar\alpha)\wedge\sigma_0 - \mathrm{i}\,\beta\wedge\bar\beta + \tfrac{1}{2}\mathrm{i}\big(\sigma_0^2\wedge\bar\eta - \overline{\sigma_0^2}\wedge\eta\big)\right)\wedge\theta\\
&-\mathrm{i}\left(\mathrm{d}b - (2\alpha+\bar\alpha)\,b + \tfrac{3}{2}\,\overline{\sigma_0^2}\right)\wedge\eta\wedge\theta\\
&-\mathrm{i}\left(\mathrm{d}\bar b - (2\bar\alpha+\alpha)\,\bar b + \tfrac{3}{2}\,\sigma_0^2\right)\wedge\bar\eta\wedge\theta
\end{aligned}
$$

The imaginary part of (3.23) implies that

$$
(3.24) \qquad \mathrm{d}b \equiv (2\alpha+\bar\alpha)\,b - \tfrac{3}{2}\,\overline{\sigma_0^2} \quad \mathrm{mod}\ \theta,\eta,\bar\eta,
$$

which implies that, on each fiber of $B_2 \to B_1$, an equation of the form $\mathrm{d}b = \tfrac{3}{2}\,\overline{\mathrm{d}s^2}$ holds.

In particular, the equation $b = 0$ defines a G_3-structure $B_3 \subset B_2$ on B_1, where G_3 is the 1-dimensional subgroup of G_2 defined by the equation $s^2 = 0$.

Now pull back all of the forms and functions involved to B_3. The structure equations take the form

$$
(3.25) \qquad \mathrm{d}\begin{pmatrix}\theta\\\eta\\\alpha\\\beta\end{pmatrix} = -\begin{pmatrix}0&0&0&0\\0&0&0&0\\\sigma_0&0&0&0\\0&\sigma_0&0&0\end{pmatrix}\begin{pmatrix}\theta\\\eta\\\alpha\\\beta\end{pmatrix} + \begin{pmatrix}-(\alpha+\bar\alpha)\wedge\theta + \mathrm{i}\,\eta\wedge\bar\eta\\-\beta\wedge\theta-\alpha\wedge\eta\\-\mathrm{i}\,\beta\wedge\bar\eta-2\mathrm{i}\,\bar\beta\wedge\eta\\-\beta\wedge\bar\alpha-\sigma_0^2\wedge\theta\end{pmatrix},
$$

where σ_0^2 is now basic. From the imaginary part of (3.23), this form σ_0^2 must satisfy $\big(\overline{\sigma_0^2}\wedge\eta + \sigma_0^2\wedge\bar\eta\big)\wedge\theta = 0$. This implies $\sigma_0^2\wedge\theta = (r\eta + s\bar\eta)\wedge\theta$, where r and s are real and complex functions, respectively, on B_3. By adding $r\theta$ to σ_0 and calling the result σ, it can be arranged that $r = 0$, and the structure equations become

$$
(3.26) \qquad \mathrm{d}\begin{pmatrix}\theta\\\eta\\\alpha\\\beta\end{pmatrix} = -\begin{pmatrix}0&0&0&0\\0&0&0&0\\\sigma&0&0&0\\0&\sigma&0&0\end{pmatrix}\begin{pmatrix}\theta\\\eta\\\alpha\\\beta\end{pmatrix} + \begin{pmatrix}-(\alpha+\bar\alpha)\wedge\theta + \mathrm{i}\,\eta\wedge\bar\eta\\-\beta\wedge\theta-\alpha\wedge\eta\\-\mathrm{i}\,\beta\wedge\bar\eta-2\mathrm{i}\,\bar\beta\wedge\eta\\-\beta\wedge\bar\alpha-s\,\bar\eta\wedge\theta\end{pmatrix}
$$

where, now, σ is uniquely specified by these conditions.

Thus, B_3 is endowed with a canonical $\{e\}$-structure and this constitutes Cartan's solution of the equivalence problem.

3.3.1. *Identities.* To complete the structure equations, however, a formula for $\mathrm{d}\sigma$ is needed. The $\mathrm{d}(\mathrm{d}\alpha) = 0$ equation now yields

$$(3.27) \qquad 0 = \big(\mathrm{d}\sigma - (\alpha + \bar\alpha)\wedge\sigma - \mathrm{i}\,\beta\wedge\bar\beta\,\big)\wedge\theta$$

so that $\mathrm{d}\sigma = (\alpha + \bar\alpha)\wedge\sigma + \mathrm{i}\,\beta\wedge\bar\beta + \rho\wedge\theta$ where ρ is a real 1-form.

