Minimal Resolutions of Monomial Ideals

by

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Dissertation submitted in partial fulfillment of the
requirements for the degree of Doctor of Philosophy
in the Department of Mathematics
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ABSTRACT

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Abstract

It has been an open problem since the early 1960s to construct free resolutions of monomial ideals. Beginning with the 1966 Taylor resolution, the first resolution for arbitrary monomial ideals, there have since been many constructions of free resolutions of monomial ideals, satisfying some of the following properties: canonical, minimal, universal, closed form, and combinatorial. The goal is for such a construction to satisfy all of the desired properties. The constructions given so far each satisfy some subset of these properties. This dissertation gives a full solution to the problem, satisfying all of the desired properties, over a field of characteristic 0 and most positive characteristics, with these positive characteristics depending on the ideal. The differential is a weighted sum over lattice paths in $\mathbb{Z}^n$ that come from analogues of spanning trees in simplicial complexes that are indexed by the lattice. Over a field of any characteristic, noncanonical sylvan resolutions are defined. The noncanonical resolutions are minimal, universal, closed form, and combinatorial. The differentials sum over choices of these generalized spanning trees at points along the lattice paths. Finally, a combinatorial description of the canonical three-variable case is given, and noncanonical sylvan resolutions are used to produce planar graph resolutions in the three-variable case and a minimal resolution of the Stanley–Reisner ideal of the min-

imal triangulation of $\mathbb{RP}^2$ in characteristic 2. Substantial portions of the results are based on joint work with John Eagon and Ezra Miller.

Contents

List of Tables

List of Figures

Acknowledgements

First, I would like to thank my advisor, Ezra Miller, for his advice and mentoring, for making time to meet with me any time I needed, and for letting me join such an exciting research project.

Next, I would like to thank Jack Eagon, for the many patient emails, visits to Durham, and encouragement he has given me throughout my graduate career.

I want to thank my husband Sam for supporting me during the last five years, putting his career goals on hold, and moving to Durham with me so that I could get my Ph.D. at Duke. I could not have done it without him.

I would like to thank my mom, my dad, Teddie, C.J., and Mickey for supporting my education, letting me move across the country, and for giving me a much-needed break when I visited home.

Next, I would like to thank my math teachers and professors, especially my undergraduate research advisors, Kendra Killpatrick and David Strong, for supporting my math education and encouraging my plans to go to graduate school, and Kevin Iga, for teaching me topology in the summer after I graduated.

I want to thank Paul Aspinwall, Leslie Saper, and Patricia Hersh for serving on my committee and for their flexibility during the coronavirus outbreak.

Lastly, I want to thank my classmates at Duke, for helping me get through my courses, for giving advice, and for being great friends.

Chapter 1

Introduction

It has been an open problem since the early 1960s to construct free resolutions of monomial ideals. It is believed that the problem was first posed by Irving Kaplansky, who kept lists of problems for his students at the University of Chicago. One of his students, Diana Taylor, produced the first resolution for arbitrary monomial ideals in her 1966 Ph.D. thesis [Tay66]. The Taylor resolution has a differential induced by the boundary map of the simplex whose faces are indexed by the least common multiples of subsets of the generators of the monomial ideal.

Ideally, the resolution would be universal (meaning it works for arbitrary monomial ideals), canonical (meaning there are no choices involved), minimal (meaning there is no redundancy), combinatorial, and closed form (meaning there is an explicit formula for the differential). It is not clear which of these properties, if any, Kaplansky explicitly listed. The requirement for the resolution to be explicit and minimal is stated at least as far back as 1982 ([CEP82], see the Open Problem on page 29), and many of the partial solutions have combinatorial descriptions. The Taylor resolution is universal, canonical, combinatorial, and closed form, but it is usually non-minimal.

Since then, there have been numerous constructions of monomial resolutions satisfying some subset of the desired properties. For example, the hull resolution [BS98] is nonminimal, the Buchberger resolution [OW16] and the Lyubeznik resolution [Lyu88] are noncanonical and nonminimal, and the Scarf [BPS98, MSY00], planar graph [Mil02], and Eliahou-Kervaire resolutions [EK90] are nonuniversal.

Formulas for the ranks of the free modules in a minimal free resolution of I, called the Betti numbers, have been known since a 1977 result of Hochster [Hoc77]: the i^{th} Betti number in degree $\mathbf{b}$ is $\beta_{i,\mathbf{b}}(I) = \dim_{\Bbbk} \widetilde{H}_{i-1}(K^{\mathbf{b}}I; \Bbbk)$, where $K^{\mathbf{b}}I$ is the Koszul simplicial complex $K^{\mathbf{b}}I = \{\text{squarefree } \tau \mid \mathbf{x}^{\mathbf{b}-\tau} \in I\}$. Therefore the i^{th} free module in a minimal free resolution of I can be written $F_i = \bigoplus_{\mathbf{b}\in\mathbb{N}^n} \widetilde{H}_{i-1}(K^{\mathbf{b}}I; \Bbbk) \otimes \Bbbk[\mathbf{x}](-\mathbf{b})$.

What is left is to define explicit differentials between these modules. This can be done by defining a morphism on homology $\widetilde{H}_i K^{\mathbf{b}}I \to \bigoplus_{\mathbf{a}\preceq\mathbf{b}} \widetilde{H}_{i-1} K^{\mathbf{a}}I$ such that the induced maps give a free resolution of I. The resolution is automatically minimal because the free modules have the correct ranks by construction.

The combinatorial description arises from higher-dimensional analogues of spanning trees in the simplicial complexes $K^{\mathbf{b}}I$. The differential sums over all saturated, decreasing lattice paths in $\mathbb{N}^n$ between $\mathbf{b}$ and $\mathbf{a}$ and over all chain-link fences. The chain-link fences are sequences of faces linked to each other via the higher-dimensional spanning trees, along these lattice paths.

1.1 Wall complexes

The Koszul complex of the monomial ideal I can be decomposed into a bicomplex whose vertical, horizontal, and total differentials are Koszul differentials. By taking spectral sequence associated to the filtration by columns of the bicomplex, the first page (given by the vertical homology) has modules $E^1_{p,q} = \bigoplus_{|\mathbf{a}|=p} \widetilde{H}_{p+q-1} K^{\mathbf{a}} I \otimes \Bbbk[\mathbf{x}]$. To produce the differential in a minimal free resolution of I, one must produce maps $\widetilde{H}_i K^{\mathbf{b}} I \otimes \Bbbk[\mathbf{x}](-\mathbf{b}) \to \bigoplus_{\mathbf{a} \preceq \mathbf{b}} \widetilde{H}_{i-1} K^{\mathbf{a}} I \otimes \Bbbk[\mathbf{x}](-\mathbf{a})$. The pages of the spectral sequence only produce maps between subquotients of the appropriate modules.

The Wall complex is introduced in [Eag90] as a way to lift these maps on subquotients to the modules themselves. This is done by selecting a splitting of the vertical differential in the bicomplex. The main theorem of [Eag90] states that the spectral sequence associated to the filtration by columns of the Wall complex is the same as the spectral sequence for the original bicomplex. In discussions and a preprint in the early 1990s, John Eagon and Joel Roberts applied this theorem to the Koszul bicomplex. They found that the Wall complex associated to the Koszul bicomplex gives a free resolution of an associated graded module of I. From joint work with John Eagon and Ezra Miller, Proposition 3.2.8 in this dissertation shows that the Wall complex is a minimal free resolution of I, and not another associated graded module. This result is also shown by Eisenbud, Fløystad, and Schreyer ([EFS03], Lemma 3.5), but none of the authors of [EMO19] were aware of that lemma until they were informed of it in response to the posting of [EMO19] on the arXiv. The paper does not cite [Eag90].

[Eag90] suggests using the Moore–Penrose pseudoinverse as the splitting for the Wall complex, but the idea had not been explored further.

1.2 The Moore–Penrose pseudoinverse

To get a minimal resolution that is canonical, the Moore–Penrose pseudoinverse is used as the splitting in the natural Wall complex, as it allows constructions with no choices. If a combinatorial description of the Moore–Penrose pseudoinverse is available, then following it through the Wall complex construction gives combinatorial differentials for monomial resolutions. Theorem 4.3.2 in this thesis presents a combinatorial formula for the pseudoinverse, as a weighted average of splittings associated to higher-dimensional analogues of spanning trees. Spanning trees (and spanning forests) have been studied extensively by [Kal83, DKM09, DKM11, Lyo09, Pet09]. What is called a shrubbery in this thesis is often referred to as a spanning tree or spanning forest in the literature.

The formula for the pseudoinverse uses weights of orders of torsion subgroups of subcomplexes of the Koszul simplicial complex at appropriate degree vectors. The proof of this formula uses the Higher Projection Formula of Catanzaro, Chernyak, and Klein in [CCK15]. The same formula for the Moore–Penrose pseudoinverse was proven simultaneously and independently by the same authors and published in [CCK17].

This formula is very similar to a formula for the pseudoinverse in [Ber86]. That paper surprisingly has few citations and was unknown to the author of this thesis and the authors of [CCK17] at the time of publication of [CCK17].

1.3 Sylvan resolutions

By using the Moore–Penrose pseudoinverse as the splitting of the vertical differential in the natural Wall complex, a canonical minimal free resolution of an arbitrary monomial ideal is obtained. The combinatorial formula for the Moore–Penrose pseudoinverse gives way to a combinatorial formula for the differential in the canonical sylvan resolution. The main result of this dissertation is Theorem 5.4.5, which gives this combinatorial description of the differential. With this result, the canonical sylvan resolution gives a full solution to Kaplansky's problem that satisfies all of the desired properties.

The formula for the differential in the canonical sylvan resolution simplifies in the case of monomial ideals in three variables. Theorems 6.1.1 and 6.2.16 give formulas for the differential with simplified weights along each lattice path.

The proof of the main theorem, Theorem 5.4.5, uses Lemma 5.3.1, which expresses the Wall differentials as sums over lattice paths in $\mathbb{N}^n$ for an arbitrary splitting of the vertical differential. By selecting a splitting that comes from choices of higher-dimensional analogues of spanning trees, noncanonical sylvan resolutions are obtained. The differentials in these noncanonical minimal resolutions have a simpler combinatorial formula given in Theorem 5.3.2.

It is of interest to know whether existing minimal free resolutions can be expressed as noncanonical sylvan resolutions with some choice of splitting. Theorem 7.1.1 shows that planar graph resolutions are sylvan.

Finally, Section 7.2 gives a sylvan resolution of the Stanley–Reisner ideal of the minimal triangulation of $\mathbb{RP}^2$ in characteristic 2.

Substantial portions of the work presented in this thesis are the result of joint work with John Eagon and Ezra Miller [EMO19].

Chapter 2

Background

This chapter gives the definition of free resolution and cellular resolution, which is a type of free resolution with a combinatorial and topological description. Section 2.2 gives an overview of some prominent past constructions of resolutions of monomial ideals.

2.1 Preliminaries

Let $\Bbbk$ be a field, and let $\mathbf{x}$ represent the variables $x_1, x_2, \ldots, x_n$. The *monomial* $\mathbf{x}^{\mathbf{b}} = x_1^{b_1} x_2^{b_2} \cdots x_n^{b_n} \in \Bbbk[\mathbf{x}]$ is identified with its degree vector $\mathbf{b} = (b_1, b_2, \ldots, b_n) \in \mathbb{N}^n$. A monomial is *squarefree* if each b_i is 0 or 1. The polynomial ring $\Bbbk[\mathbf{x}]$ is $\mathbb{N}^n$-*graded*; it is a direct sum $\Bbbk[\mathbf{x}] = \bigoplus_{\mathbf{b} \in \mathbb{N}^n} \Bbbk\{\mathbf{x}^{\mathbf{b}}\}$ of $\Bbbk$-vector spaces, each generated by $\mathbf{x}^{\mathbf{b}}$.

2.1.1 Free resolutions

Let I be a *monomial ideal* in $\Bbbk[\mathbf{x}]$, meaning it is an ideal generated by monomials. A complex

$$\mathcal{F}. : 0 \leftarrow F_0 \xleftarrow{\phi_1} F_1 \xleftarrow{\phi_2} F_2 \leftarrow \cdots \leftarrow F_{\ell-1} \xleftarrow{\phi_\ell} F_\ell \leftarrow 0$$

of free $\Bbbk[\mathbf{x}]$-modules F_i is *acyclic* if it is exact everywhere except at the 0^{th} stage. It is a *free resolution* of I if it is acyclic and the homology at the 0^{th} stage is isomorphic to I. The resolution is $\mathbb{N}^n$*-graded* if the maps are degree-preserving. If the rank $\beta_{i,\mathbf{b}}$ of $F_i = \bigoplus_{\mathbf{b}\in\mathbb{N}^n} S(-\mathbf{b})^{\beta_{i,\mathbf{b}}}$ in degree $\mathbf{b}$ is minimized, the resolution is *minimal.* In this case, $\beta_{i,\mathbf{b}}$ is the i^{th} *Betti number* of I in degree $\mathbf{b}$.

If bases are selected for the free modules F_i, the differentials in a free resolution of I can be written down using *monomial matrices* [MS05]:

$$\bigoplus_q S(-\mathbf{a}_q) \xleftarrow{\begin{array}{c|ccc|} & \cdots & \mathbf{a}_p & \cdots \\ \vdots & & & \\ \mathbf{a}_q & & \lambda_{pq} & \\ \vdots & & & \end{array}} \bigoplus_p S(-\mathbf{a}_p),$$

where the entry λ_{qp} means that the basis vector of $S(-\mathbf{a}_p)$ maps to $\lambda_{qp} \cdot \mathbf{x}^{\mathbf{a}_p - \mathbf{a}_q}$ times the basis vector of $S(-\mathbf{a}_q)$. One of the problems addressed in this dissertation is how to write down canonical resolutions without selecting bases; see Section 5.1.

The problem of constructing minimal free resolutions of monomial ideals can be understood topologically and combinatorially via simplicial complexes associated to the degree vectors $\mathbf{b} \in \mathbb{N}^n$.

Definition 2.1.1. The *Koszul simplicial complex* of I in degree $\mathbf{b}$ is the set

$$K^{\mathbf{b}}I = \{\text{squarefree } \tau \mid \mathbf{x}^{\mathbf{b}-\tau} \in I\}.$$

Example 2.1.2. The staircase diagram for $I = \langle xz, yz, x^2y, xy^2 \rangle$ is given in Figure 2.1 (see Section 2.2.8 for the definition of staircase diagram). At degree 221 (marked by the red dot in the diagram), one can move back in the xy-, xz-, and yz-directions and still remain on the staircase surface. This means that xy, xz, and yz are the facets of $K^{221}I$. Similarly, the facets of $K^{211}I$ are xy and z. At the degree vector of a generator (marked by black dots in the diagram), one can only move back in the direction of the 0-vector and still remain on the staircase surface. The 0-vector corresponds to the unique face of dimension -1, called the empty face. Thus, for example, $K^{101}I = \{\varnothing\}$, the set containing the empty face.

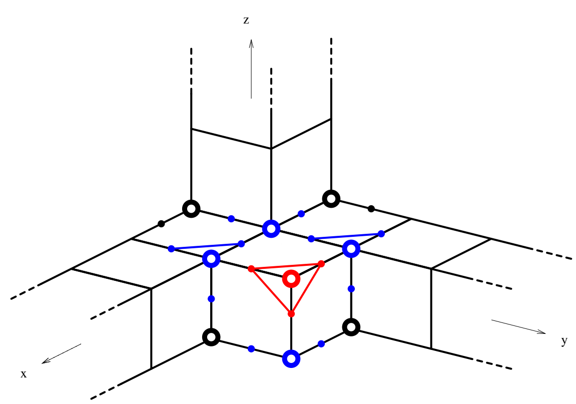


Figure 2.1: Staircase diagram for $I = \langle xz, yz, x^2y, xy^2 \rangle$

The Betti numbers can be computed by taking the homology of the Koszul simplicial complex, by a result of Hochster.

Theorem 2.1.3 (Hochster's formula [Hoc77], see [MS05], Theorem 1.34)**.** *The i^{th} Betti number of I in degree* $\mathbf{b}$ *can be computed*

$$\beta_{i,\mathbf{b}}(I) = \dim_{\Bbbk} \widetilde{H}_{i-1}(K^{\mathbf{b}}I; \Bbbk).$$

Example 2.1.4. Consider again $I = \langle xz, yz, x^2y, xy^2 \rangle$, whose staircase diagram is given in Figure 2.1. The Koszul simplicial complex $K^{221}I$ has facets xy, xz, and yz. By Hochster's formula, $\beta_{2,221}(I) = \dim_{\Bbbk} \widetilde{H}_1(K^{221}I; \Bbbk) = 1$. Similarly, $\beta_{1,211}(I) = \dim_{\Bbbk} \widetilde{H}_0 \ (K^{211}I; \Bbbk) = 1$, and $\beta_{0,101}(I) = \dim_{\Bbbk} \widetilde{H}_{-1}(K^{101}I; \Bbbk) = 1$.

Hochster's formula implies that each free module in a minimal resolution of I is a direct sum, over $\mathbf{b} \in \mathbb{N}^n$, of its component $\widetilde{H}_{i-1}(K^{\mathbf{b}}I; \Bbbk) \otimes \Bbbk[\mathbf{x}](-\mathbf{b})$ generated in degree $\mathbf{b}$, which is naturally spanned by the Koszul homology. What remains is to define explicit maps between these modules. This can be done by defining a map on the reduced chains,

$$\widetilde{C}_i(K^{\mathbf{b}}I; \Bbbk) \to \bigoplus_{\mathbf{a} \in \mathbb{N}^n} \widetilde{C}_{i-1}(K^{\mathbf{a}}I; \Bbbk)$$

that induces a homomorphism on homology

$$\widetilde{H}_i(K^{\mathbf{b}}I;\Bbbk) \to \bigoplus_{\mathbf{a}\in\mathbb{N}^n} \widetilde{H}_{i-1}(K^{\mathbf{a}}I;\Bbbk)$$

such that the induced $\Bbbk[\mathbf{x}]$-module homomorphisms

$$\widetilde{H}_i(K^{\mathbf{b}}I;\Bbbk)\otimes\Bbbk[\mathbf{x}](-\mathbf{b}) \to \bigoplus_{\mathbf{a}\in\mathbb{N}^n} \widetilde{H}_{i-1}(K^{\mathbf{a}}I;\Bbbk)\otimes\Bbbk[\mathbf{x}](-\mathbf{a})$$

give a free resolution of I. In order for a map on the chains to induce a homomorphism on homology, the map must send cycles to cycles and boundaries to boundaries. The resulting resolution will automatically be minimal, by the construction of the modules: the rank of the i^{th} free module in degree $\mathbf{b}$ equals the Betti number $\beta_{i,\mathbf{b}}(I)$.

2.1.2 Cellular resolutions

Cellular resolutions provide the framework for many of the combinatorial descriptions of resolutions of monomial ideals. The theory is developed for monomial modules in [BS98], and this thesis focuses on the case of monomial ideals. The resolutions constructed in this thesis are not cellular, so the reader who is looking for only novel information can skip this subsection.

A polyhedral cell complex X is a finite set of faces (convex polytopes) that is closed under taking subfaces and intersections of faces. Given orientations of the faces, the boundary map for a polyhedral cell complex is given by $\partial(F) = \sum_{\text{facets } F' \subset F} \text{sign}(F', F)F'$. The sign of F' in F, $\text{sign}(F', F)$, is ± 1, depending on whether the orientation of F' is induced by that of F or is instead opposite.

The generators of I give the cell complex X an $\mathbb{N}^n$-grading. The *labeled polyhedral cell complex* is obtained by labeling the vertices of X with the monomial generators of I and labeling each face with the least common multiple of the labels on the vertices of that face. The *cellular (free) complex* $\mathcal{F}_X$ *supported on* X is given by the resolution where the rank of the i^{th} free module in degree $\mathbf{b}$ is equal to the number of i-faces of X with label $\mathbf{x}^{\mathbf{b}}$ and whose monomial matrices have entries given by the boundary maps of X.

Let $X_{\preceq \mathbf{b}}$ be the subcomplex of X containing all faces with label $\mathbf{x}^{\mathbf{a}}$ such that $\mathbf{a} \preceq \mathbf{b}$. The following proposition describes when the cellular complex gives a resolution of I.

Proposition 2.1.5 ([BS98], Proposition 1.2)**.** *The cellular complex* $\mathcal{F}_X$ *is a free resolution of* I *if and only if* $X_{\preceq \mathbf{b}}$ *is acyclic over* $\Bbbk$ *for all* $\mathbf{b}$*. In this case, the complex* $\mathcal{F}_X$ *is called a* cellular resolution *of* I*.*

2.2 Past constructions

Since the construction of the Taylor resolution in 1966, there have been numerous other constructions of free resolutions of monomial ideals. These constructions each give a partial solution to Kaplansky's problem, with the added desired characteristics of being minimal, canonical, universal, combinatorial, and closed form.

2.2.1 The Taylor resolution

The Taylor resolution was the first construction of a free resolution for an arbitrary monomial ideal I [Tay66]. Let X be the labeled polyhedral cell complex given by the full simplex whose vertices are the monomial generators of I. The Taylor resolution is the cellular complex $\mathcal{F}_X$ supported on X. Since, in each degree $\mathbf{b}$, $X_{\preceq \mathbf{b}}$ is acyclic (as it is a simplex), $\mathcal{F}_X$ is a free resolution of I by Proposition 2.1.5. This resolution has a canonical, closed-form differential with a combinatorial description. However, it is usually nonminimal.

2.2.2 The Lyubeznik resolution

The Lyubeznik resolution is a more efficient version of the Taylor resolution. Instead of being supported on the cell complex whose faces are indexed by the least common multiples of all subsets of the generators, the resolution is supported on the labeled cell complex

$$\Delta = \{\mathrm{lcm}(m_{i_1}, \dots m_{i_s}) \mid m_q \nmid \mathrm{lcm}(m_{i_t}, \dots, m_{i_s}) \text{ for all } t < s, q < i_t\},$$

where m_j is a monomial generator of I for all j [Lyu88]. Therefore this resolution is a subcomplex of the Taylor resolution, but it is still generally nonminimal.

2.2.3 Wall resolutions

In [Eag90], John Eagon shows that the Wall complex gives minimal free resolutions of the Stanley–Reisner ideal of triangulations of spheres. These resolutions are non-canonical.

Eagon also notes that, by selecting a splitting, the Wall complex can give a minimal free resolution of an arbitrary monomial ideal, and he suggests using the Moore–Penrose pseudoinverse. This thesis is the result of exploring that idea and proving that the Wall complex of the Koszul bicomplex does in fact give a minimal free resolution of I, with a combinatorial description guided by the combinatorics of the Moore–Penrose pseudoinverse.

2.2.4 The Eliahou–Kervaire resolution for stable ideals

The Eliahou–Kervaire resolution has a canonical, closed-form differential with a combinatorial description but is only minimal for a certain class of monomial ideals. For a monomial $\mathbf{x}^\mathbf{a}$, where $\mathbf{a} = (a_1, \ldots, a_n) \in \mathbb{N}^n$, define $\max(\mathbf{x}^\mathbf{a}) = \max\{i \mid a_i \neq 0\}$. A monomial ideal I is *stable* if for all monomials $\mathbf{x}^\mathbf{a} \in I$ and all $1 \leq i < \max(\mathbf{x}^\mathbf{a}) =: r$, $\mathbf{x}^\mathbf{a} x_i/x_r \in I$. The Eliahou–Kervaire resolution minimally resolves any stable ideal I [EK90]. In [Mer10], it is shown that the resolution is supported on a cell complex.

2.2.5 The Scarf complex for generic ideals

The Scarf (simplicial) complex was introduced in an economic context by Herbert Scarf in [Sca86], providing the inspiration for the algebraic Scarf complex in [BPS98]. The resolution gives a canonical, closed-form differential with a combinatorial description. Suppose $I = \langle m_1, \ldots, m_s \rangle$ is a monomial ideal with monomial generators $m_1, \ldots, m_s$. I is *generic* if whenever two generators m_i and m_j agree in the degree of a variable x_ℓ appearing in both, there is a third generator m_k that strictly divides $\mathrm{lcm}(m_i, m_j)$. The Scarf (simplicial) complex Δ_I of I is the set of subsets of $\{m_1, \ldots, m_s\}$ whose least common multiple is unique.

The *algebraic Scarf complex* is a complex of free modules whose differentials are derived from the boundary map of Δ_I. More precisely, it is a cellular free complex supported on Δ_I. It is generally non-minimal, except in the case where I is generic; if I is generic, then the algebraic Scarf complex is a minimal free resolution of S/I.

2.2.6 The hull resolution

The construction of the hull resolution is given in [BS98]. The hull resolution is a generalization of the Scarf complex, since the Scarf complex is a subcomplex of the hull resolution for any monomial ideal I. This resolution is canonical, combinatorial, and universal, as well as respecting symmetry. It is constructed by rescaling the degree vectors of the monomial generators of I in a certain way and taking the convex hull of the vectors in $\mathbb{R}^n$. The complex hull(I) is comprised of the bounded faces of the polyhedron. The hull resolution is the cellular free complex supported on hull(I), and it gives a free resolution of I. If I is generic, the Scarf complex and the hull resolution coincide and are both minimal [BS98].

2.2.7 Minimal Taylor resolutions

Sergey Yuzvinsky describes a construction of a resolution of certain homogeneous polynomial ideals (including monomial ideals) that is universal and minimal [Yuz99]. He notes that every minimal resolution is a subcomplex of the Taylor resolution and describes how to project this resolution onto a subcomplex that is minimal. The resolution is given by algorithm, and the paper does not give a closed-form differential.

The construction on its own does not give a canonical resolution, but by using the Moore–Penrose pseudoinverse as the splitting, the resulting differential will be canonical and likely combinatorial. Writing down a closed-form differential for this construction and describing the result combinatorially when the Moore–Penrose pseudoinverse is used as the splitting is an open problem.

2.2.8 Planar graph resolutions

For monomial ideals in the polynomial ring $\Bbbk[x, y, z]$ in three variables, a staircase diagram can be drawn that represents the monomial ideal in $\mathbb{R}^3$. Let $\mathbf{a}$ be the degree vector of a monomial generator $\mathbf{x}^{\mathbf{a}}$. The union $\bigcup_{\mathbf{a}}(\mathbf{a}+\mathbb{R}^3_{\geq 0})$ contains all degree vectors of monomials in I, and the rest of $\mathbb{R}^3_{\geq 0}$ contains all degree vectors of monomials not in I. The interface between these two regions is called the *staircase surface*.

Every monomial ideal $I \subset \Bbbk[x, y, z]$ has a minimal resolution by planar graph [MS99], where the planar graph has $\beta_{i,\mathbf{b}}(I)$ i-faces with label $\mathbf{b}$, and the graph has an embedding into the staircase surface. These resolutions are noncanonical. In this thesis, it is shown that resolutions by planar graph can be obtained from what are called noncanonical sylvan resolutions with a choice of splitting.

2.2.9 Buchberger resolutions

The Buchberger resolution was first given in the three-variable case [MS05], and it was constructed for monomial ideals in any number of variables in [OW16].

Define the Buchberger complex of I to be the set of subsets of generators such that no generator strictly divides their least common multiple. The cellular complex supported on the Buchberger complex of I gives a free resolution of I.

This resolution is minimal if, whenever two subsets of generators have the same least common multiple, there is a generator that strictly divides those least common multiples [OW16].

2.2.10 Algorithmic canonical resolutions

In a 2019 preprint, Alexandre Tchernev constructs canonical resolutions of monomial and toric ideals by explicit recursive algorithms [Tch19]. He describes how to get a minimal resolution by starting with a chain complex and using dynamical systems theory to obtain a smaller complex. These resolutions currently do not have closed-form expressions, and they are combinatorial in the sense that the algorithm is combinatorial. It is an open problem to give a closed-form expression for the differential and describe this expression combinatorially.

Chapter 3

Sylvan resolutions from Wall complexes

Sylvan resolutions are constructed by taking the Wall complex of a bicomplex whose differentials are induced by the Koszul complex. Much of this work in this chapter summarizes the work in [Eag90] and preprints written by John Eagon and Joel Roberts in the early 1990s. The contributions of this thesis, from joint work with John Eagon and Ezra Miller, are Proposition 3.2.4, which corrects a circular argument in the proof of the main theorem in [Eag90], and Proposition 3.2.8, which proves that the resolution resolves I, and not another associated graded module of I.

3.1 The Koszul bicomplex

Let V be an n-dimensional vector space with variables $z_1, \ldots, z_n$ having degree $e_1, \ldots, e_n$. Given $\tau \subseteq \{1, \ldots, n\}$, let $\mathbf{z}^\tau$ be the exterior product of the $\mathbf{z}$-variables corresponding to τ. All tensor products are over the field $\Bbbk$ unless otherwise noted.

Definition 3.1.1. The Koszul complexes in the $\mathbf{x}$-variables and the $\mathbf{y}$-variables are given by

$$\mathbb{K}^{\mathbf{x}}_{\bullet} = \textstyle\bigwedge^{\bullet} V \otimes \Bbbk[\mathbf{x}] \quad \text{and} \quad \mathbb{K}^{\mathbf{y}}_{\bullet} = \textstyle\bigwedge^{\bullet} V \otimes \Bbbk[\mathbf{y}]$$

with $\mathbb{N}^n$-graded differentials

$$\mathbf{z}^\tau \otimes \mathbf{x}^\mathbf{b} \mapsto \sum_{k\in\tau} (-1)^{e_k \subset \tau} \mathbf{z}^{\tau - e_k} \otimes \mathbf{x}^{\mathbf{b}+e_k}.$$

Let $I^\mathbf{y}$ be the copy of the ideal I in the ring $\Bbbk[\mathbf{y}]$.

Definition 3.1.2. The Koszul complex of the ideal I is given by

$$\mathbb{K}^\mathbf{y}_\bullet \otimes_{\Bbbk[\mathbf{y}]} I^\mathbf{y} = \bigwedge^\bullet V \otimes \Bbbk[\mathbf{y}] \otimes_{\Bbbk[\mathbf{y}]} I^\mathbf{y}$$

with differential induced by the differential of the Koszul complex in $\mathbf{y}$:

$$\mathbf{z}^\tau \otimes \mathbf{y}^\mathbf{b} \otimes_{\Bbbk[\mathbf{y}]} \mathbf{y}^\mathbf{a} \mapsto \sum_{k\in\tau} (-1)^{e_k \subset \tau} \mathbf{z}^{\tau - e_k} \otimes \mathbf{x}^{\mathbf{b}+e_k} \otimes_{\Bbbk[\mathbf{y}]} \mathbf{y}^\mathbf{a}.$$

Let $\mathbf{x}+\mathbf{y}$ denote the sequence $x_1+y_1, x_2+y_2, \ldots, x_n+y_n$ in $\Bbbk[\mathbf{x},\mathbf{y}]$.

Lemma 3.1.3. $\bigwedge^\bullet V \otimes \Bbbk[\mathbf{x}] \otimes \Bbbk[\mathbf{y}] \otimes_{\Bbbk[\mathbf{y}]} I^\mathbf{y}$ *is a free resolution of* I*.*

Proof. $\bigwedge^\bullet V \otimes \Bbbk[\mathbf{x}] \otimes \Bbbk[\mathbf{y}] \otimes_{\Bbbk[\mathbf{y}]} I^\mathbf{y} = \bigwedge^\bullet V \otimes \Bbbk[\mathbf{x},\mathbf{y}] \otimes_{\Bbbk[\mathbf{y}]} I^\mathbf{y} = \bigwedge^\bullet V \otimes \Bbbk[\mathbf{x},\mathbf{y}] I^\mathbf{y}$. Since $\Bbbk[\mathbf{x},\mathbf{y}] = (\Bbbk[\mathbf{y}])[\mathbf{x}]$, $\mathbf{x}+\mathbf{y}$ is a regular sequence in $\Bbbk[\mathbf{x},\mathbf{y}] = \Bbbk[\mathbf{y}] \otimes \Bbbk[\mathbf{x}]$. Therefore the complex is acyclic, and $H_0(\bigwedge^\bullet V \otimes \Bbbk[\mathbf{x}] \otimes \Bbbk[\mathbf{y}] \otimes_{\Bbbk[\mathbf{y}]} I^\mathbf{y}) \cong I$. □

The modules $\bigwedge^\bullet V \otimes \Bbbk[\mathbf{x}] \otimes \Bbbk[\mathbf{y}] \otimes_{\Bbbk[\mathbf{y}]} I^\mathbf{y}$ can be decomposed into a bicomplex with differentials given by the Koszul complexes in the $\mathbf{x}$-, $\mathbf{y}$-, and $(\mathbf{x}+\mathbf{y})$-variables.

