Essays on Information Economics

by

Yasha Miami

Department of Economics
Duke University

Date: ________________________________

Approved:

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Attila Ambrus, Supervisor

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Curtis Taylor

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Huseyin Yildirim

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David McAdams

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Dissertation submitted in partial fulfillment of the
requirements for the degree of Doctor of Philosophy
in the Department of Economics
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ABSTRACT

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Abstract

This thesis contains essays on the economics of information. In particular it focuses on specific environments where groups of individuals are faced with both uncertainty and having their main source of information on the underlying state of the world is controlled by outside parties with their own agenda. The goal of this thesis is to characterize the equilibrium behavior and examine the welfare implications of having third parties controlling the group's information structure. The first chapter studies the coordination and free-riding problem commonly found in the private provision of a public good and how a fundraiser can help alleviate the issues by designing information structures that determine the donors' behavior. And the second chapter studies a duopoly model where the firms invest in advertising to divert the consumers to adopting their product. The group's welfare can be improved even if their interests are not closely aligned with the outside parties' interests as long the information gains are high enough.

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Chapter 1

Introduction

The second chapter models a fundraising event where an organizer is tasked with soliciting as many charitable contributions possible from a group of donors. The donors are participating with the goal of successfully providing a public good through their donations. However, the organizer must overcome the usual complications prevailing in environments of private provisions of public goods, mainly with problems arising from under-provision and coordination efforts. To address these complications, the organizer utilizes the fundraising event as a platform to disseminate information among the donors that help them to coordinate and boost their perceived contributional value of donating. The chapter characterizes the emerging equilibrium behavior. The organizer designs information structures that both inform donors of potential donations from others and persuade them into believing they are crucial to the provision of the good. Persuading each donor into believing they are pivotal is key to coordinating and increasing the total number of donations.

The third chapter considers two competing firms selling different products who race to dominate the market. In this environment, the standard herding archetype is adopted where buyers arrive sequentially and must decide on which one of the two goods to purchase with only observing a private signal and the history of past purchases. The firms invest in advertising resources that determine the private signal distribution of the buyers. The paper characterizes the optimal firms' advertising levels and its impact on consumer welfare. It's shown that as the intensity of competition increases consumers become worse off as their signal distributions become less informative, potentially leaving consumers with completely uninformative signals. Thus implicating that advertising is harmful and wasteful, at least from the perspective of consumers' welfare.

Chapter 2

Fundraising and Persuasion

2.1 Introduction

This chapter studies the benefit a fundraiser has on the private provision of a public good from organizing an event that brings donors together. In particular, this paper looks to address the question by adopting tools from information design theory. This provides the opportunity to examine both the benefit in coordination among the donors as well as the gain from donors becoming more informed about the quality of the charity.

The organizer is faced with the task of soliciting as many donations possible regardless of the quality of the charity. However, although the donors do care about successfully raising the minimum funds necessary for the public good they also have private benefits that result in free-riding in addition to caring about the quality. The organizer needs to both coordinate the donors actions to prevent the zero contribution equilibrium while minimizing any free-riding effects.

The main result here is that donations are always higher and the likelihood of the public good being provided also increases in the presence of a fundraiser. This is due to the fact that, in addition to the coordination gains, the fundraiser is able to privately persuade donors. The ability to privately persuade donors is crucial in soliciting donations past the minimum. This is because donors are more willing to contribute as they believe their marginal benefit from contributions are higher and as a result believe they are more pivotal resulting in also mitigating any free-riding effects.

Most the focus is on the symmetric case where donors all have the same payoffs and costs to donating. This results in the fundraiser being able to solicit donations from the entire group. However, with heterogeneity introduced into the model the fundraiser has to now strategically target certain subgroups. The fundraiser ends up soliciting the j easiest to persuade donors and completely ignoring the remaining donors. However, the amount of donations solicited may be non-monotonic in how easy they are persuaded. This is because being easier to persuade implies that the donor does not need to feel as pivotal as compared to a donor harder to persuade; and the way to make a donor feel more pivotal is to increase the likelihood of them being in pivotal groups.

In relation to the charitable giving literature[1], this paper follows Andreoni (1998) in modeling charity events with donors making simultaneous contributions to a privately funded public good. Similar works include Name-Correa and Yildirim (2013), Bag and Roy (2007), and Vesterlund (2003). Name-Correa and Yildirim (2013) offers a similar insight on targeting donors; however, in this paper the organizer targets donors based on how easily they are to persuade. And in Vesterlund (2003) and Bag and Roy (2007) announcing previous donations help donors learn more about the charity's quality and improve free-riding. But unlike these two there is no private information and this paper is not restricted to only two donors.

The persuasion model here borrows from the bayesian persuasion/information design literature.[2] The most closely related papers are the multiple receiver extensions of the novel work by Kamenica and Gentzkow (2011): Chan et al. (2019), Arieli and Babichenko (2019), Wang (2013), and Bardhi and Guo (2018). However, none of these papers incorporate free-riding. Arieli and Babichenko (2019) has the receivers

[1]Andreoni and Payne (2013) provides a great overview of this literature

[2]E.g. see Bergemann and Morris (2019) for an overview.

only ever all strictly preferring the same single action over the alternative. And in Chan et al. (2019) the sender does not receive any additional gains from having more than the minimum k receivers act in the sender's favor and as a result only needs to focus on persuading pivotal groups of receivers as the likelihood of soliciting a vote is the highest when an individual believes he is pivotal. Due to the lack of free-riding, these papers offer conclusions different from this model's results. In particular all three of these conclude that, at least when receivers are symmetric, public persuasion is the optimal information structure for the information designer. But with free-riding present, there is still value in private persuasion—the donors are not sure exactly how many others were recommended to donate and as a result believe that their marginal contributional benefit is higher than it might otherwise be. Other notably related papers from are Doval and Ely (2019) and Galperti and Perego (2019). Doval and Ely (2019) relates to the two-stage mechanism presented in this paper as their research studies the dynamic information structures in extensive form. However, their dynamics differ since in this model donors of the second stage observe and can learn from the initial donations, but in Doval and Ely (2019) the actions in the initial stage is only observed by the designer. This results in the designer in the present model needing to create contingent-based information structures depending on the history. And Galperti and Perego (2019) offer dual perspective in approaching and solving for information design games similar to the perspective offered in the general cost case here. They interpret a designer that can buy probabilities of a signal realization given a budget to be used efficiently similarly to how the organizer here can purchase higher probability of donations by offering the donor to be more pivotal; however, the organizer has a finite budget since only so many donors can believe they are pivotal at a given time.

2.2 Model

There is a fundraising coordinator (she) that organizes an event to solicit contributions from N identical donors (he) indexed by $i \in \mathcal{I}$. The total funds collected goes towards supplying a public good. The organizer will need to raise a minimum provision level of $0 < k \leq N$ from the individuals in order to reap the benefits of the public good. Here k is fixed and known by everyone. Each individual simultaneously takes action $a_i \in A_i := \{0, 1\}$ where $a_i = 1$ denotes i contributing and $a_i = 0$ for not. Denote $a \in A := \bigtimes_{i=1}^{n} A_i$ to be the action profile. The donors' cost of contributing is $0 < c < 1$. There are two states of the world, $\theta \in \Theta := \{L, H\}$, with state-dependent payoffs for the individuals $u_i : A \times \Theta \to \mathbb{R}$ given by:

$$u_i(a_i, a_{-i}; \theta) = \begin{cases} \mathbb{1}_{\theta=H} g(a) - a_i \cdot c & \text{if } \sum_m a_m \geq k \\ -a_i \cdot c & \text{if } \sum_m a_m < k \end{cases}$$

where $g \geq 0, g' \geq 0, g'' \leq 0$, and $g = 0$ when $\sum_m a_m < k$. Here the donations are assumed to be sunk.

The organizer, independent of the state, only cares about maximizing total donations:

$$v(a) = \sum_i a_i$$

The true state of the world is unknown to everyone, but there is a common prior $\mu_0 := \mathbb{P}(\theta = H)$. The organizer can design a charity fundraising event in order to coordinate donors' actions and provide information about the underlying state. She can provide information by designing an experiment resulting in a private recommendation for each individual. Formally, the organizer has a finite set of signals to generate for individual i denoted by S_i where $S := \bigtimes_{i=1}^{n} S_i$. She chooses a state-conditional probability distribution over all possible signals—that is, chooses a distribution $\pi \in \Pi$ where $\pi : \Theta \to \Delta(S)$. Denote $\pi_\theta(s) := \pi(s \mid \theta)$.

Upon receiving signal s_i individual i updates his belief on θ according to Bayes' Law to posterior $\mu_\pi := \mathbb{P}(\theta = H|s_i) = \pi_H(s_i)/(\pi_H(s_i) + \pi_L(s_i))$ and maximizes his conditional expected utility $U_i(a, s_i; \theta) := \mathbb{E}_{\theta\sim\mu_\pi}[u_i(a;\theta)|s_i]$. Define a_i^* as

$$a_i^*(\pi) := \arg\max_{a_i \in A_i} U_i(a_i, a_{-i}, s_i; \theta)$$

where $a_i^*(\pi)$ is i's maximizing action given a signaling policy π and its realization s_i. And the organizer maximizes the ex-ante donations

$$V(a^*) := \max_{\pi\in\Pi} \mathbb{E}_{\theta\sim\mu_0}\left[\mathbb{E}_{\theta\sim\mu_\pi} v(a^*(\pi))\right].$$

For each signal $s_i \in S_i$, individual i takes action a_i if and only if for each $a_i' \in A_i$

$$\mu_\pi u_i(a_i, s_i; H) + (1-\mu_\pi)u_i(a_i, s_i; L) \geq \mu_\pi u_i(a_i', s_i, H) + (1-\mu_\pi)u_i(a_i', s_i; L) \quad (2.1)$$

for a given $\pi \in \Pi$ and $k \leq N$. This is known as i's obedience constraints.

To simplify the problem, we can without loss of generality restrict the signaling space to be the action space, $S = A$ (see e.g. Bergemann and Morris (2016)). The signaling policy can now be interpreted as a distribution over recommendations with $\pi_\theta(s)$ as the probability of recommending to donate given state θ. The organizer's problem is to solve the following optimization.

$$\begin{aligned} \max_{\pi\in\Pi} \quad & \sum_{i\in\mathcal{I}} \sum_{\substack{s\in S \\ s_i=1}} \pi(s) \\ \text{s.t.} \quad & (2.1) \end{aligned} \quad (2.2)$$

The timing of events are as follows. As usual in the literature, the organizer is assumed to be able to fully commit to a signaling policy at the beginning of the game which is observed by the donors. She first designs a policy mapping the states to the distribution of recommendations over joint actions. State θ is then realized (but unknown). Then each individual receives a recommendation as a realization of the signaling policy where the recommendations are all private. Finally, the donors participate in a simultaneous move game and in equilibrium follows their recommendations. Payoffs are then realized. The solution concept is a Bayesian Nash equilibrium of the incomplete information game described above. There are many equilibria and it's assumed that the organizer is allowed to choose among the set of equilibria.

2.2.1 Characterization

As a benchmark, suppose that

$$g(a) = \begin{cases} \sum_i a_i & \text{if } \sum_i a_i \geq k \\ 0 & \text{o.w.} \end{cases}$$

with common prior $\mu_0 = 1/2$. In the basic game (simultaneous move game without the organizer) there is always a zero contribution Nash equilibrium for all values of c. And there is also an equilibrium where everyone contributes provided $c < N/2$. However, introducing the organizer can help eliminate the zero equilibrium outcome.

Define π_θ^i to be the marginal probability the policy recommends donor i to contribute in state θ. The following proposition (due to Arieli and Babichenko (2019)) describes the optimal policy.

Proposition 1. *(Arieli & Babichenko 2019)*

Fix $k \leq N$. $\pi_H^{i*} = 1$ *for each* $i \in \mathcal{I}$ *and* $0 < c < 1$. *And* $\pi_L^{i*} = \min\{1, (1-c)/c\}$ *if* $k < N$ *and* $\pi_L^{i*} = \min\{1, (N-c)/c\}$ *for* $k = N$.

The optimal policy simply has everyone contributing with probability 1 conditional on $\theta = H$ since both the organizer and the individuals' incentives align perfectly. However for $\theta = L$, the incentives no longer align and the policy obfuscates information in order to continue to solicit donations. The policy is still able to fully extract contributions in the low state when c is small enough since the donors benefit from following the recommendations due to policy also acting as a coordination device.

However, the model specified in Arieli and Babichenko (2019) does not capture any free-riding problems that typically arise in private provision of public goods. And this particular $g(\cdot)$ specification fails to capture free-riding as well since the marginal benefit of contributing always exceeds the cost (conditional on $\theta = H$). The remainder of this chapter assumes that payoffs are simply

$$u_i(a_i, a_{-i}; \theta) = \begin{cases} \mathbb{1}_{\theta=H} - a_i \cdot c & \text{if } \sum_m a_m \geq k \\ -a_i \cdot c & \text{if } \sum_m a_m < k \end{cases}$$

where there is a discrete jump in payoffs to $g = 1$ when the good is provided but does not increase in further donations past k. This is not too restrictive since any g such that $g' > 0, g'' < 0$ (e.g. $g = \sqrt{\sum_i a_i}$) will produce similar optimal policies without adding much additional insight except to introduce cumbersome cases. The important features are that donors do want to contribute when they're pivotal in the provision of the good and that once the minimal funds are acquired that free-riding does eventually occur.

In order to simplify the organizer's problem, define S^j, P, and P_i as

$$S^j := \{s \in S \mid \sum_m a_m = j\} \tag{2.3}$$

$$P := \{s \in S \mid \sum_m a_m \geq k\} \tag{2.4}$$

$$P_i := \{s \in S \mid s_i = 1 \text{ and } \sum_m a_m = k\} \tag{2.5}$$

where $S^j \subseteq S$ is the set of signals that generates $j = 0, 1, \ldots, N$ donations. That is, the set such that the policy recommends exactly j donors to contribute obediently resulting in j contributions. Similarly, $P \subseteq S$ is the set of signals that lead to the good being provided (given everyone is obedient and follows their recommendations) and $P_i \subseteq P$ is the set of signals such that i's contribution is pivotal. Then the obedience constraints from (2.1) can be rewritten as

$$\frac{\sum_{s \in P, s_i=1} \pi_H(s)}{\sum_{s \in S, s_i=1} \sum_{\theta \in \Theta} \pi_\theta(s)} - c \geq \frac{\sum_{s \in P \setminus P_i, s_i=1} \pi_H(s)}{\sum_{s \in S, s_i=1} \sum_{\theta \in \Theta} \pi_\theta(s)} \tag{2.6}$$

$$\frac{\sum_{s \in P, s_i=0} \pi_H(s)}{\sum_{s \in S, s_i=0} \sum_{\theta \in \Theta} \pi_\theta(s)} \geq \frac{\sum_{s \in P \cup P_i, s_i=0} \pi_H(s)}{\sum_{s \in S, s_i=0} \sum_{\theta \in \Theta} \pi_\theta(s)} - c. \tag{2.7}$$

where (2.6) and (2.7) are the conditions such that individual i follows the recommendation to contribute and to not contribute respectively. That is the left-hand side (2.6) is the payoff for following the recommendation to contribute (given everyone else also follows their recommendations) and the right-hand side is the payoff for deviating (again where everyone else remains to follow). Lemma 1 helps simplify the organizer's problem. All proofs are found in Appendix A.

Lemma 1. *Let π^* be the optimal signaling policy for given $k \leq N$. Then for all* $0 < c < 1$

i. (2.7) *is non-binding.*

ii. $\pi^*_H(S^N) + \pi^*_H(S^k) = 1$.

