

Monopoles on AC 3-manifolds

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Abstract

We construct monopoles on asymptotically conical (AC) 3-manifold X with vanishing second Betti number $b^2(X)$. For sufficiently large mass, our construction covers an open set in the moduli space of monopoles.

We also give a more general construction of Dirac monopoles in any AC manifold, which may be useful for generalizing our result to the case when $b^2(X) \neq 0$.

Let (X^3, g) be a 3 dimensional Riemannian manifold, $P \rightarrow X$ an $SU(2)$ -bundle and \mathfrak{g}_P the associated adjoint bundle, equipped with an Ad -invariant metric. A pair (A, Φ) where A is a connection on P and $\Phi \in \Omega^0(X, \mathfrak{g}_P)$ called the Higgs field is said to be a monopole if they satisfy

$$(0.1) \quad *F_A - \nabla_A \Phi = 0.$$

In this paper we shall focus on asymptotically conical (AC) 3 manifolds (X, g) . These are noncompact and it is helpful to think of them as the interior of a manifold \bar{X} with boundary $\Sigma = \partial\bar{X}$. Then, along Σ the metric g is modeled on a cone metric on $\mathbb{R}^+ \times \Sigma$, see definition 2 for the details. Denote by Σ_i , for $i \in \{1, \dots, l\}$ the connected components of Σ . We shall refer to the i -th end of X as the one modeled on $\mathbb{R}^+ \times \Sigma_i$, and let ρ denote a radial coordinate which at each end is identified with the radial distance along the cone. Suppose that along the end, the connection A converges uniformly to some connection A_∞ pulled back from a bundle over Σ . In this situation, a monopole (A, Φ) is said to have finite mass if at each end, i say, there is $m_i \in \mathbb{R}^+$ with $|\Phi|$ converging uniformly in Σ_i to m_i . Under such conditions one can prove [6], [10]

$$(0.2) \quad |\Phi| = m_i - \frac{k_i^O}{2\rho} + O(\rho^{-2}),$$

for some $m_i \in \mathbb{R}$ and $k_i^O \in \mathbb{Z}$ (in fact more is known regarding the asymptotic behavior of finite mass monopoles and we refer the reader to previous references). We shall call the tuples $\vec{k}^O = (k_1^O, \dots, k_l^O)$ and $\vec{m} = (m_1, \dots, m_l) \in \mathbb{R}^l$ the charge and the mass of the monopole (A, Φ) respectively. The integer $k = k_1^O + \dots + k_l^O$ will be called the total charge.

Remark 1. *In fact one can prove, [10] that under the assumption of finite mass the Higgs field converges to a parallel section of \mathfrak{g}_P at each end. Moreover, if it does not vanish, i.e. $m_i \neq 0$, then the connection A_∞ , to which A converges at the ends, is reducible and Yang-Mills. Having this in mind we remark that each of the k_i^O is the degree of the complex line bundles over the Σ_i 's associated with each of such reductions.*

Definition 1. *Denote the space of smooth pairs (A, Φ) with total charge $k \in \mathbb{Z}$ and mass $\vec{m} \in \mathbb{Z}^l$ by $C_{k, \vec{m}}(X, g)$, and by $M_{k, \vec{m}}$ those $(A, \Phi) \in C_{k, \vec{m}}$ solving the monopole equation 0.1.*

Let \mathcal{G} be the space of gauge transformations $g \in \text{Aut}(P)$. Equation 0.1 is invariant under the \mathcal{G} -action and we define the moduli space of total charge k and mass m monopoles to be

$$\mathcal{M}_{k,m}(X, g) = M_{k,m}(X, g)/\mathcal{G}.$$

The nomenclature of definition 1 is in agreement with that of [7], and should be thought of as moduli space of unframed monopoles. The main result of this paper is