Definition 3.1.4. The Koszul bicomplex $\mathbb{K}_{\bullet\bullet}(I)$ of I is the bicomplex in Figure 3.1, with vertical differentials induced by those of $\mathbb{K}^{\mathbf{y}}_{\bullet}(I)$, horizontal differentials induced by those of $\mathbb{K}^{\mathbf{x}}_{\bullet}$, and total differentials induced by those of $\mathbb{K}^{\mathbf{x}+\mathbf{y}}_{\bullet}$.

$$
\begin{array}{|lllll}
\hline
\bigoplus\limits_{|\mathbf{a}|=0} \mathbb{K}^{\mathbf{y}}_0 \otimes I^{\mathbf{y}}_{\mathbf{a}} \otimes \Bbbk[\mathbf{x}] & \bigoplus\limits_{|\mathbf{a}|=0} \mathbb{K}^{\mathbf{y}}_1 \otimes I^{\mathbf{y}}_{\mathbf{a}} \otimes \Bbbk[\mathbf{x}] & \bigoplus\limits_{|\mathbf{a}|=0} \mathbb{K}^{\mathbf{y}}_2 \otimes I^{\mathbf{y}}_{\mathbf{a}} \otimes \Bbbk[\mathbf{x}] & \bigoplus\limits_{|\mathbf{a}|=0} \mathbb{K}^{\mathbf{y}}_3 \otimes I^{\mathbf{y}}_{\mathbf{a}} \otimes \Bbbk[\mathbf{x}] & \\
 & \downarrow & \downarrow & \downarrow & \\
 & \bigoplus\limits_{|\mathbf{a}|=1} \mathbb{K}^{\mathbf{y}}_0 \otimes I^{\mathbf{y}}_{\mathbf{a}} \otimes \Bbbk[\mathbf{x}] & \bigoplus\limits_{|\mathbf{a}|=1} \mathbb{K}^{\mathbf{y}}_1 \otimes I^{\mathbf{y}}_{\mathbf{a}} \otimes \Bbbk[\mathbf{x}] & \bigoplus\limits_{|\mathbf{a}|=1} \mathbb{K}^{\mathbf{y}}_2 \otimes I^{\mathbf{y}}_{\mathbf{a}} \otimes \Bbbk[\mathbf{x}] & \cdots \\
 & & \downarrow & \downarrow & \\
 & & \bigoplus\limits_{|\mathbf{a}|=2} \mathbb{K}^{\mathbf{y}}_0 \otimes I^{\mathbf{y}}_{\mathbf{a}} \otimes \Bbbk[\mathbf{x}] & \bigoplus\limits_{|\mathbf{a}|=2} \mathbb{K}^{\mathbf{y}}_1 \otimes I^{\mathbf{y}}_{\mathbf{a}} \otimes \Bbbk[\mathbf{x}] & \\
 & & & \downarrow & \\
 & & \ddots & \vdots &
\end{array}
$$

Figure 3.1: The Koszul bicomplex $\mathbb{K}_{\bullet\bullet}(I)$

Lemma 3.1.5. *The vertical homology of* $\mathbb{K}_{\bullet\bullet}(I)$ *is* $H_{-q}(\mathbb{K}_{p\bullet}(I)) = \bigoplus\limits_{|\mathbf{a}|=p} \widetilde{H}_{p+q-1}(K^{\mathbf{a}}I;\Bbbk)$.

Proof. By Hochster's formula, $H_i\big(\mathbb{K}^{\mathbf{y}}_{\bullet}(I^{\mathbf{y}})\big)_{\mathbf{b}} \cong \widetilde{H}_{i-1}(K^{\mathbf{b}}I;\Bbbk)$. Then $H_i\big(\big(\mathbb{K}^{\mathbf{y}}_{\bullet}(I^{\mathbf{y}})\big)_{\mathbf{b}} \otimes \Bbbk[\mathbf{x}]\big) \cong \widetilde{H}_{i-1}(K^{\mathbf{b}}I;\Bbbk) \otimes \Bbbk[\mathbf{x}]$. $\square$

$$
\begin{array}{|lllll}
\hline
\bigoplus\limits_{|\mathbf{a}|=0} \widetilde{H}_{-1}K^{\mathbf{a}}I \otimes \Bbbk[\mathbf{x}] & \bigoplus\limits_{|\mathbf{a}|=1} \widetilde{H}_{0}K^{\mathbf{a}}I \otimes \Bbbk[\mathbf{x}] & \bigoplus\limits_{|\mathbf{a}|=2} \widetilde{H}_{1}K^{\mathbf{a}}I \otimes \Bbbk[\mathbf{x}] & \bigoplus\limits_{|\mathbf{a}|=3} \widetilde{H}_{2}K^{\mathbf{a}}I \otimes \Bbbk[\mathbf{x}] & \\
 & \bigoplus\limits_{|\mathbf{a}|=1} \widetilde{H}_{-1}K^{\mathbf{a}}I \otimes \Bbbk[\mathbf{x}] & \bigoplus\limits_{|\mathbf{a}|=2} \widetilde{H}_{0}K^{\mathbf{a}}I \otimes \Bbbk[\mathbf{x}] & \bigoplus\limits_{|\mathbf{a}|=3} \widetilde{H}_{1}K^{\mathbf{a}}I \otimes \Bbbk[\mathbf{x}] & \cdots \\
 & & \bigoplus\limits_{|\mathbf{a}|=2} \widetilde{H}_{-1}K^{\mathbf{a}}I \otimes \Bbbk[\mathbf{x}] & \bigoplus\limits_{|\mathbf{a}|=3} \widetilde{H}_{0}K^{\mathbf{a}}I \otimes \Bbbk[\mathbf{x}] & \\
 & & \ddots & \vdots &
\end{array}
$$

Figure 3.2: Vertical homology of the Koszul bicomplex

Proposition 3.1.6. *The spectral sequence associated to the filtration by columns of the Koszul bicomplex* $\mathbb{K}_{\bullet\bullet}(I)$ *of* I *converges to* $H_{p+q}\mathbb{K}_{\bullet}^{\mathbf{x}+\mathbf{y}}(I)$.

Proof. The Koszul bicomplex is a fourth-quadrant spectral sequence. In particular, since the polynomial ring is in n variables, $\bigwedge^i V$ is nonzero only when $0 \leq i \leq n$. Therefore the bicomplex is concentrated in the fourth-quadrant strip between the diagonals $p + q = 0$ and $p + q = n$.

Let F be the filtration by columns of $\text{Tot}(\mathbb{K}_{\bullet\bullet}(I))$. Then the p^{th} column of $\mathbb{K}_{\bullet\bullet}(I)$ is equal to $F_p(\text{Tot}(\mathbb{K}_{\bullet\bullet}(I))/F_{p-1}(\text{Tot}(\mathbb{K}_{\bullet\bullet}(I)))$. Along the diagonal $p + q = i$, $H_{p+q}(\mathbb{K}_{p\bullet}(I)) = H_i(\mathbb{K}_{p\bullet}(I)) = \bigoplus_{|\mathbf{a}|=p} \widetilde{H}_{i-1}K^{\mathbf{a}}I \otimes \Bbbk[\mathbf{x}]$. Taking the vertical homology of the Koszul bicomplex $\mathbb{K}_{\bullet\bullet}(I)$ gives the bicomplex in Figure 3.2. Thus $E^1_{p,q} = H_{p+q}(F_p\text{Tot}(\mathbb{K}_{\bullet\bullet}(I))/F_{p-1}\text{Tot}(\mathbb{K}_{\bullet\bullet}(I)))$. Therefore $E^1_p \Rightarrow H(\text{Tot}(\mathbb{K}_{\bullet\bullet}(I))$. □

The maps of the spectral sequence only give maps between subquotients of the appropriate modules. By taking the Wall complex of the Koszul bicomplex, the maps are lifted to homomorphisms between the homology modules themselves.

3.2 The Wall complex of the Koszul bicomplex

The maps can be lifted to homomorphisms between the homology modules themselves by selecting a splitting of the vertical differential in the Koszul bicomplex.

Definition 3.2.1. Given a bicomplex $C_{\bullet\bullet}$ with vertical differential d and horizontal differential d_1, a vertical splitting is a differential

$$d^+_{pq} : C_{pq} \to C_{p,q+1}$$

such that $dd^+d = d$ and $d^+dd^+ = d^+$.

A vertical splitting is equivalent to a direct sum decomposition

$$C_{pq} = B_{pq} \oplus H_{pq} \oplus B'_{p,q-1},$$

where $B_{pq} = \operatorname{im}(d_{p,q+1})$, $H_{pq} \cong H_q(C_{p\bullet})$, and $B'_{p,q-1}$ is isomorphic to $\operatorname{im}(d_{p,q})$, the boundaries in $C_{p,q-1}$. The maps dd^+, d^+d, and $I - dd^+ - d^+d$ are the projections of C_{pq} onto B_{pq}, $B'_{p,q-1}$, and H_{pq}, respectively.

Definition 3.2.2. Let $W_{pq} = H_q(C_{p\bullet}) \cong H_{pq}$, the homology of the vertical differential d in $C_{\bullet\bullet}$. Suppose only finitely many W_{pq} are nonzero. Define

$$W_i = \bigoplus_{p+q=i} W_{pq} \text{ and } \omega_j = (I - dd^+ - d^+d)d_1(d^+d_1)^{j-1} : W_{pq} \to W_{p-j,q+j-1}.$$

The modules W_i together with the maps $D_i = \sum_{j=0}^{\infty} \omega_j : W_i \to W_{i-1}$ give a Wall complex of $C_{\bullet\bullet}$.

Proposition 3.2.3 ([Eag90]). *The total complex of $W_{\bullet\bullet}$ has a filtration by columns.*

The spectral sequence associated to this filtration is the same as the spectral sequence associated to the filtration by columns of $C_{\bullet\bullet}$.

The proof of equation (2.3.1) in the proof of the above proposition in [Eag90] contains a circular argument. The proof is corrected here:

Proposition 3.2.4. *Let* $x \in \overline{Z}^r_{p,q}$*. Define* $y \in Tot(E)$ *inductively downward by*

$$y_i = (-1)^i x_i - d^+ d_1 y_{i+1} \text{ for all } i.$$

(all terms are zero for $i > p$*). Then*

$$d(y_i) + d_1(y_{i+1}) = 0 \text{ for } p \geq i > p - r. \tag{2.3.1}$$

Lemma 3.2.5. $y_i = (-1)^i \sum_{j=0}^{p-i} (d^+ d_1)^j x_{i+j}$.

Proof. By downward induction. Base case: $i = p$: $y_p = (-1)^p x_p$ by (2.3.3).

Assume $y_{i+1} = (-1)^{i+1} \sum_{j=0}^{p-i-1} (d^+ d_1)^j x_{i+j+1}$. Then

$$\begin{aligned} y_i &= (-1)^i x_i - d^+ d_1 y_{i+1} \\ &= (-1)^i x_i - (-1)^{i+1} \sum_{j=0}^{p-i-1} (d^+ d_1)^{j+1} x_{i+j+1} \\ &= (-1)^i (x_i + \sum_{j=1}^{p-i} (d^+ d_1)^j x_{i+j}) \end{aligned}$$

$$= (-1)^i(\sum_{j=0}^{p-i}(d^+d_1)^j x_{i+j}) \qquad \square$$

Lemma 3.2.6. *If* $d(y_i) + d_1(y_{i+1}) = 0$*, then* $d^+dd_1d^+d_1(y_{i+1}) = 0$.

Proof.

$$\begin{aligned} d^+dd_1d^+d_1(y_{i+1}) &= -d^+d_1dd^+d_1(y_{i+1}) \\ &= d^+d_1dd^+d(y_i) \\ &= d^+d_1d(y_i) \\ &= -d^+d_1d_1(y_{i+1}) \\ &= 0. \end{aligned} \qquad \square$$

Proof of (2.3.1). By downward induction. Base case: $i = p$:

$$d(y_p) + d_1(y_{p+1}) = (-1)^p d(x_p) = 0$$

by the definition of $\overline{Z}^r_{p,q}$. Now assume that $d(y_i) + d_1(y_{i+1}) = 0$.

$$\begin{aligned} d(y_{i-1}) + d_1(y_i) &= (-1)^{i-1}d(x_{i-1}) - dd^+d_1(y_i) + d_1(y_i) \\ &= (I - dd^+)d_1(y_i) \\ &= (I - dd^+)d_1(-1)^i(x_i) - (I - dd^+)d_1d^+d_1(y_{i+1}) \\ &= (-1)^i(I - dd^+ - d^+d)d_1(x_i) - (I - dd^+ - d^+d)d_1d^+d_1(y_{i+1}) \\ &= (-1)^i(I - dd^+ - d^+d)(d_1(x_i) + d_1\sum_{j=0}^{p-i-1}(d^+d_1)^{j+1}x_{i+j+1}) \end{aligned}$$

$$= (-1)^i(I - dd^+ - d^+d)(d_1(x_i) + d_1 \sum_{j=1}^{p-i} (d^+d_1)^j x_{i+j}$$

$$= (-1)^i(I - dd^+ - d^+d)(d_1 \sum_{j=0}^{p-i} (d^+d_1)^j x_{i+j})$$

which is equal to 0 by the definition of $\overline{Z}^r_{p,q}$. □

Remark 3.2.7. Since $\bigwedge^i V \otimes \Bbbk[\mathbf{y}] \otimes_{\Bbbk[\mathbf{y}]} I^{\mathbf{y}} \otimes \Bbbk[\mathbf{x}] \cong \bigwedge^i V \otimes I^{\mathbf{y}} \otimes \Bbbk[\mathbf{x}]$, the modules $\mathbb{K}_{pq}(I)$ have a $\Bbbk$-linear basis consisting of elements $\mathbf{z}^\tau \otimes \mathbf{y}^{\mathbf{b}-\tau} \otimes \mathbf{x}^{\mathbf{a}}$. Under the isomorphism $(\mathbb{K}_\bullet \otimes I)_{\mathbf{b}} \cong \widetilde{C}_\bullet K^{\mathbf{b}} I$, the element $\mathbf{z}^\tau \otimes \mathbf{y}^{\mathbf{b}-\tau} \otimes 1 \in \bigwedge^i V \otimes I^{\mathbf{y}} \otimes \Bbbk[\mathbf{x}]$ is identified with $\tau \otimes \mathbf{x}^{\mathbf{b}} \in \widetilde{H}_{i-1}(K^{\mathbf{b}} I; \Bbbk) \otimes \langle \mathbf{x}^{\mathbf{b}} \rangle$.

Proposition 3.2.8. *The total complex of the Wall complex of the Koszul bicomplex $\mathbb{K}_{\bullet\bullet}(I)$ is a minimal free resolution of I.*

Proof. By Lemma 3.1.3 and Proposition 3.1.6, the complex is acyclic.

The modules W_{pq} are given by the vertical homology of the Koszul bicomplex. By Lemma 3.1.5, $W_{pq} = \bigoplus_{|\mathbf{a}|=p} \widetilde{H}_{p+q-1} K^{\mathbf{a}} I \otimes \Bbbk[\mathbf{x}]$. Therefore $W_i = \bigoplus_{\mathbf{a} \in \mathbb{N}^n} \widetilde{H}_{i-1} K^{\mathbf{a}} I \otimes \Bbbk[\mathbf{x}]$. Since this is a direct sum of free $\Bbbk[\mathbf{x}]$-modules whose ranks in degree $\mathbf{a}$ and dimension i are $\dim_\Bbbk(\widetilde{H}_{i-1} K^{\mathbf{a}} I) = \beta_{i,\mathbf{a}}(I)$, the resolution is minimal.

It remains to show that the resolution resolves I, not $\mathrm{gr}(I)$ for a filtration of I. Consider the sequence

$$\Bbbk[\mathbf{x}] \xleftarrow{D_0} W_0 \xleftarrow{D_1} W_1.$$

The map D_0 is defined as follows: Let $\mathrm{gen}(I)$ be the set of degree vectors of minimal monomial generators of I. Then W_0 is the $\Bbbk$-linear span of elements of the form $1 \otimes \mathbf{y}^{\mathbf{g}} \otimes \mathbf{x}^{\mathbf{a}}$, where $\mathbf{g} \in \mathrm{gen}(I)$. Define $D_0(1 \otimes \mathbf{y}^{\mathbf{g}} \otimes \mathbf{x}^{\mathbf{a}}) = \mathbf{x}^{\mathbf{g}+\mathbf{a}}$. Elements of W_1 are linear combinations w of the elements $z_k \otimes \mathbf{y}^{\mathbf{b}} \otimes \mathbf{x}^{\mathbf{a}}$ such that $d(w) = 0$. Note that $D_0(d(z_k \otimes \mathbf{y}^{\mathbf{b}} \otimes \mathbf{x}^{\mathbf{a}})) = D_0(1 \otimes \mathbf{y}^{\mathbf{b}+e_k} \otimes \mathbf{x}^{\mathbf{a}}) = \mathbf{x}^{\mathbf{b}+\mathbf{a}+e_k} = D_0(d_1(z_k \otimes \mathbf{y}^{\mathbf{b}} \otimes \mathbf{x}^{\mathbf{a}}))$, and since $d(w) = 0$, $D_0(d_1(w)) = D_0(d(w)) = 0$. Note also that $d(x) = 0$ for any x in W_0. Therefore

$$
\begin{aligned}
D_0(D_1(w)) &= \sum_j D_0(I - dd^+ - d^+d)(d_1d^+)^{j-1}d_1(w) \\
&= \sum_j (D_0(d_1d^+)^{j-1}d_1 - D_0dd^+(d_1d^+)^{j-1}d_1 - D_0d^+d(d_1d^+)^{j-1}d_1)(w) \\
&= \sum_j (D_0(d_1d^+)^{j-1}d_1 - D_0dd^+(d_1d^+)^{j-1}d_1)(w) \\
&= (D_0d - D_0dd^+d_1 + \sum_{j\geq 2} D_0dd^+(d_1d^+)^{j-2}d_1 - D_0dd^+(d_1d^+)^{j-1}d_1)(w) \\
&= D_0d(w) \\
&= 0,
\end{aligned}
$$

where the second-to-last equality is due to the telescoping sum and the fact that the sum is finite, since only finitely many ω_j in the sum for D are nonzero. The image of D_0 is clearly equal to I.

Since $\mathrm{im}(D_1) \subseteq \ker(D_0)$ and $\mathrm{im}(D_0) \cong W_0/\ker(D_0)$, there is a surjection $I = \mathrm{im}(D_0) \leftarrow F_0/\mathrm{im}(D_1) = H_0(W_\bullet)$. Since the $E^\infty_{p,-p}$ terms give a filtration $\cdots \subset F_{p-1}I \subset F_pI \subset F_{p+1}I \subset \cdots$ of I and the Hilbert series $HS(I)$ of I is equal to the sum $\sum_p HS(F_pI/F_{p-1}I)$, the Hilbert series of I and the Hilbert series of $H_0(W_\bullet)$ are the same. Therefore the surjection $I \leftarrow H_0(W_\bullet)$ is also an injection. □

Note that the input of the maps ω_j is an element of the submodule H_{pq} which is isomorphic to the vertical homology of the bicomplex via the projection $I - dd^+ - d^+d$. In a homomorphism $\widetilde{\omega}_j : \widetilde{H}_{pq} \to \widetilde{H}_{p-j,q+j-1}$ on the homology groups, the input is a homology class, represented by a cycle. In order to ensure that cycles representing the same homology class get sent to the same homology class, the cycles must first be projected onto H_{pq} via $I - dd^+$, since dd^+ is the projection onto the boundaries. Since the output of the maps is again a representative of a homology class, it must be a cycle in $Z_{p-j,q+j-1}$. Therefore the output should be projected onto the cycles, via $I - d^+d$, and there is no need to subtract the projection onto the boundaries.

Definition 3.2.9. The *natural Wall complex* is the bicomplex $\widetilde{H}_{\bullet\bullet}$ with homology groups in the place of the homology submodule H_{pq} and differentials $\widetilde{\omega}_0 = 0$ and

$$\widetilde{\omega}_j = (I - d^+d)(d_1d^+)^{j-1}d_1(I - dd^+) : Z_{pq} \to Z_{p-j,q+j-1} \qquad \text{for } j \geq 1.$$

Proposition 3.2.10. *The total complex of the natural Wall complex is a minimal free resolution of I.*

Proof. The maps $\sum_j \widetilde{\omega}_j$ are differentials, since

$$(\sum_j \widetilde{\omega}_j)^2 = \sum_{i,j}(I - d^+d)(d_1d^+)^{i-1}d_1(I - dd^+)(I - d^+d)(d_1d^+)^{j-1}d_1(I - dd^+)$$

$$= \sum_{i,j}(I - d^+d)(d_1d^+)^{i-1}(I - dd^+ - d^+d)(d_1d^+)^{j-1}d_1(I - dd^+).$$

Since $D^2 = 0$, $\sum_{i,j}(I-dd^+-d^+d)(d_1d^+)^{i-1}(I-dd^+-d^+d)(d_1d^+)^{j-1}d_1 = 0$. Therefore $\sum_{i,j}(I - d^+d)(d_1d^+)^{i-1}(I - dd^+ - d^+d)(d_1d^+)^{j-1}d_1(I - dd^+)$ is a boundary. Since the output is a representative of a homology class, it is meant to be understood modulo the boundaries. So $\sum_j \widetilde{\omega}_j$ is a differential. By the discussion above, they define the same differential as the one in the original Wall complex. □

Chapter 4

Moore–Penrose pseudoinverses as weighted averages of splittings

Any vertical splitting of the boundary maps of the simplicial complexes $K^{\mathbf{b}}I$ can be used to construct minimal free resolutions of I. The Moore–Penrose pseudoinverse is used to get a minimal free resolution that is canonical since, by Theorem 4.3.2, it is a weighted average of all splittings that come from higher-dimensional analogues of spanning trees. The proof of this formula uses the Higher Projection Formula of Catanzaro, Chernyak, and Klein in [CCK15]. The work presented here is the result of joint work with John Eagon and Ezra Miller. The same formula for the Moore–Penrose pseudoinverse was proven simultaneously and independently by Catanzaro, Chernyak, and Klein and published in [CCK17]. Another similar formula for the pseudoinverse is given in [Ber86], which surprisingly was unknown to the community, including the author of this thesis and the authors of [CCK17] at the time of publication.

4.1 Shrubberies, stake sets, and hedges

Let X be a simplicial complex with boundary map $d_i : C_i(X) \to C_{i-1}(X)$.

Definition 4.1.1. A *shrubbery* T_i of $C_\bullet(X;\Bbbk)$ is a set of i-faces of X such that $d(T_i)$ is a basis for the boundaries $d(C_i)$.

Definition 4.1.2. A *stake set* S_{i-1} is a set of $(i-1)$-faces of X such that $\widetilde{C}_{i-1}(X;\Bbbk) = \Bbbk\{\bar{S}_{i-1}\} \oplus \widetilde{B}_{i-1}(X;\Bbbk)$, where $\bar{S}_{i-1}$ is the complement of the stake set in the set of $(i-1)$-faces of X.

Definition 4.1.3. A *hedge* ST_i of dimension i is a pair (S_{i-1}, T_i) consisting of a stake set of dimension $i-1$ and a shrubbery of dimension i.

4.2 Hedge splittings

Definition 4.2.1. For a hedge ST_i, define $d^+_{ST_i} : \widetilde{C}_{i-1}(X;\Bbbk) \to \widetilde{C}_i(X;\Bbbk)$ as the linear map such that:

1. $d^+_{ST_i}(d(t)) = t$ for all $t \in T_i$
2. $d^+_{ST_i}(s) = 0$ for all $s \in \bar{S}_{i-1}$

Definition 4.2.2. A *community* is a sequence $ST_\bullet = (ST_0, ST_1, ST_2, \ldots)$ of hedges such that $T_i \cap S_i = \emptyset$ for all i.

Proposition 4.2.3. *Given a community $ST_\bullet$, the maps $d^+_{ST_i} : \widetilde{C}_{i-1}(X;\Bbbk) \to \widetilde{C}_i(X;\Bbbk)$ give a vertical splitting of the boundary map $d_i : \widetilde{C}_i(X;\Bbbk) \to \widetilde{C}_{i-1}(X;\Bbbk)$.*

Proof. The first property to prove is that the maps $d^+_{ST_i}$ give a differential. Let $x \in \widetilde{C}_{i-1}(X;\Bbbk)$. Since $\widetilde{C}_{i-1}(X;\Bbbk) = \Bbbk\{\bar{S}_{i-1}\} \oplus \widetilde{B}_{i-1}(X;\Bbbk)$, x can be written uniquely as $x = \sum_k c_k s_k + \sum_\ell c_\ell d(t_\ell)$, where $s_k \in \bar{S}_{i-1}$ and $t_\ell \in T_i$. Then

$$d^+_{ST_{i+1}} d^+_{ST_i}(x) = d^+_{ST_{i+1}}(\sum_\ell c_\ell t_\ell).$$

Since $T_i \cap S_i = \emptyset$, $t_\ell \in \bar{S}_i$ for all ℓ, so $d^+_{ST_{i+1}}(\sum_\ell c_\ell t_\ell) = 0$. Therefore $d^+_{ST_i}$ gives a differential.

Next, it must be shown that $dd^+_{ST_i}d = d$. For any $x \in \widetilde{C}_i(X;\Bbbk)$, $d(x)$ can be written uniquely as $\sum_\ell c_\ell d(t_\ell)$, where $t_\ell \in T_i$. Then

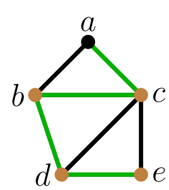
$$\begin{aligned} dd^+_{ST_i}d(x) &= d\sum_\ell c_\ell d^+_{ST_i}(d(t_\ell)) \\ &= d(\sum_\ell c_\ell t_\ell) \\ &= d(x). \end{aligned}$$

Lastly, $d^+_{ST_i}dd^+_{ST_i} = d^+_{ST_i}$:

$$\begin{aligned} d^+_{ST_i}dd^+_{ST_i}\big(\sum_k c_k s_k + \sum_\ell c_\ell d(t_\ell)\big) &= d^+_{ST_i}\big(\sum_\ell c_\ell d(t_\ell)\big) \\ &= \sum_\ell c_\ell t_\ell \\ &= d^+_{ST_i}\big(\sum_k c_k s_k + \sum_\ell c_\ell d(t_\ell)\big). \end{aligned}$$

□

Example 4.2.4. When X is one-dimensional, each T_1 is a spanning forest in the usual sense for a graph, meaning it is a spanning tree on each connected component. For each stake set S_0, $\bar{S}_0$ is a set of vertices, one for each connected component of X. Let v be a vertex of X, and let v_0 be the vertex in $\bar{S}_0$ that is in the same connected component as v. For a hedge ST_1, $d^{+}_{ST_1}(v)$ is the linear combination of the edges in the unique path in T_1 from v_0 to v, oriented in the direction of v_0 to v. See Figure 4.1.



$T_1 = \{ac, bc, bd, de\}$
$S_0 = \{b, c, d, e\}$
$ST_1 = (S_0, T_1)$ $\qquad d^{+}_{ST_1}(b) = ac - bc$

Figure 4.1: Shrubbery, stake set, and hedge of dimension 1

Let $X^{(i-1)}$ be the $(i-1)$-skeleton of X. Define $\langle T_i \rangle$ to be the simplicial complex whose faces are the faces in T_i and all faces of $X^{(i-1)}$. The rest of this section gives results crucial to the proof combinatorial Moore–Penrose pseudoinverse formula in Section 4.3.

Definition 4.2.5 ([CCK15], Definition 1.4)**.** Let T_i be a shrubbery of dimension i. Define ϕ_{T_i} to be the linear transformation

$$\phi_{T_i} : \widetilde{C}_i(X; \Bbbk) \to \widetilde{Z}_i(X; \Bbbk)$$

such that:

1. $\phi_{T_i}(x) = 0$ for all $x \in T_i$

2. $\phi_{T_i}(x) = c/\langle c, x\rangle$ if $x \notin T_i$, where c is a generator of $\widetilde{H}_i(\langle T_i \cup \{x\}\rangle; \mathbb{Z}) = \widetilde{Z}_i(\langle T_i \cup \{x\}\rangle; \mathbb{Z})$ and $\langle c, x\rangle$ is the coefficient of x in c.

The following lemma says that the map ϕ_{T_i} is the projection onto cycles given by a vertical splitting $d^+_{ST_i}$ for the shrubbery T_i and any stake set S_{i-1}.

Lemma 4.2.6. *Let ST_i be a hedge of dimension i. Then $\phi_{T_i} = I - d^+_{ST_i} d$.*

Proof. First, suppose $x \in T_i$. Then $(I - d^+_{ST_i} d)(x) = x - x = 0 = \phi_{T_i}(x)$.

If $x \notin T_i$, then $\phi_{T_i}(x) = c/\langle c, x\rangle$, where c is a generator of $\widetilde{H}_i(\langle T_i \cup \{x\}\rangle; \mathbb{Z})$. Note that $c = \langle c, x\rangle x + \sum_j a_j e_j$, where $e_j \in T_i$ for all j. Since $d(c/\langle c, x\rangle) = 0$, $d(x) = -\sum_j \frac{a_j}{\langle c,x\rangle} d(e_j)$. Then $(I - d^+_{ST_i} d)(x) = x + \sum_j \frac{a_j}{\langle c,x\rangle} d^+_{ST_i} d(e_j) = x + \sum_j \frac{a_j}{\langle c,x\rangle} e_j = c/\langle c, x\rangle$. □

Lemma 4.2.7. *Let S_i be a stake set of dimension i. For each $x \in S_i$, there exists a unique boundary $B^{S_i}_x \in \Bbbk\{\bar{S}_i \cup \{x\}\} \cap \widetilde{B}_i(X; \Bbbk)$ with coefficient 1 on x.*

Proof. Since $\widetilde{C}_i(X; \Bbbk) = \Bbbk\{\bar{S}_i\} \oplus \widetilde{B}_i(X; \Bbbk)$, x can be written uniquely as $x = \sum_k c_k s_k + \sum_\ell c_\ell d(t_\ell)$ with $s_k \in \bar{S}_i$ and $t_\ell \in T_{i+1}$. Define $B^{S_i}_x := \sum_\ell c_\ell d(t_\ell) = x - \sum_k c_k s_k \in \Bbbk\{\bar{S}_i \cup \{x\}\}$. □

Lemma 4.2.8. *Let x be an i-face in X, and let $B^{S_i}_x = \sum_\ell c_\ell d(t_\ell)$ be the unique boundary from Lemma 4.2.7. Then $d^+_{ST_i}(x) = d^+_{ST_i}(B^{S_i}_x) = \sum_\ell c_\ell t_\ell$.*

Proof. As before, $x = \sum_k c_k s_k + \sum_\ell c_\ell d(t_\ell)$, with $s_k \in \bar{S}_{i-1}$ and $t_\ell \in T_i$. Then

$$\begin{aligned} d^+_{ST_i}(x) &= \sum_k c_k d^+_{ST_i}(s_k) + \sum_\ell c_\ell d^+_{ST_i} d(t_\ell) \\ &= \sum_\ell c_\ell t_\ell. \end{aligned}$$

□

Lemma 4.2.9. *Let $x \in S_i$. For any $x' \in \bar{S}_i$ such that the coefficient of x' in $B^{S_i}_x$ is nonzero, $S'_i := S_i \cup \{x'\} \setminus \{x\}$ is a stake set.*

Proof. Again write $B^{S_i}_x = x - \sum_k c_k s_k$. Since the coefficient of x' in $B^{S_i}_x$ is nonzero, x' must be equal to some s_k. Therefore $B = x - \langle B, x'\rangle x' - \sum_k c_k s_k$, where the appropriate s_k is removed. Then

$$x' = \frac{1}{\langle B, x'\rangle}(x - \sum_k c_k s_k - B^{S_i}_x),$$

meaning $x' \in \Bbbk\{\bar{S}'_i\} \oplus \widetilde{B}_i(X;\Bbbk)$. Since x was uniquely written as $x = \langle B, x'\rangle + \sum_k c_k s_k + B^{S_i}_x$ in $\Bbbk\{\bar{S}_i\} \oplus \widetilde{B}_i(X;\Bbbk)$, x' is uniquely written as above in $\Bbbk\{\bar{S}'_i\} \oplus \widetilde{B}_i(X;\Bbbk)$. Therefore $\widetilde{C}_i(X;\Bbbk) = \Bbbk\{\bar{S}'_i\} \oplus \widetilde{B}_i(X;\Bbbk)$. □

Lemma 4.2.10 ([CCK15], Lemma 2.6)**.** *Let $x \in \bar{T}_i$. Then the homology class $[d(x)]$ generates a torsion element of $\widetilde{H}_{i-1}(\langle T_i\rangle;\mathbb{Z})$.*

Definition 4.2.11. Define $\theta_{S_{i-1}}$ and θ_{T_i} as the orders of the torsion subgroups of $\widetilde{H}_{i-2}(\langle \bar{S}_{i-1}\rangle;\mathbb{Z})$ and $\widetilde{H}_{i-1}(\langle T_i\rangle;\mathbb{Z})$, respectively. Let $\Delta := \sum_{ST_i} \theta^2_{S_{i-1}}\theta^2_{T_i}$.