The first part of Lemma 1 states that given an optimal policy π^* the obedient constraint for recommending to not contribute (2.7) is always slack. And the second part gives the familiar result in the literature that in the high state, the organizer will never find it optimal to recommend any individual to not donate. This differs from Proposition 1 where $\pi^*_H(S^N) = 1$ due to the lack of diminishing returns on contributions. As much as the organizer would like to have $\pi^*_H(S^N) = 1$, each individual will deviate from their recommendation since it's known that this policy recommends all the other $N-1$ donors to donate thus he would rather free-ride. Moreover, she would never design a policy that will put any weight on signals that generate j investments for $j = 1, \ldots, k-1, k+1, \ldots, N-1$. The reason is that if there was any positive measure for $j < k$ she can always shift all of $\pi^*_H(S^j)$ to $\pi^*_H(S^k)$ since there is no conflict of interest here. This increases contributions while respecting the obedience constraint since no donor would ever want to contribute if the good is not going to be provided. Similarly, for $k < j < N$ she can shift all of $\pi^*_H(S^j)$ to $\pi^*_H(S^N)$ since the donors are indifferent between S^j and S^N—they want to free-ride given that they're not pivotal. Given Lemma 1 the obedience constraints can now be simplified to

$$\frac{1-c}{c} \sum_{s \in P_i} \pi_H(s) \geq \sum_{\substack{s \in S \setminus P_i \\ s_i = 1}} \pi_H(s) + \sum_{\substack{s \in S \\ s_i = 1}} \pi_L(s) \tag{2.8}$$

where the organizer's problem is now

$$\begin{aligned} \max_{\pi \in \Pi} \quad & \sum_{i \in \mathcal{I}} \sum_{\substack{s \in S \\ s_i = 1}} \sum_{\theta \in \Theta} \pi_\theta(s) \\ \text{s.t.} \quad & (2.8) \text{ and } \pi_H(S^N) + \pi_H(S^k) = 1. \end{aligned} \tag{2.9}$$

The left-hand side of (2.8) indicates that individual i must believe he is somewhat critical in generating enough funds in order to follow any recommendations to donate. Thus in order to solicit contributions the organizer must choose policies such that each individual believes he is pivotal. The constraint slackens when he believes he's more pivotal, and tightens when he thinks otherwise.

Due to the symmetry of the problem, the remainder of this section will focus on the symmetric equilibrium where, conditional on θ, the probability of recommending $j \leq N$ individuals to invest is the same across across all $\binom{N}{j}$ combinations. The organizer cannot do better with any other equilibria. The following proposition characterizes the Bayesian Nash equilibrium of the game.

Proposition 2. *For* $0 < k \leq N$ *the optimal signaling policy involves*

$$\begin{aligned}
\pi_H^*(S^N) &= \begin{cases} \frac{k-(N+k)c}{k+(N-k)c} & \text{for } 0 < c \leq c^* \\ 0 & \text{for } c^* < c \end{cases} \\
\pi_H^*(S^k) &= 1 - \pi_H^*(S^N) \\
\pi_L^{i*} &= \begin{cases} 1 & \text{for } 0 < c \leq c^* \\ \frac{1-c}{c}\frac{k}{N} & \text{for } c^* < c \end{cases} \\
c^* &= \frac{k}{N+k}
\end{aligned} \tag{2.10}$$

Proposition 2 states that when c is low enough ($c < c^*$), the organizer finds it easier to coordinate everyone's actions to donate. But as c increases the individuals are more tempted to free-ride therefore the policy must ensure that they continue to believe they are pivotal; it's impossible to solicit a donation from a donor when he is never pivotal under the policy. This is maintained by moving weight away from $\pi_H^*(S^N)$ to $\pi_H^*(S^k)$ when c increases. However, given $\theta = H$, once the threshold is reached ($c = c^*$) the policy can no longer persuade groups larger than k individuals. At this point everyone recommended to invest is pivotal in the high state and remains this way since preferences align—individuals never want to invest when part of a failing coalition of less than k donors and the organizer would never want less than k contributions. Similar to Proposition 1, as c continues to increase the policy becomes more informative to maintain credibility. Up until now $\pi_L^*(S^N) = 1$ implying that the policy has been obfuscating most the information regarding the underlying state—so far an individual only ever receives a recommendation to not donate in the high state when he is non-pivotal. Since no one is ever pivotal in the low state organizer designs any policy such that the marginal probability of donating for donor i is $\pi_L^{i*} = (1-c)k/(cN)$ to maintain the obedience constraints which of course decreases with c.

An important feature of this policy is that even with having symmetric donors a public signaling policy is sub-optimal for the organizer (when $k < N$ since everyone is pivotal when $k = N$). It's possible to solicit more donations by leading on donors into believing they are pivotal as shown in Example 1. All examples are worked out in Appendix B.

Example 1. Consider the public signaling policy $\phi \in \Pi$ where a signal realization is the joint recommendation that is observed and shared by all players. Note that $\sum_\theta \sum_{j>k} \phi_\theta(S^j) = 0$ for $0 < c < 1$ since everyone would rather free-ride knowing exactly how many others are asked to contribute in this public signal:

$$\phi_H^*(S^k) = 1$$
$$\phi_L^{i*} = \min\left\{\frac{k}{N}, \frac{k}{N}\frac{1-c}{c}\right\}$$

It's clear that π^* is better for $c \leq 1/2$ and coincides with ϕ^* for higher c since then the marginal probabilities are equal. The difference in ex-ante total contributions is

$$\begin{cases} \frac{N-k}{2}\left(\frac{k-(N+k)c}{k+(N-k)c}+1\right) & \text{for } 0 < c \leq c^* \\ \frac{1}{2}(N - \frac{1-c}{c}k) & \text{for } c^* < c \leq 1/2 \\ 0 & \text{for } 1/2 < c. \end{cases}$$

The larger the gap between N and k the larger the benefit of private persuasion is since the $N-k$ donations are unattainable in the public policy. And for this very same reason the receivers prefer the public signaling policy since they are guaranteed to be pivotal when recommended to donate.

Proposition 2 has multiple equilibria since all that matters is that the policy respects each donor's marginal probability in the low state. Example 1 constructs two different equilibrium policies.

Example 2. For $c > c^* = k/(N+k)$,

i.

$$\pi_L^*(S^N) = \frac{1-c}{c}\frac{k}{N}$$
$$\pi_L^*(S^0) = 1 - \pi_L^*(S^N).$$

ii. for $j = 1, \ldots, N$

$$\pi_L^*(S^{N-j+1}) = \frac{k - c(N+k-j)}{c} \quad \text{for } c_j^* < c \leq c_{j+1}^*$$
$$\pi_L^*(S^{N-j}) = 1 - \pi_L^*(S^{N-j+1})$$
$$\text{where } c_j^* = \frac{k}{N+k-j+1} \text{ and } c_{j+1}^* = \frac{k}{N+k-j}.$$

The first one simply shifts weight between having everyone investing to no one investing. And the second policy has a daisy chain-like pattern where it only ever has weights on persuading two coalitions at a time shifting weight from S^{N-j+1} donors to S^{N-j} and continues until it reaches to S^0. Both the organizer and donors are indifferent between the policies since they are identical when $c \leq c^*$ and for $c > c^*$ each individual always receives a recommendation with the same marginal probability, π_L^{i*}.

2.2.2 Efficiency

So far the organizer has been soliciting donations by designing a signaling policy that respects the donors' obedience constraints. The set of outcomes that the organizer can induce is known as the Bayes correlated equilibrium (BCE)—an incomplete information extension of Aumann (1987)'s complete information correlated equilibrium. One of the biggest hurdles of private provision of public goods is coordinating the actions of the participants. Cavaliere (2001) show that efficiency is always attainable under coordinated equilibria. Efficiency characterized by the public good being provided when (i) there's a surplus benefit of creating it (ii) as cost-efficient as possible. Here efficiency will always be referred to with respect to the donors' total ex-ante surplus and not pertaining to the organizer's surplus.

Proposition 3. *(Cavaliere 2001)*
There exists a Bayes correlated equilibrium outcome that is efficient for all $0 < c < 1$.

Cavaliere (2001) focuses on the complete information case; however, it can easily be extended to the current model. Figure 2.1 shows the BCE distribution over actions corresponding to the efficient outcome for $N = 2$ and $k = 1$.

The efficient BCE maximizes the group's surplus where it coordinates exactly k

	D	N
D	0	0
N	0	1

$\theta = \theta_L$

	D	N
D	0	1/2
N	1/2	0

$\theta = \theta_H$

Figure 2.1: Efficient symmetric BCE for $N = 2$ and $k = 1$ where D is the player donating.

people to donate in the high state, e.g. the symmetric BCE has equal probability over the $\binom{N}{k}$ combinations in the high state resulting in each individual contributing with probability k/N. Note that the organizer's policy is never efficient because although she successfully coordinates the donors, she induces (i) too many donations and (ii) unnecessary donations in the low state. But from Proposition 2 the optimal signaling policy is asymptotically efficient.

Corollary 1. *Let* $0 < k \leq N$. *Then*

i. $\lim_{c \to 1} \pi^*_L(S^0) = 1$ *and* $\pi^*_H(S^k) = 1$.

ii. $\lim_{c \to 0} \pi^*_H(S^N) = 1$ *and* $\pi^*_L(S^N) = 1$.

The first part is simply due to the fact that the policy must be increasing in informativeness with c. And that the obedience constraints are tightening with c ensuring the individuals become more pivotal. And of course the second part is efficient in the limit as costs vanishes. This implies that there is a non-monotonic relationship in the difference between the organizer's policy and the efficient outcome.

For a policy $\pi \in \Pi$ define its efficiency as the difference in donors' surplus from the efficient outcome. Total surplus for the efficient symmetric BCE is $\sum_{i \in \mathcal{I}} \frac{1}{2}(1 - \frac{k}{N}c) = (N - kc)/2$. And total surplus for policy π is

$$\sum_{i \in \mathcal{I}} U_i = \sum_{i \in \mathcal{I}} \frac{1}{2} - \pi^i c$$

where π^i is the probability i contributes under the policy where it's understood that the good is always provided in the high state since it's true for all equilibrium policies. Donor i's payoff under equilibrium π^* is

$$U_i = \begin{cases} \frac{1}{2}\frac{k+(N-3k)c}{k+(N-k)c}, & c \leq c^* \\ \frac{1}{2}\left(1-\frac{k}{N}\right), & c^* < c \end{cases} \tag{2.11}$$

and the welfare difference between π^* and the efficient BCE is

$$N\left(\frac{1}{2}-\frac{k}{2N}c-U_i\right) = \begin{cases} \frac{kNc}{2}\left(\frac{2}{k+(N-k)c}-\frac{1}{N}\right) & c \leq c^* \\ \frac{k}{2}(1-c) & c^* < c \end{cases} \tag{2.12}$$

where it's increasing for $c \leq c^*$ and decreasing in $c > c^*$. Therefore the most inefficient case is where $c = c^*$. For $c \leq c^*$ the difference is increasing because although investing becomes more efficient in the high state it is outweighed by the loss in utility from the increase in costs as can be seen from $|\partial U_i/\partial c| > k/N$. And for $c > c$ the difference is decreasing since investing becomes more efficient in the low state while $|\partial U_i/\partial c| = 0$. Utility is constant here because although costs are increasing they're offset by the gain in informativeness as can be seen by the posterior belief always equaling c.

2.2.3 Two-stage mechanism

A common component of fundraising events is the announcement strategy where past contributions are announced to all donors still participating in the fundraiser. This section focuses on two-stage mechanisms where the organizer has the option of splitting the donors into two groups and soliciting donations from each group sequentially. This is then compared to the single-stage model above.

The timing of this game is similar to the one-stage model. First the organizer commits to a signaling policy in advance and then the state is realized. In the first stage, a subset of $m < k$ donors (leaders) receive their private recommendations then proceed to engage in the simultaneous move contribution game among each other (these m donors do not continue to contribute in the following stage). Then in the second stage the remaining $N - m$ donors (followers) observe the total donations and then receive their own private recommendations and engage in the final simultaneous move contribution game. Payoffs are then realized.

The two most notable differences now are that (i) the second stage donors learn more information about the underlying state and (ii) the first stage donors now have incentives to free-ride and back-load any responsibility to the later donors. The organizer's policy will have to take into account these nuances while remaining to solicit as many donations are possible.

Denote the two periods as $t = 0, 1$ and h_t as the sum of donations made prior to period t. And define π_t as the policy designed for period t and μ_t as the belief $\theta = H$ in period t after the donors observe their signals and any past history, h_t. Here π_1 is contingent on history h_1, that is the organizer designs $\pi_1(s; h_1) \in \Pi$ for each h_1: $\pi_1(h_1) : \Theta \to \Delta(S)$ for each h_1. Continue to assume that the common prior remains at $\mu_0 = 1/2$. Solving for the optimal policy is done in two steps. First take an $m > 0$ as given and solve for the policy, $\pi^*(m)$, using backward induction. Then find the maximizing m among the $\pi^*(m)$'s.

The obedience constraint, given π_0, for the followers $i > m$ after observing h_1 is

$$\begin{aligned} \mu_1 \frac{1-c}{c} \sum_{s \in P_i} \pi_{H,1}(s) &\geq (1 - \mu_1) \sum_{\substack{s \in S \setminus P_i \\ s_i = 1}} \pi_{H,1}(s) + \sum_{\substack{s \in S \\ s_i = 1}} \pi_{L,1}(s) \\ \mu_1 = \mathbb{P}(\theta = H \mid h_1) &= \frac{\pi_{H,0}(h_1)}{\pi_{H,0}(h_1) + \pi_{L,0}(h_1)} \end{aligned} \tag{2.13}$$

where $\pi_{\theta,0}(h_1)$ is the probability, conditional on θ, of observing h_0 given that the $i \leq m$ leaders followed their recommendations. This constraint is the same as before but now incorporates the beliefs and Proposition 2 can easily be extended to include beliefs.

Corollary 2. *For $0 < k \leq N$ and belief $0 < \mu < 1$ the optimal signaling policy in the single-stage game involves*

$$\begin{aligned}
\pi_H^*(S^N) &= \begin{cases} \frac{k-\left(\frac{1-\mu}{\mu}N+k\right)c}{k+(N-k)c} & \textit{for } 0 < c \leq c^* \\ 0 & \textit{for } c^* < c \end{cases} \\
\pi_H^*(S^k) &= 1 - \pi_H^*(S^N) = \frac{\frac{1}{\mu}Nc}{k+(N-k)c} \\
\pi_L^*(S^N) &= 1 \quad \textit{for } 0 < c \leq c^* \\
\pi_L^{i*} &= \frac{\mu}{1-\mu}\frac{1-c}{c}\frac{k}{N} \quad \textit{for } c^* < c \\
c^* &= \frac{k}{\frac{1-\mu}{\mu}N+k}.
\end{aligned} \tag{2.14}$$

Corollary 2 verifies that the equilibrium behavior remains the same as before. The only difference is that the organizer is able to solicit more donations the more optimistic donors are. The threshold simply shifts upward with the belief. This is also the Nash equilibrium policy for the followers thus replacing (N, k, μ) with $(N-m, k-h_1, \mu_1)$ gives the optimal policy for the followers, $\pi_1^*(h_1)$.

Given $\pi_1^*(h_1)$ the obedience constraint for the leaders $i \leq m$ is

$$\sum_{\substack{s\in S \\ s_i=1}} \pi_{H,0}(s) \sum_{s\in P} [\pi_{H,1}(s;h_1) - \pi_{H,1}(s;h_1')] \geq c \sum_{\substack{s\in S \\ s_i=1}} \sum_{\theta\in\Theta} \pi_{\theta,0}(s) \tag{2.15}$$

where h_1 is the obedient history and h_1' is the dis-obedient history where donor i deviated from his recommendation to contribute and all other leading donors followed their recommendation. The organizer is facing a trade-off when implementing the two-stage mechanism over the single-stage. For one she is faced with a tighter obedience constraint from the leaders for the additional free-riding. Also the followers' constraint has two effects: (i) becomes tighter since they believe they are less pivotal than compared to the single-stage and (ii) slackens from becoming more optimistic from observing the leaders following through on the recommendation to contribute. Thus in order to benefit from having leaders the optimism of the followers must be high enough to outweigh the tightening effects. To see the dynamics of the two-stage mechanism consider the case of a single leader in the following example.

Example 3. Let $N = 3$, $k = 2$, $m = 1$, and $\mu_0 = 1/2$. The last period, the followers observe whether the leader, after receiving his recommendation, made a donation or not. Denote the probability the leader was recommended to donate in θ as $\pi^L_{\theta,0}$ and π_1 as the contingent policy for the followers. Denote $\pi_{\theta,1}(S^j; h_1)$ for $j \in \{1, 2\}$ and as the probability of recommending j followers to contribute given h_1. Then the followers update their priors to μ_1 as defined in (2.13). The leader's obedience constraint is

$$\frac{\pi^L_{H,0}}{\pi^L_{H,0} + \pi^L_{L,0}} - c \geq \frac{\pi^L_{H,0}\pi_{H,1}(S^2; 0)}{\pi^L_{H,0} + \pi^L_{L,0}}$$
$$\iff p_H(1 - c - \pi_{H,1}(S^2; 0)) \geq cp_L$$

where the left-hand side still remains to be how pivotal he is. It's tighter than before due to the additional free-riding incentives of back-loading. It follows that the organizer can set $p_H = 1$ if $\pi_{H,1}(S^2; 0) < 1 - c$ since otherwise the leader will never donate. However, the organizer can create a policy such that the leader always contributes in the high state. And since followers' beliefs are formed based on the leader being obedient this implies that $\mu_1 = 0$ when $h_1 = 0$ hence the followers never contribute when $h_0 = 0$ (in particular $\pi_{H,1}(S^2, 0) = 0$ and the leader can no longer back-load any responsibilities).