Using this equation, the identity $\mathrm{d}(\mathrm{d}\beta) = 0$ expands to

$$(3.28) \qquad 0 = \mathrm{d}(\mathrm{d}\beta) = \rho\wedge\eta\wedge\theta - \big(\mathrm{d}s - (3\bar\alpha + \alpha)s\big)\wedge\bar\eta\wedge\theta,$$

from which it follows that there are complex functions u, p, and q on B_3 so that

$$(3.29) \qquad \mathrm{d}s = (3\bar\alpha + \alpha)s + u\,\theta + p\,\eta + q\,\bar\eta,$$

whence $\rho\wedge\theta = -(p\,\bar\eta + \bar p\,\eta)\wedge\theta$, so that

$$(3.30) \qquad \mathrm{d}\sigma = (\alpha + \bar\alpha)\wedge\sigma + \mathrm{i}\,\beta\wedge\bar\beta - (p\,\bar\eta + \bar p\,\eta)\wedge\theta.$$

The final Bianchi identity will follow from $\mathrm{d}\big(\mathrm{d}\sigma\big) = 0$, and this expands to give the statement that there exist functions a, r, and v on B_3, with r being real valued, so that

$$(3.31) \qquad \mathrm{d}p = (3\bar\alpha + 2\alpha)p - \mathrm{i}s\,\bar\beta + a\theta + r\eta + v\bar\eta.$$

3.4. **Conclusions.** Several conclusions can be drawn from these calculations.

3.4.1. *Automorphisms of CR-structures.* First of all, since the group of symmetries of a nondegenerate CR-structure on a 3-manifold embeds into the group of symmetries of an $\{e\}$-structure on an 8-manifold, it follows that the group of symmetries of such a CR-structure is a Lie group of dimension at most 8. Moreover, this maximum dimension can be reached only if the local symmetry group of the $\{e\}$-structure on B_3 acts with open orbits on B_3.

3.4.2. *Maximal symmetry.* However, by the structure equations, such open orbits exist if and only if the functions s and p are locally constant. The structure equation for $\mathrm{d}s$, however, shows that s cannot be locally constant unless it vanishes (which implies, in turn, that p vanishes as well).

In this case, the equations

$$(3.32) \qquad \begin{aligned}
\mathrm{d}\theta &= -(\alpha + \bar\alpha)\wedge\theta + \mathrm{i}\,\eta\wedge\bar\eta \\
\mathrm{d}\eta &= -\beta\wedge\theta - \alpha\wedge\eta \\
\mathrm{d}\alpha &= -\sigma\wedge\theta - \mathrm{i}\,\beta\wedge\bar\eta - 2\mathrm{i}\,\bar\beta\wedge\eta \\
\mathrm{d}\beta &= -\sigma\wedge\eta + \bar\alpha\wedge\beta \\
\mathrm{d}\sigma &= (\alpha + \bar\alpha)\wedge\sigma + \mathrm{i}\,\beta\wedge\bar\beta
\end{aligned}$$

are the structure equations of a Lie group of dimension 8.

Naturally, the reader will want to know which one. The simplest way to identify the group is to notice that there are no $\alpha\wedge\bar\alpha$ terms on the right hand side of these equations, but that α appears in the right hand side of all the equations except that of $\mathrm{d}\alpha$. This implies that the vector fields X and Y dual to the real and imaginary parts of α form a maximal torus of dimension 2 in the Lie algebra of infinitesimal symmetries of the coframing.

For any form ϕ in the coframing, define its X- and Y-weights by the formulae

$$(3.33) \qquad \begin{aligned}
w_X(\phi)\,\phi &= X \lrcorner \mathrm{d}\phi, \\
\mathrm{i}\,w_Y(\phi)\,\phi &= Y \lrcorner \mathrm{d}\phi.
\end{aligned}$$

Then, plotting the pairs $\big(w_X(\phi), w_Y(\phi)\big)$ in the plane as ϕ ranges over the basis $(\theta, \eta, \bar{\eta}, \alpha, \bar{\alpha}, \beta, \bar{\beta}, \sigma)$ reveals the characteristic hexagon of the roots of A_2. Moreover, because the roots are 'half-real', the actual real form of A_2 represented must be $\mathfrak{su}(2,1)$. Thus, the group must be $\mathrm{SU}(2,1)$.