4.3 A combinatorial simplicial Moore–Penrose pseudoinverse formula

Let $\Bbbk$ be a subfield of the real numbers $\mathbb{R}$.

Definition 4.3.1. The *Moore–Penrose pseudoinverse* of the map $d_{i+1} : C_{i+1}(X; \Bbbk) \to C_i(X; \Bbbk)$ is the unique map $d_i^+ : C_i(X; \Bbbk) \to C_{i+1}(X; \Bbbk)$ satisfying the following four properties:

1. $dd^+d = d$
2. $d^+dd^+ = d^+$
3. $(dd^+)^* = dd^+$
4. $(d^+d)^* = d^+d$

Furthermore, $(d^+)^2 = 0$, so d^+ is a vertical splitting of the boundary map d.

The following theorem, which gives a combinatorial formula for the Moore–Penrose pseudoinverse as a weighted average of hedge splittings, is the main result of this chapter. When used as the splitting in the natural Wall complex, it gives way to a combinatorial formula for the differential in a canonical minimal free resolution for an arbitrary monomial ideal.

Theorem 4.3.2. *The Moore–Penrose pseudoinverse* $d^+_{i-1} : \widetilde{C}_{i-1}(X;\Bbbk) \to \widetilde{C}_i(X;\Bbbk)$ *is a weighted average of splittings from hedges* ST_i*:*

$$d^+_{i-1} = \frac{1}{\Delta}\sum_{ST_i} \theta^2_{S_{i-1}}\theta^2_{T_i} d^+_{ST_i}.$$

Proof. The sum on the right satisfies the four properties of the pseudoinverse:

1. $dd^+d = d$
2. $d^+dd^+ = d^+$
3. $(dd^+)^* = dd^+$
4. $(d^+d)^* = d^+d$

Property 1:

$$\begin{aligned} d\Big(\frac{1}{\Delta}\sum_{ST_i}\theta^2_{S_{i-1}}\theta^2_{T_i}d^+_{ST_i}\Big)d &= \frac{1}{\Delta}\sum_{ST_i}\theta^2_{S_{i-1}}\theta^2_{T_i}dd^+_{ST_i}d \\ &= \frac{1}{\Delta}\sum_{ST_i}\theta^2_{S_{i-1}}\theta^2_{T_i}d \\ &= \frac{\Delta}{\Delta}d \\ &= d. \end{aligned}$$

Property 2:

It is first shown that if $ST_i = (S_{i-1}, T_i)$ and $ST'_i = (S'_{i-1}, T'_i)$, then $d^+_{ST'_i}dd^+_{ST_i} = d^+_{ST''_i}$, where $ST''_i = (S_{i-1}, T'_i)$.

Let $x \in S_{i-1}$, and let $B_x^{S_{i-1}}$ be the unique boundary from Lemma 4.2.7. Applying d to the result in Lemma 4.2.8, we get $dd^+_{ST_i}(x) = B_x^{S_{i-1}}$. Since $B_x^{S_{i-1}} \in \widetilde{B}_i(X;\Bbbk)$, it can be written uniquely as $B_x^{S_{i-1}} = \sum_\ell c_\ell d(t'_\ell)$, where $t'_\ell \in T'_i$. Then $d^+_{ST'_i}(B_x^{S_{i-1}}) = \sum_\ell c_\ell t'_\ell$.

To compute $d^+_{ST''_i}(x)$, write $x = \sum_j c_j s_j + B_x^{S_{i-1}}$, where $s_j \in \bar{S}_{i-1}$. Then

$$\begin{aligned} d^+_{ST''_i}(x) &= d^+_{ST''_i}(B_x^{S_{i-1}}) \\ &= \sum_\ell c_\ell t'_\ell \\ &= d^+_{ST'_i} dd^+_{ST_i}(x). \end{aligned}$$

If $x \in \bar{S}_{i-1}$, then $d^+_{ST_i}(x) = 0 = d^+_{ST''_i}(x)$.

Therefore

$$\begin{aligned} \left(\frac{1}{\Delta}\sum_{ST_i}\theta^2_{S_{i-1}}\theta^2_{T_i}d^+_{ST_i}\right)(d)\left(\frac{1}{\Delta}\sum_{ST_i}\theta^2_{S_{i-1}}\theta^2_{T_i}d^+_{ST_i}\right) &= \left(\frac{1}{\Delta}\right)^2 \sum_{ST_i,ST'_i} \theta^2_{S_{i-1}}\theta^2_{T_i}\theta^2_{S'_{i-1}}\theta^2_{T'_i} d^+_{ST_i} dd^+_{ST'_i} \\ &= \left(\frac{1}{\Delta}\right)^2 \sum_{ST_i,ST'_i} \theta^2_{S_{i-1}}\theta^2_{T_i}\theta^2_{S'_{i-1}}\theta^2_{T'_i} d^+_{ST''_i} \\ &= \left(\frac{1}{\Delta}\right)^2 \sum_{(S'_{i-1},T_i)} \theta^2_{S'_{i-1}}\theta^2_{T_i} \sum_{ST''_i} \theta^2_{S_{i-1}}\theta^2_{T'_i} d^+_{ST''_i} \\ &= \frac{1}{\Delta}\sum_{ST''_i}\theta^2_{S_{i-1}}\theta^2_{T_i}d^+_{ST''_i} \\ &= \frac{1}{\Delta}\sum_{ST_i}\theta^2_{S_{i-1}}\theta^2_{T_i}d^+_{ST_i}. \end{aligned}$$

Property 3:

This property is equivalent to the property $\langle dd^+(x), x'\rangle = \langle x, dd^+(x')\rangle$ for all x, x'.

Let $x \in S_{i-1}$, $x' \in \bar{S}_{i-1}$. Define $S'_{i-1} := S_{i-1} \setminus \{x\} \cup \{x'\}$ (thus $\bar{S}'_{i-1} = \bar{S}_{i-1} \setminus \{x'\} \cup \{x\}$). Consider $\langle \bar{S}_{i-1} \rangle$ and $\langle \bar{S}'_{i-1} \rangle$. Let $B_x^{S_{i-1}}$ be the unique boundary from Lemma 4.2.7 in $\Bbbk\{\bar{S}_{i-1} \cup \{x\}\}$. If $\langle B_x^{S_{i-1}}, x' \rangle \neq 0$, then by Lemma 4.2.9, S'_{i-1} is a stake set. Define $Y_{S_{i-1}}^{x,x'} := \bar{S}_{i-1} \cup \{x\} = \bar{S}'_{i-1} \cup \{x'\}$. Note that for a pair (x, x'), $Y_{S_{i-1}}^{x,x'} = Y_{S'_{i-1}}^{x',x}$.

Since $\widetilde{C}_{i-1} = \mathbb{Q}\{\bar{S}_{i-1}\} \oplus \widetilde{B}_{i-1}(X;\mathbb{Q})$, $\widetilde{B}_{i-2}(\langle \bar{S}_{i-1} \rangle;\mathbb{Q}) = \widetilde{B}_{i-2}(X;\mathbb{Q})$. Therefore the homology class $[d(x)]$ is torsion in $\widetilde{H}_{i-2}(\langle \bar{S}_{i-1} \rangle;\mathbb{Z})$.

By Lemma 4.2.7, $\widetilde{Z}_{i-1}(\langle Y_{S_{i-1}}^{x,x'} \rangle;\mathbb{Q}) \cap \widetilde{B}_{i-1}(X;\mathbb{Q})$ is infinite cyclic. Therefore $\widetilde{Z}_{i-1}(\langle Y_{S_{i-1}}^{x,x'} \rangle;\mathbb{Z}) \cap \widetilde{B}_{i-1}(X;\mathbb{Z})$ is also infinite cyclic. Choose B to be a generator.

Next it is shown that $Y_{S_{i-1}}^{x,x'}$ has an i-boundary with coefficient γ on x if and only if $\gamma d(x) = 0$ in $\widetilde{H}_{i-2}(\langle \bar{S}_{i-1} \rangle;\mathbb{Z})$:

$$\begin{aligned} \gamma d(x) \in \widetilde{B}_{i-2}(\langle \bar{S}_{i-1} \rangle;\mathbb{Z}) &\Leftrightarrow \gamma d(x) = d(w) \text{ for some } w \in \bar{S}_{i-1} \\ &\Leftrightarrow \gamma d(x) - d(w) = 0 \\ &\Leftrightarrow d(\gamma x - w) = 0. \end{aligned}$$

Since $\widetilde{C}_{i-1}(X;\mathbb{Q}) = \mathbb{Q}\{\bar{S}_{i-1}\} \oplus \widetilde{B}_{i-1}(X;\mathbb{Q})$, we can write $\gamma x - w = w' + b$, where $w' \in \bar{S}_{i-1}$ and $b \in \widetilde{B}_{i-1}(X;\mathbb{Q})$. Since $x \in S_{i-1}$, the coefficient on x in b is γ. Note that $b = \gamma x - w - w' \in Y_{S_{i-1}}^{x,x'}$. Therefore the smallest positive γ such that $\gamma d(x) = 0$ is the smallest positive γ such that $Y^{x,x'}_{S_{i-1}}$ has an i-boundary with coefficient γ on x. Since B is a generator for $\widetilde{Z}_{i-1}(\langle Y_{S_{i-1}}\rangle;\mathbb{Z}) \cap \widetilde{B}_{i-1}(X;\mathbb{Z})$, the smallest such γ is $|\langle B, x\rangle|$. So the order of the torsion element $[d(x)]$ in $\widetilde{H}_{i-2}(\langle \bar{S}_{i-1}\rangle;\mathbb{Z})$ is $|\langle B, x\rangle|$.

Then

$$0 \to \mathbb{Z}/|\langle B, x\rangle|\mathbb{Z} \to \widetilde{H}_{i-2}(\langle \bar{S}_{i-1}\rangle;\mathbb{Z}) \to \widetilde{H}_{i-2}(\langle Y^{x,x'}_{S_{i-1}}\rangle;\mathbb{Z}) \to 0$$

is an exact sequence. Define $\theta_{Y_{S_{i-1}}}$ as the order of the torsion subgroup of $\widetilde{H}_{i-2}(\langle Y^{x,x'}_{S_{i-1}}\rangle;\mathbb{Z})$. Therefore $|\langle B, x\rangle|\theta_{Y_{S_{i-1}}} = \theta_{S_{i-1}}$. Similarly, $|\langle B, x'\rangle|\theta_{Y_{S_{i-1}}} = \theta_{S'_{i-1}}$.

Next it must be shown that $\langle \sum_{ST_i} \theta^2_{S_{i-1}} dd^+_{ST_i}(x), x'\rangle = \langle x, \sum_{ST_i} \theta^2_{S_{i-1}} dd^+_{ST_i}(x')\rangle$. Write $B = \langle B, x\rangle x + \sum_k c_k s_k$, with $s_k \in \bar{S}_{i-1}$. Then $dd^+_{ST_i}(B) = \langle B, x\rangle dd^+_{ST_i}(x)$ by Lemma 4.2.8. Fix a shrubbery T_i. $B = \sum_\ell c_\ell d(t_\ell)$ with $t_\ell \in T_i$, $dd^+_{ST_i}(B) = \sum_\ell c_\ell dd^+_{ST_i} d(t_\ell) = \sum_\ell d(t_\ell) = B$. Therefore $dd^+_{ST_i}(x) = \frac{1}{\langle B, x\rangle} B$ (similarly, $dd^+_{ST'_i}(x') = \frac{1}{\langle B, x'\rangle} B)$, which is used below:

$$\begin{aligned}
\theta^2_{S_{i-1}}\langle dd^+_{ST_i}(x), x'\rangle &= \theta^2_{S_{i-1}}\langle B/\langle B, x\rangle, x'\rangle \\
&= \theta^2_{S_{i-1}}\Big(\frac{1}{\langle B, x\rangle}\Big)\langle B, x'\rangle \\
&= |\langle B, x\rangle|^2\theta^2_{Y_{S_{i-1}}}\Big(\frac{1}{\langle B, x\rangle}\Big)\langle B, x'\rangle \\
&= \theta^2_{Y_{S_{i-1}}}\langle B, x\rangle\langle B, x'\rangle \\
&= \theta^2_{Y_{S_{i-1}}}\langle B, x\rangle\frac{|\langle B, x'\rangle|^2}{\langle B, x'\rangle}
\end{aligned}$$

$$= \theta^2_{S'_{i-1}} \frac{\langle B, x\rangle}{\langle B, x'\rangle}$$

$$= \theta^2_{S'_{i-1}} \langle x, dd^+_{ST'_i}(x')\rangle.$$

Let $\mathbb{T}_{x,x'}$ be the set of all hedges ST_i such that $\langle dd^+_{ST_i}(x), x'\rangle \neq 0$. Since $dd^+_{ST_i}(x) = \frac{1}{\langle B,x\rangle} B$ and $dd^+_{ST'_i}(x') = \frac{1}{\langle B,x'\rangle} B$, this is in bijection with the set $\mathbb{T}_{x',x}$ of hedges ST'_i such that $\langle x, dd^+_{ST'_i}(x')\rangle \neq 0$.

Summing over all $ST_i \in \mathbb{T}_{x,x'}$ gives

$$\sum_{ST_i \in \mathbb{T}_{x,x'}} \theta^2_{S_{i-1}} \langle dd^+_{ST_i}(x), x'\rangle = \sum_{ST'_i \in \mathbb{T}_{x',x}} \theta^2_{S'_{i-1}} \langle x, dd^+_{ST'_i}(x')\rangle.$$

So, for any pair (x, x'),

$$\langle \sum_{ST_i} \theta^2_{S_{i-1}} dd^+_{ST_i}(x), x'\rangle = \sum_{ST_i} \theta^2_{S_{i-1}} \langle dd^+_{ST_i}(x), x'\rangle$$

$$= \sum_{ST_i \in \mathbb{T}_{x,x'}} \theta^2_{S_{i-1}} \langle dd^+_{ST_i}(x), x'\rangle$$

$$= \sum_{ST'_i \in \mathbb{T}_{x',x}} \theta^2_{S'_{i-1}} \langle x, dd^+_{ST'_i}(x')\rangle$$

$$= \sum_{ST_i} \theta^2_{S_{i-1}} \langle x, dd^+_{ST_i}(x')\rangle$$

$$= \langle x, \sum_{ST_i} \theta^2_{S_{i-1}} dd^+_{ST_i}(x')\rangle.$$

Thus $\langle \sum_{ST_i} \theta^2_{S_{i-1}} \theta^2_{T_i} dd^+_{ST_i}(x), x'\rangle = \langle x, \sum_{ST_i} \theta^2_{S_{i-1}} \theta^2_{T_i} dd^+_{ST_i}(x')\rangle$.

Property 4:

This property is equivalent to $\langle d^+d(x), x'\rangle = \langle x, d^+d(x')\rangle$ for all x, x'.

By Lemma 4.2.6, $d^+_{ST_i}d = I - (I - d^+_{ST_i}d) = I - \phi_{T_i}$. Consider $\langle T_i\rangle$. Let $x \in \bar{T}_i$, and let c be a generator of $\widetilde{H}_i(\langle T_i \cup \{x\}\rangle; \mathbb{Z})$. Let $x' \in T_i$ such that $\langle c, x'\rangle \neq 0$. Define T_i' as the shrubbery T_i with x substituted in for x'. By Lemma 2.5 of [CCK15], T_i' is a shrubbery. Let $ST_i' = (S_{i-1}, T_i')$. By Corollary 2.9 of [CCK15], $\theta^2_{T_i}\langle -\phi_{T_i}(x), x'\rangle = \theta^2_{T_i'}\langle x, -\phi_{T_i'}(x')\rangle$.

Therefore $\theta^2_{S_{i-1}}\theta^2_{T_i}\langle \phi_{T_i}(x), x'\rangle = \theta^2_{S_{i-1}}\theta^2_{T_i'}\langle x, \phi_{T_i'}(x')\rangle$.

Now let $\mathbb{T}_{x,x'}$ be the set of hedges ST_i such that $\langle \phi_{T_i}(x), x'\rangle \neq 0$. Note that this is in bijection with the set $\mathbb{T}_{x',x}$ of hedges such that $\langle x, \phi_{T_i'}(x')\rangle \neq 0$.

Then

$$\begin{aligned}
\left\langle \sum_{ST_i} \theta^2_{S_{i-1}}\theta^2_{T_i}(I - \phi_{T_i})(x), x'\right\rangle &= \left\langle (\sum_{ST_i} \theta^2_{S_{i-1}}\theta^2_{T_i} x - \sum_{ST_i} \theta^2_{S_{i-1}}\theta^2_{T_i}\phi_{T_i}(x)), x'\right\rangle \\
&= \left\langle \sum_{ST_i} \theta^2_{S_{i-1}}\theta^2_{T_i} x, x'\right\rangle - \left\langle \sum_{ST_i} \theta^2_{S_{i-1}}\theta^2_{T_i}\phi_{T_i}(x), x'\right\rangle \\
&= \left\langle \sum_{ST_i} \theta^2_{S_{i-1}}\theta_{T_i} x, x'\right\rangle - \left\langle \sum_{ST_i \in \mathbb{T}_{x,x'}} \theta^2_{S_{i-1}}\theta^2_{T_i}\phi_{T_i}(x), x'\right\rangle \\
&= \left\langle x, \sum_{ST_i} \theta^2_{S_{i-1}}\theta^2_{T_i} x'\right\rangle - \left\langle x, \sum_{ST_i' \in \mathbb{T}_{x',x}} \theta^2_{S_{i-1}}\theta^2_{T_i'}\phi_{T_i'}(x')\right\rangle \\
&= \left\langle x, \sum_{ST_i} \theta^2_{S_{i-1}}\theta^2_{T_i} x'\right\rangle - \left\langle x, \sum_{ST_i} \theta^2_{S_{i-1}}\theta^2_{T_i}\phi_{T_i'}(x')\right\rangle \\
&= \left\langle x, \sum_{ST_i} \theta^2_{S_{i-1}}\theta^2_{T_i}(I - \phi_{T_i})(x')\right\rangle. \qquad \square
\end{aligned}$$

Chapter 5

Combinatorial description of the Wall differentials

A minimal free resolution for an arbitrary monomial ideal can now be written down over a field of any characteristic by taking the Wall complex of the Koszul bicomplex, with an appropriate choice of splitting. Over a field of characteristic 0 and all but finitely many positive characteristics, the Moore–Penrose pseudoinverse gives a canonical resolution, called the *sylvan resolution*. What is left is to determine a combinatorial description of the canonical sylvan resolution—where the Moore–Penrose pseudoinverse is the splitting—in Theorem 5.4.5, the main result of this dissertation. A combinatorial description of noncanonical sylvan resolutions is given in Theorem 5.3.2. These theorems express the differential as a sum over lattice paths in $\mathbb{N}^n$ of weights of sequences of faces associated to each other via shrubberies, stake sets, and hedges. The work presented here is the product of joint work with John Eagon and Ezra Miller.

5.1 Sylvan matrices

One problem that arises when writing down the maps of the resolution canonically is that the homology vector spaces $\widetilde{H}_i(K^{\mathbf{b}}I;\Bbbk)$ do not have canonical bases. For instance, in the ideal $I = \langle xy, xz, yz\rangle \subseteq \Bbbk[x,y,z]$, $K^{111}I$ is isomorphic to three vertices, x, y, and z. Writing down a matrix for the map $\bigoplus_{\mathbf{a}\preceq 111} \widetilde{H}_{-1}(K^{\mathbf{a}}I;\Bbbk) \leftarrow \widetilde{H}_0(K^{111}I;\Bbbk)$ with the appropriate dimensions requires a choice of basis. Since the dimension of $\widetilde{H}_0(K^{111}I;\Bbbk)$ as a $\Bbbk$-vector space is 2 and $\widetilde{H}_0(K^{111}I;\Bbbk)$ is spanned by differences of pairs of vertices, namely $x-y$, $x-z$, and $y-z$, there is no canonical basis.

The chains $\widetilde{C}_i(K^{\mathbf{b}}I;\Bbbk)$, however, have canonical bases given by the i-dimensional faces in $K^{\mathbf{b}}I$. The differentials in the sylvan resolution are defined on the reduced chains. In order to induce a homomorphism on homology, they need to send cycles to cycles and boundaries to boundaries. This fact follows from the definition of the maps $\widetilde{\omega}_j$ (Definition 3.2.9); the first map applied is the projection $I - dd^+$, which is the identity map minus the projection onto the boundaries. This ensures that boundaries are sent to boundaries; in particular, boundaries are sent to zero. The last map applied in $\widetilde{\omega}_j$ is the projection $I - d^+d$ onto cycles, ensuring that the output is a cycle.

The resolutions are written down using sylvan matrices, where the rows and columns are indexed by the bases for the reduced chains.

Definition 5.1.1. The *sylvan matrix* for the map $\widetilde{H}_{i-1}K^{\mathbf{a}}I \leftarrow \widetilde{H}_i K^{\mathbf{b}}I$ of subquotients of the chain groups is the matrix

$$\widetilde{H}_{i-1}K^{\mathbf{a}}\otimes\langle\mathbf{x}^{\mathbf{a}}\rangle \xleftarrow{\quad \begin{array}{c c} & \begin{array}{cccc} \tau_1 & \tau_2 & \cdots & \tau_r \end{array} \\ \begin{array}{c} \sigma_1 \\ \sigma_2 \\ \vdots \\ \sigma_s \end{array} & \left[\begin{array}{cccc} & & & \\ & & & \\ & & & \\ & & & \end{array}\right] \end{array} \quad} \widetilde{H}_i K^{\mathbf{b}}\otimes\langle\mathbf{x}^{\mathbf{b}}\rangle$$

where τ_ℓ is an i-face in $K^{\mathbf{b}}I$, σ_ℓ is an $(i-1)$-face in $K^{\mathbf{a}}I$, and the entry $D^{\mathbf{ab}}_{\sigma\tau}$ is the coefficient of $\sigma \otimes \mathbf{x}^{\mathbf{a}} \cdot \mathbf{x}^{\mathbf{b}-\mathbf{a}}$ in $D(\tau \otimes \mathbf{x}^{\mathbf{b}})$. See Remark 3.2.7 for an explanation of this notation.

5.2 Chain-link fences

The combinatorial formula for the differential in the sylvan resolution is given as a sum of weights of chain-link fences, which are sequences of faces that are related to each other via shrubberies, stake sets, and hedges.

Lemma 5.2.1. *For any i-face τ and any shrubbery T_i, there is a unique cycle $\zeta_{T_i}(\tau) = \tau - t \in \Bbbk\{T_i \cup \{\tau\}\}$.*

Proof. The boundary $d(\tau)$ can be written uniquely as $\sum_k c_k d(t_k) = d(\sum_k c_k t_k) = d(t)$. Then $d(\tau - t) = d(\tau) - d(t) = 0$. $\square$

Lemma 5.2.2. *Given a hedge ST_i and a face $\sigma \in S_{i-1}$, there is a unique $s \in \Bbbk\{T_i\}$ such that $\langle d(s), \sigma \rangle = 1$ and $\langle d(s), \sigma' \rangle = 0$ for all $\sigma' \in S_{i-1}$, $\sigma' \neq \sigma$.*

Proof. This is immediate from Lemma 4.2.7. $\square$

Lemma 5.2.3. *Given a stake set S_i and an i-face ρ, there is a unique $r \in \Bbbk\{\bar{S}_i\}$ such that $\rho - r \in \widetilde{B}_i(X; \Bbbk)$. If $\rho \neq r$, then $\rho - r = B_\rho^{S_i}$ from Lemma 4.2.7.*

Proof. This is immediate from Lemma 4.2.7. $\square$

Definition 5.2.4. Fix a shrubbery T_{i-1}.

1. The *circuit* of an $(i-1)$-face σ is the cycle $\zeta_{T_{i-1}}(\sigma)$ in Lemma 5.2.1.

2. σ is *cycle-linked* to any $(i-1)$-face σ' if $\langle \zeta_{T_{i-1}}(\sigma), \sigma' \rangle \neq 0$.

Definition 5.2.5. Fix a hedge ST_i.

1. The *shrub* of a stake $\sigma \in S_{i-1}$ is the chain s in Lemma 5.2.2.

2. σ is *chain-linked* to an i-face τ if $\langle \tau, s \rangle \neq 0$.

Definition 5.2.6. Fix a stake set S_{i-1}.

1. The *hedge rim* of an i-face ρ is the chain r from Lemma 5.2.3.

2. ρ is *boundary-linked* to an i-face ρ' if $\langle r, \rho' \rangle \neq 0$.

Example 5.2.7. Consider the simplicial complex of dimension 1 given in Figure 5.1. Let ST_1 be the hedge (S_0, T_1), where $S_0 = \{b, c, d, e\}$ and $T_1 = \{ac, bc, bd, de\}$. Then the circuit of the edge cd is $\zeta_{T_1}(cd) = cd + bc - bd$, the shrub of the vertex b is $s(b) = bc - ac$, and the hedge rim of b is $r(b) = a$. This means that cd is cycle-linked to cd, bc, and bd. The vertex b is chain-linked to the edges bc and ac, and it is boundary-linked to the vertex a.

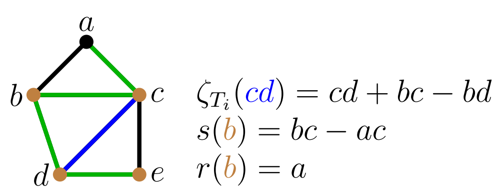


Figure 5.1: Example of a circuit, shrub, and hedge rim

Definition 5.2.8. Given a lattice path $\lambda = (\mathbf{a} = \mathbf{b}_j, \mathbf{b}_{j-1}, \ldots, \mathbf{b}_1, \mathbf{b}_0 = \mathbf{a}) \in \Lambda(\mathbf{a}, \mathbf{b})$, a *hedgerow* ST_i^λ on λ consists of a stake set $S_i^{\mathbf{b}} \subseteq K^{\mathbf{b}}I$, a hedge $ST_i^{\mathbf{c}} \subseteq K^{\mathbf{c}}I$ for each degree $\mathbf{c} = \mathbf{b}_1, \ldots, \mathbf{b}_{j-1}$, and a shrubbery $T_{i-1}^{\mathbf{a}} \subseteq K_{i-1}^{\mathbf{a}}I$.

Definition 5.2.9. Given a lattice path $\lambda \in \Lambda(\mathbf{a}, \mathbf{b})$, a *chain-link fence* ϕ from an i-face τ to an $(i-1)$-face σ along λ consists of a hedgerow ST_i^λ and a sequence of faces

$$
\begin{array}{ccccccccc}
 & \tau_{j-1} & & \cdots & & \tau_1 & & \tau_0 - \tau & \\
 & / \quad \backslash & / & & \backslash & / & \backslash & / & \\
\sigma - \sigma_j & & \sigma_{j-1} & & \sigma_2 & & \sigma_1 & &
\end{array}
$$

in which $\tau_\ell \in K_i^{\mathbf{b}_\ell} I$, $\sigma_\ell \in K_{i-1}^{\mathbf{b}_\ell} I$, and

- τ is boundary-linked to τ_0 in the stake set $S_i^{\mathbf{b}}$,
- $\sigma_\ell \in S_{i-1}^{\mathbf{b}_\ell}$ is chain-linked to τ_ℓ for all $\ell = 1, \ldots, j-1$,
- $\sigma_\ell = \tau_{\ell-1} - e_{k_\ell}$ for all $\ell = 1, \ldots, j-1$, where $e_{k_\ell} = \mathbf{b}_{\ell-1} - \mathbf{b}_\ell$, and
- σ_j is cycle-linked to σ in the shrubbery $T_{i-1}^{\mathbf{a}}$.

The chain-link fence ϕ is *subordinate* to the hedgerow ST_i^λ, denoted $\phi \vdash ST_i^\lambda$. Define $\Phi_{\sigma\tau}(\lambda)$ as the set of chain-link fences from τ to σ along the lattice path λ.

Each edge in a chain-link fence is assigned a weight, which is the coefficient of the succeeding face in the circuit, shrub, or hedge rim of the preceding face.

Definition 5.2.10. The simple weight of the boundary link is $\langle \tau_0, r(\tau) \rangle$, the simple weight of the chain-link between σ_ℓ and τ_ℓ is $\langle \tau_\ell, s(\sigma_\ell) \rangle$, and the simple weight of the cycle-link is $\langle \sigma, \zeta_{T_{i-1}^{\mathbf{a}}}(\sigma_j) \rangle$. The simple weight of the edge between $\tau_{\ell-1}$ and σ_ℓ is $(-1)^{e_{k_\ell}}$. The *simple weight* of the chain-link fence ϕ is the product w_ϕ of the weights on its edges.

5.3 Combinatorial descriptions of noncanonical sylvan resolutions

The maps in the natural Wall complex can be decomposed into sums over lattice paths $\lambda \in \Lambda(\mathbf{a}, \mathbf{b})$ of weights of chain-link fences. The lemma below is used to prove the main two results of this chapter, Theorems 5.4.5 and 5.3.2.

Lemma 5.3.1. *Given any splittings $d^{\mathbf{b}+}$ of the boundary maps $d^{\mathbf{b}}$ in $K^{\mathbf{b}}I$, the Wall complex differential $D : \widetilde{H}_i K^{\mathbf{b}} I \to \widetilde{H}_{i-1} K^{\mathbf{a}} I$ can be written*

$$D = \sum_{\lambda \in \Lambda(\mathbf{a},\mathbf{b})} (I^{\mathbf{a}} - d_i^{\mathbf{a}+} d_i^{\mathbf{a}}) d_1^{k_j} \Big(\prod_{\ell=1}^{j-1} d_i^{\mathbf{b}_\ell +} d_1^{k_\ell} \Big) (I^{\mathbf{b}} - d_{i+1}^{\mathbf{b}} d_{i+1}^{\mathbf{b}+}),$$

where $d_1^k(\tau) = (-1)^{e_k \subset \tau} \tau \setminus k$ if $k \in \tau$ and $= 0$ otherwise.

Here, each $\lambda = (\mathbf{a} = \mathbf{b}_j, \mathbf{b}_{j-1}, \ldots, \mathbf{b}_1, \mathbf{b}_0 = \mathbf{b})$, where $\mathbf{b}_{i-1} - \mathbf{b}_i = e_{k_i}$.

Proof. Recall that the differential for the natural Wall complex is given by $D = \sum_j \widetilde{\omega}_j$, where $\widetilde{\omega}_j = (I - d^+ d)(d_1 d^+)^{j-1} d_1 (I - dd^+) : \widetilde{H}_{pq} \to \widetilde{H}_{p-j,q+j-1}$. The modules $\mathbb{K}_{pq}(I)$ have a $\Bbbk$-linear basis consisting of elements $\mathbf{z}^\tau \otimes \mathbf{y}^{\mathbf{b}-\tau} \otimes \mathbf{x}^{\mathbf{a}}$. The vertical differential d and the horizontal differential d_1 are given by

$$d(\mathbf{z}^\tau \otimes \mathbf{y}^{\mathbf{b}-\tau} \otimes \mathbf{x}^{\mathbf{a}}) = \sum_{k \in \tau} (-1)^{e_k \subset \tau} \mathbf{z}^{\tau - e_k} \otimes \mathbf{y}^{\mathbf{b}+e_k} \otimes \mathbf{x}^{\mathbf{a}}$$

and

$$d_1(\mathbf{z}^\tau \otimes \mathbf{y}^{\mathbf{b}-\tau} \otimes \mathbf{x}^{\mathbf{a}}) = \sum_{k \in \tau} (-1)^{e_k \subset \tau} \mathbf{z}^{\tau - e_k} \otimes \mathbf{y}^{\mathbf{b}} \otimes \mathbf{x}^{\mathbf{a}+e_k}.$$

Since $(\mathbb{K}^{\mathbf{y}}_{\bullet}(I^{\mathbf{y}}))_{\mathbf{b}} \cong \widetilde{C}_{\bullet} K^{\mathbf{b}} I$, the element $\mathbf{z}^\tau \otimes \mathbf{y}^{\mathbf{b}-\tau} \in \bigwedge^{|\tau|} V \otimes I^{\mathbf{y}}_{\mathbf{b}-\tau}$ is identified with the face $\tau \in K^{\mathbf{b}} I$. Thus the vertical differential d can be identified with the boundary map of $K^{\mathbf{b}} I$. The horizontal differential d_1 can be decomposed as $d_1 = \sum_k d_1^k$, where $d_1^k(\mathbf{z}^\tau \otimes \mathbf{y}^{\mathbf{b}} \otimes \mathbf{x}^{\mathbf{a}}) = (-1)^{e_k \subset \tau} \mathbf{z}^{\tau - e_k} \otimes \mathbf{y}^{\mathbf{b}} \otimes \mathbf{x}^{\mathbf{a}+e_k}$ if $k \in \tau$ and $= 0$ otherwise. Again because of the isomorphism $(\mathbb{K}^{\mathbf{y}}_{\bullet}(I^{\mathbf{y}}))_{\mathbf{b}} \cong \widetilde{C}_{\bullet} K^{\mathbf{b}} I$, d_1^k takes the face $\tau \in K^{\mathbf{b}} I$ to the face $\tau \setminus \{k\} \in K^{\mathbf{b}-e_k} I$. Thus every time d_1^k is applied, the Koszul degree moves back by e_k. Since d^+ is a splitting of d, it is induced by a splitting of the boundary map ∂ of $K^{\mathbf{b}} I$. Therefore it does not alter the Koszul degree.