With a threshold of $c^* = 1/3$ the optimal sequential signaling policy is

$$\pi^{L*}_{H,0} = 1, \quad \pi^{L*}_{L,0} = \min\left\{1, \frac{1-c}{c}\right\}$$

$$\mu^*_1 = \max\{c, 1/2\}$$

$$\pi^*_H(S^1) = \begin{cases} \frac{4c}{1+c} & \text{for } 0 < c \leq 1/3 \\ 1 & \text{for } 1/3 < c \end{cases}$$

$$\pi^*_H(S^2) = 1 - \pi^*_H(S^1)$$

$$\pi^*_L(S^2) = 1 \quad \text{for } 0 < c \leq 1/3$$

$$\pi^{i*}_L = \frac{1}{2} \quad \text{for } 1/3 < c.$$

Figure 2.2 represents this game in extensive form displaying the equilibrium path.

In this example the two-stage out-performs the single-stage when $c > 1/2$. It's conjectured that more generally the organizer finds it optimal to always recommend the m leaders to contribute in the high state since it results in back-loading to only appear on the off-the-equilibrium paths.

2.3 Heterogeneous Donors

Extending the first section we now suppose that there are N players ordered by increasing costs: $0 < c_1 < \ldots < c_N < 1$. And define $l_i := c_i/(1 - c_i)$ so that

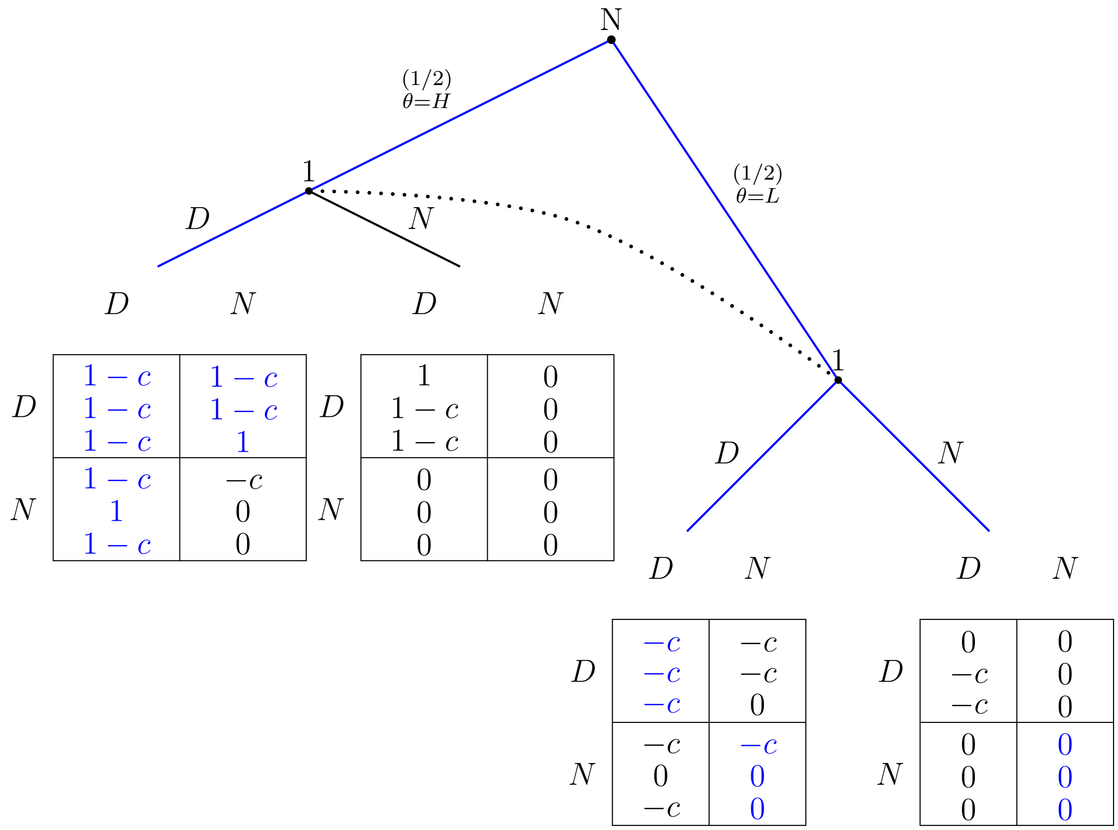


Figure 2.2: Extensive form game tree of the two-stage mechanism. The blue path and text highlights the equilibrium outcomes induced by the optimal policy. This uses the policy where either both or neither of the followers contribute in the low state (see Example 2(i)).

individual i's obedience constraint can be written out as

$$\frac{1}{l_i} \cdot \sum_{\substack{s \in P_i \\ s_i = 1}} \pi_H(s) \geq \sum_{\substack{s \in S \setminus P_i \\ s_i = 1}} \pi_H(s) + \sum_{\substack{s \in S \\ s_i = 1}} \pi_L(s). \tag{2.16}$$

Similar to the symmetric case where $\pi_L^i = 1$ when $c \leq c^*$, there is an analogous result summarized in the following proposition.

Proposition 4. *For $\sum_{i=1}^{N} l_i \leq k$ then the optimal signaling policy involves*

$$\pi_L^{i*} = 1 \; \forall i$$
$$\pi_H^*(S^N) = \frac{k - \sum_{i=1}^N l_i}{k + \sum_{i=1}^N l_i}$$
$$\pi_L^*(S^k) = 1 - \pi_H^*(S^N) = \frac{2\sum_{i=1}^N l_i}{k + \sum_{i=1}^N l_i}$$

We see again that when the group's costs fall below the threshold the optimal policy focuses first on moving around π_H^* in order to maintain the obedience constraints. And for $\sum_i l_i > k$ it must be that $\pi_H^*(S^k) = 1$. The obedience constraints can then be rewritten as

$$\pi_H^i \geq l_i \pi_L^i. \quad (2.17)$$

Lemma 2 provides key results and intuition in solving for the rest of the equilibrium.

Lemma 2. *If $\sum_{i=1}^N l_i > k$ the optimal policy has the following properties.*

i. $\pi_H^*(S^k) = 1$ *and* $\sum_i \pi_H^{i*} = k$.

ii. *If* $\pi_H^{i*} > 0$ *then* $\pi_L^{i*} > 0$.

iii. $\pi_L^{i*} \geq \pi_L^{j*}$ *for* $i < j$.

iv. *Define the m-th individual s.t.* $\sum_{i \leq m} l_i \leq k$ *and* $\sum_{i \leq m+1} l_i > k$. *Then* $\pi_L^{i*} = \min\{1, 1/l_i\}$ *for* $i \leq m$.

The first part of this lemma picks up from Proposition 2 stating that once the group's costs are past the threshold the seller can no longer solicit more than k contributions in the high state. Moreover we can consider a new interpretation where the seller has a budget of $\sum_i \pi_H^{i*} = k$ (the sum of the marginals in the high state) and uses it to purchase contributions in the low state. She can manage her budget by shifting around the values of the π_H^{i*}'s so long as they sum to k. The idea is that the higher π_H^{i*} is, the higher i believes he's pivotal thus relaxing his constraint as seen from (2.17) allowing her to increase π_L^i but comes at the opportunity cost of relaxing others' constraints. The second part shows that the seller never wastes any of the budget, that is if she is going to spend any resources on individual i that she will ensure some level of contributions in the low state. The third part explains that she uses her budget efficiently such that she solicits more contributions the easier the individual is to persuade. And the final part provides a sufficient condition of when the seller can isolate a group of the easiest people to persuade and solicit maximum contributions from them.

With this interpretation pinning down the equilibrium is straightforward. The seller follows a simple procedure highlighted in this example.

Example 4. Let $N = 4$ and $k = 2$ and suppose that $l_1 + l_2 < 2$ and $l_1 + l_2 + l_3 > 2$.

From Lemma 2 we know that we have $\pi_L^{1*} = \pi_L^{2*} = 1$. Writing out the constraints gives

$$\pi_H^1 \geq l_1$$
$$\pi_H^2 \geq l_2$$
$$\pi_H^3 \geq l_3 \pi_L^3$$
$$\pi_H^4 \geq l_4 \pi_L^4$$

where it immediately follows that $\pi_H^{1*} = l_1$ and $\pi_H^{2*} = l_2$ since there is no benefit of increasing either of them. The seller has a remaining budget of $2 - l_1 - l_2$ for 3 and 4. She proceeds to max out individual 3 first and if there still remains a budget will continue to 4.

$$\pi_H^{3*} = \min\{2 - l_1 - l_2, 1\}, \quad \pi_L^{3*} = \frac{\pi_H^{3*}}{l_3}$$
$$\pi_H^{4*} = \max\{1 - l_1 - l_2, 0\}, \quad \pi_L^{4*} = \frac{\pi_H^{4*}}{l_4}$$

Below summarizes the full equilibrium.

Proposition 5. *Define the m-th individual s.t.* $\sum_{i \leq m} l_i \leq k$ *and* $\sum_{i \leq m+1} l_i > k$*. For* $k > \sum_{i=1}^{N} l_i$ *the optimal signaling policy involves*

$$\pi_L^{i*} = \frac{\pi_H^{i*}}{l_i} \ \textit{where} \tag{2.18}$$

i. for $i \leq m$*,*

$$\pi_H^{i*} = \min\{1, l_i\} \tag{2.19}$$

ii. for $m < i \leq m + k$*,*

$$\pi_H^{i*} = \min\left\{1, \max\left\{k - \sum_{j=1}^{i-1} \pi_H^{i*}, 0\right\}\right\} \tag{2.20}$$

iii. for $m + k < i$*,*

$$\pi_H^{i*} = 0 \tag{2.21}$$

iv. and for $m = 0$ $(l_1 > k)$*,*

$$\pi_H^{i*} = 1 \quad \textit{for } 1 \leq i \leq k \tag{2.22}$$
$$\pi_H^{i*} = 0 \quad \textit{for } k < i \tag{2.23}$$

The seller finds it optimal to split the group into three: the first $m \geq 0$ individuals, the next k individuals afterwards, and lastly any remaining people. The seller fully extracts contributions in the low state of the first m easiest individuals. She then proceeds to use the remaining budget to target the next easiest to persuade individual and attempts to solicit as much contributions as possible. And continues until the budget is exhausted. After the m, she can at most target the next k individuals thus always ignoring the hardest $N - m - k$ individuals at the very least.

2.4 Discussion

This paper introduces information design to the rich field of public goods research and shows its importance by offering additional insights to the field. With the tools of information design it's easier to break down and understand what role and benefits a fundraiser offers in the provision of public goods. By both coordinating and increasing perceived value of each donor's contribution, a charitable fundraising event can address many of the common issues that might otherwise obstruct the provision of the good.

This paper only demonstrates a glimpse of the usefulness of intersecting public goods and information design theory. There are many ways into extending this research and gaining new insights on the various topics not presently covered.

Chapter 3

Advertising Contests

3.1 Introduction

This paper studies the optimal advertising strategies of two competing firms in a market where consumers rely on past purchases when deciding between the two experience goods. Consumers have imperfect knowledge of the good and can only make inferences based on the past purchases as well as their own private knowledge. For example, a consumer in the market for a new cell phone might be deciding between an iOS or Android device. And for those looking for a new laptop might be weighing between Windows or macOS. This paper mainly examines situations where one product emerges as the victor capturing the entire market[1].

The social learning environment brings out an information externality. However, it's well known in the literature that the social learning process does not necessarily help the consumers to aggregate enough information to learn the true state of the world. As studied in e.g. Banerjee (1992), Bikhchandani et al. (1992), and Smith and Sørensen (2000) the learning process can converge to the true state but it may also converge far from the truth (e.g. when signals have bounded strength). Much of the literature focused on understanding how and why informational cascades and herding occurs. These are not generic results and hence papers have focused on studying which environments these phenomena occur.

A common assumption in the literature is that the agents' signal structure is

[1] E.g. format wars (Blu-ray vs HD-DVD, VHS vs Betamax, etc.), war of the currents, competing substitutable technologies.

exogenously given. The main objective of this paper is to understand how profit maximizing firms can strategically endogenize the signal distribution and its implications on the learning process and welfare. Here consumers rely on past purchases as well as the signals they receive to formulate an expected valuation of the quality of the experience goods. Naturally, the firms have an incentive to manipulate the signal distribution in order to direct the public belief in their favor.

In this paper, firms look to influence the signal distribution through advertising. The higher the level of investment, the more likely consumers will receive a favorable signal. Here firms compete in a contest that determines the signal distribution. The contest is modeled with a variant of the standard Tullock contest where firms are given exogenous head starts to their level of advertising. Franke et al. (2016) also examines the Tullock lottery contest with heterogeneous head starts. Just like in their model, the head starts act as perfect substitutes to the level of advertising. That is, as a firm's head start increases, its investment also decreases by the same amount.

The main result of this paper is that advertising is wasteful and reduces total welfare. The reason is that competition ends up (weakly) reducing the information available to the consumers. As competition intensifies, the endogenized signal distribution becomes less informative, up to the point where signals are completely uninformative (this occurs when the high-quality firm's head start is small). And because competition obscures the information, the likelihood that consumers end up herding on the low-quality product increases. This inefficiency stems from the fact that consumers have imperfect knowledge of the goods and only have bounded beliefs and as a result are not always able to aggregate information that will converge to the true valuation. Thus the low-quality firm can exploit their bounded beliefs and increase the likelihood that consumers herd on the wrong product.

3.1.1 Related Literature

Other papers in the herding literature have also examined firms controlling the social learning process. E.g. Bose et al. (2006, 2008) analyze a model where a monopolist dynamically chooses prices in order to extract the rent from the current buyer and control the information being diffused to the future buyers. They show that the monopolist typically delays herding but still induces herding to occur almost surely. However, these papers focus on the role of prices and keep the consumers' signal structure exogenous. In this paper, prices are fixed but signals are manipulated to control the information.

This paper also relates to the rent-seeking literature, specifically on contests with head starts and the tug-of-war battles. Dasgupta and Nti (1998) and Amegashie (2006) study models with contestants of equal head starts. Franke et al. (2016) is closer related to this paper as they examine heterogeneous starts. Head starts play similar roles, but Franke et al. (2016) focus on a designer creating optimal rules to extract the most revenue from the contestants. However, in this paper it is socially optimal to reduce competition rather than maximize total investment.

In the tug-of-war models a series of battles are taken place until one player achieves enough wins to claim the winning prize. Konrad and Kovenock (2005) and Agastya and McAfee (2006) characterize the equilibria where each battle is represented as an all-pay first price auction. More closely related to this paper is Harris and Vickers (1987). They characterize the equilibria in the tug-of-war model with the Tullock contest rather than an all-pay auction. In the dynamic setting where firms are investing in each period, this paper becomes a special case of Harris and Vickers (1987) when both firms have no head starts. They show that the winning firm is always investing more than the losing firm, but this need not be true when they have head starts.

Other papers also have similar results regarding the inefficiencies of competition. In the information manipulation context, Mayzlin (2006) and Dellarocas (2006) model firms strategically manipulating online opinions in a single period (no learning takes place). Mayzlin (2006) examines an environment where competing firms can strategically post biased messages (alongside the available unbiased messages) to consumers that will affect their beliefs before deciding on which good to purchase. In equilibrium, firms are manipulating the online opinion where the low-quality firm puts more resources into manipulation, i.e. the information value decreases for consumers. And in the bargaining context, Yildirim (2007) studies a sequential bargaining model where agents are competing to become the proposer by expending costly efforts that are socially wasteful. However, effort levels can be reduced in the presence of tough bargainers, similar to this paper where the high-quality firm can reduce the advertising of the low-quality firm with a large enough head start.

3.2 Model

The consumers are modeled the same way as in Bikhchandani et al. (1992) with a slight reinterpretation. In their model, each consumer has to make the choice of adopting some behavior (and receiving a payoff of either 1 or 0) or not. Adopting costs the consumer 1/2 and choosing not to adopt leaves the consumer with an outside option of 0. The difference with the model below is that consumers are now deciding on purchasing between two goods where one good has a payoff of 1 and the other of 0, but the consumers can't tell which good would yield the higher payoff. The price of both goods is fixed at 1/2.

