A Strategy and Honesty Based Comparison of Preferential Ballot Voting Methods

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Abstract:

This paper presents an analysis of various preferential ballot voting systems based on the idea that voters should be encouraged to vote honestly and independently of the other votes cast. Random votes are simulated in three and four candidate elections with N voters, while a block of votes of size b, all of which are all the same, represents the votes of a subset of the electorate with a given preference. Given b and N, we examine the likelihood P that, for a variety of voting methods, it benefits this body of voters to cast a block of votes that does not represent their true preferences. We then view P as a function of the single variable $b/\sqrt{N}$, and compare the function P for various preferential ballot voting methods, noting which methods are more likely to encourage dishonest or strategic voting under different circumstances.

1. Acknowledgements:

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I would also like to thank the Duke mathematics department for the attention, guidance, opportunities, and education I have received as a student of mathematics for the past four years.

Finally, I'd like to thank my family and friends for their continued support during this project.

2. Introduction: The Imperfections of Voting Methods

Various preferential ballot voting methods have been examined as potential ways to determine the outcome of a many candidate election. These methods aim to gain information from the electorate in order to choose a candidate as the winner. It is not always clear how to interpret voter preferences, and various election methods satisfy different properties regarding these preferences. Arrow's theorem shows that no election method satisfies the criteria of universality, monotonicity, independence of irrelevant alternatives, citizen sovereignty, and non-dictatorship. As such, various methods are compared based on their strengths, weaknesses, and various properties they do satisfy, as there is no perfect method.

Recent years have seen various multiple elections with more than two candidates of great influence. The three candidate US presidential elections of 1992 and 2000 demonstrate the importance of the voting method chosen. In both cases, the candidate receiving the least votes still received enough votes to potentially influence the outcome of the election. From this it is clear that the marginal preferences of these voters among the two more popular candidates is important and should be accounted for. While the Single Vote Plurality method, used in each state for US presidential elections, does not take these preferences into account, other preferential ballot method do, and do not always yield the same results. Previous approaches compare methods based on properties they satisfy, while this paper attempts to approach the problem from an angle of strategy.

3. Overview of Voting Methods and Properties

This section will give a brief overview of the five voting methods we examine in this paper. For more complete descriptions with examples and properties of these methods, the reader should look to Wright [1] and Tabachnik [2], each of which spends a great deal of time defining and deriving properties of these methods and others.

3.1. Single Vote Plurality

Single Vote Plurality is perhaps the most well known voting method. Among any number of candidates, each voter chooses exactly one, and casts their vote. The votes are tallied and the candidate with the most (first place) votes wins.

3.2. Ranked Pairs

The ranked pairs method, also called the Tideman method, is a margin of victory matrix method. Given a margin of victory matrix M, where $M_{i,j}$ is given by the number of voters who prefer candidate i to candidate j, each pairwise matchup is ranked by their margin of victory. In order, each matchup is mandated unless it would cause an intransitive cycle, in which case it is removed. Once completed, the method leaves a clear ordering, as no cycle is allowed to exist. The first candidate in the ordering is declared the winner.

3.3. Borda Count

Borda count is a quantitative ballot method where candidates are given points based on where they were rated on each voter's ballot. In an election with N candidates, a voter's top choice will receive N points, their next choice, N-1, and their last choice one point. It should be noted that any point system given by an arithmetic progression will yield the same results. Borda count is used in the weekly Associated Press Poll to determine the top 25 college teams in various sports.

3.4. Instant Runoff Voting

Instant runoff voting is another commonly known voting method, using an iterative approach. Voters use a preferential ballot to rank their preferences. At each iteration, the votes are tallied by taking each voter's first choice that has not already been eliminated. The candidate receiving the fewest votes is then eliminated, and the votes are recounted excluding that candidate. The method continues until only one candidate remains, who is then declared the winner.

3.5. Instant Runoff Borda Count

Instant runoff borda count is simply the instant runoff iteration idea applied to the borda method. In each iteration, the candidate with the fewest points is eliminated from the election, and the votes are recalculated. The last remaining candidate is the winner.

4. The Function P: The Likelihood of Strategic Voting

4.1. Assumptions and Definition

The main aim of this paper is to compare different voting methods based on how often they encourage dishonest voting behavior. We now pose the question more precisely. Given:

- An election with K candidates
- N total Voters
- A preferential ballot voting method M
- A block X of b identical votes, with one honest preference or vote
- N – b randomly generated votes

We define $P^{C}_{M}(b,N)$ to be the probability that a dishonest vote from the block X could result in a winner that X prefers to the winner that would result from X voting honestly. Due to candidate symmetry, we will assume without loss of generality that X has an honest preference of the candidates in order. For example, in an election with 3 candidates A, B, and C, X will prefer A first, B second, and C last.