Plugging the formula $d_1 = \sum_k d_1^k$ into the formula for $\widetilde{\omega}_j$ and splitting it into sums of terms $(I - d^+ d)(d_1^{k_j} d^+ d_1^{k_{j-1}} d^+ \cdots d_1^{k_1})(I - dd^+)$ gives a formula for $\widetilde{\omega}_j$ as a sum over lattice paths $(\mathbf{a} = \mathbf{b}_j, \mathbf{b}_{j-1}, \ldots, \mathbf{b}_1, \mathbf{b}_0 = \mathbf{b})$, where $\mathbf{b}_{i-1} - \mathbf{b}_i = e_{k_i}$. □

The following theorem says that the coefficients from the map D are the sums of the simple weights of the chain-link fences along all lattice paths $\lambda \in \Lambda(\mathbf{a}, \mathbf{b})$, giving a combinatorial description of the differentials in noncanonical sylvan resolutions. This theorem, together with Theorem 5.4.5, is the main result of this chapter.

Theorem 5.3.2. *Let the vertical splitting d^+ be induced by the splittings associated to hedges in a community for $K^{\mathbf{b}}I$ in each degree $\mathbf{b} \in \mathbb{N}^n$. Then the sylvan matrix has entries*

$$D^{\mathbf{ab}}_{\sigma\tau} = \sum_{\lambda \in \Lambda(\mathbf{a},\mathbf{b})} \sum_{\substack{\phi \in \Phi_{\sigma\tau}(\lambda) \\ \phi \vdash ST_i^{\lambda}}} w^{\Bbbk}_{\phi}.$$

Proof. The chain-link fences come from selecting one face with nonzero coefficient in the result of applying each map in $\widetilde{\omega}_j$. Applying the formula in Lemma 5.3.1 and starting with a face $\tau \in K^{\mathbf{b}}I$, the projection $I - dd^+$ is applied. One face with nonzero coefficient in $(I - dd^+)(\tau)$, the hedge rim of τ, is chosen as τ_0, and the weight along the edge between τ and τ_0 in the chain-link fence is the coefficient of τ_0 in $(I - dd^+)(\tau)$. Next the map $d_1^{k_1}$ is applied. This takes the face $\tau_0 \in K^{\mathbf{b}}I$ to $\sigma_1 = \tau_0 \setminus k_1 \in K^{\mathbf{b}-e_k}I$. The weight on the edge between these faces in the chain-link fence is the coefficient of σ_1 in $d_1^{k_1}(\tau_0)$, which is $(-1)^{e_{k_1} \subset \tau_0}$. Next d^+ is applied to σ_1. This takes σ_1 to the shrub of σ_1 in $ST_i^{\mathbf{b}}$, a linear combination of faces in $K^{\mathbf{b}-e_{k_1}}I$. One of these faces is selected as τ_1, and the weight along the edge between σ_1 and τ_1 is the coefficient of τ_1 in $d^+(\sigma_1)$. This process continues until a face σ is selected from $(I - d^+d)(\sigma_j)$, the circuit of σ_j in $T^{\mathbf{b}}_{i-1}$, and the coefficient of σ in $(I - d^+d)(\sigma_j)$ is the weight of the edge between these two faces in the chain-link fence. Thus taking the product of the weights along the edges in the chain-link fence and summing over all chain-link fences with initial post τ and terminal post σ along all lattice paths from $\mathbf{b}$ to $\mathbf{a}$ in $\mathbb{N}^n$ gives the coefficient of σ in $D^{\mathbf{ab}}(\tau)$. $\square$

5.4 Combinatorial description of the canonical sylvan resolution

The differential for the canonical sylvan resolution uses the Moore–Penrose pseudoinverse as the splitting of the boundary map d, as the pseudoinverse is a weighted average of all hedge splittings. New weights are introduced for the edges in a chain-link fence, coming from the coefficients in the formula for the pseudoinverse, from Theorem 4.3.2. This gives rise to the combinatorial description of the differential in the canonical sylvan resolution in Theorem 5.4.5.

Definition 5.4.1. Given a chain-link fence

$$\begin{array}{ccccccccccc} & & \tau_{j-1} & & \cdots & & \tau_1 & & \tau_0 & — & \tau \\ & / & & \backslash \quad / & & \backslash & / & \backslash & / & & \\ \sigma & — & \sigma_j & & \sigma_{j-1} & & \sigma_2 & & \sigma_1 & & \end{array}$$

along a lattice path λ, the following weights are defined:

1. The weight of the boundary-link is $\theta^2_{S_i^{\mathbf{b}}}\langle \tau_0, r(\tau)\rangle$.

2. The weight of the chain-link between σ_ℓ and τ_ℓ is $\theta^2_{S^{\mathbf{b}_\ell}_{i-1}} \theta^2_{T^{\mathbf{b}_\ell}_i} \langle \tau_\ell, s(\sigma_\ell) \rangle$.

3. The weight of the cycle-link is $\theta^2_{T^{\mathbf{a}}_{i-1}} \langle \sigma, \zeta_{T^{\mathbf{a}}_{i-1}}(\sigma_j) \rangle$.

4. The weight of the edge between $\tau_{\ell-1}$ and σ_ℓ is $(-1)^{e_{k\ell}}$.

Definition 5.4.2. Let X be a simplicial complex. Define

- $\Delta_i^T X = \sum_{T_i} \theta^2_{T_i}$
- $\Delta_{i-1}^S X = \sum_{S_{i-1}} \theta^2_{S_{i-1}}$
- $\Delta_i^{ST_i} X = \sum_{ST_i} \theta^2_{S_{i-1}} \theta^2_{T_i}$.

Given a lattice path $\lambda = (\mathbf{a} = \mathbf{b}_j, \mathbf{b}_{j-1}, \ldots, \mathbf{b}_1, \mathbf{b}_0 = \mathbf{b})$, define

$$\Delta_{i,\lambda} I = (\Delta_{i-1}^T K^{\mathbf{a}} I) \prod_{\ell=1}^{j-1} \Delta_i^{ST_i} K^{\mathbf{b}_\ell} I (\Delta_i^S K^{\mathbf{b}} I).$$

Proposition 5.4.3. *Let d^+ be the Moore–Penrose pseudoinverse of the boundary map d. Then the projection $I - dd^+$ of C_i onto $H_i \oplus B'_{i-1}$ has the formula*

$$I - dd^+ = \frac{1}{\sum_{S_i} \theta^2_{S_i}} \sum_{S_i} \theta^2_{S_i} (I - B^{S_i}),$$

where B^{S_i} is the map that sends $x \in S_i$ to $B^{S_i}_x$, the unique boundary from Lemma 4.2.7, and sends $x \in \bar{S}_i$ to 0. Note that $(I - B^{S_i})(x)$ is equal to the hedge rim r.

Proof. By Lemma 4.2.8, $dd^+_{ST_{i+1}} = B^{S_i}$. Then, using the formula for d^+ from Theorem 4.3.2,

$$\begin{aligned} I - dd^+ &= I - \frac{1}{\Delta} \sum_{ST_{i+1}} \theta^2_{S_i} \theta^2_{T_{i+1}} dd^+_{ST_{i+1}} \\ &= \frac{1}{\Delta} \big(\sum_{ST_{i+1}} \theta^2_{S_i} \theta^2_{T_{i+1}} - \sum_{ST_{i+1}} \theta^2_{S_i} \theta^2_{T_{i+1}} B^{S_i} \big) \\ &= \frac{1}{\sum_{S_i} \theta^2_{S_i} \sum_{T_{i+1}} \theta^2_{T_{i+1}}} \big(\sum_{S_i} \theta^2_{S_i} \sum_{T_{i+1}} \theta^2_{T_{i+1}} - \sum_{T_{i+1}} \theta^2_{T_{i+1}} \sum_{S_i} \theta^2_{S_i} B^{S_i} \big) \\ &= \frac{1}{\sum_{S_i} \theta^2_{S_i}} \sum_{S_i} \theta^2_{S_i} (I - B^{S_i}). \end{aligned}$$

□

Proposition 5.4.4. *Let d^+ be the Moore–Penrose pseudoinverse of the boundary map d. Then the projection $I - d^+d$ of C_{i-1} onto Z_{i-1} has the formula*

$$I - d^+d = \frac{1}{\sum_{T_{i-1}} \theta^2_{T_{i-1}}} \sum_{T_{i-1}} \theta^2_{T_{i-1}} \zeta_{T_{i-1}},$$

where $\zeta_{T_{i-1}}$ is the map that sends an i-face x to its circuit $\zeta_{T_{i-1}}(x)$ given the shrubbery T_{i-1}.

Proof. Theorem A in [CCK15] gives the formula $\frac{1}{\sum_{T_{i-1}} \theta^2_{T_{i-1}}} \sum_{T_{i-1}} \theta^2_{T_{i-1}} \phi_{T_{i-1}}$ for the orthogonal projection $C_{i-1} \to Z_{i-1}$ onto cycles. The orthogonal projection onto cycles is equal to $I - d^+d$. By Lemma 4.2.6, $\phi_{T_{i-1}} = I - d^+_{ST_{i-1}} d$. Thus the projection can be written

$$I - d^+d = \frac{1}{\sum_{T_{i-1}} \theta^2_{T_{i-1}}} \sum_{T_{i-1}} \theta^2_{T_{i-1}} (I - d^+_{ST_{i-1}} d).$$

Note that

$$\begin{aligned}(I - d^+_{ST_{i-1}} d)(\tau) &= \tau - d^+_{ST_{i-1}} \sum_k c_k d(t_k) \\ &= \tau - d^+_{ST_{i-1}} d(t) \\ &= \tau - t \\ &= \zeta_{T_{i-1}}(\tau).\end{aligned}$$

This gives

$$I - d^+d = \frac{1}{\sum_{T_{i-1}} \theta^2_{T_{i-1}}} \sum_{T_{i-1}} \theta^2_{T_{i-1}} \zeta_{T_{i-1}}.$$ □

The following theorem is the main result of this dissertation, giving a combinatorial formula for the differential in the canonical sylvan resolution. With this formula, the canonical sylvan resolution satisfies all of the desired properties for a solution to Kaplansky's problem.

Theorem 5.4.5. *The canonical sylvan homomorphism* $D^{\mathbf{ab}} : \widetilde{C}_i K^{\mathbf{b}} I \to \widetilde{C}_{i-1} K^{\mathbf{a}} I$ *is* *given by its sylvan matrix, with entries*

$$D_{\sigma\tau}^{\mathbf{ab}} = \sum_{\lambda \in \Lambda_{\mathbf{ab}}} \frac{1}{\Delta_{i,\lambda} I} \sum_{\phi \in \Phi_{\sigma\tau}(\lambda)} w_\phi.$$

Proof. Plug the formula from Theorem 4.3.2 into the formula for D in Lemma 5.3.1. By Proposition 5.4.3, this splits the first projection applied into the weighted sum over all stake sets S_i of hedge rims. By Theorem 4.3.2, each $d_i^{\mathbf{b}_\ell +}$ is split into a weighted sum over all hedges ST_i of shrubs. And finally, by Proposition 5.4.4, the last projection applied is split into a weighted sum over all shrubberies T_{i-1} of circuits. Factor out all of the denominators from the formulas in Propositions 5.4.3 and 5.4.4 and Theorem 4.3.2 along each lattice path. In total, this splits the entire differential into a sum over all lattices paths of $\frac{1}{\Delta_{i,\lambda} I}$ times the sum over all combinations of choices of circuits at lattice point $\mathbf{a}$, hedges at interior points $\mathbf{b}_{j-1}, \ldots, \mathbf{b}_1$, and stake sets at point $\mathbf{b}$.

Along a lattice path λ, the first projection sends an i-face τ to the weighted sum over all hedges ST_i of its hedge rim $r(\tau)$ by Proposition 5.4.3. The hedge rim is a linear combination of faces that are boundary-linked to τ, and the coefficient of each face in $r(\tau)$ is equal to the weight of the boundary-link between τ and the corresponding face in the hedge rim. Applying the next map to just one of these faces τ_0 gives the face $\sigma_1 := \tau_0 - \lambda_1$. The coefficient on σ_1 is the weight of the boundary link between τ

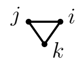
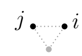
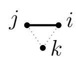
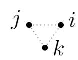
and τ_0 times $(-1)^{\lambda_1}$. By Lemma 4.2.8, applying $d_i^{\mathbf{b}_1+}$ gives the shrub of σ_1 times the weights previously mentioned. Select one of the faces in the shrub of σ_1 to get the face τ_1, which is chain-linked to σ_1. The coefficient on τ_1 in this composition of maps is the product of the two weights already applied and the weight of the chain-link between σ_1 and τ_1. Iterate this process until arriving at a face σ_j in $K^{\mathbf{a}}I$. Applying the last projection gives the circuit of σ_j. Select just one of these faces to get the face σ, which is cycle-linked to σ_j. The coefficient of σ in the circuit of σ_j is the weight of the cycle-link between σ_j and σ. The total coefficient on σ in this sequence of maps applied to τ is w_ϕ. Thus the coefficient of σ in $D^{\mathbf{ab}}(\tau)$ along this lattice path is the sum of all weights of chain-link fences along λ with initial post τ and terminal post σ. □

Chapter 6

The three-variable case

The combinatorial formula for the differential in the canonical sylvan resolution simplifies in the three-variable case, since there is a small number of cases for $K^{\mathbf{b}}I$, and all torsion numbers are 1. The two main results of this chapter are Theorem 6.1.1 and Theorem 6.2.16, which specify the first and second boundary maps in the canonical sylvan resolution of a monomial ideal in three variables. The first of these, Theorem 6.1.1 in Section 6.1, describes all values $D^{\mathbf{ab}}_{\varnothing v}$, where v is a vertex in $K^{\mathbf{b}}I$. As a corollary to Theorem 6.1.1, Corollary 6.1.6 expresses each entry $D^{\mathbf{ab}}_{\varnothing v}$ as the number of lattice paths satisfying certain properties in a grid, all with multiplicity 1, divided by the total number of lattice paths. The second main result, Theorem 6.2.16 in Section 6.2.4, combinatorially expresses each entry $D^{\mathbf{ab}}_{ve}$, where v is a vertex in $K^{\mathbf{a}}I$ and e is an edge in $K^{\mathbf{b}}I$. Section 6.3 gives more examples of canonical sylvan resolutions in three variables, using the results proven in this chapter.

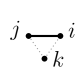
6.1 Coefficients in maps from $\widetilde{H}_0(K^{\mathbf{b}}I;\Bbbk)$ to $\widetilde{H}_{-1}(K^{\mathbf{c}}I;\Bbbk)$

Let x_1, x_2, and x_3 be indeterminants. In the three-variable case, $K^{\mathbf{b}}I$ is always a subcomplex of the simplex . Since the simplicial complexes $K^{\mathbf{b}}I$ are too small to have any torsion, $\Delta_{i,\lambda}I$ is the number of hedgerows along λ. If $\beta_{1,\mathbf{b}} = \dim_{\Bbbk}(\widetilde{H}_0(K^{\mathbf{b}}(I);\Bbbk) \neq 0$, then $K^{\mathbf{b}}(I)$ is isomorphic to either two vertices , a vertex and an edge , or three vertices . The main result of this section is Theorem 6.1.1, which describes all values $D^{\mathbf{ab}}_{\varnothing v}$. See Theorem 6.2.16 for the entries $D^{\mathbf{ab}}_{ve}$.

Theorem 6.1.1. *Let* $\mathbf{b}$ *and* $\mathbf{a}$ *be degree vectors such that* $\mathbf{a} \prec \mathbf{b}$*,* $dim_{\Bbbk}(\widetilde{H}_0 K^{\mathbf{b}}I;\Bbbk) \neq 0$*, and* $\mathbf{a}$ *is the degree vector of a monomial generator of* I*. Then the values* $D^{\mathbf{ab}}_{\varnothing v}$ *are given below.*

1. *If* i *is an isolated vertex in* $K^{\mathbf{b}}I$ *and* $\mathbf{a}$ *is the unique generator that lies behind* $\mathbf{b}$ *in the* e_i*-direction, then* $D^{\mathbf{ab}}_{\varnothing i} = 1$*.*

2. *If* $\mathbf{a}$ *does not lie behind* $\mathbf{b}$ *in the direction of the connected component of* v*, then* $D^{\mathbf{ab}}_{\varnothing v} = 0$*.*

3. *Suppose* $K^{\mathbf{b}}I$ *is*  *and* $\mathbf{a}$ *lies behind* $\mathbf{b}$ *in the direction of the edge. Given a lattice path* $\lambda \in \Lambda(\mathbf{a}, \mathbf{b})$*, let* $\mathbf{b}_\lambda$ *be the degree vector closest to* $\mathbf{b}$ *along* λ *such*

that $K^{\mathbf{b}_\lambda} I$ contains one or fewer vertices. Let n be the length of λ. Then

$$D^{\mathbf{ab}}_{\varnothing i} = \frac{1}{2^n} \sum_{\lambda \in \Lambda(\mathbf{a},\mathbf{b})} 2^{|\mathbf{b}_\lambda - \mathbf{a}|}.$$

Proof. These claims are Lemmas 6.1.2, 6.1.3, and 6.1.4. □

Lemma 6.1.2. *Let i be an isolated vertex in $K^{\mathbf{b}} I$, and let $\mathbf{a}$ be the unique generator that lies behind $\mathbf{b}$ in the e_i-direction. Then $D^{\mathbf{ab}}_{\varnothing i} = 1$.*

Proof. Note that there can only be one saturated, decreasing lattice path in the staircase from $\mathbf{b}$ to $\mathbf{c}$. For every degree vector $\mathbf{b}'$ between $\mathbf{b}$ and $\mathbf{c}$, $K^{\mathbf{b}'} I$ is isomorphic to the single vertex i or the empty face. There is only one hedgerow along this lattice path:

λ :	$\mathbf{c}$	——	$\mathbf{b}_{j-1}$	$\cdots$	$\mathbf{b}_1$	——	$\mathbf{b}$
ST^{λ}_0 :	$T_{-1} = \{\}$		$T_0 = \{x_i\}$		$T_0 = \{x_i\}$		$S_0 = \{\}$
			$S_{-1} = \{\varnothing\}$		$S_{-1} = \{\varnothing\}$		

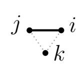
Since x_i does not appear with nonzero coefficient in the boundary of any edges of $K^{\mathbf{b}} I$, the hedge rim r of x_i in $S_0 = \{\}$ is x_i, giving $x_i - r = x_i - x_i = 0 \in \widetilde{B}_0(K^{\mathbf{b}} I; \Bbbk)$. The lattice path moves back in the x_i-direction only. This means that each 0-face τ in the chain-link fence must be x_i, since $\sigma_\ell = \tau_{\ell-1} \setminus e_i$. Finally, $\sigma_\ell = \varnothing$, and since $\varnothing$ is a boundary and $T_{-1} = \{\}$, $\varnothing$ is cycle-linked to itself. The weight of each linkage is 1. Thus, along this hedgerow, there is also only one possible chain-link fence beginning with x_i and ending with σ:

$$\begin{array}{ccccccccc} & & x_i & & x_i & & x_i & \overset{1}{—} & x_i \\ & /_1 & & \cdots & & \backslash^1 & /_1 & & \\ \varnothing \overset{1}{—} \varnothing & & & & \varnothing & & & & \end{array}$$

Note that it is also the only chain-link fence along the hedgerow, regardless of the initial and terminal faces. Then $D^{\mathbf{cb}}_{\varnothing x_i} = \sum_{\lambda \in \Lambda(\mathbf{c},\mathbf{b})} \frac{1}{\Delta_{i,\lambda} I} \sum_{\phi \in \Phi(\varnothing, x_i)} w_\phi = 1.$ $\square$

Lemma 6.1.3. *If* $\mathbf{a}$ *does not lie behind* $\mathbf{b}$ *in the direction of the connected component of a vertex* v*, then* $D^{\mathbf{ab}}_{\varnothing v} = 0$.

Proof. A vertex can only be boundary-linked to a vertex in the same connected component. If $\mathbf{a}$ does not lie behind $\mathbf{b}$ in the direction of the connected component of v, then no lattice path λ from $\mathbf{b}$ to $\mathbf{a}$ will move back in the direction of any vertex in the connected component of v. Therefore the chain-link fence will end. $\square$

Lemma 6.1.4. *Suppose that* $K^{\mathbf{b}}(I)$ *is* 

and $\mathbf{a}$ *lies behind* $\mathbf{b}$ *in the direction of the edge. Given a lattice path* $\lambda \in \Lambda(\mathbf{c}, \mathbf{b})$*, let* $\mathbf{b}_\lambda$ *be the degree vector closest to* $\mathbf{b}$ *along* λ *such that the Koszul simplicial complex contains one or fewer vertices. Let* n

be the length of λ. *Then*

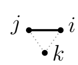
$$D^{\mathbf{ab}}_{\varnothing i} = \frac{1}{2^n} \sum_{\lambda \in \Lambda(\mathbf{a},\mathbf{b})} 2^{|\mathbf{b}_\lambda - \mathbf{a}|}.$$

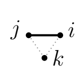
Proof. Given $\lambda \in \Lambda(\mathbf{c}, \mathbf{b})$, a hedgerow must be chosen. In degree $\mathbf{b}$, the stake set S_0 can be $\{x_i\}$ or $\{x_j\}$. At each interior lattice point $\mathbf{a}$, a hedge of dimension 0 must be chosen. The stake set S_{-1} must be the set consisting of the empty face $\{\varnothing\}$, since $\varnothing$ is a dimension -1 boundary. Any vertex in $K^{\mathbf{a}}I$ can be chosen for the shrubbery T_0, and $K^{\mathbf{a}}I$ can have at most two vertices. Once the degree vector $\mathbf{b}_\lambda$ is reached in the lattice path, there is only one choice of hedge for each subsequent degree vector along λ, since there is only one (or fewer) vertex in the Koszul simplicial complex. In degree $\mathbf{c}$, the only shrubbery of dimension -1 is $T_{-1} = \{\,\}$. Therefore $\Delta_{\lambda,0} I = 2^{|\mathbf{b} - \mathbf{b}_\lambda|}$.

Next it is shown that hedgerows that yield chain-link fences are in bijection with lattice paths and with chain-link fences with initial post x_i. Suppose in degree $\mathbf{b}$, $S_0 = \{x_i\}$. Since the chain-link fence must start with x_i and $x_i - x_j$ is a boundary, x_i is only boundary-linked to x_j. Therefore the only vertex boundary-linked to x_i is x_j, and the weight of the linkage is 1. The chain-link fence ends unless the lattice path moves back in the x_j-direction. Similarly, if $S_0 = \{x_j\}$, x_i is boundary-linked to x_i with weight 1, and the chain-link fence ends unless the lattice path moves back in the x_i-direction, also with weight 1. Now the face $\varnothing \in K^{\mathbf{b}_1} I$ is chain-linked only to the vertex in T_0, with weight 1. The chain-link fence ends unless the lattice path moves back in the direction of that vertex. Once $\mathbf{b}_\lambda$ is reached, there is only one choice of hedge for each subsequent lattice point. The vertex in the shrubbery is the same as the direction in which the lattice path must move back. The chain-link fence ends with $\sigma \in K^{\mathbf{c}}I$, which is cycle-linked to itself with weight 1. Therefore there is only one hedgerow along a given lattice path that yields a chain-link fence, and it yields exactly one chain-link fence. The weight $w_\phi = 1$ for all ϕ. Thus

$$\begin{aligned} D^{\mathbf{cb}}_{\varnothing x_i} &= \sum_{\lambda \in \Lambda(\mathbf{c},\mathbf{b})} \frac{1}{2^{|\mathbf{b}-\mathbf{b}_\lambda|}} \sum_{\phi \in \Phi_{\varnothing x_i}(\lambda)} 1 \\ &= \sum_{\lambda \in \Lambda(\mathbf{c},\mathbf{b})} \frac{1}{2^{|\mathbf{b}-\mathbf{b}_\lambda|}} \\ &= \sum_{\lambda \in \Lambda(\mathbf{c},\mathbf{b})} \frac{2^{|\mathbf{b}_\lambda-\mathbf{c}|}}{2^{|\mathbf{b}_\lambda-\mathbf{c}|}2^{|\mathbf{b}-\mathbf{b}_\lambda|}} \\ &= \frac{1}{2^n} \sum_{\lambda \in \Lambda(\mathbf{c},\mathbf{b})} 2^{|\mathbf{b}_\lambda-\mathbf{c}|}. \end{aligned}$$

□

Definition 6.1.5. Suppose $K^{\mathbf{b}}I$ is , and let $\{\mathbf{a}_\ell\}_{\ell=1}^{s}$ be the set of degree vectors of generators of I that lie behind $\mathbf{b}$ in the direction of the edge. Order the degree vectors $\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_s$ so that the j^{th} component of $\mathbf{a}_{\ell+1}$ is greater than the j^{th} component of $\mathbf{a}_\ell$ for $\ell = 1, \ldots, s-1$. Then the *neighboring syzygies* of $\mathbf{a}_\ell$ are the degree vectors $\mathrm{lcm}(\mathbf{a}_\ell, \mathbf{a}_{\ell-1})$ and $\mathrm{lcm}(\mathbf{a}_\ell, \mathbf{a}_{\ell+1})$.

Corollary 6.1.6. *Suppose that $K^{\mathbf{b}}(I)$ is* *. Let $\{\mathbf{a}_\ell\}_\ell$ be the set of degree vectors* *of generators that lie behind* $\mathbf{b}$ *in the* e_ie_j*-direction. Then* $D_{\varnothing i}^{\mathbf{a}_\ell \mathbf{b}}$ *can be computed with the following algorithm:*

1. *Let* $m := max_{\mathbf{a}_\ell, \lambda \in \Lambda(\mathbf{a}_\ell, \mathbf{b})}\{|\mathbf{b} - \mathbf{b}_\lambda|\}$.

2. *Draw an* $m \times m$ *grid with a diagonal line running from the bottom left corner to the top right corner. Label the generators and syzygies where they would fall on the staircase surface, with* $\mathbf{b}$ *in the bottom right corner.*

3. $D_{\varnothing i}^{\mathbf{a}_\ell \mathbf{b}}$ *is the number of saturated, decreasing lattice paths that start at* $\mathbf{b}$*, pass either between neighboring syzygies or through a neighboring syzygy and leaving in the direction of* $\mathbf{a}_\ell$*, and end on the diagonal divided by the total number of lattice paths from* $\mathbf{b}$ *to the diagonal (*2^m*).*

Proof. Lemma 6.1.4 says that the coefficients $D_{\varnothing v}^{\mathbf{ab}}$ are sums over all saturated, decreasing lattice paths from $\mathbf{b}$ to $\mathbf{a}$ of the weight associated to the lattice path, divided by 2^n. This weight is $2^{|\mathbf{b}_\lambda - \mathbf{a}|}$. This is the same as the number of lattice paths that follow along λ until the lattice point $\mathbf{b}_\lambda$ and then branch off in any (decreasing) direction for a length of $|\mathbf{b}_\lambda - \mathbf{a}|$, i.e., until they reach the diagonal drawn in the $n \times n$ grid. The lattice paths only contribute to $D_{\varnothing v}^{\mathbf{ab}}$ if the original lattice path λ ends at degree $\mathbf{a}$. Therefore on the grid shown, the lattice paths must pass between neighboring syzygies of $\mathbf{a}$ or pass through the neighboring syzygies in the direction of $\mathbf{a}$.

An $m \times m$ grid can be drawn instead of an $n \times n$ grid because 2^{n-m} will divide every weight and can thus be canceled from the terms in the sum in 6.1.4. □

Example 6.1.7. Consider the ideal $I = \langle x^3z, xyz, y^2z, x^3y^2, x^2y^3 \rangle$ whose staircase diagram is given in Figure 6.1. $K^{321}I$ has facets xy and z. To use Corollary 6.1.6, note $m = \max_{\mathbf{a},\lambda \in \Lambda(\mathbf{a},321)}\{|321 - 321_\lambda|\} = 3$. To compute the entries of $D^{301,321}$, $D^{111,321}$, and $D^{021,321}$, draw a 3×3 grid and plot the intermediate degree vectors as in Figure 6.2. There are five decreasing lattice paths that start at **321**, pass through **311** and move up, pass between **311** and **121**, or pass through **121** and move left and then end on the diagonal, as shown in Figure 6.2. Therefore $D^{111,321}_{\varnothing,x} = D^{111,321}_{\varnothing,y} = \frac{5}{2^3}$ by Corollary 6.1.6.

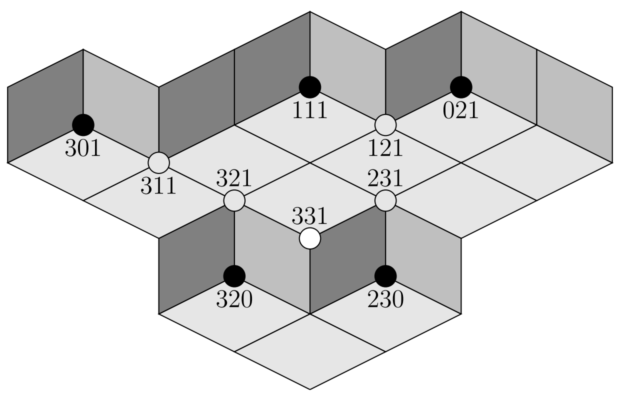


Figure 6.1: The staircase diagram for $I = \langle x^3z, xyz, y^2z, x^3y^2, x^2y^3 \rangle$ with syzygy degrees marked

Similarly, $D^{301,321}_{\varnothing x} = D^{301,321}_{\varnothing y} = \frac{1}{4}$ and $D^{021,321}_{\varnothing x} = D^{021,321}_{\varnothing y} = \frac{1}{8}$.

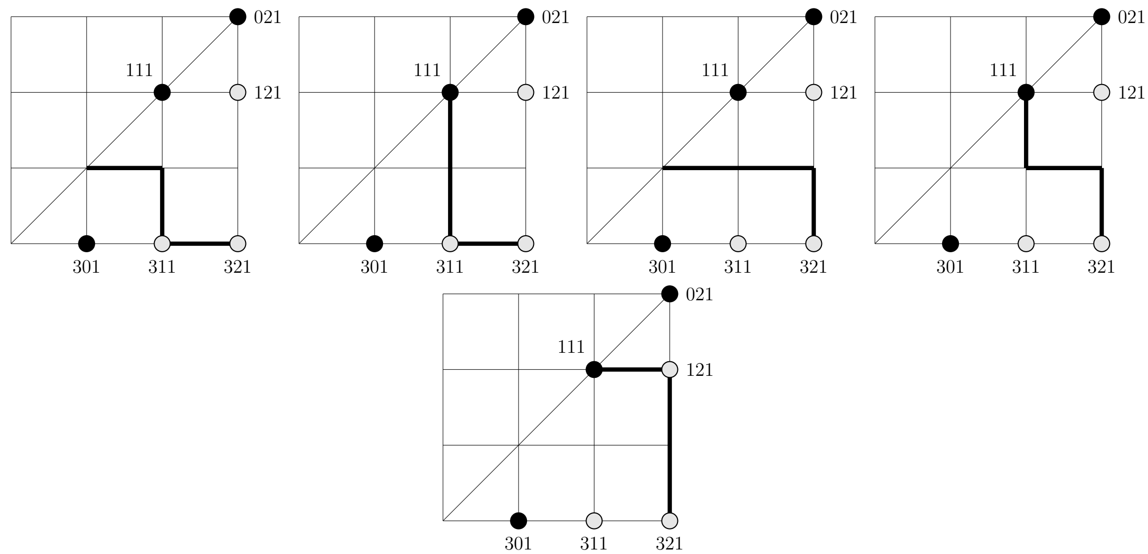
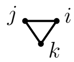


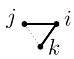
Figure 6.2: All lattice paths contributing to the numerator of $D_{\varnothing x}^{301,321}$

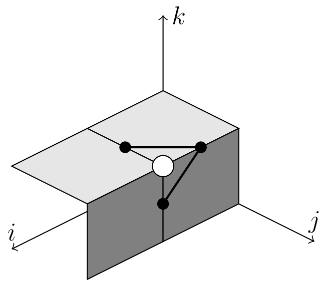
6.2 Coefficients in the maps from $\widetilde{H}_1(K^{\mathbf{b}}I;\Bbbk)$ to $\widetilde{H}_0(K^{\mathbf{a}}I;\Bbbk)$

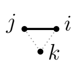
The coefficients in the maps $\widetilde{H}_0(K^{\mathbf{a}}I;\Bbbk) \leftarrow \widetilde{H}_1(K^{\mathbf{b}}I;\Bbbk)$ are determined by going through cases of Koszul complexes that can occur along a lattice path from $\mathbf{b}$ to $\mathbf{a}$. These cases are given in Section 6.2.1. The coefficients of the sylvan matrix are obtained by adding the contributions along all lattice paths from $\mathbf{b}$ to $\mathbf{a}$. The lemmas in Sections 6.2.2 and 6.2.3 contribute to the proof of the main result of this section, Theorem 6.2.16. This theorem describes all values $D_{ve}^{\mathbf{ab}}$ in the three-variable case, where v is a vertex and e is an edge.