There is an infinite horizon time period $t = 1, 2, 3, \ldots$, where in each period a new consumer chooses between two goods, A and B. The consumer chooses to purchase one of the two goods (but not both). Consumers do not know the true valuations of the goods prior to making a purchase, but they do know that good $i \in \{A, B\}$ is high-valued at $v^i = 1$ and good $j \neq i$ is low-valued at $v^j = 0$. It's common knowledge that either good is equally likely, that is $\mathbb{P}(\theta = A) = \mathbb{P}(\theta = B) = 1/2$ where $\theta = i$ is the state where good i is the high-valued good. Before making a purchase, each consumer receives an imperfect private signal and is able to observe all the past purchases made by previous consumers. Using this information, the consumer will decide on which good to purchase.

Formally the consumers, indexed by time, are Bayesian rational, risk-neutral expected utility maximizers and have identical unit demands for each of the two goods. Their payoffs are $u_t^i = \mathbb{E}_t(v^i \mid s_t) - 1/2$ for $i \in \{A, B\}$ where $1/2$ represents the price of the good and $\mathbb{E}_t(v^i \mid s_t) = \mathbb{E}(v^i \mid s_t, h_t)$ is the expectation conditional on receiving signal $s_t \in \{A, B\}$ and on observing the public history of past actions, h_t. Good i is chosen when $u_t^i > u_t^j$ for $i \neq j$, or equivalently $\mathbb{P}_t(\theta = i \mid s_t) > \mathbb{P}_t(\theta = j \mid s_t)$. When indifferent, consumers will purchase the good favored by the signal.

Each consumer receives an independent signal $s \in \{A, B\}$ drawn from the following symmetric binary distribution for $\mu \geq 1/2$.

	$\mathbb{P}(s = A \mid \theta)$	$\mathbb{P}(s = B \mid \theta)$
$\theta = A$	μ	$1 - \mu$
$\theta = B$	$1 - \mu$	μ

For the consumer who receives signal $s_t = A$, their updated posterior is

$$
\begin{aligned}
\mathbb{P}_t(\theta = A | s_t = A) &= \frac{\mathbb{P}_t(s_t = A \mid \theta = A)\mathbb{P}_t(\theta = A)}{\mathbb{P}_t(s_t = A \mid \theta = A)\mathbb{P}_t(\theta = A) + \mathbb{P}_t(s_t = A \mid \theta = B)\mathbb{P}_t(\theta = B)} \\
&= \frac{\mu \cdot \mathbb{P}_t(\theta = A)}{\mu \cdot \mathbb{P}_t(\theta = A) + (1 - \mu) \cdot \mathbb{P}_t(\theta = B)}
\end{aligned}
$$

where the likelihood ratio is

$$\frac{\mathbb{P}_t(s_t = A \mid \theta = A)}{\mathbb{P}_t(s_t = A \mid \theta = B)} = \frac{\mu}{1-\mu}.$$

This likelihood ratio is greater than 1 if and only if $\mu > 1/2$. Thus when $\mu > 1/2$ receiving a signal of A is good news for good A, that is, the updated probability of A being the high-valued good increases (analogous for B). And if $\mu = 1/2$ then the signals provide no information at all.

Bikhchandani et al. (1992) calculate the perfect Bayesian equilibrium of this game where μ is exogenously given, and it can be summarized by an absorbing Markov chain. Suppose that $\mu > 1/2$ and that the probability θ is either A or B is $1/2$. Then given the tie-breaking rule of following own signal when indifferent between the goods, the equilibrium is described in Figure 1. Here the game starts at the middle state denoted by R (the reset state). The first consumer begins indifferent between the good, thus if he receives signal A he will purchase that good and the state transitions to the left to state A. The probability of receiving a favorable signal for A is p_A and therefore this is also the probability of transitioning to state A.

If the second consumer also receives a signal of A, she will purchase good A and the state transitions left again to the absorbing state W_A. Again this will happen with probability p_A. What is interesting about this absorbing state is that an informational cascade occurs. From here on, no new information can be learned from consumers' actions and so the public belief stays forever fixed in favor of good A. As a result each consumer will simply follow the herd. That is, each new consumer from here on, regardless of their signal realization, will choose to purchase good A. And if the second consumer receives signal B then the state transitions to the reset state R. From here, the third consumer's decision problem will be identical to the first (the

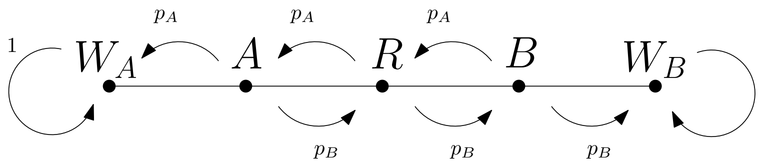


Figure 3.1: Absorbing Markov chain summarizing equilibrium behavior of consumers.

differing signals of the first two consumers effectively cancel each other out).

The timing of the game is as follows. There are two firms, A and B, with differing qualities. With equal probability, Nature chooses one of them to be a high-quality firm which is common knowledge between the firms. Then the firms invest in a simultaneous, once-and-for-all level of advertising that will determine the signal distribution for the entire game. The signal distribution is fixed and identical for all consumers. Consumers then sequentially decide on which good to purchase. They are unable to observe the firms' investment decisions; however, they form the correct beliefs of the signal distribution in equilibrium.

Both firms are assumed to produce the goods at constant marginal costs normalized to 0. They compete over the market share, trying to achieve the informational cascade in their favor. With prices fixed at 1/2 the firms engage in an advertising war that will determine the consumers' signal distribution. Formally, for advertising levels $c_i \geq 0$, the signal distribution is determined by the following contest success function

$$\begin{aligned} p_i(c_i, c_j) &= \mathbb{P}(s = i) = \frac{c_i + k_i}{\sum_{l=1}^{2} c_l + k_l} \quad \text{where } k_i \geq 0 \text{ for each } i \in \{A, B\} \\ p_j(c_i, c_j) &= \mathbb{P}(s = j) = 1 - p_i(c_i, c_j) \\ p_i(c_i, c_j) &= 1/2 \quad \text{when } c_i = c_j = k_i = k_j = 0. \end{aligned} \tag{3.1}$$

The parameters k_i, k_j measure the head starts of each firm. For example they might capture consumer loyalty or the degree of alliance formed with the advertising platform. If $k_i \geq k_j$ then firm i is said to have an advantage over firm j, where $k_i/(k_i+k_j)$ is the relative advantage. It's assumed that the high-quality firm will always have the advantage over the low-quality firm.

Firms are risk-neutral profit maximizers. They have linear costs equal to their investment levels and seek to maximize the infinite sum of expected discounted profits with discount factor $\delta \in (0, 1)$. They anticipate the consumers adopting the herding strategy described previously so that firms are in the game described by the Markov chain in Figure 1. For each purchase of good i the firm makes 1/2 in revenue. And once the absorbing state W_i is reached (and all subsequent consumers herd on product i) firm i earns the infinite discounted stream of 1/2's, $\frac{1}{2(1-\delta)}$, and j earns 0. Let $\pi_i(\omega)$ be i's expected revenue starting from state $\omega \in \{W_A, A, R, B, W_B\}$. Then $\pi_i(\omega)$ is defined by the system of equations.

$$\pi_i(W_i) = \frac{1}{2(1-\delta)}$$

$$\pi_i(W_j) = 0$$

$$\pi_i(i) = p_i(c_i, c_j)[1/2 + \delta\pi_i(W_i)] + [1 - p_i(c_i, c_j)]\delta\pi_i(R)$$

$$\pi_i(j) = p_i(c_i, c_j)[1/2 + \delta\pi_i(R)] + [1 - p_i(c_i, c_j)]\delta\pi_i(W_j)$$

$$\pi_i(R) = p_i(c_i, c_j)[1/2 + \delta\pi_i(i)] + [1 - p_i(c_i, c_j)]\delta\pi_i(j)$$

The game starts at state R thus a firm's payoff is $\pi_i(R) - c_i$ where

$$\pi_i(R) = \frac{\frac{1+\delta}{2}p_i(c_i, c_j) + \frac{\delta^2}{2(1-\delta)}p_i(c_i, c_j)^2}{1 - 2\delta^2 p_i(c_i, c_j)[1 - p_i(c_i, c_j)]}.$$

Each firm simultaneously chooses c_i in order to maximize profits:

$$\max_{c_i \geq 0} \pi_i(R) - c_i.$$

The next proposition characterizes the perfect Bayesian equilibrium where $p_i^* := p_i(c_i^*, c_j^*)$ is the equilibrium signal distribution. All proofs are found in the appendix.

Proposition 1. *Define* $k^* = 1/[4(1-\delta)(2-\delta^2)]$. *For* $0 < \delta < 1$ *and* $0 \leq k_j \leq k_i$ *there exists a* $k^{**} > k^*$ *and a* $\hat{k} < k^*$ *such that*

i. *for* $k_i \in (0, k^*]$, $c_i^* = k^* - k_i$, $c_j^* = k^* - k_j$, *and* $p_i^* = 1/2$.

ii. *for* $k_i \in (k^*, k^{**})$ *and* $k_j \leq \hat{k}$, $c_i^* = 0, c_j^* = \hat{k} - k_j$, *and* $p_i^* = k_i/(k_i + \hat{k})$ *where* c_j^* *is continuously decreasing in* k_i. *And* $p_i^* \in (1/2, k_i/(k_i + k_j))$ *is increasing in* k_i.

iii. *for* $k_i \in [k^{**}, \infty)$, $c_i^* = c_j^* = 0$ *and* $p_i^* = k_i/(k_i + k_j)$.

iv. $\hat{k} = \hat{k}(k_i)$ *is decreasing in* $k_i \in [k^*, k^{**}]$. *Define* $\delta^* = \sqrt{1 - \sqrt{2}/2}$. *For* $\delta \leq \delta^*$, $k^{**} = k^{**}(k_j)$ *is decreasing in* k_j. *For* $\delta > \delta^*$, k^{**} *is initially increasing in* k_j *then decreasing.*

To understand the proposition, when both firms are symmetric ($k_j = k_i = \kappa \geq 0$) they optimally want to invest in k^*, $c_i^* = c_j^* = \max\{k^* - \kappa, 0\}$. Here head starts are substituted in place of investing; firms only invest when their head start does not account for all of k^*.

When firms are no longer symmetric ($0 \leq k_j < k_i$) the same result holds when i's head start is small ($k_i \leq k^*$). Thus the two firms are effectively symmetric, $c_i^* + k_i = c_j^* + k_j = k^*$, where the advantaged firm earns higher profits from having to invest less. Here a higher head start only decreases own investment but does not

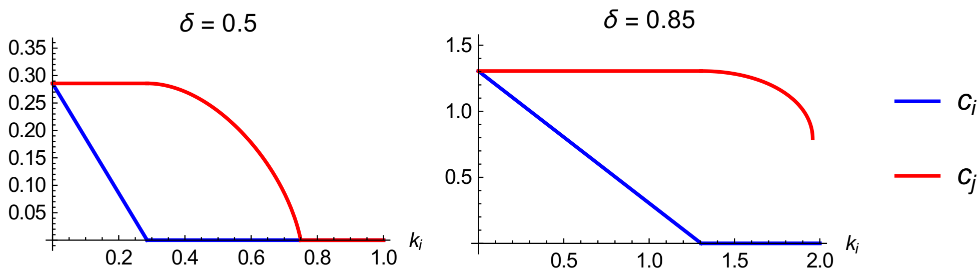


Figure 3.2: Numerical solutions of Proposition 1 plotted for $\delta = 0.5$ and $\delta = 0.85$ when $k_j = 0$.

affect the other firm's level of investment. But as i's advantage begins to grow beyond k^*, any improvements in i's head start no longer decreases c_i (since $c_i^* = 0$) but rather i is able to extract j's rent and lower j's incentives to invest. i is able to continue extracting the rent until j no longer finds it profitable to invest at all. Therefore when the advantage is very large ($k_i \geq k^{**}$) $c_j^* = 0$. Here $\hat{k}$ is firm j's optimal level of total investment ($c_j + k_j = \hat{k}$). For $k_i \leq k^*$, $\hat{k} = k^*$. But as i's advantage increases past k^* and j falls further behind j can no longer keep up with i but instead does its best to minimize the disadvantage. j finds it optimal to not invest when either $k_j \geq \hat{k}$ (since k_j remains as a perfect substitute to advertising) or i's advantage is just too large ($k_i \geq k^{**}$).

As for part (*iv*), k_j has different effects on k^{**} depending on δ. The reason is that for $\delta < \delta^*$, the expected revenue function is strictly concave. But for $\delta > \delta^*$ it is S-shaped (for low k_j) and so firm j's marginal revenue is initially increasing in advertising. Therefore substituting its early stages of investment with its head start increases the additional profits made from investing in $c_j^* = \hat{k} - k_j$, resulting in a higher k^{**}.

Figure 3.2 summarizes the results of Proposition 1 by plotting numerical solutions for the case of $k_B = 0$. The reason for the discontinuity when $\delta = 0.85$ is because $0.85 > \delta^*$.

An implication of Proposition 1 is that competition can lead to wasteful advertising. For example if firms are symmetric and $k_j = k_i = \kappa$ is small enough (i.e. $\kappa \leq k^*$) then both firms will advertise and earn profits of $(1+\delta)/(4(2-\delta^2)) + \kappa$ which is Pareto dominated by both firms not investing in any advertising and earning $1/(4(1-\delta))$. Moreover, when firms are no longer symmetric competition can not only decrease firms' profits but it can also harm consumers. That is, for all $k_j < k_i < k^{**}$, competition will decrease the information available to the consumers. If the firms chose not to invest ($c_j = c_i = 0$) then $p_i = k_i/(k_i + k_j) > 1/2$. But in equilibrium the firms will compete and lower p_i^* to some value strictly less than $k_i/(k_i + k_j)$ for all $k_j < \hat{k}$. And the extreme case is when $k_i < k^*$ where competition makes signals completely uninformative preventing consumers from learning any new information. Here the high-type firm not only does not have a large enough of an advantage to prevent the low-type from engaging in the advertising war, but it's also not high enough to separate itself from the low-type.

Define h_i to be the probability of herding on good i and $\mathcal{T}$ the expected time for consumers to begin herding. The next result discusses some comparative statics.

Proposition 2. *For all* $0 < \delta < 1$ *and* $0 \leq k_j < k_i$,

i. *h_i is weakly increasing (decreasing) in k_i (k_j). Strictly when $c_i^* = 0$, or equivalently when $k_i > k^*$. Analogous for h_j.*

ii. *$\mathcal{T}$ is maximized when $p_i^* = 1/2$ and decreases as p_i^* increases. Thus $\mathcal{T}$ is (weakly) decreasing in k_i and (weakly) increasing k_j.*

iii. *As δ increases, p_i^* decreases towards 1/2. As a result, h_i is decreasing and $\mathcal{T}$ is increasing.*

The first part states that increasing the head start of firm i (weakly) increases the probability of consumers herding on their good. The second part shows that competition increases the length of the game. That is, as the high-type firm's relative advantage over the low-type begins to shrink, the probability moves closer to $p_i^* = 1/2$. As for the last part, δ increases the competition among the firms. The intuition is that as δ increases both firms would value future consumers' purchases more (i.e. herding is more valuable). Thus both firms are incentivized to invest more heavily as δ increases. And as a result the high-type's head start has less of an impact as firms become more willing to invest.

3.2.1 Monopoly

Returning to the effects of competition, consider the monopolist case as a benchmark. Suppose there is a monopolist that is either a high type (k_H) or a low type (k_L) where $0 \leq k_L \leq k_H$. The monopolist is selling a single good and consumers must decide on whether or not to purchase the good after receiving a signal. Consumers' payoffs are the same. Similar to before, the signal distribution is now determined by

$$p_i(c_i) = \mathbb{P}(s = i) = \frac{c_i + k_i}{c_i + k_i + k_j}$$

for $k_i, k_j \in \{k_L, k_H\}$ and $k_j \neq k_i$ and $c_i \geq 0$.

The monopolist's problem is the same as before where now $c_j = 0$:

$$\max_{c_i} \pi_i(R) - c_i.$$

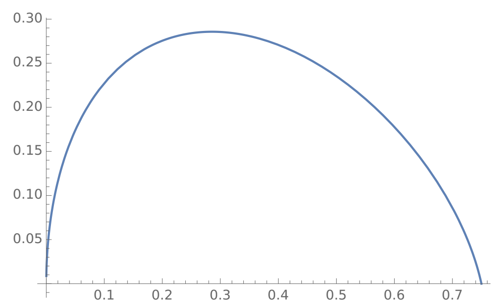
When $k_H \geq k^*$ the equilibrium is identical to the duopoly case above.