Theorem 1. *Let $k \in \mathbb{Z}$ and (X, g) be an asymptotically conical 3-manifold with $b^2(X) = 0$ (notice this implies $l = 1$). Then, there is $\mu \in \mathbb{R}$, such that if $m \geq \mu$ and $X^k(m) \subset X^k$ denotes the open set defined by*

$$X^k(m) = \left\{ (p_1, \dots, p_k) \in X^k \mid \text{dist}(p_i, p_j) > 4m^{-\frac{1}{2}}, \text{ for } i \neq j \right\},$$

while $\check{\mathbb{T}}^{k-1} = \{(e^{i\theta_1}, \dots, e^{i\theta_k}) \in \mathbb{T}^k \mid e^{i(\theta_1 + \dots + \theta_k)} = 1\}$, there is a local diffeomorphism

$$(0.3) \quad h : X^k(m) \times H^1(X, \mathbb{S}^1) \times \check{\mathbb{T}}^{k-1} \rightarrow \mathcal{M}_{k,m}.$$

Remark 2. 1. *We may interpret the monopoles in the image of the map h as being formed by gluing k separated monopoles. Then one may think of the parameters in $X^k(m) \times H^1(X, \mathbb{S}^1) \times \check{\mathbb{T}}^{k-1}$ used in their construction as follows: The points $(p_1, \dots, p_k) \in X^k(m)$ denote the location of the k monopoles, $\check{\mathbb{T}}^{k-1}$ denotes local phases assigned to each of these and $H^1(X, \mathbb{S}^1)$ a possible twist with a reducible flat connection, i.e. a usual vector potential whose strength (magnetic field) vanishes inside the superconductor filling in X .*

2. *In [7], Kottke computes the virtual dimension of the moduli space of monopoles with fixed mass on an AC 3-manifold and obtains the formula $\dim(\mathcal{M}_{m,k}) = 4k + \frac{1}{2}b^1(\Sigma) - b^0(\Sigma)$. The long exact sequence on cohomology $H_c(X) \rightarrow H(X) \rightarrow H(\Sigma)$ and the duality $H_c^* \cong (H^{3-*})^*$, can be used to rewrite Kottke's formula as*

$$\dim(\mathcal{M}_{m,k}) = 4k + b^1(X) - b^2(X) - 1,$$

and the construction described here gives a good geometric interpretation of all these parameters in an open set in $\mathcal{M}_{m,k}$. Even though it will be evident that the elements of $b^2(X)$ obstruct the gluing construction, the author believes the analysis here can be changed to take these into account and extend theorem 1 to the case of $b^2(X) \neq 0$.

We shall now give a short outline of the paper and of the proof of the results above. The first result of the paper is a construction of finite mass Dirac monopoles, as in definition 5, on AC manifolds. This requires definition 7 which constructs $X^k(m) \subset X^k$ as the intersection of a $3k - b^2(X)$ dimensional submanifold with a big open set in X^k . Then, for each element in $X^k(m) \times H^1(X, \mathbb{S}^1)$ theorem 2 constructs a Dirac monopole. Later in section section 2.1, proposition 4, these Dirac Monopoles are smoothed out around the singular points $p_i \in X$. To smooth each singular point requires an element of \mathbb{S}^1 and this construction gives a map

$$(0.4) \quad H : X^k(m) \times H^1(X, \mathbb{S}^1) \times \check{\mathbb{T}}^{k-1} \rightarrow C_{k,m}(X, g),$$

whose image consists of approximate solutions to the Bogomolny equations. Still in in section 2.1 we estimate the error term $e_0 = e(A_0, \Phi_0)$ of a configuration in the image of the map H . Section 3 constructs a general setup to solve the monopole equation by deforming an approximate solution (A_0, Φ_0) in the image of the map

H. More precisely, section 3.1 constructs function spaces depending on the approximate solution (A_0, Φ_0) . These are specially adapted to uniformly invert the linearised equation, make the error term e_0 small, and handle the nonlinearities. Then, in section 3.2 the linearised equation is shown to admit a right inverse with uniformly bounded norm on the previously defined function spaces. Finally, section 3.3 solves the monopole equation, see proposition 6, by using a special version of the contraction mapping principle.

To the author's knowledge the first reference in the literature investigating monopoles on a large class of 3-manifolds is Braam's paper [1], which explores monopoles in asymptotically hyperbolic manifolds. Monopoles on asymptotically Euclidean 3-manifolds have been investigated by Floer [3] and Ernst [2]. The main motivation for all these is the possibility of using the moduli space of monopoles on noncompact 3-manifolds to attack problems in 3-dimensional topology. This tries to imitate the way instantons have shed light on 4 dimensional topology and this possibility have remained unexplored. In this direction here we have extended Floer's and Ernst to the more general class of asymptotically conical manifolds where we are able to describe an open set of the moduli space.