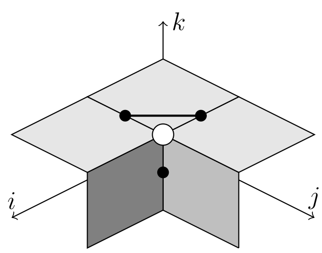
6.2.1 Lattice path cases

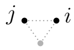
Suppose $K^{\mathbf{b}}I$ is the empty triangle  (for the staircase diagram, see Figure 6.3). Let $\mathbf{b}'$ such that $\mathbf{c} \preceq \mathbf{b}' \preceq \mathbf{b}$ and $\dim_{\Bbbk}(\widetilde{H}_0(K^{\mathbf{c}}I;\Bbbk))$ is nonzero. Suppose further that $\mathbf{b}'$ is a degree vector in a lattice path from $\mathbf{b}$ to such a $\mathbf{c}$ where the first edge is in the e_i-direction. (By symmetry, the results also hold for lattice paths that initially move back in the e_j- or e_k-direction, exchanging each i and j or k, respectively.) Then there are several cases for $K^{\mathbf{b}'}I$:

-  the edges ij and ik

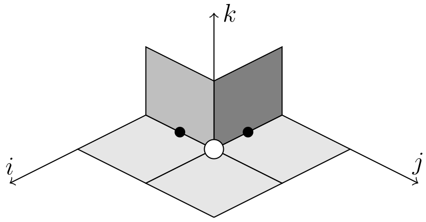


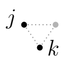
-  the edge ij and the vertex k



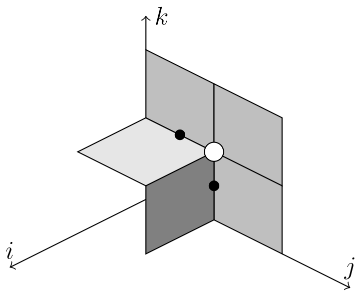
- 

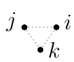
the vertices i and j



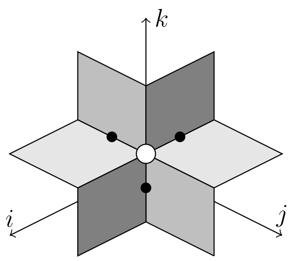
- 

the vertices j and k



- 

the vertices i, j, and k



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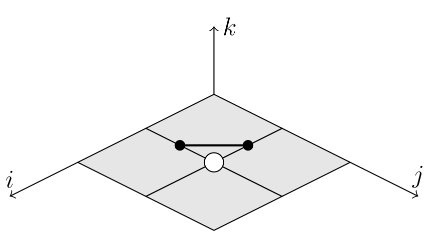
the edge ij



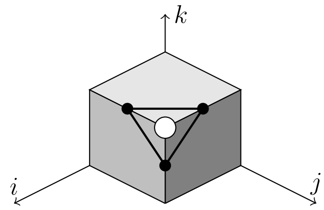


Figure 6.3: Staircase diagram with degree vector (labeled by white dot) such that $K^{\mathbf{b}}I$ is the empty triangle

The lemmas below are used to determine in which order the cases can occur along a lattice path on the staircase surface.

Lemma 6.2.1. *If* $\mathbf{b}$ *and* $\mathbf{c}$ *are on the staircase and* $\mathbf{c} \preceq \mathbf{b}$*, then* $F := \mathit{supp}(\mathbf{b} - \mathbf{c}) \in K^{\mathbf{b}}I$.

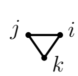
Proof. $K^{\mathbf{b}}I = \{\text{supp}(\mathbf{b} - \mathbf{a}) \mid \mathbf{a} \text{ is on the staircase and } \mathbf{a} \preceq \mathbf{b}\}$. □

Lemma 6.2.2. *If* F *is a facet of* $K^{\mathbf{b}}I$*, then* $K^{\mathbf{c}}I \subseteq F$.

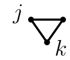
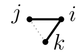
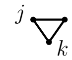
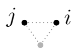
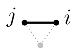
Proof. Let $\text{supp}(\mathbf{b}-\mathbf{a}) \in K^{\mathbf{c}}I$. Since $F = \text{supp}(\mathbf{b}-\mathbf{c})$ is a facet of $K^{\mathbf{b}}I$, $\text{supp}(\mathbf{b}-\mathbf{a}) \preceq \text{supp}(\mathbf{b}-\mathbf{c})$ for all $\mathbf{a} \preceq \mathbf{b}$ on the staircase. Since $\text{supp}(\mathbf{c}-\mathbf{a}) \preceq \text{supp}(\mathbf{b}-\mathbf{a})$, $\text{supp}(\mathbf{c}-\mathbf{a})$ is a subface of F. □

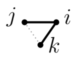
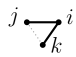
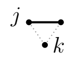
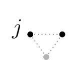
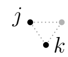
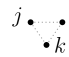
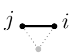
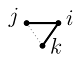
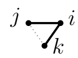
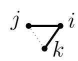
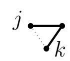
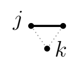
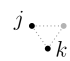
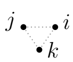
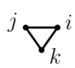
Lemma 6.2.3. $K^{\mathbf{c}}I \subseteq \mathit{star}(F, K^{\mathbf{b}}I)$*, the faces of* $K^{\mathbf{b}}I$ *containing* F.

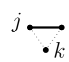
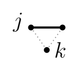
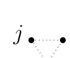
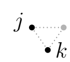
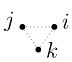
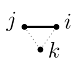
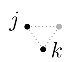
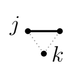
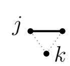
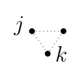
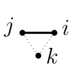
Proof. Moving back from $\mathbf{c}$ by G is the same as moving back from $\mathbf{b}$ by $F \cup G$. □

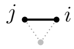
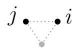
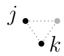
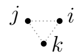
Proposition 6.2.4. *The diagram in Figure 6.4 shows all cases for lattice paths starting at the Koszul simplicial complex* 

and ending at a degree vector $\mathbf{a}$ *such that* $\widetilde{H}_0(K^{\mathbf{a}}I; \Bbbk) \neq 0$.

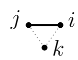
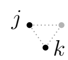
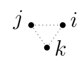
Proof. Start at a degree vector $\mathbf{b}$ such that $K^{\mathbf{b}}I$ is  and move back in the e_i-direction to degree $\mathbf{b}'$. Then $F = \text{supp}(\mathbf{b} - \mathbf{b}') = e_i$. By Lemma 6.2.3, $K^{\mathbf{b}'}I \subseteq \text{star}(F, K^{\mathbf{b}}I)$, so $K^{\mathbf{b}'}I$ must be a subcomplex of  . Since ik is in $K^{\mathbf{b}}I$, the degree vectors $\mathbf{b} - e_i - e_k$ and $\mathbf{b} - e_i$ are on the staircase, which means that $\text{supp}((\mathbf{b} - e_i) - (\mathbf{b} - e_i - e_k)) = e_k \in K^{\mathbf{b-e_i}}I = K^{\mathbf{b}'}I$ by Lemma 6.2.1. Therefore  cannot lead to  or  .

The complex  can lead to any of  ,  ,  ,  ,  , or  . Moving back along a single edge (in the e_i-, e_j-, or e_k-direction) from $\mathbf{b}$ to $\mathbf{c}$, $F = i$, j, or k. Since $K^{\mathbf{c}}I \subseteq \text{star}(F, K^{\mathbf{b}}I)$,  can lead to  in the e_i-direction only. Similarly, any time j and k are both in $K^{\mathbf{c}}I$, the lattice path can only have moved back in the e_i-direction. This is true when  leads to  ,  ,  , and  . By an argument for  above, $K^{\mathbf{b-e_i}}I$ must contain j and k, so the lattice path can only get to  or  by moving back in the e_j-direction.

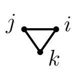
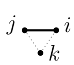
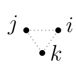
Simplicial complexes that are subcomplexes of  are  ,  ,  ,  , or  .  cannot lead to  because if the lattice path moves back in the e_i- or e_j-direction, $F = i$ or j, and then $K^{\mathbf{c}}I$ cannot contain k by Lemma 6.2.3. Similarly, if the lattice path moves back in the e_k-direction, $K^{\mathbf{c}}I$ cannot contain i or j. By this same argument (and symmetry),  also cannot lead to  or  .  can lead to  or  .

Subcomplexes of  are either  or .  can lead to either of these. By Lemma 6.2.3, , , and  cannot lead to any case where the reduced 0^{th} homology is nonzero. □

6.2.2 Sylvan matrix entries $D_{ve}^{\mathbf{ab}}$ when $\mathbf{a} = \mathbf{b} - ne_i$

When $K^{\mathbf{b}}I$ is the empty triangle, $K^{\mathbf{a}}I$ is , , or , and $\mathbf{a} \preceq \mathbf{b}$, there is one saturated, decreasing lattice path from $\mathbf{b}$ to $\mathbf{a}$. This is because the lattice path must move back in the e_i-direction each time (see Figure 6.4). The lemmas in this section give the entries $D_{ve}^{\mathbf{ab}}$ of the sylvan matrices, where e is an edge and v is a vertex. These lemmas contribute to the proof of Theorem 6.2.16.

Let $(-1)^{i,\prod_{i_l} i_l} := \text{sign}(x_i, \prod_{i_l} x_{i_l})$.

Lemma 6.2.5. *Suppose* $\mathbf{b}$ *and* $\mathbf{a} := \mathbf{b} - e_i$ *are degree vectors such that* $K^{\mathbf{b}}I$ *is*  *and* $K^{\mathbf{a}}I$ *is*  *or* . *Then the sylvan matrix is:*

$$\widetilde{H}_0 K^{\mathbf{a}} \otimes \langle \mathbf{x^a} \rangle \xleftarrow{\begin{array}{c|ccc} & x_ix_j & x_ix_k & x_jx_k \\ \hline x_i & \frac{(-1)^{j,ij}}{3} & \frac{(-1)^{k,ik}}{3} & 0 \\ x_j & \frac{2(-1)^{i,ij}}{3} & \frac{(-1)^{k,ik}}{3} & 0 \\ x_k & \frac{(-1)^{j,ij}}{3} & \frac{2(-1)^{i,ik}}{3} & 0 \end{array}} \widetilde{H}_1 K^{\mathbf{b}} \otimes \langle \mathbf{x^b} \rangle$$

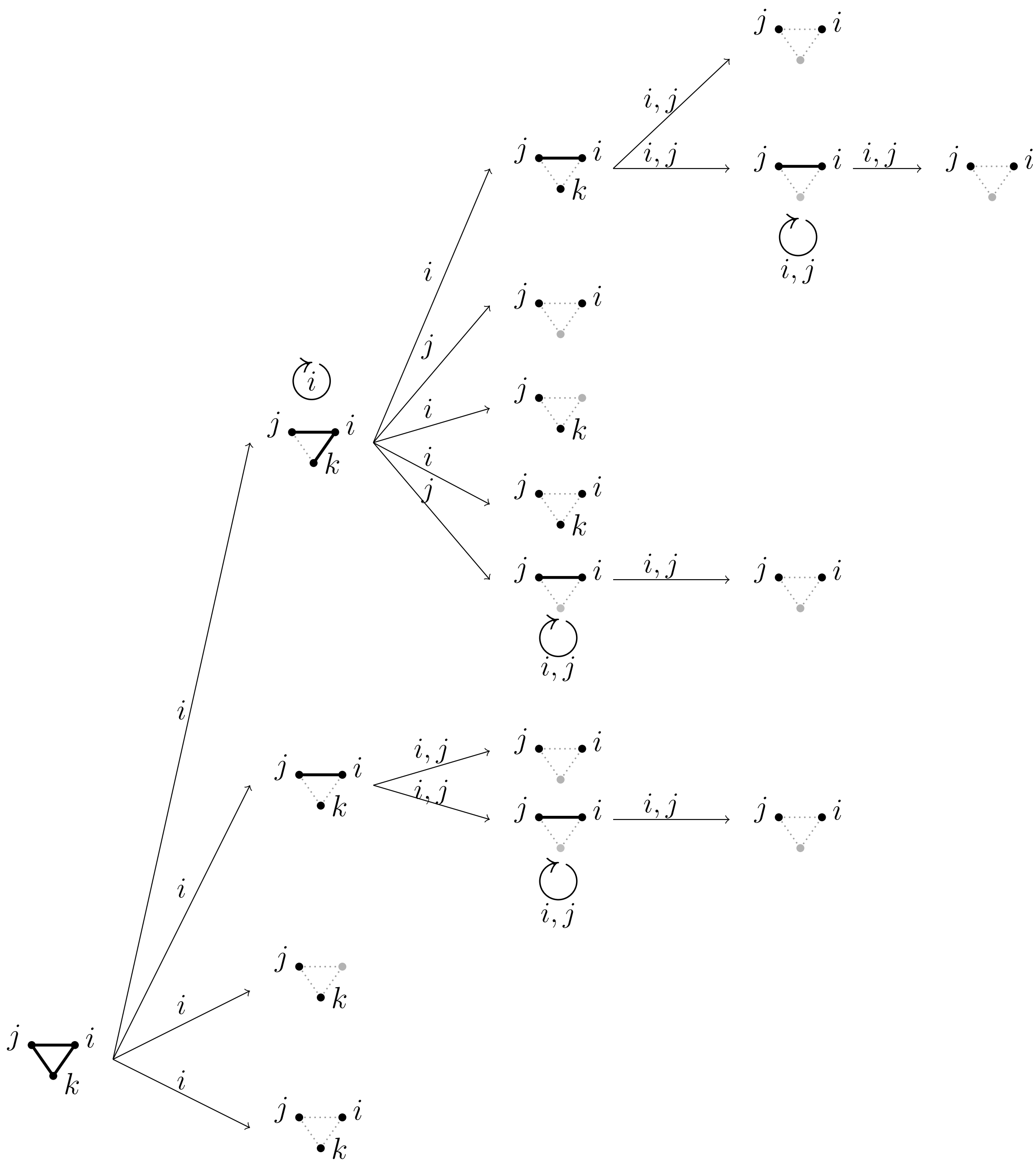
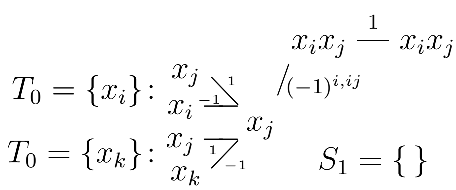
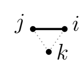
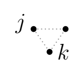
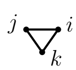
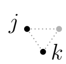


Figure 6.4: All cases for Koszul complexes along lattice paths that begin by moving back in the e_i-direction contributing to the maps $F_1 \leftarrow F_2$

Proof. A stake set $S_1^{\mathbf{b}}$ and a shrubbery $T_0^{\mathbf{a}}$ are chosen. $S_1^{\mathbf{b}}$ is forced to be the empty set $\{\}$, since there are no 1-boundaries. The shrubbery $T_0^{\mathbf{a}}$ can contain any of the vertices x_i, x_j, or x_k, and there is only one lattice path from $\mathbf{b}$ to $\mathbf{a}$. Therefore $\Delta_{1,\lambda} I = 3$. All chain-link fences starting at $x_i x_j$ are given below.



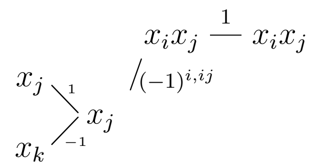
The lattice paths with initial post $x_i x_k$ are the same, swapping every j with k. There are no chain-link fences with initial post $x_j x_k$, since the lattice path moves back in the e_i-direction. The reason the matrices are the same for  and  is that the final two faces in the chain-link fence depend only on the vertices that appear in $K^{\mathbf{a}} I$, and these cases all contain three vertices. □

Lemma 6.2.6. *Suppose* $\mathbf{b}$ *and* $\mathbf{a} := \mathbf{b} - e_i$ *are degree vectors such that* $K^{\mathbf{b}} I$ *is*  *and* $K^{\mathbf{a}} I$ *is* . *Then the sylvan matrix is given below.*

$$\widetilde{H}_0 K^{\mathbf{a}} \otimes \langle \mathbf{x}^{\mathbf{a}} \rangle \xleftarrow{\begin{array}{c} \quad x_i x_j \qquad x_i x_k \qquad x_j x_k \\ \begin{array}{c} x_j \\ x_k \end{array} \begin{bmatrix} \frac{(-1)^{i,ij}}{2} & \frac{(-1)^{k,ik}}{2} & 0 \\ \frac{(-1)^{j,ij}}{2} & \frac{(-1)^{i,ik}}{2} & 0 \end{bmatrix} \end{array}} \widetilde{H}_1 K^{\mathbf{b}} \otimes \langle \mathbf{x}^{\mathbf{b}} \rangle$$

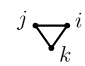
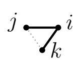
Proof. The stake set $S_1^{\mathbf{b}}$ is forced to be the empty set $\{\}$, and $T_0^{\mathbf{a}}$ can be either vertex

x_j or x_k. With the stake set $S_1^{\mathbf{b}}$, edge x_ix_j is only boundary-linked to itself. All chain-link fences with initial post x_ix_j are shown below.



$$T_0 = \{x_k\} \qquad S_1 = \{\}$$

The chain-link fences with initial post x_ix_k are the same, swapping each j and k. Since the lattice path moves back in the e_i-direction, there are no chain-link fences with initial post x_jx_k. $\square$

Lemma 6.2.7. *If* $\mathbf{b}$ *and* $\mathbf{a} := \mathbf{b} - r(e_i)$ *are degree vectors such that* $K^{\mathbf{b}}I$ *is*  *, and in the lattice path* $\lambda = (\mathbf{a} = \mathbf{b}_r, \mathbf{b}_{r-1}, \ldots, \mathbf{b}_1, \mathbf{b})$*, all degree vectors* $\mathbf{b}_1, \ldots, \mathbf{b}_r$ *have* $K^{\mathbf{b}_\ell}I$  *, there are* 3^r *chain-link fences with initial post* x_ix_j*. The number of these chain-link fences that end with* x_ix_j *is* $\lceil \frac{3^r}{2} \rceil$*, with weight* $\lceil \frac{3^r}{2} \rceil$*, and the number that end with* x_ix_k *is* $\lfloor \frac{3^r}{2} \rfloor$*, with weight* $(-1)^{i,ijk+1} \lfloor \frac{3^r}{2} \rfloor$*. The contribution to* $\Delta_{1,\lambda}I$ *is* 3^r*.*

Proof. The stake set $S_1^{\mathbf{b}}$ must be the empty set. For $\mathbf{b}_\ell$, $\ell = 1, \ldots, r$, $T_1^{\mathbf{b}_\ell} = \{x_ix_j, x_ix_k\}$, and $S_0^{\mathbf{b}_\ell}$ can be any pair of vertices, giving three options for each $\mathbf{b}_\ell$.

In degree $\mathbf{b}_\ell$, the stake x_j is chain-linked to x_ix_j if $S_0^{\mathbf{b}_\ell} = \{x_j, x_k\}$, to x_ix_j and x_ix_k if $S_0^{\mathbf{b}_\ell} = \{x_j, x_i\}$. If $S_0^{\mathbf{b}_\ell} = \{x_i, x_k\}$, then x_j is not chain-linked to any face, since it is not a stake. Since the Koszul complex is symmetric, the stake x_k is chain-linked to x_ix_k if $S_0^{\mathbf{b}_\ell} = \{x_k, x_j\}$ and to x_ix_k and x_ix_j if $S_0^{\mathbf{b}_\ell} = \{x_k, x_i\}$.

When the lattice path moves back in the e_i-direction again, removing x_i from the edges produces two chain-link fences temporarily ending at x_j and one temporarily ending at x_k. Each x_j is chain-linked to x_ix_j in two more chain-link fences and is chain-linked to x_ix_k in one more. Similarly, each x_k is chain-linked to x_ix_k in two more chain-link fences and to x_ix_j in one more.

Proof by induction. Base case: after moving back in the e_i-direction once, there is one x_j chain-linked to $2 = \lceil \frac{3^1}{2} \rceil$ x_ix_j's and $\lfloor \frac{3^1}{2} \rfloor$ x_ix_k's.

Let A_n be the number of chain-link fences ending at edge x_ix_j in degree $\mathbf{b}_n$, and let B_n be the number of chain-link fences ending at x_ix_k in degree $\mathbf{b}_n$. Assume $A_n = \lceil \frac{3^n}{2} \rceil$ and $B_n = \lfloor \frac{3^n}{2} \rfloor$. Since, after moving back again in the e_i-direction, the vertex x_i is removed from each edge, there will be A_n chain-link fences ending in x_j and B_n ending in x_k. This produces $2(A_n) + B_n$ chain-link fences ending with x_ix_j and $A_n + 2(B_n)$ ending with x_ix_k. Since 3^n is odd, $\lceil \frac{3^n}{2} \rceil = \frac{3^n+1}{2}$. This gives

$$\begin{aligned} A_{n+1} &= 2(A_n) + B_n \\ &= 2\lceil \frac{3^n}{2} \rceil + \lfloor \frac{3^n}{2} \rfloor \\ &= \lceil \frac{3^n}{2} \rceil + 3^n \end{aligned}$$

$$= \frac{3^n + 1 + 2(3^n)}{2}$$

$$= \frac{3^{n+1} + 1}{2}$$

$$= \lceil \frac{3^{n+1}}{2} \rceil.$$

Since the total number of chain-link fences is 3^n, $B_{n+1} = \lfloor \frac{3^{n+1}}{2} \rfloor$.

Next, it is shown that the weight of the chain-link from x_j to $x_i x_k$ is $(-1)^{i,ij}(-1)^{i,ijk+1}$. Note that

$$\begin{aligned} d((-1)^{i,ij}((-1)^{i,ijk+1}x_i x_k + x_i x_j)) &= (-1)^{i,ij}(-1)^{i,ijk+1}((-1)^{i,ik}x_k + (-1)^{k,ik}x_i) \\ &\quad + (-1)^{i,ij}(-1)^{i,ij}x_j + (-1)^{i,ij}(-1)^{j,ij}x_i \\ &= (-1)^{i,ij}(-1)^{i,ijk+1}((-1)^{i,ik}x_k + (-1)^{k,ik}x_i) \\ &\quad + x_j - x_i. \end{aligned}$$

If in the order of the variables, $i > j,k$ or $i < j,k$, then $(-1)^{i,ij}(-1)^{k,ik} = -1$ and $(-1)^{i,ijk+1} = -1$. If $j > i > k$ or $k > i > j$, then $(-1)^{i,ij}(-1)^{k,ik} = 1$ and $(-1)^{i,ijk+1} = 1$. Therefore $(-1)^{i,ij}(-1)^{i,ijk+1}(-1)^{k,ik} = 1$. Then

$$d((-1)^{i,ij}((-1)^{i,ijk+1}x_i x_k + x_i x_j)) = (-1)^{i,ij}(-1)^{i,ijk+1}(-1)^{i,ik}x_k + x_j.$$

Thus the weight of the chain-link between x_j and $x_i x_k$ is $(-1)^{i,ij}(-1)^{i,ijk+1}$. By symmetry, the weight of the chain-link between x_k and $x_i x_j$ is $(-1)^{i,ik}(-1)^{i,ijk+1}$.

There are four cases for segments of the chain-link fence:

$$\begin{array}{ccc} x_ix_j & & x_ix_j \\ {\scriptstyle (-1)^{i,ij}}\backslash & & /{\scriptstyle (-1)^{i,ij}} \\ & x_j & \end{array}$$

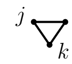
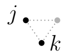
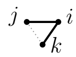
$$\begin{array}{ccc} x_ix_k & & x_ix_k \\ {\scriptstyle (-1)^{i,ik}}\backslash & & /{\scriptstyle (-1)^{i,ik}} \\ & x_k & \end{array}$$

$$\begin{array}{ccc} x_ix_k & & x_ix_j \\ \backslash^{a} & & /{\scriptstyle (-1)^{i,ij}} \\ & x_j & \end{array}$$

$$\begin{array}{ccc} x_ix_j & & x_ix_k \\ \backslash^{b} & & /{\scriptstyle (-1)^{i,ik}} \\ & x_k & \end{array}$$

Here, $a = (-1)^{i,ij}(-1)^{i,ijk+1}$ and $b = (-1)^{i,ik}(-1)^{i,ijk+1}$. The signs in the first two cases cancel. In the third and fourth cases, the weights cancel to $(-1)^{i,ijk+1}$.

Starting the chain-link fence with x_ix_j, each time the edge switches from x_ix_j to x_ix_k or vice versa, the total weight is multiplied by $(-1)^{i,ijk+1}$. If the edge switches an odd number of times, the chain-link fence ends with x_ix_k and has weight $(-1)^{i,ijk+1}$. If the edge switches an even number of times, the chain-link fence ends with x_ix_j and has weight 1. Therefore the weight of all chain-link fences ending with edge x_ix_j is $\lceil \frac{3^r}{2} \rceil$, and the weight of all chain-link fences ending with edge x_ix_k is $(-1)^{i,ijk+1}\lfloor \frac{3^r}{2} \rfloor$. $\square$

Lemma 6.2.8. *If* $\mathbf{b}$ *and* $\mathbf{a}$ *are degree vectors such that* $K^{\mathbf{b}}I$ *is*  *and* $K^{\mathbf{a}}I$ *is*  *, and in the lattice path* $\lambda = (\mathbf{a}, \mathbf{b}_r, \mathbf{b}_{r-1}, \ldots, \mathbf{b}_1, \mathbf{b})$*, all degree vectors* $\mathbf{b}_1, \ldots, \mathbf{b}_r$ *have Koszul complexes*  *, then the contribution of* λ *to the sylvan matrix is given below.*

$$\widetilde{H}_0K^{\mathbf{a}} \otimes \langle \mathbf{x}^{\mathbf{a}} \rangle \xleftarrow{\begin{array}{c|ccc} & x_ix_j & x_ix_k & x_jx_k \\ \hline x_j & \frac{(-1)^{i,ij}}{2} & \frac{(-1)^{k,ik}}{2} & 0 \\ x_k & \frac{(-1)^{j,ij}}{2} & \frac{(-1)^{i,ik}}{2} & 0 \end{array}} \widetilde{H}_1K^{\mathbf{b}} \otimes \langle \mathbf{x}^{\mathbf{b}} \rangle$$

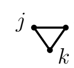
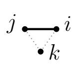
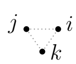
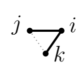
Proof. In the set of chain-link fences along λ from $\mathbf{a}$ to $\mathbf{b}_r$ with initial post x_ix_j, there are $\lceil \frac{3^r}{2} \rceil$ that end with x_ix_j by Lemma 6.2.7. After applying $d_1^{e_i}$, these are linked to x_j with weight $(-1)^{i,ij}$. There are $\lfloor \frac{3^r}{2} \rfloor$ that end with x_ix_k, and these are linked to x_k with weight $(-1)^{i,ik}$. A shrubbery of dimension 0 in $K^{\mathbf{a}}I$ can be $\{x_j\}$ or $\{x_k\}$, meaning $\Delta_{1,\lambda}I = 2 \cdot 3^r$. If $T_0^{\mathbf{a}} = \{x_k\}$, then x_j is cycle-linked to x_j with weight 1 and to x_k with weight -1. If $T_0^{\mathbf{a}} = \{x_j\}$, then x_k is cycle-linked to x_k with weight 1 and to x_j with weight -1. Thus there are $\lceil \frac{3^r}{2} \rceil$ chain-link fences along λ with terminal post x_j with weight $(-1)^{i,ij}$ and $\lfloor \frac{3^r}{2} \rfloor$ with weight $-(-1)^{i,ik}(-1)^{i,ijk+1}$, by Lemma 6.2.7. This gives $D^{\mathbf{ab}}_{x_j,x_ix_j} = \frac{1}{2\cdot 3^r}((-1)^{i,ij}\lceil \frac{3^r}{2} \rceil - (-1)^{i,ik}(-1)^{i,ijk+1}\lfloor \frac{3^r}{2} \rfloor)$. Note that $(-1)^{i,ij}(-1)^{i,ik}(-1)^{i,ijk+1} = -1$ (see proof of Lemma 6.2.7). Then

$$\begin{aligned}\frac{1}{2\cdot 3^r}\left((-1)^{i,ij}\lceil\frac{3^r}{2}\rceil - (-1)^{i,ik}(-1)^{i,ijk+1}\lfloor\frac{3^r}{2}\rfloor\right) &= \frac{(-1)^{i,ij}}{2\cdot 3^r}\left(\lceil\frac{3^r}{2}\rceil + \lfloor\frac{3^r}{2}\rfloor\right)\\ &= \frac{(-1)^{i,ij}3^r}{2\cdot 3^r}\\ &= \frac{(-1)^{i,ij}}{2}.\end{aligned}$$

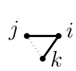
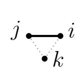
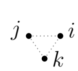
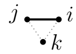
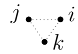
Thus $D^{\mathbf{ab}}_{x_j,x_ix_j} = \frac{(-1)^{i,ij}}{2}$.

Similarly, there are $\lceil\frac{3^r}{2}\rceil$ chain-link fences that end with x_k with weight $-(-1)^{i,ij}$ and $\lfloor\frac{3^r}{2}\rfloor$ with weight $(-1)^{i,ik}(-1)^{i,ijk+1}$. This gives $D^{\mathbf{ab}}_{x_k,x_ix_j} = \frac{(-1)^{j,ij}}{2}$.

The other entries follow by symmetry. $\square$

Lemma 6.2.9. *If* $\mathbf{b}$ *and* $\mathbf{a}$ *are degree vectors such that* $K^{\mathbf{b}}I$ *is*  *and* $K^{\mathbf{a}}I$ *is*  *or* , *and in the lattice path* $\lambda = (\mathbf{a}, \mathbf{b}_r, \mathbf{b}_{r-1}, \ldots, \mathbf{b}_1, \mathbf{b})$, *all degree vectors* $\mathbf{b}_1, \ldots, \mathbf{b}_r$ *have Koszul complexes* , *then the contribution of* λ *to the sylvan matrix is*

$$\widetilde{H}_0K^{\mathbf{a}}\otimes\langle\mathbf{x}^{\mathbf{a}}\rangle \xleftarrow{\begin{array}{c|ccc} & x_ix_j & x_ix_k & x_jx_k\\ \hline x_i & \frac{(-1)^{j,ij}}{3^{r+1}} & \frac{(-1)^{k,ik}}{3^{r+1}} & 0\\ x_j & (-1)^{i,ij}\frac{3^r+\lceil\frac{3^r}{2}\rceil}{3^{r+1}} & (-1)^{k,ik}\frac{3^r+\lfloor\frac{3^r}{2}\rfloor}{3^{r+1}} & 0\\ x_k & (-1)^{j,ij}\frac{3^r+\lfloor\frac{3^r}{2}\rfloor}{3^{r+1}} & (-1)^{i,ik}\frac{3^r+\lceil\frac{3^r}{2}\rceil}{3^{r+1}} & 0\end{array}} \widetilde{H}_1K^{\mathbf{b}}\otimes\langle\mathbf{x}^{\mathbf{b}}\rangle$$

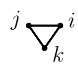
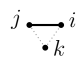
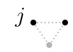
Proof. To get from  to  or , the chain-link fence must move back in the e_i-direction. The reason  and  are treated the same is that they both have three vertices, and the vertices are what determine the shrubbery $T_0^{\mathbf{a}}$. A chain-link fence ending in x_ix_j as described in Lemma 6.2.7 will go to x_j. There are three options for $T_0^{\mathbf{a}}$: $\{x_i\}$, $\{x_j\}$, and $\{x_k\}$. If $T_0^{\mathbf{a}} = \{x_i\}$, then x_j is cycle-linked to x_j with weight 1 and to x_i with weight -1. If $T_0^{\mathbf{a}} = \{x_k\}$, then x_j is cycle-linked to x_j with weight 1 and to x_k with weight -1. If $T_0^{\mathbf{a}} = \{x_j\}$, then x_j is not cycle-linked to any vertex. A chain-link fence ending in x_ix_k as described in Lemma 6.2.7 will go to x_k. If $T_0^{\mathbf{a}} = \{x_i\}$, then x_k is cycle-linked to x_k with weight 1 and to x_i with weight -1. If $T_0^{\mathbf{a}} = \{x_j\}$, then x_k is cycle-linked to x_k with weight 1 and to x_j with weight -1.