In fact, when s vanishes identically, the structure equations can be written in the form $\mathrm{d}\gamma = -\gamma \wedge \gamma$ where

$$(3.34) \qquad \gamma = \begin{pmatrix} -\frac{1}{3}(2\alpha + \bar{\alpha}) & -\mathrm{i}\,\bar{\beta} & -\mathrm{i}\,\sigma \\ \eta & \frac{1}{3}(\alpha - \bar{\alpha}) & \mathrm{i}\,\beta \\ -\mathrm{i}\,\theta & \bar{\eta} & \frac{1}{3}(\alpha + 2\bar{\alpha}) \end{pmatrix}.$$

Note that γ takes values in $\mathfrak{su}(2,1)$, where the model of $\mathrm{SU}(2,1)$ being used is the subgroup of $\mathrm{SL}(3,\mathbb{C})$ that fixes the Hermitian form H in three variables

$$(3.35) \qquad H(Z) = Z_3\,\overline{Z_1} - Z_2\,\overline{Z_2} + Z_1\,\overline{Z_3}\,.$$

In particular, if M^3 is simply connected, there is a smooth map $F : B_3 \to \mathrm{SU}(2,1)$ that satisfies $F^*(g^{-1}\,\mathrm{d}g) = \gamma$. As the structure equations show, F maps each fiber of $B_3 \to M$ to a left coset of the parabolic subgroup $P \subset \mathrm{SU}(2,1)$ consisting of the upper triangular matrices in $\mathrm{SU}(2,1)$, i.e., the subgroup that fixes the H-null line $L_0 \subset \mathbb{C}^3$ defined by $Z_2 = Z_3 = 0$.

Now, $\mathrm{SU}(2,1)/P$ is naturally identified with the hypersurface $N^3 \subset \mathbb{CP}^2$ of H-null lines in $\mathbb{CP}^2$. Thus, F covers a map $f : M^3 \to N^3$ that is a local equivalence of CR-structures.

The conclusion is that every CR-structure with 8-dimensional infinitesimal symmetry algebra on a simply connected 3-manifold has a 'developing map' to N^3 that is unique up to composition with a CR-automorphism of this 'flat' structure.[8]

3.4.3. *The non-flat case.* In the general case, s is the coefficient of a tensor field that is well-defined on M.

The simplest such expression involving s is perhaps $Q = s\bar{s}\,\theta^4$, which is a well-defined section of $S^4\big(D^\perp\big)$ on M. This well-definedness follows since Q is manifestly semibasic and a computation using the structure equations reveals that, for any vertical vector field Y for the projection $B_3 \to M$, the Lie derivative of Q with respect to Y vanishes.

Also the expression $S = s\,\bar{\eta} \otimes \bar{\eta} \otimes \theta$ can be interpreted as a well-defined section of the bundle $S^{0,2}(D) \otimes D^\perp$ over M, i.e., the bundle of complex anti-linear quadratic forms on D with values in $D^\perp$.

Other combinations of the functions on B_3 make well-defined tensors on M as well, but have to be treated with more care. For example, the expression

$$(3.36) \qquad E = s\,\bar{\eta}^2 \circ \theta + 2\mathrm{i}p\,\bar{\eta} \circ \theta^2 \quad \mathrm{mod}\ \theta^3$$

describes a well-defined section of the quotient bundle $S^3(T^*M)/(D^\perp)^3$.

The verification of these statements will be left to the reader.

3.4.4. *Structure reduction in the non-flat case.* In the case where $Q = s\bar{s}\,\theta^4$ is nonvanishing on M, there is a canonical reduction of B_3 to a $\mathbb{Z}_2$-structure $B_4 \to M$ defined by the equations $s = -1$, $p = 0$, $u + \bar{u} = 0$.

This follows from the formulae for $\mathrm{d}s$ and $\mathrm{d}p$ together with the formula

$$(3.37) \qquad \mathrm{d}u \equiv (4\bar{\alpha} + 2\alpha)u + p\,\beta + 4s\sigma \quad \mathrm{mod}\ \theta, \eta, \bar{\eta},$$

---

[8]Explicitly computing this developing map requires solving a Lie equation of the form $\mathrm{d}g = g\gamma$ where γ is a known 1-form with values in $\mathfrak{su}(2,1)$.

which is derived from the identity $\mathrm{d}(\mathrm{d}s) = 0$.

