Algebraic De Rham Theory for Completions of Fundamental Groups of Moduli Spaces of Elliptic Curves

by

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Dissertation submitted in partial fulfillment of the requirements for the degree of
Doctor of Philosophy in the Department of Mathematics
in the Graduate School of Duke University
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Abstract

To study periods of fundamental groups of algebraic varieties, one requires an explicit algebraic de Rham theory for completions of fundamental groups. This thesis develops such a theory in two cases. In the first case, we develop an algebraic de Rham theory for unipotent fundamental groups of once punctured elliptic curves over a field of characteristic zero using the universal elliptic KZB connection of Calaque-Enriquez-Etingof and Levin-Racinet. We use it to give an explicit version of Tannaka duality for unipotent connections over an elliptic curve with a regular singular point at the identity. In the second case, we develop an algebriac de Rham theory for relative completion of the fundamental group of the moduli space of elliptic curves with one marked point. This allows the construction of iterated integrals involving modular forms of the *second kind*, whereas previously Brown and Manin only studied iterated integrals of *holomorphic* modular forms.

To my parents

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1

Introduction

This work is focused on studying periods of fundamental groups of algebraic varieties defined over $\mathbb{Q}$. To get a theory for periods of fundamental groups, one needs an algebraic de Rham theory for their various completions, such as unipotent completion and relative (unipotent) completion. Although there are abstract results establishing the existence of their algebraic de Rham structures [13], one needs concrete constructions and explicit formulas to study the periods. Such explicit constructions were previously known only for unipotent completions of fundamental groups of hyperplane complements. In this thesis, we develop an explicit algebraic de Rham theory for unipotent completions of fundamental groups of once punctured elliptic curves, and for the relative completion of the modular group.

1.1 Periods

Periods can be regarded as coefficients of a comparison isomorphism

$$\mathrm{comp} : V^{\mathrm{dR}} \otimes_{\mathbb{Q}} \mathbb{C} \xrightarrow{\approx} V^{\mathrm{B}} \otimes_{\mathbb{Q}} \mathbb{C}$$

between the complexifications of two finite dimensional $\mathbb{Q}$-vector spaces: a de Rham vector space $V^{\mathrm{dR}}_{/\mathbb{Q}}$ and a Betti vector space $V^{\mathrm{B}}_{/\mathbb{Q}}$. If $\gamma \in \check{V}^{\mathrm{B}}_{\mathbb{Q}} = \mathrm{Hom}(V^{\mathrm{B}}, \mathbb{Q})$ and $\omega \in V^{\mathrm{dR}}_{\mathbb{Q}}$, then $\langle \mathrm{comp}(\omega), \gamma \rangle$ is a period. Classically, when $X_{/\mathbb{Q}}$ is a smooth variety over $\mathbb{Q}$, $V^{\mathrm{dR}} = H^m_{\mathrm{dR}}(X_{/\mathbb{Q}})$ is the algebraic de Rham cohomology of X and $V^{\mathrm{B}} = H^m(X(\mathbb{C}); \mathbb{Q})$ is the Betti (i.e. singular) cohomology of its corresponding complex analytic variety $X(\mathbb{C})$. Grothendieck's algebraic de Rham theorem provides the comparison isomorphism, which is induced by integration. If $\gamma \in \check{V}^{\mathrm{B}}$ and $\omega \in V^{\mathrm{dR}}$, then we have

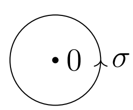
$$\mathrm{comp} : \omega \mapsto (\gamma \mapsto \int_\gamma \omega),$$

and $\displaystyle\int_\gamma \omega$ is a period.

Example 1.1.1. Let $\mathbb{G}_m = \mathrm{Spec}\, \mathbb{Q}[x, x^{-1}]$. Consider its algebraic de Rham cohomology and Betti cohomology:

$$V^{\mathrm{dR}} = H^1_{\mathrm{dR}}(\mathbb{G}_m) = \mathbb{Q} \cdot \frac{dx}{x}$$
$$V^{\mathrm{B}} = H^1_{\mathrm{B}}(\mathbb{G}_m(\mathbb{C}); \mathbb{Q}) = \mathbb{Q} \cdot \check{\sigma}$$

where $\mathbb{G}_m(\mathbb{C}) = \mathbb{C}^*$ denotes the corresponding complex analytic variety, and $\sigma \in \check{V}^{\mathrm{B}}$ is the loop wrapping around 0.



In this case, Grothendieck's comparison isomorphism takes $\frac{dx}{x}$ to $2\pi i \check{\sigma}$. The period is

$$\langle \mathrm{comp}\left(\frac{dx}{x}\right), \sigma \rangle = \int_\sigma \frac{dx}{x} = 2\pi i.$$

Other interesting examples of periods include elliptic integrals such as

$$\int_0^1 \frac{dx}{\sqrt{x(x-1)(x-t)}} \quad \text{and} \quad \int_0^1 \frac{x dx}{\sqrt{x(x-1)(x-t)}} \quad \text{where } t \in \overline{\mathbb{Q}} \text{ and } t \neq 0, 1,$$

special values of L-functions, and multiple zeta values

$$\zeta(n_1,\cdots,n_r)=\sum_{0<k_1<\cdots<k_r}\frac{1}{k_1^{n_1}\cdots k_r^{n_r}}\quad\text{where } n_j\geqslant 1, n_r\geqslant 2.$$

In particular, multiple zeta values can be viewed as periods of fundamental groups and written as iterated integrals, which we now recall.

1.2 Iterated Integrals

Let X be a complex manifold. Given a path $\gamma:[0,1]\to X$ and 1-forms ω_j on X, $j=1,\cdots,r$, the iterated integral $\int\omega_1\omega_2\cdots\omega_r$ takes the value

$$\int_\gamma\omega_1\omega_2\cdots\omega_r=\int_{0<t_1<\cdots<t_r<1}f_1(t_1)\cdots f_r(t_r)dt_1\cdots dt_r,$$

on the path γ, where $\gamma^*\omega_j=f_j(t)dt$, $j=1,\cdots,r$. If an iterated integral's value on a path only depends on its homotopy class relative to end points, then we say this iterated integral is "closed". Chen's π_1-de Rham theorem states that

$$\{\text{closed iterated integrals of smooth 1-forms on } X\}\xrightarrow{\approx}V^{\mathrm{B}}\otimes\mathbb{C},$$

where

$$V^{\mathrm{B}}=\varinjlim_n\operatorname{Hom}(\mathbb{Q}\pi_1(X,x)/I^n,\mathbb{Q}),$$

with I being the augmentation ideal of the group algebra $\mathbb{Q}\pi_1(X,x)$. Note that V^{B} is a commutative Hopf algebra. The corresponding affine group, defined over $\mathbb{Q}$, is the unipotent completion of the fundamental group $\pi_1(X,x)$. We will denote it by $\pi_1^{\mathrm{un}}(X,x)$.

When X is a smooth algebraic variety defined over $\mathbb{Q}$, let

$$V^{\mathrm{dR}}=\{\text{closed iterated integrals of "suitably chosen" algebraic 1-forms on } X\}.$$

Then we have a comparison isomorphism

$$\text{comp} : V^{\text{dR}} \otimes \mathbb{C} \xrightarrow{\approx} V^{\text{B}} \otimes \mathbb{C}.$$

Therefore, periods of fundamental groups are given by algebraic iterated integrals. However, finding the correct set of algebraic iterated integrals is non-trivial, even when X is a once punctured elliptic curve, as we shall explain in Chapter 2.

Example 1.2.1. Multiple Zeta Values as Iterated Integrals. Let $X_{/\mathbb{Q}} = \mathbb{P}^1 - \{0, 1, \infty\} = \mathbb{G}_m - \{\text{id}\}$. We have its algebraic de Rham cohomology $H^1_{\text{dR}}(X_{/\mathbb{Q}}) = \mathbb{Q} \cdot \omega_0 \oplus \mathbb{Q} \cdot \omega_1$ with basis

$$\omega_0 = \frac{dx}{x}, \quad \omega_1 = \frac{dx}{1-x}.$$

The $\mathbb{Q}$-vector space of periods of the path torsor in X from $\partial/\partial x \in T_0\mathbb{P}^1$ to $-\partial/\partial x \in T_1\mathbb{P}^1$ is the $\mathbb{Q}$-vector space spanned by multiple zeta values, for example,

$$\zeta(n) = \int_0^1 \omega_1 \underbrace{\omega_0 \cdots \omega_0}_{n-1},$$

and

$$\zeta(n, m) = \int_0^1 \omega_1 \underbrace{\omega_0 \cdots \omega_0}_{n-1} \omega_1 \underbrace{\omega_0 \cdots \omega_0}_{m-1}.$$

1.3 Algebraic De Rham Theory for Completions of Fundamental Groups

Let X be a smooth algebraic variety defined over $\mathbb{Q}$, $x \in X(\mathbb{Q})$ or a tangential base point. Denote by $\pi_1(X, x)$ its fundamental group of the corresponding complex analytic variety. To study its periods, we first linearize $\pi_1(X, x)$ using a suitable algebraic completion. The Betti incarnation of the completion of $\pi_1(X, x)$ is usually well known, constructed by group theoretic terms. It is an affine group defined over $\mathbb{Q}$, which we denote by $\mathcal{G}^{\text{B}}$. Its coordinate ring $V^{\text{B}} = \mathcal{O}(\mathcal{G}^{\text{B}})$ is a $\mathbb{Q}$-vector space.

What one needs to find is its de Rham analogue, a $\mathbb{Q}$-group $\mathcal{G}^{\mathrm{dR}}_{/\mathbb{Q}}$, and its coordinate ring $V^{\mathrm{dR}} = \mathcal{O}(\mathcal{G}^{\mathrm{dR}})$. Moreover, there should be a comparison isomorphism

$$\mathrm{comp} : V^{\mathrm{dR}} \otimes \mathbb{C} \xrightarrow{\approx} V^{\mathrm{B}} \otimes \mathbb{C}.$$

We will construct these explicitly when X is a once punctured elliptic curve in Chapter 2, and the modular curve in Chapter 3.

1.4 The Genus 0 Case

To motivate our results, we first review the genus 0 case. Let $X = \mathbb{P}^1 - \{0, 1, \infty\}$. Since fundamental groups depend on its base points, it is natural to consider a vector bundle

$$\begin{array}{ccc} \mathfrak{p}_x & \subset & \mathcal{P} \\ \Big| & & \Big| \\ x & \in & X \end{array}$$

where the fiber $\mathfrak{p}_x$ over $x \in X$ is the Lie algebra of the unipotent completion $\pi_1^{\mathrm{un}}(X, x)$ of the fundamental group $\pi_1(X, x)$. Since the fundamental group $\pi_1(X, x)$ is a free group with two generators, $\mathfrak{p}_x$ is a (completed) free Lie algebra of rank 2. This is the Betti side.

Now to the de Rham side. Let $\mathbb{Q}\langle\langle \mathbf{e}_0, \mathbf{e}_1 \rangle\rangle$ be the algebra of the formal power series in noncommuting indeterminants $\mathbf{e}_0$, $\mathbf{e}_1$, and $\mathbb{L}_{\mathbb{Q}}(\mathbf{e}_0, \mathbf{e}_1)^\wedge$ the (completed) free Lie algebra generated by $\mathbf{e}_0$, $\mathbf{e}_1$. The 1-form

$$\omega_{KZ} = \omega_0 \mathbf{e}_0 + \omega_1 \mathbf{e}_1 \in H^0(\Omega^1_{\mathbb{P}^1/\mathbb{Q}}(\log\{0, 1, \infty\})) \hat{\otimes} \mathbb{Q}\langle\langle \mathbf{e}_0, \mathbf{e}_1 \rangle\rangle$$

defines a flat connection with regular singularities at $0, 1, \infty$ on the trivial bundle

$$\mathbb{Q}\langle\langle \mathbf{e}_0, \mathbf{e}_1 \rangle\rangle \times \mathbb{P}^1_{/\mathbb{Q}} \to \mathbb{P}^1_{/\mathbb{Q}}$$

by the formula $\nabla f = df - f\omega_{KZ}$. This connection is called the KZ connection. Its parallel transport function

$$T_{KZ} : \{\text{paths in } \mathbb{P}^1 - \{0,1,\infty\}\} \to \{\text{group-like elements of } \mathbb{C}\langle\langle \mathbf{e}_0, \mathbf{e}_1 \rangle\rangle\}$$

is given by the formula

$$\gamma \mapsto T_{KZ}(\gamma) := 1 + \int_\gamma \omega_{KZ} + \int_\gamma \omega_{KZ}\omega_{KZ} + \int_\gamma \omega_{KZ}\omega_{KZ}\omega_{KZ} + \cdots .$$

This induces a comparison isomorphism

$$\mathrm{comp} : \mathrm{Lie}(\pi_1^{\mathrm{un}}(X, x)) \otimes_{\mathbb{Q}} \mathbb{C} \to \mathbb{L}_{\mathbb{Q}}(\mathbf{e}_0, \mathbf{e}_1)^\wedge \otimes \mathbb{C}$$

and corresponding comparison isomorphisms for both $\pi_1^{\mathrm{un}}(X, x)$ and its coordinate ring $\mathcal{O}(\pi_1^{\mathrm{un}}(X, x))$.

When studying motives unramified over $\mathbb{Z}$, it is necessary to use tangential base points and regularized periods. For example, one often chooses the tangent vector $-\partial/\partial x$ of $\mathbb{P}^1$ at $x = 1$ and the tangent vector $\partial/\partial x$ at $x = 0$, which we denote by $\vec{\mathsf{v}}$ and $\vec{\mathsf{w}}$ respectively. Brown's work [2] shows that $\mathcal{O}(\pi_1^{\mathrm{un}}(\mathbb{P}^1 - \{0,1,\infty\}, \vec{\mathsf{v}}))$ generates the category $\mathsf{MTM}(\mathbb{Z})$ of mixed Tate motives over $\mathbb{Z}$. Periods of $\pi_1^{\mathrm{un}}(\mathbb{P}^1 - \{0,1,\infty\}, \vec{\mathsf{v}})$, and thus periods of all objects in $\mathsf{MTM}(\mathbb{Z})$, are linear combinations of multiple zeta values. These multiple zeta values are also the coefficients of the Drinfeld associator $\Phi(\mathbf{e}_0, \mathbf{e}_1) \in \mathbb{C}\langle\langle \mathbf{e}_0, \mathbf{e}_1 \rangle\rangle$, which is the regularized value of T_{KZ} on the unique straight line ("dch") from 0 to 1 (from $\vec{\mathsf{w}}$ to $\vec{\mathsf{v}}$ to be precise, cf. Example 1.2.1).

1.5 The Elliptic Case

A once punctured elliptic curve $E' := E - \{\mathrm{id}\}$ is given by an affine equation $y^2 = 4x^3 - ux - v$, such that $u^3 - 27v^2 \neq 0$. It is defined over the field $\mathbb{K} = \mathbb{Q}(u, v)$. Then 1-forms $\frac{dx}{y}$ and $\frac{x\,dx}{y}$ form an $\mathbb{K}$-basis of $H^1_{\mathrm{dR}}(E_{/\mathbb{K}})$. One can mimic the genus 0 case naively, using the algebraic 1-form

$$\omega_{\text{naive}} = -\frac{xdx}{y}\mathsf{T} + \frac{dx}{y}\mathsf{S} \in H^0(\Omega^1_{E'}) \otimes \mathbb{L}(\mathsf{S},\mathsf{T})^\wedge$$

to define a flat connection on the trivial bundle $\mathbb{L}(\mathsf{S},\mathsf{T})^\wedge \times E' \to E'$. We call this the *naive connection*.

Define a regular connection on E to be a meromorphic connection that is holomorphic on E' and has a regular singularity at the identity. Since the form $\frac{x\,dx}{y}$ has a double pole at the identity[1], the naive connection is not a regular connection on E. In fact, this naive connection is not algebraically gauge equivalent to any regular connection, as we show in Section 2.8.1.

Instead of the naive connection, one should work with the correct elliptic analogue of the KZ connection, which is called the elliptic KZB connection. As we show in Section 2.9, if E is an elliptic curve defined over a field $\mathbb{K}$ of characteristic 0, then this connection is defined over $\mathbb{K}$, and has a regular singularity at the identity. In the analytic setting, this connection was studied by Calaque, Enriquez and Etingof [5], and by Levin and Racinet [22]. The latter one also provides algebraic formulas but with irregular singularities at the identity of the elliptic curve E.

The complication is on the bundle that this elliptic KZB connection lives. Deligne's canonical extension of this bundle, unlike in the case of $\mathbb{P}^1 - \{0, 1, \infty\}$, is not a trivial bundle as the corresponding monodromy representation fails a Hodge theoretic restriction (see [16]). One has to trivialize this bundle on an open cover of E. We choose the cover $\{E', E''\}$, where E'', containing the identity, is the complement of three non-trivial 2-torsion points of E (the trivial one being the identity). In Sections 2.8 and 2.9, we write down algebraic formulas for the elliptic KZB connection

[1] Let $\tau \in \mathfrak{h}$, the map $z \mapsto [\wp_\tau(z), \wp'_\tau(z), 1]$ induces an embedding of the elliptic curve $E_\tau = \mathbb{C}/\Lambda_\tau$ into $\mathbb{P}^2(\mathbb{C})$ via the Weierstrass $\wp$-function $\wp_\tau(z) = \frac{1}{z^2} + \text{holo. in } z$. The pull-back of $\frac{x\,dx}{y}$ through this map is $\wp_\tau(z)dz$, which has a double pole at $z = 0$.

explicitly on E' and E''. For instance, on E', the connection is given by the 1-form

$$\omega_{KZB} = -\frac{xdx}{y}\mathsf{T} + \frac{dx}{y}\mathsf{S} + \sum_{n=2}^{\infty} q_n(x,y)\frac{dx}{y}\mathsf{T}^n \cdot \mathsf{S} \in H^0(\Omega^1_{E'}) \otimes \mathbb{L}(\mathsf{S},\mathsf{T})^{\wedge},$$

where $q_n(x,y)$ are explicit polynomials in $\mathbb{K}[x,y]$ that express Eisenstein elliptic functions in terms of Weierstrass $\wp$-function and its derivative. While this 1-form still has a double pole at the identity, we show explicitly that the connection can be algebraically gauge transformed into a connection that has regular singularity at the identity on E''. This completes the algebraic de Rham theory in the elliptic case, and allows one to discuss and study (regularized) periods of $\pi_1^{\mathrm{un}}(E',\vec{\mathsf{v}})$, where $\vec{\mathsf{v}}$ is a tangential base point at the identity. We can then define

$$V^{\mathrm{dR}} = \{\text{coefficients of the transport function } T_{KZB}\}$$

where

$$T_{KZB} = 1 + \int \omega_{KZB} + \int \omega_{KZB}\omega_{KZB} + \int \omega_{KZB}\omega_{KZB}\omega_{KZB} + \cdots .$$

When $\mathbb{K} \hookrightarrow \mathbb{C}$, there is a comparison isomorphism

$$\mathrm{comp} : V^{\mathrm{dR}} \otimes_{\mathbb{K}} \mathbb{C} \xrightarrow{\approx} V^{\mathrm{B}} \otimes_{\mathbb{Q}} \mathbb{C},$$

with $V^{\mathrm{B}} = \mathcal{O}(\pi_1^{\mathrm{un}}(E',\vec{\mathsf{v}}))$, as required to define periods.

In the spirit of results of Brown [2, 3], when $(u,v) \in \mathbb{Q}^2$, one can ask whether $\mathcal{O}(\pi_1^{\mathrm{un}}(E',\vec{\mathsf{v}}))$ generates the conjectural tannakian category $\mathsf{MEM}(E)$ of mixed elliptic motives of $E_{/\mathbb{Q}}$.

1.6 The Modular Case

In the modular case, we view $\mathrm{SL}_2(\mathbb{Z})$ as the (orbifold) fundamental group of the moduli space $\mathcal{M}_{1,1}$ of elliptic curves. Since the unipotent completion of $\mathrm{SL}_2(\mathbb{Z})$

is trivial, we consider instead the relative completion of $\mathrm{SL}_2(\mathbb{Z})$ with respect to the inclusion $\mathrm{SL}_2(\mathbb{Z}) \hookrightarrow \mathrm{SL}_2(\mathbb{Q})$. In fact, the relative completions of $\mathrm{SL}_2(\mathbb{Z})$ and modular groups are particularly interesting, because they form a bridge between modular forms and categories of admissible variations of mixed Hodge structures over modular curves.

More precisely, we regard $\mathrm{SL}_2(Z)$ as the fundamental group $\pi_1(\mathcal{M}_{1,1}, \partial/\partial q)$, where the base point $\partial/\partial q$ is a unit tangent vector at the cusp. The relative completion of $\mathrm{SL}_2(\mathbb{Z})$, which we denote by $\mathcal{G}^{\mathrm{rel}}$, is an affine group defined over $\mathbb{Q}$. It is an extension of SL_2 by a prounipotent group $\mathcal{U}^{\mathrm{rel}}$. The Lie algebra $\mathfrak{u}^{\mathrm{rel}}$ of $\mathcal{U}^{\mathrm{rel}}$ is freely topologically generated by

$$\prod_{n\geqslant 0} H^1(\mathrm{SL}_2(\mathbb{Z}), S^{2n}H)^* \otimes S^{2n}H,$$

where H is the standard representation of SL_2, and $S^{2n}H$ its $2n$-th symmetric power.

Brown [3] defines multiple modular values to be periods of the coordinate ring $\mathcal{O}(\mathcal{G}^{\mathrm{rel}})$. In order to study and compute multiple modular values, one needs an explicit $\mathbb{Q}$-de Rham theory for $\mathcal{G}^{\mathrm{rel}}$. To do this, the first step is to construct an explicit $\mathbb{Q}$-de Rham structure on $H^1(\mathrm{SL}_2(\mathbb{Z}), S^{2n}H)$. Recall that there is a mixed Hodge structure on $H^1(\mathrm{SL}_2(\mathbb{Z}), S^{2n}H)$, which has weight and Hodge filtrations defined over $\mathbb{Q}$ [31]:

$$\begin{aligned} W_{2n+1}H^1(\mathrm{SL}_2(\mathbb{Z}), S^{2n}H) &= H^1_{\mathrm{cusp}}(\mathrm{SL}_2(\mathbb{Z}), S^{2n}H); \\ W_{4n+2}H^1(\mathrm{SL}_2(\mathbb{Z}), S^{2n}H) &= H^1(\mathrm{SL}_2(\mathbb{Z}), S^{2n}H), \\ F^{2n+1}H^1(\mathrm{SL}_2(\mathbb{Z}), S^{2n}H) &\cong \{\textit{holomorphic}\text{ modular forms}\}. \end{aligned}$$

The $\mathbb{Q}$-structure for the *holomorphic* part $F^{2n+1}H^1(\mathrm{SL}_2(\mathbb{Z}), S^{2n}H)$ is classically well-known; in this *holomorphic* part, all $\mathbb{Q}$-de Rham classes correspond to classical modular forms of weight $2n+2$ with rational Fourier coefficients [15, §21]. To obtain a complete $\mathbb{Q}$-de Rham basis of $H^1(\mathrm{SL}_2(\mathbb{Z}), S^{2n}H)$, one needs to consider what we call modular forms of the *second kind*. In Section 3.5.2, we find representatives of all these $\mathbb{Q}$-de Rham classes in $H^1(\mathrm{SL}_2(\mathbb{Z}), S^{2n}H)$. Our representatives have at worst logarithmic singularities at the cusp and may have singularities with trivial residue at other points. This differs from the traditional approach using weakly modular forms, which allows arbitrary poles at the cusp (cf. Brown–Hain [4]).

Denote the relative completion of $\pi_1(\mathcal{M}_{1,1}, x)$ with respect to its inclusion $\rho_x : \pi_1(\mathcal{M}_{1,1}, x) \hookrightarrow \mathrm{SL}_2(\mathbb{Q})$ by $\mathcal{G}_x$. Denote the Lie algebra of its unipotent radical by $\mathfrak{u}_x$. They are isomorphic to $\mathcal{G}^{\mathrm{rel}}$ and $\mathfrak{u}^{\mathrm{rel}}$ respectively. We thus have a Betti vector bundle $\boldsymbol{\mathfrak{u}}_{\mathrm{B}} \to \mathcal{M}_{1,1}$ whose fiber over x is the Lie algebra $\mathfrak{u}_x$.

To construct the de Rham analogue $\boldsymbol{\mathfrak{u}}_{\mathrm{dR}} \to \overline{\mathcal{M}}_{1,1}$, we start from representatives of all $\mathbb{Q}$-de Rham classes in $H^1(\mathrm{SL}_2(\mathbb{Z}), S^{2n}H)$ found earlier. We then apply a Čech-de Rham version of Chen's method of power series connections [6] to find a canonical flat connection, defined over $\mathbb{Q}$ and with regular singularity at the cusp, on the vector bundle $\boldsymbol{\mathfrak{u}}_{\mathrm{dR}}$. This requires us to trivialize $\boldsymbol{\mathfrak{u}}_{\mathrm{dR}}$ on the open cover of $\overline{\mathcal{M}}_{1,1}$ consisting of $\overline{\mathcal{M}}_{1,1} - \{[i]\}$ and $\overline{\mathcal{M}}_{1,1} - \{[\rho]\}$, and to provide an inductive algorithm for constructing the connection on both opens, and for finding the gauge transformation on their intersection. We carry this algorithm out explicitly up to $\boldsymbol{\mathfrak{u}}_{\mathrm{dR}}/\mathbb{L}^3(\boldsymbol{\mathfrak{u}}_{\mathrm{dR}})$ in Section 3.7.2. This connection is analogous to the KZB connection in the elliptic case §1.5, but is more general. This enables us to explicitly construct iterated integrals of algebraic modular forms, involving both *holomorphic* modular forms and modular forms of the *second kind*. This also allows us to define

$$V^{\mathrm{dR}} = \mathcal{O}(\mathcal{G}^{\mathrm{rel}}_{\mathrm{dR}}) = \{\text{“closed” iterated integrals of algebraic modular forms}\},$$

and a comparison isomorphism

$$\mathrm{comp} : V^{\mathrm{dR}} \otimes \mathbb{C} \xrightarrow{\approx} V^{\mathrm{B}} \otimes \mathbb{C},$$

with $V^{\mathrm{B}} = \mathcal{O}(\mathcal{G}^{\mathrm{rel}})$. These newly constructed iterated integrals provide all multiple modular values, whereas previously only those multiple modular values that are iterated integrals of *holomorphic* modular forms have been studied by Brown [3] and Manin [25, 26].

2

Part I: Algebraic De Rham Theory for Unipotent Fundamental Groups of Elliptic Curves

2.1 Introduction

A once punctured elliptic curve is an elliptic curve with it identity removed. In this part, we describe an algebraic de Rham theory for unipotent fundamental groups of once punctured elliptic curves.

The analytic version of this story is described by Calaque, Enriquez and Etingof [5] and by Levin and Racinet [22]. For a family $\mathscr{X} \to T$ where each fiber X_t over $t \in T$ is a once punctured elliptic curve,

$$\begin{array}{ccc} X_t & \subset & \mathscr{X} \\ \big| & & \big| \\ t & \in & T \end{array}$$

there is a vector bundle $\mathcal{P}$ of prounipotent groups over $\mathscr{X}$, endowed with a flat connection. For a point $x \in \mathscr{X}$ that lies over t, i.e. $x \in X_t$, the fiber of $\mathcal{P}$ over x is the unipotent fundamental group $\pi_1^{\mathrm{un}}(X_t, x)$. We are particularly interested in the case when $\mathscr{X} = \mathcal{E}'$ and $T = \mathcal{M}_{1,1}$. In this case, the flat connection is called the universal elliptic KZB[1] connection[2]. The bundle $\mathcal{P}$ extends naturally by Deligne's canonical extension $\overline{\mathcal{P}}$ to $\overline{\mathcal{E}}$, and the universal elliptic KZB connection on $\overline{\mathcal{P}}$ has regular singularities around boundary divisors: the identity section of the universal elliptic curve $\mathcal{E} \to \mathcal{M}_{1,1}$, and the nodal cubic. One can restrict the bundle $\mathcal{P}$ to a single fiber of $\mathcal{E}' \to \mathcal{M}_{1,1}$, i.e. a once punctured elliptic curve $E' := E - \{\mathrm{id}\}$, and obtain Deligne's canonical extension $\overline{\mathcal{P}}$ of it to E. It is endowed with a unipotent connection on E, having regular singularity at the identity. We call this the elliptic KZB connection on E.

Working algebraically, Levin and Racinet have shown that there is a $\mathbb{K}$-structure on the bundle $\mathcal{P}$ over a punctured elliptic curve X defined over a field $\mathbb{K}$ of characteristic zero, and a $\mathbb{Q}$-structure for the bundle $\mathcal{P}$ over $\mathcal{M}_{1,2/\mathbb{Q}}$. However, their algebraic formulas for the (universal) elliptic KZB connection is neither explicit nor having regular singularity at the identity (section). By resolving these issues for the case of $\mathcal{M}_{1,2}$, we will prove that

Theorem. *There is a $\mathbb{Q}$-de Rham structure $\overline{\mathcal{P}}_{\mathrm{dR}}$ on the bundle $\overline{\mathcal{P}}$ over $\overline{\mathcal{M}}_{1,2}$ with the universal elliptic KZB connection, which has regular singularities along boundary divisors, the identity section and the nodal cubic.*

Restricting to a single elliptic curve, we get

Corollary. *If E is an elliptic curve defined over a field $\mathbb{K}$ of characteristic zero, then there is a $\mathbb{K}$-de Rham structure $\overline{\mathcal{P}}_{\mathrm{dR}}$ on the bundle $\overline{\mathcal{P}}$ over E with an elliptic KZB connection, which has regular singularity at the identity.*

The tricky part is that Deligne's canonical extension of $\mathcal{P}_{\mathrm{dR}}$, the de Rham realization $\overline{\mathcal{P}}_{\mathrm{dR}}$ of $\overline{\mathcal{P}}$, is not a trivial bundle as the corresponding monodromy representation

[1] Named after physicists Knizhnik, Zamoldchikov and Bernard.

[2] The general universal elliptic KZB connection is the flat connection on the bundle $\mathcal{P}$ over $\mathcal{M}_{1,n+1}$ whose fiber over $[E; 0, x_1, \ldots, x_n]$ is the unipotent fundamental group of the configuration space of n points on E' with base point $(x_1, \ldots, x_n)$. Calaque et al [5] write down the universal elliptic KZB connection for all $n \geqslant 1$.

fails a Hodge theoretic restriction (see [16]). We trivialize the bundle $\overline{\mathcal{P}}_{\mathrm{dR}}$ on two open subsets of the (universal) elliptic curve, and write down algebraic connection formulas according to these trivializations, with suitable gauge transformation on their intersection. These two opens are E' and E'', where E'', containing the identity, is the complement of three non-trivial 2-torsion points of E (the trivial one being the identity).

For a single elliptic curve $E_{/\mathbb{K}}$, this bundle $\overline{\mathcal{P}}_{\mathrm{dR}}$ is a universal object of the tannakian category

$$\mathcal{C}_{\mathbb{K}}^{\mathrm{dR}} := \left\{ \begin{array}{c} \text{unipotent vector bundles } \overline{\mathcal{V}} \text{ over } E \text{ defined over } \mathbb{K} \\ \text{with a flat connection } \nabla \text{ that has regular singularity} \\ \text{at the identity with nilpotent residue} \end{array} \right\}.$$

This allows us to compute the tannakian fundamental group $\pi_1(\mathcal{C}_{\mathbb{K}}^{\mathrm{dR}}, \omega)$ of this category, where the fiber functor ω is the fiber over the identity. It is a free pronilpotent group of rank 2 defined over $\mathbb{K}$. This tannakian formalism implies that the connection ∇ that we have computed explicitly on the algebraic vector bundle $\overline{\mathcal{P}}_{\mathrm{dR}}$ can be viewed as a universal unipotent connection over $E_{/\mathbb{K}}$. Similar results have been recently obtained by Enriquez–Etingof [12] for the configuration space of n points in an elliptic curve E with ground field $\mathbb{C}$.

In Section 2.8.1, we show that the naive elliptic analogue of the KZ connection, which we call the naive connection, cannot be algebraically gauge transformed to a connection on any Zariski open subset of E that has regular singularity at the identity. This means the naive connection does not belong to the category $\mathcal{C}_{\mathbb{K}}^{\mathrm{dR}}$.

2.2 Moduli Spaces of Elliptic Curves

Here we quickly review the construction of moduli spaces of elliptic curves and their Deligne-Mumford compactifications. Full details can be found in [14], or the first section of [15].

2.2.1 Moduli spaces as algebraic stacks

Denote the moduli stack over $\mathbb{Q}$ of elliptic curves with n marked points and r non-zero tangent vectors by $\mathcal{M}_{1,n+\vec{r}}$. Note that $\mathcal{M}_{1,n+\vec{r}}$ is a scheme if $n > 4$ or $r > 0$. The Deligne-Mumford compactification of $\mathcal{M}_{1,n}$ will be denoted by $\overline{\mathcal{M}}_{1,n}$.

One can define $\mathcal{M}_{1,n+1}$ to be the stack quotient of $\mathcal{M}_{1,n+\vec{1}}$ by the $\mathbb{G}_m$-action:

$$\lambda : [E; x_1, \cdots, x_n; \omega] \mapsto [E; x_1, \cdots, x_n; \lambda\omega],$$

where a moduli point $[E; x_1, \cdots, x_n; \omega] \in \mathcal{M}_{1,n+\vec{1}}$ is represented by tuples

$$(E; x_1, \cdots, x_n; \omega),$$

an elliptic curve E with n marked points and the differential form ω that is dual to the marked tangent vector. For example, the moduli space $\mathcal{M}_{1,\vec{1}}$ over $\mathbb{Q}$ is the scheme

$$\mathcal{M}_{1,\vec{1}} := \mathbb{A}^2_{\mathbb{Q}} - D,$$

where D is the discriminant locus $\{(u,v) \in \mathbb{A}^2 : \Delta = u^3 - 27v^2 = 0\}$. The point (u,v) corresponds to the once punctured elliptic curve (the plane cubic) $y^2 = 4x^3 - ux - v$ with the abelian differential dx/y. The moduli stack $\mathcal{M}_{1,1}$ is defined as the quotient of $\mathcal{M}_{1,\vec{1}}$ by the $\mathbb{G}_m$-action:

$$\lambda \cdot (u,v) = (\lambda^{-4}u, \lambda^{-6}v).$$

Its compactification, the moduli stack $\overline{\mathcal{M}}_{1,1}$, is the quotient of $Y := \mathbb{A}^2 - \{(0,0)\}$[3] by the same $\mathbb{G}_m$-action above.