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1 Preliminary Remarks

1.1 Linearized Operator

Let (∇_A, Φ) be a connection and Higgs field, not necessarily satisfying the Bogomolny equations. For such a configuration the quantity $e_0 = *F_A - \nabla_A \Phi$ may be nonzero. The linearized Bogomolny equation fits into a sequence

$$(1.1) \quad \Omega^0(\mathfrak{su}(P)) \xrightarrow{d_1} \Omega^1(\mathfrak{su}(P)) \oplus \Omega^0(\mathfrak{su}(P)) \xrightarrow{d_2} \Omega^1(\mathfrak{su}(P)),$$

with $d_1 \xi = (-\nabla_A \xi, -[\Phi, \xi])$ and

$$(1.2) \quad d_2(a, \phi) = *d_A a - \nabla_A \phi - [a, \Phi].$$

Their formal adjoints are given by $d_1^*(a, \phi) = -\nabla_A^* a + [\Phi, \phi]$ and $d_2^* a = (*d_A a + [a, \Phi], -\nabla_A^* a)$. If (A, Φ) is a monopole then the sequence in 1.1 is actually an elliptic complex and so the operator $D = d_2 \oplus d_1^*$ acting on sections of $(\Lambda^1 \oplus \Lambda^0)(\mathfrak{su}(P))$ is elliptic. Its formal adjoint is $D^* = d_2^* \oplus d_1$ and these can be written as

$$D = \begin{pmatrix} *d_A & -\nabla_A \\ -d_A^* & 0 \end{pmatrix} + [\Phi, \cdot], \quad D^* = D - 2[\Phi, \cdot].$$

Remark 3. Fix a monopole $(A, \Phi) \in C_{k, \bar{m}}$, for the topology being considered in the moduli space of monopoles (definition 1), we shall take \mathcal{G} to consist of those $g \in \text{Aut}(P)$ with $g^{-1} \nabla_A g$ in the Sobolev space $H_{1, -1/2}^2$ as in definition 9.

The moduli space of framed monopoles, as defined in that reference, can be obtained by taking the quotient of $M_{k, \bar{m}}(X, g)$ by the action of those gauge transformations such that $g^{-1} \nabla_A g \in H_{1, \beta}^2$ for $\beta \leq -1$. We further remark that the weight α here corresponds to the weight $-\gamma - 3/2$ in reference [7].

We now state some standard Weitzenböck formulas which will be useful later. The proofs of such are straightforward computations.

Lemma 1. (Standard Weitzenböck) Let ∇_A be a connection and $u \in \Omega^1(\mathfrak{su}(2)) \oplus \Omega^0(\mathfrak{su}(2))$, then

$$(1.3) \quad \Delta_A u = \nabla_A^* \nabla_A u + F^W(u) + Ric^W(u).$$

Where $F^W(a, \phi) = (*[*F_A \wedge a], 0)$ and $Ric^W(a, \phi) = (Ric(a), 0)$.

Lemma 2. (Monopole Weitzenböck) Let (∇_A, Φ) be a connection and an Higgs Field. Let $u \in \Omega^1(\mathfrak{su}(2)) \oplus \Omega^0(\mathfrak{su}(2))$, then

$$(1.4) \quad DD^*u = \nabla_A^* \nabla_A u - [[u, \Phi], \Phi] + Ric^W(u) + \epsilon_0^W(u)$$

$$(1.5) \quad D^*Du = DD^*u + 2(\nabla_A \Phi)^W(u).$$

Where $b^W(a, \phi) = (*[a \wedge b] - [b, \phi], \langle b, a \rangle)$ and b is either $\epsilon_0 = *F_A - \nabla_A \Phi$, Ric or $(\epsilon_0 + 2d_A \Phi)$.