Pulling all the given quantities back to B_4, writing $u = 2im$ where m is real and replacing q by $8\bar{q}$ for notational convenience, this results in equations

$$(3.38) \qquad \begin{aligned} \alpha &= im\,\theta - 3q\,\eta + \bar{q}\,\bar{\eta} \\ \beta &= ia\,\theta + iv\,\eta + ir\,\bar{\eta}, \end{aligned}$$

and in structure equations of the form

$$(3.39) \qquad \begin{aligned} \mathrm{d}\theta &= 2(q\eta + \bar{q}\bar{\eta}) \wedge \theta + i\,\eta \wedge \bar{\eta} \\ \mathrm{d}\eta &= t\,\eta \wedge \theta + \bar{q}\,\eta \wedge \bar{\eta} - ir\,\bar{\eta} \wedge \theta \end{aligned}$$

for some function t constructed out of the other invariants.

Under the $\mathbb{Z}_2$-action on the double cover $B_4 \to M$, the form θ is even while η is odd. Thus, the coframining (θ, η) is well-defined on M up to a replacement of the form $(\theta, \eta) \mapsto (\theta, -\eta)$. It also follows that t and r are even while q is odd.

In particular, it follows from this discussion that the group of symmetries of a nondegenerate CR-structure for which $Q \neq 0$ is a Lie group of dimension at most 3 and that this upper bound is reached only for homogeneous structures, in which case, the functions q, r, and t must be constants.

Indeed, if one assumes that these functions are constants, then computing the exterior derivatives of the above equations yields that $t + \bar{t} = 0$, so that $t = ib$, for some real constant b, and the equation $rq + b\bar{q} = 0$.

Conversely, any solution $(r, b, q) \in \mathbb{R}^2 \times \mathbb{C}$ of $rq + b\bar{q} = 0$ defines a homogeneous CR-structure. Cartan used this fact in his 1932 papers to classify the homogeneous CR-hypersurfaces in $\mathbb{C}^2$.

4. Generalizations

In this brief section, some remarks will be made about generalizations of these constructions to higher dimensions.

The reader may wonder just how much of this geometry for real hypersurfaces in unimodular complex surfaces can be generalized to higher dimensions. In fact, a great deal of it can be. Such generalizations are not the focus of this article, but the following comments may be a useful guide to the reader who wants to explore this further.

As the reader will no doubt realize, the geometric structures induced on hypersurfaces discussed here are quite different from the ones considered by Huisken and Klingenberg [7, 10]. Consequently, the corresponding flows are unrelated to the flows that they consider.

4.1. Hypersurfaces in unimodular complex manifolds. Let $n \geq 1$ be fixed and let X be a complex manifold of dimension $n{+}1$ and Υ be a nowhere-vanishing holmorphic $(n{+}1, 0)$-form on X. The pair (X, Υ) will be referred to as a *unimodular complex $(n{+}1)$-manifold*.

Let $M^{2n+1} \subset X$ be a real hypersurface that is smoothly embedded in X. (The immersed case does not differ substantially and lower regularity will suffice for the constructions to be carried out in this subsection, but these refinements will be left to the interested reader.) Let $\Phi = M^*\Upsilon$ be the pullback of Υ to M.

By linear algebra, there exists, at least locally on M, a nonvanishing 1-form $\theta = \bar{\theta}$ such that $\theta \wedge \Phi = 0$. This 1-form is unique up to a real multiple. The kernel of θ is a codimension 1 plane field $D \subset TM$ that is, by definition the complex tangent space

of M. One says that M is *nondegenerate* if $\mathrm{d}\theta$ is a nondegenerate 2-form on D, i.e., if $\theta \wedge (\mathrm{d}\theta)^n$ is nowhere-vanishing on the domain of definition of θ. In any case, $\mathrm{d}\theta$ restricts to D to be a real $(1,1)$-form. (Of course, this is, up to multiples, the Levi form of M.) One says that M is *strictly pseudoconvex* if it is possible to choose θ such that $\mathrm{d}\theta$ is a positive definite $(1,1)$-form on D. This latter requirement, if satisfiable, determines θ up to a positive multiple.