Similarly, define the moduli stack $\mathcal{M}_{1,2}$ over $\mathbb{Q}$ to be the quotient of the scheme

$$\mathcal{M}_{1,1+\vec{1}} := \{(x,y,u,v) \in \mathbb{A}^2 \times \mathbb{A}^2 : y^2 = 4x^3 - ux - v, \text{ and } u^3 - 27v^2 \neq 0\}$$

[3] One may regard Y as a partial compactification of $\mathcal{M}_{1,\vec{1}}$.

by the $\mathbb{G}_m$-action

$$\lambda : (x, y, u, v) \mapsto (\lambda^{-2}x, \lambda^{-3}y, \lambda^{-4}u, \lambda^{-6}v).$$

Here the point $(x, y, u, v) \in \mathcal{M}_{1,1+\vec{1}}$ corresponds to the point (x, y) on the punctured elliptic curve $y^2 = 4x^3 - ux - v$ with the abelian differential dx/y. Note that $\mathcal{M}_{1,2}$ is $\mathcal{E}'$, the universal elliptic curve $\mathcal{E}$ over $\mathcal{M}_{1,1}$ with its identity section removed. We define its compactification $\overline{\mathcal{M}}_{1,2}$ as the quotient of the scheme

$$\{(x, y, u, v) \in \mathbb{A}^2 \times \mathbb{A}^2 : y^2 = 4x^3 - ux - v, (u, v) \neq (0, 0)\}$$

by the same $\mathbb{G}_m$-action above. Note that $\overline{\mathcal{M}}_{1,2}$ is the compactification $\overline{\mathcal{E}}$ of the universal elliptic curve $\mathcal{E}$ whose restriction to the q-disk is the Tate curve $\mathcal{E}_{\mathrm{Tate}} \to \operatorname{Spec} \mathbb{Z}[[q]]$ (see [15, §1]).

2.2.2 Moduli spaces as complex analytic orbifolds

Working complex analytically, we can define moduli spaces as complex orbifolds

$$\mathcal{M}_{1,1}^{\mathrm{an}} := \mathbb{G}_m \backslash\backslash \mathcal{M}_{1,\vec{1}}^{\mathrm{an}}, \quad \mathcal{M}_{1,2}^{\mathrm{an}} := \mathbb{G}_m \backslash\backslash \mathcal{M}_{1,1+\vec{1}}^{\mathrm{an}},$$

where $\mathcal{M}_{1,\vec{1}}^{\mathrm{an}} := \mathcal{M}_{1,\vec{1}}(\mathbb{C})$ and $\mathcal{M}_{1,1+\vec{1}}^{\mathrm{an}} := \mathcal{M}_{1,1+\vec{1}}(\mathbb{C})$ are complex analytic manifolds.

The moduli space $\mathcal{M}_{1,1}^{\mathrm{an}}$ can also be defined as the orbifold quotient $\mathrm{SL}_2(\mathbb{Z}) \backslash\backslash \mathfrak{h}$ of the upper half plane $\mathfrak{h}$ by the standard $\mathrm{SL}_2(\mathbb{Z})$ action:

$$\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} : \tau \mapsto \gamma\tau = \frac{a\tau + b}{c\tau + d}$$

where $\gamma \in \mathrm{SL}_2(\mathbb{Z})$ and $\tau \in \mathfrak{h}$. The map

$$\mathfrak{h} \to \mathcal{M}_{1,\vec{1}}^{\mathrm{an}} = \{(u, v) \in \mathbb{C}^2 : u^3 - 27v^2 \neq 0\}$$

$$\tau \mapsto (20G_4(\tau), \frac{7}{3}G_6(\tau))$$

induces an isomorphism of orbifolds $\mathrm{SL}_2(\mathbb{Z})\backslash\!\backslash\mathfrak{h} \cong \mathbb{G}_m\backslash\!\backslash\mathcal{M}^{\mathrm{an}}_{1,\vec{1}}$, where $G_{2n}(\tau)$ is the normalized Eisenstein series of weight $2n$ (see Section 2.4.1 for definition).

A point $\tau \in \mathfrak{h}$ corresponds to the framed elliptic curve $E_\tau := \mathbb{C}/\Lambda_\tau$ where $\Lambda_\tau := \mathbb{Z} \oplus \mathbb{Z}\tau$, with a basis $\mathbf{a}, \mathbf{b}$ of $H_1(E_\tau; \mathbb{Z})$ that corresponds to $1, \tau$ of the lattice Λ_τ via the identification $H_1(E_\tau; \mathbb{Z}) \cong \Lambda_\tau$.

There is a canonical family of elliptic curve $\mathcal{E}_\mathfrak{h}$ over the upper half plane $\mathfrak{h}$, called the universal framed family of elliptic curves in [14], whose fiber over $\tau \in \mathfrak{h}$ is E_τ. It is the quotient of the trivial bundle $\mathbb{C} \times \mathfrak{h} \to \mathfrak{h}$ by the $\mathbb{Z}^2$-action:

$$(m, n) : (\xi, \tau) \mapsto \left(\xi + \begin{pmatrix} m & n \end{pmatrix} \begin{pmatrix} \tau \\ 1 \end{pmatrix}, \tau\right).$$

The universal elliptic curve $\mathcal{E}^{\mathrm{an}}$ over $\mathcal{M}^{\mathrm{an}}_{1,1}$ is the orbifold quotient of $\mathbb{C} \times \mathfrak{h}$ by the semi-direct product[4] $\mathrm{SL}_2(\mathbb{Z}) \ltimes \mathbb{Z}^2$, where $\mathbb{Z}^2$ acts on $\mathbb{C} \times \mathfrak{h}$ as above, and $\gamma \in \mathrm{SL}_2(\mathbb{Z})$ acts as follows:

$$\gamma : (\xi, \tau) \mapsto \big((c\tau + d)^{-1}\xi, \gamma\tau\big).$$

The universal elliptic curve $\mathcal{E}^{\mathrm{an}}$ can also be obtained as the orbifold quotient of $\mathcal{E}_\mathfrak{h} \to \mathfrak{h}$ by $\mathrm{SL}_2(\mathbb{Z})$. It is an orbifold family of elliptic curves whose fiber over a moduli point $[E] \in \mathcal{M}_{1,1}$ is an elliptic curve isomorphic to E. If we remove all the lattice points $\Lambda_\mathfrak{h} := \{(\xi, \tau) \in \mathbb{C} \times \mathfrak{h} : \xi \in \Lambda_\tau\}$ from $\mathbb{C} \times \mathfrak{h}$, then take the orbifold quotient of the same $\mathrm{SL}_2(\mathbb{Z}) \ltimes \mathbb{Z}^2$-action above, we obtain another analytic description of the moduli space $\mathcal{M}^{\mathrm{an}}_{1,2}$. To relate the two descriptions, there is a map

$$\mathbb{C} \times \mathfrak{h} - \Lambda_\mathfrak{h} \to \mathcal{M}^{\mathrm{an}}_{1,1+\vec{1}} = \{(x, y, u, v) \in \mathbb{C}^2 \times \mathbb{C}^2 : y^2 = 4x^3 - ux - v, (u, v) \neq (0, 0)\}$$

$$(\xi, \tau) \mapsto (P_2(\xi, \tau), -2P_3(\xi, \tau), 20G_4(\tau), \frac{7}{3}G_6(\tau))$$

[4] The semi-product structure is induced from the right action of $\mathrm{SL}_2(\mathbb{Z})$ on $\mathbb{Z}^2$: $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$: $\begin{pmatrix} m & n \end{pmatrix} \mapsto \begin{pmatrix} m & n \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix}$.

that induces an isomorphism $\mathrm{SL}_2(\mathbb{Z}) \ltimes \mathbb{Z}^2 \backslash\!\backslash (\mathbb{C} \times \mathfrak{h} - \Lambda_{\mathfrak{h}}) \cong \mathbb{G}_m \backslash\!\backslash \mathcal{M}^{\mathrm{an}}_{1,1+\vec{1}}$, where $P_2(\xi, \tau)$ and $P_3(\xi, \tau)$ are, up to a constant, Weierstrass $\wp$-function $\wp_\tau(\xi)$ and its derivative $\wp'_\tau(\xi)$ (see Section 2.4.2 for definition).

2.3 The Local System $\mathbb{H}$ with its Betti and $\mathbb{Q}$-de Rham Realizations

The local system $\mathbb{H}$ over $\mathcal{M}_{1,1}$ is a "motivic local system" $R^1\pi_*\mathbb{C}$ associated to the universal elliptic curve $\pi : \mathcal{E} \to \mathcal{M}_{1,1}$. It has a set of compatible realizations: Betti, $\mathbb{Q}$-de Rham, Hodge and l-adic described in [20, §5]. In this section we will follow [20, §5] closely and describe its Betti realization $\mathbb{H}^{\mathrm{B}}$ and $\mathbb{Q}$-de Rham realization $\mathcal{H}^{\mathrm{dR}}$, and the comparison between these two.

We will denote the pull back of $\mathbb{H}$ (resp. $\mathbb{H}^{\mathrm{B}}$, $\mathcal{H}^{\mathrm{dR}}$) to $\mathcal{M}_{1,n+\vec{r}}$ by $\mathbb{H}_{n+\vec{r}}$ (resp. $\mathbb{H}^{\mathrm{B}}_{n+\vec{r}}$, $\mathcal{H}^{\mathrm{dR}}_{n+\vec{r}}$), so that $\mathbb{H}_1$ (resp. $\mathbb{H}^{\mathrm{B}}_1$, $\mathcal{H}^{\mathrm{dR}}_1$) is the same as $\mathbb{H}$ (resp. $\mathbb{H}^{\mathrm{B}}$, $\mathcal{H}^{\mathrm{dR}}$).

2.3.1 Betti realization $\mathbb{H}^{\mathrm{B}}$

The Betti realization $\mathbb{H}^{\mathrm{B}}$ of $\mathbb{H}$ is the local system $R^1\pi^{\mathrm{an}}_*\mathbb{Q}$ over $\mathcal{M}^{\mathrm{an}}_{1,1}$ associated to the universal elliptic curve $\pi^{\mathrm{an}} : \mathcal{E}^{\mathrm{an}} \to \mathcal{M}^{\mathrm{an}}_{1,1}$. We identify it, via Poincaré duality $H^1(E) \to H_1(E)$ fiberwise, with the local system over $\mathcal{M}^{\mathrm{an}}_{1,1}$ whose fiber over $[E] \in \mathcal{M}_{1,1}$ is $H_1(E; \mathbb{Q})$.

There is a natural $\mathrm{SL}_2(\mathbb{Z})$ action

$$\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} : \begin{pmatrix} \mathbf{b} \\ \mathbf{a} \end{pmatrix} \mapsto \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} \mathbf{b} \\ \mathbf{a} \end{pmatrix},$$

where $\mathbf{a}, \mathbf{b}$ is the basis of $H_1(E_\tau; \mathbb{Z})$ that corresponds to the basis $1, \tau$ of Λ_τ. The sections $\mathbf{a}, \mathbf{b}$ trivialize the pullback $\mathbb{H}_{\mathfrak{h}}$ of $\mathbb{H}^{\mathrm{B}}$ to $\mathfrak{h}$.

Denote the dual basis of $H^1(E_\tau; \mathbb{Q}) \cong \mathrm{Hom}(H_1(E_\tau), \mathbb{Q})$ by $\check{\mathbf{a}}, \check{\mathbf{b}}$. Then, under Poincaré duality,

$$\check{\mathbf{a}} = -\mathbf{b} \text{ and } \check{\mathbf{b}} = \mathbf{a}.$$

And the corresponding $\mathrm{SL}_2(\mathbb{Z})$-action on this dual basis is

$$\gamma : \begin{pmatrix}\mathbf{a} & -\mathbf{b}\end{pmatrix} \mapsto \begin{pmatrix}\mathbf{a} & -\mathbf{b}\end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

One can construct the local system $\mathbb{H}^{\mathrm{B}}$ by taking the orbifold quotient of the local system $\mathbb{H}_{\mathfrak{h}} \to \mathfrak{h}$ by the above $\mathrm{SL}_2(\mathbb{Z})$-action.

2.3.2 $\mathbb{Q}$-de Rham realization $\mathcal{H}^{\mathrm{dR}}$

The $\mathbb{Q}$-de Rham realization $\mathcal{H}^{\mathrm{dR}}$ is a vector bundle on $\mathcal{M}_{1,1/\mathbb{Q}}$. Recall that $\mathcal{M}_{1,1} := \mathbb{G}_m \backslash\!\backslash \mathcal{M}_{1,\vec{1}}$, so to work with $\mathcal{M}_{1,1}$ is to work $\mathbb{G}_m$-equivariantly with $\mathcal{M}_{1,\vec{1}}$. Define a vector bundle

$$\mathcal{H}^{\mathrm{dR}}_{\vec{1}} := \mathcal{O}_{\mathcal{M}_{1,\vec{1}}}\mathsf{S} \oplus \mathcal{O}_{\mathcal{M}_{1,\vec{1}}}\mathsf{T}$$

over $\mathcal{M}_{1,\vec{1}}$ with a $\mathbb{G}_m$-action:

$$\lambda \cdot \mathsf{S} = \lambda^{-1}\mathsf{S} \quad \text{and} \quad \lambda \cdot \mathsf{T} = \lambda \mathsf{T},$$

where the sections S and T represent algebraic differential forms xdx/y and dx/y respectively. This $\mathbb{G}_m$-action extends the action on $\mathcal{M}_{1,\vec{1}}$ to the bundle $\mathcal{H}^{\mathrm{dR}}_{\vec{1}}$ over it.

Define a connection

$$\nabla_0 = d + \left(-\frac{1}{12}\frac{d\Delta}{\Delta}\mathsf{T} + \frac{3\alpha}{2\Delta}\mathsf{S}\right)\frac{\partial}{\partial \mathsf{T}} + \left(-\frac{u\alpha}{8\Delta}\mathsf{T} + \frac{1}{12}\frac{d\Delta}{\Delta}\mathsf{S}\right)\frac{\partial}{\partial \mathsf{S}}, \qquad (2.3.1)$$

where $\alpha = 2udv - 3vdu$ and $\Delta = u^3 - 27v^2$ (cf. [15] Prop 19.6). This connection is $\mathbb{G}_m$-invariant, and defined over $\mathbb{Q}$. Therefore, the bundle $\mathcal{H}^{\mathrm{dR}}_{\vec{1}}$ with connection ∇_0 over $\mathcal{M}_{1,\vec{1}}$ descends to a bundle $\mathcal{H}^{\mathrm{dR}}$ over $\mathcal{M}_{1,1}$.

The canonical extension $\overline{\mathcal{H}}^{\mathrm{dR}}_{\vec{1}}$ of $\mathcal{H}^{\mathrm{dR}}_{\vec{1}}$ to $Y := \mathbb{A}^2_{\mathbb{Q}} - \{(0,0)\}$ is a vector bundle

$$\overline{\mathcal{H}}^{\mathrm{dR}}_{\vec{1}} := \mathcal{O}_Y\mathsf{S} \oplus \mathcal{O}_Y\mathsf{T}$$

with the same connection ∇_0 above. Since the connection has regular singularities along the discriminant locus $D = \{\Delta = 0\}$, and recall that $\overline{\mathcal{M}}_{1,1} = \mathbb{G}_m \backslash\backslash Y$ from Section 2.2.1, the bundle $\overline{\mathcal{H}}_{\mathbb{1}}^{\mathrm{dR}} \to Y$ descends to a bundle $\overline{\mathcal{H}}^{\mathrm{dR}}$ over $\overline{\mathcal{M}}_{1,1}$ with regular singularity at the cusp. It is Deligne's canonical extension of $\mathcal{H}^{\mathrm{dR}}$ to $\overline{\mathcal{M}}_{1,1}$.

2.3.3 The flat vector bundle $\mathcal{H}^{\mathrm{an}}$ and the comparison between $\mathcal{H}^{\mathrm{an}}$ and $\mathcal{H}^{\mathrm{dR}}$

One can put a complex structure on $\mathbb{H}$. Denote the corresponding holomorphic vector bundle over $\mathcal{M}_{1,1}^{\mathrm{an}}$ by $\mathcal{H}^{\mathrm{an}}$. The pullback of $\mathcal{H}^{\mathrm{an}}$ to $\mathfrak{h}$ (using the quotient map $\mathfrak{h} \to \mathcal{M}_{1,1}^{\mathrm{an}} = \mathrm{SL}_2(\mathbb{Z})\backslash\backslash\mathfrak{h}$) is the vector bundle

$$\mathcal{H}_{\mathfrak{h}}^{\mathrm{an}} = \mathcal{O}_{\mathfrak{h}}\mathbf{a} \oplus \mathcal{O}_{\mathfrak{h}}\mathbf{b},$$

where the sections $\mathbf{a}$ and $\mathbf{b}$ are flat.

Define a holomorphic section $\mathbf{w}$ of $\mathcal{H}_{\mathfrak{h}}^{\mathrm{an}}$ by

$$\mathbf{w}(\tau) = w_\tau = 2\pi i(\check{\mathbf{a}} + \tau\check{\mathbf{b}}) = 2\pi i(\tau\mathbf{a} - \mathbf{b}),$$

where w_τ is the class in $H^1(E_\tau;\mathbb{C})$ represented by the holomorphic differential $2\pi i\, d\xi$.

The sections $\mathbf{a}$ and $\mathbf{w}$ trivialize the pull back

$$\mathcal{H}_{\mathbb{D}^*}^{\mathrm{an}} := \mathcal{O}_{\mathbb{D}^*}\mathbf{a} \oplus \mathcal{O}_{\mathbb{D}^*}\mathbf{w}$$

of $\mathcal{H}^{\mathrm{an}}$ to $\mathbb{D}^*$ via the map

$$\mathfrak{h} \to \mathbb{D}^*, \qquad \tau \mapsto q := e^{2\pi i\tau},$$

as they are invariant under $\tau \mapsto \tau + 1$ (with γ being $\pm\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$), and thus invariant under the monodromy action on the punctured q-disk $\mathbb{D}^*$.

An easy computation [20, §5.2] shows that the connection on $\mathcal{H}_{\mathbb{D}^*}^{\mathrm{an}}$ in terms of this framing is

$$\nabla_0^{\mathrm{an}} = d + \mathbf{a}\frac{\partial}{\partial\mathbf{w}}\frac{dq}{q}.$$

Since this connection has a regular singularity at the cusp $q = 0$, we can extend $\mathcal{H}^{\mathrm{an}}_{\mathbb{D}^*}$ to the q-disk $\mathbb{D}$ by defining

$$\mathcal{H}^{\mathrm{an}}_{\mathbb{D}} := \mathcal{O}_{\mathbb{D}}\mathbf{a} \oplus \mathcal{O}_{\mathbb{D}}\mathbf{w}.$$

Therefore, we obtain Deligne's canonical extension $\overline{\mathcal{H}}^{\mathrm{an}}$ of $\mathcal{H}^{\mathrm{an}}$ to $\overline{\mathcal{M}}^{\mathrm{an}}_{1,1}$, endowed with a connection ∇^{an}_0 that has a regular singularity at the cusp.

To relate Betti and de Rham sections of $\mathcal{H}^{\mathrm{an}}$, we pull back the bundle $\mathcal{H}^{\mathrm{an}}_{\vec{1}}$ to $\mathfrak{h}$ via the map $\mathfrak{h} \to \mathcal{M}^{\mathrm{an}}_{1,\vec{1}}$ in Section 2.2.2, and compare it with $\mathcal{H}^{\mathrm{an}}_{\mathfrak{h}}$.

Proposition 2.3.1. *There is a natural isomorphism*

$$(\overline{\mathcal{H}}^{\mathrm{an}}, \nabla^{\mathrm{an}}_0) \cong (\overline{\mathcal{H}}^{\mathrm{dR}}, \nabla_0) \otimes_{\mathcal{O}_{\overline{\mathcal{M}}_{1,1/\mathbb{Q}}}} \mathcal{O}_{\overline{\mathcal{M}}^{\mathrm{an}}_{1,1}}$$

induced from the pull back. The sections T *and* S *that correspond to* dx/y *and* xdx/y *respectively, after being pulled back, become* $\mathsf{T} = \mathbf{w}/2\pi i$, *and* $\mathsf{S} = (\mathbf{a} - 2G_2(\tau)\mathbf{w})/2\pi i$.

Remark 2.3.2. Our formulas for T and S differ from those in Proposition 5.2 of [20] by a factor of $2\pi i$. The reason is that the cup product of dx/y and xdx/y is $2\pi i$, and we have multiplied their Poincaré duals by $(2\pi i)^{-1}$ to obtain a $\mathbb{Q}$-de Rham basis of the first homology [15, §20], such that $\mathsf{T} = \check{\mathsf{S}}$ and $\mathsf{S} = -\check{\mathsf{T}}$. More explanations are provided in Section 2.3.4 below.

2.3.4 The fiber of $\mathbb{H}$ at the cusp

To better understand various Betti and de Rham sections of $\mathcal{H}^{\mathrm{an}}$, we observe the fiber $H := H_{\partial/\partial q}$ at the cusp associated to the tangent vector $\partial/\partial q$. One can compute the limit mixed Hodge structure on H (computed in [20, §5.4]), which is isomorphic to $\mathbb{Q}(0) \oplus \mathbb{Q}(-1)$, with Betti realization $H^{\mathrm{B}} = \mathbb{Q}\mathbf{a} \oplus \mathbb{Q}\mathbf{b}$, and de Rham realization $H^{\mathrm{dR}} = \mathbb{Q}\mathbf{a} \oplus \mathbb{Q}\mathbf{w}$. Note that on H, $-\mathbf{b} = \check{\mathbf{a}} = \mathbf{w}/2\pi i$ spans $\mathbb{Q}(-1)$ and $\mathbf{a} = \check{\mathbf{b}}$ spans $\mathbb{Q}(0)$.

One can think of H as the cohomology $H^1(E_{\partial/\partial q})$. It is better to work with first homology, which is the abelian quotient of the fundamental group. We use Poincaré duality to identify $H_1(E)$ with $H^1(E)(1)$. Therefore, we have $H_1(E_{\partial/\partial q}) = H(1) = \mathbb{Q}(1) \oplus \mathbb{Q}(0)$, with Betti realization $\mathbb{Q}\mathbf{a} \oplus \mathbb{Q}\mathbf{b}$ and de Rham realization $\mathbb{Q}\mathsf{A} \oplus \mathbb{Q}\mathsf{T}$, where

$$\mathsf{A} := \mathbf{a}/2\pi i \text{ and } \mathsf{T} := \mathbf{w}/2\pi i.$$

Note that on $H(1)$, $\mathbf{a} = 2\pi i\,\mathsf{A}$ spans $\mathbb{Q}(1)$ and $-\mathbf{b} = \mathsf{T}$ spans $\mathbb{Q}(0)$.

By Proposition 2.3.1, we can write S in terms of this de Rham framing A, T of $\mathbb{H}$ (or in fact $\mathbb{H}(1)$)

$$\mathsf{S} = \mathsf{A} - 2G_2(\tau)\mathsf{T}.$$

We will use these sections A, S and T to write down the universal elliptic KZB connection in later sections.

2.4 Eisenstein Elliptic Functions and the Jacobi Form $F(\xi, \eta, \tau)$

2.4.1 Eisenstein series

A modular form[5] of weight k is a holomorphic function $f(\tau)$ on the upper half plane $\mathfrak{h}$ that satisfies

$$f(\gamma\tau) = (c\tau + d)^k f(\tau), \qquad \text{where } \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z})\ ,\ \tau \in \mathfrak{h},\ \text{and } \gamma\tau = \frac{a\tau + b}{c\tau + d}.$$

Since $f(\tau + 1) = f(\tau)$, it has a Fourier expansion of the form

$$f(\tau) = \sum_{n\in\mathbb{Z}} a_n q^n \qquad \text{where } q = e^{2\pi i \tau}.$$

Note that this q-series starting with terms of nonnegative q powers is equivalent to the condition that a modular form is holomorphic at the cusp. Moreover, a modular form is called a cusp form if the leading coefficient a_0 of its q-series is 0.

[5] In this paper, we will only consider modular forms of level one, i.e. those with respect to the entire modular group $\mathrm{SL}_2(\mathbb{Z})$.

Example 2.4.1. Weight k Eisenstein Series e_k. For $\tau \in \mathfrak{h}$, define

$$e_k(\tau) := \sum_{\substack{n,m \\ (n,m)\neq(0,0)}} (n\tau + m)^{-k}.$$

Note that $e_k = 0$ if k is odd.

Example 2.4.2. Normalized Weight k Eisenstein Series G_k. We will normalize the Eisenstein series following Zagier [30]. Define G_k to be zero when k is odd. For $k \geqslant 1$, $G_{2k}(\tau) := \frac{1}{2}\frac{(2k-1)!}{(2\pi i)^{2k}} e_{2k}(\tau)$, it has Fourier expansion

$$G_{2k}(\tau) = -\frac{B_{2k}}{4k} + \sum_{n=1}^{\infty} \sigma_{2k-1}(n) q^n$$

where B_k's are Bernoulli numbers[6] and $\sigma_k(n) = \sum_{d|n} d^k$, with

$$G_{2k}|_{q=0} = -\frac{B_{2k}}{4k} = \frac{(2k-1)!}{(2\pi i)^{2k}} \zeta(2k).$$

When $k > 2$ is even, the series for G_k is absolutely convergent. It is holomorphic on the upper half plane, satisfying $G_k(\gamma\tau) = (c\tau + d)^k G_k(\tau)$, and it is holomorphic at the cusp. Therefore, it is a modular form of weight k. When $k = 2$, the series needs to be added in some specific order (cf. [29]), and it satisfies (cf. [30])

$$G_2(\gamma\tau) = (c\tau + d)^2 G_2(\tau) + ic(c\tau + d)/4\pi.$$

It is well known that the ring of all normalized (Hecke eigen) modular forms is the polynomial ring $\mathbb{Q}[G_4, G_6]$.

2.4.2 Eisenstein elliptic functions

For $k \geqslant 2$, $\tau \in \mathfrak{h}$ and $\xi \in \mathbb{C}$, define Eisenstein elliptic functions [29] by

$$E_k(\xi, \tau) := \sum_{n,m} (\xi + n\tau + m)^{-k}.$$

[6] One can define Bernoulli numbers B_n by $\frac{x}{e^x - 1} = \sum_{n=0}^{\infty} B_n \frac{x^n}{n!}$. Note that $B_0 = 1$, $B_1 = -1/2$ and that $B_{2k+1} = 0$ when $k > 0$.

We will need the following formula, which is adapted from equation (9) of Chap. IV in [29].

Formula 2.4.3.

$$2\pi i\frac{\partial E_1}{\partial \tau} = E_3 - E_1E_2.$$

Now we define functions $P_k(\xi,\tau)$ for $k \geqslant 2$

$$P_k(\xi,\tau) := (2\pi i)^{-k}(E_k(\xi,\tau) - e_k(\tau)).$$

Note that up to a scalar, P_2 and P_3 are the Weierstrass $\wp$-function $\wp_\tau(\xi)$ and its derivative respectively.

These P_k's satisfy recurrence relations (cf. [29]): for $m \geqslant 3$, $n \geqslant 3$,

$$\begin{aligned} P_mP_n - P_{m+n} = &\frac{(-1)^n}{(n-1)!}\sum_{h=1}^{m-2}\frac{2}{h!}G_{n+h}P_{m-h} \\ &+ \frac{(-1)^m}{(m-1)!}\sum_{k=1}^{n-2}\frac{2}{k!}G_{m+k}P_{n-k} \\ &+ (-1)^m\frac{2(m+n)}{m!n!}G_{m+n} \end{aligned}$$

The same relation also holds for $m = 2, n \geqslant 2$.

Therefore, the algebra generated by Eisenstein elliptic functions is the ring $\mathbb{K}[P_2, P_3]$ with coefficients in $\mathbb{K} := \mathbb{Q}[G_4, G_6]$.

Remark 2.4.4. If variable τ is fixed, one can use P_2, P_3 to embed the elliptic curve E_τ into a cubic in $\mathbb{P}^2$ (see Section 2.7), then P_k's are algebraic functions on this elliptic curve. In particular, if the elliptic curve E_τ is defined over a field $\mathbb{K}$ of characteristic 0, then there is an embedding with $G_4, G_6 \in \mathbb{K}$, so that $G_k \in \mathbb{K}$ for all $k \geqslant 4$. Therefore, P_k's are polynomials of P_2, P_3 with coefficients in $\mathbb{K}$, i.e. P_k's are in the coordinate ring $\mathcal{O}(E_{\tau/\mathbb{K}})$, which is a $\mathbb{K}$-algebra generated by P_2, P_3.

2.4.3 The Jacobi forms $F(\xi,\eta,\tau)$ and $F^{\mathrm{Zag}}(u,v,\tau)$

There are two different versions of the Jacobi form F: one $F(\xi,\eta,\tau)$ used by Levin–Racinet [22], and another $F^{\mathrm{Zag}}(u,v,\tau)$ used by Zagier [30]. They are related to each other by

$$F(\xi,\eta,\tau)=2\pi i F^{\mathrm{Zag}}(2\pi i\xi,2\pi i\eta,\tau).$$

In this paper, we will use F^{Zag} exclusively.

2.4.4 Some useful formulas

In this section, we provide some formulas that will be used in later sections.

First, we express the Jacobi form F^{Zag} in terms of Eisenstein elliptic functions $P_k=(2\pi i)^{-k}(E_k-e_k)$ for $k\geqslant 2$ and $(2\pi i)^{-1}(E_1-e_1)=(2\pi i)^{-1}E_1$.

Formula 2.4.5.

$$TF^{\mathrm{Zag}}(2\pi i\xi,T,\tau)=\exp\Big(-\sum_{k=1}^{\infty}\frac{(-T)^k}{k}P_k(\xi,\tau)\Big).$$

Proof. By [30] p456, (viii),

$$F^{\mathrm{Zag}}(u,v,\tau)=\frac{u+v}{uv}\exp\Big(\sum_{k>0}\frac{2}{k!}[u^k+v^k-(u+v)^k]G_k(\tau)\Big).$$

Multiplying v, then take logarithm on both sides, we get

$$\begin{aligned}\log(vF^{\mathrm{Zag}}(u,v,\tau))&=\log(1+\frac{v}{u})+\sum_{k>0}\frac{2}{k!}[u^k+v^k-(u+v)^k]G_k(\tau)\\&=-\sum_{k=1}^{\infty}\frac{(-v/u)^k}{k}+\sum_{k=1}^{\infty}\frac{2v^k}{k!}[1+(\frac{u}{v})^k-(1+\frac{u}{v})^k]G_k(\tau)\end{aligned}$$

Let $u = 2\pi i\xi$, $v = T$, and rescale G_k back to e_k, we have

$$
\begin{aligned}
&\log(TF^{\mathrm{Zag}}(2\pi i\xi, T, \tau)) \\
&= -\sum_{k=1}^{\infty} \frac{(-T/2\pi i\xi)^k}{k} + \sum_{k=1}^{\infty} \frac{2T^k}{k!}\left[1 + (\frac{2\pi i\xi}{T})^k - (1 + \frac{2\pi i\xi}{T})^k\right] \frac{1}{2}\frac{(k-1)!}{(2\pi i)^k} e_k(\tau) \\
&= -\sum_{k=1}^{\infty} \frac{(-T)^k}{k}(2\pi i\xi)^{-k} - \sum_{l=1}^{\infty} \frac{T^l}{l}(2\pi i)^{-l} \sum_{k=1}^{l-1} \binom{l}{k} (\frac{2\pi i\xi}{T})^{l-k} e_l(\tau) \\
&= -\sum_{k=1}^{\infty} \frac{(-T)^k}{k}(2\pi i)^{-k}\frac{1}{\xi^k} - \sum_{k=1}^{\infty} \frac{T^k}{k}(2\pi i)^{-k} \sum_{l=k+1}^{\infty} \frac{k}{l}\binom{l}{k} \xi^{l-k} e_l(\tau) \\
&= -\sum_{k=1}^{\infty} \frac{(-T)^k}{k}(2\pi i)^{-k}\frac{1}{\xi^k} - \sum_{k=1}^{\infty} \frac{(-T)^k}{k}(2\pi i)^{-k}(-1)^k \sum_{l=k+1}^{\infty} \binom{l-1}{k-1} \xi^{l-k} e_l(\tau) \\
&= -\sum_{k=1}^{\infty} \frac{(-T)^k}{k}(2\pi i)^{-k}\left[\frac{1}{\xi^k} + (-1)^k \sum_{l=k+1}^{\infty} \binom{l-1}{k-1} \xi^{l-k} e_l(\tau)\right] \\
&= -\sum_{k=1}^{\infty} \frac{(-T)^k}{k}(2\pi i)^{-k}(E_k - e_k) = -\sum_{k=1}^{\infty} \frac{(-T)^k}{k} P_k(\xi, \tau)
\end{aligned}
$$

where the last line follows from equation (10) of Chap. III in [29], and the facts that $\binom{l-1}{k-1} = 0$ for $l < k$ and that $e_k(\tau) = 0$ for odd k. After taking exponential on both sides, (2.4.5) follows. □

Taking partial derivative with respect to T, we have

Formula 2.4.6.

$$
T\frac{\partial F^{\mathrm{Zag}}}{\partial T}(2\pi i\xi, T, \tau) = \exp\Big(-\sum_{k=1}^{\infty} \frac{(-T)^k}{k} P_k(\xi, \tau)\Big)\Big(\sum_{k=1}^{\infty} (-T)^{k-1} P_k(\xi, \tau) - \frac{1}{T}\Big).
$$

2.5 Unipotent Completion of a Group and its Lie Algebra

Given a finitely generated group Γ, and a field $\mathbb{K}$ of characteristic 0. The group algebra $\mathbb{K}\Gamma$ is naturally a Hopf algebra with coproduct, antipode and augmentation given by

$$\Delta : g \mapsto g \otimes g, \quad i : g \mapsto g^{-1}, \quad \epsilon : g \mapsto 1.$$

Note that it is cocommutative but not necessarily commutative, and thus not corresponding to the coordinate ring of an (pro-)algebraic group. It is natural to consider its (continuous) dual, which is commutative. We define the unipotent completion Γ^{un} of Γ over $\mathbb{K}$, an (pro-)algebraic group, by its coordinate ring

$$\mathcal{O}(\Gamma^{\mathrm{un}}_{/\mathbb{K}}) = \mathrm{Hom}_{\mathrm{cts}}(\mathbb{K}\Gamma, \mathbb{K}) := \varinjlim_{n} \mathrm{Hom}(\mathbb{K}\Gamma/I^n, \mathbb{K}),$$

where we give $\mathbb{K}\Gamma$ a topology by powers of its augmentation ideal $I := \ker \epsilon$. The set of its $\mathbb{K}$-rational points $\Gamma^{\mathrm{un}}(\mathbb{K})$ is in 1-1 correspondence with the set of

$$\{\text{ring homomorphisms } \mathcal{O}(\Gamma^{\mathrm{un}}_{/\mathbb{K}}) \to \mathbb{K}\}.$$

For example, any $\gamma \in \Gamma$ gives a ring homomorphism $\mathcal{O}(\Gamma^{\mathrm{un}}_{/\mathbb{K}}) \to \mathbb{K}$ by evaluating $\mathcal{O}(\Gamma^{\mathrm{un}}_{/\mathbb{K}})$ at γ, thus determines a $\mathbb{K}$-point $\gamma \in \Gamma^{\mathrm{un}}(\mathbb{K})$.