If $(A, \Phi) = (A_0 + a, \Phi_0 + \phi)$ for suitable $u = (a, \phi) \in \Omega^1(\mathfrak{su}(P)) \oplus \Omega^0(\mathfrak{su}(P))$ is a monopole, then

$$(1.6) \quad \epsilon_0 + D(u) + Q(u, u) = 0,$$

where the operator D is as above and $Q(u, u) = \begin{pmatrix} *[a \wedge a] - [a, \phi] \\ 0 \end{pmatrix}$, i.e. the nonlinear terms appearing here in the gauge-fixed monopole equation are zero order and quadratic.

We turn now to one other very important property of monopoles. This is the scale invariance of the Bogomolny equation, inherited from the conformal invariance of the ASD equations in 4 dimensions. The precise result is

Proposition 1. Let (∇_A, Φ) be a monopole on (M^3, g) , where M^3 is a Riemannian 3 manifold. Then $(\nabla_A, \delta^{-1}\Phi)$ is a monopole for $(M^3, \tilde{g} = \delta^2 g)$.

Proof. In general, if ω is a k form and $\tilde{*}$ the Hodge operator for the metric \tilde{g} , then $\tilde{*}\omega = \delta^{n-2k} * \omega$ ($n = 3$). This implies that $\tilde{*}F_A = \delta^{-1} * F_A = \delta^{-1} \nabla_A \Phi$, and the result follows. \square

The Yang-Mills-Higgs (YMH) Energy is defined on a precompact set $U \subset X$ as

$$(1.7) \quad E_U = \frac{1}{2} \int_U |\nabla_A \Phi|^2 + |F_A|^2.$$

The Euler Lagrange equations are $d_A^* F_A = [d_A \Phi, \Phi]$, $\Delta_{d_A} \Phi = 0$. It is well known that monopoles not only solve these, but also minimize the YMH energy on X . The energy over a precompact set U with smooth boundary is given by the flux $\int_{\partial U} \langle \Phi, F_A \rangle$. On an AC manifold finite mass monopoles have finite energy given by $E_U = 4\pi \sum_{i=1}^k m_i k_i^O$, see [10] for a proof.

1.2 Analytical Preliminaries on AC Manifolds

Definition 2. A 3 dimensional Riemannian manifold (X^n, g) is called asymptotically conical (AC) with rate $\nu < 0$, if there is a compact set $K \subset X$, a Riemann surface (Σ^2, g_Σ) and a diffeomorphism

$$\varphi : (1, \infty) \times \Sigma \rightarrow X \setminus K,$$

such that the metric $g_C = dr^2 + r^2 g_\Sigma$ on $(1, \infty) \times \Sigma$ satisfies $|\nabla^j(\varphi^*g - g_C)|_C = O(r^{\nu-j})$, for all $j \in \mathbb{N}_0$. Here ∇ is the Levi Civita connection of g_C . A radius function will be any positive function $\rho : X \rightarrow \mathbb{R}_+$, such that in $X \setminus K$, $\rho = r \circ \varphi^{-1}$.

In particular Σ may be not be connected. In that case we can write Σ as the disjoint union of its connected components, $\Sigma = \cup_{i=1}^l \Sigma_i$. Then we shall refer to $\varphi((1, +\infty) \times \Sigma_i)$ as the i -th end of X .

The rest of this section contains a brief discussion of the Lockhart-McOwen conically weighted spaces, see chapter 4 in [9] for more details. For $p > 0$, $n \in \mathbb{N}$ and $\beta \in \mathbb{R}$ define the norms

$$\|f\|_{L_{n,\beta}^p}^p = \sum_{i \leq n} \int_X |\rho^{-\beta-i} \nabla^i f|^p \rho^{-3} d\text{vol}.$$

Definition 3. Let $L_{n,\beta}^p$ denote the Banach space completion of the smooth compactly supported sections in the norm $\|\cdot\|_{L_{n,\beta}^p}$ defined above.

Later in section 3.1 we shall define Hilbert spaces $H_{n,\beta}$ such that $d_2 : H_{1,\beta} \rightarrow H_{0,\beta-1}$ is surjective and admits a bounded right inverse for some $\beta \in \mathbb{R}$. To prove that this is the case we shall need to analyze the operator

$$(1.8) \quad d + d^* : L_{1,\beta}^2 \rightarrow L_{0,\beta-1}^2.$$

Lemma 3. Let (X^3, g) be AC and denote by g_Σ the metric on the link Σ at the conical end of X , then the following statements hold.