For simplicity, only the strictly pseudoconvex case will be discussed any further here. The generalization of Proposition 1 to all cases $n \geq 1$ is then

Proposition 6 (The canonical $\mathrm{SU}(n)$-structure)**.** *Let $M \subset X$ be a strictly pseudo-convex real hypersurface in a unimodular complex manifold (X^{n+1}, Υ) and let $\Phi = M^*\Upsilon$. Then there is a unique $\mathrm{SU}(n)$-structure on M whose sections $(\theta, \eta_1, \ldots, \eta_n)$ satisfy $\theta = \overline{\theta}$ and*

(1) $\Phi = \theta \wedge \eta_1 \wedge \ldots \wedge \eta_n$,
(2) $\mathrm{d}\theta = \mathrm{i}\big(\eta_1 \wedge \overline{\eta_1} + \cdots + \eta_n \wedge \overline{\eta_n}\big)$.

Proof. The proof follows exactly the same lines as that of Proposition 1, so there is no need to repeat it here. $\qquad\square$

Of course, Proposition 6 implies that there is a canonical normal associated to a strictly pseudoconvex real hypersurface, so there will be an evolution equation for such hypersurfaces generalizing the $n = 1$ case.

There is also a canonical volume form defined as $\frac{2^{-n}}{n!}\theta \wedge (\mathrm{d}\theta)^n$, which is the volume form associated to the canonical metric

$$(4.1) \qquad \mathrm{d}s^2 = \theta^2 + \eta_1 \circ \overline{\eta_1} + \cdots + \eta_n \circ \overline{\eta_n}.$$

As in the $n = 1$ case, the primary invariants are found by computing the derivatives of the remaining forms in a coframing of the $\mathrm{SU}(n)$-structure. Computation (i.e., expanding the identities $\mathrm{d}(\mathrm{d}\theta) = \mathrm{d}(\theta \wedge \eta_1 \wedge \cdots \wedge \eta_n) = 0$ and applying exterior algebra identities) yields that, for such a coframing $(\theta, \eta_1, \ldots, \eta_n)$, there are structure equations

$$(4.2) \qquad \begin{aligned} \mathrm{d}\theta &= \mathrm{i}\big(\eta_1 \wedge \overline{\eta_1} + \cdots + \eta_n \wedge \overline{\eta_n}\big), \\ \mathrm{d}\eta_j &= -\phi_{j\bar{k}} \wedge \eta_k + 2\mathrm{i}\,\theta \wedge \big(a\,\eta_j + b_{jk}\,\overline{\eta_k}\big), \end{aligned}$$

for some unique real-valued function $a = \bar{a}$ (independent of the choice of coframing), some unique complex functions $b_{ij} = b_{ji}$, and some unique 1-forms $\phi_{j\bar{k}} = -\overline{\phi_{k\bar{j}}}$ satisfying $\phi_{1\bar{1}} + \cdots + \phi_{n\bar{n}} = 0$.

The invariant a, which is fourth order, has the same 'mean curvature' interpretation relative to the canonical volume form as it does in the $n = 1$ case.

The quadratic expression $B = \overline{b_{jk}}\,\eta_j \circ \eta_k$ is also fourth order and well-defined, independent of choice of coframing. When B vanishes identically, one finds that a is constant. Moreover, the quadratic form $\mathrm{d}\sigma^2 = \eta_1 \circ \overline{\eta_1} + \cdots + \eta_n \circ \overline{\eta_n}$ descends to the leaf space Z of the Reeb vector field T on M to define a Kähler-Einstein metric on Z with associated Kähler form $\frac{1}{2}\mathrm{d}\theta$ and Ricci form $\mathrm{Ric}(\mathrm{d}\sigma^2) = 4na\,\mathrm{d}\sigma^2$. When $a \neq 0$, the original M can be recovered (up to a covering) as the circle bundle of $(n,0)$-forms of a fixed norm with respect to $\mathrm{d}\sigma^2$, regarded as a real hypersurface in the total space of the canonical bundle of Z (endowed with its tautological holomorphic $(n+1)$-form). When $a = 0$ and h is a (local) Kähler potential for the Ricci-flat Kähler metric $\mathrm{d}\sigma^2$, one can recover M locally as the hypersurface in $\mathbb{C} \times Z$

defined by $\mathrm{Im}(z^0) = h$, where the holomorphic $(n+1)$-form on $\mathbb{C} \times Z$ is $\Upsilon = \mathrm{d}z^0 \wedge \Psi$ and Ψ is a (locally defined) $\mathrm{d}\sigma^2$-parallel holomorphic volume form on Z.

Just as in the case $n = 1$, the function a and the tensor B are all of the fourth order invariants of the strictly pseudoconvex hypersurface under the action of the unimodular biholomorphism pseudogroup. However, when $n > 1$, these invariants and the invariants derived from them do not constitute a complete set of invariants.