4.2. An Example and Some Properties of P

We first give an example of an election where a dishonest strategic vote by the block X will result in a preferable outcome for X. Consider a 3 candidate plurality election with 100 total voters and a block X of five votes. In other words, let K = 3, N = 100, M = single vote plurality, and b = 5. Now consider the following outcome assuming X votes honestly:

Candidate	Votes
A	10
B	43
C	47

In this result, candidate C wins the election, which is the worst possible outcome according to the preferences of X. Since X voted honestly, we know that five votes were cast in favor of candidate A. Now suppose X had voted in favor of its second choice candidate, B. The five votes would be taken from A and given to B, yielding the following result:

Candidate	Votes
A	5
B	48
C	47

Now candidate B is the winner, which is a better outcome for X. This example is similar in nature to the aforementioned presidential elections of 1992 and 2000, where candidate A is the least popular candidate. In both cases, if even a relatively modest portion of candidate A's supporters with the same second choice voted for their second choice over candidate A, they may have been able to influence the election in their favor.

While examples of these situations abound for all of the voting methods we study, we intuitively note two qualitative properties the situation must satisfy. In order to have influence, b must be sufficiently large relative to N, lest X be drowned out by the variance in the random votes, but b must also not be too large, as X could then simply dominate the election with an honest vote. In particular, it is clear that for every voting method M which satisfies the majority criterion:

$P^K_M(b,N) = 0$ for $b > N/2$, for any K.

This is simple to verify, since X may simply vote honestly, and candidate A will receive the majority of the votes, forcing a victory for A, the best outcome for X.

Then we may infer that for $P^K_M(b,N)$ to achieve a high value, b must be significant relative to N, but not too large. This inference is affirmed in our experimental calculations of $P^K_M(b,N)$.

4.3. Calculation of P by Simulation

It is very difficult to work with $P^K_M(b,N)$ analytically, as it depends on 3 variables and which voting method is used. Because it is difficult to make general statements about P, we look to computational method to allow us to estimate its value for various parameters and methods. We therefore estimate $P^K_M(b,N)$ using the following procedure:

1. Simulate N-b random votes
2. Compute the winner using method M using the honest vote for X.
3. Compute the winner using method M for all other possible votes for X.
4. If the any of the winners from step 3 are preferable to the winner using the honest vote, then it is advantageous for X to vote dishonestly
5. Repeat many times (in our calculations, 10,000 times), and compute $P^K_M(b,N)$ as the proportion of times that it was advantageous for X to vote dishonestly

As an example, Figure 1 shows the results of such a calculation for Ranked Pairs and Plurality voting, where N and K are fixed and several b are chosen. The authors performed these calculations in MATLAB. We will see in the next section why it is appealing to view P as a curve in terms of the block size b, rather than attempting to examine it as a surface of both b and N. Our initial suspicions about the impact of block size are affirmed in this chart, as P increases with b initially, but then decreases past a certain point, essentially disappearing for the larger b. We note that Ranked Pairs seems to encourage dishonest voting less than Plurality for small b, but as b increases, the opposite becomes true.

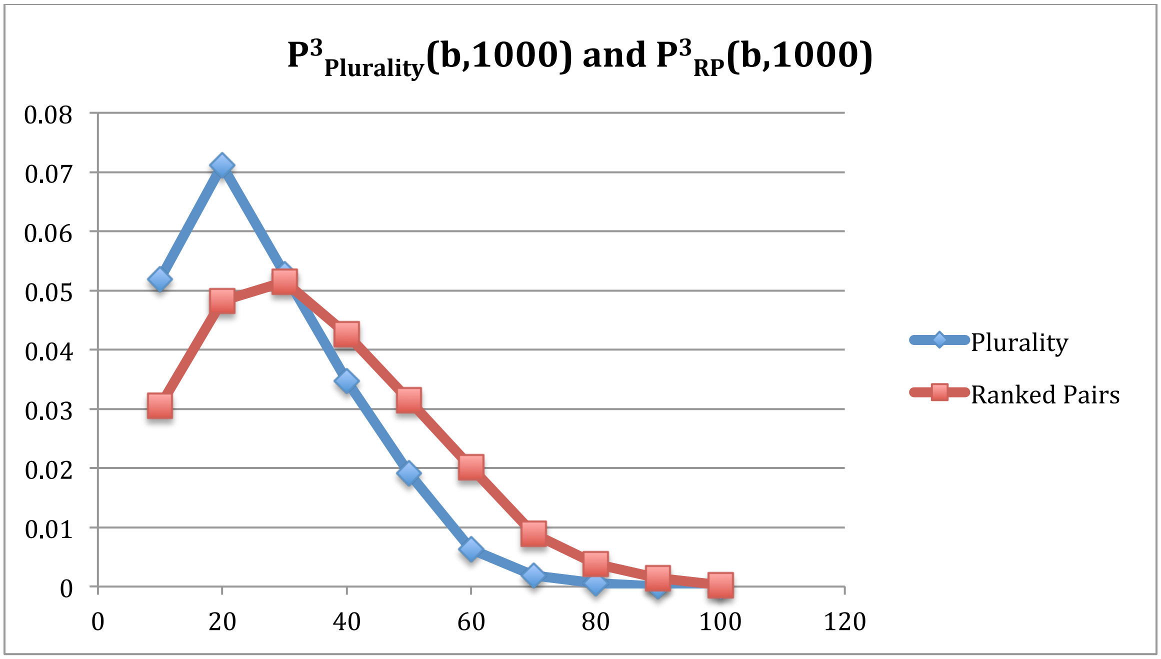

Figure 1: Graphs of P for Ranked Pairs and Plurality voting. N = 1000, K = 3, b = 10, 20, … 100. 10,000 simulations.