For the purpose of this paper, we only need to consider the case of Γ being a free group.

2.5.1 The unipotent completion of a free group

Suppose that Γ is the free group $\langle x_1, \ldots, x_n \rangle$ generated by the set $\{x_1, \ldots, x_n\}$. The coordinate ring $\mathcal{O}(\Gamma^{\mathrm{un}})$ of its unipotent completion Γ^{un} over $\mathbb{K}$ is a $\mathbb{K}$ vector space spanned by a basis $\{a_I\}$ indexed by tuples $I = (i_1, i_2, \cdots, i_r)$, where $i_j \in \{1, 2, \cdots, n\}$. If the index is empty, then $a_\varnothing \equiv 1$; if the index tuple only consists of one number $I = (i)$, we will simply write a_I as a_i. The product structure on $\mathcal{O}(\Gamma^{\mathrm{un}})$ is induced by shuffle product $ш$ and linearity

$$a_I \cdot a_J = \sum_{K \in \, I ш J} a_K.$$

For each $\mathbb{K}$-point $\gamma \in \Gamma^{\mathrm{un}}(\mathbb{K})$, "coordinate function" a_I takes value $a_I(\gamma)$ in $\mathbb{K}$, and

$$a_I(\gamma) \cdot a_J(\gamma) = \sum_{K \in I ш J} a_K(\gamma). \tag{2.5.1}$$

Note that it is natural to define $\{a_1, \cdots, a_n\}$ as the dual basis of $\{x_1, \cdots, x_n\}$, so that $a_i(x_j) = \delta_{ij}$.

To determine the structure of Γ^{un}, consider the ring $\mathbb{K}\langle\langle X_1, \ldots, X_n\rangle\rangle$ of formal power series in the non-commuting indeterminants X_j. It is a Hopf algebra with each X_j being primitive, and its augmentation ideal is the maximal ideal $I = (X_1, \ldots, X_n)$.

There is a unique group homomorphism

$$\theta : \Gamma \to \mathbb{K}\langle\langle X_1, \ldots, X_n\rangle\rangle$$
$$\gamma \mapsto \sum_I a_I(\gamma) X_I$$

that takes x_j to $\exp(X_j)$, where for $I = (i_1, i_2, \cdots, i_r)$, define $X_I := X_{i_1} X_{i_2} \cdots X_{i_r}$.

For any $\mathbb{K}$-point $\gamma \in \Gamma^{\mathrm{un}}(\mathbb{K})$, the element $\sum_I a_I(\gamma) X_I$ is group-like by (2.5.1). This induces a continuous isomorphism

$$\hat{\theta} : \Gamma^{\mathrm{un}}(\mathbb{K}) \to \{\text{group-like elements in } \mathbb{K}\langle\langle X_1, \ldots, X_n\rangle\rangle^\wedge\},$$

where $\mathbb{K}\langle\langle X_1, \ldots, X_n\rangle\rangle^\wedge$ is completed from $\mathbb{K}\langle\langle X_1, \ldots, X_n\rangle\rangle$ with respect to its augmentation ideal.

It is easy to use universal mapping properties to prove:

Proposition 2.5.1. *The homomorphism $\hat{\theta}$ is an isomorphism of complete Hopf algebras.* □

Corollary 2.5.2. *The restriction of $\hat{\theta}$ induces a natural isomorphism*

$$d\hat{\theta} : \mathrm{Lie}(\Gamma^{\mathrm{un}}(\mathbb{K})) \to \mathbb{L}_{\mathbb{K}}(X_1, \ldots, X_n)^\wedge$$

of topological Lie algebras.

Proof. This follows immediately from the fact that $\hat{\theta}$ induces an isomorphism on primitive elements and the well-known fact that the set of primitive elements of the power series algebra $\mathbb{K}\langle\langle X_1, \ldots, X_n\rangle\rangle$ is the completed free Lie algebra $\mathbb{L}_{\mathbb{K}}(X_1, \ldots, X_n)^\wedge$. □

Remark 2.5.3. By the Baker-Campbell-Hausdorff formula, the exponential map

$$\exp : \mathbb{L}_{\mathbb{K}}(X_1, \ldots, X_n)^\wedge \to \{\text{group-like elements in } \mathbb{K}\langle\langle X_1, \ldots, X_n\rangle\rangle^\wedge\},$$

is a group isomorphism. Therefore, $\Gamma^{\mathrm{un}}(\mathbb{K})$ and its Lie algebra $\mathrm{Lie}(\Gamma^{\mathrm{un}}(\mathbb{K}))$ are isomorphic as groups.

2.6 Universal Elliptic KZB Connection—Analytic Formula

In this section, we describe the main object to be studied in this paper. There is a canonical vector bundle $\mathcal{P}$ (resp. $\mathfrak{p}$) over $\mathcal{M}_{1,2}$ whose fiber over a moduli point $[E', x]$ is the unipotent fundamental group $\pi_1^{\mathrm{un}}(E', x)$ (resp. $\mathrm{Lie}(\pi_1^{\mathrm{un}}(E', x))$).[7] This vector bundle comes with an integrable connection, which is called the universal elliptic KZB connection. Analytic formulas for this connection have been given in different forms by Levin and Racinet [22] and by Calaque, Enriquez and Etingof [5].

The universal elliptic KZB connection for the bundle $\mathcal{P}$ over $\mathcal{E}'$ actually lives on $\mathcal{E}$, even $\overline{\mathcal{E}}$. Since $\mathcal{P}$ is a unipotent vector bundle, using Deligne's canonical extension, we obtain $\overline{\mathcal{P}}$ over $\overline{\mathcal{E}}$ by extending $\mathcal{P}$ across the boundary divisors, the identity section and the nodal cubic. The universal elliptic KZB connection has regular singularities around these divisors, as is shown in [15, §12, §13].

By Section 2.5.1, the fiber of $\mathcal{P}$ over a point $[E', x]$ is the Lie algebra $\mathrm{Lie}(\pi_1^{\mathrm{un}}(E', x))$ of its unipotent fundamental group $\pi_1^{\mathrm{un}}(E', x)$, which can be identified with $\mathbb{L}_{\mathbb{C}}(\mathsf{A}, \mathsf{T})^\wedge$, where A and T are the sections defined in §2.3.4.

[7] From Remark 2.5.3, the unipotent completion of a group and its Lie algebra are isomorphic, we will regard this bundle as a local system of both unipotent fundamental groups and Lie algebras, whichever is appropriate.

We now write the connection form in terms of analytic coordinates (ξ, τ) on $\mathbb{C} \times \mathfrak{h}$. It is shown to be $\mathrm{SL}_2(\mathbb{Z}) \ltimes \mathbb{Z}^2$-invariant and flat in [15, §9]. Therefore, it descends to a flat connection on the bundle $\mathcal{P}$ over the orbifold $\mathcal{E}$.

The connection is defined by

$$\nabla^{\mathrm{an}} f = df + \omega f,$$

with a 1-form

$$\omega \in \Omega^1(\mathbb{C} \times \mathfrak{h} \log \Lambda_{\mathfrak{h}}) \otimes \mathrm{Der}\, \mathbb{L}_{\mathbb{C}}(\mathsf{A}, \mathsf{T})^\wedge,$$

whose analytic formula is given by

$$\omega = 2\pi i\, d\tau \otimes \mathsf{A} \frac{\partial}{\partial \mathsf{T}} + \psi + \nu,$$

with

$$\psi = \sum_{m \geqslant 1} \frac{G_{2m+2}(\tau)}{(2m)!} 2\pi i\, d\tau \otimes \sum_{\substack{j+k=2m+1 \\ j,k>0}} (-1)^j [\mathrm{ad}_{\mathsf{T}}^j(\mathsf{A}), \mathrm{ad}_{\mathsf{T}}^k(\mathsf{A})] \frac{\partial}{\partial \mathsf{A}},$$

and

$$\nu = \nu_1 + \nu_2 = \mathsf{T} F^{\mathrm{Zag}}(2\pi i \xi, \mathsf{T}, \tau) \cdot \mathsf{A} \otimes 2\pi i\, d\xi + (\frac{1}{\mathsf{T}} + \mathsf{T} \frac{\partial F^{\mathrm{Zag}}}{\partial \mathsf{T}}(2\pi i \xi, \mathsf{T}, \tau)) \cdot \mathsf{A} \otimes 2\pi i\, d\tau.$$

Here, we view $\mathbb{L}_{\mathbb{C}}(\mathsf{A}, \mathsf{T})^\wedge$ as a Lie subalgebra of $\mathrm{Der}\, \mathbb{L}_{\mathbb{C}}(\mathsf{A}, \mathsf{T})^\wedge$ via the adjoint action, and T^n acts on $\mathbb{L}_{\mathbb{C}}(\mathsf{A}, \mathsf{T})^\wedge$ as $\mathrm{ad}_{\mathsf{T}}^n$.

In this part, we describe an algebraic de Rham structure $\overline{\mathcal{P}}_{\mathrm{dR}}$ on the restriction of the canonical bundle $\overline{\mathcal{P}}$ to a single elliptic curve E. We essentially reproduce and then complete the unfinished work of Levin–Racinet [22, §5]. In particular, their connection, though being algebraic, has an irregular singularity at the identity of the elliptic curve. Moreover, their formula is not explicit.

We compute explicitly the restriction of the universal elliptic KZB connection to a single elliptic curve in terms of its algebraic coordinates. We resolve the issue of irregular singularities at the identity in the connection formula by trivializing the bundle $\overline{\mathcal{P}}_{\mathrm{dR}}$ on two open subsets of E, one of which contains a neighborhood of the identity where the connection has a regular singularity. Therefore, we have constructed a de Rham structure $\overline{\mathcal{P}}_{\mathrm{dR}}$ on $\overline{\mathcal{P}}$ over E. Trivializing it on different open subsets is necessary because Deligne's canonical extension $\overline{\mathcal{P}}$ of $\mathcal{P}$ from E' (elliptic curve E punctured at the identity) to E, unlike the genus zero case (of $\mathbb{P}^1$), is not trivial as an algebraic vector bundle.

2.7 Elliptic Curves as Algebraic Curves

Fix $\tau \in \mathfrak{h}$ and an elliptic curve $E = E_\tau$. Using the Weierstrass $\wp$-function

$$\wp_\tau(\xi) := E_2(\xi, \tau) - e_2(\tau),$$

we can embed a punctured elliptic curve E' into $\mathbb{P}^2$ as follows:

$$\xi \mapsto [(2\pi i)^{-2}\wp_\tau(\xi), (2\pi i)^{-3}\wp'_\tau(\xi), 1].^8$$

This satisfies an affine equation

$$y^2 = 4x^3 - ux - v,$$

where $u = g_2(\tau) = 20G_4(\tau), v = g_3(\tau) = \frac{7}{3}G_6(\tau)$. It is defined over $\mathbb{K} := \mathbb{Q}(u, v)$. The identity of E is at the infinity. The equation $y = 0$ picks out three nontrivial order 2 elements in E (the trivial one being the identity), we define

$$E'' := E - \{y = 0\}.$$

Note that $\mathrm{id} \in E''$, and $\{E', E''\}$ form an open cover of E.

By pulling back through the above embedding, one can identify algebraic functions and forms with their analytic counterparts, which is how we will turn the

[8] We choose this embedding so that powers of $2\pi i$ will not appear in our algebra formulas of KZB connections later.

analytic formula of the connection into an algebraic formula. For example, coordinate functions $x, -\frac{y}{2}$ pull back to P_2, P_3 defined in Section 2.4.2, and the differential $\frac{dx}{y}$ pulls back to $2\pi i\, d\xi$. Note from Remark 3.6.1 that for $k \geqslant 2$, P_k can be expressed by a polynomial of P_2, P_3, i.e. $P_k = P_k(x, y) \in \mathbb{K}[x, y]$.

2.8 Algebraic Connection Formula over E'

Fix $\tau \in \mathfrak{h}$, an elliptic curve $E = E_\tau$ defined over a field $\mathbb{K}$ of characteristic zero, and its algebraic embedding as in the last section. The elliptic KZB connection restricted from the universal one to the once punctured elliptic curve $E' = E - \{\mathrm{id}\}$ is

$$\nabla^{\mathrm{an}} = d + \nu_1 = d + \mathsf{T}F^{\mathrm{Zag}}(2\pi i\xi, \mathsf{T}, \tau) \cdot \mathsf{A} \otimes 2\pi i\, d\xi. \tag{2.8.1}$$

Note that when described in terms of sections A and T, the bundle $\mathcal{P}$ has factors of automorphy [15, §6]. We would like to make sections of $\mathcal{P}$ elliptic (i.e. periodic with respect to the lattice Λ_τ) by a gauge transformation so that the connection form would also be elliptic and can be expressed in terms of algebraic coordinate functions x, y, and P_k's. Following Levin–Racinet [22, §5] and using Formula 2.4.5, the connection tranforms under the gauge $g_{\mathrm{alg}}(\xi) = \exp(-\frac{1}{2\pi i}E_1\mathsf{T})$ into $\nabla = d + \nu_1^{\mathrm{alg}}$, with 1-form

$$\begin{aligned}
\nu_1^{\mathrm{alg}} &= -dg_{\mathrm{alg}} \cdot g_{\mathrm{alg}}^{-1} + g_{\mathrm{alg}}\, \nu_1\, g_{\mathrm{alg}}^{-1} \\
&= -\frac{1}{2\pi i}E_2\mathsf{T}\, d\xi + \exp\Big(-\frac{E_1}{2\pi i}\mathsf{T}\Big)\exp\Big(-\sum_{k=1}^{\infty}\frac{(-\mathsf{T})^k}{k}P_k(\xi,\tau)\Big) \cdot \mathsf{A} \otimes 2\pi i\, d\xi \\
&= -(2\pi i)^{-2}(E_2 - e_2)\mathsf{T} \otimes 2\pi i d\xi \\
&\qquad + \exp\Big(-\sum_{k=2}^{\infty}\frac{(-\mathsf{T})^k}{k}P_k(\xi,\tau)\Big) \cdot (\mathsf{A} - (2\pi i)^{-2}e_2\mathsf{T}) \otimes 2\pi i\, d\xi
\end{aligned}$$

$$= -\frac{xdx}{y}\mathsf{T} + \exp\Big(-\sum_{k=2}^{\infty} \frac{(-\mathsf{T})^k}{k} P_k(x,y)\Big) \cdot \mathsf{S} \otimes 2\pi i \, d\xi$$

$$= -\frac{xdx}{y}\mathsf{T} + \frac{dx}{y}\mathsf{S} + \sum_{n=2}^{\infty} q_n(x,y)\frac{dx}{y}\mathsf{T}^n \cdot \mathsf{S}.$$

Here $\nu_1^{\text{alg}} \in \Omega^1(E'_{/\mathbb{K}}) \otimes \mathbb{L}_{\mathbb{K}}(\mathsf{S},\mathsf{T})^\wedge$ and

$$q_n(x,y) = \sum_{2a_2+3a_3+\cdots+na_n=n} \frac{1}{a_2!a_3!\cdots a_n!} \prod_{k=2}^{n} \left(\frac{(-1)^{k+1}P_k(x,y)}{k} \right)^{a_k} \in \mathcal{O}(E'_{/\mathbb{K}}),$$

where $\mathcal{O}(E'_{/\mathbb{K}}) = \mathbb{K}[x,y]/(y^2 - 4x^3 + ux + v)$. Note that the above sum is indexed by the partitions of integer n with summands at least 2. For example, 5 has 2 such partitions: 5 and $2+3$, so $q_5 = \frac{P_5}{5} + (-\frac{P_2}{2}) \cdot \frac{P_3}{3} = \frac{1}{5}P_5 - \frac{1}{6}P_2P_3$.

Remark 2.8.1. One can use the recurrence relations of P_k's described in Section 2.4.2 to find relations among q_n's.

Note that the form ν_1^{alg} is defined over $\mathbb{K}$, so we have constructed an algebraic vector bundle $(\mathcal{P}_{\text{dR}}, \nabla)$ over E' whose fibers can be identified with $\mathbb{L}_{\mathbb{K}}(\mathsf{S},\mathsf{T})^\wedge$. This algebraic bundle is defined over $\mathbb{K}$, with its connection ∇ also defined over $\mathbb{K}$. It provides us with a $\mathbb{K}$-structure $\mathcal{P}_{\text{dR}}$ on $\mathcal{P}$ over E'. Since the form ν_1^{alg} has irregular singularity ($\frac{xdx}{y}$ having a double pole) at the identity, we cannot extend it naively across the identity to obtain Deligne's canonical extension. To construct the canonical extension, we have to change gauge on a Zariski open neighborhood E'' of the identity. We do this in Section 2.9.

2.8.1 The naive connection vs. the elliptic KZB connection

Before we do this, we consider the naive connection on the trivial bundle

$$\mathbb{L}_{\mathbb{K}}(\mathsf{S},\mathsf{T})^\wedge \times E' \to E'$$

which is defined by $\nabla' = d + \nu_1^{\text{naive}}$, where

$$\nu_1^{\text{naive}} = -\frac{xdx}{y}\mathsf{T} + \frac{dx}{y}\mathsf{S}.^9$$

This flat connection is defined over $\mathbb{K}$, whose monodromy also induces an isomorphism between $\operatorname{Lie}\pi_1^{\text{un}}(E', x) \otimes_{\mathbb{K}} \mathbb{C}$ and $\mathbb{L}_{\mathbb{C}}(\mathsf{S}, \mathsf{T})^\wedge$. Since the elliptic KZB connection $\nabla = d + \nu_1^{\text{alg}}$ induces the same isomorphism (up to an inner automorphism of the fiber), one might expect that it gives the same $\mathbb{K}$-structure as the naive connection. However, this is not the case. The first step is the following lemma.

Lemma 2.8.2. *Fix a field $\mathbb{K}$ of characteristic zero. The elliptic KZB connection $\nabla = d + \nu_1^{\text{alg}}$ and the naive connection $\nabla' = d + \nu_1^{\text{naive}}$ are not algebraically gauge equivalent over any Zariski open subset of $E_{/\mathbb{K}}$.*

Proof. Suppose they were gauge equivalent, there would be a gauge transformation $g : E \dashrightarrow \exp \mathbb{L}(\mathsf{S}, \mathsf{T})^\wedge \subset \operatorname{Aut} \mathbb{L}(\mathsf{S}, \mathsf{T})^\wedge$, with coefficients in $\mathbb{K}(E)$, field of fractions of $\mathcal{O}(E'_{/\mathbb{K}})$, such that

$$\nu_1^{\text{alg}} = -dg \cdot g^{-1} + g\nu_1^{\text{naive}} g^{-1},$$

or equivalently

$$dg = g\nu_1^{\text{naive}} - \nu_1^{\text{alg}} g. \tag{2.8.2}$$

This is an equation of 1-forms on E with values in $\operatorname{Der} \mathbb{L}(\mathsf{S}, \mathsf{T})^\wedge$. Identify $\operatorname{Der} \mathbb{L}(\mathsf{S}, \mathsf{T})^\wedge$ with the universal enveloping algebra of $\mathbb{L}(\mathsf{S}, \mathsf{T})^\wedge$, and let

$$\begin{aligned} g = 1 + \alpha\mathsf{T} + \beta\mathsf{S} + \gamma\mathsf{T}^2 + \lambda\mathsf{ST} + \mu\mathsf{TS} + \delta\mathsf{S}^2 \\ + \sigma\mathsf{T}^3 + \zeta\mathsf{T}^2\mathsf{S} + \eta\mathsf{TST} + \xi\mathsf{TS}^2 + \tau\mathsf{ST}^2 + \kappa\mathsf{STS} + \epsilon\mathsf{S}^2\mathsf{T} + \iota\mathsf{S}^3 + \cdots \end{aligned}$$

where coefficients $\alpha, \beta, \gamma, \cdots \in \mathbb{K}(E)$ should be regarded as rational functions on the elliptic curve $E_{/\mathbb{K}}$. We substitute g into the above equation (3.6.2). Now we equate

[9] The "$-$"-sign appears as we regard $-\mathsf{T}$ and S as a basis for $H_1(E)$, dual to $\mathsf{S} = xdx/y$ and $\mathsf{T} = dx/y$ in $H^1(E)$, see Remark 2.3.2 before Section 2.3.4.

the coefficients up to the third degree of derivations in $\operatorname{Der}\mathbb{L}(\mathsf{S},\mathsf{T})^\wedge$. We have

$$d\alpha = 0, \quad d\beta = 0, \quad d\gamma = 0, \quad d\delta = 0, \quad d\sigma = 0 \tag{2.8.3}$$

$$d\mu = \alpha\,\frac{dx}{y} + \beta\,\frac{x\,dx}{y} \tag{2.8.4}$$

$$d\lambda = -\beta\,\frac{x\,dx}{y} - \alpha\,\frac{dx}{y} \tag{2.8.5}$$

$$d\zeta = \gamma\,\frac{dx}{y} + \mu\,\frac{x\,dx}{y} + \frac{1}{2}\frac{x\,dx}{y} \tag{2.8.6}$$

$$d\eta = -\mu\,\frac{x\,dx}{y} + \lambda\,\frac{x\,dx}{y} \tag{2.8.7}$$

$$d\xi = \mu\,\frac{dx}{y} + \delta\frac{x\,dx}{y} \tag{2.8.8}$$

$$\cdots \tag{2.8.9}$$

From (2.8.3), we know that α and β are constants. Taking cohomology classes on both sides of (2.8.4), we have $\alpha\,[\frac{dx}{y}] + \beta\,[\frac{x\,dx}{y}] = 0$, and easily get $\alpha = \beta = 0$. Thus $d\mu = 0$, and μ is a constant. For the same reason, λ is a constant.

Now taking cohomology classes on both sides of (2.8.6) and of (2.8.7), we get $\gamma = 0$, and $\lambda = \mu = -\frac{1}{2}$. Similarly, taking cohomology classes on both sides of (2.8.8), we get $\mu = \delta = 0$. However, μ cannot be $-\frac{1}{2}$ and 0 at the same time! □

Proposition 2.8.3. *The naive connection provides a different* $\mathbb{K}$*-structure than the canonical* $\mathbb{K}$*-structure given by the elliptic KZB connection.*

Proof. Note that the elliptic KZB connection has regular singularity at the identity (already known analytically and will be shown algebraically in the next two sections). Suppose the $\mathbb{K}$-structures on the vector bundles over E' given by both connections were the same, then their canonical extensions à la Deligne from E' to E would be the same up to a gauge transformation. In particular, working over $\mathbb{C}$, there would be a morphism between their sheaves of flat sections of these vector bundles over

E, which is compatible with the connections and meromorphic at the identity. This contradicts Lemma 2.8.2 when the field is $\mathbb{C}$. □

Remark 2.8.4. Since the monodromy representations induced by both connections are the same (up to conjugacy) over $\mathbb{C}$, there is a complex analytic gauge transformation between the connections. Our discussion above in the proof shows that this complex analytic gauge transformation cannot be meromorphic at the identity, so it must have an essential singularity there.

Remark 2.8.5. Working over $\mathbb{C}$, define a regular connection over E to be a meromorphic connection over E that is holomorphic on E' and has at most a simple pole at the identity under some meromorphic gauge transformation. It follows from the previous lemma that the naive connection is not a regular connection. Otherwise, by Riemann-Hilbert correspondence,

$$\left\{ \begin{array}{c} \text{local systems} \\ \mathbb{V} \text{ over } E' \end{array} \right\} \leftrightarrow \left\{ \begin{array}{c} \text{vector bundles } \mathcal{V} \\ \text{over } E' \text{ with a flat} \\ \text{connection } \nabla \end{array} \right\} \leftrightarrow \left\{ \begin{array}{c} \text{vector bundles } \overline{\mathcal{V}} \\ \text{over } E \text{ with a regular} \\ \text{connection } \nabla \end{array} \right\},$$

the naive connection and the elliptic KZB connection over E', having the same monodromy representation up to conjugacy, would extend to the same regular connection over E up to gauge equivalence, which contradicts the lemma.

2.9 Algebraic Connection Formula over E''

Recall that we have an analytic local system $(\mathcal{P}, \nabla^{\text{an}})$ of (Lie algebras of) unipotent fundamental groups over E'. The elliptic KZB connection ∇^{an} is obtained by restricting the universal elliptic KZB connection to E'. Note that the analytic formula of the elliptic KZB connection ∇^{an} has regular singularity at the identity with pronilpotent residue. The elliptic KZB connection thus extends naturally from E' to E, and we obtain Deligne's canonical extension $(\overline{\mathcal{P}}, \nabla^{\text{an}})$ of $(\mathcal{P}, \nabla^{\text{an}})$. It is not immediately clear that $(\overline{\mathcal{P}}, \nabla^{\mathrm{an}})$ has an algebraic de Rham structure. The question is to determine whether this canonical extension is defined over $\mathbb{K}$, the field of definition of E. In this section, we show that the elliptic KZB connection ∇^{an} is gauge equivalent to its algebraic counterpart ∇ defined over $\mathbb{K}$, which has regular singularity at the identity with pronilpotent residue. It follows that Deligne's canonical extension $\overline{\mathcal{P}}$ of $\mathcal{P}$ to E is defined over $\mathbb{K}$.

We start with the algebraic connection $\nabla = d + \nu_1^{\mathrm{alg}}$, which is defined to be gauge equivalent to the analytic one ∇^{an} in the last section. Since ν_1^{alg} has irregular singularities at the identity of E, we would like to apply another gauge transformation to make it regular. The reason ν_1^{alg} has irregular singularity is that we introduced a gauge transformation involving E_1, which has a pole at the identity. To cancel this effect and make the connection regular at the identity, we apply a gauge transformation $g_{\mathrm{reg}} = \exp(-\frac{2x^2}{y}\mathsf{T})$. Then the connection becomes $\nabla = d + \nu_1^{\mathrm{reg}}$, with 1-form

$$\begin{aligned}
\nu_1^{\mathrm{reg}} &= -dg_{\mathrm{reg}} \cdot g_{\mathrm{reg}}^{-1} + g_{\mathrm{reg}}\, \nu_1^{\mathrm{alg}}\, g_{\mathrm{reg}}^{-1} \\
&= \left(d\left(\frac{2x^2}{y}\right) - \frac{x dx}{y} \right) \mathsf{T} + \exp\Big(-\frac{2x^2}{y}\mathsf{T} - \sum_{k=2}^{\infty} \frac{(-\mathsf{T})^k}{k} P_k(x,y) \Big) \cdot \mathsf{S} \otimes 2\pi i \, d\xi \\
&= \left(d\left(\frac{2x^2}{y}\right) - \frac{x dx}{y} \right) \mathsf{T} + \frac{dx}{y}\mathsf{S} + \sum_{n=1}^{\infty} r_n(x,y) \frac{dx}{y} \mathsf{T}^n \cdot \mathsf{S}.
\end{aligned}$$

Here $\nu_1^{\mathrm{reg}} \in \Omega^1(E'' \log\{\mathrm{id}\}) \otimes \mathbb{L}_{\mathbb{K}}(\mathsf{S},\mathsf{T})^{\wedge}$ and we have rational functions

$$r_n(x,y) = \sum_{a_1+2a_2+3a_3+\cdots+na_n=n} \frac{1}{a_1!a_2!a_3!\cdots a_n!} \prod_{k=1}^{n} \left(\frac{(-1)^{k+1} P_k(x,y)}{k} \right)^{a_k} \in \mathcal{O}(E'' - \{\mathrm{id}\}),$$

where $P_1(x,y) := -\frac{2x^2}{y}$ and $\mathcal{O}(E'' - \{\mathrm{id}\}) = \mathcal{O}(E'_y) = \mathcal{O}(E')[y^{-1}]$. Note that the sum for r_n is indexed by the partitions of integer n with no restrictions of the summands.

For example, 4 has 5 partitions: 4, 3 + 1, 2 + 2, 2 + 1 + 1, 1 + 1 + 1 + 1, so

$$r_4 = (-\frac{P_4}{4}) + \frac{1}{1!1!}(\frac{P_3}{3})\cdot(\frac{P_1}{1}) + \frac{1}{2!}(-\frac{P_2}{2})^2 + \frac{1}{2!1!}(\frac{P_1}{1})^2\cdot(-\frac{P_2}{2}) + \frac{1}{4!}(\frac{P_1}{1})^4$$
$$= -\frac{1}{4}P_4 + \frac{1}{3}P_3P_1 + \frac{1}{8}P_2^2 - \frac{1}{4}P_2P_1^2 + \frac{1}{24}P_1^4$$

Remark 2.9.1. One can use the recurrence relations of P_k's described in Section 2.4.2 to find relations among r_n's.

In the next section, we will check that

Lemma 2.9.2. *The connection $\nabla = d + \nu_1^{\text{reg}}$ has a regular singularity at the identity with pronilpotent residue.*

Therefore, this connection a priori living on $E'' - \{\text{id}\}$, can be extended naturally across the identity. It is an algebraic connection defined over $\mathbb{K}$ on an algebraic vector bundle $\overline{\mathcal{P}}_{\text{dR}}|_{E''}$ over the open subset E'' of E. This is one part of a vector bundle $\overline{\mathcal{P}}_{\text{dR}}$ over E. The other part $\overline{\mathcal{P}}_{\text{dR}}|_{E'} = \mathcal{P}_{\text{dR}}$ was constructed using the connection $\nabla = d + \nu_1^{\text{alg}}$ in the last section. Now we have trivialized $\overline{\mathcal{P}}_{\text{dR}}$ on an open cover of two different subsets of E. By gluing two trivializations together in terms of the gauge transformation, we have constructed an algebraic vector bundle $\overline{\mathcal{P}}_{\text{dR}}$ over E.

Summarizing results in this part, we get

Theorem 2.9.3 (The algebraic de Rham structure $\overline{\mathcal{P}}_{\text{dR}}$ on $\overline{\mathcal{P}}$ over E). *Suppose that $\mathbb{K}$ is a field of characteristic 0, embeddable in $\mathbb{C}$. Let E be an elliptic curve defined over $\mathbb{K}$. Then for each embedding $\sigma : \mathbb{K} \hookrightarrow \mathbb{C}$, we have an algebraic vector bundle $\overline{\mathcal{P}}_{\text{dR}}$ over $E_{/\mathbb{K}}$ endowed with connection ∇, and an isomorphism*

$$(\overline{\mathcal{P}}_{\text{dR}}, \nabla) \otimes_{\mathbb{K},\sigma} \mathbb{C} \approx (\overline{\mathcal{P}}, \nabla^{\text{an}}).$$

The algebraic bundle $\overline{\mathcal{P}}_{\text{dR}}$ and its connection ∇ are both defined over $\mathbb{K}$. The $\mathbb{K}$-de Rham structure $(\overline{\mathcal{P}}_{\text{dR}}, \nabla)$ on $(\overline{\mathcal{P}}, \nabla^{\text{an}})$ is explicitly given by the connection formulas *for ν_1^{alg} on E' and ν_1^{reg} on E'' above. In particular, the connection ∇ has a regular singularity at the identity.*

2.10 Regular Singularity and Residue at the Identity

In this section, we prove Lemma 2.9.2 by showing that ν_1^{reg} has regular singularity at the identity, and we compute its residue there.

It is easy to check that analytically $d\left(\frac{2x^2}{y}\right) - \frac{xdx}{y}$ is holomorphic at the identity. So we are left to check that

$$1 + \sum_{n=1}^{\infty} r_n(x,y)\mathsf{T}^n = \exp\Big(- \sum_{k=1}^{\infty} \frac{(-\mathsf{T})^k}{k} P_k(x,y)\Big) \tag{2.10.1}$$

has a regular singularity at the identity.

Let ξ be the complex coordinate near the identity. Analytically, we need to calculate (in terms of ξ) the principal parts of $P_1 = -\frac{2x^2}{y}$ and P_k's ($k \geqslant 2$). The principal part of $P_1 = -\frac{2x^2}{y}$ is $\frac{1}{2\pi i \xi}$; the principal part of P_k ($k \geqslant 2$) is $\frac{1}{(2\pi i\xi)^k}$, since

$$\begin{aligned}
P_k &= (2\pi i)^{-k}(E_k - e_k) \\
&= (2\pi i)^{-k} \sum_{m,n} (\xi + m\tau + n)^{-k} - \sum_{m,n}{}' (m\tau + n)^{-k} \\
&= \frac{1}{(2\pi i)^k \xi^k} + (2\pi i)^{-k} \sum_{m,n}{}' \left(\frac{1}{(\xi + m\tau + n)^k} - \frac{1}{(m\tau+n)^k}\right) \\
&= \frac{1}{(2\pi i)^k \xi^k} + (2\pi i)^{-k} \sum_{m,n}{}' \frac{1}{(m\tau+n)^k} \sum_{l=1}^{\infty} (-1)^l \binom{l+k-1}{k-1} \frac{\xi^l}{(m\tau+n)^l} \\
&= \frac{1}{(2\pi i \xi)^k} + \sum_{l=1}^{\infty} (-1)^l \binom{l+k-1}{k-1} (2\pi i)^{-(k+l)} e_{k+l}\, (2\pi i \xi)^l.
\end{aligned}$$

Therefore, (2.10.1) is of the following form near $\xi = 0$,

$$\exp\Big(-\sum_{k=1}^{\infty}\frac{(-\mathsf{T}/(2\pi i\xi))^k}{k} + \text{holo. in } \xi\Big)$$

$$= \exp\left(\ln(1+\mathsf{T}/(2\pi i\xi))\right)\exp\Big(\sum_{n=0}^{\infty} a_n(\mathsf{T})(2\pi i\xi)^n\Big)$$

$$= (1+\mathsf{T}/(2\pi i\xi))\exp\Big(\sum_{n=1}^{\infty} a_n(\mathsf{T})(2\pi i\xi)^n\Big),$$

which has a regular singularity at the identity. Here $\forall n \geqslant 0, a_n(\mathsf{T}) \in \mathbb{K}[\mathsf{T}]$ and $a_0(\mathsf{T}) = 0$.