This gives, in total, $\lceil \frac{3^r}{2} \rceil$ ending with x_i with weight $-(-1)^{i,ij}$ and $\lfloor \frac{3^r}{2} \rfloor$ with weight $-(-1)^{i,ik}(-1)^{i,ijk+1} = (-1)^{i,ij}$, giving a total weight of $-(-1)^{i,ij}(\lceil \frac{3^r}{2} \rceil - \lfloor \frac{3^r}{2} \rfloor) = 1(-1)^{i,ij} = (-1)^{j,ij}$. It gives $2 \cdot \lceil \frac{3^r}{2} \rceil$ ending with x_j with weight $(-1)^{i,ij}$ and $\lfloor \frac{3^r}{2} \rfloor$ with weight $-(-1)^{i,ik}(-1)^{i,ijk+1} = (-1)^{i,ij}$, giving a total weight of $(-1)^{i,ij}(2 \cdot \lceil \frac{3^r}{2} \rceil + \lfloor \frac{3^r}{2} \rfloor) = (-1)^{i,ij}(3^r + \lceil \frac{3^r}{2} \rceil)$. Finally, it gives $\lceil \frac{3^r}{2} \rceil$ ending with x_k with weight $-(-1)^{i,ij}$ and $2 \cdot \lfloor \frac{3^r}{2} \rfloor$ with weight $(-1)^{i,ik}(-1)^{i,ijk+1} = (-1)^{j,ij}$, giving a total weight of $-(-1)^{i,ij}(\lceil \frac{3^r}{2} \rceil + 2 \cdot \lfloor \frac{3^r}{2} \rfloor) = (-1)^{j,ij}(3^r + \lfloor \frac{3^r}{2} \rfloor)$.

$\Delta_{1,\lambda} I = 3^{r+1}$. □

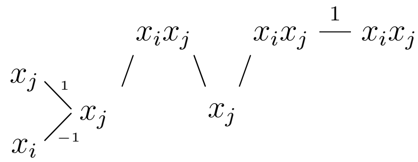
6.2.3 Contributions of individual lattice paths to the sylvan matrix

The lemmas in this subsection give the contributions of each lattice path to the sylvan matrix. These lemmas contribute to the proof of Theorem 6.2.16. To get an entry $D^{\mathbf{ab}}_{ve}$, add the entries (v, e) of the matrices corresponding to all lattice paths from $\mathbf{b}$ and $\mathbf{a}$.

Lemma 6.2.10. *Suppose* $\mathbf{b}$, $\mathbf{b}_1 := \mathbf{b} - e_i$, *and* $\mathbf{a} := \mathbf{b}_1 - e_i$ *or* $\mathbf{a} := \mathbf{b}_1 - e_j$ *are degree vectors such that* $K^{\mathbf{b}}I$ *is*  , $K^{\mathbf{b}_1}I$ *is*  , *and* $K^{\mathbf{a}}I$ *is*  . *Then the contribution to the sylvan matrix from* $\lambda = (\mathbf{a}, \mathbf{b}_1, \mathbf{b})$ *is given below.*

$$\widetilde{H}_0 K^{\mathbf{a}} \otimes \langle \mathbf{x}^{\mathbf{a}} \rangle \xleftarrow{\begin{array}{c} \quad x_ix_j \quad x_ix_k \quad x_jx_k \\ \begin{array}{c} x_i \\ x_j \end{array} \begin{bmatrix} \frac{(-1)^{j,ij}}{4} & 0 & 0 \\ \frac{(-1)^{i,ij}}{4} & 0 & 0 \end{bmatrix} \end{array}} \widetilde{H}_1 K^{\mathbf{b}} \otimes \langle \mathbf{x}^{\mathbf{b}} \rangle$$

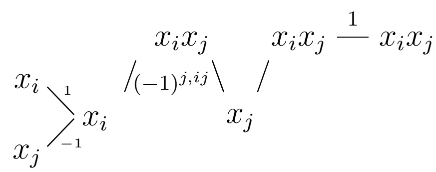
Proof. $S^{\mathbf{b}}_1$ must be the empty set, $T^{\mathbf{b}_1}_1$ must be $\{x_ix_j\}$ since there is only one edge in $K^{\mathbf{b}_1}I$, $S^{\mathbf{b}_1}_0$ can be either $\{x_j\}$ or $\{x_i\}$, and $T^{\mathbf{a}}_0$ can be $\{x_i\}$ or $\{x_j\}$. All chain-link fences for $\mathbf{a} = \mathbf{b}_1 - e_i$ with initial post x_ix_j are shown below.



$$T_0 = \{x_i\} \qquad S_0 = \{x_j\}$$

The unlabeled links have weight $(-1)^{i,ij}$.

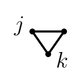
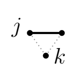
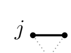
All chain-link fences for $\mathbf{a} = \mathbf{b}_1 - e_j$ with initial post $x_i x_j$ are shown below:



$$T_0 = \{x_j\} \qquad S_0 = \{x_j\}$$

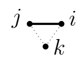
The unlabeled links again have weight $(-1)^{i,ij}$.

Note that in either case, the initial post $x_i x_k$ cannot give any chain-link fences, since after moving back in the e_i-direction, the next face is x_k, which is not a vertex of any $S_0^{\mathbf{b}_1}$. □

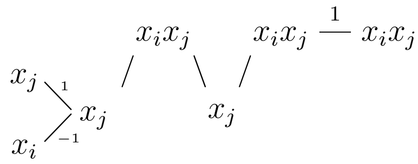
Lemma 6.2.11. *Suppose* $\mathbf{b}$ *and* $\mathbf{a}$ *are degree vectors such that* $K^{\mathbf{b}} I$ *is*  , $K^{\mathbf{a}} I$ *is*  , *and a lattice path* λ *from* $\mathbf{b}$ *to* $\mathbf{a}$ *passes through*  *and then*  r *times.*

Then the contribution to the sylvan matrix along the lattice path is given below.

$$\widetilde{H}_0 K^{\mathbf{a}} \otimes \langle \mathbf{x}^{\mathbf{a}} \rangle \xleftarrow{\begin{array}{c c} & \begin{array}{ccc} x_i x_j & x_i x_k & x_j x_k \end{array} \\ \begin{array}{c} x_i \\ x_j \end{array} & \begin{bmatrix} \frac{(-1)^{j,ij}}{2^{r+2}} & 0 & 0 \\ \frac{(-1)^{i,ij}}{2^{r+2}} & 0 & 0 \end{bmatrix} \end{array}} \widetilde{H}_1 K^{\mathbf{b}} \otimes \langle \mathbf{x}^{\mathbf{b}} \rangle$$

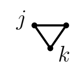
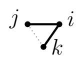
Proof. The choices of 1-dimensional hedges for  and  are the same, as T_1 must be the edge $x_i x_j$, and S_0 must be a vertex with nonzero coefficient in the boundary of the edge (either x_i or x_j). In , T_0 can be either $\{x_i\}$ or $\{x_j\}$. Thus $\Delta_{1,\lambda} = 2^{r+2}$, where r is the number of lattice points $\mathbf{b}_\ell$ such that $K^{\mathbf{b}_\ell} I$ is .

Starting with initial post $x_i x_j$, the only 1-dimensional face that can appear in any chain-link fence is $x_i x_j$. The 0-faces that are not the terminal post must be x_i or x_j, depending on which direction the lattice path moves back. The weight of the link between $x_i x_j$ and the vertex is the same as the weight of the chain-link between that same vertex and $x_i x_j$, as both are the coefficient of the vertex in the boundary of the edge. Therefore the signs will cancel. If the last direction the lattice path moves back in is the e_i-direction, then the signs that remain are $(-1)^{i,ij}$ and 1 if the terminal post is x_j or $(-1)^{i,ij}$ and -1 if the terminal post is x_i (see below). The same argument applies for the lattice path moving back in the e_j-direction last, where i and j are swapped.



$$T_0 = \{x_i\} \qquad S_0 = \{x_j\}$$

□

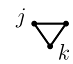
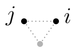
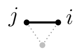
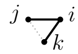
Lemma 6.2.12. *If* $\mathbf{b}$ *and* $\mathbf{a}$ *are degree vectors such that* $K^{\mathbf{b}}I$ *is*  *and* $K^{\mathbf{a}}I$ *is* *, and in the lattice path* $\lambda = (\mathbf{a}, \mathbf{b}_r, \mathbf{b}_{r-1}, \ldots, \mathbf{b}_1, \mathbf{b})$*, all degree vectors* $\mathbf{b}_1, \ldots, \mathbf{b}_r$ *have Koszul complexes*  *, then the contribution of* λ *to the sylvan matrix is*

$$\widetilde{H}_0 K^{\mathbf{a}} \otimes \langle \mathbf{x}^{\mathbf{a}} \rangle \xleftarrow{\begin{array}{c} \quad x_ix_j \qquad\qquad x_ix_k \qquad x_jx_k \\ \begin{array}{c} x_i \\ x_j \end{array} \begin{bmatrix} \frac{(-1)^{j,ij}}{2\cdot 3^r}\lceil \frac{3^r}{2} \rceil & \frac{(-1)^{i,ik}}{2\cdot 3^r}\lfloor \frac{3^r}{2} \rfloor & 0 \\ \frac{(-1)^{i,ij}}{2\cdot 3^r}\lceil \frac{3^r}{2} \rceil & \frac{(-1)^{k,ik}}{2\cdot 3^r}\lfloor \frac{3^r}{2} \rfloor & 0 \end{bmatrix} \end{array}} \widetilde{H}_1 K^{\mathbf{b}} \otimes \langle \mathbf{x}^{\mathbf{b}} \rangle$$

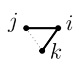
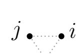
Proof. The lattice path must have $\mathbf{b}_r - \mathbf{a} = e_j$, so only the chain-link fences in Lemma 6.2.7 ending in x_ix_j will contribute to the entries of the sylvan matrix. Applying $d_1^{e_j}$, there are now $\lceil \frac{3^r}{2} \rceil$ chain-link fences that end in x_i with weight $(-1)^{j,ij}$. The shrubbery $T_0^{\mathbf{a}}$ can be $\{x_i\}$ or $\{x_j\}$. If $T_0^{\mathbf{a}} = \{x_j\}$, then x_i is cycle-linked to x_i with weight 1 and to x_j with weight -1. Thus $D^{\mathbf{ab}}_{x_i,x_ix_j} = \frac{(-1)^{j,ij}}{2\cdot 3^r}\lceil \frac{3^r}{2} \rceil$ and $D^{\mathbf{ab}}_{x_j,x_ix_j} = -\frac{(-1)^{j,ij}}{2\cdot 3^r}\lceil \frac{3^r}{2} \rceil$.

If the chain-link fence begins with x_ix_k, then by Lemma 6.2.7 there are $\lfloor \frac{3^r}{2} \rfloor$ chain-link fences ending in x_ix_j. By the argument above, $D^{\mathbf{ab}}_{x_i,x_ix_k} = \frac{(-1)^{j,ij}(-1)^{i,ijk+1}}{2\cdot 3^r}\lfloor \frac{3^r}{2} \rfloor$ and

$D^{\mathbf{ab}}_{x_j,x_ix_k} = -\frac{(-1)^{j,ij}(-1)^{i,ijk+1}}{2\cdot 3^r}\lfloor \frac{3^r}{2} \rfloor.$ □

Lemma 6.2.13. *Suppose* $\mathbf{b}$ *and* $\mathbf{a}$ *are degree vectors such that* $K^{\mathbf{b}}I$ *is*  *and* $K^{\mathbf{a}}I$ *is* . *Suppose also that* $\lambda = (\mathbf{a}, \mathbf{c}_s, \ldots, \mathbf{c}_1, \mathbf{b}_r, \ldots, \mathbf{b}_1, \mathbf{b})$ *is a lattice path such that* $K^{\mathbf{c}_\ell}I$ *is* , $K^{\mathbf{b}_\ell}I$ *is*  . *Then the contribution to the sylvan matrix is given as follows:*

$$\widetilde{H}_0K^{\mathbf{a}}\otimes\langle\mathbf{x}^{\mathbf{a}}\rangle \xleftarrow{\begin{array}{c|ccc} & x_ix_j & x_ix_k & x_jx_k \\ \hline x_i & (-1)^{j,ij}\frac{\lceil \frac{3^r}{2}\rceil}{3^r2^{s+1}} & (-1)^{i,ik}\frac{\lfloor \frac{3^r}{2}\rfloor}{3^r2^{s+1}} & 0 \\ x_j & (-1)^{i,ij}\frac{\lceil \frac{3^r}{2}\rceil}{3^r2^{s+1}} & (-1)^{k,ik}\frac{\lfloor \frac{3^r}{2}\rfloor}{3^r2^{s+1}} & 0 \end{array}} \widetilde{H}_1K^{\mathbf{b}}\otimes\langle\mathbf{x}^{\mathbf{b}}\rangle$$

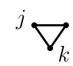
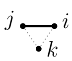
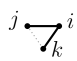
Proof. By Lemma 6.2.7, there are $\lceil \frac{3^r}{2} \rceil$ chain-link fences that end in x_ix_j with weight 1 after moving through  r times. $\mathbf{b}_r - \mathbf{c}_1 = e_j$, so only these chain-link fences will produce chain-link fences of full length. At each degree $\mathbf{c}_\ell$, $T_1 = \{x_ix_j\}$ and $S_0 = \{x_i\}$ or $\{x_j\}$. Along the segment of λ from $\mathbf{c}_1$ to $\mathbf{c}_s$, there is only one hedgerow that yields a chain-link fence of full length, since the direction the lattice path moves back in determines the stake that must be selected for S_0, since the vertex in the chain-link fence must be in the stake set in order for the fence to continue. As in the proof of Proposition 6.2.11, this segment of the chain-link fence has weight 1. Finally, after moving back in the e_i-direction to , $T_0 = \{x_i\}$ or $\{x_j\}$, and x_j is cycle-linked to x_j with weight 1 and to x_i with weight -1 if $T_0 = \{x_i\}$. By Lemma

6.2.7, there are $\lceil \frac{3^r}{2} \rceil$ chain-link fences ending with x_j with total weight $(-1)^{i,ij}$ and $\lceil \frac{3^r}{2} \rceil$ ending with x_i with total weight $-(-1)^{i,ij}$.

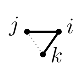
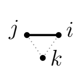
Similarly, if $\mathbf{c}_s - \mathbf{a} = e_j$, there are $\lceil \frac{3^r}{2} \rceil$ ending with x_i with total weight $(-1)^{j,ij} = -(-1)^{i,ij}$ and $\lceil \frac{3^r}{2} \rceil$ ending with x_j with total weight $-(-1)^{j,ij} = (-1)^{i,ij}$. Note that $\Delta_{1,\lambda} I = 3^r 2^s 2$.

For chain-link fences beginning with $x_i x_k$, there are $\lfloor \frac{3^r}{2} \rfloor$ ending with $x_i x_j$ with weight $(-1)^{i,ijk+1}$ from Lemma 6.2.7. Multiply by the partial weights described above.

□

Lemma 6.2.14. *Suppose* $\mathbf{b}$ *and* $\mathbf{a}$ *are degree vectors such that* $K^{\mathbf{b}} I$ *is*  *and* $K^{\mathbf{a}} I$ *is*  *. Suppose also that* $\lambda = (\mathbf{a}, \mathbf{c}, \mathbf{b}_r, \ldots, \mathbf{b}_1, \mathbf{b})$ *is a lattice path such that* $K^{\mathbf{c}} I$ *is*  *,* $K^{\mathbf{b}_\ell} I$ *is*  *. Then the contribution to the sylvan matrix is given as follows:*

$$\widetilde{H}_0 K^{\mathbf{a}} \otimes \langle \mathbf{x}^{\mathbf{a}} \rangle \xleftarrow{\begin{array}{c} \quad x_i x_j \qquad\qquad x_i x_k \qquad x_j x_k \\ \begin{array}{c} x_i \\ x_j \end{array} \begin{bmatrix} (-1)^{j,ij} \frac{\lceil \frac{3^r}{2} \rceil}{4 \cdot 3^r} & (-1)^{i,ik} \frac{\lfloor \frac{3^r}{2} \rfloor}{4 \cdot 3^r} & 0 \\ (-1)^{i,ij} \frac{\lceil \frac{3^r}{2} \rceil}{4 \cdot 3^r} & (-1)^{k,ik} \frac{\lfloor \frac{3^r}{2} \rfloor}{4 \cdot 3^r} & 0 \end{bmatrix} \end{array}} \widetilde{H}_1 K^{\mathbf{b}} \otimes \langle \mathbf{x}^{\mathbf{b}} \rangle$$

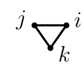
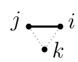
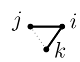
Proof. After moving through  r times, there are $\lceil \frac{3^r}{2} \rceil$ chain-link fences ending with $x_i x_j$ and $\lfloor \frac{3^r}{2} \rfloor$ ending with $x_i x_k$ by Lemma 6.2.7. After moving back in the e_i-direction to , the chain-link fences from $x_i x_k$ will end, since x_k cannot be a stake in degree $\mathbf{c}$. The shrubbery $T_1^{\mathbf{c}} = \{x_i x_j\}$, and the stake set $S_0^{\mathbf{c}} = \{x_i\}$ or

$\{x_j\}$. Following the proof of Lemma 6.2.10, if $\mathbf{c} - \mathbf{a} = e_i$, then there are in total $\lceil \frac{3^r}{2} \rceil$ chain-link fences ending in x_j with weight $(-1)^{i,ij}$ and $\lceil \frac{3^r}{2} \rceil$ ending in x_i with weight $-(-1)^{i,ij}$.

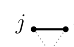
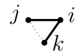
If $\mathbf{c} - \mathbf{a} = e_j$, then there are in total $\lceil \frac{3^r}{2} \rceil$ ending in x_i with weight $(-1)^{j,ij} = -(-1)^{i,ij}$ and $\lceil \frac{3^r}{2} \rceil$ ending in x_j with weight $-(-1)^{j,ij} = (-1)^{i,ij}$.

Similarly, there are $\lfloor \frac{3^r}{2} \rfloor$ beginning with $x_i x_k$ and ending with $x_i x_j$ at $\mathbf{b}_r$. These have weight $(-1)^{i,ijk+1}$ by Lemma 6.2.7.

Note $\Delta_{1,\lambda} = 2 \cdot 2 \cdot 3^r$. □

Lemma 6.2.15. *Suppose* $\mathbf{b}$ *and* $\mathbf{a}$ *are degree vectors such that* $K^{\mathbf{b}}I$ *is*  *and* $K^{\mathbf{a}}I$ *is* . *Suppose also that* $\lambda = (\mathbf{a}, \mathbf{a}_s, \ldots, \mathbf{a}_1, \mathbf{c}, \mathbf{b}_r, \ldots, \mathbf{b}_1\mathbf{b})$ *is a lattice path such that* $K^{\mathbf{a}_\ell}I$ *is* , $K^{\mathbf{c}}I$ *is* , $K^{\mathbf{b}_\ell}I$ *is* . *Then the contribution to the sylvan matrix is given as follows:*

$$\widetilde{H}_0 K^{\mathbf{a}} \otimes \langle \mathbf{x}^{\mathbf{a}} \rangle \xleftarrow{\begin{array}{c|ccc} & x_ix_j & x_ix_k & x_jx_k \\ \hline x_i & (-1)^{j,ij}\frac{\lceil \frac{3^r}{2} \rceil}{3^r \cdot 2^{s+2}} & (-1)^{i,ik}\frac{\lfloor \frac{3^r}{2} \rfloor}{3^r \cdot 2^{s+2}} & 0 \\ x_j & (-1)^{i,ij}\frac{\lceil \frac{3^r}{2} \rceil}{r \cdot 2^{s+2}} & (-1)^{k,ik}\frac{\lfloor \frac{3^r}{2} \rfloor}{3^r \cdot 2^{s+2}} & 0 \end{array}} \widetilde{H}_1 K^{\mathbf{b}} \otimes \langle \mathbf{x}^{\mathbf{b}} \rangle$$

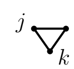
Proof. Segments of chain-link fences where the Koszul complex is  are in bijection with lattice path segments and hedgerows. The vertex must be the vertex in the stake set S_0, and it is the vertex not equal to the vertex whose degree is the direction the lattice path moved back in. The weights of the linkages cancel, as in the proof of Proposition 6.2.11. Therefore the only difference between this case and the case in Proposition 6.2.14 is $\Delta_{1,\lambda} I$. Here, $\Delta_{1,\lambda} I = 3^r \cdot 2 \cdot 2^s \cdot 2$, where r is the number of times the lattice path passes through 

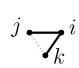
and s is the number of times the lattice path passes through 

. □

6.2.4 Results that apply to all cases

Let $D_{ve}^{\mathbf{ab},\lambda}$ be the contribution to $D_{ve}^{\mathbf{ab}}$ along the lattice path λ, where v is a vertex and e is an edge. The following theorem, together with Theorem 6.1.1, is the main result of this section and describes all entries $D_{ve}^{\mathbf{ab}}$ in the three-variable case. See Theorem 6.1.1 for a description of the entries $D_{\varnothing v}^{\mathbf{ab}}$.

Theorem 6.2.16. *Let* $\mathbf{b}$ *and* $\mathbf{a}$ *be degree vectors such that* $\mathbf{a} \preceq \mathbf{b}$*,* $K^{\mathbf{b}} I$ *is* 

, and $\dim_{\Bbbk} \widetilde{H}_0(K^{\mathbf{a}} I; \Bbbk) \neq 0$*. Let* $\lambda = (\mathbf{a} = \mathbf{b}_j, \ldots, \mathbf{b}_1, \mathbf{b})$ *be a saturated, decreasing lattice path such that* $\mathbf{b} - \mathbf{b}_1 = e_i$*, and let* r *be the number of times the lattice path* λ *passes through a degree vector* $\mathbf{c}$ *such that* $K^{\mathbf{c}} I$ *is* 

. Then

$$D_{ve}^{\mathbf{ab},\lambda} = s_{ve}^{\lambda} \frac{c_{ve}^{\lambda}}{\Delta_{1,\lambda} I},$$

where

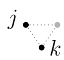
- $s_{i,ij}^{\lambda} := (-1)^{j,ij}$,

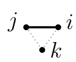
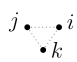
- $s^{\lambda}_{j,ij} := (-1)^{i,ij}$, *and*

- $s^{\lambda}_{k,ij} := (-1)^{j,ij}$,

and the values c^{λ}_{ve} *are given below. Note that if there is only one lattice path from* $\mathbf{b}$ *to* $\mathbf{a}$, c^{λ}_{ve} *is just given as* c_{ve}.

1. $c_{v,jk} = 0$ *for all* v.

2. *If* $K^{\mathbf{a}}I$ *is* , *then* $c_{v,ij} = 3^r$ *for any vertex* $v \in K^{\mathbf{a}}I$.

3. *If* $K^{\mathbf{a}}I$ *is*  *or* , *then* $c_{i,ij} = 1$, $c_{j,ij} = 3^r + \lceil \frac{3^r}{2} \rceil$, *and* $c_{k,ij} = 3^r + \lfloor \frac{3^r}{2} \rfloor$.

4. *If* $K^{\mathbf{a}}I$ *is* , *then* $c^{\lambda}_{v,ij} = \lceil \frac{3^r}{2} \rceil$ *and* $c^{\lambda}_{v,ik} = \lfloor \frac{3^r}{2} \rfloor$ *for any* $v \in K^{\mathbf{a}}I$.

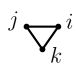
Finally, $\Delta_{1,\lambda}I = v_j \prod_{\ell=1}^{j-1}(e_\ell + 1)$, *where* v_j *is the number of vertices in* $K^{\mathbf{b}_j}I$ *and* e_ℓ *is the number of edges in* $K^{\mathbf{b}_\ell}I$.

Lemma 6.2.17. *Along a lattice path* $\lambda = (\mathbf{b}_j, \mathbf{b}_{j-1}, \ldots, \mathbf{b}_1, \mathbf{b}_0 = \mathbf{b})$, $\Delta_{1,\lambda}I$ *can be computed as*

$$\Delta_{1,\lambda}I = v_j \prod_{\ell=1}^{j-1}(e_\ell + 1),$$

where v_j *is the number of vertices in* $K^{\mathbf{b}_j}I$ *and* e_ℓ *is the number of edges in* $K^{\mathbf{b}_\ell}I$.

Proof. In the three-variable case, $\theta_{T_i} = \theta_{S_{i-1}} = 1$. Therefore $\Delta_0^T K^{\mathbf{b}_j}I = \sum_{T_0} \theta^2_{T_0}$ is the number dimension-0 shrubberies of $K^{\mathbf{b}_j}I$. Similarly, $\Delta_1^{ST_1} K^{\mathbf{b}_\ell}I = \sum_{ST_1} \theta^2_{S_0}\theta^2_{T_1}$ is the number of 1-dimensional hedges of $K^{\mathbf{b}_\ell}I$. Finally, $\Delta_1^S K^{\mathbf{b}}I = \sum_{S_1} \theta_{S_1}^2$ is the number of 1-dimensional stake sets of $K^{\mathbf{b}}I$.

In a hedgerow along λ, a shrubbery $T_0^{\mathbf{b}_j}$ of dimension 0 is chosen in degree $\mathbf{b}_j$. This can be any of the vertices in $K^{\mathbf{b}_j}I$, so $\Delta_0^T K^{\mathbf{b}_j}I = v_j$. a stake set $S_1^{\mathbf{b}}$ of dimension 1 is chosen at degree $\mathbf{b}$. A hedge $ST_1^{\mathbf{b}_\ell}$ is chosen at each interior lattice point $\mathbf{b}_1, \ldots, \mathbf{b}_{j-1}$. There are no 1-cycles in $K^{\mathbf{b}_\ell}I$ for $\ell = 1, \ldots, j-1$, so $T_1^{\mathbf{b}_\ell}$ must include all edges in $K^{\mathbf{b}_\ell}I$. If there are no edges in $K^{\mathbf{b}_\ell}I$, then there are no 0-boundaries, so $S_0^{\mathbf{b}_\ell} = \{\}$. If there is one edge in $K^{\mathbf{b}_\ell}I$, then there are two choices for $S_0^{\mathbf{b}_\ell}$: the sets including either vertex in the boundary of the edge. If there are two edges in $K^{\mathbf{b}_\ell}I$, then $S_0^{\mathbf{b}_\ell}$ can be any pair of vertices, giving three options. Therefore $\Delta_1^{ST_1} K^{\mathbf{b}_\ell}I = e_j + 1$. Since $K^{\mathbf{b}}I$ is 

and has no 1-boundaries, $S_1^{\mathbf{b}}$ is forced to be the empty set $\{\}$. Therefore $\Delta_1^S K^{\mathbf{b}}I = 1$.

Then $\Delta_{1,\lambda}I = (\Delta_0^T K^{\mathbf{b}_j}I \prod_{\ell=1}^{j-1} \Delta_1^{ST_1} K^{\mathbf{b}_\ell}I(\Delta_1^S K^{\mathbf{b}}I) = v_j \prod_{\ell=1}^{j-1}(e_\ell + 1)$. □

Lemma 6.2.18. *Let $\lambda = (\mathbf{a} = \mathbf{b}_j, \mathbf{b}_{j-1}, \ldots, \mathbf{b}_1, \mathbf{b}_0 = \mathbf{b})$ be a lattice path such that $K^{\mathbf{b}}I$ is . If an edge e does not appear in any Koszul complex $K^{\mathbf{b}_\ell}I$ except for $\ell = 0$, then $D_{v,e}^{\mathbf{ab}} = 0$ for all vertices v.*

Proof. Beginning a chain-link fence at e, the fence moves back in the i-direction to a vertex which is isolated from any edges in $K^{\mathbf{b}_1}I$ (since x_jx_k is not an edge of $K^{\mathbf{b}_1}I$). Therefore the vertex cannot be chain-linked to any edges, and the chain-link fence terminates. □

Proof of 6.2.16. Statement 1. follows from Lemma 6.2.18, and the last sentence of the theorem is proven in Lemma 6.2.17. Statement 2. follows from Lemmas 6.2.6 and 6.2.8. Statement 3. follows from Lemmas 6.2.5 and 6.2.9. Statement 4. follows from Lemmas 6.2.10, 6.2.11, 6.2.12, 6.2.14, and 6.2.15. □

6.3 More examples of sylvan resolutions in three variables

Example 6.3.1. The staircase diagram for $I = \langle xy, xz, yz\rangle$ is given in Figure 6.5. Its canonical sylvan resolution is given in Figure 6.6.

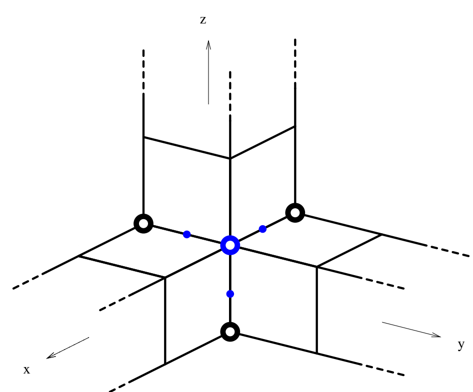


Figure 6.5: Staircase diagram for $I = \langle xy, xz, yz\rangle$

The entries of the sylvan matrix can be computed by using the combinatorial

$$
\begin{array}{c}
\widetilde{H}_{-1}K^{110}\otimes\langle xy\rangle \\
\oplus \\
\widetilde{H}_{-1}K^{101}\otimes\langle xz\rangle \\
\oplus \\
\widetilde{H}_{-1}K^{011}\otimes\langle yz\rangle
\end{array}
\xleftarrow{\begin{array}{c c} & \begin{array}{ccc} x & y & z \end{array} \\ \begin{array}{c}\varnothing\\ \varnothing\\ \varnothing\end{array} & \begin{bmatrix} [0 & 0 & 1] \\ [0 & 1 & 0] \\ [1 & 0 & 0] \end{bmatrix} \end{array}}
\widetilde{H}_0K^{111}\otimes\langle xyz\rangle
$$

Figure 6.6: Sylvan resolution for $I = \langle xy, xz, yz\rangle$

formula for the differential given in Theorem 5.4.5 or by using Theorem 6.1.1. Since $K^{111}I$ is three vertices, Theorem 6.1.1 gives $D^{011,111}_{\varnothing x} = D^{101,111}_{\varnothing y} = D^{110,111}_{\varnothing z} = 1$. Again by Theorem 6.1.1, all other entries of the sylvan matrix are 0.

Example 6.3.2. The staircase diagram for $I = \langle xy, y^3, z\rangle$ is given in Figure 6.7.