Here we see that for b = 20, approximately 7% of elections will result in a scenario where X should vote dishonestly using the plurality method. For b = 30, using Ranked Pairs, we see that about 5% of elections will results in a scenario where X is incented to vote dishonestly.

5. Block Size in Units of $\sqrt{N}$

5.1. Motivation

In order to simplify our analysis of P, we attempt to relate the impact of b to its value relative to N. Intuitively; P should depend somehow on the size of b relative to N, rather than depending on either absolutely on either of them. But how do we measure the importance of b votes among N votes? There are many options. We could consider b as a percentage of N, for example. In the next section, we provide a relatively simple rationale for measuring b in units of the square root of N, thus providing a sensible way of comparing the relative values of b and N that is unanimously supported by our data.

5.2. Three Candidate Plurality

In a Single Vote Plurality Election, only the first choice on each ballot is considered, with one vote going to that choice. Consider a 3 candidate election with N-b randomly generated votes, we may then consider the votes received by each candidate as a Trinomial random variable with N-b trials with $p_1 = p_2 = p_3 = 1/3$. If we denote the number of votes received by candidate A as #A, we have:

$$\begin{bmatrix} \#A \\ \#B \\ \#C \end{bmatrix} \sim Trinomial\left(N-b, \begin{bmatrix} 1/3 \\ 1/3 \\ 1/3 \end{bmatrix}\right)$$

Here, #A, #B, #C are correlated binomials with N-b trials and p = 1/3. Therefore we can compute the expectation and variance of each, as well as their covariances:

$$E(\#A) = \frac{N-b}{3}$$

$$Var(\#A) = \frac{2(N-b)}{9}$$

$$COV(\#A, \#B) = -\frac{N-b}{9}$$

Note that all three candidates have the same expected value and variance in the number of votes they receive, by symmetry, and all pairs of different candidates have the same covariance. When considering the influence of the block in determining the winner of a Plurality election, we aim to know the probability that b is greater than the difference between the number of votes received first and second place candidates. Since we are considering differences of votes, we compute:

$$E(\#A - \#B) = E(\#A) - E(\#B) = \frac{N-b}{3} - \frac{N-b}{3} = 0$$

$$Var(\#A - \#B) = Var(\#A) + Var(\#B) - 2COV(\#A, \#B) = \frac{2(N-b)}{3}$$

Again, we note that the we obtain the same results for any pair of candidates. Therefore we note that the difference of votes between two candidates is binomial with mean 0 and standard deviation proportional to $\sqrt{N-b}$. Then, as n goes to infinity, we use the normal approximation to obtain:

$$(\#C - \#A, \#C - \#B) \sim MVN\left(\begin{pmatrix} 0 \\ 0 \end{pmatrix}, \frac{N-b}{3}\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}\right)$$

Where MVN is the multivariate normal distribution (bivariate in this case) with mean vector and covariance matrix given by our calculations above. Now let:

$$z = b/\sqrt{N}$$

Then we have:

$$(\#C - \#A, \#C - \#B) \sim MVN\left(\binom{0}{0}, \frac{N - z\sqrt{N}}{3}\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}\right) \approx MVN\left(\binom{0}{0}, \frac{N}{3}\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}\right)$$

We may max the second approximation since as N tends to infinity, it will dominate any factor of its square root. Now consider the case when X is incented to vote dishonestly. In terms of the votes cast before V, this is when C beats A by more than b votes, and C beats B by less than B votes. Then V will cast its ballots for B, its second choice, and B will win over C, V's third choice. That is:

$$\#C - \#A > b\ , b > \#C - \#B > 0$$

Finally, we calculate p for large N:

$$p^3_{Plurality}(b, N) = \int_b^\infty \int_0^b \varphi_{\binom{0}{0}, \frac{N}{3}\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}}(x, y)dydx$$

$$= \int_{z\sqrt{N}}^\infty \int_0^{z\sqrt{N}} \varphi_{\binom{0}{0}, \frac{N}{3}\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}}(x, y)dydx = \int_z^\infty \int_0^z \varphi_{\binom{0}{0}, \frac{1}{3}\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}}(x, y)dydx$$

$$= \phi_{\binom{0}{0}, \frac{N}{3}\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}}(0, -z) - \phi_{\binom{0}{0}, \frac{N}{3}\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}}(-z, -z)$$