The 1-forms $\phi_{j\bar{k}}$ are the connection forms (relative to the chosen coframing) of a canonical connection on the $\mathrm{SU}(n)$-structure. To generate a complete set of invariants in Cartan's sense, one must include, along with a and B and their covariant derivatives, the curvature tensor of the connection $\phi = (\phi_{j\bar{k}})$ (a tensor that is of fifth order) and its covariant derivatives.

The $\mathrm{SU}(n)$-structure and its canonical connection constitute the solution of the equivalence problem (in Cartan's sense) for strictly pseudoconvex real hypersurfaces in unimodular complex manifolds. Higher order invariants can be defined and studied by differentiating these equations.

There is also a generalization of the embedding results derived in the $n = 1$ case: Every real-analytic $\mathrm{SU}(n)$-structure on M^{2n+1} that satisfies the first order structure equations (4.2) (i.e., whose intrinsic torsion has this form) arises as the canonical $\mathrm{SU}(n)$-structure associated to a real-analytic, strictly pseudoconvex real hypersurface in a unimodular complex manifold of dimension $n+1$ and this realization is essentially unique. The proof is a straightforward application of the Cartan-Kähler Theorem and will be left to the reader.

4.2. Hypersurfaces in complex manifolds with a specified volume form.

Finally, it should be pointed out that one can even generalize the construction to cover the case real hypersurfaces in a complex $(n+1)$-manifold X endowed with a positive, real $(n+1, n+1)$-form, i.e., a volume form Ω (in the usual sense) on X.

The point is that, for a real hypersurface $M \subset X$ with definite Levi form, one can use Ω to make a canonical choice of θ as a real-valued 1-form on M whose kernel is the complex tangent space of M and whose associated volume form $\theta \wedge (\mathrm{d}\theta)^n$ is determined by Ω.

More precisely, let θ_0 be any 1-form on an open set $U \subset M$ whose kernel is the complex tangent bundle $D \subset TM$ and such that $\mathrm{d}\theta_0$ restricts to D to be a positive $(1,1)$-form. These conditions determine θ_0 up to multiplication by a positive function f on U. Let T_0 be the Reeb vector field on M associated to θ_0, i.e., $\theta_0(T_0) = 1$ and $\mathrm{d}\theta_0(T_0, Y) = 0$ for all tangent vector fields Y on U. Let f be any positive smooth function on U. Then the Reeb vector field T associated to $\theta = f\,\theta_0$ is of the form $T = (1/f)\,T_0 + S$ where S is a section of D. Since $\theta \wedge (\mathrm{d}\theta)^n = f^{n+1}\,\theta_0 \wedge (\mathrm{d}\theta_0)^n$ and since $M^*\big((\mathrm{i}T)\lrcorner\Omega\big) = f^{-1}\,M^*\big((\mathrm{i}T_0)\lrcorner\Omega\big)$, it follows that there is a unique choice of $f > 0$ such that $\theta \wedge (\mathrm{d}\theta)^n = M^*\big((\mathrm{i}T)\lrcorner\Omega\big)$. Once θ is fixed, its Reeb vector field T is determined, yielding a canonical normal vector field $\mathrm{i}T$ along M, which can then be used to define an evolution for such hypersurfaces that is invariant under the pseudogroup of biholomorphisms that preserve Ω.

However, the reader should be aware that this is, in some sense, not much of a generalization. In the first place, the pseudogroup of local biholomorphisms of X that preserve Ω will, generally, be 'smaller' than the unimodular biholomorphism group because the pseudogroup of biholomorphisms that preserves Ω must also preserve the $(1,1)$-form $\mathrm{Ric}(\Omega)$. Usually, this $(1,1)$-form will be nonzero, forcing the pseudogroup to preserve a (usually singular) complex foliation. On the other

hand, when $\mathrm{Ric}(\Omega)$ vanishes identically, the pseudogroup perserves a holomorphic volume form up to a constant multiple, so the pseudogroup is only slightly larger than the unimodular biholomorphism pseudogroup and the picture of the invariants is essentially the same as in the unimodular case. In the second place, the analysis of the local invariants is rendered somewhat more complicated because the geometry of the form $\mathrm{Ric}(\Omega)$ must be taken into account in writing down local normal forms and computing evolution identities. Most likely, this complication is more of a nuisance than a serious difficulty in studying the geometry of the evolution equation in this more general case, but this will be left for the interested reader to pursue.

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