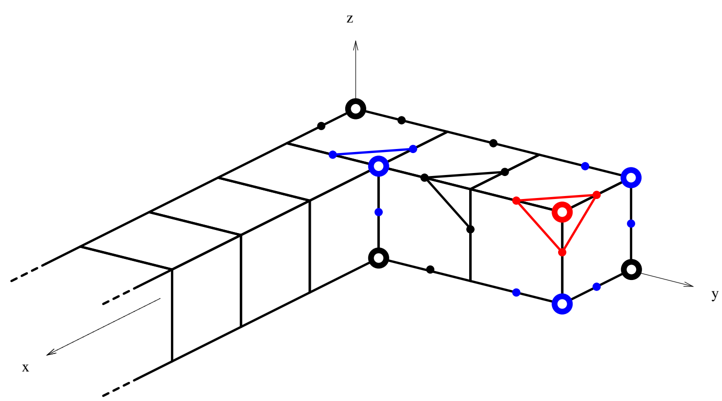
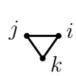
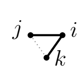
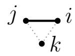


Figure 6.7: Staircase diagram for $I = \langle xy, y^3, z\rangle$

Figure 6.8 gives the sylvan resolution for I. To compute $D^{111,131}_{x,zy}$, note that there is only one lattice path from 131 to 111, each time moving back in the y-direction. $K^{131}I$ is , $K^{121}I$ is the edges yx and zy, i.e., , and $K^{111}I$ is the edge yx and the vertex z, i.e., . Then by Theorem 6.2.16, $D^{111,131}_{x,zy} = (-1)^{z \subset zy}\left(\frac{3^1+\lfloor\frac{3^1}{2}\rfloor}{3^2}\right) = \frac{4}{9}$.

$$
\begin{array}{c}
\widetilde{H}_{-1}K^{110} \otimes \langle xy \rangle \\ \oplus \\ \widetilde{H}_{-1}K^{030} \otimes \langle y^3 \rangle \\ \oplus \\ \widetilde{H}_{-1}K^{001} \otimes \langle z \rangle
\end{array}
\xleftarrow{\begin{array}{c} \begin{array}{cccc} & x\,y\,z & x\,y & y\,z \\ \varnothing & & & \\ \varnothing & & & \\ \varnothing & & & \end{array} \left[\begin{array}{ccc} [0\;0\;1] & [0\;1] & [0\;0] \\ [0\;0\;0] & [1\;0] & [0\;1] \\ [1\;1\;0] & [0\;0] & [1\;0] \end{array}\right] \end{array}}
\begin{array}{c}
\widetilde{H}_{0}K^{111} \otimes \langle xyz \rangle \\ \oplus \\ \widetilde{H}_{0}K^{130} \otimes \langle xy^3 \rangle \\ \oplus \\ \widetilde{H}_{0}K^{031} \otimes \langle y^3z \rangle
\end{array}
\xleftarrow{\begin{array}{c} \begin{array}{cccc} & zy & yx & xz \end{array} \\ \begin{array}{c} x \\ y \\ z \\ x \\ y \\ y \\ z \end{array} \left[\begin{array}{c} \left[\begin{array}{ccc} \frac{4}{9} & \frac{5}{9} & 0 \\ \frac{1}{9} & -\frac{1}{9} & 0 \\ -\frac{5}{9} & -\frac{4}{9} & 0 \end{array}\right] \\ \left[\begin{array}{ccc} -\frac{1}{2} & 0 & -\frac{1}{2} \\ \frac{1}{2} & 0 & \frac{1}{2} \end{array}\right] \\ \left[\begin{array}{ccc} 0 & -\frac{1}{2} & -\frac{1}{2} \\ 0 & \frac{1}{2} & \frac{1}{2} \end{array}\right] \end{array}\right] \end{array}}
\widetilde{H}_{1}K^{131} \otimes \langle xy^3z \rangle
$$

Figure 6.8: Sylvan resolution for $I = \langle xy, y^3, z \rangle$

Example 6.3.3. Let $I = \langle yz, xz, xy^2, x^2y \rangle$, whose staircase diagram is given in Figure 6.9.

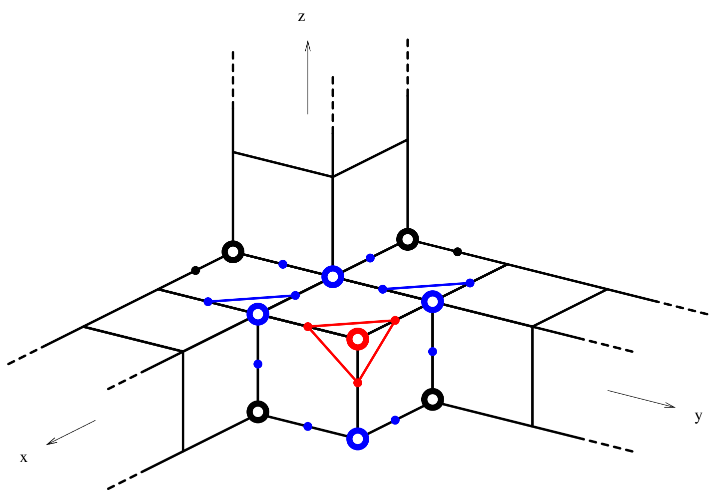
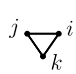
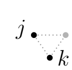


Figure 6.9: Staircase diagram for $I = \langle yz, xz, xy^2, x^2y \rangle$

To compute $D^{220,221}_{x,zy}$, note that there is one lattice path from 221 to 220, moving back in the z-direction. $K^{221}I$ is , and $K^{220}I$ is the vertices x and y, i.e., .

By Theorem 6.2.16, $D^{220,221}_{x,zy} = (-1)^{y \subset zy} \binom{3^0}{2} = -\frac{1}{2}$.

The sylvan resolution for I is given in Figure 6.10.

$$
\begin{matrix} \widetilde{H}_{-1}K^{011} \otimes \langle yz \rangle \\ \oplus \\ \widetilde{H}_{-1}K^{101} \otimes \langle xz \rangle \\ \oplus \\ \widetilde{H}_{-1}K^{120} \otimes \langle xy^2 \rangle \\ \oplus \\ \widetilde{H}_{-1}K^{210} \otimes \langle x^2y \rangle \end{matrix}
\xleftarrow{\begin{matrix} & x\ y & x\ y\ z & x\ y\ z & x\ y \\ \varnothing & \begin{bmatrix} 0 & 0 \end{bmatrix} & \begin{bmatrix} \frac{3}{4} & \frac{3}{4} & 0 \end{bmatrix} & \begin{bmatrix} \frac{1}{4} & \frac{1}{4} & 0 \end{bmatrix} & \begin{bmatrix} 1 & 0 \end{bmatrix} \\ \varnothing & \begin{bmatrix} 0 & 0 \end{bmatrix} & \begin{bmatrix} \frac{1}{4} & \frac{1}{4} & 0 \end{bmatrix} & \begin{bmatrix} \frac{3}{4} & \frac{3}{4} & 0 \end{bmatrix} & \begin{bmatrix} 0 & 1 \end{bmatrix} \\ \varnothing & \begin{bmatrix} 1 & 0 \end{bmatrix} & \begin{bmatrix} 0 & 0 & 1 \end{bmatrix} & \begin{bmatrix} 0 & 0 & 0 \end{bmatrix} & \begin{bmatrix} 0 & 0 \end{bmatrix} \\ \varnothing & \begin{bmatrix} 0 & 1 \end{bmatrix} & \begin{bmatrix} 0 & 0 & 0 \end{bmatrix} & \begin{bmatrix} 0 & 0 & 1 \end{bmatrix} & \begin{bmatrix} 0 & 0 \end{bmatrix} \end{matrix}}
\begin{matrix} \widetilde{H}_{0}K^{220} \otimes \langle x^2y^2 \rangle \\ \oplus \\ \widetilde{H}_{0}K^{121} \otimes \langle xy^2z \rangle \\ \oplus \\ \widetilde{H}_{0}K^{211} \otimes \langle x^2yz \rangle \\ \oplus \\ \widetilde{H}_{0}K^{111} \otimes \langle xyz \rangle \end{matrix}
\xleftarrow{\begin{matrix} & zy & yx & xz \\ x & -\frac{1}{2} & 0 & -\frac{1}{2} \\ y & \frac{1}{2} & 0 & \frac{1}{2} \\ x & 0 & \frac{1}{3} & -\frac{1}{3} \\ y & 0 & -\frac{2}{3} & -\frac{1}{3} \\ z & 0 & \frac{1}{3} & \frac{2}{3} \\ x & \frac{1}{3} & \frac{2}{3} & 0 \\ y & \frac{1}{3} & -\frac{1}{3} & 0 \\ z & -\frac{2}{3} & -\frac{1}{3} & 0 \\ x & 0 & \frac{1}{2} & 0 \\ y & 0 & -\frac{1}{2} & 0 \end{matrix}}
\widetilde{H}_{1}K^{221} \otimes \langle x^2y^2z \rangle
$$

Figure 6.10: Sylvan resolution for $I = \langle yz, xz, xy^2, x^2y \rangle$

Chapter 7

Noncanonical sylvan resolutions

Instead of using the Moore–Penrose pseudoinverse for the splitting in the Wall complex, any splittings of the boundary maps of the Koszul simplicial complexes can be used to obtain a minimal free resolution for an arbitrary monomial ideal. Any choice of community in $K^{\mathbf{b}}I$ for each degree $\mathbf{b} \in \mathbb{N}^n$ gives rise to a splitting of the boundary map in each degree that yields a minimal free resolution. Such resolutions are called sylvan, and in order to see where sylvan resolutions fit in the set of existing resolutions of monomial ideals, it is important to know which existing resolutions are sylvan. This chapter shows that planar graph resolutions are sylvan; Theorem 7.1.1 shows that certain choices of communities yield resolutions by planar graph for monomial ideals in three variables, and an example is given in Section 7.1.1. It is believed that the Eliahou–Kervaire resolution is sylvan, but the details are left for future work. Section 7.2 gives a minimal free resolution of the Stanley–Reisner ideal of the minimal triangulation of $\mathbb{RP}^2$ in characteristic 2.

7.1 Resolutions by planar graph

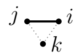
In the polynomial ring in three variables, a minimal resolution of a monomial ideal can be obtained by drawing a planar graph having exactly $\dim_{\Bbbk}\widetilde{H}_{i-1}(K^{\mathbf{b}}I;\Bbbk)$ i-faces with label $\mathbf{x}^{\mathbf{b}}$. The resolution is then supported on the cell complex of dimension 2 obtained from the vertices, edges, and regions of the corresponding planar map. Every monomial ideal in three variables has a minimal free resolution by planar graph ([Mil02], Theorem 11.1; see [MS05], Theorem 3.17). The following theorem is the main result of this chapter.

Theorem 7.1.1. *Any minimal free resolution by planar graph is sylvan. The resolution by planar graph is obtained by embedding the planar graph in the staircase surface in the following way.*

1. *Each edge must consist of two saturated, decreasing lattice paths starting at the degree vector corresponding to the label of the edge and ending at the degree vectors corresponding to the endpoints of the edge (one lattice path per endpoint).*

2. *Lattice paths moving back in the direction of the same connected component can only intersect either at the endpoints of the corresponding edges or must agree after the point of intersection (see Figures 7.3 and 7.4 for examples).*

Proof. There are three cases for $K^{\mathbf{b}}I$ such that $\dim_{\Bbbk}\widetilde{H}_0(K^{\mathbf{b}}I;\Bbbk) \geq 1$.

Case one: $K^{\mathbf{b}}I$ is . In this case $\dim_{\Bbbk}\widetilde{H}_0(K^{\mathbf{b}}I;\Bbbk) = 1$, so the planar graph has one edge with label $\mathbf{x}^{\mathbf{b}}$. This edge has two endpoints, one with label $\mathbf{x}^{\mathbf{a}}$ in the direction of the edge ij and one with label $\mathbf{x}^{\mathbf{c}}$ in the direction of k. The formula in Proposition 5.3.1 is used to compute $D^{\mathbf{ab}}_{\varnothing v}$. Consider the hedgerow given below for the edge whose embedding contains the lattice path $\lambda = (\mathbf{a} = \mathbf{b}_j, \mathbf{b}_{j-1}, \ldots, \mathbf{b}_1, \mathbf{b}_0 = \mathbf{b})$, identified with $(\lambda_j, \lambda_{j-1}, \ldots, \lambda_1)$, where $\lambda_\ell = \mathbf{b}_{\ell-1} - \mathbf{b}_\ell$. Note that λ_ℓ is either e_i or e_j, so $\mathbf{x}^{\lambda_\ell}$ is either x_i or x_j. Let v be the vertex in the edge $x_i x_j$ not equal to $\mathbf{x}^{\lambda_1}$.

$$
\begin{array}{cccccccc}
\lambda : & \mathbf{a} & \text{——} & \mathbf{b}_{j-1} & \cdots & \mathbf{b}_1 & \text{——} & \mathbf{b} \\
ST_0^\lambda : & T_{-1} = \{\} & & T_0 = \{\mathbf{x}^{\lambda_j}\} & & T_0 = \{\mathbf{x}^{\lambda_2}\} & & S_0 = \{v\} \\
 & & & S_{-1} = \{\varnothing\} & & S_{-1} = \{\varnothing\} & &
\end{array}
$$

The hedgerow above yields the following chain-link fences only:

$$
\begin{array}{ccccccc}
 & & \mathbf{x}^{\lambda_j} & \mathbf{x}^{\lambda_2} & & \mathbf{x}^{\lambda_1} \overset{1}{\text{—}} v \\
 & /_1 & & \cdots & \backslash^1 & /_1 \\
\varnothing \overset{1}{\text{—}} \varnothing & & & & \varnothing &
\end{array}
$$

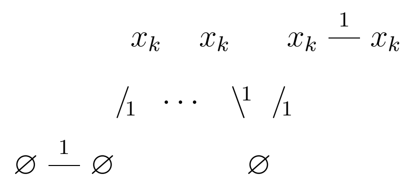
$$
\begin{array}{ccccccc}
 & & \mathbf{x}^{\lambda_j} & \mathbf{x}^{\lambda_2} & & \mathbf{x}^{\lambda_1} \overset{1}{\text{—}} \mathbf{x}^{\lambda_1} \\
 & /_1 & & \cdots & \backslash^1 & /_1 \\
\varnothing \overset{1}{\text{—}} \varnothing & & & & \varnothing &
\end{array}
$$

Given $S_0^{\mathbf{b}}$, a vertex in the edge ij can only be boundary-linked to the vertex in $\bar{S}_0$, and the vertices v and $\mathbf{x}^{\lambda_1}$ are both boundary-linked to $\mathbf{x}_{\lambda_1}$ if $\bar{S}_0 = \{\mathbf{x}^{\lambda_1}\}$. A saturated, decreasing lattice path from $\mathbf{b}$ to $\mathbf{a}$ can only initially move back in the direction of $\mathbf{x}^{\lambda_1}$ or v. If the lattice path moves back in the direction of v, the chain-link fence will terminate here, and the contribution of this lattice path to $D_{\varnothing v}^{\mathbf{ab}}$ or $D_{\varnothing \mathbf{x}^{\lambda_1}}$ is 0. By selecting $\mathbf{x}^{\lambda_\ell}$ as the vertex in $T_0^{\mathbf{b}_{\ell-1}}$, the chain-link fence will terminate unless the next direction the lattice path moves back in is λ_ℓ. This is because the shrub of $\varnothing$ is the vertex in $T_0^{\mathbf{b}_{\ell-1}}$. Therefore the only lattice path that contributes to $D_{\varnothing i}^{\mathbf{ab}}$ or $D_{\varnothing j}^{\mathbf{ab}}$ is λ. Finally, the empty set must be chosen as $T_{-1}^{\mathbf{a}}$, and $\varnothing$ is only cycle-linked to itself. Each of these linkages has weight 1. Thus $D_{\varnothing i}^{\mathbf{ab}} = D_{\varnothing j}^{\mathbf{ab}} = 1$.

To compute $D_{\varnothing k}^{\mathbf{cb}}$, note that there is only one lattice path from $\mathbf{b}$ to $\mathbf{c}$ and only one possible hedgerow, given below (note that $S_0^{\mathbf{b}}$ was selected in the discussion above).

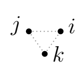
$\lambda :$	$\mathbf{c}$	——	$\mathbf{b}_{j-1}$	$\cdots$	$\mathbf{b}_1$	——	$\mathbf{b}$
$ST_0^{\lambda} :$	$T_{-1} = \{\}$		$T_0 = \{x_k\}$		$T_0 = \{x_k\}$		$S_0 = \{v\}$
			$S_{-1} = \{\varnothing\}$		$S_{-1} = \{\varnothing\}$		

along λ, yielding a single chain-link fence



Thus $D_{\varnothing k}^{\mathbf{cb}} = 1$.

Case two: $K^{\mathbf{b}}I$ is . Since $\dim_{\Bbbk}\widetilde{H}_0(K^{\mathbf{b}}I;\Bbbk) = 1$, there is again one edge in the planar graph with label $\mathbf{x}^{\mathbf{b}}$. The hedges and chain-link fences are given as the ones above for the computation of $D_{\varnothing k}^{\mathbf{cb}}$, replacing k with either i or j. Therefore $D_{\varnothing i}^{\mathbf{a}_i\mathbf{b}} = D_{\varnothing j}^{\mathbf{a}_j\mathbf{b}} = 1$, where $\mathbf{a}_i$ is the degree vector of the generator lying behind $\mathbf{b}$ in the e_i-direction, and $\mathbf{a}_j$ is the degree vector of the generator lying behind $\mathbf{b}$ in the e_j-direction.

Case three: $K^{\mathbf{b}}I$ is . This time $\dim_{\Bbbk}\widetilde{H}_0(K^{\mathbf{b}}I;\Bbbk) = 2$, so there are two edges in the planar graph with label $\mathbf{x}^{\mathbf{b}}$. The hedges and chain-link fences are again the same as in case two, giving $D_{\varnothing i}^{\mathbf{a}_i\mathbf{b}} = D_{\varnothing j}^{\mathbf{a}_j\mathbf{b}} = D_{\varnothing k}^{\mathbf{a}_k\mathbf{b}} = 1$. Here $\mathbf{a}_\ell$ is the degree vector that lies behind $\mathbf{b}$ in the x_ℓ-direction.

The monomial matrices that give the maps in the resolution by planar graph can be written down by picking a basis for $\widetilde{H}_0(K^{\mathbf{b}}I;\Bbbk)$. Again there are three cases for $K^{\mathbf{b}}I$ such that $\widetilde{H}_0(K^{\mathbf{b}}I;\Bbbk)$ is nonzero.

One basis element for $\widetilde{H}_0(K^{\mathbf{b}}I;\Bbbk)$ is chosen for each edge $\mathbf{x}^{\mathbf{b}}$ in the planar graph. For each edge, choose the basis element to be the difference of two vertices v and v' such that $\mathbf{x}^{\mathbf{b}}$ contains vertices whose degrees lie behind $\mathbf{b}$ in the direction of the connected components of v and v'. Choose the signs to match the orientation of the edge in the planar graph, i.e., if $d(\mathbf{x}^{\mathbf{b}}) = \mathbf{x}^{\mathbf{a}_v} - \mathbf{x}^{\mathbf{a}_{v'}}$, then the basis element is $v - v'$.

It is not possible to choose a community so that the edges of the embedding of the planar graph intersect and then diverge. This is because the edges in the embedding are lattice paths that follow the direction of the vertex in T_0. Once T_0 is selected in the degree of an interior lattice point $\mathbf{a}$, any lattice path that passes through degree $\mathbf{a}$ must move back in the direction of the vertex in T_0.

The proof above takes care of the map $F_0 \leftarrow F_1$ in the resolution. Once the map $F_0 \leftarrow F_1$ is determined, the map $F_1 \leftarrow F_2$ is fixed up to scalar multiple. This is because the kernel of the map $\bigoplus_{\mathbf{c}} \widetilde{H}_{-1}(K^{\mathbf{c}}I;\Bbbk) \leftarrow \bigoplus_{\mathbf{b}} \widetilde{H}_0(K^{\mathbf{b}}I;\Bbbk)$ is free and generated in incomparable degrees, so it has a unique basis up to scalar multiple. Thus the image of $\bigoplus_{\mathbf{b}} \widetilde{H}_0(K^{\mathbf{b}}I;\Bbbk) \leftarrow \bigoplus_{\mathbf{w}} \widetilde{H}_1(K^{\mathbf{w}}I;\Bbbk)$ has a unique basis up to scalar multiple, so it must match the image of the map in the planar resolution up to scalar multiple. If the resolution is constructed over $\mathbb{Z}$ instead of $\Bbbk$, then the image of $\bigoplus_{\mathbf{b}} \widetilde{H}_0(K^{\mathbf{b}}I;\mathbb{Z}) \leftarrow \bigoplus_{\mathbf{w}} \widetilde{H}_1(K^{\mathbf{w}}I;\mathbb{Z})$ will match the image of the planar resolution up to a multiple of ± 1. Tensoring the resolution over $\Bbbk$ gives the same maps, and multiplying the image by -1 is equivalent to reversing the orientation of the 2-dimensional faces in the planar graph. This is fine, since an arbitrary orientation on the faces of the planar graph gives a resolution of I. $\square$

Corollary 7.1.2. *The monomial matrices of a given planar graph resolution can be obtained from the sylvan matrices from Theorem 7.1.1 by choosing a basis for* $\widetilde{H}_0(K^{\mathbf{b}}I;\Bbbk)$*. For each edge in the planar graph with label* $\mathbf{x}^{\mathbf{b}}$ *and endpoints* $\mathbf{x}^{\mathbf{a}}$ *and* $\mathbf{x}^{\mathbf{c}}$*, choose a basis element* $\pm(v-v')$*, where* $\mathbf{a}$ *lies behind* $\mathbf{b}$ *in the direction of the connected component of* v *and* $\mathbf{c}$ *lies behind* $\mathbf{b}$ *in the direction of the connected component of* v'*. Choose the sign to match the orientation of the planar graph.*

7.1.1 Examples of sylvan planar graph resolutions

Example 7.1.3. The ideal $I = \langle x^3y^2, x^3z, x^2y^3, xyz, y^2z\rangle$ has a minimal resolution supported on the planar graph in Figure 7.1.

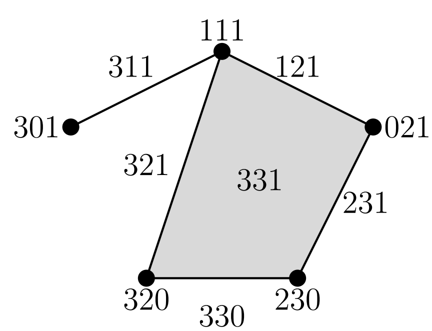


Figure 7.1: Planar graph that supports a minimal free resolution of I

Suppose the planar graph is oriented so that the minimal free resolution of I is given in Figure 7.2. Figure 7.3 gives one possible embedding of the planar graph that yields the resolution by planar graph as a sylvan resolution.

$$\mathbb{k}[\mathbf{x}]^5 \xleftarrow{\begin{array}{c|ccccc} & x^3yz & xy^2z & x^2y^3z & x^3y^3 & x^3y^2z \\ \hline x^3z & -1 & 0 & 0 & 0 & 0 \\ xyz & 1 & 1 & 0 & 0 & -1 \\ y^2z & 0 & -1 & 1 & 0 & 0 \\ x^3y^2 & 0 & 0 & 0 & -1 & 1 \\ x^2y^3 & 0 & 0 & -1 & 1 & 0 \end{array}} \mathbb{k}[\mathbf{x}]^5 \xleftarrow{\begin{array}{c|c} & x^3y^3z \\ \hline x^3yz & 0 \\ xy^2z & 1 \\ x^2y^3z & 1 \\ x^3y^3 & 1 \\ x^3y^2z & 1 \end{array}} \mathbb{k}[\mathbf{x}]$$

Figure 7.2: Planar graph resolution of $I = \langle x^3y^2x^3z, x^2y^3, x^3y^2\rangle$

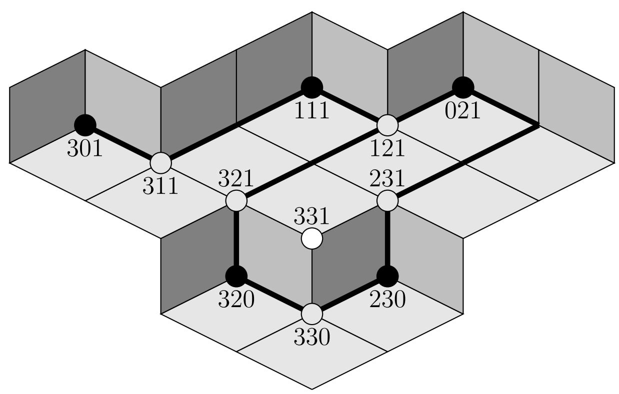


Figure 7.3: An embedding of the planar graph in Figure 7.1 that yields the planar graph resolution as sylvan

The embedding in Figure 7.4 does not satisfy the desired properties, because the edges with labels 321 and 231 meet at degree 121 but diverge afterwards to meet the vertices with labels 111 and 021. Therefore there is no choice of community that would yield these lattice paths.

The first embedding shown is used to construct the planar graph resolution as a sylvan resolution. The hedgerow corresponding to the lattice path from degree 321

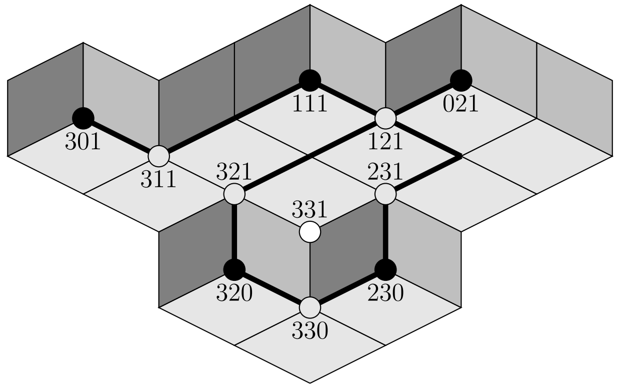


Figure 7.4: An embedding of the planar graph that does not correspond to a choice of hedges for a sylvan resolution

to 111 is given as:

$$\begin{array}{cccccccc}
\lambda : & 111 & \text{———} & 121 & \text{———} & 221 & \text{———} & 321 \\
ST_0^\lambda : & T_{-1} = \{\} & & T_0 = \{y\} & & T_0 = \{x\} & & S_0 = \{y\} \\
 & & & S_{-1} = \{\varnothing\} & & S_{-1} = \{\varnothing\} & &
\end{array}$$

giving the chain-link fences

$$\begin{array}{ccccccc}
 & & y & & x & & x \overset{1}{\text{—}} y \\
 & & /_1 & \backslash^1 & /_1 & \backslash^1 & /_1 \\
\varnothing \overset{1}{\text{—}} \varnothing & & \varnothing & & \varnothing & &
\end{array}$$

$$\begin{array}{ccccccc}
 & & y & & x & & x \overset{1}{\text{—}} x \\
 & & /_1 & \backslash^1 & /_1 & \backslash^1 & /_1 \\
\varnothing \overset{1}{\text{—}} \varnothing & & \varnothing & & \varnothing & &
\end{array}$$

The map $F_0 \leftarrow F_1$ in the sylvan resolution corresponding to the choice of community is given in Figure 7.5.

$$
\begin{array}{c}
\widetilde{H}_{-1}K^{301}\otimes\langle x^3z\rangle \\ \oplus \\
\widetilde{H}_{-1}K^{111}\otimes\langle xyz\rangle \\ \oplus \\
\widetilde{H}_{-1}K^{021}\otimes\langle y^2z\rangle \\ \oplus \\
\widetilde{H}_{-1}K^{320}\otimes\langle x^3y^2\rangle \\ \oplus \\
\widetilde{H}_{-1}K^{230}\otimes\langle x^2y^3\rangle
\end{array}
\xleftarrow{\begin{array}{c}
\begin{array}{ccccc} x\,y & x\,y & x\,y\,z & x\,y & x\,y\,z \end{array} \\
\begin{array}{c} \varnothing \\ \varnothing \\ \varnothing \\ \varnothing \\ \varnothing \end{array}
\begin{bmatrix}
[0\,1] & [0\,0] & [0\,0\,0] & [0\,0] & [0\,0\,0] \\
[1\,0] & [0\,1] & [0\,0\,0] & [0\,0] & [1\,1\,0] \\
[0\,0] & [1\,0] & [1\,1\,0] & [0\,0] & [0\,0\,0] \\
[0\,0] & [0\,0] & [0\,0\,0] & [0\,1] & [0\,0\,1] \\
[0\,0] & [0\,0] & [0\,0\,1] & [1\,0] & [0\,0\,0]
\end{bmatrix}
\end{array}}
\begin{array}{c}
\widetilde{H}_{0}K^{311}\otimes\langle x^3yz\rangle \\ \oplus \\
\widetilde{H}_{0}K^{121}\otimes\langle xy^2z\rangle \\ \oplus \\
\widetilde{H}_{0}K^{231}\otimes\langle x^2y^3z\rangle \\ \oplus \\
\widetilde{H}_{0}K^{330}\otimes\langle x^3y^3\rangle \\ \oplus \\
\widetilde{H}_{0}K^{321}\otimes\langle x^3y^2z\rangle
\end{array}
$$

Figure 7.5: The map $F_0 \leftarrow F_1$ in the sylvan resolution corresponding to the choice of community in Example 7.1.3

7.2 A minimal resolution of the Stanley–Reisner ideal of the minimal triangulation of $\mathbb{RP}^2$ in characteristic 2

Example 7.2.1. Consider the Stanley–Reisner ideal of the minimal triangulation of $\mathbb{RP}^2$, $I_\Delta = \langle abc, ace, abf, ade, adf, bcd, bde, bef, cdf, cef\rangle$. A minimal free resolution of I_Δ over a field $\Bbbk$ such that $\mathrm{char}(\Bbbk) \neq 2$ via the cell complex X in Figure 7.6 (see [MS05], Section 4.3.5), but X is not acyclic when $\mathrm{char}(\Bbbk) = 2$.

When $\mathrm{char}(\Bbbk) = 2$, the sylvan resolution for any choice of community gives a

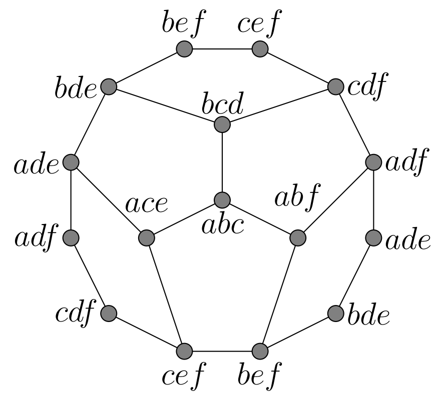


Figure 7.6: Cell complex supporting a minimal free resolution of the Stanley–Reisner ideal of the six-point triangulation of $\mathbb{RP}^2$ in $\mathrm{char}(\Bbbk) \neq 2$

Table 7.1: Nonzero homology modules for the Stanley–Reisner ring of the six-point triangulation of $\mathbb{RP}^2$ in characteristic 2

homological degree i	degrees $\mathbf{b}$ such that $\widetilde{H}_{i-1}(K^{\mathbf{b}}I;\Bbbk) \neq 0$
0	$abc, ace, abf, ade, adf, bcd, bde, bef, cdf, cef$
1	$abcd, abce, abcf, abdf, acdf, acef, adef, cdef,$ $abef, bdef, abde, bcdf, acde, bcde, bcef$
2	$abcdef, abcde, abcdf, abcef, abdef, acdef, bcdef$
3	$abcdef$

minimal free resolution of I_Δ.

The Betti numbers in characteristic 2 are $1, 10, 15, 7$, and 1. The Koszul simplicial complexes with nonzero homology in the corresponding degrees are in Table 7.1.

$K^{abcdef}I$ is isomorphic to the six-point triangulation of $\mathbb{RP}^2$, as shown in Figure 7.7. The facets of $K^{abcdef}I$ are the triangles $def, bdf, cde, bcf, bce, aef, acf, acd, abe, abd$. To determine the coefficients of the map $F_2 \leftarrow F_3$, a stake set of dimension 2 in degree $abcdef$ and a shrubbery of dimension 1 in degrees $abcde$, $abcdf$, $abcef$, $abdef$, $acdef$,

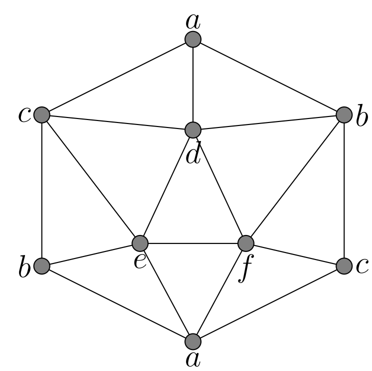


Figure 7.7: The simplicial complex $K^{abcdef}I$

Table 7.2: Hedgerows for lattice paths from $abcdef$ to degrees preceding $abcdef$ by a standard basis vector

$abcdef$	$S_2 = \{\}$
$abcde$	$T_1 = \{de, bd, bc, ae, ac\} \setminus \{de\}$
$abcdf$	$T_1 = \{df, cd, bc, af, ab\} \setminus \{df\}$
$abcef$	$T_1 = \{ef, bf, ce, ac, ab\} \setminus \{ef\}$
$abdef$	$T_1 = \{de, bf, be, af, ad\} \setminus \{de\}$
$acdef$	$T_1 = \{df, cf, ce, ae, ad\} \setminus \{df\}$
$bcdef$	$T_1 = \{ef, cf, cd, be, bd\} \setminus \{ef\}$

and $bcdef$ are chosen, as shown below. For each of the degrees where a shrubbery of dimension 1 is chosen, the Koszul simplicial complex is isomorphic to a hollow pentagon.