1. There is a discrete set

$$D(d + d^*) = \{\beta \in \mathbb{R} \mid (\beta + 1)(3 - 1 + \beta) \in \text{Spec}(\Sigma)_0\},$$

such that if $\beta \in \mathbb{R} \setminus D(d + d^*)$ the operator 1.8 on 1-forms is Fredholm. Here $\text{Spec}(\Sigma)_0$ denotes the spectrum of the Laplacian Δ_{g_Σ} on functions.

2. For $\beta \leq -\frac{1}{2}$, $\ker(d + d^*)_\beta = \ker(\Delta)_\beta$ and for 1-forms, there is an isomorphism $\ker(\Delta)_{-\frac{3}{2}} \cong H_c^1(X, \mathbb{R}) \cong H^2(X, \mathbb{R})^*$.

Proof. For the first item see from [9], chapter 6. The weights appearing in $D(d + d^*)$ correspond to the rates of the homogeneous closed and coclosed 1-forms on the metric cone $(\mathbb{R}^+ \times \Sigma, g_C = dr^2 + r^2 g_\Sigma)$. The second item follows from the fact that $L_{0,-3/2}^2 = L^2$ and the statement that there is an isomorphism between the space of L^2 harmonic 1-forms and the compactly supported cohomology, see [8]. \square

Corollary 1. Let (X^3, g) be AC with $b^2(X) = 0$, then there is a constant $c > 0$ such that for $\alpha < -1$ and all $u \in L_{1,\alpha}^2$

$$\|(d + d^*)u\|_{L_{0,\alpha-1}^2} \geq c \|u\|_{L_{1,\alpha}^2}$$

Proof. It follows from the fact that $d + d^*$ is Fredholm for those $\alpha \notin D(d + d^*)$, that $L_{0,\alpha-1}^2 = (d + d^*)(L_{1,\alpha}^2) \oplus W_{\alpha-1}$, with $W_{\alpha-1} \cong \ker(d + d^*)_{-2-\alpha}$. Since $d + d^*$ has closed image for $\alpha \notin D(d + d^*)$, there is $c > 0$ such that

$$\|(d + d^*)u\|_{L_{0,\alpha-1}^2} \geq c\|u\|_{L_{1,\alpha}^2},$$

for all $u \in \ker(d + d^*)_{\alpha}^{\perp}$. Moreover, for $\alpha \leq -\frac{3}{2} + 1$, we can integrate by parts and so $\ker(\Delta)_{\alpha} = \ker(d + d^*)_{\alpha}$. This together with the second item in lemma 3 gives $\ker(d + d^*)_{\alpha} \subset \ker(\Delta)_{-\frac{3}{2}} \cong H^2(X, \mathbb{R})^*$, for all $\alpha \leq -\frac{3}{2}$. Then, the assumption that $b^2(X) = 0$ finally gives $\ker(d + d^*)_{\alpha} = 0$ for all $\alpha \leq -\frac{3}{2}$. However, as the Laplacian Δ_{Σ} has no negative eigenvalues, the first item in lemma 3 gives that there are no critical rates α in the interval $(-2, -1)$. Hence, one can increase α up until (but excluding) -1 . \square

The following two results will also be used later during the construction of monopoles on AC 3 manifolds and it is convenient to have them stated now.

Lemma 4. *Let ∇_A be a metric compatible connection on an Hermitian vector bundle E over an AC manifold (X^3, g) . Then, for all $\alpha \in [1, 3]$, there is a constant $c_K(\alpha) > 0$, such that*

$$\left(\int_X |\rho^{\frac{1}{2}} u|^{2\alpha} \frac{d\text{vol}_g}{\rho^3} \right)^{\frac{1}{2\alpha}} \leq c_K(\alpha) \left(\int_X |\nabla_A u|^2 \right)^{\frac{1}{2}}$$

for all smooth and compactly supported section u . In particular for $\alpha = 3, 1$ one has respectively $\|u\|_{L_6}^2 \leq c_K \|\nabla_A u\|_{L_2}^2$ and $\|\rho^{-1} u\|_{L_2}^2 \leq c_K \|\nabla_A u\|_{L_2}^2$.