Proposition 3. *Let* $0 < \delta < 1$ *and* $0 < k_L < k_H < k^*$. *Then*

i. The low type monopolist weakly invests more than the high type.

ii. c_i^* *is increasing in* k_j.

iii. Ex-ante, the probability consumers herd on the correct good is higher in the case of a monopolist with unknown type than with two competing firms with different types.



The figure above is the numerical best response plot for $\delta = 1/2$ with k_i on the horizontal axis and $c_j + k_j$ on the vertical axis. The maximum is obtained at $k_i = k^*$. Thus the best response is identical to the previous graphs from $k^* \leq k_i$. From Proposition 3, the best response is increasing in the other type's head start. So in the duopolistic environment, competition is incentivizing both firms to invest more heavily. This will continue until both invests in $k^* - k_l$ as discussed in Proposition 1. As a result, the consumers are not able to aggregate as much information in the duopoly case since the firms drive down the signal informativeness.

3.2.2 Dynamic Model

This section looks at the dynamic model where the competing firms are repeatedly investing in each period before the new consumer receives the signal and makes a purchase. For tractability, it is assumed that the firms make binary investment decisions, $c_i(t) \in \{0, 1\}$ for firm $i \in \{A, B\}$ in period t, with costs of 1 for investing. Firms remain to be differentiated by head-starts, $k := (k_i, k_j) \in \mathbb{R}_+^2$ hence the contest success function is similar to equation (3.1) other than restricting the investment space. Since the public belief of the consumers follow the Markov chain from Figure 1, strategies can be simplified to $c(\omega) = (c_i(\omega), c_j(\omega))$ for $\omega \in \Omega := \{A, R, B\}$. Therefore this section will focus on stationary Markov perfect equilibrium.

For convenience define $c := \{(c_i(i), c_j(i))_i, (c_i(R), c_j(R))_R, (c_i(j), c_j(j))_j\}$ and re-write the expected revenue from state ω for firm i as

$$\pi_i(c, \omega) = p_i(c, \omega)\gamma_i(c, \omega) + \delta\pi_i(c, \omega; k) - 1$$
$$\gamma_i(c(\omega); k) := 1/2 + \delta\pi_i(c(\omega_i^+); k) - \delta\pi_i(c(\omega_i^-); k)$$

where $\gamma_i(c, \omega; k)$ is the net value from winning. That is, $\gamma_i(c, \omega) := 1/2 + \delta\pi_i(c, \omega_i^+) - \delta\pi_i(c, \omega_i^-)$ where ω_i^+ represents transitioning from ω to the new state resulting from i winning and to ω_i^- from i losing. Firm i's optimization problem in state ω is then

$$\max_{c_i(\omega) \in \{0,1\}} \pi_i(c; \omega).$$

Firm i invests in state ω if and only if the marginal gain in the net value from winning exceeds the marginal cost

$$[p_i(1, c_j(\omega)) - p_i(0, c_j(\omega))]\gamma_i(c, \omega) \geq 1$$

or, equivalently, if and only if the net value from winning is high enough

$$\gamma_i(c, \omega) \geq \frac{1}{[p_i(1, c_j(\omega)) - p_i(0, c_j(\omega))]}$$

For a given strategy c, $\gamma_i(\omega)$ is increasing in δ and therefore incentives for investing increases with δ. Define

$$\underline{\delta}_k := \inf\{\delta \mid 0 < \delta < 1 \text{ and } c_i(\omega) = 1\ \forall i \\ \text{and } \forall \omega \in \Omega \text{ for a given pair } (k_i, k_j) \geq (0,0)\}.$$

It's the smallest δ such that all firms are investing in each period for a given level of head starts. It's nonempty since δ can be chosen close enough to 1 such that both firms are investing. This is because the payoff from winning the market, $1/(2(1-\delta))$, is unbounded as $\delta \to 1$.

The following proposition characterizes the symmetric stationary Markov perfect equilibrium.

Proposition 4. *Let $\delta > \underline{\delta}_k$ and $k_i = k_j = \kappa \geq 0$. Define $\beta_i(j;\kappa)$ to be the probability that firm i invests in state j given κ. Then*

i. $c^* = c_1 := \{(1,1)_i, (1,1)_R, (1,1)_j\}$ *when* $\kappa < \kappa_1^*$

ii. $c^* = c_2 := \{(1, \beta_j(i))_i, (1,1)_R, (\beta_i(j), 1)_j\}$ *when* $\kappa_1^* \leq \kappa < \kappa_2^*$

iii. $c^* = c_3 := \{(1,0)_i, (1,1)_R, (0,1)_j\}$ *when* $\kappa_2^* \leq \kappa < \kappa_3^*$

iv. $c^* = c_4 := \{(1,0)_i, (0,0)_R, (0,1)_j\}$ *when* $\kappa_3^* \leq \kappa < \kappa_4^*$

v. $c^* = c_5 := \{(0,0)_i, (0,0)_R, (0,0)_j\}$ *when* $\kappa_4^* \leq \kappa$

where $0 < \kappa_1^* < \kappa_2^* < \kappa_3^* < \kappa_4^*$ *are given from the indifference conditions*

$$\gamma_i(c_1, j) = 2(1 + 2\kappa_1^*) \tag{3.2}$$

$$\beta_i(j; \kappa_2^*) = 0 \tag{3.3}$$

$$\gamma_i(c_3, j) = 2(1 + 2\kappa_3^*) \tag{3.4}$$

$$\gamma_i(c_4, j) = 2(1 + 2\kappa_4^*) \tag{3.5}$$

$$\tag{3.6}$$

and

$$\beta_i(j; \kappa) := \frac{2(6 + 28\kappa + 32\kappa^2 - \delta(3 + 22\kappa + 32\kappa^2)}{\delta^2(2 - 3\delta + 8(1-\delta)\kappa)} + \frac{-\delta^2(2 + 10\kappa + 16\kappa^2) + \delta^3(-2 + 3\kappa + 16\kappa^2))}{\delta^2(2 - 3\delta + 8(1-\delta)\kappa)} \tag{3.7}$$

where $\beta_i(j;\kappa)$ *is continuously decreasing in* κ *with* $\beta_i(j;\kappa_1^*) = 1$ *and* $\beta_i(j;\kappa_2^*) = 0$. *And for* $0 < \delta \leq \underline{\delta}_k$ *define the thresholds as*

$$\begin{aligned} \tilde{k}_l^* &= \max\{k_l^*, 0\} \text{ for } l = 1,2,3,4 \\ 0 &\leq \tilde{k}_1^* \leq \tilde{k}_2^* \leq \tilde{k}_3^* \leq \tilde{k}_4^* \end{aligned} \tag{3.8}$$

and the above equilibrium remains with these new thresholds.

When firms are investing in all states ($\delta > \underline{\delta}_k$) the trailing firm is the first to go inactive as κ increases. The trailing firm, now indifferent from investing, will begin mixing. Eventually head-starts increase high enough for the trailing firm to go completely inactive. The leading firm remains investing, and as κ increases firms will soon restrain from investing in the reset state. And finally the leading firm will best last to go inactive as $\kappa \geq \kappa_4^*$. This equilibrium is analogous to the equilibrium analyzed in Harris and Vickers (1987) where they also find that the leading firm exerts more resources than the trailing firm.

The thresholds are decreasing in δ and may fall to 0 for $\delta \leq \underline{\delta}_k$. This is due to the fact that incentives decrease as (i) firms become more impatient and (ii) firms begin with higher head-starts. But for a fixed δ, the equilibrium behavior over varying head-starts remain the same—the leading firm (weakly) invests more than the trailing firm. However, Proposition 5 shows that the leading firm doesn't necessarily invest more than the trailing firm in the case of $k_i > k_j = 0$. Therefore if the high-type has a big enough advantage

Proposition 5. *Let $\delta > \underline{\delta}_k$ and $k_i > k_j = 0$. Define $\beta_l(R;k)$ to be the probability that firm l invests in state R given k. Then*

i. $c^* = c_1 := \{(1,1)_i, (1,1)_R, (1,1)_j\}$ *when* $k_i^* < k_1^*$

ii. $c^* = c_2 := \{(0,1)_i, (1,1)_R, (1,1)_j\}$ *when* $k_1^* \leq k_i < k_2^*$

iii. $c^* = c_3 := \{(0,0)_i, (1,1)_R, (1,1)_j\}$ *when* $k_2^* \leq k_i < k_3^*$

iv. $c^* = c_4 := \{(0,0)_i, (\beta_i(R;k_i), \beta_j(R;k_i))_R, (1,1)_j\}$ *when* $k_3^* \leq k_i < k_4^*$

v. $c^* = c_5 := \{(0,0)_i, (0,0)_R, (0,0)_j\}$ *when* $k_4^* \leq k_i$

where $0 < \kappa_1^* < \kappa_2^* < \kappa_3^* < \kappa_4^*$ *are given from the indifference conditions*

$$\gamma_i(c_1, i) = (1 + k_1^*)(2 + k_1^*) \tag{3.9}$$

$$\gamma_j(c_2, i) = (2 + k_2^*) \tag{3.10}$$

$$\gamma_i(c_3, i) = (1 + k_3^*)(2 + k_3^*) \tag{3.11}$$

$$\beta_j(R; \kappa_4^*) = 0 \tag{3.12}$$

And for $0 < \delta \leq \underline{\delta}_k$ *define the thresholds as*

$$\begin{aligned} \tilde{k}_l^* &= \max\{k_l^*, 0\} \text{ for } l = 1,2,3,4 \\ 0 \leq \tilde{k}_1^* \leq \tilde{k}_2^* &\leq \tilde{k}_3^* \leq \tilde{k}_4^* \end{aligned} \tag{3.13}$$

and the above equilibrium remains with these new thresholds.

Proposition 5 shows that, at least in the extreme case ($k_i > k_j = 0$), the leading firm doesn't always invest more than the trailing firm. Moreover, when the low-type firm is trailing it (weakly) invests more resources than the high-type. This coincides with the previous section where the low-type is also investing more.

The equilibrium for the remaining $k \in \mathbb{R}_+^2$ still need to be solved. It's conjectured that the equilibrium is some mix of the two propositions. E.g. it can be shown that for any $k_i > k_j > 0$ that investing must first stop in state $\omega = i$ (high-type is leading). Define $\Delta_k = k_i - k_j$, then for $\Delta_k > \Delta^*$ (Δ^* monotonic in δ) the high-type is first to go inactive and when $\Delta_k < \Delta^*$ the low-type is first to go inactive. Therefore when the high-type has a large enough lead it will go inactive first in the high-state (high-type is leading). Otherwise, the first to go inactive is the low-type again in the high-state.

Chapter 4

Conclusion

This paper has argued that competition results in advertising becoming a wasteful resource. The firms end up becoming more aggressive in their investment levels yet fail to distinguish themselves when consumers make their decision. As a result, competition confounds the learning process of the consumers. Buyers find it more difficult to aggregate information correctly as competition becomes more intense. And at the extreme case, consumers are left with completely insignificant signals portraying zero information.

However, this opens up further research into analyzing social learning models and the implications competition has over the signal distribution. For example, the complementarity between advertising and the learning process may vary depending on the specification of the learning process as well as the contest technology adopted.

Appendix A (Proofs to Chapter 2)

Proof of Proposition 1:

This result is immediate from Arieli & Babichenko Lemma 1. The setup is the same and their result requires organizer's payoff to be supermodular which is satisfied here. i's obedience constraint for following a recommendation to donate is

$$\frac{\pi_H^i N}{\pi_H^i + \pi_L^i} - c \geq \frac{\pi_H^i (N-1)}{\pi_H^i + \pi_L^i}$$
$$\frac{\pi_H^i N}{\pi_H^i + \pi_L^i} - c \geq 0$$

for $k < N$ and $k = N$ respectively. It's clear that the organizer simply chooses $\pi_H^{i*} = 1$ and π_L^{i*} comes from the constraints binding. Equivalently the persuasion level

$$a_i = \min\left\{1, \frac{u_i(1,H) - u_i(1,L)}{u_i(0,L) - u_i(1,L)}\right\}$$

as defined in Arieli & Babichenko will yield identical results.

□

Proof of Lemma 1:

i. Fix $\pi^* \in \Pi$ to be the seller's optimal policy. For sake of contradiction, suppose that (2.7) is binding. Then (2.6) and (2.7) can be rewritten as

$$\sum_{\substack{s \in P_i \\ s_i = 1}} \pi_H^*(s) > c \sum_{\substack{s \in S \\ s_i = 1}} \sum_{\theta \in \Theta} \pi_\theta^*(s) \tag{4.1}$$

$$c \sum_{\substack{s \in S \\ s_i = 0}} \sum_{\theta \in \Theta} \pi_\theta^*(s) = \sum_{\substack{s \in P_i \\ s_i = 0}} \pi_H^*(s) \tag{4.2}$$

But the seller can always relax (4.2) by lowering $\pi_H^*(s_i = 0, s \in P_i)$ by $\epsilon > 0$

and increasing $\pi^*_H(s_i = 1, s \in S \setminus P_i)$ by $\epsilon > 0$. For small enough $\epsilon > 0$ both (4.1) and (4.2) are slack. But this is an improvement in the seller's payoff since she only values $s_i = 1$ contradicting π^* being optimal.

ii. Again fix $\pi^* \in \Pi$ and suppose instead that $\pi^*_H(S^N) + \pi^*_H(S^k) < 1$. Then the seller can always increase profits by simply shifting all of $\sum_{j=k+1}^{N} \pi^*_H(S^j)$ to $\pi^*_H(S^N)$ and shifting all of $\sum_{j=1}^{k-1} \pi^*_H(S^j)$ to $\pi^*_H(S^k)$ while still respecting the obedience constraint thus contradicting π^* being optimal.

□

Proof of Proposition 2:

Fix $\pi^* \in \Pi$. First note that for $c > 0$ small enough the optimal policy satisfies $\pi^*_L(S^N) = 1$. This is immediate for $k = N$ and for $k < N$ suppose that it didn't, hence $\pi^*_L(S^N) < 1$ then there exists an i such that

$$\frac{1-c}{c} \sum_{\substack{s \in P \\ s_i = 1}} \pi^*_H(s) = \pi^*_H(S^N) + \sum_{\substack{s \in S \\ s_i = 1}} \pi^*_L(s)$$

where the last term is less than 1. But if the seller decreases $\pi^*_H(S^N)$ by $\epsilon > 0$ and increases $\pi^*_H(S^k)$ then at the very least i's constraint slackens. She can add $(1 + (1-c)/c)\epsilon$ to $\sum_{s_i=1} \pi^*_L(s)$. The seller therefore loses $(N-k)\epsilon$ but gains more

$$(N-k)\epsilon < \left(1 + \frac{1-c}{c}\right)\epsilon \iff c < \frac{1}{N-k}$$

improving her total payoff provided $c > 0$ satisfies the inequality.

Define $\pi_\theta(S^k) := \binom{N}{k} p_{k\theta}$, then i's constraint is

$$\begin{aligned} \frac{1-c}{c} \sum_{\substack{s \in P \\ s_i = 1}} \pi_H^*(s) &= \pi_H^*(S^N) + 1 \\ \iff \frac{1-c}{c} \binom{N-1}{k-1} p_{kH} &= p_{NH} + 1 \end{aligned} \tag{4.3}$$

and combining this with $p_{NH} + \binom{N}{k} p_{kH} = 1$ gives [1]

$$\binom{N}{k} p_{kH}^* = \frac{2Nc}{k + (N-k)c}$$

thus the optimal policy for $0 \leq c^*$ is

$$\begin{aligned} \pi_L^*(S^N) &= p_{NL} = 1 \\ \pi_H^*(S^k) &= \binom{N}{k} p_{kH}^* = \frac{2Nc}{k + (N-k)c} \\ \pi_H^*(S^N) &= 1 - \pi_H^*(S^k) = \frac{k - (N+k)c}{k + (N-k)c} \end{aligned} \tag{4.4}$$

where c^* is determined by

$$p_{NH}^* = 0 \iff \binom{N}{k} p_{kH}^* = 1 \implies c^* = \frac{k}{N+k}. \tag{4.5}$$

And for $c^* < c$ it must be that $\pi_H^*(S^N) = 0$. The new constraint is

$$\frac{1-c}{c} \frac{k}{N} \geq \pi_L^{i*} \tag{4.6}$$

thus we simply equate the two.

□

Proof of Proposition 3:

[1] Throughout the proof a useful identity is used: $\binom{N-1}{k-1} = \frac{k}{N}\binom{N}{k}$.