Now it's easy to calculate the residue. Note that $\frac{2x^2}{y}$ is an odd function in ξ, and when expressed in terms of ξ, it has constant term 0 in the holomorphic part; so does each of the P_k's according to their expansions above. Therefore, we know that the holomorphic part in ξ also has constant term 0, and the residue at the identity we are looking for is then

$$\frac{\mathsf{T}}{2\pi i}\exp(0)\cdot\mathsf{S}(2\pi i) = \mathsf{T}\cdot\mathsf{S} = \mathrm{ad}_{[\mathsf{T},\mathsf{S}]},$$

which is pronilpotent in $\operatorname{Der}\mathbb{L}(\mathsf{S},\mathsf{T})^{\wedge}$. Note that $(2\pi i)$ at the end of the first expression above comes from $dx/y = 2\pi i d\xi$.

2.11 Tannaka Theory and a Universal Unipotent Connection over E

Recall that a unipotent object in a tensor category $\mathcal{C}$ with the identity object $\mathbb{1}_{\mathcal{C}}$ is an object V with a filtration in $\mathcal{C}$

$$0 = V_0 \subseteq \cdots \subseteq V_n = V$$

such that each quotient V_j/V_{j-1} is isomorphic to $\mathbb{1}_{\mathcal{C}}^{\oplus k_j}$ for some $k_j \in \mathbb{N}$.

Let E be an elliptic curve defined over $\mathbb{K}$ and $E' = E - \{\mathrm{id}\}$. Consider the following tensor categories:

1. **Unipotent Local Systems**

$$\mathcal{C}_F^{\mathrm{B}} := \{\text{unipotent local systems } \mathbb{V}_F \text{ over } E'(\mathbb{C})\},$$

where F is a field of characteristic 0, and the identity object $\mathbb{1}_{\mathcal{C}_F^{\mathrm{B}}}$ is the constant sheaf $F_{E'}$ on $E'(\mathbb{C})$;

2. **Algebraic de Rham**

$$\mathcal{C}_{\mathbb{K}}^{\mathrm{dR}} := \left\{ \begin{array}{c} \text{unipotent vector bundles } \overline{\mathcal{V}} \text{ over } E_{/\mathbb{K}} \text{ defined over } \mathbb{K} \\ \text{with a flat connection } \nabla \text{ that has regular singularity} \\ \text{at the identity with nilpotent residue} \end{array} \right\},$$

where the identiy object $\mathbb{1}_{\mathcal{C}_{\mathbb{K}}^{\mathrm{dR}}}$ is the trivial vector bundle $\mathcal{O}_E$ with the trivial connection given by the exterior differential d;

3. **Analytic de Rham**

$$\mathcal{C}^{\mathrm{an}} := \left\{ \begin{array}{c} \text{unipotent flat vector bundles } \overline{\mathcal{V}}^{\mathrm{an}} \text{ over } E^{\mathrm{an}} \text{ with a connection} \\ \text{that is holomorphic over } E'(\mathbb{C}) \text{, meromorphic over } E(\mathbb{C}) \text{ and} \\ \text{has regular singularity at the identity with nilpotent residue} \end{array} \right\},$$

where $E^{\mathrm{an}} = E(\mathbb{C})$ is the analytic variety associated to $E_{/\mathbb{K}}$, and the identiy object $\mathbb{1}_{\mathcal{C}^{\mathrm{an}}}$ is the trivial vector bundle $\mathcal{O}_{E^{\mathrm{an}}}$ with the trivial connection given by the exterior differential d.

One can define fiber functors for these tensor categories so that they become neutral tannakian categories. Taking the fiber over $x \in E'(\mathbb{C})$ of any object in $\mathcal{C}_F^{\mathrm{B}}$ provides a fiber functor ω_x of $\mathcal{C}_F^{\mathrm{B}}$. By Tannaka duality and the universal property of unipotent completion, the tannakian fundamental group of $\mathcal{C}_F^{\mathrm{B}}$ with respect to the fiber functor ω_x, which we denote by $\pi_1(\mathcal{C}_F^{\mathrm{B}}, \omega_x)$, is the unipotent fundamental group $\pi_1^{\mathrm{un}}(E', x)_F$ over F. We will denote $\pi_1^{\mathrm{un}}(E', x)_{\mathbb{Q}}$ simply by $\pi_1^{\mathrm{un}}(E', x)$.

In the same way, one can define a fiber functor ω_x of $\mathcal{C}^{\mathrm{an}}$ for any $x \in E(\mathbb{C})$, and a fiber functor ω_x of $\mathcal{C}_{\mathbb{K}}^{\mathrm{dR}}$ for any $x \in E(\mathbb{K})$. Note that we can take x to be the identity id $\in E(\mathbb{K})$. We denote their corresponding tannakian fundamental groups by $\pi_1(\mathcal{C}^{\mathrm{an}}, \omega_x)$ and $\pi_1(\mathcal{C}^{\mathrm{dR}}_{\mathbb{K}}, \omega_x)$ respectively. Our objective is to establish a natural comparison isomorphism between $\pi_1(\mathcal{C}^{\mathrm{an}}, \omega_x)$ and $\pi_1(\mathcal{C}^{\mathrm{dR}}_{\mathbb{K}}, \omega_x) \times_{\mathbb{K}} \mathbb{C}$ for any $x \in E(\mathbb{K})$.

2.11.1 Extension groups in $\mathcal{C}^{\mathrm{B}}_F$, $\mathcal{C}^{\mathrm{an}}$ and $\mathcal{C}^{\mathrm{dR}}_{\mathbb{K}}$

We start with a general setting. Let K be a field of characterisitic zero. Let $\mathcal{C}$ be a neutral tannakian category over K with a fiber functor ω all of whose objects are unipotent. Denote its identity object by $\mathbb{1}_{\mathcal{C}}$. The tannakian fundamental group of $\mathcal{C}$ with respect to ω, which we denote by $\mathfrak{U}$, is a prounipotent group defined over K. Denote its Lie algebra by $\mathfrak{u}$, viewed as a topological Lie algebra. Since the category of $\mathfrak{U}$-modules is equivalent to the category of continuous $\mathfrak{u}$-modules, we have

$$H^m_{\mathrm{cts}}(\mathfrak{u}) \cong H^m(\mathfrak{U}) \cong \mathrm{Ext}^m_{\mathcal{C}}(\mathbb{1}_{\mathcal{C}}, \mathbb{1}_{\mathcal{C}}).$$

The following is standard.

Proposition 2.11.1. *Let $\mathfrak{u}$ be a pronilpotent Lie algebra, and denote its abelianization by $H_1(\mathfrak{u})$. Then*

$$H_1(\mathfrak{u}) \cong \mathrm{Hom}(H^1_{\mathrm{cts}}(\mathfrak{u}), K).$$

If $H^2(\mathfrak{u}) = 0$, then $\mathfrak{u}$ is a free Lie algebra.

Therefore, if $\mathrm{Ext}^2_{\mathcal{C}}(\mathbb{1}_{\mathcal{C}}, \mathbb{1}_{\mathcal{C}}) = 0$, then the Lie algebra $\mathfrak{u}$ of the tannakian fundamental group of $\mathcal{C}$ is freely generated by $\mathrm{Ext}^1_{\mathcal{C}}(\mathbb{1}_{\mathcal{C}}, \mathbb{1}_{\mathcal{C}})^*$, the K-dual of $\mathrm{Ext}^1_{\mathcal{C}}(\mathbb{1}_{\mathcal{C}}, \mathbb{1}_{\mathcal{C}})$.

Now we compute extension groups in categories $\mathcal{C}^{\mathrm{B}}_F$, $\mathcal{C}^{\mathrm{an}}$ and $\mathcal{C}^{\mathrm{dR}}_{\mathbb{K}}$.

Lemma 2.11.2.

$$\mathrm{Ext}^1_{\mathcal{C}}(\mathbb{1}_{\mathcal{C}}, \mathbb{1}_{\mathcal{C}}) \cong \begin{cases} H^1(E(\mathbb{C}); F) & \textit{when } \mathcal{C} = \mathcal{C}^{\mathrm{B}}_F, \\ H^1(E(\mathbb{C}); \mathbb{C}) & \textit{when } \mathcal{C} = \mathcal{C}^{\mathrm{an}}, \\ H^1_{\mathrm{dR}}(E_{/\mathbb{K}}) & \textit{when } \mathcal{C} = \mathcal{C}^{\mathrm{dR}}_{\mathbb{K}}. \end{cases}$$

Proof. The first two cases are well known. The third case can be easily obtained by a GAGA argument; instead, we provide another proof by using our algebraic connection formulas on $\overline{\mathcal{P}}_{\mathrm{dR}}$. Given a global 1-form ω on E, we can define a connection

$$\nabla = d + \begin{pmatrix} 0 & \omega \\ 0 & 0 \end{pmatrix},$$

on the trivial bundle $\mathcal{O}_E \oplus \mathcal{O}_E$. This defines an extension in $\mathrm{Ext}^1_{\mathcal{C}^{\mathrm{dR}}_{\mathbb{K}}}(\mathbb{1}_{\mathcal{C}^{\mathrm{dR}}_{\mathbb{K}}}, \mathbb{1}_{\mathcal{C}^{\mathrm{dR}}_{\mathbb{K}}})$ and gives rise to a map

$$e : H^0(E, \Omega^1_E) \to \mathrm{Ext}^1_{\mathcal{C}^{\mathrm{dR}}_{\mathbb{K}}}(\mathbb{1}_{\mathcal{C}^{\mathrm{dR}}_{\mathbb{K}}}, \mathbb{1}_{\mathcal{C}^{\mathrm{dR}}_{\mathbb{K}}}),$$

which is injective. To see this, we first tensor both sides of this map with $\mathbb{C}$. One can then identify the extension group as $H^1(E; \mathbb{C})$ by using monodromy. And the map becomes the inclusion of holomorphic 1-forms on E into $H^1(E; \mathbb{C})$, which is injective.

Suppose we have a vector bundle $(\mathcal{V}, \nabla) \in \mathcal{C}^{\mathrm{dR}}_{\mathbb{K}}$, which is an extension of $(\mathcal{O}_E, d)$ by $(\mathcal{O}_E, d)$. By forgeting the connections on all these bundles, this extension determines a class in $\mathrm{Ext}^1_E(\mathcal{O}_E, \mathcal{O}_E) \cong H^1(E, \mathcal{O}_E)$. This gives rise to a map f and the following sequence

$$0 \to H^0(E, \Omega^1_E) \xrightarrow{e} \mathrm{Ext}^1_{\mathcal{C}^{\mathrm{dR}}_{\mathbb{K}}}(\mathbb{1}_{\mathcal{C}^{\mathrm{dR}}_{\mathbb{K}}}, \mathbb{1}_{\mathcal{C}^{\mathrm{dR}}_{\mathbb{K}}}) \xrightarrow{f} H^1(E, \mathcal{O}_E) \to 0.$$

The result follows if this is a short exact sequence.

Suppose a vector bundle $(\mathcal{V}, \nabla)$ represents a class in $\ker f$, then we have a split extension (without connection)

$$0 \to \mathcal{O}_E \to \mathcal{V} \to \mathcal{O}_E \to 0.$$

Fixing a splitting on $\mathcal{V}$, the connection can be written as

$$\nabla = d + \begin{pmatrix} 0 & \omega \\ 0 & 0 \end{pmatrix},$$

where ω is a global 1-form on E. So we have

$$\ker f = \operatorname{im} e.$$

To show f is surjective, we provide here explicitly a vector bundle with connection that corresponds to a nontrivial extension class in $H^1(E, \mathcal{O}_E)$. Recall that the connection ∇ on $\overline{\mathcal{P}}_{\mathrm{dR}}$ is given by algebraic connection formulas

$$\nabla = \begin{cases} d + \nu_1^{\mathrm{alg}} = d - \frac{xdx}{y}\mathsf{T} + \frac{dx}{y}\mathsf{S} + \cdots & \text{on } E', \\ d + \nu_1^{\mathrm{reg}} = d + \left(d\left(\frac{2x^2}{y}\right) - \frac{xdx}{y}\right)\mathsf{T} + \frac{dx}{y}\mathsf{S} + \cdots & \text{on } E''. \end{cases}$$

The leading terms recorded here provides a connection on the abelianization of $\overline{\mathcal{P}}_{\mathrm{dR}}$. This gives a nontrivial extension of $\mathcal{O}_E$ by $\mathcal{O}_E$, thus corresponds to a nontrivial class in $H^1(E, \mathcal{O}_E)$. □

2.11.2 The de Rham tannakian fundamental group $\pi_1(\mathcal{C}^{\mathrm{dR}}_{\mathbb{K}}, \omega_x)$

It is well known that there is an equivalence of categories

$$\mathcal{C}^{\mathrm{B}}_{\mathbb{C}} \rightleftarrows \mathcal{C}^{\mathrm{an}}.$$

The right arrow is the functor that takes a unipotent local system $\mathbb{V}$ over E' to Deligne's canonical extension $(\mathcal{V}, \nabla)$ of $\mathbb{V} \otimes \mathcal{O}^{\mathrm{an}}_{E'}$, where

$$\nabla : \mathcal{V} \to \mathcal{V} \otimes \Omega^1_E(\log\{\mathrm{id}\}).$$

The left arrow is the functor obtained by taking locally flat sections of $\mathcal{V}$ over E'. By this equivalence, we obtain an isomorphism between their tannakian fundamental groups

$$\mathrm{comp}_{\mathrm{an,B}} : \pi_1(\mathcal{C}^{\mathrm{an}}, \omega_x) \xrightarrow{\cong} \pi_1(\mathcal{C}^{\mathrm{B}}_{\mathbb{C}}, \omega_x) = \pi_1^{\mathrm{un}}(E', x) \times_{\mathbb{Q}} \mathbb{C} \tag{2.11.1}$$

for each $x \in E'(\mathbb{C})$. By Section 2.5.1, as an abstract group, the unipotent fundamental group $\pi_1^{\mathrm{un}}(E', x)_{\mathbb{C}}$ can be identified with its Lie algebra $\mathbb{L}_{\mathbb{C}}(\mathsf{A}, \mathsf{T})^\wedge$, which is the same as $\mathbb{L}_{\mathbb{C}}(\mathsf{S}, \mathsf{T})^\wedge$, where A, S and T are the sections defined in §2.3.4.

The local system $\mathcal{P}$ over E' is a pro-object in $\mathcal{C}^{\mathrm{B}}_{\mathbb{C}}$, which is equivalent to an action of the tannakian fundamental group on the fiber of $\mathcal{P}$ over x

$$\pi_1(\mathcal{C}^{\mathrm{B}}_{\mathbb{C}}, \omega_x) \to \mathrm{Aut}\, \mathbb{L}_{\mathbb{C}}(\mathsf{A}, \mathsf{T})^\wedge. \tag{2.11.2}$$

This corresponds to the adjoint action of $\mathbb{L}_{\mathbb{C}}(\mathsf{A},\mathsf{T})^\wedge$ on itself

$$\mathrm{ad} : \mathbb{L}_{\mathbb{C}}(\mathsf{A},\mathsf{T})^\wedge \to \mathrm{Der}\, \mathbb{L}_{\mathbb{C}}(\mathsf{A},\mathsf{T})^\wedge. \tag{2.11.3}$$

There is another equivalence of categories

$$\mathcal{C}_{\mathbb{C}}^{\mathrm{dR}} \rightleftarrows \mathcal{C}^{\mathrm{an}},$$

where the right arrow is the obvious one, and the left arrow exists by GAGA: since $E^{\mathrm{an}} = E(\mathbb{C})$ is projective, the category of analytic sheaves over E^{an} is equivalent to its algebraic counterpart over $\mathbb{C}$. By this equivalence, we have an isomorphism of tannakian fundamental groups

$$\pi_1(\mathcal{C}^{\mathrm{an}}, \omega_x) \to \pi_1(\mathcal{C}_{\mathbb{C}}^{\mathrm{dR}}, \omega_x)$$

for any $x \in E(\mathbb{C})$. It is not immediately clear whether one can get a $\mathbb{K}$-structure on $\pi_1(\mathcal{C}_{\mathbb{C}}^{\mathrm{dR}}, \omega_x)$, or a natural comparison isomorphism between $\pi_1(\mathcal{C}^{\mathrm{an}}, \omega_x)$ and $\pi_1(\mathcal{C}_{\mathbb{K}}^{\mathrm{dR}}, \omega_x) \times_{\mathbb{K}} \mathbb{C}$ for each $x \in E(\mathbb{K})$.

Proposition 2.11.3. *There is a natural comparison isomorphism*

$$\mathrm{comp}_{\mathrm{an,dR}} : \pi_1(\mathcal{C}^{\mathrm{an}}, \omega_x) \to \pi_1(\mathcal{C}_{\mathbb{K}}^{\mathrm{dR}}, \omega_x) \times_{\mathbb{K}} \mathbb{C}$$

for any $x \in E(\mathbb{K})$.

Proof. This map $\mathrm{comp}_{\mathrm{an,dR}}$ is induced from the functor of tensoring with $\mathbb{C}$:

$$\mathcal{C}_{\mathbb{K}}^{\mathrm{dR}} \otimes \mathbb{C} \to \mathcal{C}_{\mathbb{C}}^{\mathrm{dR}} \simeq \mathcal{C}^{\mathrm{an}}.$$

We study it by working with a special object. In Section 2.9, we constructed such an object: an algebraic vector bundle $(\overline{\mathcal{P}}_{\mathrm{dR}}, \nabla)$ over E with a connection ∇ defined over $\mathbb{K}$. It is a pro-object in $\mathcal{C}_{\mathbb{K}}^{\mathrm{dR}}$, and corresponds to an action

$$\pi_1(\mathcal{C}_{\mathbb{K}}^{\mathrm{dR}}, \omega_x) \to \mathrm{Aut}\, \mathbb{L}_{\mathbb{K}}(\mathsf{S},\mathsf{T})^\wedge \tag{2.11.4}$$

of the tannakian fundamental group on the fiber over x. Recall from Theorem 2.9.3 that

$$(\overline{\mathcal{P}}_{\mathrm{dR}}, \nabla) \otimes_{\mathbb{K}} \mathbb{C} \approx (\overline{\mathcal{P}}, \nabla^{\mathrm{an}}),$$

where $\overline{\mathcal{P}}$ is Deligne's canonical extension of $\mathcal{P}$ over E' to E. Therefore, after tensoring with $\mathbb{C}$, we obtain the object $\mathcal{P}$ in $\mathcal{C}^{\mathrm{an}}$, which by (2.11.1), (2.11.2) is equivalent to an action

$$\pi_1(\mathcal{C}^{\mathrm{an}}, \omega_x) \to \mathrm{Aut}\, \mathbb{L}_{\mathbb{C}}(\mathsf{S}, \mathsf{T})^\wedge.$$

This action factors through the action given by (2.11.4) $\times_{\mathbb{K}} \mathbb{C}$, that is, we have a diagram

$$\begin{array}{ccc} \pi_1(\mathcal{C}^{\mathrm{an}}, \omega_x) & \xrightarrow{\mathrm{comp}_{\mathrm{an,dR}}} & \pi_1(\mathcal{C}^{\mathrm{dR}}_{\mathbb{K}}, \omega_x) \times_{\mathbb{K}} \mathbb{C} \\ & \searrow & \downarrow \\ & & \mathrm{Aut}\ \mathbb{L}_{\mathbb{C}}(\mathsf{S}, \mathsf{T})^\wedge \end{array}$$

Note that the functor that takes a unipotent group to its Lie algebra is an equivalence of categories between the category of unipotent $\mathbb{K}$-groups and the category of nilpotent Lie algebras over $\mathbb{K}$. The above diagram is thus equivalent to the following diagram

$$\begin{array}{ccc} \mathbb{L}_{\mathbb{C}}(\mathsf{A}, \mathsf{T})^\wedge & \longrightarrow & \mathfrak{u}(\mathcal{C}^{\mathrm{dR}}_{\mathbb{K}}, \omega_x) \otimes_{\mathbb{K}} \mathbb{C} \\ & \overset{\mathrm{ad}}{\searrow} & \downarrow \\ & & \mathrm{Der}\ \mathbb{L}_{\mathbb{C}}(\mathsf{A}, \mathsf{T})^\wedge \end{array}$$

where $\mathfrak{u}(\mathcal{C}^{\mathrm{dR}}_{\mathbb{K}}, \omega_x)$ denotes the Lie algebra of $\pi_1(\mathcal{C}^{\mathrm{dR}}_{\mathbb{K}}, \omega_x)$. Since the adjoint action $\mathrm{ad} : \mathbb{L}_{\mathbb{C}}(\mathsf{A}, \mathsf{T})^\wedge \to \mathrm{Der}\, \mathbb{L}_{\mathbb{C}}(\mathsf{A}, \mathsf{T})^\wedge$ from (2.11.3) is injective, the top row of the previous diagram

$$\mathrm{comp}_{\mathrm{B,dR}} : \pi_1(\mathcal{C}^{\mathrm{an}}, \omega_x) \to \pi_1(\mathcal{C}^{\mathrm{dR}}_{\mathbb{K}}, \omega_x) \times_{\mathbb{K}} \mathbb{C}$$

must also be injective. The surjectivity of this map follows from the fact that the Lie algebra $\mathfrak{u}(\mathcal{C}^{\mathrm{dR}}_{\mathbb{K}}, \omega_x)$ is generated by the $\mathbb{K}$-dual $H^1_{\mathrm{dR}}(E_{/\mathbb{K}})^*$ of $H^1_{\mathrm{dR}}(E_{/\mathbb{K}})$, see discussion in Section 2.11.1. $\square$

Remark 2.11.4. One can always choose x to be the identity. The corresponding fiber functor ω_{id} is obtained by taking the fiber at the identity (or the unit tangent vector at the identity to be precise).

Corollary 2.11.5. *There is an isomorphism of groups over* $\mathbb{K}$

$$\pi_1(\mathcal{C}_{\mathbb{K}}^{\mathrm{dR}}, \omega_x) \cong \mathbb{L}_{\mathbb{K}}(\mathsf{S}, \mathsf{T})^{\wedge}.$$

Remark 2.11.6. One can establish the isomorphism in a different way. By Deligne [9, Cor. 10.43], in the unipotent case, the tannakian fundamental groupoid is compatible with extension of scalars. In particular, for our category $\mathcal{C}_{\mathbb{K}}^{\mathrm{dR}}$, given any $x \in E(\mathbb{K})$, restricting its tannakian fundamental groupoid to a diagonal point (x, x) gives a tannakian fundamental group defined over $\mathbb{K}$, which is also compatible with extension of scalars, i.e. $\pi_1(\mathcal{C}_{\mathbb{K}}^{\mathrm{dR}}) \times_{\mathbb{K}} \mathbb{C} \cong \pi_1(\mathcal{C}_{\mathbb{C}}^{\mathrm{dR}})$. Therefore, one obtains an isomorphism

$$\pi_1(\mathcal{C}^{\mathrm{an}}, \omega_x) \to \pi_1(\mathcal{C}_{\mathbb{K}}^{\mathrm{dR}}, \omega_x) \times_{\mathbb{K}} \mathbb{C}.$$

2.11.3 Universal unipotent connection over an elliptic curve $E_{/\mathbb{K}}$

Using the explicit universal connection ∇ on $\overline{\mathcal{P}}_{\mathrm{dR}}$, we provide an explicit construction of the $\mathbb{K}$-connection on a unipotent representation

$$\rho : \mathbb{L}_{\mathbb{K}}(\mathsf{S}, \mathsf{T})^{\wedge} \to \mathrm{Aut}(V),$$

which by Cor. 2.11.5 corresponds to vector bundle $\mathcal{V}$ over $E_{/\mathbb{K}}$ in $\mathcal{C}_{\mathbb{K}}^{\mathrm{dR}}$. This is achieved by composing the universal connection forms with the representation ρ.

Given a unipotent vector bundle $\mathcal{V}$ over $E_{/\mathbb{K}}$ in $\mathcal{C}_{\mathbb{K}}^{\mathrm{dR}}$. Choose the fiber functor ω_{id} at the identity $\mathrm{id} \in E(\mathbb{K})$ and denote by $V := V_{\mathrm{id}}$ the fiber over the identity. This vector bundle $\mathcal{V}$ corresponds to a unipotent representation

$$\rho : \mathbb{L}_{\mathbb{K}}(\mathsf{S}, \mathsf{T})^{\wedge} \to \mathrm{Aut}(V),$$

and equivalently a Lie algebra homomorphism

$$\log \rho : \mathbb{L}_{\mathbb{K}}(\mathsf{S},\mathsf{T})^\wedge \to \mathrm{End}(V).$$

Recall that we have defined 1-forms

$$\nu_1^{\mathrm{alg}} \in \Omega^1(E') \otimes \mathbb{L}_{\mathbb{K}}(\mathsf{S},\mathsf{T})^\wedge \quad \text{and} \quad \nu_1^{\mathrm{reg}} \in \Omega^1(E'' \log\{\mathrm{id}\}) \otimes \mathbb{L}_{\mathbb{K}}(\mathsf{S},\mathsf{T})^\wedge$$

in Section 2.8 and Section 2.9, respectively. They are gauge equivalent on $E' \cap E''$ via the transformation

$$g_{\mathrm{reg}} : E' \cap E'' \to \exp \mathbb{L}_{\mathbb{K}}(\mathsf{S},\mathsf{T})^\wedge \subset \mathrm{Aut}\, \mathbb{L}_{\mathbb{K}}(\mathsf{S},\mathsf{T})^\wedge.$$

Define 1-forms

$$\Omega'_V := (1 \otimes \log \rho) \circ \nu_1^{\mathrm{alg}} \in \Omega^1(E') \otimes \mathrm{End}(V)$$

and

$$\Omega''_V := (1 \otimes \log \rho) \circ \nu_1^{\mathrm{reg}} \in \Omega^1(E'' \log\{\mathrm{id}\}) \otimes \mathrm{End}(V).$$

Over E' and E'', we endow trivial bundles

$$\begin{array}{ccc} V \times E' & \quad\text{and}\quad & V \times E'' \\ \downarrow & & \downarrow \\ E' & & E'' \end{array}$$

with connections $\nabla = d + \Omega'_V$ and $\nabla = d + \Omega''_V$, respectively. Define

$$g_V : E' \cap E'' \to \mathrm{Aut}(V)$$

by $g_V := \exp(\log \rho \circ \log g_{\mathrm{reg}})$, then Ω'_V and Ω''_V are gauge equivalent on $E' \cap E''$ via g_V. After gluing these two trivial bundles by the gauge transformation g_V, we obtain a connection ∇ on $\mathcal{V}$ defined over $\mathbb{K}$ such that

$$\nabla : \mathcal{V} \to \mathcal{V} \otimes \Omega^1_E(\log\{\mathrm{id}\}).$$

2.12 Algebraic Connection Formula over $\mathcal{E}'$

Levin–Racinet [22, §5] sketched a proof to show that the bundle $\mathcal{P}$ over $\mathcal{E}$ and its connection, the universal elliptic KZB connection, are defined over $\mathbb{Q}$. However, just as in the case of a single elliptic curve, their work is incomplete in that their connection formula has irregular singularities along the identity section of $\mathcal{E}$.

We show that after an algebraic change of gauge, the universal elliptic KZB connection has regular singularities along the identity section of $\mathcal{E}$ and the nodal cubic. Since all these data are defined over $\mathbb{Q}$, we have completed the work.

Similar to the previous part, we compute explicitly the connection formula in terms of algebraic coordinates on $\mathcal{E}$. We resolve the issue of irregular singularities by trivializing the bundle on two open subsets $\mathcal{E}'$ and $\mathcal{E}''$ of $\mathcal{E}$, where $\mathcal{E}'$ is obtained from $\mathcal{E}$ by removing the identity section[10], and $\mathcal{E}''$ is obtained from $\mathcal{E}$ by removing three sections that correspond to three nontrivial order 2 elements on each fiber. On both open subsets, the algebraic connection formulas are defined over $\mathbb{Q}$, and the one on $\mathcal{E}''$ has regular singularities along the identity section. Note that the singularities around the nodal cubic are regular on both open subsets, and the gauge transformation on their intersection is compatible with the canonical extension $\overline{\mathcal{P}}$ of $\mathcal{P}$ over $\mathcal{E}$ to $\overline{\mathcal{E}}$. One can think of the universal elliptic KZB connection as an algebraic connection on an algebraic vector bundle $\overline{\mathcal{P}}_{\mathrm{dR}}$ over $\overline{\mathcal{E}}$, which is defined over $\mathbb{Q}$ with regular singularities along boundary divisors. Therefore, we have constructed a $\mathbb{Q}$-de Rham structure $\overline{\mathcal{P}}_{\mathrm{dR}}$ on $\overline{\mathcal{P}}$ over $\overline{\mathcal{E}}$.

In Section 2.6, we defined the universal elliptic KZB connection ∇^{an} on the bundle $\mathcal{P}$ over $\mathcal{E}'$ analytically. This bundle $\mathcal{P}$ can be pulled back to a bundle over $\mathcal{M}^{\mathrm{an}}_{1,1+\vec{1}}$ with connection, which we also denote by ∇^{an}. In this section, we will write this connection in terms of algebraic coordinates x, y, u, v on $\mathcal{M}_{1,1+\vec{1}}$ (defined in Section

[10] It is $\mathcal{M}_{1,2}$ as defined in Section 2.2.2.

2.2.1). The connection is $\mathbb{G}_m$-invariant, and is trivial on each fiber of $\mathcal{M}_{1,1+\vec{1}} \to \mathcal{M}_{1,2}$, thus descends to a connection on $\mathcal{M}_{1,2} = \mathcal{E}'$.

As in the case of a single elliptic curve, fiber by fiber, we apply the gauge transformation of $g_{\text{alg}}(\xi, \tau) = \exp(-\frac{E_1}{2\pi i}\mathsf{T})$ with both g_{alg} and E_1 having the extra variable τ. After the gauge transformation, the connection

$$\nabla^{\text{an}} = d + \omega$$

transforms into

$$\nabla = d + \omega_{\text{alg}} = d - dg_{\text{alg}} \cdot g_{\text{alg}}^{-1} + g_{\text{alg}}\, \omega\, g_{\text{alg}}^{-1}.$$

So using Formulas 2.4.3, 2.4.5, 2.4.6 and Lemma 9.3 in [15] we have

$$\omega_{\text{alg}} = -dg_{\text{alg}} \cdot g_{\text{alg}}^{-1} + g_{\text{alg}} \cdot \left(2\pi i\, d\tau \otimes \mathsf{A}\frac{\partial}{\partial \mathsf{T}}\right) + g_{\text{alg}}\,\psi\, g_{\text{alg}}^{-1} + g_{\text{alg}}\,\nu\, g_{\text{alg}}^{-1}$$

$$= \frac{1}{2\pi i}(\frac{\partial E_1}{\partial \xi}d\xi + \frac{\partial E_1}{\partial \tau}d\tau)\mathsf{T} + 2\pi i\, d\tau \otimes \mathsf{A}\frac{\partial}{\partial \mathsf{T}} + \psi$$

$$+ \frac{1-\exp(-\frac{E_1}{2\pi i}\mathsf{T})}{\mathsf{T}} \cdot \mathsf{A} \otimes 2\pi i d\tau + \exp(-\frac{E_1\mathsf{T}}{2\pi i})\mathsf{T}F^{\text{Zag}}(2\pi i\xi, \mathsf{T}, \tau) \cdot \mathsf{A} \otimes d\xi$$

$$+ \exp(-\frac{E_1\mathsf{T}}{2\pi i})\big(\frac{1}{\mathsf{T}} + \mathsf{T}\frac{\partial F^{\text{Zag}}}{\partial \mathsf{T}}(2\pi i\xi, \mathsf{T}, \tau)\big) \cdot \mathsf{A} \otimes 2\pi i d\tau$$

$$= \frac{1}{2\pi i}(-E_2 d\xi + \frac{1}{2\pi i}(E_3 - E_1E_2)d\tau)\mathsf{T} + 2\pi i\, d\tau \otimes \mathsf{A}\frac{\partial}{\partial \mathsf{T}} + \psi$$

$$+ \frac{1-\exp(-\frac{E_1}{2\pi i}\mathsf{T})}{\mathsf{T}} \cdot \mathsf{A} \otimes 2\pi i d\tau + \exp\big(-\sum_{k=2}^{\infty}\frac{(-\mathsf{T})^k}{k}P_k(\xi,\tau)\big) \cdot \mathsf{A} \otimes 2\pi i d\xi$$

$$+ 2\pi i d\tau \otimes \Big[\frac{\exp(-\frac{E_1}{2\pi i}\mathsf{T})}{\mathsf{T}}$$

$$+ \exp(-\sum_{k=2}^{\infty}\frac{(-\mathsf{T})^k}{k}P_k(\xi,\tau))(\sum_{k=1}^{\infty}(-\mathsf{T})^{k-1}P_k(\xi,\tau) - \frac{1}{\mathsf{T}})\Big] \cdot \mathsf{A}$$

$$= \Big(-(2\pi i)^{-2}(E_2 - e_2)\mathsf{T} + \exp(-\sum_{k=2}^{\infty}\frac{(-\mathsf{T})^k}{k}P_k(\xi,\tau)) \cdot \mathsf{S}\Big) \otimes 2\pi i(d\xi + \frac{1}{2\pi i}E_1 d\tau)$$

$$+ (2\pi i)^{-3}E_3\mathsf{T} \otimes 2\pi i d\tau + 2\pi i\, d\tau \otimes \mathsf{A}\frac{\partial}{\partial \mathsf{T}} + \psi$$

$$+ \Big[\exp(-\sum_{k=2}^{\infty}\frac{(-\mathsf{T})^k}{k}P_k(\xi,\tau))(\sum_{k=2}^{\infty}(-\mathsf{T})^{k-1}P_k(\xi,\tau) - \frac{1}{\mathsf{T}}) + \frac{1}{\mathsf{T}}\Big] \cdot \mathsf{S} \otimes 2\pi i d\tau$$

Recall the map from Section 2.2.2

$$\mathbb{C} \times \mathfrak{h} - \Lambda_{\mathfrak{h}} \to \mathcal{M}_{1,1+\vec{1}}^{\text{an}} = \{(x,y,u,v) \in \mathbb{C}^2 \times \mathbb{C}^2 : y^2 = 4x^3 - ux - v, (u,v) \neq (0,0)\}$$

$$(\xi,\tau) \mapsto (P_2(\xi,\tau), -2P_3(\xi,\tau), 20G_4(\tau), \frac{7}{3}G_6(\tau))$$

that induces an isomorphism $\mathrm{SL}_2(\mathbb{Z}) \ltimes \mathbb{Z}^2 \backslash\!\backslash (\mathbb{C} \times \mathfrak{h} - \Lambda_{\mathfrak{h}}) \cong \mathbb{G}_m \backslash\!\backslash \mathcal{M}_{1,1+\vec{1}}^{\text{an}}$. By pulling back through this map, we identify some algebraic forms with their analytic counterparts appeared in the formula above in the following

Lemma 2.12.1. *Set* $\alpha = 2udv - 3vdu$, $\Delta = u^3 - 27v^2$. *Then* $2\pi i\, d\tau = \frac{3\alpha}{2\Delta}$, *and* $2\pi i(d\xi + \frac{1}{2\pi i}E_1 d\tau) = \frac{dx}{y} - \frac{6x^2-u}{y}\frac{\alpha}{2\Delta} - \frac{1}{6}\frac{d\Delta}{\Delta}\frac{x}{y}$.