Where φ and ϕ are the probability density function and the cumulative distribution function for the multivariate normal, respectively. The second to last equality is due to a simple change of variables, very much the one used to standardize the normal. We now have p written as a function only of z for large N. This approach may be generalized to plurality for any number of candidates, yielding:

$$\lim_{n \to \infty} p^k_{Plurality}(z\sqrt{N}, N) = (k - 2) * I_1 - \sum_{i=2}^{k} I_i$$

Where:

$$I_j = \phi_{\binom{0}{0}, \frac{N}{3}\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}}(0,0, \ldots 0, -z, -z, \ldots)$$

Where the first k-j-1 components are 0, and the rest (the other j components) are –z.

5.3. Transforming the Data. P as a function of b/$\sqrt{N}$

In the last section, we showed for large N that we may view P as a function of a single variable b/$\sqrt{N}$ in the case of plurality. We now look to the data to check the validity of this approximation.

Figure 2 shows 3 curves of P for a 3 candidate Plurality election in terms of b, each for a fixed N. We could plot them as cross sections of a P surface, but we instead attempt to standardize the magnitude of the effect of each block size for each N dividing it by $\sqrt{N}$.

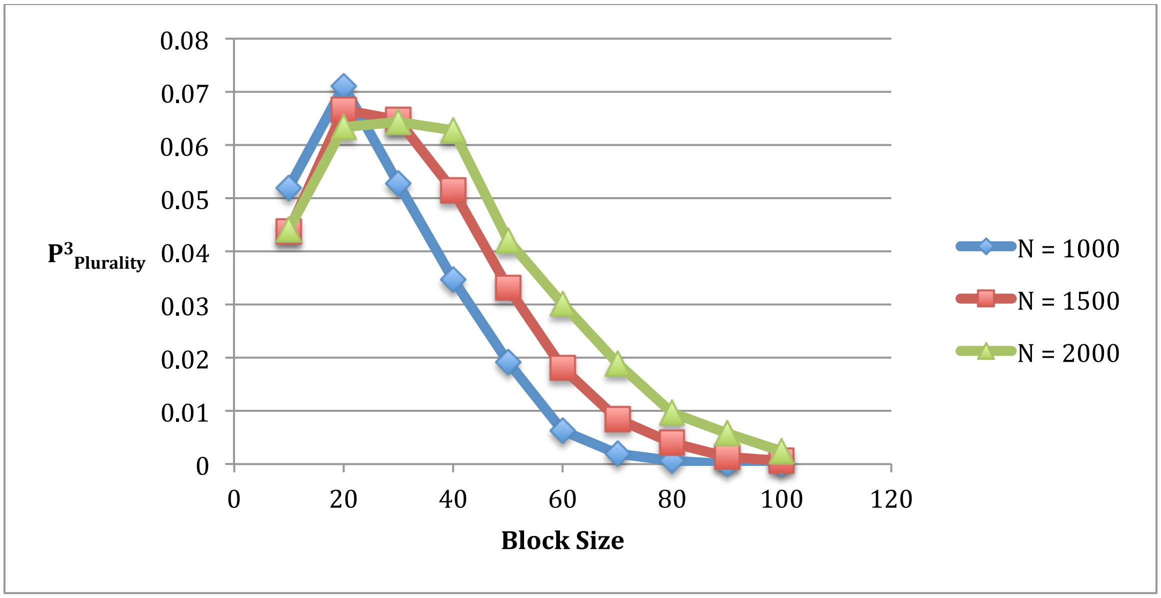

Figure 2: Graphs of P for Plurality voting. N = 1000, 1500, 2000, K = 3, b = 10, 20, … 100. 10,000 simulations.

Figure 3 shows the same data as figure 2, but with each curve rescaled so the X axis is represented by our measure b/$\sqrt{N}$. Under this transformation, the curves become nearly identical! This is encouraging, as it allows us to easily calculate P for any block size and number of votes, however large, by simply calculating P for a b and N with where b has the same ratio to $\sqrt{N}$. This is a valuable tool, as it is computationally very expensive to simulate elections with millions of voters or more, as is often the case in real world applications. We will see in the following sections that this property empirically holds for all voting methods considered in this paper. We also note our theoretically obtained formula for p as a function of z is nearly identical to our simulated results, as seen in figure 4. Figure 5 contains plots of our analytic solution for a plurality election with many candidates. For large numbers of candidates, these curves are computationally expensive to simulate, as the number of different ballots cast we must check is k!.