The symmetric efficient BCE is $\pi_L^*(S^0) = 1$ and $\pi_H^{i*} = k/N$ for $i \in \mathcal{I}$ and for each $0 < c < 1$. And this clearly satisfies i's obedience constraint

$$\frac{1-c}{c} \sum_{\substack{s \in P \\ s_i = 1}} \pi_H^*(s) = \pi_H^*(S^N) + \sum_{\substack{s \in S \\ s_i = 1}} \pi_L^*(s) \iff \frac{1-c}{c} \frac{k}{N} \geq 0.$$

□

Proof of Corollary 1:

This is immediate from Proposition 2.

$$\lim_{c \to 1} \pi_L^*(S^0) = \lim_{c \to 1} \frac{1-c}{c} \frac{k}{n} = 0$$
$$\lim_{c \to 1} \pi_H^*(S^k) = \lim_{c \to 1} 1 = 1$$

And when $c = 0$ it's efficient if the good is provided with certainty.

□

Proof of Corollary 2:

The obedience constraint for $0 < \mu < 1$ is

$$\frac{\mu}{1-\mu} \sum_{s \in P_i} \pi_H(s) \geq \frac{\mu}{1-\mu} \sum_{\substack{s \in S \setminus P_i \\ s_i = 1}} \pi_H(s) + \sum_{\substack{s \in S \\ s_i = 1}} \pi_L(s) \tag{4.7}$$

and all the same logic and steps are identical as from the proof of Proposition 2 above. Therefore for very low c, $\pi_L^*(S^N) = 1$ and

$$\frac{1-c}{c} \frac{\mu}{1-\mu} \frac{k}{N} \pi_H(S^k) = \frac{\mu}{1-\mu} \pi_H(S^N) + 1$$
$$\pi_H(S^N) + \pi_H(S^k) = 1$$
$$\implies \pi_H^*(S^k) = \frac{Nc/\mu}{k + (N-k)c}$$

with threshold

$$\pi_H^*(S^k) = 1 \implies c^* = \frac{k}{\frac{1-\mu}{\mu}N + k}$$

and after the threshold the obedience constraint immediately gives π_L^{i*}.

□

Proof of Proposition 5:

Suppose the organizer chooses $\pi_{H,0}(S^m) = 1$. As a result the followers, when observing a donation, would form belief

$$\mu_1 = \frac{1}{1 + \pi_{L,0}(S^m)} \tag{4.8}$$

and $\mu_1 = 0$ when observing no donation. Then a leading donor's obedience constraint is

$$\begin{aligned} \frac{1-c}{c} &\geq \pi_{L,0}^i = \pi_{L,0}(S^m) \\ \implies \pi_{L,0}^*(S^m) &= \max\{1, (1-c)/c\} \\ \implies \mu_1^* &= \max\{c, 1/2\}. \end{aligned} \tag{4.9}$$

and the corresponding follower's obedience constraint is

$$\begin{aligned} &\frac{1-c}{c}\frac{k}{N}\pi_{H,1}(S^{k-m}) \geq \pi_{H,1}(S^{N-m}) + \pi_{L,1}^i \quad \text{for } c \leq 1/2 \\ &\frac{k}{N}\pi_{H,1}(S^{k-m}) \geq \frac{c}{1-c}\pi_{H,1}(S^{N-m}) + \pi_{L,1}^i \quad \text{for } c > 1/2 \end{aligned} \tag{4.10}$$

where it follows that the organizer finds it best to set $\pi_{L,1}^i = 1$ in both cases whenever $\pi_{H,1}(S^{N-m}) > 0$. Therefore with Corollary 2 the followers' policy, when the leaders have donate, is

$$\pi^*_{H,1}(S^{N-m}) = \begin{cases} \frac{k-m-(N+k-2m)c}{k-m+(N-k)c} & \text{for } 0 < c \leq c^* \\ 0 & \text{for } c^* < c \end{cases}$$

$$\pi^*_{H,1}(S^{k-m}) = 1 - \pi^*_{H,1}(S^{N-m}) \tag{4.11}$$

$$\pi^{i*}_{L,1} = \begin{cases} 1 & \text{for } 0 < c \leq c^* \\ \frac{1-c}{c}\frac{k-m}{N-m} & \text{for } c^* < c \leq 1/2 \\ \frac{k-m}{N-m} & \text{for } 1/2 < c \end{cases} \quad c^* = \frac{k-m}{N+k-2m}$$

and when no donation is observed simply set $\mu^*_1 = 0 \implies \pi_{H,1}(S^0) = \pi_{L,1}(S^0)$.

Proof of Proposition 5:

Similar argument as before, for l_i's small enough the seller chooses $\pi^{i*}_L = 1$ for all $i \in \mathcal{I}$. To see why, suppose instead that there exists another policy $\phi \in \Pi$ such that $V(\phi) > V(\pi^*)$ and $\phi^j_L < 1$ for at least one j and therefore $\phi_H(S^N) > \pi^*_H(S^N)$. Note that all constraints are binding under both policies. Thus summing over the constraints under ϕ gives

$$k\phi(S^k) = \sum_{i=1}^{N} l_i(\phi_H(S^N) + \phi^i_L)$$

and combining this with $\phi_H(S^N) + \phi_H(S^k) = 1$ gives

$$\phi_H(S^N) = \frac{k - \sum_{i=1}^N l_i \phi^i_L}{k + \sum_{i=1}^N l_i}.$$

Therefore the benefit of adopting ϕ over π^* is gaining more contributions in the high state, $(N-k)(\phi_H(S^N) - \pi^*_H(S^N))$, but comes with the cost of losing some contributions in the low state, $\sum_i(1-\phi_L^i)$. Then

$$(N-k)(\phi_H(S^N)-\pi_H^*(S^N)) = (N-k)\frac{\sum_{i=1}^N(1-\phi_L^i)l_i}{k+\sum_{i=1}^N l_i} < \sum_{i=1}^N(1-\phi_L^i)$$

because

$$\begin{aligned}(N-k)\sum_{i=1}^N(1-\phi_L^i)l_i &< (N-k)\left(k-\sum_{i=1}^N \phi_L^i l_i\right)\\ &< (N-k)\left(k+\sum_{i=1}^N l_i\right)\\ &< \sum_{i=1}^N(1-\phi_L^i)\left(k+\sum_{i=1}^N l_i\right)\end{aligned}$$

where the first and last inequalities use $\sum_i l_i < k$.

To calculate $\pi_H^*(S^N)$ sum the constraints to get

$$k\pi_H(S^k) = (1+\pi_H(S^N))\sum_{i=1}^N l_i$$

and combining this with $\pi_H(S^N)+\pi_H(S^k)=1$ gives

$$\pi_H^*(S^N) = \frac{k-\sum_{i=1}^N l_i}{k+\sum_{i=1}^N l_i} \tag{4.12}$$

$$\pi_L^*(S^N) = 1-\pi_H^*(S^N) = \frac{2\sum_{i=1}^N l_i}{k+\sum_{i=1}^N l_i}. \tag{4.13}$$

□

Proof of Lemma 2:

Fix $\pi^* \in \Pi$.

i. This follows immediately for $k = N$ so consider $0 < k < N$. Suppose instead that $\pi^*_H(S^k) < 1$ thus $\pi^*_H(S^N) := p^*_N > 0$. Then i's constraint is

$$\pi^{i*}_H \geq l_i(p^*_N + \pi^{i*}_L) \quad \forall i.$$

Moreover there exists at least one j such that $\pi^{j*}_L < 1$ since otherwise summing over all the constraints give

$$k\pi^*_H(S^k) = k(1 - p^*_N) \geq \sum_i l_i(1 + p^*_N) \iff p^*_N \leq \frac{k - \sum_i l_i}{N + \sum_i l_i} < 0$$

but p^*_N must be between 0 and 1. Also all the constraints must bind since if it was slack for j such that $\pi^{j*}_L < 1$ then we can always increase π^{j*}_L until it's either equal to 1 or the constraint binds. And for i such that $\pi^{i*}_L = 1$ and is slack then we can always[2] lower π^{i*}_H and increase π^{j*}_H for $\pi^{j*}_L < 1$ now making j's constraint slack thus allowing us to increase π^{j*}_L. Therefore all have to bind. The seller's payoffs can now be simplified to

$$\begin{aligned} V &= N\pi^*_H(S^N) + k\pi_H(S^k) + \sum_{i=1}^{N} \pi^{i*}_L \\ &= Np^*_N + k(1 - p^*_N) + \sum_{i=1}^{N} \left(\frac{\pi^{i*}_H}{l_i} - p^*_N \right) \\ &= k(1 - p^*_N) + \sum_{i=1}^{N} \frac{\pi^{i*}_H}{l_i} \end{aligned}$$

where it's decreasing in p^*_N leading to the seller improving her payoffs by choosing $p^*_N = 0$.

And now that $\pi^*_H(S^k) = 1$, it follows immediately that $\sum_{i=1}^N \pi^{i*}_H = k$ since any

[2]Simply re-index all the i's to j's as this will maintain everyone else's constraint. For example let $N = 5$ and $k = 3$ and let $i = 2$ and $j = 4$. Then shifting $p_{134} \in P_j \setminus P_i$ to $p_{123} \in P_i \setminus P_j$ doesn't affect the obedience constraints for individuals 1,3, and 5.

$\pi^*_H(s)$ such that $s \in S^k$ is counted k times in the sum for each i that receives the recommendation.

ii. If $\pi^{i*}_L < \min\{1, \pi^{i*}_H/l_i\}$ then the constraint is slack, $\pi^{i*}_H > l_i \pi^*_L$. But the seller can always set $\pi^{i*}_L = \min\{1, \pi^{i*}_H/l_i\}$ by increasing $\pi^*(s \mid s_i = 1 \text{ and } s_j = 0 \text{ for } j \neq i)$ maintaining i's constraint and not affecting anyone else's.

iii. Suppose instead that there exists a $j > i$ such that $\pi^{j*}_L > \pi^{i*}_L$. Writing both of their obedience constraints

$$\pi^{i*}_H \geq l_i \pi^{i*}_L$$
$$\pi^{j*}_H \geq l_j \pi^{j*}_L$$

where $\pi^{j*}_L > \pi^{i*}_L \implies \pi^{j*}_L > 0 \implies \pi^{j*}_H > 0$ (or $P_j \neq \emptyset$). Recall $\pi^{j*}_H := \sum_{s \in P_j} \pi^*_H(s)$, and now we consider a few cases. If $P_i \supseteq P_j$, then $\pi^{i*}_H \geq \pi^{j*}_H$ but this implies that the seller can improve her payoff by simply setting $\pi^{i*}_L = \pi^{j*}_L$ since $l_i < l_j$. For $P_i \not\supseteq P_j$, given that $\pi^{j^*}_L > 0$ consider lowering π^{j*}_L by $\epsilon > 0$ and increasing π^{i*}_L by ϵ giving the following constraints[3]

$$\pi^{i*}_H + l_j \epsilon > l_i(\pi^{i*}_L + \epsilon)$$
$$\pi^{j*}_H - l_j \epsilon = l_j(\pi^{j*}_L - \epsilon)$$

where π^{j*}_H is lowered by $l_j \epsilon$ in order to maintain j's constraint (and shifted to P_i). This is always feasible (for small enough $\epsilon > 0$) since we can always choose an $s \in P_j \setminus P_i$ and re-index j to i while not changing anyone else's constraints. Note that i's constraint is now slack since $l_i \epsilon < l_j \epsilon$. We can continue this until

[3] We look at the case where both constraints binds since otherwise if j is slack we can always lower π^{j*}_H until it binds. The rest of the argument works for either case of i's constraint.

$\pi_H^{j^*} = \pi_L^{i*}$ but then this implies that $\pi_L^{i*} \geq \pi_L^{j*}$ since $l_i < l_j$.

iv. Suppose that instead there exists i such that $\pi_L^{i*} < \min\{1, 1/l_i\} \leq 1$. We can use similar logic above to find an improvement. Note that since $\sum_{n=1}^{N} \pi_H^{n*} = k$ at least one of the $\pi_H^{j*} > 0$ (hence $\pi_L^{j*} > 0$) for some $j > m$ thus

$$\pi_H^{i*} \geq l_i \pi_L^{i*}$$
$$\pi_H^{j*} \geq l_j \pi_L^{j*}.$$

But we can apply the same procedure from the previous part and find a strict improvement by moving weight from π_H^{j*} to π_H^{i*}. Continue this until π_L^{i*} is maximized, i.e. $\pi_L^{i*} = \min\{1, 1/l_i\}$. This is always feasible since

$$\sum_{i=1}^{m} \pi_H^{i*} = \sum_{i=1}^{m} l_i \pi_L^{i*} \leq \sum_{i=1}^{m} l_i \leq k$$

hence never violating $\sum_i \pi_H^{i*} = k$.

□

Proof of Proposition 3:

The first part is directly from Lemma 2(iv). Note that $\sum_{i=1}^{m} \pi_H^{i*} \leq \sum_{i=1}^{m} l_i < k$ thus there is always some budget for at the very least individual $m+1$. First suppose that $m > 0$ such that $m + k < N$. Now comparing $m+1$ with some $j > m+1$ the constraints are

$$\pi_H^{m+1} = l_{m+1} \pi_L^{m+1}$$
$$\pi_H^{j} = l_j \pi_L^{j}$$

where they bind in equilibrium since otherwise the seller can always lower π_H^i and use it to solicit more contributions from other individuals. Using the budget surplus either π_H^{m+1} can be increased or j. But since $l_{m+1} < l_j$ it is always more efficient to maximize individual $m+1$ before any j. Formally, if $\epsilon > 0$ is used to increase π_H^{m+1} then

$$\pi_H^{m+1} + \epsilon = l_{m+1}\left(\pi_L^{m+1} + \frac{\epsilon}{l_{m+1}}\right)$$

resulting in an increase of ϵ/l_{m+1}. And if ϵ was used towards individual j then contributions increase by ϵ/l_j. But $\epsilon/l_{m+1} > \epsilon/l_j$ hence the optimal policy should maximize π_H^{m+1} before any resources go into $j > m+1$. Either the seller has enough of a budget to maximize $\pi_H^{m+1} = 1$, or she allocates the entire budget:

$$\pi_H^{(m+1)*} = \min\left\{1, k - \sum_{i=1}^{m} \pi_H^{i*}\right\}.$$

If $\pi_H^{(m+1)*} = 1$ and there still remains a budget ($\sum_{i=1}^{m+1} \pi_H^{i*} < k$) then using the same argument proceed to allocate as much as possible to individual $m+2$. This process continues until the budget is exhausted. Note that $\pi_H^{j*} = 0$ for $j > m+k$ since otherwise that would imply at least that $\pi_H^{(m+k+1)*} > 0$ hence $\pi_H^{(m+k)*} = 1$ but then

$$\sum_{i=1}^{m+k} \pi_H^{i*} = \sum_{i=1}^{m} \pi_H^{i*} + \sum_{i=m+1}^{m+k} \pi_H^{i*} = \sum_{i=1}^{m} l_i + \sum_{i=m+1}^{m+k} 1 = \sum_{i=1}^{m} l_i + k > k$$

exceeding the budget of k proving parts 2 and 3. The last part also comes immediate from using the same argument from the beginning: $\pi_H^{i*} \geq \pi_H^{j*}$ for $i < j$ and the seller proceeds to maximize the first k easiest to persuade individuals.