Proof. Direct computation from [22, Prop. 5.2.3]. □

Recall from Remark 3.6.1 that $P_k(\xi,\tau) = (2\pi i)^{-k}(E_k - e_k)$, $k \geqslant 2$ can be written as rational polynomials of $x = P_2(\xi,\tau)$, $y = -2P_3(\xi,\tau)$, $u = 20G_4(\tau)$ and $v = \frac{7}{3}G_6(\tau)$, i.e. for all $k \geqslant 2$, it can be written as $P_k(x,y,u,v) \in \mathbb{Q}[x,y,u,v]$. Combining this with Lemma 2.12.1, we only need to show that in terms of basis elements T and S,

$$d + 2\pi i\, d\tau \otimes \mathsf{A}\frac{\partial}{\partial \mathsf{T}} + \psi$$

is algebraic. But with respect to the above framing, $d + 2\pi i\, d\tau \otimes \mathsf{A}\frac{\partial}{\partial \mathsf{T}}$ transforms to

$$d + \left(-\frac{1}{12}\frac{d\Delta}{\Delta}\mathsf{T} + \frac{3\alpha}{2\Delta}\mathsf{S}\right)\frac{\partial}{\partial \mathsf{T}} + \left(-\frac{u\alpha}{8\Delta}\mathsf{T} + \frac{1}{12}\frac{d\Delta}{\Delta}\mathsf{S}\right)\frac{\partial}{\partial \mathsf{S}}, \tag{2.12.1}$$

(cf. [15] Prop 19.6), and ψ transforms to

$$\sum_{m\geqslant 1}\frac{1}{(2m)!}\frac{3\alpha}{2\Delta}p_{2m+2}(u,v) \otimes \sum_{\substack{j+k=2m+1\\ j,k>0}}(-1)^j[\mathrm{ad}_{\mathsf{T}}^j(\mathsf{S}), \mathrm{ad}_{\mathsf{T}}^k(\mathsf{S})]\frac{\partial}{\partial \mathsf{S}}, \tag{2.12.2}$$

where G_{2m+2} is replaced by $p_{2m+2}(u,v) \in \mathbb{Q}[u,v]$ ($p_{2m}(u,v)$'s are polynomials defined by $G_{2m}(\tau) = p_{2m}(20G_4(\tau), 7G_6(\tau)/3)$, where G_{2m} is a normalized Eisenstein series of weight $2m$).

So the algebraic 1-form of the universal elliptic KZB connection is given by

$$\begin{aligned}
\omega_{\mathrm{alg}} = & \left(-\frac{1}{12}\frac{d\Delta}{\Delta}\mathsf{T}+\frac{3\alpha}{2\Delta}\mathsf{S}\right)\frac{\partial}{\partial\mathsf{T}}+\left(-\frac{u\alpha}{8\Delta}\mathsf{T}+\frac{1}{12}\frac{d\Delta}{\Delta}\mathsf{S}\right)\frac{\partial}{\partial\mathsf{S}} \\
& +\Big(-\frac{xdx}{y}+\frac{1}{4}\frac{\alpha}{\Delta}\frac{ux+3v}{y}+\frac{1}{6}\frac{d\Delta}{\Delta}\frac{x^2}{y}\Big)\mathsf{T} \\
& +\exp\Big(-\sum_{k=2}^{\infty}\frac{(-\mathsf{T})^k}{k}P_k(x,y,u,v)\Big)\cdot\mathsf{S}\left(\frac{dx}{y}-\frac{6x^2-u}{y}\frac{\alpha}{2\Delta}-\frac{1}{6}\frac{d\Delta}{\Delta}\frac{x}{y}\right) \\
& +\Big[\exp\Big(-\sum_{k=2}^{\infty}\frac{(-\mathsf{T})^k}{k}P_k(x,y,u,v)\Big)\Big(\sum_{k=2}^{\infty}(-\mathsf{T})^{k-1}P_k(x,y,u,v)-\frac{1}{\mathsf{T}}\Big)+\frac{1}{\mathsf{T}}\Big]\cdot\mathsf{S}\frac{3\alpha}{2\Delta} \\
& +\sum_{m\geqslant 1}\frac{1}{(2m)!}\frac{3\alpha}{2\Delta}p_{2m+2}(u,v)\otimes\sum_{\substack{j+k=2m+1\\ j,k>0}}(-1)^j[\mathrm{ad}_{\mathsf{T}}^j(\mathsf{S}),\mathrm{ad}_{\mathsf{T}}^k(\mathsf{S})]\frac{\partial}{\partial\mathsf{S}} \\
= & \left(-\frac{1}{12}\frac{d\Delta}{\Delta}\mathsf{T}+\frac{3\alpha}{2\Delta}\mathsf{S}\right)\frac{\partial}{\partial\mathsf{T}}+\left(-\frac{u\alpha}{8\Delta}\mathsf{T}+\frac{1}{12}\frac{d\Delta}{\Delta}\mathsf{S}\right)\frac{\partial}{\partial\mathsf{S}} \\
& +\Big(-\frac{xdx}{y}+\frac{1}{4}\frac{\alpha}{\Delta}\frac{ux+3v}{y}+\frac{1}{6}\frac{d\Delta}{\Delta}\frac{x^2}{y}\Big)\mathsf{T}+\left(\frac{dx}{y}-\frac{6x^2-u}{y}\frac{\alpha}{2\Delta}-\frac{1}{6}\frac{d\Delta}{\Delta}\frac{x}{y}\right)\mathsf{S} \\
& +\sum_{n\geqslant 2}\left(\frac{dx}{y}-\frac{6x^2-u}{y}\frac{\alpha}{2\Delta}-\frac{1}{6}\frac{d\Delta}{\Delta}\frac{x}{y}+(n-1)\frac{3\alpha}{2\Delta}\right)q_n(x,y,u,v)\ \mathsf{T}^n\cdot\mathsf{S} \\
& +\sum_{m\geqslant 1}\frac{1}{(2m)!}\frac{3\alpha}{2\Delta}p_{2m+2}(u,v)\otimes\sum_{\substack{j+k=2m+1\\ j,k>0}}(-1)^j[\mathrm{ad}_{\mathsf{T}}^j(\mathsf{S}),\mathrm{ad}_{\mathsf{T}}^k(\mathsf{S})]\frac{\partial}{\partial\mathsf{S}}
\end{aligned}$$

where $\Delta = u^3 - 27v^2$, $\alpha = 2udv - 3vdu$, $q_n(x,y,u,v) \in \mathbb{Q}[x,y,u,v]$ $(n \geqslant 2)$ are essentially the same polynomials as in Section 2.8 but with two more variables u, v (previously u, v are fixed as the elliptic curve is fixed) and $p_{2m}(u,v) \in \mathbb{Q}[u,v]$ are polynomials we just defined.

2.13 Algebraic Connection Formula over $\mathcal{E}''$

As in the single elliptic curve case, we apply the gauge transformation $g_{\text{reg}} = \exp(-\frac{2x^2}{y}\mathsf{T})$ to the previous formula for the algebriac 1-form, and obtain the algebraic 1-form:

$$
\begin{aligned}
\omega_{\text{reg}} =& - dg_{\text{reg}} \cdot g_{\text{reg}}^{-1} + g_{\text{reg}}\, \omega_{\text{alg}}\, g_{\text{reg}}^{-1} \\
=& \left(-\frac{1}{12}\frac{d\Delta}{\Delta}\mathsf{T} + \frac{3\alpha}{2\Delta}\mathsf{S}\right)\frac{\partial}{\partial \mathsf{T}} + \left(-\frac{u\alpha}{8\Delta}\mathsf{T} + \frac{1}{12}\frac{d\Delta}{\Delta}\mathsf{S}\right)\frac{\partial}{\partial \mathsf{S}} \\
&+ \left[\left(d\left(\frac{2x^2}{y}\right) - \frac{xdx}{y}\right) + \frac{1}{4}\frac{\alpha}{\Delta}\frac{ux+3v}{y}\right]\mathsf{T} + \left(\frac{dx}{y} + \frac{1}{y}\frac{u\alpha}{2\Delta} - \frac{1}{6}\frac{d\Delta}{\Delta}\frac{x}{y}\right)\mathsf{S} \\
&+ \sum_{n\geqslant 1}\left(\frac{dx}{y} + \frac{1}{y}\frac{u\alpha}{2\Delta} - \frac{1}{6}\frac{d\Delta}{\Delta}\frac{x}{y} + (n-1)\frac{3\alpha}{2\Delta}\right) r_n(x,y,u,v)\mathsf{T}^n \cdot \mathsf{S} \\
&+ \sum_{m\geqslant 1}\frac{1}{(2m)!}\frac{3\alpha}{2\Delta}p_{2m+2}(u,v) \otimes \sum_{\substack{j+k=2m+1 \\ j,k>0}} (-1)^j [\mathrm{ad}_{\mathsf{T}}^j(\mathsf{S}), \mathrm{ad}_{\mathsf{T}}^k(\mathsf{S})]\frac{\partial}{\partial \mathsf{S}}
\end{aligned}
$$

where $r_n(x,y,u,v) \in \mathbb{Q}(x,y,u,v)$ $(n \geqslant 2)$ are essentially the same rational functions as in Section 2.9 but with two more variables u, v.

Note that the $\mathbb{G}_m$-action of λ multiplies T by λ, and S by λ^{-1}. It is easy to check that both connection forms ω_{alg} and ω_{reg} are $\mathbb{G}_m$-invariant. One can also show that the latter connection form ω_{reg} has regular singularity along the identity section, and along the nodal cubic; the residue of the connection around the identity section is $\mathrm{ad}_{[\mathsf{T},\mathsf{S}]}$, which is pronilpotent.

Just like the single elliptic curve case, we can use both connections ω_{alg} and ω_{reg} with the gauge tranformation g_{reg} between them to construct a vector bundle $\mathcal{P}_{\text{dR}}$ over $\mathcal{E}_{/\mathbb{Q}}$. Since both connection forms are defined over $\mathbb{Q}$ and have regular singularities along the nodal cubic, we can extend $\mathcal{P}_{\text{dR}}$ to $\overline{\mathcal{E}}_{/\mathbb{Q}}$ and obtain an algebraic vector bundle $\overline{\mathcal{P}}_{\text{dR}}$.

Let $(\overline{\mathcal{P}}, \nabla^{\text{an}})$ be Deligne's canonical extension to $\overline{\mathcal{E}}$ of the bundle $\mathcal{P}$ of (Lie algebras of) unipotent fundamental groups over $\mathcal{E}'$. We have

Theorem 2.13.1 (The $\mathbb{Q}$-de Rham structure $\overline{\mathcal{P}}_{\mathrm{dR}}$ on $\overline{\mathcal{P}}$ over $\overline{\mathcal{E}}$). *There is an algebraic vector bundle $\overline{\mathcal{P}}_{\mathrm{dR}}$ over $\overline{\mathcal{E}}_{/\mathbb{Q}}$ endowed with connection ∇, and an isomorphism*

$$(\overline{\mathcal{P}}_{\mathrm{dR}}, \nabla) \otimes_{\mathbb{Q}} \mathbb{C} \approx (\overline{\mathcal{P}}, \nabla^{\mathrm{an}}).$$

The algebraic bundle $\overline{\mathcal{P}}_{\mathrm{dR}}$ and its connection ∇ are both defined over $\mathbb{Q}$. The $\mathbb{Q}$-de Rham structure $(\overline{\mathcal{P}}_{\mathrm{dR}}, \nabla)$ on $(\overline{\mathcal{P}}, \nabla^{\mathrm{an}})$ is explicitly given by the connection formulas for ω_{alg} on $\mathcal{E}'$ and ω_{reg} on $\mathcal{E}''$ above. In particular, the connection ∇ has regular singularities along boundary divisors, the identity section and the nodal cubic.

3

Part II: Algebraic De Rham Theory for the Relative Completion of $\mathrm{SL}_2(\mathbb{Z})$

3.1 Introduction

Brown [3] defines multiple modular values to be periods of the coordinate ring $\mathcal{O}(\mathcal{G}^{\mathrm{rel}})$ of the relative completion $\mathcal{G}^{\mathrm{rel}}$ of $\mathrm{SL}_2(\mathbb{Z})$. In order for this period definition of multiple modular values to make sense, one needs an explicit $\mathbb{Q}$-de Rham theory for $\mathcal{G}^{\mathrm{rel}}$. In this part, we provide such a theory, which enables us to explicitly construct iterated integrals of modular forms possibly of the *second kind* (see below) that may have singularities away from the cusp around which there is no monodromy. These newly constructed iterated integrals provide all multiple modular values, whereas previously only those multiple modular values that are iterated integrals of *holomorphic* modular forms have been studied by Brown [3] and Manin [25, 26].

The relative completion $\mathcal{G}^{\mathrm{rel}}$ of $\mathrm{SL}_2(\mathbb{Z})$ with respect to the inclusion $\rho : \mathrm{SL}_2(\mathbb{Z}) \hookrightarrow \mathrm{SL}_2(\mathbb{Q})$ is an extension of SL_2 by a prounipotent group $\mathcal{U}^{\mathrm{rel}}$. The Lie algebra $\mathfrak{u}^{\mathrm{rel}}$ of $\mathcal{U}^{\mathrm{rel}}$ is freely topologically generated by

$$\prod_{n\geqslant 0} H^1(\mathrm{SL}_2(\mathbb{Z}), S^{2n}H)^* \otimes S^{2n}H,$$

where H is the standard representation of SL_2, and $S^{2n}H$ its $2n$-th symmetric power.

The first step is to construct an explicit $\mathbb{Q}$-de Rham structure on $H^1(\mathrm{SL}_2(\mathbb{Z}), S^{2n}H)$. Recall that there is a mixed Hodge structure on $H^1(\mathrm{SL}_2(\mathbb{Z}), S^{2n}H)$, which has weight and Hodge filtrations defined over $\mathbb{Q}$ [31]:

$$W_{2n+1}H^1(\mathrm{SL}_2(\mathbb{Z}), S^{2n}H) = H^1_{\mathrm{cusp}}(\mathrm{SL}_2(\mathbb{Z}), S^{2n}H);$$
$$W_{4n+2}H^1(\mathrm{SL}_2(\mathbb{Z}), S^{2n}H) = H^1(\mathrm{SL}_2(\mathbb{Z}), S^{2n}H),$$
$$F^{2n+1}H^1(\mathrm{SL}_2(\mathbb{Z}), S^{2n}H) \cong \{\textit{holomorphic} \text{ modular forms}\}.$$

Now we consider its $\mathbb{Q}$-de Rham structure $H^1_{\mathrm{dR}}(\mathcal{M}_{1,1/\mathbb{Q}}, S^{2n}\mathcal{H})$, where $\mathcal{H}$ is the relative de Rham cohomology of the universal elliptic curve over $\mathcal{M}_{1,1}$ with Gauss-Manin connection

$$\nabla_0 : \mathcal{H} \to \mathcal{H} \otimes \Omega^1_{\overline{\mathcal{M}}_{1,1}}(\log P).$$[1]

The $\mathbb{Q}$-structure for the *holomorphic* part $F^{2n+1}H^1(\mathrm{SL}_2(\mathbb{Z}), S^{2n}H)$ is classically well-known. In this *holomorphic* part, all $\mathbb{Q}$-de Rham classes correspond to classical modular forms of weight $2n+2$ with rational Fourier coefficients [15, §21]. To obtain a complete $\mathbb{Q}$-de Rham basis of $H^1_{\mathrm{dR}}(\mathcal{M}_{1,1/\mathbb{Q}}, S^{2n}\mathcal{H})$, one needs to consider modular forms of the *second kind*. In Section 3.5.2, we find representatives of all $\mathbb{Q}$-de Rham classes in $H^1_{\mathrm{dR}}(\mathcal{M}_{1,1/\mathbb{Q}}, S^{2n}\mathcal{H})$. These classes correspond to modular forms of the *second kind*. Their representatives have at worst logarithmic singularities at the cusp and may have singularities with trivial residue at other points. This differs from the traditional approach using weakly modular forms, which allows arbitrary poles at the cusp (cf. Brown–Hain [4]).

For each choice of a base point x of the moduli space $\mathcal{M}_{1,1}$ of elliptic curves, we identify $\mathrm{SL}_2(\mathbb{Z})$ with the (orbifold) fundamental group $\pi_1(\mathcal{M}_{1,1}, x)$. Denote the relative completion of $\pi_1(\mathcal{M}_{1,1}, x)$ with respect to the inclusion $\rho_x : \mathrm{SL}_2(\mathbb{Z}) \hookrightarrow \mathrm{SL}_2(\mathbb{Q})$ by $\mathcal{G}_x$. It is isomorphic to $\mathcal{G}^{\mathrm{rel}}$. Denote the Lie algebra of its unipotent radical by $\mathfrak{u}_x$.

[1] Here P denotes the cusp.

One can construct canonical mixed Hodge structures, depending on the base point x, on $\mathcal{O}(\mathcal{G}_x)$ and on $\mathfrak{u}_x$ that are compatible with their algebraic structures [18, 19]. It is achieved by finding a canonical flat connection on the Betti bundle $\boldsymbol{\mathfrak{u}}_{\mathrm{B}} \to \mathcal{M}_{1,1}$ whose fiber over x is the Lie algebra $\mathfrak{u}_x$. This connection is more general than the KZB connection in the elliptic curve case [5].

To provide a $\mathbb{Q}$-de Rham theory for $\mathcal{G}^{\mathrm{rel}}$, we construct in Section 3.6 a $\mathbb{Q}$-de Rham version of the canonical flat connection on $\boldsymbol{\mathfrak{u}}_{\mathrm{dR}} \to \overline{\mathcal{M}}_{1,1}$ with a regular singularity at the cusp as follows. Starting from representatives of $H^1_{\mathrm{dR}}(\mathcal{M}_{1,1/\mathbb{Q}}, S^{2n}\mathcal{H})$ found earlier, we define a canonical flat connection, defined over $\mathbb{Q}$, on a vector bundle $\boldsymbol{\mathfrak{u}}_{1,\mathrm{dR}}$ over $\mathcal{M}_{1,1}$ whose fibers are the abelianizations of fibers of $\boldsymbol{\mathfrak{u}}^{\mathrm{dR}}$. From this $\mathbb{Q}$-connection on $\boldsymbol{\mathfrak{u}}_{1,\mathrm{dR}}$, we apply a Čech-de Rham version of Chen's method of power series connections [6] to obtain a sequence of canonical flat connections that converges to a canonical flat connection, defined over $\mathbb{Q}$ and with regular singularity at the cusp, on the vector bundle $\boldsymbol{\mathfrak{u}}_{\mathrm{dR}}$. In other words, we trivialize $\boldsymbol{\mathfrak{u}}_{\mathrm{dR}}$ on the open cover of $\overline{\mathcal{M}}_{1,1}$ consisting of $\overline{\mathcal{M}}_{1,1} - \{[i]\}$ and $\overline{\mathcal{M}}_{1,1} - \{[\rho]\}$, and provide an inductive algorithm for constructing the connections on both opens, and for finding the gauge transformation on their intersection. By using this constructed de Rham bundle $\boldsymbol{\mathfrak{u}}_{\mathrm{dR}}$ with connection, it is routine to construct a $\mathbb{Q}$-de Rham structure on $\mathcal{O}(\mathcal{G}^{\mathrm{rel}})$ [18, §7.6].

One of the main applications for this $\mathbb{Q}$-de Rham theory is the construction of all closed iterated integrals of modular forms possibly of the *second kind*, which enables us to provide all multiple modular values. Previously Manin [25, 26] and Brown [3] only studied those multiple modular values that are (regularized) iterated integrals of *holomorphic* modular forms. In the final section, we illustrate with some explicit examples of how to construct iterated integrals that provide the remaining multiple modular values in length two. This is achieved by carrying out the algorithm for constructing connections up to $\boldsymbol{\mathfrak{u}}_{\mathrm{dR}}/\mathbb{L}^3(\boldsymbol{\mathfrak{u}}_{\mathrm{dR}})$.

3.2 The Moduli Space $\overline{\mathcal{M}}_{1,1}$ and Its Open Cover

3.2.1 The moduli stack $\overline{\mathcal{M}}_{1,1}$ of stable elliptic curves

The moduli stack $\overline{\mathcal{M}}_{1,1}$ of stable elliptic curves with one marked point is the stack quotient of

$$\overline{Y} := \mathbb{A}^2 - \{(0,0)\}^2$$

by a $\mathbb{G}_m$-action

$$\lambda \cdot (u, v) = (\lambda^4 u, \lambda^6 v).$$

This $\mathbb{G}_m$-action is equivalent to a grading on the coordinate ring

$$\mathcal{O}(\overline{Y}) := \mathbb{k}[u, v] = \bigoplus_d \mathrm{gr}_d \mathcal{O}(\overline{Y})$$

given by $\deg(u) = 4$ and $\deg(v) = 6$.

The discriminant function

$$\Delta := u^3 - 27v^2$$

has weight 12. The moduli stack $\mathcal{M}_{1,1}$ of elliptic curves with one marked point is the stack quotient of

$$Y := \mathbb{A}^2 - D$$

by the same $\mathbb{G}_m$-action, where D is the discriminant locus defined by $\Delta = 0$.

Remark 3.2.1. One could think of $\overline{\mathcal{M}}_{1,1}$ as a projective space, which would make our later discussions on $\overline{\mathcal{M}}_{1,1}$ more natural. In the next section, we will define an "affine" open cover of $\overline{\mathcal{M}}_{1,1}$ analogous to that of a projective space, and later use this open cover of $\overline{\mathcal{M}}_{1,1}$ to compute Čech cohomology with twisted coefficients.

[2] The reason we choose this notation $\overline{Y}$, indicating the space being viewed as projective instead of affine throughout this paper, will be evident in Section 3.4, Remark 3.4.7.

3.2.2 An open cover of $\overline{\mathcal{M}}_{1,1}$

Define

$$Y_0 := \operatorname{Spec} \Bbbk[u, v, u^{-1}], \quad Y_1 := \operatorname{Spec} \Bbbk[u, v, v^{-1}].$$

Since $\overline{Y} = \mathbb{A}^2 - \{(0,0)\}$, we have $\overline{Y} = Y_0 \cup Y_1$. The $\mathbb{G}_m$-action on $\overline{Y}$: $\lambda \cdot (u, v) = (\lambda^4 u, \lambda^6 v)$, restricts to act on both Y_0 and Y_1. Note that for $i = 0, 1$, the coordinate rings

$$\mathcal{O}(Y_i) = \bigoplus_d \operatorname{gr}_d \mathcal{O}(Y_i)$$

are graded with u, v, u^{-1}, v^{-1} having weights 4, 6, -4, -6, respectively. Define

$$U_0 := \mathbb{G}_m \backslash\!\backslash Y_0, \quad U_1 := \mathbb{G}_m \backslash\!\backslash Y_1 \tag{3.2.1}$$

to be the stack quotients of the $\mathbb{G}_m$-action. Then

$$\mathfrak{U} = \{U_0, U_1\}$$

forms an open cover of $\overline{\mathcal{M}}_{1,1}$.

Remark 3.2.2. The cusp $P \in \overline{\mathcal{M}}_{1,1}$, which corresponds to the isomorphism class of a nodal cubic, is in both U_0 and U_1.

Let V_0 be the affine subscheme of $\overline{Y}$ defined by $u = 1$. Its coordinate ring is $\mathcal{O}(V_0) = \mathcal{O}(\overline{Y})/I_u$, where I_u is the graded ideal of $\mathcal{O}(\overline{Y})$ generated by $u - 1$. Since u has weight 4, the affine group scheme $\boldsymbol{\mu}_4$ acts on $V_0 \simeq \mathbb{A}^1$. Similarly, define $V_1 := \operatorname{Spec} \mathcal{O}(\overline{Y})/I_v$, where I_v is the graded ideal generated by $v - 1$. Since v has weight 6, the affine group scheme $\boldsymbol{\mu}_6$ acts on $V_1 \simeq \mathbb{A}^1$. Using the same argument of Lemma 3.2 in [4], we have

Proposition 3.2.3. *The inclusions $V_0 \hookrightarrow Y_0$ and $V_1 \hookrightarrow Y_1$ induce isomorphisms of stacks over $\mathbb{Q}$*

1. $\boldsymbol{\mu}_4 \backslash\!\backslash V_0 \xrightarrow{\simeq} \mathbb{G}_m \backslash\!\backslash Y_0 = U_0,$

2. $\boldsymbol{\mu}_6 \backslash\backslash V_1 \xrightarrow{\simeq} \mathbb{G}_m \backslash\backslash Y_1 = U_1$.

Remark 3.2.4. Let Z be the affine subscheme of $\overline{Y}$ defined by $\Delta = 1$. Its projective closure is an elliptic curve that is isomorphic to the Fermat curve. Since Δ has weight 12, the affine group scheme $\boldsymbol{\mu}_{12}$ acts on Z, Lemma 3.2 in [4] shows that $\boldsymbol{\mu}_{12} \backslash\backslash Z \simeq \mathbb{G}_m \backslash\backslash Y = \mathcal{M}_{1,1}$. One could use Z and its compactification to develop an algebraic de Rham theory for $\mathcal{M}_{1,1}$ and $\overline{\mathcal{M}}_{1,1}$, see [4].

To develop a $\mathbb{Q}$-de Rham theory, we will use descriptions (3.2.1) of the open cover $\mathfrak{U} = \{U_0, U_1\}$ of $\overline{\mathcal{M}}_{1,1}$ and work $\mathbb{G}_m$-equivariantly on Y_0, Y_1 and $\overline{Y}$. Other descriptions provided for these stacks involve roots of unity, which could be complicated to handle.

3.3 Vector Bundles $S^{2n}\mathcal{H}$ on $\mathcal{M}_{1,1}$ and Their Canonical Extensions $S^{2n}\overline{\mathcal{H}}$ on $\overline{\mathcal{M}}_{1,1}$

Since $\mathcal{O}(\overline{Y})$ is a graded ring, one associates graded $\mathcal{O}(\overline{Y})$-modules to coherent sheaves/vector bundles on $\overline{\mathcal{M}}_{1,1}$.

3.3.1 The Gauss-Manin connection on a rank two vector bundle $\mathcal{H}$ over $\mathcal{M}_{1,1}$

Define a trivial rank two vector bundle $\overline{\mathcal{H}}$ on $\overline{Y}$ by

$$\overline{\mathcal{H}} := \mathcal{O}_{\overline{Y}}\mathsf{S} \oplus \mathcal{O}_{\overline{Y}}\mathsf{T},$$

where the multiplicative group $\mathbb{G}_m$ acts on it by

$$\lambda \cdot \mathsf{S} = \lambda \mathsf{S}, \quad \lambda \cdot \mathsf{T} = \lambda^{-1}\mathsf{T},$$

so S and T have weights $+1$ and -1 respectively. This vector bundle $\overline{\mathcal{H}}$ and its restriction $\mathcal{H}$ to Y, descend to vector bundles $\overline{\mathcal{H}}$ over $\overline{\mathcal{M}}_{1,1}$ and $\mathcal{H}$ over $\mathcal{M}_{1,1}$. Note our abuse of notation that $\mathcal{H}$ and $\overline{\mathcal{H}}$ might denote vector bundles over $\mathcal{M}_{1,1}$ and $\overline{\mathcal{M}}_{1,1}$, or over Y and $\overline{Y}$, depending on the context.

Define the connection on $\overline{\mathcal{H}}$ and its symmetric powers

$$S^{2n}\overline{\mathcal{H}} := \operatorname{Sym}^{2n}\overline{\mathcal{H}} = \bigoplus_{s+t=2n} \mathcal{O}_{\overline{Y}}\mathsf{S}^s\mathsf{T}^t$$

by

$$\nabla_0 = d + \left(-\frac{1}{12}\frac{d\Delta}{\Delta}\mathsf{T} + \frac{3\alpha}{2\Delta}\mathsf{S}\right)\frac{\partial}{\partial\mathsf{T}} + \left(-\frac{u\alpha}{8\Delta}\mathsf{T} + \frac{1}{12}\frac{d\Delta}{\Delta}\mathsf{S}\right)\frac{\partial}{\partial\mathsf{S}}, \tag{3.3.1}$$

where $\alpha = 2udv - 3vdu$ and $\Delta = u^3 - 27v^2$.[3] It is $\mathbb{G}_m$-invariant, and has regular singularities along the discriminant locus D. Note also that this connection ∇_0, when pulled back to each $\mathbb{G}_m$-orbit, is the trivial connection d. Therefore, it descends to a connection on $\overline{\mathcal{H}}$, and its symmetric powers

$$S^{2n}\overline{\mathcal{H}} := \operatorname{Sym}^{2n}\overline{\mathcal{H}}$$

over $\overline{\mathcal{M}}_{1,1}$. These bundles are the canonical extensions of $\mathcal{H}$ and $S^{2n}\mathcal{H} := \operatorname{Sym}^{2n}\mathcal{H}$ over $\mathcal{M}_{1,1}$.

3.3.2 The local system $\mathbb{H}$ over $\mathcal{M}_{1,1}^{\mathrm{an}}$

Let $\mathbb{H} := R^1\pi_*\mathbb{C}$, where $\pi : \mathcal{E}^{\mathrm{an}} \to \mathcal{M}_{1,1}^{\mathrm{an}}$ is the universal elliptic curve. For any field $\Bbbk \subset \mathbb{C}$, algebraic de Rham theorem induces a natural isomorphism

$$\mathcal{H} \otimes_{\Bbbk} \mathbb{C} \cong \mathbb{H}_{\mathrm{B}} \otimes_{\mathbb{Q}} \mathbb{C},$$

of bundles with connections over $Y^{\mathrm{an}} := Y(\mathbb{C})$, where $\mathbb{H}_{\mathrm{B}} := R^1\pi_*\mathbb{Q}$ denotes the Betti realization of $\mathbb{H}$, being endowed with the Gauss-Manin connection. For each n, define $2n$-th symmetric power $S^{2n}\mathbb{H} := \operatorname{Sym}^{2n}\mathbb{H}$ of $\mathbb{H}$ over $\mathcal{M}_{1,1}^{\mathrm{an}}$, then $S^{2n}\mathcal{H}$ over $\mathcal{M}_{1,1}$ is its de Rham realization.

For any point τ in the upper half plane $\mathfrak{h}$, define a lattice $\Lambda_\tau := \mathbb{Z} \oplus \mathbb{Z}\tau$, and an elliptic curve $E_\tau := \mathbb{C}/\Lambda_\tau$. Removing all the lattice points $\Lambda_{\mathfrak{h}} := \{(\xi,\tau) \in \mathbb{C} \times \mathfrak{h} : \xi \in$

[3] In [4, §2.4], one forms $\psi = \frac{1}{12}\frac{d\Delta}{\Delta}$ and $\omega = \frac{3\alpha}{2\Delta}$ are defined, and being used to write the same connection in a different form.

$\Lambda_\tau\}$ from $\mathbb{C} \times \mathfrak{h}$, there is a map

$$\psi : \mathbb{C} \times \mathfrak{h} - \Lambda_{\mathfrak{h}} \to \mathcal{M}^{\mathrm{an}}_{1,1+\vec{1}} = \{(x, y, u, v) \in \mathbb{C}^2 \times Y^{\mathrm{an}} : y^2 = 4x^3 - ux - v\}$$

$$(\xi, \tau) \mapsto ((2\pi i)^{-2}\wp_\tau(\xi), (2\pi i)^{-3}\wp'_\tau(\xi), g_2(\tau), g_3(\tau))$$

that induces an isomorphism

$$\mathrm{SL}_2(\mathbb{Z}) \ltimes \mathbb{Z}^2 \backslash\!\backslash (\mathbb{C} \times \mathfrak{h} - \Lambda_{\mathfrak{h}}) \cong \mathbb{G}_m \backslash\!\backslash \mathcal{M}^{\mathrm{an}}_{1,1+\vec{1}} = \mathcal{M}^{\mathrm{an}}_{1,2},$$

where $\wp_\tau(\xi)$ and $\wp'_\tau(\xi)$ are the Weierstrass $\wp$-function and its derivative, $g_2(\tau)$ and $g_3(\tau)$ are normalized Eisenstein series of weight 4 and 6.[4] On the left hand side, $\mathbb{Z}^2$ acts on $\mathbb{C} \times \mathfrak{h}$ by:

$$(m, n) \in \mathbb{Z}^2 : (\xi, \tau) \mapsto \left(\xi + \begin{pmatrix} m & n \end{pmatrix} \begin{pmatrix} \tau \\ 1 \end{pmatrix}, \tau\right),$$

and $\mathrm{SL}_2(\mathbb{Z})$ acts compatibly[5] by:

$$\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}) : (\xi, \tau) \mapsto \big((c\tau + d)^{-1}\xi, \gamma\tau\big).$$

On the right hand side, the multiplicative group $\mathbb{G}_m$ acts on $\mathcal{M}^{\mathrm{an}}_{1,1+\vec{1}}$ by

$$\lambda \cdot (x, y, u, v) = (\lambda^2 x, \lambda^3 y, \lambda^4 u, \lambda^6 v).$$

One can pull back the local system $\mathbb{H}$ through $\pi : \mathcal{E}^{\mathrm{an}} \to \mathcal{M}^{\mathrm{an}}_{1,1}$ to $\mathcal{E}^{\mathrm{an}}$, and then restrict to $\mathcal{M}^{\mathrm{an}}_{1,2}$. One thus obtains a local system $\mathbb{H}$ over $\mathcal{M}^{\mathrm{an}}_{1,2}$ and its de Rham realization $\mathcal{H}$ using the same process. The sections S and T of $\mathcal{H}$ over $\mathcal{M}_{1,2}$ correspond to de Rham classes represented by algebraic forms xdx/y and dx/y respectively (cf. [24, Prop. 2.1]). In particular, T corresponds to the abelian differential of an elliptic curve, which pulls back to $2\pi i d\xi$ on E_τ under the map ψ.