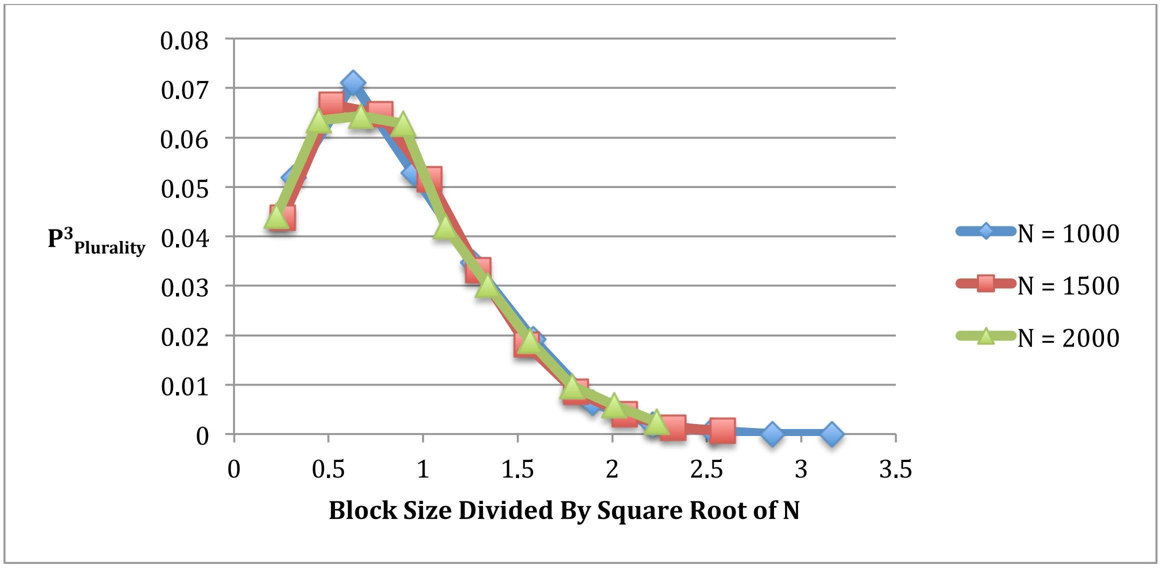

Figure 3: Graphs of P for Plurality voting. N = 1000, 1500, 2000, K = 3, b = 10, 20, ... 100. 10,000 simulations. X values for each curve scaled by the square root of N

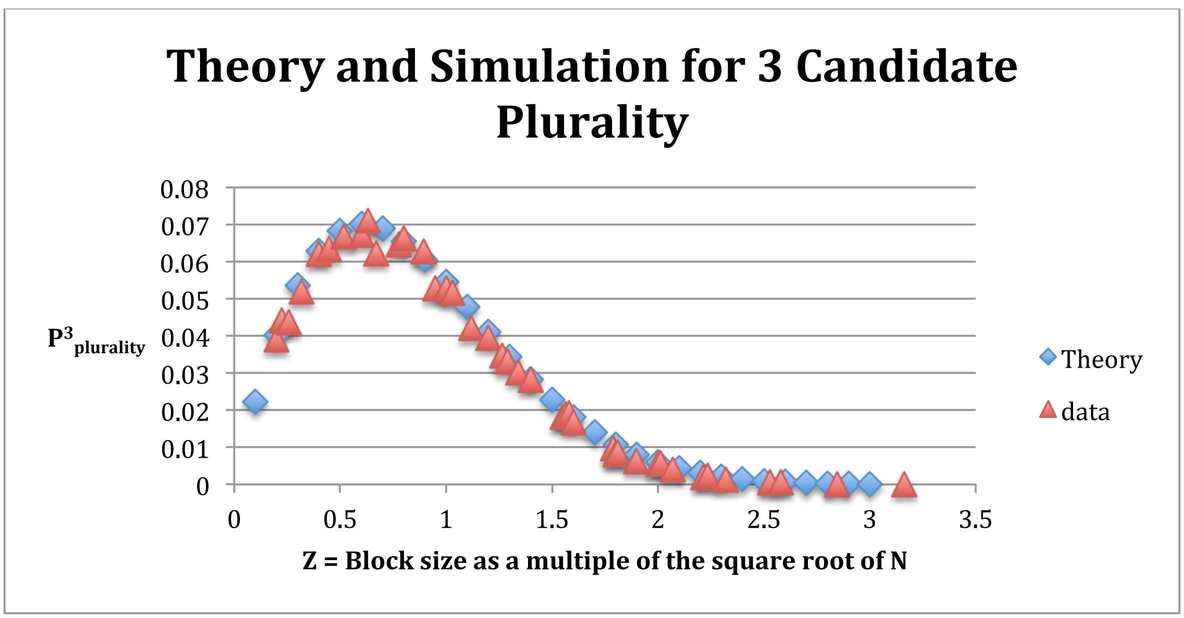

Figure 4: Graphs of derived formula for p and simulated data for 3 candidate plurality. N = 500, 1000, 1500, 2000. b = 10, 20, ... 100. 10,000 simulations,