□

Appendix B (Examples to Chapter 2)

Example 2:

The first part simply only ever recommends all N individuals simultaneously and clearly it respects each person's marginal $\pi_L^{i*} = ((1-c)/c)k/N$. As for the second part, let $c_1^* = c^*$. The seller can maximize investments by initially only putting weight on p_{NL} and $p_{(N-1)L}$ but as c increases she no longer can support p_{NL} in her policy thus puts weight on $p_{(N-1)L}$ and $p_{(N-2)L}$. More precisely, for $c_1^* < c \leq c_2^*$ the policy is pinned by

$$\frac{1-c}{c}\frac{k}{N} = p_{NL} + (N-1)p_{(N-1)L}$$
$$p_{NL} + Np_{(N-1)L} = 1$$

therefore the optimal policy for $c_1^* < c \leq c_2^*$ is

$$\begin{aligned} \pi_L^*(S^N) = p_{NL}^* &= \frac{k-(N+k-1)c}{c} \\ \pi_H^*(S^k) = 1 - \pi_L^*(S^N) = Np_{(N-1)L}^* &= \frac{(N+k)c-k}{c} \\ \text{where } c_2^* &= \frac{k}{N+k-1} \end{aligned} \tag{4.14}$$

and in general it's $c_j^* < c \leq c_{j+1}^*$ for $j = 1, \ldots, N$ where the seller is choosing the probabilities of $p_{(N-(j-1))L}$ and $p_{(N-j)L}$:

$$\frac{1-c}{c}\frac{k}{N} = \binom{N-1}{N-j}p_{(N-j+1)L} + \binom{N-1}{N-j-1}p_{(N-j)L}$$
$$\binom{N}{N-j+1}p_{(N-j+1)L} + \binom{N}{N-j}p_{(N-j)L} = 1$$

reducing to

$$\begin{aligned}\pi_L^*(S^{N-j+1}) &= \binom{N}{N-j+1} p_{(N-j+1)L}^* = \frac{k-c(N+k-j)}{c} \\ \pi_L^*(S^{N-j}) = 1-\pi_L^*(S^{N-j+1}) &= \binom{N}{N-j} p_{(N-j)L}^* = \frac{c(N+k-j+1)-k}{c}\end{aligned} \tag{4.15}$$

where the thresholds are

$$\begin{aligned}\binom{N}{N-j+1} p_{(N-j+1)L}^* = 1 &\implies c_j^* = \frac{k}{N+k-j+1} \\ \binom{N}{N-j} p_{(N-j)L}^* = 1 &\implies c_{j+1}^* = \frac{k}{N+k-j}\end{aligned} \tag{4.16}$$

□

Appendix C (Proofs to Chapter 3)

Recall the expected revenue function is

$$\pi_i(c_i, c_j; k_i, k_j, \delta) = \frac{\frac{1+\delta}{2} p_i(c_i, c_j) + \frac{\delta^2}{2(1-\delta)} p_i(c_i, c_j)^2}{1 - 2\delta^2 p_i(c_i, c_j)[1 - p_i(c_i, c_j)]}.$$

The following lemma describes the concavity of the expected revenue function which will be important for the proofs below.

Lemma 3. *Fix $k_i = 0$. For $c_j + k_j > 0$, there exists a $\delta^* = \sqrt{1 - \sqrt{2}/2}$ such that $\pi_i(c_i, \cdot)$ is strictly concave for $\delta \leq \delta^*$ and it begins strictly convex then becomes strictly concave for $\delta > \delta^*$. δ^* increases as k_i increases.*

The lemma reduces the problem for solving the Nash equilibrium investment quantities to simply comparing the corner solution ($c_i = 0$) with the highest quantity satisfying the FOC (see Figure 4.1).

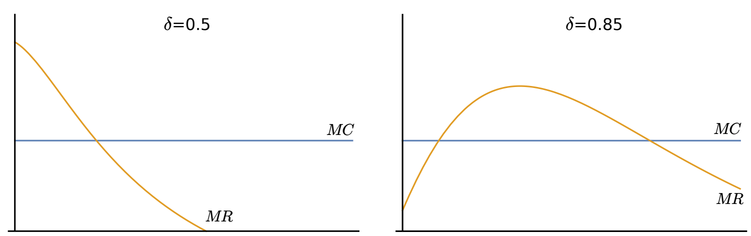


Figure 4.1: Graphs of marginal revenue and marginal cost for $\delta = 0.5$ and $\delta = 0.85$.

It will be useful to define

$$\Pi_i(c_i, c_j; k_i, k_j, \delta) = \pi_i(c_i, c_j; k_i, k_j, \delta) - c_i - \pi_i(0, c_j; k_i, k_j, \delta)$$

so that the solution to firm i's optimization, c_i^*, is a best-response to c_j when $\Pi_i(c_i^*, c_j; k_i, k_j, \delta) \geq 0$. That is, firm i finds the benefit from reducing costs by deviating to $c_i = 0$ and only depending on k_i to be outweighed by the loss of having a lower expected revenue from the lowered investment.

Proof. The second derivative of $\pi_i(c_i, \cdot)$ with respect to c_i is

$$\frac{\tilde{c}_j[-(1-\delta^2)\tilde{c}_i^3 - 3\tilde{c}_j\tilde{c}_i^2 - 3(1-\delta^2)\tilde{c}_j^2\tilde{c}_i - (1-4\delta^2+2\delta^4)\tilde{c}_j^3]}{(1-\delta)[\tilde{c}_i^2 + 2(1-\delta^2)\tilde{c}_j\tilde{c}_i + \tilde{c}_j^2]^3} \tag{4.17}$$

where $\tilde{c}_l = c_l + k_l$. The denominator is strictly positive for all $c_i, c_j > 0$ and $0 < \delta < 1$. Note that for $c_i = 0$ and $c_j > 0$, the only way for the above equation to equal 0 is if $(1 - 4\delta^2 + 2\delta^4) = 0$.

From the quadratic formula,

$$\delta = \pm\sqrt{1 \pm \frac{\sqrt{2}}{2}}$$

where the only feasible root is $\delta^* = \sqrt{1 - \sqrt{2}/2}$. It's clear from Figure 4.2 that $(1 - 4\delta^2 + 2\delta^4) \geq 0$ for $\delta \leq \delta^*$. Thus for all $c_i > 0$ the numerator is negative for these values of δ proving the strict concavity of the expected revenue function. And when $\delta > \delta^*$, $(1 - 4\delta^2 + 2\delta^4) < 0$. But for a fixed $c_j > 0$, the numerator is monotonically decreasing in c_i. Thus the numerator will eventually switch signs from positive to negative when c_i increases high enough.

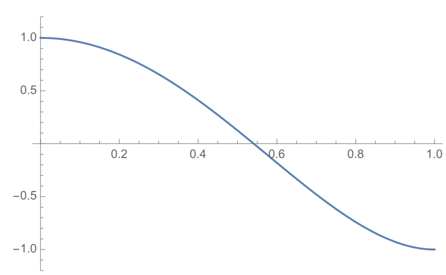


Figure 4.2: Graph of $f(\delta) = 1 - 4\delta^2 + 2\delta^4$.

□

Proof of Proposition 1:

Firm j's optimization problem is

$$\max_{c_j \geq 0} \Pi(c_j, c_i; k_j, k_i, \delta)$$

with FOC

$$\frac{(c_i + k_i)[(1 + \delta - 2\delta^2)(c_j + k_j)^2 + 2(c_i + k_i)(c_j + k_j) + (1 - \delta)(c_i + k_i)^2)]}{2(1 - \delta)[(c_j + k_j)^2 + 2(1 - \delta^2)(c_i + k_i)(c_j + k_j) + (c_i + k_i)^2]^2} = 1 \tag{4.18}$$

where the marginal revenue equals the marginal cost. Here firms are symmetric for all $k_i, k_j < k^*$. To see this, define $\tilde{c}_l = c_l + k_l$. Then the optimization problem $\max_{\tilde{c}_j \geq 0} \Pi_i(\tilde{c}_j, \tilde{c}_i; k_j, k_i, \delta) + k_j$ is symmetric and will yield identical FOC and value functions of the original optimization problem. Thus by symmetry, $\tilde{c}_i = \tilde{c}_j = k^*$. Plugging this into the FOC gives $k^* = 1/[4(1 - \delta)(2 - \delta^2)]$. The non-negativity constraints do not bind when $k_j \leq k_i \leq k^*$.

i. To verify that $c_l^* = k^* - k_l$ is a Nash equilibrium, it must be checked that $\Pi_j(c_j^*, c_i^*; k_j, k_i, \delta) \geq 0$ and $\Pi_i(c_i^*, c_j^*; k_i, k_j, \delta) \geq 0$ for all $k_j \leq k_i \leq k^*$ and for all $0 < \delta < 1$.

$$\begin{aligned}\Pi_j &= \pi_j(c_j^*, c_i^*; k_j, k_i, \delta) - c_j^* - \pi_j(0, c_i^*; k_j, k_i, \delta) \\ &= \frac{(1 + \delta + 4(2 - \delta^2)k_j)(1 - k_j/k^*)^2}{4(2 - \delta^2)[1 + 8(1 - \delta)^2(2 + 2\delta - \delta^2 - \delta^3)k_j + 16(2 - 2\delta - \delta^2 + \delta^3)^2 k_j^2]}\end{aligned}$$

where it's equal to 0 for $k_j = k^*$ and positive for all $0 \leq k_j < k^*$ and for all $0 < \delta < 1$ since each of the terms is positive. By symmetry, $\Pi_i(c_i^*, c_j^*; \cdot)$ is also non-negative.

ii. First fix $k_j = 0$. Define $\hat{k}$ to be the largest value that solves firm j's FOC equation given $c_i^* = 0$ and $k_i \geq k^*$. Thus $\hat{k}$ is implicitly defined as the highest positive root of a quartic polynomial. As shown previously in part (i), $\hat{k} = k^*$ when $k_i = k^*$. However, as k_i increases, the marginal revenue from investing in c_j is decreasing (when $c_j < k^*$). To see this, the derivative of the marginal revenue of c_j with respect to k_i is

$$\frac{(1-\delta^2)[c_j^4 - k_i^4] + 2(1+2\delta^2-\delta^4)[k_i c_j^3 - k_i^3 c_j]}{2(1-\delta)[c_j^2 + 2(1-\delta^2)k_i c_j + k_i^2]^3}.$$

The derivative is negative if and only if $k_i > c_j$. Since it's equal to 0 when $c_i = k_i = k^*$, as k_i increases it must be that $\hat{k}$ is decreasing. Moreover, because the roots of a polynomial depend continuously on its coefficients, $\hat{k}$ continuously decreases in k_i. And as k_i increases, the additional profits from $c_j^* = \hat{k}$ shrinks until it's no longer profitable to invest ($c_j^* = 0$). Define such a k_i to be k^{**}. That is, $k_i = k^{**}$ implies $\Pi_j(c_j^* = \hat{k}, c_i^* = 0, \cdot) = 0$ therefore $c_j^* = 0$ for all $k_i \geq k^{**}$.

Now let $k_j > 0$. The only thing that changes is that $\hat{k}$ is the largest positive root for $\tilde{c}_j = c_j + k_j$ (the function simply shifts left). Therefore $c_j^* = \hat{k} - k_j$. Again, k_i decreases $\hat{k}$ hence decreases c_j^* until $\Pi_j(c_j^* = \hat{k} - k_i, 0, \cdot) = 0$. k^{**} varies depending on the value of k_j.

It remains to verify that firm i's best-response is $c_i^* = 0$ for all $k^* < k_i < k^{**}$. From part (i), i's best-response is $c_i^* = 0$ when $k_i = k_j = k^*$ where the marginal revenue equals the marginal cost only when $c_i = 0$ and is below the marginal cost for all $c_i > 0$. It suffices to show that the marginal revenue curve is lower for all $c_i \geq 0$ when k_i increases or when k_j decreases. First fix $k_j = k^*$. Since k_i increasing is equivalent to the marginal revenue curve shifting left, $c_i = 0$ remains a best-response for all $k^* < k_i$. Now fix $k_i = k^*$. The derivative of the marginal revenue with respect to k_j is given by equation (4.17) by simply setting $\tilde{c}_i = k^*$ and $\tilde{c}_j = \hat{k}$. The numerator is negative for all $0 \leq k_j < k_i$ and $0 < \delta < 1$:

$$\begin{aligned}
\text{numerator} &= \hat{k}[-(1-\delta^2)k_i^3 - 3\hat{k}k_i^2 - 3(1-\delta^2)\hat{k}^2 k_i - (1 - 4\delta^2 + 2\delta^4)\hat{k}^3] \\
&\leq \hat{k}[-(1-\delta^2)\hat{k}^3 - 3\hat{k}^3 - 3(1-\delta^2)\hat{k}^3 - (1 - 4\delta^2 + 2\delta^4)\hat{k}^3] \\
&= \hat{k}^4[-8 + 7\delta^2 - 2\delta^4] \\
&< 0
\end{aligned}$$

since $\hat{k} \leq k_i$. Thus the marginal revenue curve is lowered (resulting in marginal cost always exceeding the revenue). Therefore i's best-response is $c_i = 0$ for all $k_i \geq k^*$.

iii. This part follows immediately since $c_i^* = c_j^* = 0$ is optimal when $k_i = k^{**}$ and any higher k_i simply lowers the incentives for both firms to invest therefore it remains that $c_i^* = c_j^* = 0$. Same reasoning when $k_j > \hat{k}$.

iv. See part (ii) for $\hat{k}$ decreasing in k_i. For $\delta \leq \delta^*$, the expected revenue function is strictly concave hence as k_j increases the marginal revenue of c_j is lower for all $c_j \geq 0$. Thus k_i doesn't need to increase as much to vanish j's additional profits. And when $\delta > \delta^*$, as k_j initially increases (from 0) the marginal revenue from $c_j = 0$ is increasing (due to the initial convexity of the expected profits function). Therefore k^{**} must increase because there are more additional profits that i can extract. However, as k_j continues to increase eventually j's additional profits from investing will begin to decrease until it becomes 0. The analysis here is then identical to the case of $\delta \leq \delta^*$. Refer to Figure 4.1 where k_j increasing simply shifts j's marginal revenue curves to the left.

$\square$

Proof of Proposition 2

The transition matrix of the Markov chain in the canonical form is

$$\boldsymbol{P} = \begin{array}{c} \\ A \\ R \\ B \\ W_A \\ W_B \end{array} \begin{array}{c} \begin{array}{ccc|cc} A & R & B & W_A & W_B \end{array} \\ \left(\begin{array}{ccc|cc} 0 & p_B & 0 & p_A & 0 \\ p_A & 0 & p_B & 0 & 0 \\ 0 & p_A & 0 & 0 & p_B \\ \hline 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{array}\right) \end{array} = \left(\begin{array}{c|c} \boldsymbol{Q} & \boldsymbol{R} \\ \hline \boldsymbol{0} & \boldsymbol{I} \end{array}\right)$$

where $\boldsymbol{Q}$ is the transition matrix for the transient states and $\boldsymbol{R}$ for the absorbing states. The probability of a hitting an absorbing state from any state is given by the following matrix

$$(\boldsymbol{I} - \boldsymbol{Q})^{-1}\boldsymbol{R} = \frac{1}{1 - 2p_A p_B} \cdot \begin{array}{c} \\ A \\ R \\ B \end{array} \begin{array}{c} \begin{array}{cc} W_A & W_B \end{array} \\ \left(\begin{array}{cc} p_A(1 - p_A p_B) & p_B^3 \\ p_A^2 & p_B^2 \\ p_A^3 & p_B(1 - p_A p_B) \end{array}\right) \end{array}$$

Let $h_i(\omega)$ be the probability of reaching state W_i starting from state ω. Then $h_i(R) = p_i^2/(1 - 2p_i p_j)$.

Let $\boldsymbol{c}$ be a column vector of 1's. Then the expected time to get absorbed is

$$(\boldsymbol{I}-\boldsymbol{Q})^{-1}\boldsymbol{c} = \frac{1}{1-2p_Ap_B} \cdot \begin{matrix} A \\ R \\ B \end{matrix} \begin{pmatrix} p_B^2 + p_B + 1 - p_Ap_B \\ p_A + p_B + 1 \\ p_A + p_A^2 + 1 - p_Ap_B \end{pmatrix}$$

i. The probability of consumers herding to the high-quality good is $h_i(R) = p_i^2/(1-2p_i(1-p_i))$. This is monotonically increasing in $p_i \in [0,1]$ thus it's increasing for $k_i \geq k^*$ and weakly decreasing in k_j.

ii. The expected time until absorption is $2/(1-2p_i(1-p_i))$ with the maximum achieved at $p_i = 1/2$ and symmetrically decreases as p_i moves farther away from 1/2. Therefore for $k_i > k^*$, the expected time of being absorbed is monotonically decreasing until $k_i = k^{**}$.

iii. k^* is strictly increasing in δ. Therefore if $k_j \leq k_i \leq k^*$ then it remains $c_i^* + k_i = c_j^* + k_j = k^*$ and $p_H^* = 1/2$. Also, $\hat{k}$ is increasing in δ since the marginal revenue increases with δ. That is,

$$\frac{\partial^2 \pi_j(\cdot)}{\partial c_j \partial \delta} = \frac{\tilde{c}_i(m_1 + m_2 + m_3 + m_4 + m_5)}{2(1-\delta)^2(\tilde{c}_j^2 + 2(1-\delta^2)\tilde{c}_i\tilde{c}_j + \tilde{c}_i^2)^3} > 0$$

$$m_1 = (1-\delta)^2\tilde{c}_j^4$$

$$m_2 = 2(2 + 2\delta - 4\delta^2 - 2\delta^3 + 3\delta^4)$$

$$m_3 = 6(1 + 2\delta - 3\delta^2)\tilde{c}_i^2\tilde{c}_j^2$$

$$m_4 = 2(2 + 2\delta - 4\delta^2 - 2\delta^3 + 3\delta^4)\tilde{c}_i^3\tilde{c}_j$$

$$m_5 = (1-\delta)^2\tilde{c}_i^4$$

where it's positive since $m_l > 0$ for all l for all $0 \leq k_j < k_i$ and $0 < \delta < 1$. For $k^* < k_i < k^{**}$, c_j^* remains at 0 until $\hat{k}$ increases past k_j then $c_j^* = \hat{k} - k_j$ thus c_j^* is weakly increasing in δ. And similarly c_i^* is weakly increasing in δ as it becomes positive once k^* increases past k_i. Finally, k^{**} must increases with δ since otherwise k^* would eventually increase high enough such that $k^* = k^{**}$. But then $\Pi_j(c_j^* = k^* - k_j, \cdot) > 0$ for all δ thus it must be that k^{**} is higher.