Proof. Kato's inequality $|\nabla|u|| \leq |\nabla_A u|$, holds pointwise for all irreducible Hermitian connections. Then, the proof follows from combining this with corollary 1.3 in [4]. \square

Lemma 5. *In the conditions of lemma 4. Let u be a section such that $\nabla_A u \in L^2$, then there is a covariant constant limit $u|_{\Sigma} \in \Gamma(\Sigma, E|_{\Sigma})$. Moreover, on the cone $C(\Sigma_i)$ over each end there is an inequality*

$$\| |u| - u_{\Sigma_i} \|_{L_{0,-\frac{1}{2}}^{2\alpha}} \leq \|\nabla_A u\|_{L^2}.$$

Proof. This lemma is a particular case of propositions A.0.16 and A.0.17 in the Appendix A to [10]. \square

1.3 Monopoles on (\mathbb{R}^3, g_E)

In this short section we shall be interested in monopoles on \mathbb{R}^3 equipped with the Euclidean metric g_E . In the construction of the approximate solution in proposition 4 it will be important to scale these monopoles. Given $\lambda \in \mathbb{R}^+$, a monopole on (\mathbb{R}^3, g_E) , then by proposition 1 $(A, \lambda\Phi)$ is a monopole on $(\mathbb{R}^3, \lambda^{-2}g_E)$. Using the scaling map $\exp_{\lambda}(x) = \lambda x$ on \mathbb{R}^3 we define

$$(1.9) \quad \Phi_{\lambda} = \lambda \exp_{\lambda}^* \Phi, \quad A_{\lambda} = \exp_{\lambda}^* A, \quad (g_E)_{\lambda} = \lambda^{-2} \exp_{\lambda}^* g_E.$$

Since the Euclidean metric is invariant under scaling $(g_E)_{\lambda} = g_E$, and $(A_{\lambda}, \Phi_{\lambda})$ is a monopole for the Euclidean metric. However, the monopole $(A_{\lambda}, \Phi_{\lambda})$ no longer has mass m but mass λm .

Let $\underline{\mathbb{C}}^2$ denote the trivial rank 2 complex vector bundle, then an isomorphism $\eta : \underline{\mathbb{C}}^2|_{\mathbb{R}^3 \setminus \{0\}} \rightarrow H^k \oplus H^{-k}$ is called a framing. We shall fix a framing η which identifies the limiting Higgs Field and connection with the pullbacks via η of those determined by the unique $SU(2)$ invariant configurations on the Hopf bundle H . These framings are unique up to a factor of $\mathbb{S}^1/\mathbb{Z}_k$, where \mathbb{S}^1 denotes the automorphism group of H equipped

with its unique $SU(2)$ invariant connection.

We shall denote by $\mathcal{N}_{k,m}$ the moduli space of charge $k \in \mathbb{Z}$ and mass $m \in \mathbb{R}^+$ monopoles on \mathbb{R}^3 , together with a framing as above.

Example 1. Fixing a framing gives a model for the Hopf bundle H , this together with the map that to a charge one monopole assigns the zero of the Higgs field, gives a map $\mathcal{N}_{1,1} \cong \mathbb{R}^3 \times \mathbb{S}^1$. Moreover, \mathbb{R}^3 acts on monopoles by translations and we shall denote by

$$\dot{\mathcal{N}}_{1,1} \cong \mathbb{S}^1,$$

the quotient of $\mathcal{N}_{1,1}$ by this action.

Definition 4. There is a unique spherically symmetric monopole with mass 1, [11], which we shall call the BPS monopole and denote it by (A^{BPS}, Φ^{BPS}) . Moreover, one remarks that there is $R > 0$ such that $|\Phi^{BPS}| > 1/2$ outside the ball of radius R in \mathbb{R}^3 .

Then, any element of $\dot{\mathcal{N}}_{1,1}$ is a framing $\eta : \mathbb{C}^2|_{\mathbb{R}^3 \setminus \{0\}} \rightarrow H \oplus H^{-1}$ which at infinity identifies the connection A^{BPS} with the direct sum of the unique $SU(2)$ invariant connection on H . Hence, fixing a random framing η_0 any other $\eta \in \dot{\mathcal{N}}_{1,1}$ is such that $\eta \circ \eta_0^{-1}$ is multiplication by a constant function with values in \mathbb{S}^1 . This gives an isomorphism $\dot{\mathcal{N}}_{1,1} \cong \mathbb{S}^1$ and from now on we shall think of $\dot{\mathcal{N}}_{1,1}$ as being \mathbb{S}^1 .