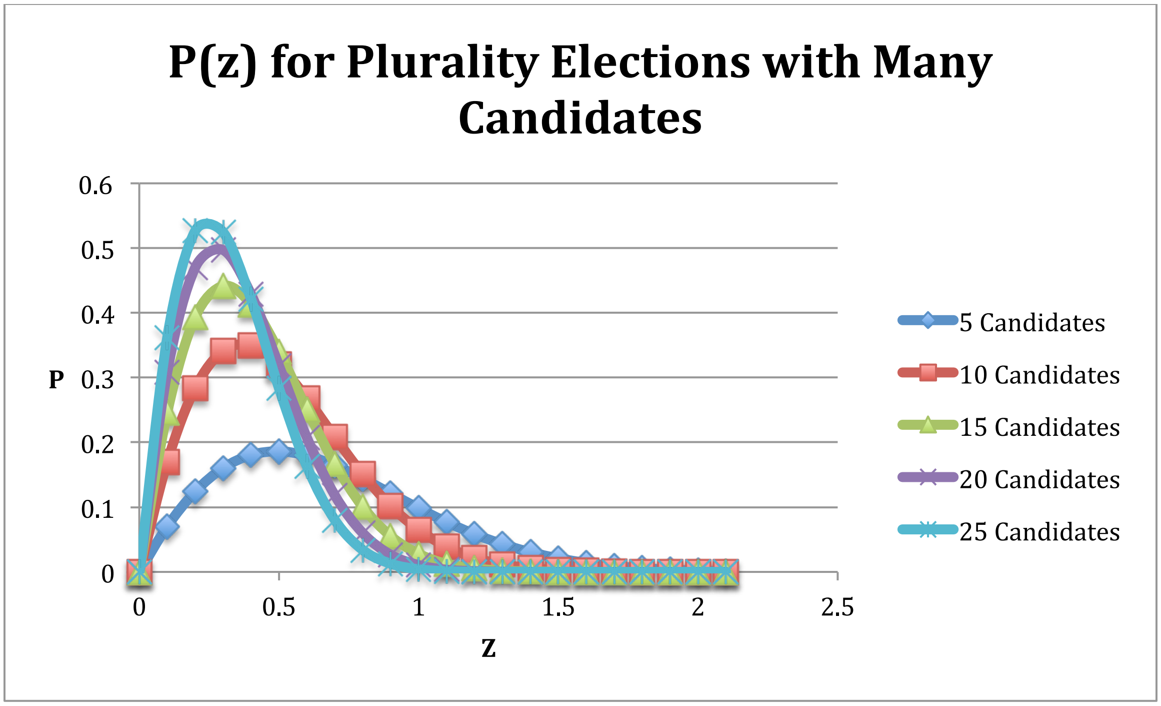

Figure 5: P for high numbers of candidates. The fast growth of k! would make it infeasible to estimate these curves (except for 5 candidates) by simulation. We are still bound computationally by our ability to compute the k-1 dimensional normal cdf.

5.4. Conjectures Concerning P as a Function of $b/\sqrt{N}$

While our data supports our reduction of P to a univariate function, we have not proven that this is always true. In particular, our observations from simulation lead us to conjecture the following:

For any method M with certain properties, we may take the following limit:

$$\lim_{n\to\infty} p_M^k\left(z\sqrt{N}, N\right) = p_M^k(z)$$

Then we are able to determine P approximately for any b, N, no matter how large, simply by determining z and using this approximation.

We note that our method of computing P theoretically was not completely specific to plurality. As we assume random voting for all methods used, we see that all P will be integrals over normal distributions, leaving us only to determine the domain over which to integrate. This fact gives more reason to believe that considering P as a univariate function of z is possible for large N, as a similar change of variables may work. For methods other than plurality, these domains become much more complicated. Further study would obtain these domains in order to determine P analytically for large N for various other voting methods.

We also wonder about the limiting behavior of P for elections with many candidates. In other words, as k tends to infinity, does $P^k(z)$ approach a constant curve? Given our analytic result for plurality, we see that the maximum value achieved by P is increasing in k, but this peak is achieved closer and closer to the origin, where P must always equal zero.

6. Results

6.1. A First Look

We are pleased to find that our consideration of b in units of $\sqrt{N}$ experimentally yields a standardized way of considering the relative influence of the block size b on the outcome of the election for all of the voting methods considered. We will refer to z as $b/\sqrt{N}$. Figure 6 shows the simulated estimates of P for three candidate elections using the five different voting methods specified. Figure 7 shows the simulated estimates of P for four candidate elections using the five different voting methods specified.

We see very similar qualitative results for both 3 and 4 candidate elections. In both cases, Borda count is most likely to yield scenarios in which X should vote dishonestly by far, for all z. The other methods are far more similar. Interestingly the curves for Instant Runoff Voting and Instant Runoff Borda are nearly identical over the entire domain. The Ranked Pairs curve approaches the IRV and IRBorda curves as z increases, and essentially coincides with them for z greater than 1.5. For z less than 1, Ranked Pairs is least likely to promote dishonest voting, followed by the instant runoff methods, and then plurality. For z greater than 1, plurality is the method which encourages the least dishonesty, followed by ranked pairs, and then the instant runoff methods. Despite this, we note that plurality has the highest maximum value for P, other than Borda. Ranked Pairs has the lowest maximum value for P.