[4] Here $g_2(\tau) = 20G_4(\tau), g_3(\tau) = \frac{7}{3}G_6(\tau)$, where we use Zagier's normalization for Eisenstein series and his notation $G_4(\tau), G_6(\tau)$, see [30].

[5] Compatible with the semi-direct product structure that is induced from the right multiplication of $\mathrm{SL}_2(\mathbb{Z})$ on $\mathbb{Z}^2$.

The relative completion $\mathcal{G}^{\mathrm{rel}}$ of $\mathrm{SL}_2(\mathbb{Z})$ with respect to the inclusion $\rho : \mathrm{SL}_2(\mathbb{Z}) \hookrightarrow \mathrm{SL}_2(\mathbb{Q})$, is an extension of SL_2 by a prounipotent group $\mathcal{U}^{\mathrm{rel}}$. The Lie algebra $\mathfrak{u}^{\mathrm{rel}}$ of $\mathcal{U}^{\mathrm{rel}}$ is freely topologically generated by

$$\prod_{n\geqslant 0} H^1(\mathrm{SL}_2(\mathbb{Z}), S^{2n}H)^* \otimes S^{2n}H,$$

where H is the standard representation of SL_2, and $S^{2n}H$ its $2n$-th symmetric power.

We identify $H^1(\mathrm{SL}_2(\mathbb{Z}), S^{2n}H)$ with $H^1(\mathcal{M}_{1,1}^{\mathrm{an}}, S^{2n}\mathbb{H})$. To develop an algebraic de Rham theory for $\mathcal{G}^{\mathrm{rel}}$, it is necessary to find an algebraic de Rham structure $H^1_{\mathrm{dR}}(\mathcal{M}_{1,1}, S^{2n}\mathcal{H})$ on $H^1(\mathcal{M}_{1,1}^{\mathrm{an}}, S^{2n}\mathbb{H})$ first.

3.4 De Rham Cohomology with Twisted Coefficients

3.4.1 Algebraic de Rham theorem

Let X be a smooth quasi-projective variety defined over $\Bbbk$. Without loss of generality one can assume $X = \overline{X} - P$, where $\overline{X}$ is smooth projective, and P is a normal crossing divisor in $\overline{X}$. Given a vector bundle $(\mathcal{V}, \nabla)$ with flat connection over X, having regular singularities along P, denote by $\mathbb{V}$ the local system of horizontal sections of $\mathcal{V}^{\mathrm{an}}$ over X^{an}. Define the twisted de Rham complex

$$\Omega^\bullet_X(\mathcal{V}) := \Omega^\bullet_X \otimes_{\mathcal{O}_X} \mathcal{V},$$

and denote its hypercohomology by $H^\bullet_{\mathrm{dR}}(X, \mathcal{V}) := \mathbb{H}^\bullet(X, \Omega^\bullet_X(\mathcal{V}))$. Deligne [8, Cor. 6.3] proved the following version of algebraic de Rham theorem for de Rham cohomology with twisted coefficients.

Theorem 3.4.1. *There is an isomorphism*

$$H^\bullet_{\mathrm{dR}}(X, \mathcal{V}) \otimes_{\Bbbk} \mathbb{C} \cong H^\bullet(X^{\mathrm{an}}, \mathbb{V}). \tag{3.4.1}$$

Remark 3.4.2. When X is affine, the de Rham structure $H^\bullet_{\mathrm{dR}}(X, \mathcal{V})$ can be computed as the cohomology $H^\bullet(\Gamma(X, \Omega^\bullet_X(\mathcal{V})))$ of global sections of the twisted de Rham complex with differential given by the connection ∇.

Remark 3.4.3. One can replace $\Omega^\bullet_X$ by any complex that is quasi-isomorphic to $\Omega^\bullet_X$ or the direct image sheaf complex $i_*\Omega^\bullet_X$ (for the inclusion $i : X \hookrightarrow \overline{X}$), for example the logarithmic de Rham complex $\Omega^\bullet_{\overline{X}}(\log P)$. Then the de Rham structure $H^\bullet_{\mathrm{dR}}(X, \mathcal{V})$ can be computed as the hypercohomology

$$\mathbb{H}^\bullet(\overline{X}, \Omega^\bullet_{\overline{X}}(\log P) \otimes_{\mathcal{O}_{\overline{X}}} \overline{\mathcal{V}})$$

of the twisted logarithmic de Rham complex

$$\Omega^\bullet_{\overline{X}}(\log P) \otimes_{\mathcal{O}_{\overline{X}}} \overline{\mathcal{V}},$$

where $\overline{\mathcal{V}}$ denotes the canonical extension of $\mathcal{V}$ to $\overline{X}$.

Example 3.4.4. Primary Example: $\mathbb{Q}$-de Rham structure $H^1_{\mathrm{dR}}(\mathcal{M}_{1,1}, S^{2n}\mathcal{H})$ on $H^1(\mathcal{M}^{\mathrm{an}}_{1,1}, S^{2n}\mathbb{H})$. Assume our field $\Bbbk = \mathbb{Q}$ and $n > 0$. Let $X = \mathcal{M}_{1,1}$, $\overline{X} = \overline{\mathcal{M}}_{1,1}$, and their coverings $Y = \mathcal{M}_{1,\vec{1}} = \mathbb{A}^2 - D$, $\overline{Y} = \mathbb{A}^2 - \{(0,0)\}$. Let P denotes the cusp in $\overline{\mathcal{M}}_{1,1}$, then

$$\overline{X} = \mathbb{G}_m \backslash\backslash \overline{Y}, \quad X = \overline{X} - P = \mathbb{G}_m \backslash\backslash (\overline{Y} - D) = \mathbb{G}_m \backslash\backslash Y,$$

where the multiplicative group $\mathbb{G}_m$ acts on Y and $\overline{Y}$ as before in Section 3.2.

Let $\mathcal{V}$ be the bundle $S^{2n}\mathcal{H}$ over $\mathcal{M}_{1,1}$ defined in Section 3.3.1, and recall that we denoted by $\overline{\mathcal{V}} = S^{2n}\overline{\mathcal{H}}$ its canonical extension to $\overline{\mathcal{M}}_{1,1}$. The $\mathbb{Q}$-de Rham structure

$$H^1_{\mathrm{dR}}(X, \mathcal{V}) = H^1_{\mathrm{dR}}(\mathcal{M}_{1,1}, S^{2n}\mathcal{H})$$

on $H^1(\mathcal{M}^{\mathrm{an}}_{1,1}, S^{2n}\mathbb{H})$, by Theorem 3.4.1 above and Remark 3.4.5 below, can be computed as the hypercohomology $\mathbb{H}^1(\overline{\mathcal{M}}_{1,1}, \mathcal{F}^\bullet_{2n})$ of the twisted logarithmic de Rham complex

$$\mathcal{F}^\bullet_{2n} := (\Omega^\bullet_{\overline{Y}}(\log D) \otimes S^{2n}\overline{\mathcal{H}})^{\mathbb{G}_m},$$

where the differential of the complex is induced by the Gauss-Manin connection ∇_0 on $\mathcal{H}$ over $\mathcal{M}_{1,1}$ given explicitly by (3.3.1) in Section 3.3.1.

Remark 3.4.5. By Theorem 3.4.1 and Remark 3.4.3, $H^1_{\mathrm{dR}}(Y, S^{2n}\mathcal{H})$ can be computed by the hypercohomology of the complex

$$\Omega^{\bullet}_{\overline{Y}}(\log D) \otimes S^{2n}\overline{\mathcal{H}}.$$

Since the differential ∇_0 is $\mathbb{G}_m$-invariant, we obtain the subcomplex $\mathcal{F}^{\bullet}_{2n} = (\Omega^{\bullet}_{\overline{Y}}(\log D)\otimes S^{2n}\overline{\mathcal{H}})^{\mathbb{G}_m}$ of $\mathbb{G}_m$-invariant forms on $\overline{Y}$, which descends to a complex $\mathcal{F}^{\bullet}_{2n}$ on $\overline{\mathcal{M}}_{1,1}$. Since $\mathbb{G}_m$ is connected, this also computes $H^1_{\mathrm{dR}}(Y, S^{2n}\mathcal{H})$. By computing the Leray spectral sequence for a $\mathbb{G}_m$-principle bundle $p : Y \to \mathcal{M}_{1,1}$, one gets natural isomorphisms

$$p^* : H^1_{\mathrm{dR}}(\mathcal{M}_{1,1}, S^{2n}\mathcal{H}) \xrightarrow{\simeq} H^1_{\mathrm{dR}}(Y, S^{2n}\mathcal{H})$$

for $n > 0$, cf. Brown–Hain [4, §3].

3.4.2 Sections of sheaves $\mathcal{F}^p_{2n}$ in the twisted de Rham complex $\mathcal{F}^{\bullet}_{2n}$

Here we prepare ourselves for explicit computations later by writing down sections of sheaves $\mathcal{F}^p_{2n}$ over open sets U_0, U_1 and their intersection U_{01}.

First, we compute global sections of sheaves $\Omega^p_{\overline{Y}}(\log D)$ in the logarithmic de Rham complex. By applying Deligne's criterion ([10, §3.1]) for being a global section, we have

Lemma 3.4.6.

$$\Gamma\big(\overline{Y}, \Omega^p_{\overline{Y}}(\log D)\big) = \begin{cases} \mathcal{O}(\overline{Y}) & p = 0, \\ \mathcal{O}(\overline{Y})\frac{\alpha}{\Delta} \bigoplus \mathcal{O}(\overline{Y})\frac{d\Delta}{\Delta} & p = 1, \\ \mathcal{O}(\overline{Y})\frac{du\wedge dv}{\Delta} & p = 2, \\ 0 & \textit{otherwise.} \end{cases}$$

Proof. In our case, a section φ is in the logarithmic complex if and only if $\Delta \cdot \varphi$ and $\Delta \cdot d\varphi$ are holomorphic on $\overline{Y}$. The only interesting case is when $p = 1$. Assume that

φ is a 1-form in the logarithmic complex, then $\phi := \Delta \cdot \varphi$ is holomorphic, we can write it as

$$\phi = f du + g dv \in \mathcal{O}(\overline{Y})du \oplus \mathcal{O}(\overline{Y})dv,$$

where f, g are polynomials in u and v as $\mathcal{O}(\overline{Y}) = \mathbb{Q}[u, v]$. Since

$$d\varphi = d\left(\frac{\phi}{\Delta}\right) = \frac{d\phi}{\Delta} + \phi \wedge \frac{d\Delta}{\Delta^2},$$

to make sure that $\Delta \cdot d\varphi$ is holomorphic, the 2-form

$$\phi \wedge d\Delta = (f du + g dv) \wedge (3u^2 du - 54v dv) = -(3u^2 g + 54vf) du \wedge dv$$

has to be a multiple of Δ, i.e. we need to have

$$(u^3 - 27v^2)\big|(3u^2 g + 54vf).$$

We call (f, g) a solution if it satisfies the above condition. One can check that $(f, g) = (-3v, 2u)$ and $(f, g) = (3u^2, -54v)$ are solutions. Denote the highest degree of v in a polynomial $f \in \mathbb{Q}[u, v]$ by $\deg_v(f)$, ignoring u or regarding u as a constant. To find other solutions, we apply a Euclidean algorithm to reduce the degrees $\deg_v(f)$ and $\deg_v(g)$ to 0. Applying Euclidean algorithm on f with $(-3v)$ from the f-component of the first solution $(-3v, 2u)$, we can write $f = (-3v)q + f_1$, where $q, f_1 \in \mathbb{Q}[u, v]$ with $\deg_v(f_1) < \deg_v(-3v) = 1$. We define $g_1 := g - 2uq$, then

$$3u^2 g_1 + 54vf_1 = 3u^2(g - 2uq) + 54v(f + 3vq) = (3u^2 g + 54vf) - 6q(u^3 - 27v^2),$$

and we reduces the problem of finding (f, g) to finding (f_1, g_1), where $\deg_v(f_1) < \deg_v(f)$ unless $\deg_v(f) = 0$. A similar process can reduce $\deg_v(g)$ by applying Euclidean algorithm on g with $(-54v)$ from the other solution $(3u^2, -54v)$. Note that we can continue this as long as $\deg_v(f)$ or $\deg_v(g)$ is positive, and $\deg_v(f) + \deg_v(g)$ keeps decreasing after every reduction step. Eventually, we would have reduced to (f,g) such that $\deg_v(f) = \deg_v(g) = 0$ and $\deg_v(3u^2g + 54vf)$ is at most 1. In this case, since on the left side $\deg_v(u^3 - 27v^2) = 2$, the right side polynomial $(3u^2g+54vf)$ has to be 0, which in turn implies $f = g = 0$. This amounts to showing that $\phi = fdu + gdv$ has to be an $\mathcal{O}(\overline{Y})$-linear combination of

$$-3vdu + 2udv \ (= \alpha) \quad \text{and} \quad 3u^2du - 54vdv \ (= d\Delta),$$

i.e.

$$\varphi = \frac{\phi}{\Delta} \in \mathcal{O}(\overline{Y})\frac{\alpha}{\Delta} \oplus \mathcal{O}(\overline{Y})\frac{d\Delta}{\Delta}.$$

□

Our next task is to apply Čech-de Rham theory to compute $\mathbb{Q}$-de Rham representatives of $H^1_{\mathrm{dR}}(\mathcal{M}_{1,1}, S^{2n}\mathcal{H})$. In preparation, we compute the $\mathbb{G}_m$-invariant sections on our open cover.

Since everything is graded by $\mathbb{G}_m$, it is easy to deduce from Lemma 3.4.6 by weight computations that global sections of $\mathcal{F}^p_{2n}$ are

$$\mathcal{F}^p_{2n}(\overline{\mathcal{M}}_{1,1}) = \Gamma\big(\overline{Y}, (\Omega^p_{\overline{Y}}(\log D) \otimes S^{2n}\overline{\mathcal{H}})^{\mathbb{G}_m}\big)$$

$$= \begin{cases} \bigoplus_{s+t=2n} (\mathrm{gr}_{t-s}\mathcal{O}(\overline{Y}))\mathsf{S}^s\mathsf{T}^t & p = 0, \\ \big(\bigoplus_{s+t=2n} (\mathrm{gr}_{t-s+2}\mathcal{O}(\overline{Y}))\frac{\alpha}{\Delta}\mathsf{S}^s\mathsf{T}^t\big) \bigoplus \big(\bigoplus_{s+t=2n} (\mathrm{gr}_{t-s}\mathcal{O}(\overline{Y}))\frac{d\Delta}{\Delta}\mathsf{S}^s\mathsf{T}^t\big) & p = 1, \\ \bigoplus_{s+t=2n} (\mathrm{gr}_{t-s+2}\mathcal{O}(\overline{Y}))\frac{du\wedge dv}{\Delta}\mathsf{S}^s\mathsf{T}^t & p = 2, \\ 0 & \text{otherwise.} \end{cases}$$

Note that for $i = 0, 1$, we have $\mathcal{O}_{\overline{Y}}(Y_i) = \mathcal{O}(Y_i)$, and sections of $\mathcal{F}^p_{2n}$ on U_i are

$$\mathcal{F}^p_{2n}(U_i) = \Gamma\big(Y_i, (\Omega^p_{\overline{Y}}(\log D) \otimes S^{2n}\overline{\mathcal{H}})^{\mathbb{G}_m}\big)$$

$$= \begin{cases} \bigoplus_{s+t=2n} (\mathrm{gr}_{t-s}\mathcal{O}(Y_i))\mathsf{S}^s\mathsf{T}^t & p = 0, \\ \big(\bigoplus_{s+t=2n} (\mathrm{gr}_{t-s+2}\mathcal{O}(Y_i))\frac{\alpha}{\Delta}\mathsf{S}^s\mathsf{T}^t\big) \bigoplus \big(\bigoplus_{s+t=2n} (\mathrm{gr}_{t-s}\mathcal{O}(Y_i))\frac{d\Delta}{\Delta}\mathsf{S}^s\mathsf{T}^t\big) & p = 1, \\ \bigoplus_{s+t=2n} (\mathrm{gr}_{t-s+2}\mathcal{O}(Y_i))\frac{du\wedge dv}{\Delta}\mathsf{S}^s\mathsf{T}^t & p = 2, \\ 0 & \text{otherwise.} \end{cases}$$

Let $Y_{01} := Y_0 \cap Y_1 = \operatorname{Spec}\mathbb{Q}[u, v, u^{-1}, v^{-1}]$, then the coordinate ring $\mathcal{O}(Y_{01}) = \mathbb{Q}[u, v, u^{-1}, v^{-1}]$ is graded with u, v, u^{-1}, v^{-1} having weights 4, 6, -4, -6, respectively. Denote degree n part of $\mathcal{O}(Y_{01})$ by $\operatorname{gr}_n\mathcal{O}(Y_{01})$. Since the $\mathbb{G}_m$-action on $\overline{Y}$ restricts to Y_{01}, and the stack quotient of this action is $U_{01} := \mathbb{G}_m\backslash\backslash Y_{01}$, we deduce similarly that

$$\mathcal{F}^p_{2n}(U_{01}) = \Gamma\big(Y_{01}, (\Omega^p_{\overline{Y}}(\log D) \otimes S^{2n}\overline{\mathcal{H}})^{\mathbb{G}_m}\big)$$

$$= \begin{cases} \bigoplus_{s+t=2n} (\operatorname{gr}_{t-s}\mathcal{O}(Y_{01}))\mathsf{S}^s\mathsf{T}^t & p = 0, \\ \big(\bigoplus_{s+t=2n} (\operatorname{gr}_{t-s+2}\mathcal{O}(Y_{01}))\frac{\alpha}{\Delta}\mathsf{S}^s\mathsf{T}^t\big) \bigoplus \big(\bigoplus_{s+t=2n} (\operatorname{gr}_{t-s}\mathcal{O}(Y_{01}))\frac{d\Delta}{\Delta}\mathsf{S}^s\mathsf{T}^t\big) & p = 1, \\ \bigoplus_{s+t=2n} (\operatorname{gr}_{t-s+2}\mathcal{O}(Y_{01}))\frac{du\wedge dv}{\Delta}\mathsf{S}^s\mathsf{T}^t & p = 2, \\ 0 & \text{otherwise.} \end{cases}$$

3.4.3 The Čech-de Rham complex $\check{C}^\bullet(\mathfrak{U}, \mathcal{F}^\bullet_{2n})$

In this section, we construct a Čech-de Rham complex $\check{C}^\bullet(\mathfrak{U}, \mathcal{F}^\bullet_{2n})$ that computes the $\mathbb{Q}$-de Rham structure $H^1_{\mathrm{dR}}(\mathcal{M}_{1,1}, S^{2n}\mathcal{H})$ for the open cover $\mathfrak{U} = \{U_0, U_1\}$:

$$\begin{array}{|ccc} \mathcal{F}^2_{2n}(U_0) \oplus \mathcal{F}^2_{2n}(U_1) & \xrightarrow{\delta} & \mathcal{F}^2_{2n}(U_{01}) \\ {\scriptstyle(\nabla_0,\nabla_0)}\uparrow & & {\scriptstyle -\nabla_0}\uparrow \\ \mathcal{F}^1_{2n}(U_0) \oplus \mathcal{F}^1_{2n}(U_1) & \xrightarrow{\delta} & \mathcal{F}^1_{2n}(U_{01}) \\ {\scriptstyle(\nabla_0,\nabla_0)}\uparrow & & {\scriptstyle -\nabla_0}\uparrow \\ \mathcal{F}^0_{2n}(U_0) \oplus \mathcal{F}^0_{2n}(U_1) & \xrightarrow{\delta} & \mathcal{F}^0_{2n}(U_{01}) \\ \hline \end{array}$$

where U_{01} is the intersection of U_0 and U_1; the horizontal differential δ is the usual one for a Čech complex, and the vertical differential is ∇_0 in the twisted de Rham complex $\mathcal{F}^\bullet_{2n}$. Let

$$D = \delta + (-1)^q\nabla_0$$

be the (total) differential of the single complex $s\check{C}^\bullet(\mathfrak{U}, \mathcal{F}^\bullet_{2n})$ associated to the Čech-de Rham double complex $\check{C}^\bullet(\mathfrak{U}, \mathcal{F}^\bullet_{2n})$.

Recall from Example 3.4.4 that the $\mathbb{Q}$-structure $H^1_{\mathrm{dR}}(\mathcal{M}_{1,1}, S^{2n}\mathcal{H})$ is computed by the hypercohomology $\mathbb{H}^1(\overline{\mathcal{M}}_{1,1}, \mathcal{F}^\bullet_{2n})$, where $\mathcal{F}^\bullet_{2n} = (\Omega^\bullet_{\overline{Y}}(\log D) \otimes S^{2n}\overline{\mathcal{H}})^{\mathbb{G}_m}$. Since

$\mathcal{F}_{2n}^p$ is coherent on $\overline{\mathcal{M}}_{1,1}$ (see [28], or [11] for an algebraic proof), and the open cover $\mathfrak{U} = \{U_0, U_1\}$ is a "good cover"[6], the Čech cohomology calculated for this open cover $\mathfrak{U}$ will give us the correct answer, and we have

$$H^1_{\mathrm{dR}}(\mathcal{M}_{1,1}, S^{2n}\mathcal{H}) = \mathbb{H}^1(\overline{\mathcal{M}}_{1,1}, \mathcal{F}_{2n}^\bullet) = H^1(\mathfrak{U}, s\check{C}^\bullet(\mathfrak{U}, \mathcal{F}_{2n}^\bullet)).$$

Remark 3.4.7. One can compute the cohomology $H^1(\overline{\mathcal{M}}_{1,1}, \mathcal{F}_{2n}^\bullet)$ of global sections of $\mathcal{F}_{2n}^\bullet$. One finds that its dimension equals that of M_{2n+2}, the $\mathbb{Q}$ vector space spanned by *holomorphic* modular forms of weight $2n+2$ with rational Fourier coefficients. It does not equal the dimension of $H^1(\mathcal{M}_{1,1}^{\mathrm{an}}, S^{2n}\mathbb{H})$. This suggests that one should not view $\overline{Y}$ with its $\mathbb{G}_m$-action as an affine variety and compute cohomology by global sections (cf. Remark 3.4.2). Instead one has to use hypercohomology of $\overline{Y}$ to compute $H^1_{\mathrm{dR}}(\mathcal{M}_{1,1}, S^{2n}\mathcal{H})$ if one wants representatives to have logarithmic singularities at the cusp.

3.5 Holomorphic Modular Forms and Modular Forms of the Second Kind

In this section, we apply the algebraic de Rham theory previously, and find explicitly all $\mathbb{Q}$-de Rham classes in the $\mathbb{Q}$-de Rham structure $H^1_{\mathrm{dR}}(\mathcal{M}_{1,1}, S^{2n}\mathcal{H})$ on $H^1(\mathcal{M}_{1,1}^{\mathrm{an}}, S^{2n}\mathbb{H})$ for every n. These $\mathbb{Q}$-de Rham classes are closely related to *holomorphic* modular forms with rational Fourier coefficients.

By Eichler–Shimura, after tensoring $H^1_{\mathrm{dR}}(\mathcal{M}_{1,1}, S^{2n}\mathcal{H})$ with $\mathbb{C}$, one obtains a natural mixed Hodge structure on $H^1(\mathcal{M}_{1,1}^{\mathrm{an}}, S^{2n}\mathbb{H})$, which has weight and Hodge filtra-

[6] In the sense of Bott and Tu [1, §8] that the (augmented) columns are exact in the Čech-de Rham complex.

tions defined over $\mathbb{Q}$:

$$W_{2n+1}H^1(\mathcal{M}_{1,1}^{\mathrm{an}}, S^{2n}\mathbb{H}) = H^1_{\mathrm{cusp}}(\mathcal{M}_{1,1}^{\mathrm{an}}, S^{2n}\mathbb{H}); \tag{3.5.1}$$

$$W_{4n+2}H^1(\mathcal{M}_{1,1}^{\mathrm{an}}, S^{2n}\mathbb{H}) = H^1(\mathcal{M}_{1,1}^{\mathrm{an}}, S^{2n}\mathbb{H}), \tag{3.5.2}$$

$$F^{2n+1}H^1(\mathcal{M}_{1,1}^{\mathrm{an}}, S^{2n}\mathbb{H}) \cong M_{2n+2} \otimes_{\mathbb{Q}} \mathbb{C}, \tag{3.5.3}$$

where M_{2n} denotes the $\mathbb{Q}$-vector space spanned by *holomorphic* modular forms of weight $2n$ with rational Fourier coefficients.

The $\mathbb{Q}$-structure for the last part $F^{2n+1}H^1(\mathcal{M}_{1,1}^{\mathrm{an}}, S^{2n}\mathbb{H})$– *holomorphic* modular forms – is well known. Every $\mathbb{Q}$-de Rham cohomology class in $F^{2n+1}H^1(\mathcal{M}_{1,1}^{\mathrm{an}}, S^{2n}\mathbb{H})$ can be represented by a *global* $\mathbb{G}_m$-invariant ∇_0-closed 1-form on $\mathcal{M}_{1,\vec{1}}$ with coefficients in $S^{2n}\mathcal{H}$. The explicit correspondence has been found in [15, §21], we record it here in our notation in the following section 3.5.1.

However, a *global* 1-form representative can not be found for the remaining $\mathbb{Q}$-de Rham classes if one insists that representatives have logarithmic singularities at the cusp (cf. Remark 3.4.7). These remaining classes are the ones, under the Eichler–Shimura isomorphism, that involve anti-holomorphic cusp forms. In Section 3.5.2, we explain how to find and represent all $\mathbb{Q}$-de Rham classes including these remaining classes using Čech cocycles in the Čech-de Rham complex $\check{C}^\bullet(\mathfrak{U}, \mathcal{F}^\bullet_{2n})$. Similar results were obtained by Brown–Hain [4] using weakly modular forms. Our representatives have the advantage of having logarithmic singularities at the cusp, which are better suited to computing regularized periods.

3.5.1 Holomorphic modular forms

Given a *holomorphic* modular form $f(\tau)$ of weight $2n+2$ with rational Fourier coefficients, it corresponds to a polynomial $h(u,v) \in \mathbb{Q}[u,v]$ of weight $2n+2$ (where u has weight 4 and v has weight 6). From Hain [15, §21], the cohomology class corresponding to $f(\tau)$ can be represented by a *global* 1-form

$$h(u,v)\frac{\alpha}{\Delta}\mathsf{T}^{2n} = h(u,v)\frac{2udv - 3vdu}{u^3 - 27v^2}\mathsf{T}^{2n} \in \left(\Omega^1_{\overline{Y}}(\log D) \otimes F^{2n}S^{2n}\mathcal{H}\right)^{\mathbb{G}_m}.$$

Analytically, the pullback of this form along the map $\mathfrak{h} \to \mathcal{M}^{\mathrm{an}}_{1,\vec{1}} = Y^{\mathrm{an}}$ defined by $\tau \mapsto ((2\pi i)^4 g_2(\tau), (2\pi i)^6 g_3(\tau))$ is up to a rational multiple ($\frac{2}{3}$ to be precise),

$$(2\pi i)^{2n} h(g_2(\tau), g_3(\tau))\mathsf{T}^{2n}\frac{dq}{q} = (2\pi i)^{2n+1} f(\tau)\mathsf{T}^{2n} d\tau, \quad \text{with } q = e^{2\pi i\tau}.$$

This 1-form is denoted by ω_f in Hain [20, §9.1], and by $\underline{f}(\tau)$ in Brown [3, §2.1] to compute periods. We will adopt Hain's notation ω_f.

Example 3.5.1. The normalized cusp form $\Delta = q - 24q^2 + 252q^3 + \cdots$ of weight 12 is a rational polynomial $\Delta = u^3 - 27v^2$. So its corresponding class is represented by the 1-form

$$\omega_\Delta = \Delta\frac{2udv - 3vdu}{u^3 - 27v^2}\mathsf{T}^{10} = (2udv - 3vdu)\mathsf{T}^{10}.$$

3.5.2 Modular forms of the second kind

In this section, we will find all $\mathbb{Q}$-de Rham classes in $H^1_{\mathrm{dR}}(\mathcal{M}_{1,1}, S^{2n}\mathcal{H})$ as promised. We start by discussing what is a 1-cocycle in the single complex $s\check{C}^\bullet(\mathfrak{U}, \mathcal{F}^\bullet_{2n})$ associated to the Čech-de Rham double complex $\check{C}^\bullet(\mathfrak{U}, \mathcal{F}^\bullet_{2n})$.

In the Čech-de Rham complex $\check{C}^\bullet(\mathfrak{U}, \mathcal{F}^\bullet_{2n})$, every 1-cochain $\widetilde{\omega}$ is of the form

$$\widetilde{\omega} = \begin{array}{|cc} (\omega^{(0)}, \omega^{(1)}) & 0 \\ 0 & l \\ \hline \end{array}$$

where $\omega^{(i)} \in \mathcal{F}^1_{2n}(U_i)$, and $l \in \mathcal{F}^0_{2n}(U_{01})$, so that $(\omega^{(0)}, \omega^{(1)}) \in \check{C}^0(\mathfrak{U}, \mathcal{F}^1_{2n})$ and $l \in \check{C}^1(\mathfrak{U}, \mathcal{F}^0_{2n})$. We often simply write it as $\widetilde{\omega} = (\omega^{(0)}, \omega^{(1)}; l)$.

A 1-cochain $\widetilde{\omega} = (\omega^{(0)}, \omega^{(1)}; l)$ in $s\check{C}^\bullet(\mathfrak{U}, \mathcal{F}^\bullet_{2n})$ is a 1-cocycle whenever $D\widetilde{\omega} = 0$; in other words, it is a cocycle if and only if $\nabla_0\omega^{(0)} = \nabla_0\omega^{(1)} = 0$ and

$$\delta(\omega^{(0)}, \omega^{(1)}) - \nabla_0 l = \omega^{(1)} - \omega^{(0)} - \nabla_0 l = 0.$$

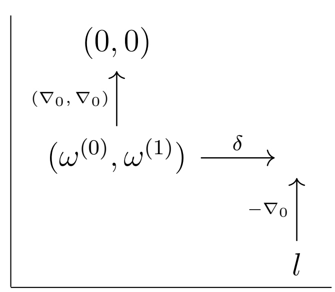
Example 3.5.2. Holomorphic modular forms as 1-cocycles. As was shown in the last section, a *holomorphic* modular form f with rational Fourier coefficients of weight $2n+2$ gives rise to a cohomology class $[\omega_f] \in H^1_{\mathrm{dR}}(\mathcal{M}_{1,1}, S^{2n}\mathcal{H})$, which can be represented by a *global* closed form ω_f. Denote by $\omega_f^{(i)}$ the restriction of ω_f to U_i, $i = 0, 1$, and define

$$\widetilde{\omega}_f := (\omega_f^{(0)}, \omega_f^{(1)}; 0),$$

then $\widetilde{\omega}_f$ is a 1-cocycle in our Čech complex $s\check{C}^\bullet(\mathfrak{U}, \mathcal{F}^\bullet_{2n})$.

To find representatives for all $\mathbb{Q}$-de Rham classes in $H^1_{\mathrm{dR}}(\mathcal{M}_{1,1}, S^{2n}\mathcal{H})$, we use the second spectral sequence of the double complex $\check{C}^\bullet(\mathfrak{U}, \mathcal{F}^\bullet_{2n})$, with the second page E_2 given by $H_{\nabla_0} H_\delta$ and differentials $d_r : E_r^{p,q} \to E_r^{p-r+1,q+r}$. Since the double complex $\check{C}^\bullet(\mathfrak{U}, \mathcal{F}^\bullet_{2n})$ concentrates in the first two column, this spectral sequence degenerates at E_3 trivially.

We now use a spectral sequence zig-zag argument. Starting from l in the lower right corner, it extends to a 1-cocycle $\widetilde{\omega} = (\omega^{(0)}, \omega^{(1)}; l)$ if it lives to E_3:



i.e. $\nabla_0 \omega^{(0)} = \nabla_0 \omega^{(1)} = 0$ and $\omega^{(1)} - \omega^{(0)} - \nabla_0 l = 0$. This is equivalent to the condition $D\widetilde{\omega} = 0$.

Given a trivial class $0 \in H^1_\delta(\mathfrak{U}, \mathcal{F}^0_{2n})$, we can always choose $l = 0$ to represent it. Then by the conditions above, we would have a *global* closed form ω, which brings us back to the previous example. In fact, when $n < 5$, the corresponding group $H^1_\delta(\mathfrak{U}, \mathcal{F}^0_{2n})$ is trivial. The first interesting case occurs at $n = 5$, which is expected, since $H^1_{\mathrm{dR}}(\mathcal{M}_{1,1}, S^{10}\mathcal{H})$ corresponds to modular forms of weight $2n+2=12$, where a cusp form appears for the first time.

Example 3.5.3. First new $\mathbb{Q}$-de Rham class in $H^1_{\mathrm{dR}}(\mathcal{M}_{1,1}, S^{2n}\mathcal{H})$. When $n=5$, one can find a nonzero class in $H^1_\delta(\mathfrak{U}, \mathcal{F}^0_{10})$, cokernel of δ on the 0-th row, represented by $\frac{1}{uv}\mathsf{S}^{10} \in \mathcal{F}^0_{10}(U_{01})$. The reason is that $\frac{1}{uv} = u^{-1}v^{-1}$, having negative powers for both u and v, cannot be the difference of two elements in $\mathbb{Q}[u,v][u^{-1}]$ and $\mathbb{Q}[u,v][v^{-1}]$ respectively, where every monomial has a nonnegative power of at least one variable.