It then follows that p_H^* is weakly decreasing in δ.

□

Monopoly case

Throughout the entire proof it's assumed that $0 < \delta < 1$ and $0 < k_j < k^*$ and $0 < \tilde{c}_i < k^*$.

i. For type i, c_i^* initially increases as k_j increases then decreases. The point where $\partial c_i^* / \partial k_j$ changes sign is $k^* = 1/[4(1-\delta)(2-\delta^2)]$. The head starts still act as perfect substitutes to investment; however, now they also affect the incentives of the other type. The case of $k_j \geq k^*$ is identical to the competitive case, so it remains to look only at $0 < k_i, k_j < k^*$ for the remainder of the proof. Again define $\tilde{c}_l = c_l^* + k_l$ and assume that $k_i > k_j$. Then $\tilde{c}_i > \tilde{c}_j$ (and hence $c_i^* > c_j^*$). To see this, suppose the opposite. From (4.17), δ must be less than δ^* in order for the solutions to be a maximizer where (assuming $\tilde{c}_j > \tilde{c}_i$) δ^* is

$$\delta^* = \frac{1}{2}\sqrt{4 + \frac{\tilde{c}_i^3}{\tilde{c}_j^3} + \frac{3\tilde{c}_i}{\tilde{c}_j} + \frac{\tilde{c}_i^2\sqrt{\tilde{c}_i^2 + 8\tilde{c}_j}}{\tilde{c}_j^3} - \frac{\sqrt{\tilde{c}_i^2 + 8\tilde{c}_j^2}}{\tilde{c}_j}}$$

where it must also be that $0 < \delta^* < 1$ (note to write out the proof). Thus

$\delta^* < 1$ implies that

$$0 > \tilde{c}_i^3 + 3\tilde{c}_i\tilde{c}_j^2 + (\tilde{c}_j^2 - \tilde{c}_i^2)\sqrt{\tilde{c}_i^2 + 8\tilde{c}_j}$$

implying that $\tilde{c}_i > \tilde{c}_j$ which is a contradiction. Thus $0 < c_j^* < c_i^*$

ii. Let $N = (1-\delta^2)\tilde{c}_i^2 + 2\tilde{c}_i\tilde{c}_j + (1-\delta^2)\tilde{c}_j^2$ and $M = \tilde{c}_i^2 + 2(1-\delta^2)\tilde{c}_i\tilde{c}_j + \tilde{c}_j^2$, then the FOC for i is

$$\frac{\tilde{c}_j N}{2(1-\delta)M^2} = 1.$$

Taking the derivative w.r.t. $\tilde{c}_j$:

$$\frac{\partial \tilde{c}_i}{\partial k_j} = \frac{(1-\delta^2)\tilde{c}_i^2 + 4k_j\tilde{c}_i + 3(1-\delta^2)k_j^2 - 8(1-\delta)M[(1-\delta^2)\tilde{c}_i + k_j]}{8(1-\delta)M[\tilde{c}_i + (1-\delta^2)k_j] - 2k_j[(1-\delta^2)\tilde{c}_i + k_j]}$$

Both the numerator and denominator are positive for all $0 < \delta < 1$ and for all $0 < k_j < \tilde{c}_i < k^*$ hence $\tilde{c}_i$ is increasing in $0 < k_j < k^*$.

Numerator is positive:

Suppose not. Then $(1-\delta^2)\tilde{c}_i^2 + 4k_j\tilde{c}_i + 3(1-\delta^2)k_j^2 < 8(1-\delta)M[(1-\delta^2)\tilde{c}_i + k_j]$. Plugging in the FOC in the RHS the inequality becomes

$$(1-\delta^2)\tilde{c}_i^2 + 4\tilde{c}_i k_j + 3(1-\delta^2)k_j^2 < \frac{4k_j N}{M}[(1-\delta^2)\tilde{c}_i + k_j].$$

Using the fact that $N/M < 1$ for all $0 < k_j < \tilde{c}_i$:

$$\begin{aligned}(1-\delta^2)\tilde{c}_i^2 &< k_j^2 + 3\delta^2 k_j^2 - 4\delta^2 k_j\tilde{c}_i \\ &< k_j^2 - \delta^2 k_j^2 \\ &= (1-\delta^2)k_j^2\end{aligned}$$

but this implies that $\tilde{c}_i < k_j$ which is a contradiction.

Denominator is positive:

Suppose not. Then $4(1-\delta)M[\tilde{c}_i + (1-\delta^2)k_j] < k_j[(1-\delta^2)\tilde{c}_i + k_j]$. Substituting the FOC in the LHS the inequality becomes

$$2N[\tilde{c}_i + (1-\delta^2)k_j] < M[(1-\delta^2)\tilde{c}_i + k_j].$$

This simplifies to

$$\begin{aligned} RHS - LHS &= -(1-\delta^2)\tilde{c}_i^3 - 4(4-2\delta^2+\delta^4)k_j\tilde{c}_i^2 - 4(1-\delta^2)k_j^2\tilde{c}_i \\ &\qquad - (1-4\delta^2+2\delta^4)k_j^3 \\ &< -(1-\delta^2)\tilde{c}_i^3 - 4(1-\delta^2)k_j^2\tilde{c}_i - (5-6\delta^2+3\delta^4)k_j^3 \\ &< 0 \end{aligned}$$

contradicting RHS > LHS.

iii. WTS

$$\underbrace{\frac{1}{2}\frac{c_H^* + k_H}{c_H^* + k_H + k_L} + \frac{1}{2}\frac{k_H}{c_L^* + k_L + k_H}}_{\text{prob. high-type wins}} \geq \underbrace{\frac{1}{2}\frac{c_L^* + k_L}{c_L^* + k_L + k_H} + \frac{1}{2}\frac{k_L}{c_H^* + k_H + k_L}}_{\text{prob. low-type wins}}.$$

Fix $k_H > k_L$. If $k_L = 0$ then $1 + k_H/(c_L^* + k_H) > c_L^*/(c_L^* + k_H)$. And when $k_L = k_H$ the above expression holds with equality. It suffices to show that

$$\frac{\partial}{\partial k_L}\left(\frac{c_H^* + k_H}{c_H^* + k_H + k_L}\right) < 0 \iff \frac{\partial c_H^*}{\partial k_L} < \frac{c_H^*}{k_L} \quad \text{for } 0 < k_L < k_H$$

to prove the above inequality. Substituting the FOC, $8(1-\delta)M = 4k_L N/M$, into $\partial c_H^*/\partial k_L$:

$$\frac{\partial c_H^*}{\partial k_L} = \frac{(1-\delta^2)c_H^* + 4k_L c_H^* + 3(1-\delta^2)k_L^2 - \frac{4k_L N}{M}[(1-\delta^2)c_H^* + k_L]}{\frac{4k_L N}{M}[c_H^* + (1-\delta^2)k_L] - 2k_L[(1-\delta^2)c_H^* + k_L]} \overset{?}{<} \frac{c_H^*}{k_L}$$

then cross multiplying

$$\begin{aligned}(1-\delta^2)(c_H^*)^2 + 4k_L c_H^* + 3(1-\delta^2)k_L^2 + 2c_H^*[(1-\delta^2)c_H^* + k_L]& \\ &< \frac{4N}{M}\Big([c_H^* + (1-\delta^2)k_L]c_H^* + [(1-\delta^2)c_H^* + k_L]k_L\Big) \\ &= \frac{4N}{M}M \\ &= 4[(1-\delta^2)(c_H^*)^2 + 2k_L c_H^* + (1-\delta^2)k_L^2]\end{aligned}$$

thus proving $\partial c_H^*/\partial k_L < c_H^*/k_L$.

Dynamic Model

Proof of Proposition 4:

Fix $\delta > \underline{\delta}_k$. By definition $\delta > \underline{\delta}_k$ implies that $c_i(\omega) = 1$ for all i and all ω. That is

$$\gamma_i(c, \omega) > 2(1 + 2\kappa)$$

for all i and ω. Consider $c = \{(c_i, c_j)_j, (c_i, c_j)_R, (c_i, c_j)_i\}$ where $(c_i, c_j)_\omega$ denotes $c_i(\omega)$ and $c_j(\omega)$ and define $\mathbf{1}_\omega := (1,1)_\omega$ and $\mathbf{0}_\omega := (0,0)_\omega$. Then $\delta > \underline{\delta}_k \implies c = c_1 := \{\mathbf{1}_i, \mathbf{1}_R, \mathbf{1}_j\}$. When both are investing in each period the expected revenue functions are

$$
\begin{aligned}
\pi_i(c_1, j) &= \frac{-12 + 6\delta + 6\delta^2 + \delta^3}{2M_1} \\
\pi_i(c_1, R) &= \frac{-6 + 7d^2}{M_1} \\
\pi_i(c_1, i) &= \frac{-12 + 10\delta + 6\delta^2 - \delta^3}{2M_1} \\
M_1 &= 2 - 2\delta - \delta^2 + \delta^3
\end{aligned}
$$

and the net gains from winning are

$$
\begin{aligned}
\gamma_i(c_1, j) &= \frac{2 - 14\delta - \delta^2 + 15\delta^3}{2M_1} \\
\gamma_i(c_1, R) &= \frac{2 - 2\delta + 3\delta^2 + \delta^3 - 2\delta^4}{2M_1} \\
\gamma_i(c_1, i) &= \frac{2 + 10\delta - \delta^2 - 13\delta^3}{2M_1} + \frac{\delta}{2(1-\delta)} \\
M_1 &= 2 - 2\delta - \delta^2 + \delta^3
\end{aligned}
$$

where it's straight forward to verify that $\min_\omega \gamma_i(c_1, \omega) = \gamma_i(c_1, j)$. Hence the first to become indifferent of investing is the trailing firm the moment $\kappa = \kappa^*$ where κ^* is given by

$$
\gamma_i(c_1, j) = 2(1 + 2\kappa^*) \implies \kappa^* = \frac{-12 + 6\delta + 6\delta^2 + \delta^3}{16M_1}.
$$

The trailing firm now begins to mix with probability $\beta_i(j)$ of investing. The leading firm remains investing provided

$$
\begin{aligned}
&\beta_i(j)\left[\frac{\gamma_j(c, j)}{2}\right] + (1 - \beta_i(j))\left[\frac{1+\kappa}{1+2\kappa}\gamma_j(c, j)\right] + \delta\pi_j(R) - 1 \\
&\qquad > \beta_i(j)\left[\frac{\kappa}{1+2\kappa}\gamma_j(c, j)\right] + (1 - \beta_i(j))\frac{\gamma_j(c, j)}{2} + \delta\pi_j(R)
\end{aligned}
$$

$$\gamma_j(c,j) > \frac{2(1+2\kappa)}{1+2\beta_i(j)} \text{ for } \kappa \geq \kappa^*$$
$$\iff \beta_i(j) > \frac{2(1+2\kappa)-\gamma_j(c,j)}{2\gamma_j(c,j)} \text{ for } \kappa \geq \kappa^*$$

which of course holds true when $c = c_1$ and $\kappa = \kappa^*$ since the RHS is negative. As κ increases beyond κ^*, $\beta_i(j)$ will be pinned down by the indifference condition

$$\gamma_i(c_\beta, j) = 2(1+2\kappa)$$
$$c_\beta := \{(\beta_i(j),1)_j, \mathbf{1}_R, (1,\beta_j(i))_i\}.$$

More precisely, the value functions are

$$\pi_i(c_\beta, j) = \frac{p_2[10\delta - 4 + 2\delta^2(2p_2 - 2 + \beta_i(j)(1-2p_2)) - \delta^3(4 + 2p_2 - \beta_i(j)(1-2p_2)]}{M_2}$$
$$\pi_i(c_\beta, R) = \frac{6 - \delta(4 - \beta_i(j)(1-2p_2)) - 2\delta^2(2-p_2)}{M_2}$$

$$\pi_i(c_\beta, i) = \frac{4(1+p_2) + 2\beta_i(j)(1-2p_2) - \delta(8 - 6p_2(1-\beta_i(j)) - 3\beta_i(j)) - +3\beta_i(j))}{M_2}$$
$$- \frac{\delta^2(4p_2(2-\beta_i(j)) + 4p_2^2(1-\beta_i(j))}{M_2}$$
$$+ \frac{\delta^3 p_2(4 + 2p_2(1-\beta_i(j)) + \beta_i(j))}{M_2}$$

$$M_2 = -2(1-\delta)[4 - \delta^2(2p_2(2-\beta_i(j)) + \beta_i(j))]$$
$$p_2 = \frac{\kappa}{1+2\kappa}$$

hence

$$\beta_i(j;\kappa) = \frac{2(6+28\kappa+32\kappa^2-\delta(3+22\kappa+32\kappa^2)}{\delta^2(2-3\delta+8(1-\delta)\kappa)}$$
$$\beta_i(j;\kappa) = \frac{-\delta^2(2+10\kappa+16\kappa^2)+\delta^3(-2+3\kappa+16\kappa^2))}{\delta^2(2-3\delta+8(1-\delta)\kappa)}$$

where $\beta_i(j) = 1$ when for $\kappa = \kappa^*$ and decreasing with $\kappa > \kappa^*$. First note that the leading firm would not stop investing as long as $\beta_i(j) \geq 0$. To see this suppose for the sake of contradiction that $\gamma_i(c_\beta, R) \geq \gamma_i(c_\beta, i) = \gamma_i(c_\beta, j)$. But $\gamma_i(c_\beta, i) = \gamma_i(c_\beta, j) \implies \pi_i(c_\beta, R) = 1/(4(1-\delta))$ which is a contradiction since $\pi_i(c, R) < 1/(4(1-\delta))$ (with equality only when $c = c_1$ with costless investing). Therefore as κ increases either $\beta_i(j) = 0$ or $\gamma_i(c_\beta, R) = \gamma_i(c_\beta, j) = 2(1+2\kappa)$. However the latter implies $\pi_i(c_\beta, R) = \pi_i(c_\beta, i) - \pi_i(c_\beta, j)$ but this is a contradiction since $\pi_i(c_\beta, R) < \pi_i(c_\beta, i) - \pi_i(c_\beta, j)$. Thus

$$\beta_i(j;\kappa^{**}) = 0$$
$$\kappa^{**} = \frac{-28+22\delta+10\delta^2-3\delta^3+\sqrt{16-80\delta+180\delta^2+32\delta^3-224\delta^4-60\delta^5+137\delta^6}}{32(2-2\delta-\delta^2+\delta^3)}$$

As κ continues to increase past κ^{**}, the next to go inactive are the firms in the reset state.

$$\gamma_i(c_2, R) = 2(1+2\kappa^{***})$$
$$c_2 = \{(1,0)_i, \mathbf{1}_R, (0,1)_j\}$$
$$\implies \kappa^{***} = \frac{-14+10\delta+5\delta^2+\sqrt{4+40\delta-40\delta^2-28\delta^3+25\delta^4}}{32(1-\delta)}.$$

Lastly, eventually the leading firm will go inactive for $\kappa \geq \kappa^{****}$:

$$\gamma_i(c_{\mathbf{0}}, i) := 2(1 + 2\kappa^{****})$$

$$c_{\mathbf{0}} = \{\mathbf{0}_j, \mathbf{0}_R, \mathbf{0}_i\}$$

$$\kappa^{****} = \frac{-28 + 30\delta + 10\delta^2 - 11\delta^3 + \sqrt{16 - 16\delta + 84\delta^2 - 128\delta^3 + 16\delta^4 + 36\delta^5 - 7\delta^6}}{32(2 - 2\delta - \delta^2 + \delta^3}$$

Define $\tilde{k} := \max\{0, k\}$. Then for any $0 < \delta \leq \delta_{\overline{k}}$ the above argument follows through for $0 \leq \tilde{\kappa}^* \leq \tilde{\kappa}^{**} \leq \tilde{\kappa}^{***} \leq \tilde{\kappa}^{****}$ since all the proofs use $0 < \delta < 1$. As δ increases each of the thresholds decreases. □

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