The P curves for four candidate elections are qualitatively almost identical, with all of the same relationships from the previous paragraph holding true. We note, however, that the values of P are much higher. A likely suspect of this is that the number of different possible ballots for four candidates is 24, rather than the 6 for 3 candidates. The greatly increased number of different votes X can choose from allows for a greater likelihood that they will find a way to strategically benefit their outcome.

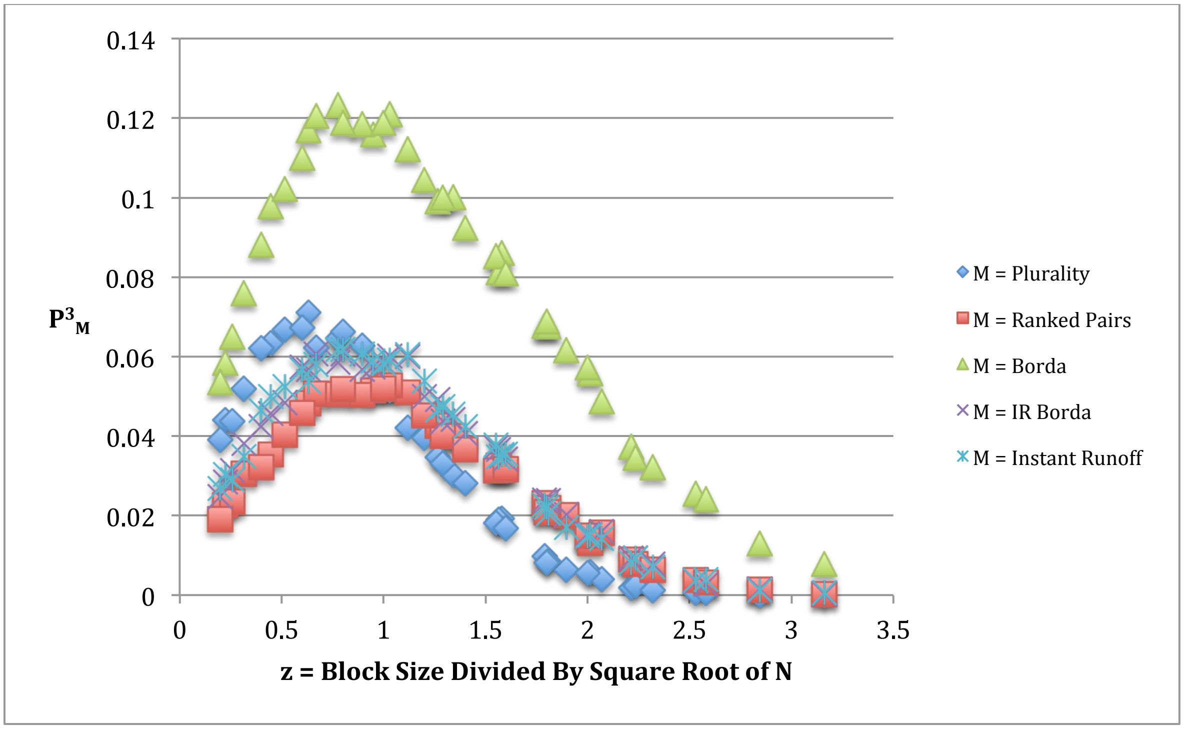

Figure 6: Graphs of P for each of 5 voting methods. N = 1000, 1500, 2000, 2500. 3 Candidates, b = 10, 20, … 100. 10,000 simulations.