To extend the nontrivial class $[\frac{1}{uv}\mathsf{S}^{10}] \in E_1^{1,0}$ through E_2, one needs to find $(\omega^{(0)}_{1,1}, \omega^{(1)}_{1,1})$ such that

$$\omega^{(1)}_{1,1} - \omega^{(0)}_{1,1} = \nabla_0 l_{1,1},$$

where $l_{1,1}$ represents the class $[\frac{1}{uv}\mathsf{S}^{10}]$. One can choose $l_{1,1}$ to be $\frac{1}{uv}\mathsf{S}^{10}$ in this case since

$$\nabla_0\left(\frac{1}{uv}\mathsf{S}^{10}\right) = -\left(\frac{9\alpha}{u^2\Delta} + \frac{u\alpha}{2v^2\Delta}\right)\mathsf{S}^{10} - \frac{5\alpha}{4v\Delta}\mathsf{S}^9\mathsf{T}$$

can be written as $\omega^{(1)}_{1,1} - \omega^{(0)}_{1,1}$ with

$$\omega^{(0)}_{1,1} = \frac{9\alpha}{u^2\Delta}\mathsf{S}^{10} \in \mathcal{F}^1_{10}(U_0),$$

and

$$\omega^{(1)}_{1,1} = -\frac{u\alpha}{2v^2\Delta}\mathsf{S}^{10} - \frac{5\alpha}{4v\Delta}\mathsf{S}^9\mathsf{T} \in \mathcal{F}^1_{10}(U_1).$$

One can easily compute and find that these two forms are ∇_0-closed, so the cochain

$$\widetilde{\omega}_{1,1} = \begin{array}{|cc} (\omega^{(0)}_{1,1}, \omega^{(1)}_{1,1}) & 0 \\ 0 & l_{1,1} \\ \hline \end{array}$$

lives to E_3 and represents a class in $H^1_{\mathrm{dR}}(\mathcal{M}_{1,1}, S^{10}\mathcal{H})$.

Using the same argument at the beginning of the above example, one easily gets

Lemma 3.5.4. *The cohomology group $H^1_\delta(\mathfrak{U}, \mathcal{F}^0_{2n})$ is spanned by classes $[\frac{1}{u^p v^q}\mathsf{S}^s\mathsf{T}^t]$ with positive integers p, q such that $4p + 6q \leqslant 2n$, and nonnegative integers $s = n + 2p + 3q$, $t = n - 2p - 3q$.*

Remark 3.5.5. Here s and t can be solved from $s + t = 2n$ ($\mathsf{S}^s\mathsf{T}^t \in S^{2n}\mathcal{H}$) and $s - t = 4p + 6q$ ($\frac{1}{u^p v^q}\mathsf{S}^s\mathsf{T}^t$ is $\mathbb{G}_m$-invariant, i.e. $\frac{1}{u^p v^q} \in \mathrm{gr}_{t-s}\mathcal{O}(Y_{01})$).

Since we already found all the *holomorphic* modular forms in $H^1_{\mathrm{dR}}(\mathcal{M}_{1,1}, S^{2n}\mathcal{H})$, and based on Eichler-Shimura (3.5.1)-(3.5.3), the remaining classes span a vector space of dimension equal to the one spanned by (anti)holomorphic cusp forms. Therefore, by Lemma 3.5.6 below, if we were to find a class $[\widetilde{\omega}_{j,k}]$ in $H^1_{\mathrm{dR}}(\mathcal{M}_{1,1}, S^{2n}\mathcal{H})$ for each positive integer pair (j, k) with $4j + 6k = 2n$, we will have found all the remaining classes.

Lemma 3.5.6. *The number of positive integer pairs (j, k) satisfying $4j + 6k = 2n$ is the same as the dimension of the space of cusp forms of weight $2n + 2$.*

Proof. The dimension of cusp forms of weight $2n + 2$ is the number of normalized Hecke eigen cusp forms of the same weight. We can choose a basis to be $\Delta \cdot u^a v^b$ where a and b are nonnegative integers, u and v are normalized Hecke eigenform of weight 4 and 6 respectively as usual. It suffices to show a bijection between the set of integer pairs (j, k) and the set of integer pairs (a, b). Clearly the weight of modular forms provides us with the restriction $4a + 6b + 12 = 2n + 2$, or equivalently $4(a + 1) + 6(b + 1) = 2n$. Therefore, $j = a + 1$ and $k = b + 1$ gives us the bijection we need. □

The following result explains how to find representatives of all the remaining $\mathbb{Q}$-de Rham classes.

Proposition 3.5.7. *Assume $n \geqslant 5$. For any pair of positive integers j, k such that $4j + 6k = 2n$, there is a class $[\widetilde{\omega}_{j,k}]$ in $H^1_{\mathrm{dR}}(\mathcal{M}_{1,1}, S^{2n}\mathcal{H})$ represented by a Čech*

1-cocycle

$$\widetilde{\omega}_{j,k} = \begin{array}{|cc} (\omega^{(0)}_{j,k}, \omega^{(1)}_{j,k}) & 0 \\ 0 & l_{j,k} \\ \hline \end{array}$$

where

$$l_{j,k} = \sum_{s+t=2n} x_{s,t} \mathsf{S}^s \mathsf{T}^t \in \mathcal{F}^0_{2n}(U_{01}),$$

with $x_{s,t} \in \mathrm{gr}_{t-s}\mathcal{O}(Y_{01})$ *and* $x_{2n,0} = \frac{1}{u^j v^k}$*. In other words, we can choose* $l_{j,k}$ *to be a* $\mathbb{Q}$*-linear combination of terms* $\frac{1}{u^p v^q}\mathsf{S}^s\mathsf{T}^t$ *starting with a term* $\frac{1}{u^j v^k}\mathsf{S}^{2n}$ *so that* $\nabla_0 l_{j,k}$ *can be expressed as* $\omega^{(1)}_{j,k} - \omega^{(0)}_{j,k}$*, with both* $\omega^{(0)}_{j,k}$ *and* $\omega^{(1)}_{j,k}$ *being* ∇_0*-closed.*

Proof. Let us order the terms in $l_{j,k}$ and $\nabla_0 l_{j,k}$ by the power of S, then S^{2n} has the highest order $2n$, while T^{2n} has the lowest order 0. By using (3.3.1), one can compute that

$$\nabla_0\left(\frac{1}{u^p v^q}\mathsf{S}^s\mathsf{T}^t\right) = \frac{3t}{2u^p v^q}\frac{\alpha}{\Delta}\mathsf{S}^{s+1}\mathsf{T}^{t-1} \tag{3.5.4}$$

$$- \left(\frac{9p}{u^{p+1}v^{q-1}}\frac{\alpha}{\Delta} + \frac{q}{2u^{p-2}v^{q+1}}\frac{\alpha}{\Delta}\right)\mathsf{S}^s\mathsf{T}^t \tag{3.5.5}$$

$$- \frac{s}{8u^{p-1}v^q}\frac{\alpha}{\Delta}\mathsf{S}^{s-1}\mathsf{T}^{t+1} \tag{3.5.6}$$

where the terms on the right hand side in (3.5.4), (3.5.5), (3.5.6) have orders $s+1$, s, $s-1$ respectively. Recall that our objective is to express $\nabla_0 l_{j,k}$ as a difference $\omega^{(1)}_{j,k} - \omega^{(0)}_{j,k}$. We call a term "bad" if its denominater has positive powers of both u and v (as we cannot write it as $\omega^{(1)} - \omega^{(0)}$). We will eliminate all bad terms appearing in $\nabla_0 l_{j,k}$ by adding ∇_0-coboundaries.

To find $l_{j,k}$, we start with $\frac{1}{u^j v^k}\mathsf{S}^{2n}$, then $\nabla_0(\frac{1}{u^j v^k}\mathsf{S}^{2n})$ has terms of order $2n$ and $2n-1$ (the $(2n+1)$-order term being 0). We can use order $2n-1$ terms $x_{2n-1,1}\mathsf{S}^{2n-1}\mathsf{T}$ to cancel the order $2n$ bad terms in $\nabla_0(\frac{1}{u^j v^k}\mathsf{S}^{2n})$ since the order $2n$ term in $\nabla_0(x_{2n-1,1}\mathsf{S}^{2n-1}\mathsf{T})$ is just a rational multiple of $x_{2n-1,1}\frac{\alpha}{\Delta}\mathsf{S}^{2n}$ by the formula in (3.5.4). Now $\nabla_0(\frac{1}{u^j v^k}\mathsf{S}^{2n} + x_{2n-1,1}\mathsf{S}^{2n-1}\mathsf{T})$ has bad terms of order at most $2n-1$.

We can repeat this process until the bad terms have order $n+7$, due to the fact that there always exist positive integer solutions for $4p+6q=2r$ with $2r>12$, which correspond to bad terms of the form $\frac{1}{u^p v^q}\mathsf{S}^{n+r}\mathsf{T}^{n-r}$. They are used to cancel rational multiples of $\frac{1}{u^p v^q}\frac{\alpha}{\Delta}\mathsf{S}^{n+r+1}\mathsf{T}^{n-r-1}$.

Now we need to vary the argument above to find the last term of order $n+5$, at which time the process terminates. We distinguish terms $\frac{1}{u^p v^q}\mathsf{S}^s\mathsf{T}^t$ and $\frac{1}{u^p v^q}\frac{\alpha}{\Delta}\mathsf{S}^s\mathsf{T}^t$ by calling them "function" and "form" respectively. By our process above,

$$\nabla_0\left(\frac{1}{u^j v^k}\mathsf{S}^{2n}+x_{2n-1,1}\mathsf{S}^{2n-1}\mathsf{T}+\cdots+x_{n+7,n-7}\mathsf{S}^{n+7}\mathsf{T}^{n-7}\right) \tag{3.5.7}$$

has bad forms of order at most $n+7$ and at least $n+6$. In fact, the only $\mathbb{G}_m$-invariant forms of order $n+7$ is a linear combination of $\frac{\alpha}{u^3\Delta}\mathsf{S}^{n+7}\mathsf{T}^{n-7}$ and $\frac{\alpha}{v^2\Delta}\mathsf{S}^{n+7}\mathsf{T}^{n-7}$, neither of which is a bad form. We are thus left with only bad forms of order $n+6$ in (3.5.7). The only bad $\mathbb{G}_m$-invariant form of order $n+6$, up to a rational multiple, is $\frac{1}{uv}\frac{\alpha}{\Delta}\mathsf{S}^{n+6}\mathsf{T}^{n-6}$. To cancel bad forms of order $n+6$, we can choose a rational multiple of $\frac{1}{uv}\mathsf{S}^{n+5}\mathsf{T}^{n-5}$ to be the last term of $l_{j,k}$. Therefore, the only possible bad terms in $\nabla_0(l_{j,k})$ comes from the order $n+5$ and $n+4$ parts of $\nabla_0(\frac{1}{uv}\mathsf{S}^{n+5}\mathsf{T}^{n-5})$. Both these parts have no bad terms, which can be easily checked by formulas (3.5.5)(3.5.6).

Eventually, we have a linear combination of bad terms

$$l_{j,k}:=\frac{1}{u^j v^k}\mathsf{S}^{2n}+x_{2n-1,1}\mathsf{S}^{2n-1}\mathsf{T}+\cdots+x_{n+7,n-7}\mathsf{S}^{n+7}\mathsf{T}^{n-7}+x_{n+5,n-5}\mathsf{S}^{n+5}\mathsf{T}^{n-5}$$

such that $\nabla_0(l_{j,k})$ has no bad terms.

After finding $l_{j,k}$, it is routine to find $\omega_{j,k}^{(0)}$ and $\omega_{j,k}^{(1)}$ by putting all terms with powers of u in the denominator into $\omega_{j,k}^{(0)}$ and putting all terms with powers of v in the denominator into $\omega_{j,k}^{(1)}$. It remains to show that both $\omega_{j,k}^{(0)}$ and $\omega_{j,k}^{(1)}$ are ∇_0-closed. This follows from [4, Cor. 3.5]. Or one sees directly from the formulas (3.5.4)-(3.5.6) that all the terms in $\omega_{j,k}^{(0)}$ and $\omega_{j,k}^{(1)}$ have α in the numerator and Δ in the denominator, and easily checks that $h\frac{\alpha}{\Delta}\mathsf{S}^s\mathsf{T}^t$ is ∇_0-closed as long as it is a $\mathbb{G}_m$-invariant form. □

Another Proof. Instead of the minimalist approach above where we only eliminate all bad terms that arise, we provide another efficient and canonical approach based on "Heads and Tails" in Brown–Hain [4, §4.1], eliminating all terms of positive order and leaving only the T^{2n} term in $\nabla_0 l_{j,k}$.

Adapting from Lemma 4.1 in [4], we can easily show that there is a unique $\mathbb{Q}$-linear map $\phi : \mathrm{gr}_{-2n}\mathcal{O}(Y_{01}) \to \mathcal{F}^0_{2n}(U_{01})$ such that if we write

$$\phi = \sum_{s+t=2n} \phi^{s,t}\mathsf{S}^s\mathsf{T}^t$$

where $\phi^{s,t} \in \mathrm{Hom}(\mathrm{gr}_{-2n}\mathcal{O}(Y_{01}), \mathrm{gr}_{t-s}\mathcal{O}(Y_{01}))$, then we have

$$\phi^{2n,0}(l) = l \quad \text{and} \quad \nabla_0\phi(l) \in \mathrm{gr}_{2n+2}\mathcal{O}(Y_{01})\frac{\alpha}{\Delta}\mathsf{T}^{2n}.$$

In fact, we have

$$\nabla_0\phi(l) = \frac{\mathcal{D}^{2n+1}(l)}{(2n)!}\frac{3\alpha}{2\Delta}\mathsf{T}^{2n},$$

where $\mathcal{D}$ is the differential operator $q\partial/\partial q = (2\pi i)^{-1}\partial/\partial\tau$. It is obvious that every function in $\mathrm{gr}_{2n+2}\mathcal{O}(Y_{01})$ contains no bad terms. Therefore, we can set $l_{j,k}$ to be

$$l_{j,k} = \phi\Big(\frac{1}{u^j v^k}\Big) = \sum_{s+t=2n} \phi^{s,t}\Big(\frac{1}{u^j v^k}\Big)\mathsf{S}^s\mathsf{T}^t,$$

and choose $\omega_{j,k}^{(0)}$, $\omega_{j,k}^{(1)}$ accordingly so that $\omega_{j,k}^{(1)} - \omega_{j,k}^{(0)} = \nabla_0 l_{j,k}$.

Note that this proof provides a different 1-cocyle, but it represents the class $[\widetilde{\omega}_{j,k}] \in H^1_{\mathrm{dR}}(\mathcal{M}_{1,1}, S^{2n}\mathcal{H})$ constructed in the previous proof. □

Remark 3.5.8. For different pairs of integers (j,k), the classes $[\widetilde{\omega}_{j,k}] \in H^1_{\mathrm{dR}}(\mathcal{M}_{1,1}, S^{2n}\mathcal{H})$ are linearly independent because of Lemma 3.5.4 and the fact that $l_{j,k}$ starts with $\frac{1}{u^j v^k}\mathsf{S}^{2n}$.

Example 3.5.9. The cocycle $\widetilde{\omega}_{2,1}$ that represents $[\widetilde{\omega}_{2,1}] \in H^1_{\mathrm{dR}}(\mathcal{M}_{1,1}, S^{14}\mathcal{H})$. We carry out the process in the above proposition when $(j,k) = (2,1)$ and $n = 7$. Starting with $\frac{1}{u^2v}\mathsf{S}^{14}$ that represents $[\frac{1}{u^2v}\mathsf{S}^{14}] \in H^1_\delta(\mathfrak{U}, \mathcal{F}^0_{14})$, we have

$$\nabla_0\left(\frac{1}{u^2v}\mathsf{S}^{14}\right) = -\left(\frac{18}{u^3}\frac{\alpha}{\Delta} + \frac{1}{2v^2}\frac{\alpha}{\Delta}\right)\mathsf{S}^{14} - \frac{7}{4uv}\frac{\alpha}{\Delta}\mathsf{S}^{13}\mathsf{T}$$

with a bad term of order $13 = n+6$, which can be eliminated by adding a multiple of the function $\frac{1}{uv}\mathsf{S}^{12}\mathsf{T}^2$ of order $n+5 = 12$. Since

$$\nabla_0\left(\frac{1}{uv}\mathsf{S}^{12}\mathsf{T}^2\right) = \frac{3}{uv}\frac{\alpha}{\Delta} - \left(\frac{9}{u^2}\frac{\alpha}{\Delta} + \frac{u}{2v^2}\frac{\alpha}{\Delta}\right)\mathsf{S}^{12}\mathsf{T}^2 - \frac{3}{2v}\frac{\alpha}{\Delta}\mathsf{S}^{11}\mathsf{T}^3,$$

we choose the correct multiple $\frac{7}{12}$ for $\frac{1}{uv}\mathsf{T}^2\mathsf{S}^{12}$, and

$$\nabla_0\left(\frac{1}{u^2v}\mathsf{S}^{14} + \frac{7}{12}\frac{1}{uv}\mathsf{S}^{12}\mathsf{T}^2\right) = -\left(\frac{18}{u^3}\frac{\alpha}{\Delta} + \frac{1}{2v^2}\frac{\alpha}{\Delta}\right)\mathsf{S}^{14} - \left(\frac{21}{4u^2}\frac{\alpha}{\Delta} + \frac{7u}{24v^2}\frac{\alpha}{\Delta}\right)\mathsf{S}^{12}\mathsf{T}^2 - \frac{7}{8v}\frac{\alpha}{\Delta}\mathsf{S}^{11}\mathsf{T}^3$$

has no bad terms.

Define $l_{2,1} = \frac{1}{u^2v}\mathsf{S}^{14} + \frac{7}{12}\frac{1}{uv}\mathsf{S}^{12}\mathsf{T}^2$, $\omega^{(0)}_{2,1} = \frac{18}{u^3}\frac{\alpha}{\Delta}\mathsf{S}^{14} + \frac{21}{4u^2}\frac{\alpha}{\Delta}\mathsf{S}^{12}\mathsf{T}^2$, and $\omega^{(1)}_{2,1} = -\frac{1}{2v^2}\frac{\alpha}{\Delta}\mathsf{S}^{14} - \frac{7u}{24v^2}\frac{\alpha}{\Delta}\mathsf{S}^{12}\mathsf{T}^2 - \frac{7}{8v}\frac{\alpha}{\Delta}\mathsf{S}^{11}\mathsf{T}^3$, then the cocycle

$$\widetilde{\omega}_{2,1} := \left| \begin{array}{cc} (\omega^{(0)}_{2,1}, \omega^{(1)}_{2,1}) & 0 \\ 0 & l_{2,1} \end{array} \right.$$

represents the class $[\widetilde{\omega}_{2,1}]$ we are looking for.

3.6 A $\mathbb{Q}$-de Rham Structure on Relative Completion of $\mathrm{SL}_2(\mathbb{Z})$

Now we follow Hain [18, §7], and construct a $\mathbb{Q}$-de Rham structure on the relative completion $\mathcal{G}^{\mathrm{rel}}$ of $\mathrm{SL}_2(\mathbb{Z})$.

We review the Betti version first. We view $\mathrm{SL}_2(\mathbb{Z})$ as the (orbifold) fundamental group of $\mathcal{M}_{1,1}$. For any base point $x \in \mathcal{M}_{1,1}$, we have a Zariski dense monodromy representation

$$\rho_x : \pi_1(\mathcal{M}_{1,1}, x) \to \mathrm{SL}_2(\mathbb{Q}).$$

Denote the relative completion of $\pi_1(\mathcal{M}_{1,1}, x)$ with respect to ρ_x by $\mathcal{G}_x$. Denote the Lie algebra of its unipotent radical $\mathcal{U}_x$ by $\mathfrak{u}_x$. One can construct Betti $\mathbb{Q}$-structures on $\mathcal{O}(\mathcal{G}_x)$ and on $\mathfrak{u}_x$ that are compatible with their algebraic structures [19]. It is achieved by finding a (Betti) canonical flat connection

$$\nabla = \nabla_0 + \Omega$$

on the bundle $\boldsymbol{\mathfrak{u}}_{\mathrm{B}} \to \mathcal{M}_{1,1}$ whose fiber over x is the Lie algebra $\mathfrak{u}_x$.

For the de Rham version, we will find in Section 3.6.4 a Čech-de Rham 1-cochain $\widetilde{\Omega}$, interpreted as a connection form in Section 3.6.5. This provides us with connection forms $\Omega^{(j)}$ on the opens U_j, $j = 0, 1$ and a gauge transformation between the connections on the intersection U_{01}. Note that all of these are defined over $\mathbb{Q}$, and the connection forms have logarithmic poles at the cusp. From these data, we can first construct bundles on the two opens U_j with the corresponding connections

$$\nabla = \nabla_0 + \Omega^{(j)}, \qquad j = 0, 1,$$

then glue them together on the intersection U_{01} via the gauge transformation. Therefore, we have constructed a bundle $\boldsymbol{\mathfrak{u}}_{\mathrm{dR}} \to \overline{\mathcal{M}}_{1,1}$ with a (de Rham) connection that is defined over $\mathbb{Q}$ and has regular singularity at the cusp.

We start by describing the bundle $\boldsymbol{\mathfrak{u}}$ where this connection lives.

3.6.1 The bundle $\boldsymbol{\mathfrak{u}}$ of Lie algebras

Recall that the Lie algebra $\mathfrak{u}^{\mathrm{rel}}$ of $\mathcal{U}^{\mathrm{rel}}$ is freely topologically generated by

$$\prod_{n \geqslant 0} H^1(\mathrm{SL}_2(\mathbb{Z}), S^{2n}H)^* \otimes S^{2n}H,$$

where H is the standard representation of SL_2, and $S^{2n}H$ its $2n$-th symmetric power. Note that each fiber $\mathfrak{u}_x$ of $\boldsymbol{\mathfrak{u}}$ is (abstractly) isomorphic to $\mathfrak{u}^{\mathrm{rel}}$. Define

$$\boldsymbol{\mathfrak{u}}_1 := \prod_{n\geqslant 0} H^1(\mathcal{M}_{1,1}^{\mathrm{an}}, S^{2n}\mathbb{H})^* \otimes S^{2n}\mathbb{H}.$$

This is a pro-vector bundle over $\mathcal{M}_{1,1}$, whose fiber over x is the abelianization of the free Lie algebra $\mathfrak{u}_x$. Let

$$\boldsymbol{\mathfrak{u}}_n := \mathbb{L}_n(\boldsymbol{\mathfrak{u}}_1)$$

be the degree n part of the free Lie algebra generated by $\boldsymbol{\mathfrak{u}}_1$. For any $u \in \mathbb{L}(\boldsymbol{\mathfrak{u}}_1)$, we will often denote its degree n part by $(u)_n$. Set

$$\boldsymbol{\mathfrak{u}} := \varprojlim_n \bigoplus_{j=1}^n \boldsymbol{\mathfrak{u}}_j,$$

and

$$\boldsymbol{\mathfrak{u}}^N := \varprojlim_{n\geqslant N} \bigoplus_{j=N}^n \boldsymbol{\mathfrak{u}}_j,$$

the parts of degree at least N in the Lie algebra $\boldsymbol{\mathfrak{u}}$.

Note that, by what we have done in Section 3.5, $\boldsymbol{\mathfrak{u}}_1$ has its de Rham realization:

$$\boldsymbol{\mathfrak{u}}_{1,\mathrm{dR}} := \prod_{n\geqslant 0} H^1_{\mathrm{dR}}(\mathcal{M}_{1,1}, S^{2n}\mathcal{H})^* \otimes S^{2n}\mathcal{H},$$

with connection ∇_0. This gives rise to a $\mathbb{Q}$-structure $\boldsymbol{\mathfrak{u}}_{\mathrm{dR}}$ on the bundle $\boldsymbol{\mathfrak{u}}$ with the same connection ∇_0, but $(\boldsymbol{\mathfrak{u}}_{\mathrm{dR}}, \nabla_0) \otimes \mathbb{C}$ is not isomorphic to $(\boldsymbol{\mathfrak{u}}_{\mathrm{B}}, \nabla) \otimes \mathbb{C}$, where $\nabla = \nabla_0 + \Omega$ is the (Betti) canonical flat connection. In Section 3.6.4, we will see how to perturb the flat connection ∇_0 on $\boldsymbol{\mathfrak{u}}_{1,\mathrm{dR}}$ to find a (de Rham) canonical flat connection $\widetilde{\nabla} = \nabla_0 + \widetilde{\Omega}$ on $\boldsymbol{\mathfrak{u}}_{\mathrm{dR}}$. From now on, we will work exclusively with de Rham realizations such as $\boldsymbol{\mathfrak{u}}_{\mathrm{dR}}$ and $\boldsymbol{\mathfrak{u}}_{1,\mathrm{dR}}$, but we will drop the subscripts $_{\mathrm{dR}}$ on them for notation simplicity.

3.6.2 The differential graded Lie algebra $K^\bullet(\mathcal{M}_{1,1};\mathfrak{u})$

Denote by $K^\bullet(\mathcal{M}_{1,1})$ the single complex associated to the Čech-de Rham double complex $\check{C}^\bullet(\mathfrak{U},\mathcal{F}_0^\bullet)$ where

$$\mathcal{F}_0^\bullet = (\Omega^\bullet_{\overline{Y}}(\log D))^{\mathbb{G}_m},$$

and $D = \nabla_0 + \delta$ is the total differential in $\check{C}^\bullet(\mathfrak{U},\mathcal{F}_0^\bullet)$.[7] Define a product on this complex based on the convention: if $\omega \in \check{C}^p(\mathfrak{U},\mathcal{F}_0^q)$ and $\eta \in \check{C}^r(\mathfrak{U},\mathcal{F}_0^s)$, then their product $\omega \wedge \eta \in \check{C}^{p+r}(\mathfrak{U},\mathcal{F}_0^{q+s})$ is defined by

$$(\omega\wedge\eta)(U_{\alpha_0\cdots\alpha_{p+r}}) = (-1)^{qr}\omega(U_{\alpha_0\cdots\alpha_p})\wedge\eta(U_{\alpha_p\cdots\alpha_{p+r}}),$$

where on the right hand side forms are understood to be restricted to $U_{\alpha_0\cdots\alpha_{p+r}}$. Therefore, if $\widetilde{\Omega}$ and $\widetilde{\Omega}'$ are 1-cochains in $K^1(\mathcal{M}_{1,1})$ with

$$\widetilde{\Omega} = \begin{array}{|cc} (\omega^{(0)},\omega^{(1)}) & 0 \\ 0 & l \\ \hline \end{array}$$

and

$$\widetilde{\Omega}' = \begin{array}{|cc} (\omega'^{(0)},\omega'^{(1)}) & 0 \\ 0 & l' \\ \hline \end{array}$$

their product is

$$\widetilde{\Omega}\wedge\widetilde{\Omega}' := \begin{array}{|cc} (\omega^{(0)}\wedge\omega'^{(0)},\omega^{(1)}\wedge\omega'^{(1)}) & 0 \\ 0 & l\cdot\omega'^{(1)} - \omega^{(0)}\cdot l' \\ 0 & 0 \\ \hline \end{array}$$

Note that this complex $K^\bullet(\mathcal{M}_{1,1})$ does not compute $H^1_{\mathrm{dR}}(\mathcal{M}_{1,1/\mathbb{Q}})$.[8] Its first cohomology has a nontrivial class $[\frac{d\Delta}{\Delta}]$. However, it computes $H^2_{\mathrm{dR}}(\mathcal{M}_{1,1/\mathbb{Q}})$, which is all we will need in Section 3.6.4.

[7] The Čech-de Rham complex was defined in Section 3.4.3 with $n = 0$.

[8] cf. Brown–Hain [4] §3 computations before Remark 3.1.

Let

$$K^\bullet(\mathcal{M}_{1,1};\mathfrak{u}) := K^\bullet(\mathcal{M}_{1,1}) \otimes \mathfrak{u}.$$

This is a differential graded Lie algebra with differential induced from the differential D of the complex $K^\bullet(\mathcal{M}_{1,1})$. Define a product $[\cdot,\cdot]$ on $K^\bullet(\mathcal{M}_{1,1};\mathfrak{u})$ by:

$$[\widetilde{\Omega}_\beta \otimes u, \widetilde{\Omega}_\gamma \otimes v] := (\widetilde{\Omega}_\beta \wedge \widetilde{\Omega}_\gamma) \otimes [u,v],$$

where $\widetilde{\Omega}_\beta, \widetilde{\Omega}_\gamma \in K^\bullet(\mathcal{M}_{1,1})$, $u, v \in \mathfrak{u}$ with $[u,v]$ being the Lie product of u and v.

3.6.3 The canonical 1-cocycle $\widetilde{\Omega}_1$

We start with defining a canonical 1-cocycle

$$\widetilde{\Omega}_1 = (\Omega_1^{(0)}, \Omega_1^{(1)}; f_1) \in K^1(\mathcal{M}_{1,1};\mathfrak{u}_1)$$

or

$$\widetilde{\Omega}_1 = \left| \begin{array}{cc} (\Omega_1^{(0)}, \Omega_1^{(1)}) & 0 \\ 0 & f_1 \end{array} \right.$$

that represents the identity maps $H^1_{\mathrm{dR}}(\mathcal{M}_{1,1}, S^{2n}\mathcal{H}) \to H^1_{\mathrm{dR}}(\mathcal{M}_{1,1}, S^{2n}\mathcal{H})$ for every $n > 0$. It can be written as

$$\widetilde{\Omega}_1 := \prod_{n \geqslant 1} \widetilde{\Omega}_{1,2n} \in K^1(\mathcal{M}_{1,1};\mathfrak{u}_1),$$

where we define $\widetilde{\Omega}_{1,2n}$ by using cocyles found in Section 3.5,

$$\widetilde{\Omega}_{1,2n} := \sum_f \widetilde{\omega}_f \mathsf{X}_f + \sum_{\{(j,k):4j+6k=2n\}} \widetilde{\omega}_{j,k} \mathsf{X}_{j,k} \in K^1(\mathcal{M}_{1,1}; S^{2n}\mathcal{H}) \otimes H^1_{\mathrm{dR}}(\mathcal{M}_{1,1}, S^{2n}\mathcal{H})^*,$$

with the first term on the right hand side is summing over Hecke eigenforms f of weight $2n+2$, and all $\mathsf{X}_f, \mathsf{X}_{j,k} \in H^1_{\mathrm{dR}}(\mathcal{M}_{1,1}, S^{2n}\mathcal{H})^*$ form a dual basis for all $\mathbb{Q}$-de Rham classes $[\widetilde{\omega}_f]$ and $[\widetilde{\omega}_{j,k}]$ in $H^1_{\mathrm{dR}}(\mathcal{M}_{1,1}, S^{2n}\mathcal{H})$.

For our purpose, we would prefer to write $\widetilde{\Omega}_{1,2n}$ in a different way: let

$$\widetilde{\omega}_f = \widetilde{\Omega}_f \mathsf{T}^{2n} \quad \text{and} \quad \widetilde{\omega}_{j,k} = \sum_m \widetilde{\Omega}_{j,k}^m \mathsf{S}^m \mathsf{T}^{2n-m},$$

with coefficients $\widetilde{\Omega}_f, \widetilde{\Omega}_{j,k}^m \in K^1(\mathcal{M}_{1,1}), 0 \leqslant m \leqslant 2n$. Set

$$\mathbf{e}_f := \mathsf{X}_f \otimes \mathsf{T}^{2n} \quad \text{and} \quad \mathbf{e}_{j,k}^m := \mathsf{X}_{j,k} \otimes \mathsf{S}^m \mathsf{T}^{2n-m},$$

then $\mathbf{e}_f, \mathbf{e}_{j,k}^m \in \mathbf{u}_1$, and

$$\widetilde{\Omega}_{1,2n} = \sum_f \widetilde{\Omega}_f \mathbf{e}_f + \sum_{\{(j,k):4j+6k=2n\}} \left(\sum_m \widetilde{\Omega}_{j,k}^m \mathbf{e}_{j,k}^m \right).$$

3.6.4 The connection 1-cochain $\widetilde{\Omega}$

One can follow the procedure in Hain [18, §7.3] to inductively define a connection 1-cochain $\widetilde{\Omega}$ from the canonical 1-cocycle $\widetilde{\Omega}_1$. This procedure is modified from Chen's method of power series connections [6]. Given a differential graded Lie algebra

$$K^\bullet(X; \mathfrak{u}) = K^\bullet(X) \otimes \mathfrak{u},$$

where $K^\bullet(X)$ is a complex whose second cohomology vanishes, i.e. $H^2(K^\bullet(X)) = 0$; and $\mathfrak{u}$ is a free Lie algebra generated by $\mathfrak{u}_1$, whose degree n part is denoted by $\mathfrak{u}_n := \mathbb{L}_n(\mathfrak{u}_1)$, and parts of degree at least N by $\mathfrak{u}^N := \varprojlim_{n \geqslant N} \bigoplus_{j=N}^{n} \mathfrak{u}_j$. There is a bracket $[\cdot, \cdot]$ on $K^\bullet(X; \mathfrak{u})$ given by:

$$[\widetilde{\Omega}_\beta \otimes \mathbf{e}_\beta, \widetilde{\Omega}_\gamma \otimes \mathbf{e}_\gamma] := (\widetilde{\Omega}_\beta \wedge \widetilde{\Omega}_\gamma) \otimes [\mathbf{e}_\beta, \mathbf{e}_\gamma],$$

which is induced from the wedge product $\wedge$ on $K^\bullet(X)$ and the Lie bracket $[\cdot, \cdot]$ on $\mathfrak{u}$. The following result is used to inductively construct the (de Rham) connection form $\widetilde{\Omega}$.

Proposition 3.6.1. *Suppose we have a closed form*

$$\Omega_1 \in K^1(X;\mathfrak{u}_1),$$

such that $D\Omega_1 = 0$. *For any* $n \geqslant 2$, *we can find* $\Xi_n \in K^1(X;\mathfrak{u}_n)$, *and set* $\Omega_n := \Omega_{n-1} + \Xi_n$, *so that*

$$D\Omega_n + \frac{1}{2}[\Omega_n,\Omega_n] \equiv 0 \mod \mathfrak{u}^{n+1}.$$

Proof. Note that $D\Omega_1 = 0$, $[\Omega_1,\Omega_1] \in K^2(X;\mathfrak{u}_2)$ is closed. Since $H^2(K^\bullet(X)) = 0$, $[\Omega_1,\Omega_1]$ is thus exact. One can find $\Xi_2 \in K^1(X;\mathfrak{u}_2)$ such that $-D\Xi_2 = \frac{1}{2}[\Omega_1,\Omega_1]$ and $D\Omega_2 + \frac{1}{2}[\Omega_2,\Omega_2] \equiv 0 \mod \mathfrak{u}^3$.