There is only one lattice path from $abcdef$ to each of the degrees $abcde, \ldots, bcdef$. They are given in Figure 7.8. The lattice path from $abcdef$ to $abcde$ moves back in the f-direction, and in order for a chain-link fence to continue, it must arrive at the edge de. Therefore there is only one choice for the 2-face that precedes de in the chain-link fence, namely, def. The only face boundary-linked to def in degree $abcdef$

is def itself. The edge de is cycle-linked to each edge in $K^{abcde}I$. All weights are 1. The map $\widetilde{H}_1(K^{abcdef}I;\Bbbk) \otimes \langle abcdef \rangle \leftarrow \widetilde{H}_2(K^{abcdef}I;\Bbbk) \otimes \langle abcdef \rangle$ is the zero map, because the modules are generated in the same degree.

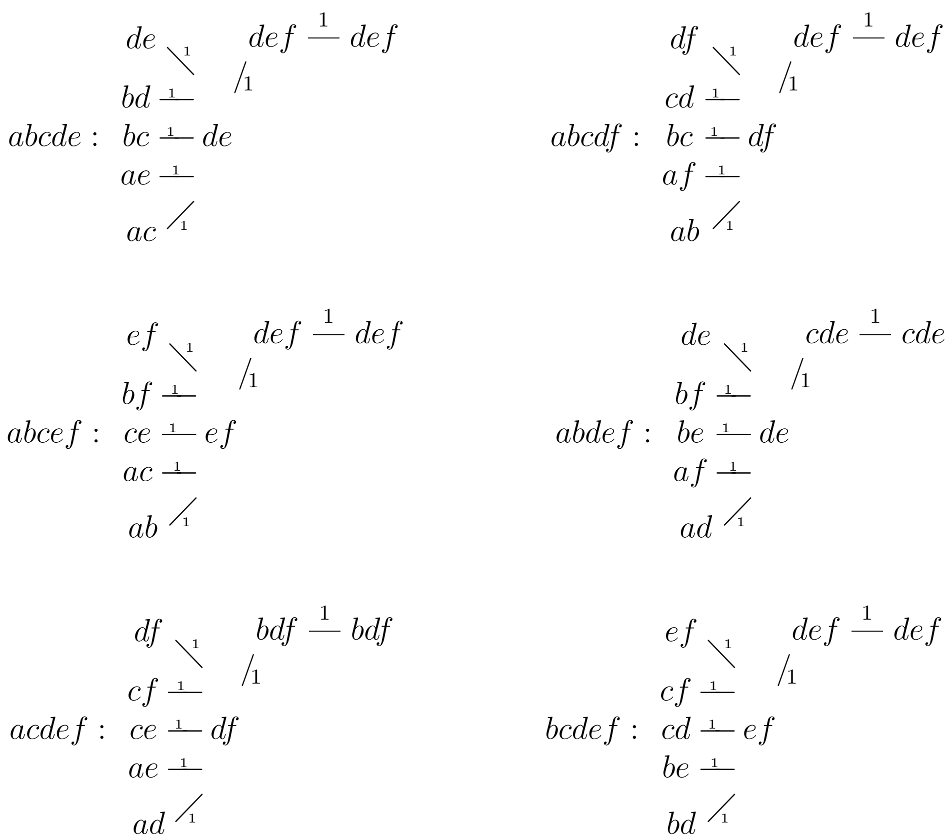


Figure 7.8: All chain-link fences along lattice paths from $abcdef$ to $abcde$, $abcdf$, $abcef$, $abdef$, $acdef$, $bcdef$

To determine the coefficients in the map $F_1 \leftarrow F_2$, a stake set of dimension 1 must be chosen for each of the degrees $abcde, abcdf, abcef, abdef, acdef$, and $bcdef$. Since for each of these degrees, the Koszul complex is isomorphic to an empty pentagon, there are no 1-boundaries. Therefore $S_1 = \{\,\}$ for each of these degrees. In addition,

Table 7.3: Shrubberies $T_0^{\mathbf{a}}$ for degrees $\mathbf{a}$ such that $\widetilde{H}_0(K^{\mathbf{a}}I;\Bbbk) \neq 0$

Degree $\mathbf{a}$	$T_0^{\mathbf{a}}$
$abcd$	$\{a,d\} \setminus \{a\}$
$abcf$	$\{c,f\} \setminus \{c\}$
$acdf$	$\{a,c\} \setminus \{a\}$
$adef$	$\{e,f\} \setminus \{e\}$
$abef$	$\{a,e\} \setminus \{a\}$
$abde$	$\{a,b\} \setminus \{a\}$
$acde$	$\{c,d\} \setminus \{c\}$
$abce$	$\{b,e\} \setminus \{b\}$
$abdf$	$\{b,d\} \setminus \{b\}$
$acef$	$\{a,f\} \setminus \{a\}$
$cdef$	$\{d,e\} \setminus \{d\}$
$bdef$	$\{d,f\} \setminus \{d\}$
$bcdf$	$\{b,f\} \setminus \{b\}$
$bcde$	$\{c,e\} \setminus \{c\}$

a shrubbery of dimension 0 must be chosen for each of the degrees $\mathbf{a}$ such that $\widetilde{H}_0(K^{\mathbf{a}}I;\Bbbk) \neq 0$. These degrees are listed in Table 7.1. Each of these Koszul complexes are isomorphic to two vertices, so one is chosen for each T_0. The choices are given in Table 7.3.

The lattice paths in each case are length one (degree $abcdef$ will be addressed later). In order for the chain-link fence to have full length and contribute to the coefficients in the map $F_1 \leftarrow F_2$, the second-to-last face in the chain-link fence must be the vertex that is not in $T_0^{\mathbf{a}}$. Otherwise, the vertex is not cycle-linked to any other vertex. The vertex not in $T_0^{\mathbf{a}}$ is cycle-linked to both vertices in $K^{\mathbf{a}}I$. Every edge in the chain-link fence is only boundary-linked to itself, and the lattice path moves back

Table 7.4: Bases for the homology groups $\widetilde{H}_{i-1}(K^{\mathbf{b}}I;\Bbbk)$

$i-1$	$\mathbf{b}$	basis for $\widetilde{H}_{i-1}(K^{\mathbf{b}}I;\Bbbk)$
-1	degree of any generator	$\varnothing$
0	any degree such that $\widetilde{H}_0 \neq 0$	sum of both vertices
1	$\{\mathbf{b} \mid \widetilde{H}_1(K^{\mathbf{b}}I) \neq 0\} \setminus \{abcdef\}$	sum of all edges
1	$abcdef$	$ac+ad+bd+be+ce$
2	$abcdef$	$abd+acd+bde+bce+ace+def+$ $aef+abf+bcf$

in a fixed direction. Therefore there are only two chain-link fences of full length for each pair of degrees. They are given in Figures 7.9 and 7.10.

To chain-link fences contributing to the map $F_0 \leftarrow F_1$, a stake set of dimension 0 is chosen at each degree $\mathbf{b}$ such that $\widetilde{H}_0(K^{\mathbf{b}}I;\Bbbk) \neq 0$. Since there are no 0-boundaries in these Koszul complexes, $S_0^{\mathbf{b}} = \{\,\}$. A shrubbery of dimension -1 is chosen at the degree of each generator. For each of these degrees $\mathbf{a}$, $T_{-1}^{\mathbf{a}} = \{\,\}$. The empty face is the only face of dimension -1, and it is cycle-linked to itself. The lattice paths have length one, and each vertex is boundary-linked to itself. The chain-link fences are given in Figures 7.11 and 7.12.

Because of the size of the sylvan matrices in this example, monomial matrices are used to write down the minimal free resolution of I_Δ. To do this, a basis must be chosen for each $\widetilde{H}_{i-1}(K^{\mathbf{b}}I;\Bbbk)$. The bases are given in Table 7.4.

Let $\bar{S}_1^{abcdef} = \{ac, ad, af, be, ce, ab\}$ and $T_2^{abcdef} = \{abd, acd, bde, bce, ace, def, aef,$

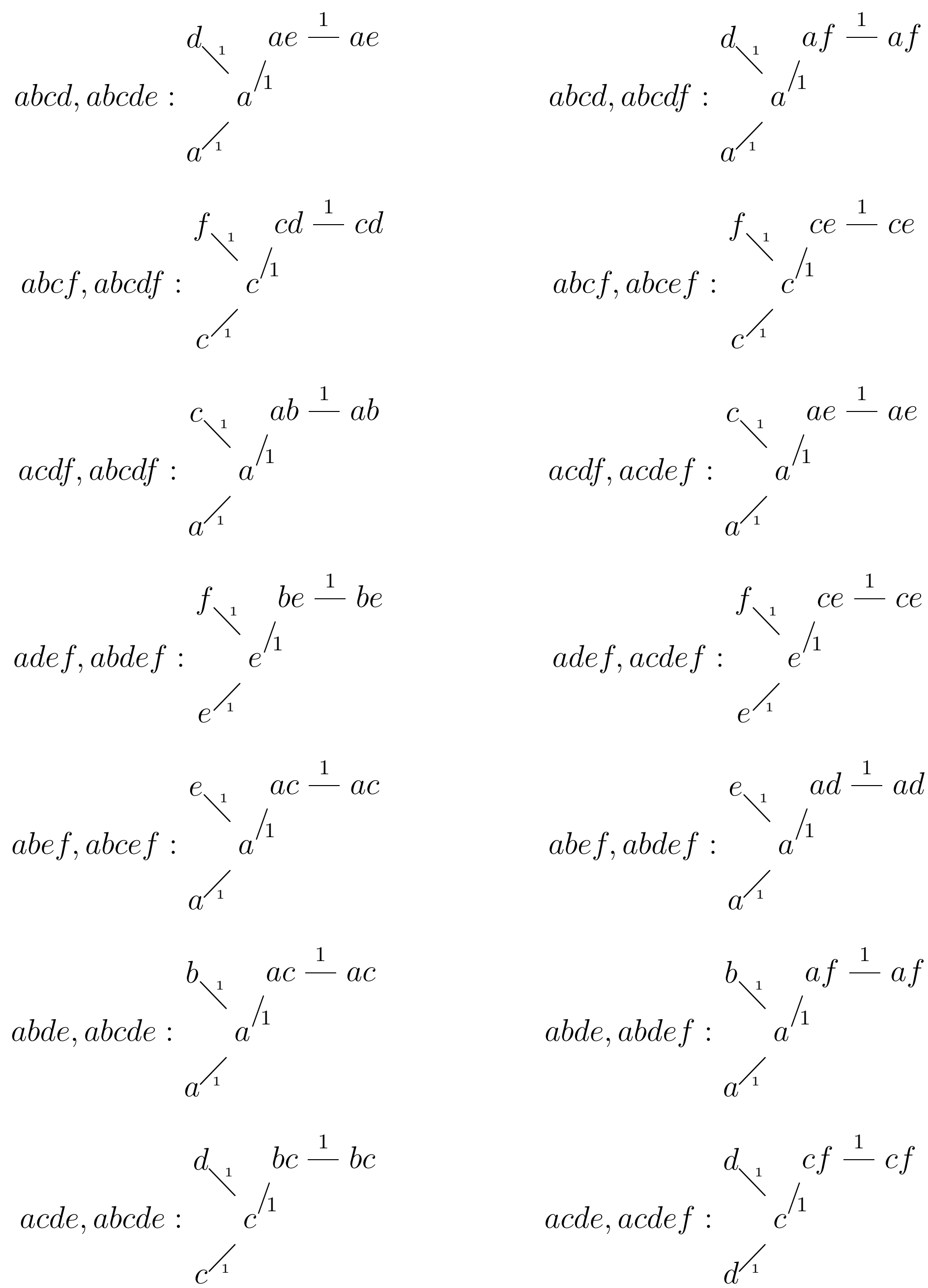


Figure 7.9: Chain-link fences contributing to the maps $F_1 \leftarrow F_2$

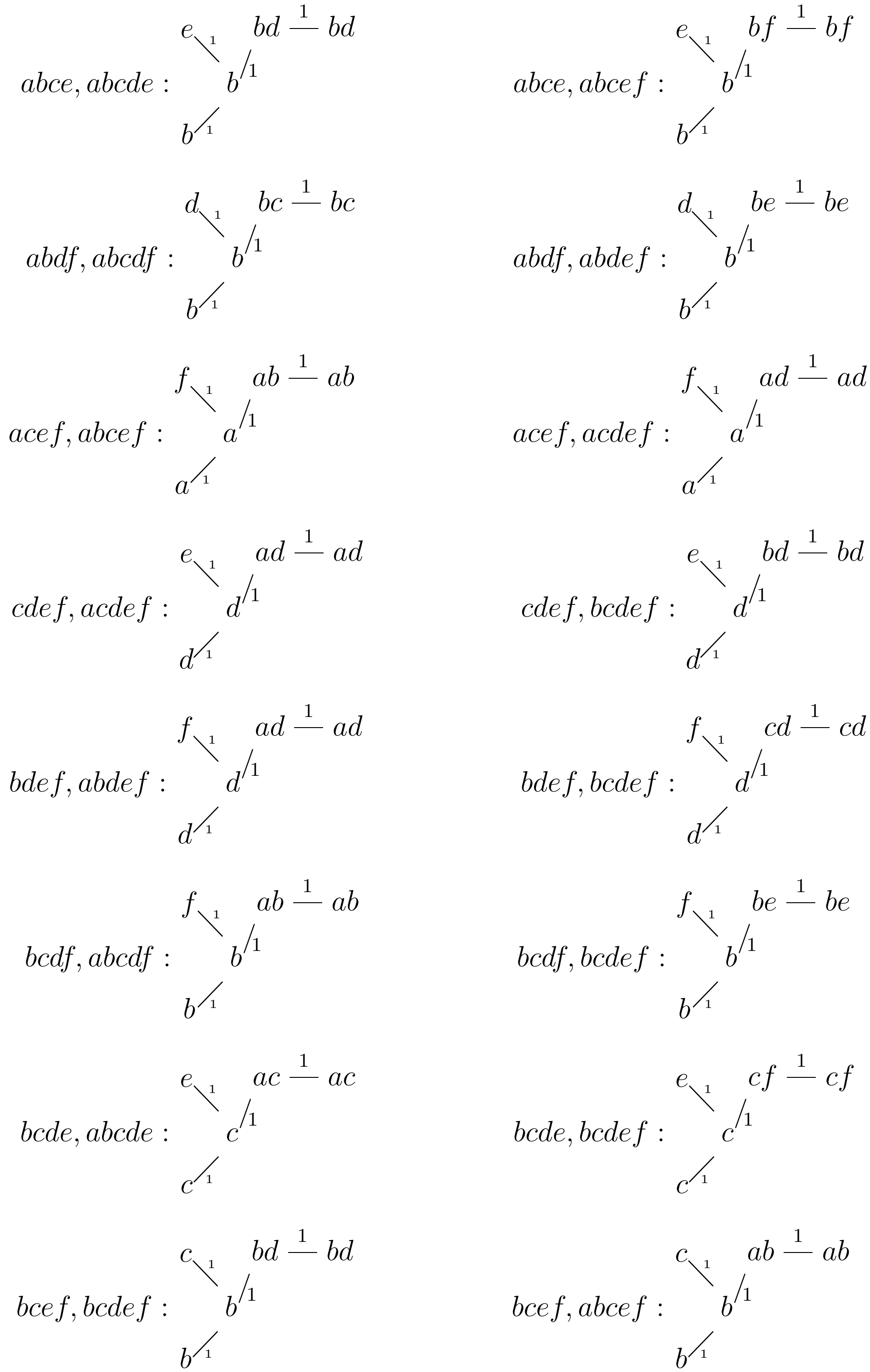


Figure 7.10: More chain-link fences contributing to the maps $F_1 \leftarrow F_2$

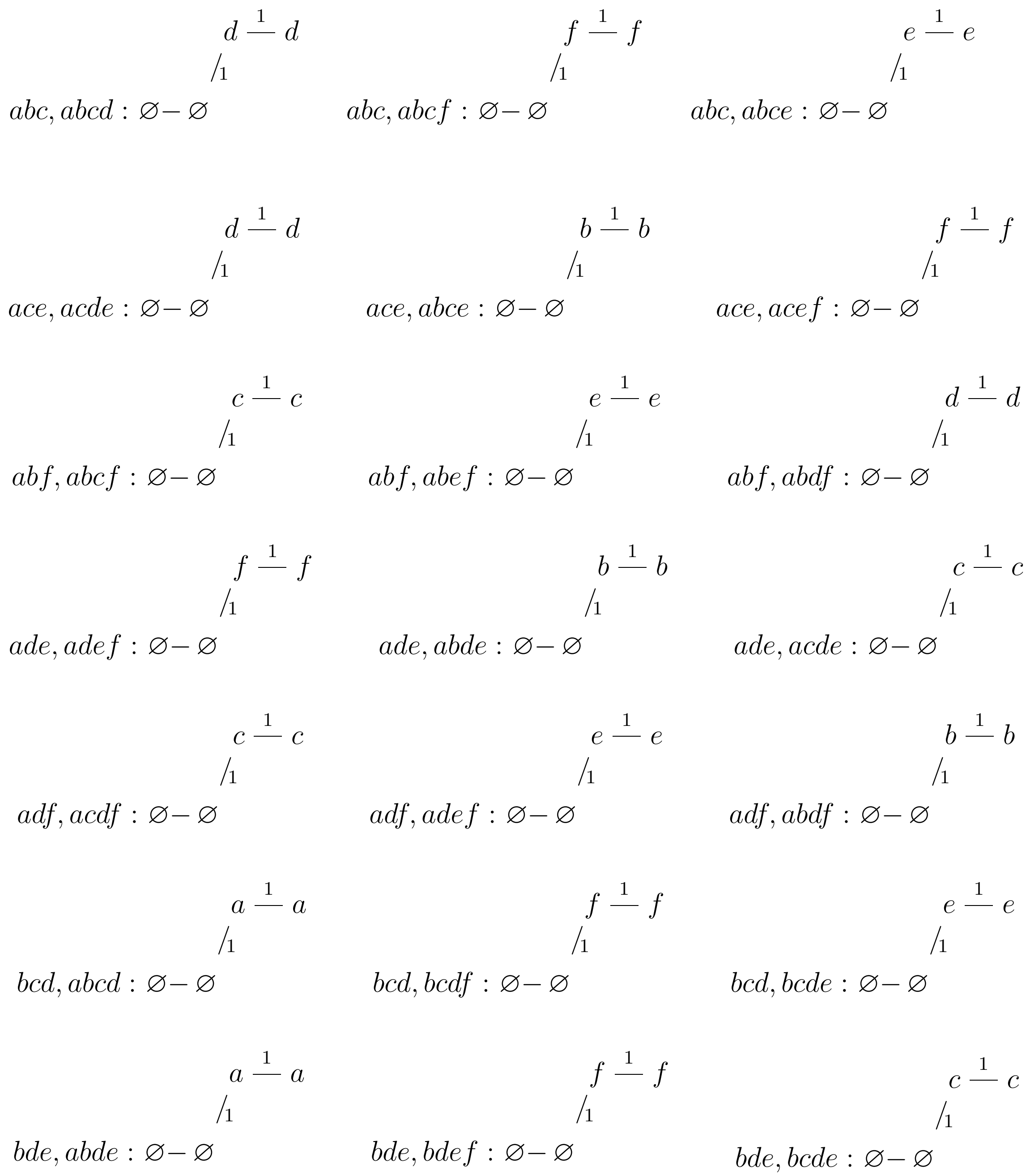


Figure 7.11: Chain-link fences contributing to the maps $F_0 \leftarrow F_1$

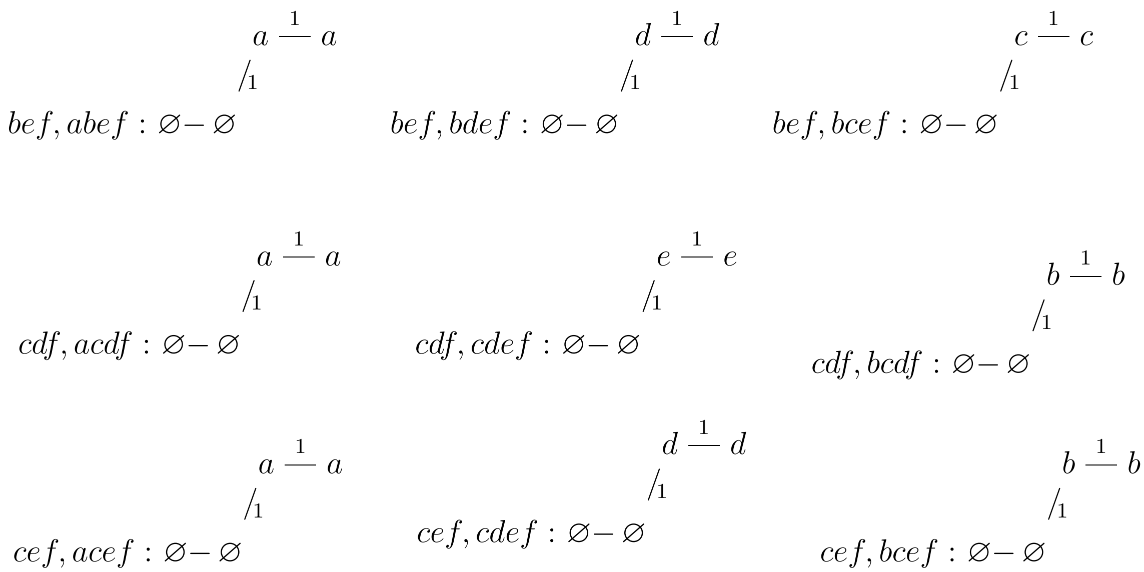


Figure 7.12: More chain-link fences contributing to the maps $F_0 \leftarrow F_1$

$abf, bcf\}$. Applying the differential D to the homology element $ac+ad+bd+be+ce$ gives the sum of $a+b$ in $K^{abde}I$, $e+f$ in $K^{adef}I$, $a+e$ in $K^{abef}I$, $d+f$ in $K^{bdef}I$, $a+e$ in $K^{acef}I$, $c+d$ in $K^{acdf}I$, and $b+c$ in $K^{bcef}I$.

Using this computation, the choice of bases for homology vector spaces, and the computation of the chain-link fences given earlier gives the resolution in Figure 7.13.

$$\widetilde{H}_{-1}K^{abc}I \otimes \langle abc\rangle \oplus \widetilde{H}_{-1}K^{ace} \otimes \langle ace\rangle \oplus \widetilde{H}_{-1}K^{abf}I \otimes \langle abf\rangle \oplus \widetilde{H}_{-1}K^{ade}I \otimes \langle ade\rangle \oplus \widetilde{H}_{-1}K^{adf}I \otimes \langle adf\rangle \oplus \widetilde{H}_{-1}K^{bcd}I \otimes \langle bcd\rangle \oplus \widetilde{H}_{-1}K^{bde}I \otimes \langle bde\rangle \oplus \widetilde{H}_{-1}K^{bef}I \otimes \langle bef\rangle \oplus \widetilde{H}_{-1}K^{cdf}I \otimes \langle cdf\rangle \oplus \widetilde{H}_{-1}K^{cef}I \otimes \langle cef\rangle$$

$\longleftarrow$

	$abcd$	$abce$	$abcf$	$abdf$	$acdf$	$acef$	$adef$	$cdef$	$abef$	$bdef$	$abde$	$bcdf$	$acde$	$bcde$	$bcef$
abc	1	1	1	0	0	0	0	0	0	0	0	0	0	0	0
ace	0	1	0	0	0	1	0	0	0	0	0	0	1	0	0
abf	0	0	1	1	0	0	0	0	1	0	0	0	0	0	0
ade	0	0	0	0	0	0	1	0	0	0	1	0	1	0	0
adf	0	0	0	1	1	0	1	0	0	0	0	0	0	0	0
bcd	1	0	0	0	0	0	0	0	0	0	0	1	0	1	0
bde	0	0	0	0	0	0	0	0	0	1	1	0	0	1	0
bef	0	0	0	0	0	0	0	0	1	1	0	0	0	0	1
cdf	0	0	0	0	1	0	0	1	0	0	0	1	0	0	0
cef	0	0	0	0	0	1	0	1	0	0	0	0	0	0	1

$$\widetilde{H}_0K^{abcd}\otimes\langle abcd\rangle \oplus \widetilde{H}_0K^{abce}\otimes\langle abce\rangle \oplus \widetilde{H}_0K^{abcf}\otimes\langle abcf\rangle \oplus \widetilde{H}_0K^{abdf}\otimes\langle abdf\rangle \oplus \widetilde{H}_0K^{acdf}\otimes\langle acdf\rangle \oplus \widetilde{H}_0K^{acef}\otimes\langle acef\rangle \oplus \widetilde{H}_0K^{adef}\otimes\langle adef\rangle \oplus \widetilde{H}_0K^{cdef}\otimes\langle cdef\rangle \oplus \widetilde{H}_0K^{abef}\otimes\langle abef\rangle \oplus \widetilde{H}_0K^{bdef}\otimes\langle bdef\rangle \oplus \widetilde{H}_0K^{abde}\otimes\langle abde\rangle \oplus \widetilde{H}_0K^{bcdf}\otimes\langle bcdf\rangle \oplus \widetilde{H}_0K^{acde}\otimes\langle acde\rangle \oplus \widetilde{H}_0K^{bcde}\otimes\langle bcde\rangle \oplus \widetilde{H}_0K^{bcef}\otimes\langle bcef\rangle$$

Figure 7.13: Monomial matrix for $F_0 \leftarrow F_1$ in the sylvan resolution of I_Δ in characteristic 2

$$
\begin{array}{c}
\widetilde{H}_0 K^{abcd} \otimes \langle abcd \rangle \\ \oplus \\
\widetilde{H}_0 K^{abce} \otimes \langle abce \rangle \\ \oplus \\
\widetilde{H}_0 K^{abcf} \otimes \langle abcf \rangle \\ \oplus \\
\widetilde{H}_0 K^{abdf} \otimes \langle abdf \rangle \\ \oplus \\
\widetilde{H}_0 K^{acdf} \otimes \langle acdf \rangle \\ \oplus \\
\widetilde{H}_0 K^{acef} \otimes \langle acef \rangle \\ \oplus \\
\widetilde{H}_0 K^{adef} \otimes \langle adef \rangle \\ \oplus \\
\widetilde{H}_0 K^{cdef} \otimes \langle cdef \rangle \\ \oplus \\
\widetilde{H}_0 K^{abef} \otimes \langle abef \rangle \\ \oplus \\
\widetilde{H}_0 K^{bdef} \otimes \langle bdef \rangle \\ \oplus \\
\widetilde{H}_0 K^{abde} \otimes \langle abde \rangle \\ \oplus \\
\widetilde{H}_0 K^{bcdf} \otimes \langle bcdf \rangle \\ \oplus \\
\widetilde{H}_0 K^{acde} \otimes \langle acde \rangle \\ \oplus \\
\widetilde{H}_0 K^{bcde} \otimes \langle bcde \rangle \\ \oplus \\
\widetilde{H}_0 K^{bcef} \otimes \langle bcef \rangle
\end{array}
\xleftarrow{\quad
\begin{array}{c|ccccccc}
 & abcdef & abcde & abcdf & abcef & abdef & acdef & bcdef \\ \hline
abcd & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\
abce & 0 & 1 & 0 & 1 & 0 & 0 & 0 \\
abcf & 0 & 0 & 1 & 1 & 0 & 0 & 0 \\
abdf & 0 & 0 & 1 & 0 & 1 & 0 & 0 \\
acdf & 1 & 0 & 1 & 0 & 0 & 1 & 0 \\
acef & 1 & 0 & 0 & 1 & 0 & 1 & 0 \\
adef & 1 & 0 & 0 & 0 & 1 & 1 & 0 \\
cdef & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\
abef & 1 & 0 & 0 & 1 & 1 & 0 & 0 \\
bdef & 1 & 0 & 0 & 0 & 1 & 0 & 1 \\
abde & 1 & 1 & 0 & 0 & 1 & 0 & 0 \\
bcdf & 0 & 0 & 1 & 0 & 0 & 0 & 1 \\
acde & 0 & 1 & 0 & 0 & 0 & 1 & 0 \\
bcde & 0 & 1 & 0 & 0 & 0 & 0 & 1 \\
bcef & 1 & 0 & 0 & 1 & 0 & 0 & 1
\end{array}
\quad}
\begin{array}{c}
\widetilde{H}_1 K^{abcdef} \otimes \langle abcdef \rangle \\ \oplus \\
\widetilde{H}_1 K^{abcde} \otimes \langle abcde \rangle \\ \oplus \\
\widetilde{H}_1 K^{abcdf} \otimes \langle abcdf \rangle \\ \oplus \\
\widetilde{H}_1 K^{abcef} \otimes \langle abcef \rangle \\ \oplus \\
\widetilde{H}_1 K^{abdef} \otimes \langle abdef \rangle \\ \oplus \\
\widetilde{H}_1 K^{acdef} \otimes \langle acdef \rangle \\ \oplus \\
\widetilde{H}_1 K^{bcdef} \otimes \langle bcdef \rangle
\end{array}
$$

Figure 7.14: Monomial matrix for $F_1 \leftarrow F_2$ in the sylvan resolution of I_Δ in characteristic 2

$$
\begin{array}{c}
\widetilde{H}_1 K^{abcdef} \otimes \langle abcdef \rangle \\
\oplus \\
\widetilde{H}_1 K^{abcde} \otimes \langle abcde \rangle \\
\oplus \\
\widetilde{H}_1 K^{abcdf} \otimes \langle abcdf \rangle \\
\oplus \\
\widetilde{H}_1 K^{abcef} \otimes \langle abcef \rangle \\
\oplus \\
\widetilde{H}_1 K^{abdef} \otimes \langle abdef \rangle \\
\oplus \\
\widetilde{H}_1 K^{acdef} \otimes \langle acdef \rangle \\
\oplus \\
\widetilde{H}_1 K^{bcdef} \otimes \langle bcdef \rangle
\end{array}
\xleftarrow{\begin{array}{r} \\ abcdef \\ abcde \\ abcdf \\ abcef \\ abdef \\ acdef \\ bcdef \end{array}\begin{array}{c} abcdef \\ \left[\begin{array}{c} 0 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{array}\right] \end{array}}
\widetilde{H}_2 K^{abcdef} \otimes \langle abcdef \rangle
$$

Figure 7.15: Monomial matrix for $F_2 \leftarrow F_3$ in the sylvan resolution of I_Δ in characteristic 2

Chapter 8

Conclusions

The canonical sylvan resolution solves Kaplansky's problem over a field of characteristic 0 and all but finitely many positive characteristics. Noncanonical sylvan resolutions give a minimal free resolution over a field of any characteristic, but they require choices of hedges at each lattice point.

There are many open questions pertaining to sylvan resolutions. One problem that was partially solved in this dissertation is to determine which existing resolutions are sylvan. Section 7.1 showed that planar graph resolutions are sylvan. Other good candidates are Scarf complexes for generic ideals and Eliahou–Kervaire resolutions of stable monomial ideals.

As mentioned in Section 2.2.7, it is an open problem to give an explicit formula for the differential in the minimal Taylor resolution when using the Moore–Penrose pseudoinverse as the splitting. The combinatorics of Yuzvinsky's construction [Yuz99] will be different from those of the canonical sylvan resolution, because this construction depends only on the poset given by least common multiples of the generators and not on its embedding in $\mathbb{N}^n$. Understanding the combinatorics of this resolution may provide a way to translate between these two fundamental resolutions and between the combinatorics of Taylor- and Koszul-type.

Another problem is to construct minimal resolutions of toric rings using a construction similar to the one for sylvan resolutions. The construction of the hull resolution and the Scarf complex for monomial ideals has been extended to resolve lattice modules [BS98]. Similarly, the construction of the sylvan resolution should extend to give a construction for a minimal resolution of an arbitrary toric ring. The combinatorial description will change as a result of quotienting by the lattice involved.

Formulas for the Moore–Penrose pseudoinverse may be helpful in solving open problems related to the combinatorial Laplacian. Many formulas for the Laplacian involve expressions that appear in the formula for the pseudoinverse.

Finally, a canonical minimal free resolution for monomial ideals should be helpful for constructing a moduli space of these resolutions, essentially describing the set of all monomial ideals, identifying any pair of ideals when their minimal resolutions are essentially the same.

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Biography

In May 2015, Erika Ordog received her Bachelor of Science in Mathematics from Pepperdine University in Malibu, California. While at Pepperdine, she did undergraduate research advised by Kendra Killpatrick and David Strong. She also received the award for Outstanding Mathematics Student of the graduating class.

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