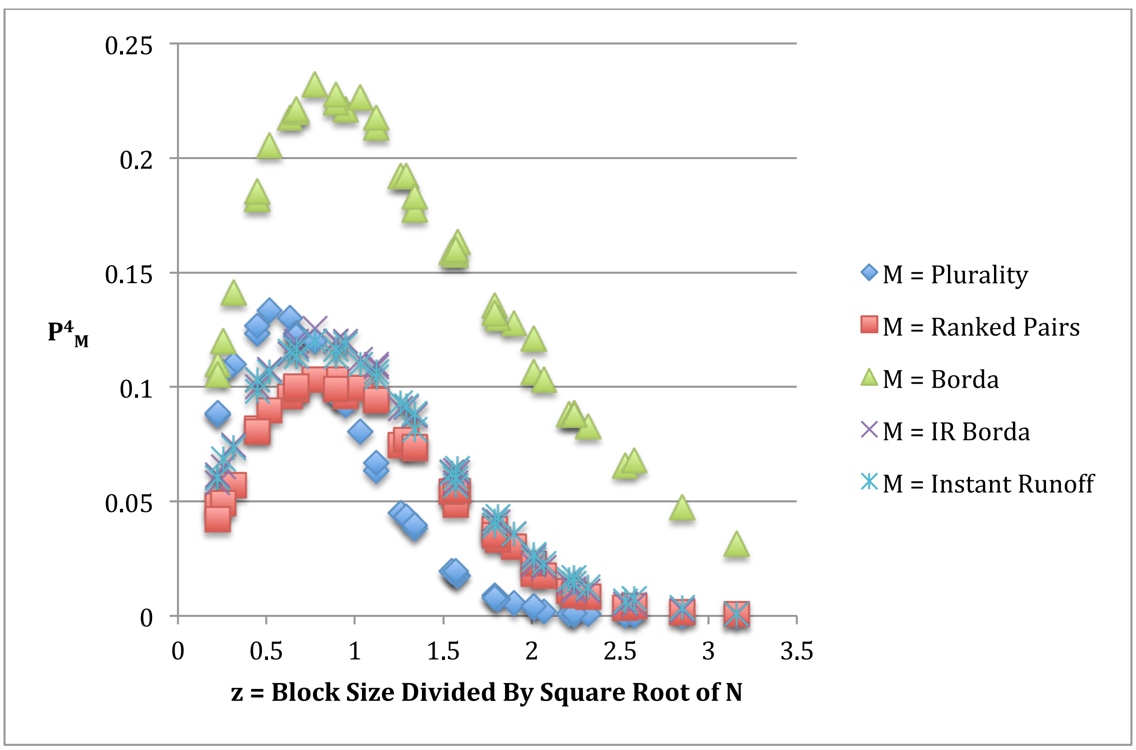

Figure 7: Graphs of P for each of 5 voting methods. N = 1000, 1500, 2000, 4 Candidates, b = 10, 20, … 100. 10,000 simulations.

6.2. Ways to Choose a Method. Metrics

Looking at our curves for P qualitatively, we may begin to see which method we might prefer to use when trying to minimize dishonest votes from an electorate. If very small voter blocks are likely to form in the electorate, then Ranked Pairs is likely to be the best choice. If larger blocks are likely to form, the Plurality may be the best performer. Among these five methods, Plurality and Ranked Pairs seem to be the only logical choices for the best method for minimizing dishonesty, as Ranked Pairs performs at least as well as all other methods across all z, except for Plurality.

For the unsure reader, we propose three simple metrics that may be used to evaluate these methods.

The first metric is the maximum metric. True to its name, this metric is given by the maximum value of P^K_M for a method M in a K candidate election. By this metric, ranked pairs is the best method, as it has the lowest maximum of all of the methods, and Plurality voting is the second worst (after Borda). Using this metric, the reader would choose Ranked Pairs, as it minimizes the maximum probability that dishonest voting would be advantageous.

Another metric we propose is the integral metric. This relies on our prior conjecture that P be at least a measurable function in z, if not continuous as we have conjectured. For purposes of calculating this metric using our simulated values, we simply linearly spline the data, and take the integral of the resulting curve.

K = 3	Maximum	Secant	Integral
Plurality	0.0711	0.112411067	0.073354
Ranked Pairs	0.0528	0.051262136	0.073386
Borda Count	0.1233	0.159096774	0.198555
IR Borda	0.0607	0.090488968	0.084329
Instant Runof	0.0618	0.079741935	0.085046

K = 4	Maximum	Secant	Integral
Plurality	0.1333	0.258333333	0.120389
Ranked Pairs	0.1033	0.115548098	0.134657
Borda Count	0.2322	0.299612903	0.397978
IR Borda	0.1255	0.161935484	0.161355
Instant Runof	0.12	0.15483871	0.160459

Figure 8: Values of three P-curve metrics for each voting method

Our last metric is the secant metric, which is given by the slope of the line segment from the origin to the maximum point of the curve P for a given method. This metric rewards methods whose maxima are smaller, and occur for larger block sizes, and are therefore more stable against smaller blocks. The results of these metrics are given in figures 8.

7. Conclusions and Further Questions

This paper examined the likelihood that a block of voters would be able to influence an election to their advantage by voting dishonestly, depending on the size of the block relative to the total electorate, and the voting method used to calculate the winner. We found that the Ranked Pairs method performed at least as well as all of the other methods except for plurality under all circumstances. Plurality was the least likely method to encourage dishonesty for large block sizes, but was among the more likely to encourage dishonest voting for small blocks.

This paper explored a new way of comparing different preferential ballot voting methods, placing importance on voters not being rewarded for voting dishonestly or penalized for voting honestly. We have found many interesting phenomena, but leave many questions unanswered. In particular, our conjectures from section 5.4., concerning P as a function of the single variable z, are especially interesting.

Overall, we hope that these methods, and any further work done on this topic may yield a better understanding of the dynamics of voter strategy, and which voting methods allow encourage more strategy rather than simple honest voting.

8. References

Barry Wright III, Objective measures of preferential ballot voting systems (2009)

Max Tabachnik, An analysis of preferential ballot voting methods (2011)