Suppose for $n \geqslant 2$ we have already found $\Xi_2,\cdots,\Xi_n$, and $\Omega_n = \Omega_1 + \sum_{i=2}^n \Xi_i$, such that $D\Omega_n + \frac{1}{2}[\Omega_n,\Omega_n] \equiv 0 \mod \mathfrak{u}^{n+1}$. We claim that the degree $(n+1)$ part

$$(D\Omega_n + \frac{1}{2}[\Omega_n,\Omega_n])_{n+1} \in K^2(X;\mathfrak{u}_{n+1})$$

is closed. In fact,

$$\begin{aligned} D(D\Omega_n + \frac{1}{2}[\Omega_n,\Omega_n]) &= \frac{1}{2}[D\Omega_n,\Omega_n] - \frac{1}{2}[\Omega_n,D\Omega_n] \\ &\equiv \frac{1}{2}[(-\frac{1}{2}[\Omega_n,\Omega_n]),\Omega_n] - \frac{1}{2}[\Omega_n,(-\frac{1}{2}[\Omega_n,\Omega_n])] \\ &= -\frac{1}{4}[[\Omega_n,\Omega_n],\Omega_n] + \frac{1}{4}[\Omega_n,[\Omega_n,\Omega_n]] = 0 \mod \mathfrak{u}^{n+2} \end{aligned}$$

where we have used Leibniz rule of D on the first line, induction hypothesis on the second line, and both terms on the last line are 0 by Jacobi identity. Since $H^2(K^\bullet(X)) = 0$, $(D\Omega_n + \frac{1}{2}[\Omega_n,\Omega_n])_{n+1} \in K^2(X;\mathfrak{u}_{n+1})$ is closed thus exact. We can find $\Xi_{n+1} \in K^1(X;\mathfrak{u}_{n+1})$ such that

$$-D\Xi_{n+1} = (D\Omega_n + \frac{1}{2}[\Omega_n,\Omega_n])_{n+1} = (\frac{1}{2}[\Omega_n,\Omega_n])_{n+1}.$$

Define $\Omega_{n+1} := \Omega_n + \Xi_{n+1}$, then it is easy to check that

$$D\Omega_{n+1} + \frac{1}{2}[\Omega_{n+1}, \Omega_{n+1}] \equiv 0 \mod \mathfrak{u}^{n+2}$$

since $D\Omega_{n+1} + \frac{1}{2}[\Omega_{n+1}, \Omega_{n+1}] \equiv D\Omega_n + \frac{1}{2}[\Omega_n, \Omega_n] \equiv 0 \mod \mathfrak{u}^{n+1}$ and the degree $(n+1)$ part on the left is $D\Xi_{n+1} + (\frac{1}{2}[\Omega_n, \Omega_n])_{n+1} = 0$. □

Example 3.6.2. Since $K^\bullet(\mathcal{M}_{1,1})$ computes the second cohomology $H^2_{\mathrm{dR}}(\mathcal{M}_{1,1/\mathbb{Q}})$ and it vanishes, the above proposition applies to our case $K^\bullet(\mathcal{M}_{1,1}; \mathfrak{u})$. Since the canonical 1-cocycle

$$\widetilde{\Omega}_1 \in K^1(\mathcal{M}_{1,1}; \mathfrak{u}_1)$$

is closed, one can recurrently define for $n \geqslant 2$,

$$\widetilde{\Omega}_n := \widetilde{\Omega}_{n-1} + \widetilde{\Xi}_n,$$

with $\widetilde{\Xi}_n \in K^1(\mathcal{M}_{1,1}; \mathfrak{u}_n)$ satisfying

$$-D\widetilde{\Xi}_n = (D\widetilde{\Omega}_{n-1} + \frac{1}{2}[\widetilde{\Omega}_{n-1}, \widetilde{\Omega}_{n-1}])_n = (\frac{1}{2}[\widetilde{\Omega}_{n-1}, \widetilde{\Omega}_{n-1}])_n, \qquad (3.6.1)$$

so that

$$D\widetilde{\Omega}_n + \frac{1}{2}[\widetilde{\Omega}_n, \widetilde{\Omega}_n] \in K^2(\mathcal{M}_{1,1}; \mathfrak{u}^{n+1}).$$

Define the connection 1-cochain

$$\widetilde{\Omega} := \varprojlim_n \widetilde{\Omega}_n \in K^1(\mathcal{M}_{1,1}; \mathfrak{u}).$$

Then it is defined over $\mathbb{Q}$ and satisfies that

$$D\widetilde{\Omega} + \frac{1}{2}[\widetilde{\Omega}, \widetilde{\Omega}] = 0.$$

To fix notations in the following sections, let

$$\widetilde{\Omega}_n := \begin{array}{|cc} (\Omega_n^{(0)}, \Omega_n^{(1)}) & 0 \\ 0 & F_n \\ \hline \end{array}$$

and write it as

$$\widetilde{\Omega}_n = \sum_{i=1}^{n} \widetilde{\Xi}_i,$$

with $\widetilde{\Xi}_i (i \geqslant 2)$ defined as before, $\widetilde{\Xi}_1 := \widetilde{\Omega}_1$, and

$$\widetilde{\Xi}_i = \begin{array}{|cc} (\Xi_i^{(0)}, \Xi_i^{(1)}) & 0 \\ 0 & f_i \\ \hline \end{array}$$

so that $\Omega_n^{(0)} = \sum\limits_{i=1}^{n} \Xi_i^{(0)}$, $\Omega_n^{(1)} = \sum\limits_{i=1}^{n} \Xi_i^{(1)}$ and $F_n = \sum\limits_{i=1}^{n} f_i$.

One can then formally write

$$\widetilde{\Omega} = \begin{array}{|cc} (\Omega^{(0)}, \Omega^{(1)}) & 0 \\ 0 & F \\ \hline \end{array}$$

as

$$\widetilde{\Omega} = \sum_{i=1}^{\infty} \widetilde{\Xi}_i,$$

where $\Omega^{(0)} = \sum\limits_{i=1}^{\infty} \Xi_i^{(0)} = \varprojlim\limits_{n} \Omega_n^{(0)}$, $\Omega^{(1)} = \sum\limits_{i=1}^{\infty} \Xi_i^{(1)} = \varprojlim\limits_{n} \Omega_n^{(1)}$ and $F = \sum\limits_{i=1}^{\infty} f_i = \varprojlim\limits_{n} F_n$.

3.6.5 Interpretation for $\widetilde{\Omega}$ as a connection form

In this section, we interpret the connection 1-cochain

$$\widetilde{\Omega} = \begin{array}{|cc} (\Omega^{(0)}, \Omega^{(1)}) & 0 \\ 0 & F \\ \hline \end{array}$$

as a connection form on the bundle $\boldsymbol{\mathfrak{u}}$. We will show that the condition

$$D\widetilde{\Omega} + \frac{1}{2}[\widetilde{\Omega}, \widetilde{\Omega}] = 0$$

amounts to the facts that the 1-forms $\Omega^{(i)}$ define flat connections

$$\nabla = \nabla_0 + \Omega^{(i)}$$

on the bundle $\mathfrak{u}$ over the open subsets U_i of $\mathcal{M}_{1,1}$, $i = 0, 1$, and that on the intersection U_{01} these two connections are algebraically gauge equivalent with gauge transformation

$$g : U_{01} \to \mathcal{U}^{\mathrm{rel}} \hookrightarrow \mathrm{Aut}(\mathfrak{u}^{\mathrm{rel}})$$

given by $g := 1 + F$.

We set up the discussion as in Hain [18, §4.1]. Suppose ∇_0 is a connection on an R-equivariant principle $\mathcal{U}$-bundle $X \times \mathcal{U} \to X$. A 1-form ω defines a connection ∇ on this bundle by

$$\nabla s = \nabla_0 s + \omega s,$$

where s is a section. This connection is flat if and only if

$$\nabla_0 \omega + \omega \wedge \omega = 0.$$

A gauge transformation $g : X \to \mathcal{U}$ changes the connection form ω to a new one by

$$\omega' = -\nabla_0 g \cdot g^{-1} + g \omega g^{-1}.$$

In our case, $X = \mathcal{M}_{1,1}$, $R = \mathrm{SL}_2(\mathbb{Z})$, and the principle $\mathcal{U}$-bundle is the bundle $\boldsymbol{\mathcal{U}}$ over $\mathcal{M}_{1,1}$ whose fiber over x is the unipotent radical $\mathcal{U}_x$ of the relative completion $\mathcal{G}_x$. Since a prounipotent group is isomorphic to its Lie algebra as a group, one can think of this bundle $\boldsymbol{\mathcal{U}}$ the same as its corresponding Lie algebra bundle $\mathfrak{u}$. The connection on $\mathfrak{u}$ is essentially given by

$$\widetilde{\nabla} = \nabla_0 + \widetilde{\Omega},$$

and provides us with a $\mathbb{Q}$-de Rham structure $\mathfrak{u}_{\mathrm{dR}}$ on $\mathfrak{u}$ (or $\boldsymbol{\mathcal{U}}_{\mathrm{dR}}$ on $\boldsymbol{\mathcal{U}}$), which we now explain.

Since $D\widetilde{\Omega} + \frac{1}{2}[\widetilde{\Omega}, \widetilde{\Omega}] \in K^2(\mathcal{M}_{1,1}; \mathfrak{u})$ a priori has two parts

$$\begin{array}{|cc} * & 0 \\ 0 & * \\ 0 & 0 \\ \hline \end{array}$$

The top part being 0 means $\nabla_0\Omega^{(0)} + \Omega^{(0)} \wedge \Omega^{(0)} = 0$ and $\nabla_0\Omega^{(1)} + \Omega^{(1)} \wedge \Omega^{(1)} = 0$, which tells us that $\Omega^{(i)}$ is flat on U_i for $i = 0, 1$. The lower part being 0 means

$$\Omega^{(1)} - \Omega^{(0)} - \nabla_0 F + F \cdot \Omega^{(1)} - \Omega^{(0)} \cdot F = 0, \tag{3.6.2}$$

where "$\cdot$" is a product on $\mathcal{F}_0^\bullet(U_{01}) \otimes \mathfrak{u}$ induced by wedge product on $\mathcal{F}_0^\bullet(U_{01})$ and Lie bracket on $\mathfrak{u}$. This equation (3.6.2) tells us that the 1-forms $\Omega^{(0)}$ and $\Omega^{(1)}$ are gauge equivalent on the intersection U_{01} of U_0 and U_1.

Proposition 3.6.3. *The function*

$$g := 1 + F \in \mathcal{U}^{\mathrm{rel}}(\mathcal{O}(U_{01}))$$

transforms $\Omega^{(1)}$ to $\Omega^{(0)}$. That is, on U_{01} we have

$$\Omega^{(0)} = -\nabla_0 g \cdot g^{-1} + g\,\Omega^{(1)}\,g^{-1}.$$

Proof. Let $g_n := 1 + F_n$, then it suffices to prove for every n, we have

$$\Omega_n^{(0)} \equiv -\nabla_0 g_n \cdot g_n^{-1} + g_n\,\Omega_n^{(1)}\,g_n^{-1} \qquad \mod \mathfrak{u}^{n+1}. \tag{3.6.3}$$

We prove this by induction. When $n = 1$, as $\nabla_0(1) = d(1) = 0$ and $g_1^{-1} \equiv 1 \mod \mathfrak{u}^1$, (3.6.3) becomes

$$\Omega_1^{(0)} = -\nabla_0 f_1 + \Omega_1^{(1)},$$

which amounts to the fact that

$$\widetilde{\Omega}_1 = \left| \begin{array}{cc} (\Omega_1^{(0)}, \Omega_1^{(1)}) & 0 \\ 0 & f_1 \end{array} \right.$$

is D-closed.

Assume that (3.6.3) holds for $(n-1)$, we have

$$\Omega_{n-1}^{(0)} \equiv -\nabla_0 g_{n-1} \cdot g_{n-1}^{-1} + g_{n-1}\,\Omega_{n-1}^{(1)}\,g_{n-1}^{-1} \qquad \mod \mathfrak{u}^n. \tag{3.6.4}$$

As for n, we only need to prove that the degree n parts on both sides of (3.6.3) are the same. On the left hand side, it is $\Xi_n^{(0)}$. One can easily show that $g_n^{-1} \equiv g_{n-1}^{-1}$ mod $\mathbf{u}^n$, so we write the right hand side as

$$\begin{aligned} &- \nabla_0 g_n \cdot g_n^{-1} + g_n \, \Omega_n^{(1)} \, g_n^{-1} \\ &\qquad = -\nabla_0 (g_{n-1} + f_n) \cdot (g_{n-1}^{-1} + u_n) + (g_{n-1} + f_n) \, (\Omega_{n-1}^{(1)} + \Xi_n^{(1)}) \, (g_{n-1}^{-1} + u_n) \end{aligned}$$

with some $u_n \in \mathbf{u}^n$. Modulo terms in $\mathbf{u}^{n+1}$, the degree n part on the right side comes from

$$-\nabla_0 g_{n-1} \cdot g_{n-1}^{-1} - \nabla_0 f_n + g_{n-1} \, \Omega_{n-1}^{(1)} \, g_{n-1}^{-1} + \Xi_n^{(1)}.$$

Or equivalently, the degree n part on the right hand side is

$$\Xi_n^{(1)} - \nabla_0 f_n + (-\nabla_0 g_{n-1} \cdot g_{n-1}^{-1} + g_{n-1} \, \Omega_{n-1}^{(1)} \, g_{n-1}^{-1})_n.$$

It remains to prove that the above is the same as $\Xi_n^{(0)}$, which is equivalent to showing that

$$\Xi_n^{(1)} - \Xi_n^{(0)} - \nabla_0 f_n = -(-\nabla_0 g_{n-1} \cdot g_{n-1}^{-1} + g_{n-1} \, \Omega_{n-1}^{(1)} \, g_{n-1}^{-1})_n. \tag{3.6.5}$$

Note that $\Omega_{n-1}^{(0)}$ has terms only of degree less than n, so one can add it to the right hand side without affecting the equality:

$$(-\nabla_0 g_{n-1} \cdot g_{n-1}^{-1} + g_{n-1} \, \Omega_{n-1}^{(1)} \, g_{n-1}^{-1})_n = (-\nabla_0 g_{n-1} \cdot g_{n-1}^{-1} + g_{n-1} \, \Omega_{n-1}^{(1)} \, g_{n-1}^{-1} - \Omega_{n-1}^{(0)})_n.$$

By the induction hypothesis (3.6.4), the form in the parenthesis on the right has terms of degree n or higher. When multiplied by g_{n-1} on the right, its degree n part remains unchanged:

$$(-\nabla_0 g_{n-1} \cdot g_{n-1}^{-1} + g_{n-1} \, \Omega_{n-1}^{(1)} \, g_{n-1}^{-1} - \Omega_{n-1}^{(0)})_n = (-\nabla_0 g_{n-1} + g_{n-1} \cdot \Omega_{n-1}^{(1)} - \Omega_{n-1}^{(0)} \cdot g_{n-1})_n.$$

Since g_{n-1}, and thus $\nabla_0 g_{n-1}$, has terms only of degree less than n, we can remove them:

$$(-\nabla_0 g_{n-1} + g_{n-1} \cdot \Omega_{n-1}^{(1)} - \Omega_{n-1}^{(0)} \cdot g_{n-1})_n = (g_{n-1} \cdot \Omega_{n-1}^{(1)} - \Omega_{n-1}^{(0)} \cdot g_{n-1})_n.$$

Plugging in $g_{n-1} = 1 + F_{n-1}$, and then removing the terms $\Omega^{(1)}_{n-1} - \Omega^{(0)}_{n-1}$ of degree less than n, we get

$$(g_{n-1} \cdot \Omega^{(1)}_{n-1} - \Omega^{(0)}_{n-1} \cdot g_{n-1})_n = (F_{n-1} \cdot \Omega^{(1)}_{n-1} - \Omega^{(0)}_{n-1} \cdot F_{n-1})_n.$$

The equation (3.6.5) now reduces to the equation

$$\Xi^{(1)}_n - \Xi^{(0)}_n - \nabla_0 f_n + (F_{n-1} \cdot \Omega^{(1)}_{n-1} - \Omega^{(0)}_{n-1} \cdot F_{n-1})_n = 0.$$

This equation holds as it is the equation (3.6.1) in the recursive definition of $\widetilde{\Xi}_n$ (cf. it also follows from the degree n part of the equation (3.6.2)). $\square$

Recall that there is a Betti vector bundle $\mathbf{u}_{\mathrm{B}} \to \mathcal{M}_{1,1}$ with its (Betti) canonical flat connection $\nabla = \nabla_0 + \Omega$. Denote by $(\overline{\mathbf{u}}_{\mathrm{B}}, \nabla)$ Deligne's canonical extension of $(\mathbf{u}_{\mathrm{B}}, \nabla)$ to $\overline{\mathcal{M}}_{1,1}$.

Theorem 3.6.4. *There is an algebraic de Rham vector bundle* $\mathbf{u}_{\mathrm{dR}}$ *over* $\overline{\mathcal{M}}_{1,1/\mathbb{Q}}$ *endowed with connection*

$$\widetilde{\nabla} = \nabla_0 + \widetilde{\Omega},$$

and an isomorphism

$$(\mathbf{u}_{\mathrm{dR}}, \widetilde{\nabla}) \otimes_{\mathbb{Q}} \mathbb{C} \approx (\overline{\mathbf{u}}_{\mathrm{B}}, \nabla) \otimes_{\mathbb{Q}} \mathbb{C}.$$

The algebraic de Rham vector bundle $\mathbf{u}_{\mathrm{dR}}$ *and its connection* $\widetilde{\nabla}$ *are both defined over* $\mathbb{Q}$*. Moreover, the connection* $\widetilde{\nabla}$ *has a regular singularity at the cusp.*

Proof. Recall that

$$\widetilde{\Omega} = \left\lfloor \begin{matrix} (\Omega^{(0)}, \Omega^{(1)}) & 0 \\ 0 & F \end{matrix} \right. .$$

We construct trivial bundles

$$\begin{array}{ccc} \mathfrak{u}^{\mathrm{rel}} \times U_0 & \text{and} & \mathfrak{u}^{\mathrm{rel}} \times U_1 \\ \downarrow & & \downarrow \\ U_0 & & U_1 \end{array}$$

with connections $\nabla = \nabla_0 + \Omega^{(0)}$ and $\nabla = \nabla_0 + \Omega^{(1)}$, respectively. Define

$$g : U_{01} \to \mathcal{U}^{\mathrm{rel}} \hookrightarrow \mathrm{Aut}(\mathfrak{u}^{\mathrm{rel}})$$

by $g = 1 + F$. After gluing these two bundles together on the intersection U_{01} via the gauge transformation g, we obtain an algebraic vector bundle $\mathbf{u}_{\mathrm{dR}}$ over $\overline{\mathcal{M}}_{1,1/\mathbb{Q}}$. It is endowed with a connection $\tilde{\nabla} = \nabla_0 + \tilde{\Omega}$. This connection is defined over $\mathbb{Q}$, and has regular singularity at the cusp. □

Corollary 3.6.5. *The de Rham vector bundle* $(\mathbf{u}_{\mathrm{dR}}, \tilde{\nabla})$ *can be used to construct* $\mathbb{Q}$*-de Rham structures on* $\mathcal{O}(\mathcal{G}^{\mathrm{rel}})$*.*

Proof. We can define tranport formula by using connection forms $\Omega^{(0)}$ and $\Omega^{(1)}$ on opens U_0 and U_1 respectively, then patching things together on their intersection via the gauge transformation g. One can then follow [18, §7.6] to find a $\mathbb{Q}$-de Rham structure on $\mathcal{O}(\mathcal{G}^{\mathrm{rel}})$ for each base point $x \in \mathcal{M}_{1,1}(\mathbb{Q})$. □

Remark 3.6.6. In particular, one can choose the base point x to be the unit tangent vector $\partial/\partial q$ at the cusp. This newly constructed $\mathbb{Q}$-de Rham structure will allow us to compute multiple modular values [3].

3.7 Twice Iterated Integrals of Modular Forms

As mentioned in the introduction, we provide in this part newly constructed closed iterated integrals of modular forms.

3.7.1 Twice iterated integrals of algebraic forms

Suppose we are given two algebraic 1-forms ω and η on a Riemann surface M. To construct a closed iterated integral of these two algebraic forms, one follows the recipe in Hain [17, §3]. Define

$$\omega \cup \eta = \sum a_j p_j \qquad a_j \in \mathbb{C},\ p_j \in M$$

if, locally, $\omega = dF$ and

$$(2\pi i)^{-1} \operatorname{Res}_{p_j} F\eta = a_j.$$

One can find another 1-form ξ such that

$$(2\pi i)^{-1} \operatorname{Res}_{p_j} \xi = -a_j.$$

Then the twice iterated integral

$$\int \omega\eta + \xi$$

is a closed iterated integral on M.

3.7.2 Twice iterated integrals of modular forms

Set $\langle \mathsf{S}, \mathsf{T} \rangle = -\langle \mathsf{T}, \mathsf{S} \rangle = -1$. This defines a skew-symmetric inner product $\langle , \rangle$ on $S^{2n}\mathcal{H}$, which induces a cup product on $H^1_{\mathrm{dR}}(\mathcal{M}_{1,1}, S^{2n}\mathcal{H})$ (cf. [4]). Now we are ready to construct our first example.

Example 3.7.1. First newly constructed iterated integral of modular forms. Recall from Example 3.5.1 that we have a *global* 1-form

$$\omega_\Delta = \alpha \mathsf{T}^{10} = (2udv - 3vdu)\mathsf{T}^{10}$$

that represents a class corresponding to the weight 12 cusp form $\Delta = u^3 - 27v^2$ in $H^1_{\mathrm{dR}}(\mathcal{M}_{1,1}, S^{10}\mathcal{H})$. In Example 3.5.3 we found in $H^1_{\mathrm{dR}}(\mathcal{M}_{1,1}, S^{10}\mathcal{H})$ another class represented by a Čech 1-cocycle

$$\widetilde{\omega}_{1,1} = \left| \begin{array}{cc} (\omega_{1,1}^{(0)}, \omega_{1,1}^{(1)}) & 0 \\ 0 & l_{1,1} \end{array} \right.$$

where $\omega_{1,1}^{(0)} = \frac{9\alpha}{u^2\Delta}\mathsf{S}^{10} \in \mathcal{F}^1_{10}(U_0)$, $\omega_{1,1}^{(1)} = -\frac{u\alpha}{2v^2\Delta}\mathsf{S}^{10} - \frac{5\alpha}{4v\Delta}\mathsf{S}^9\mathsf{T} \in \mathcal{F}^1_{10}(U_1)$, and $l_{1,1} = \frac{1}{uv}\mathsf{S}^{10} \in \mathcal{F}^0_{10}(U_{01})$. Using the interpretation in Section 3.6.5, we regard $\widetilde{\omega}_{1,1}$ as having different representatives $\omega_{1,1}^{(0)}$ and $\omega_{1,1}^{(1)}$ on two opens U_0 and U_1 of $\mathcal{M}_{1,1}$. We shall construct twice iterated integrals of ω_Δ and $\widetilde{\omega}_{1,1}$ on these two opens separately.

We first carry out the recipe in Section 3.7.1 locally on $U_0 = \mathbb{G}_m \backslash\backslash Y_0$. Contracting $S^{10}\mathcal{H}$ by inner product and extracting 1-forms from ω_Δ and $\omega_{1,1}^{(0)}$, we set

$$\omega := 2udv - 3vdu, \quad \eta^{(0)} := \frac{9\alpha}{u^2\Delta},$$

which are algebraic forms on $Y_0 = \mathbb{A}^2 - \{u = 0\}$. Note that $\omega \cup \eta^{(0)}$ can only have residue at the $\mathbb{G}_m$-orbit $u = 0$, which corresponds to $[\rho] \in \mathcal{M}_{1,1}^{\mathrm{an}}$. Now we work locally around $[\rho]$, i.e. over $\mathbb{Q}[[u]][u^{-1}]$. Take a slice $v = C$ close to $u = 0$, then we can write ω locally as

$$\omega = -3vdu = dF \quad \text{with} \quad F = -3vu,$$

regarding v as a constant. Setting $u = 0$ in $\Delta = u^3 - 27v^2$, we have

$$F\eta^{(0)} = (-3vu)\frac{9\alpha}{u^2\Delta} = -\frac{27vu\alpha}{u^2\Delta} = -\frac{27vu(-3vdu)}{u^2(-27v^2)} = -3\frac{du}{u},$$

and

$$(2\pi i)^{-1}\,\mathrm{Res}_{[\rho]}\,F\eta^{(0)} = -3.$$

Therefore,

$$\int \omega\eta^{(0)} + 3\frac{du}{u}$$

is a closed iterated integral on U_0.

Similarly on $U_1 = \mathbb{G}_m \backslash\backslash Y_1$, contracting $S^{10}\mathcal{H}$ by inner product ignores the second term $-\frac{5\alpha}{4v\Delta}\mathsf{S}^9\mathsf{T}$ in $\omega_{1,1}^{(1)}$ since $\langle \mathsf{T}^{10}, \mathsf{S}^9\mathsf{T}\rangle = 0$. Extracting 1-forms from ω_Δ and the first term $-\frac{u\alpha}{2v^2\Delta}\mathsf{S}^{10}$ in $\omega_{1,1}^{(1)}$, we set

$$\omega := 2udv - 3vdu, \quad \eta^{(1)} := -\frac{u\alpha}{2v^2\Delta},$$

which are algebraic forms on $Y_1 = \mathbb{A}^2 - \{v = 0\}$. Note that $\omega \cup \eta^{(1)}$ can only have residue at the $\mathbb{G}_m$-orbit $v = 0$, which corresponds to $[i] \in \mathcal{M}_{1,1}^{\mathrm{an}}$. Now we work locally around $[i]$, i.e. over $\mathbb{Q}[[v]][v^{-1}]$. Take a slice $u = C$ close to $v = 0$, then we can write ω locally as

$$\omega = 2udv = dF' \quad \text{with} \quad F' = 2uv,$$

regarding u as a constant. Setting $v = 0$ in $\Delta = u^3 - 27v^2$, we have

$$F'\eta^{(1)} = 2uv \cdot \left(-\frac{u\alpha}{2v^2\Delta}\right) = -\frac{2uv \cdot u\alpha}{2v^2\Delta} = -\frac{2u^2v \cdot 2udv}{2v^2 \cdot u^3} = -2\frac{dv}{v},$$

and

$$(2\pi i)^{-1} \operatorname{Res}_{[i]} F'\eta^{(1)} = -2.$$

Therefore,

$$\int \omega\eta^{(1)} + 2\frac{dv}{v}$$

is a closed iterated integral on U_1.

Remark 3.7.2. Note that $\frac{du}{u} = \frac{1}{3}\frac{d\Delta}{\Delta} + \frac{27v\alpha}{u\Delta}$ and $\frac{dv}{v} = \frac{1}{2}\frac{d\Delta}{\Delta} + \frac{u^2\alpha}{2v\Delta}$. So if one were to transfer either the residue -3 at $[\rho]$ from $\omega \cup \eta^{(0)}$ or the residue -2 at $[i]$ from $\omega \cup \eta^{(1)}$ to the cusp P, one would get the same residue -1. This is expected since we are essentially computing a cup product of $[\omega_\Delta]$ and $[\widetilde{\omega}_{1,1}]$, with both $\eta^{(0)}$ and $\eta^{(1)}$ representing the same class $[\widetilde{\omega}_{1,1}]$ on opens U_0 and U_1 of $\mathcal{M}_{1,1}$.

It turns out that we have a nice description using our Čech complex.

Example 3.7.3. Čech description of the previous example. Recall from Example 3.5.2 that the class $[\omega_\Delta] \in H^1_{\mathrm{dR}}(\mathcal{M}_{1,1}, S^{10}\mathcal{H})$ can also be represented by a Čech 1-cocycle

$$\widetilde{\omega}_\Delta = (\omega_\Delta^{(0)}, \omega_\Delta^{(1)}; 0)$$

where $\omega_\Delta^{(i)}$ is the restriction of ω_Δ on U_i.

Extracting 1-cochains of $K^\bullet(\mathcal{M}_{1,1})$ the same way as in Section 3.6.3, we write

$$\widetilde{\omega}_\Delta = \widetilde{\Omega}_\Delta \otimes \mathsf{T}^{10} \quad \text{and} \quad \widetilde{\omega}_{1,1} = \widetilde{\Omega}_{1,1}^{10} \otimes \mathsf{S}^{10} + \widetilde{\Omega}_{1,1}^{9} \otimes \mathsf{S}^9\mathsf{T},$$

where $\widetilde{\Omega}_\Delta = \begin{array}{|cc} (\alpha,\alpha) & 0 \\ 0 & 0 \\ \hline \end{array}$, $\widetilde{\Omega}^{10}_{1,1} = \begin{array}{|cc} (\frac{9\alpha}{u^2\Delta}, -\frac{u\alpha}{2v^2\Delta}) & 0 \\ 0 & \frac{1}{uv} \\ \hline \end{array}$, and $\widetilde{\Omega}^{9}_{1,1} = \begin{array}{|cc} (0, -\frac{5\alpha}{4v\Delta}) & 0 \\ 0 & 0 \\ \hline \end{array}$.

After taking the inner product, we can ignore the second term $\widetilde{\Omega}^9_{1,1} \otimes \mathsf{S}^9\mathsf{T}$ in $\widetilde{\Omega}_{1,1}$ since $\langle \mathsf{T}^{10}, \mathsf{S}^9\mathsf{T}\rangle = 0$. It suffices to follow the procedure in Prop. 3.6.1 up to $n = 2$ when constructing twice iterated integrals. By the product formula defined in Section 3.6.2 we have

$$\widetilde{\Omega}_\Delta \wedge \widetilde{\Omega}^{10}_{1,1} = \begin{array}{|cc} (0,0) & 0 \\ 0 & -\frac{\alpha}{uv} \\ 0 & 0 \\ \hline \end{array}$$

where $-\frac{\alpha}{uv} = -\frac{2udv - 3vdu}{uv} = 3\frac{du}{u} - 2\frac{dv}{v}$. One can find another 1-cochain $\widetilde{\widetilde{\Xi}}$ of the form

$$\widetilde{\widetilde{\Xi}} := \begin{array}{|cc} (\xi^{(0)}, \xi^{(1)}) & 0 \\ 0 & l \\ \hline \end{array}$$

where $\xi^{(0)} = 3\frac{du}{u}$, $\xi^{(1)} = 2\frac{dv}{v}$ and $l = 0$, so that

$$D\widetilde{\widetilde{\Xi}} + \widetilde{\Omega}_\Delta \wedge \widetilde{\Omega}^{10}_{1,1} = 0.$$

These amount to the same information in the previous example to construct twice iterated integrals.

Remark 3.7.4. In general when constructing twice iterated integrals, after extracting 1-cochains of $K^\bullet(\mathcal{M}_{1,1})$ from coefficients, one first computes their product

$$\widetilde{\Omega}_\beta \wedge \widetilde{\Omega}_\gamma = \begin{array}{|cc} (\Phi^{(0)}, \Phi^{(1)}) & 0 \\ 0 & \xi \\ 0 & 0 \\ \hline \end{array}.$$

Then one wants to find a 1-cochain

$$\widetilde{\widetilde{\Xi}} := \begin{array}{|cc} (\xi^{(0)}, \xi^{(1)}) & 0 \\ 0 & l \\ \hline \end{array}$$

so that $D\widetilde{\widetilde{\Xi}} + \widetilde{\Omega}_\beta \wedge \widetilde{\Omega}_\gamma = 0$. In our case, $\Phi^{(0)}$ and $\Phi^{(1)}$ are always 0, since all of our 1-forms involve α,[9] and the wedge product of two such 1-forms is 0. For terms in ξ, one could put those of the form $\frac{du}{u}$ into $\xi^{(0)}$ and those of the form $\frac{dv}{v}$ into $\xi^{(1)}$. The rest of the terms in ξ would be exact and can be written as dl for some l. Then $\widetilde{\widetilde{\Xi}} = (\xi^{(0)}, \xi^{(1)}; l)$ would be sufficient to construct twice iterated integrals.

[9] cf. In [4] Brown and Hain work with a complex $\Omega^\bullet(X, \mathscr{V}_n)^\omega$ that each form in their complex involves ω, which is $\frac{3\alpha}{2\Delta}$ in our notation.

4

Conclusion

Our result in Chapter 2 completes the algebraic de Rham theory for unipotent fundamental groups of once punctured elliptic curves. The (universal) elliptic KZB connection plays a vital role, since it captures the variation of algebraic de Rham structures of these unipotent fundamental groups. It is a flat, algebraic and regular connection, which we have checked explicitly by writing down the formulas. These long formulas and other complications suggest that it would not be easy in general to find an explicit algebraic de Rham theory for unipotent fundamental groups of smooth varieties over $\mathbb{Q}$, even in the cases like curves of higher genus.

Our work in Chapter 3 is far from finished. We have constructed iterated integrals of algebraic modular forms that provide all multiple modular values. The ones that involve modular forms of the *second kind* are completely new. The next step is to compute and study the periods of these newly constructed iterated integrals. Our first task is to investigate periods of twice iterated integrals of modular forms of the *second kind*. Brown [3, §8, §22] has shown that periods of twice iterated integrals of Eisenstein series involve periods of cusp forms and multiple zeta values. We expect that periods of twice, and higher length, iterated integrals of modular forms of the *second kind* should be intimately related to special values of L-functions, and that they may provide geometric explanations for the depth filtration of multiple zeta values.

It is also natural to generalize our results to relative completions of modular groups of higher level. Just like in the case of level one, in order to develop an algebraic de Rham theory for the relative completion of a modular group of higher level, we need to first find an algebraic de Rham structure on the abelianization of the Lie algebra of its unipotent radical. We should be able to do this using a similar Mayer–Viertoris type patching argument [7], and we need to consider modular forms of the *second kind* and possibly of the *third kind* [27]. Then we should be able to apply Chen's method of power series connections, and to construct algebraic de Rham structures on the relative completions of modular groups.

A long term program is to construct iterated extensions of motives associated to modular forms. We need to incorporate both Hodge theory [18] and the theory of p-adic modular forms [7] into our study of relative completions of modular groups and their periods. It is our hope that eventually this will lead to a modular construction of mixed Tate motives over cyclotomic fields [3].

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Biography

Ma Luo was born on July 30, 1990, in Shanghai, China. He entered the School of Mathematical Sciences at Peking University in 2009. He was awarded an Innovation Award and a Youth Festival Scholarship from Peking University in 2011. He earned a B.S. in Mathematics and Applied Mathematics from Peking University in 2012. He subsequently joined the graduate program in mathematics at Duke University. He was awarded Graduate School Summer Research Fellowship in 2014 and in 2015. Under the supervision of Richard Hain, he is expected to obtain a Ph.D. in Mathematics from Duke University in 2018. He has written two papers [23, 24], corresponding to the two parts in this thesis. He will continue as a Postdoctoral Research Assistant to work with Francis Brown at the Mathematical Institute, University of Oxford in 2018.