Example 2. We think of $\mathbb{S}^2 \cong SU(2)/U(1)$ via the Hopf fibration and define on \mathbb{S}^2 the $SU(2)$ -bundle

$$P = SU(2) \times_{\lambda} SU(2),$$

where $\lambda : U(1) \rightarrow SU(2)$ is given by $\lambda(e^{i\alpha}) = \text{diag}(e^{i\alpha}, e^{-i\alpha})$. Now, let T_1, T_2, T_3 be a basis of $\mathfrak{su}(2)$, with $[T_i, T_j] = 2\epsilon_{ijk}T_k$, and $\omega_1, \omega_2, \omega_3$ be its dual left invariant coframe. The unique $SU(2)$ -invariant connection on the Hopf bundle induces a connection on P whose connection 1-form is $\omega^1 \otimes T_1 \in \Omega^1(SU(2), \mathfrak{su}(2))$. We now pull this back to $\mathbb{R}^3 \setminus \{0\}$ via the framing η_0 , and denote it by A^D . Let $r : \mathbb{R}^3 \rightarrow \mathbb{R}_0^+$ be the radial distance to the origin, i.e $r(x) = |x|_{g_E}$, then one can check that (A_m^D, Φ_m^D) with

$$A_m^D = A^D, \quad \Phi_m^D = \left(\frac{1}{2r} - m \right) \otimes T_1,$$

is a monopole. In fact, the monopole above is the monopole on the $SU(2)$ bundle induced from an S^1 -monopole known as the standard mass m Dirac monopole on \mathbb{R}^3 .

Abusing notation and omitting pullbacks via η_0 , we now write the BPS monopole from definition 4. Scaling so it has mass m as in equation 1.9 it is given by

$$A_m^{BPS} = A^D + \frac{2mr}{\sinh(2mr)}(T_2 \otimes \omega^2 + T_3 \otimes \omega^3), \quad \Phi_m^{BPS} = \left(\frac{1}{2r} - \frac{m}{\tanh(2mr)} \right) \otimes T_1.$$

The following lemma is an important tool for estimating the error term of the approximate solution

Lemma 6. Let (A_m^D, Φ_m^D) and $(A_m^{BPS}, \Phi_m^{BPS})$ be the mass m Dirac and BPS monopole on \mathbb{R}^3 as above. Then, for large r

$$|\Phi_m^{BPS} - \Phi_m^D| = O(me^{-4mr}), \quad |A_m^{BPS} - A_m^D| = O(me^{-2mr}).$$

Proof. Using the explicit formulas in example 2 we compute that for large r

$$\begin{aligned} |\Phi_m^{BPS}(r) - \Phi_m^D(r)| &= m(1 - \coth(2mr))|T_1| \\ &= \frac{me^{-2mr}}{\sinh(2mr)}, \end{aligned}$$

which is of order $O(me^{-4mr})$, for r away from 0. Similarly

$$|A_m^{BPS}(r) - A_m^D(r)| = \frac{2mr}{\sinh(2mr)} |T_2 \otimes \omega^2 + T_3 \otimes \omega^3| = O(me^{-2mr}).$$

□

2 Dirac Monopoles

This section contains the linear analysis necessary for the construction of Dirac monopoles (see definition 5) with a specified mass. The main result comes in the form of theorem 2. We start by defining Dirac monopoles

Definition 5. Let (X^3, g) be a 3-dimensional Riemannian manifold. A pair (A, Φ) is said to be a Dirac monopole on X if there is $k \in \mathbb{N}$ and $(p_1, \dots, p_k) \in X^k$, such that

1. (A, Φ) is a monopole, reducible to an S^1 bundle over $X \setminus \{p_1, \dots, p_k\}$.

2. For each $i \in \{1, \dots, k\}$,

$$\lim_{r_i \rightarrow 0} r_i |\Phi| \neq 0 \text{ and } \lim_{r_i \rightarrow 0} r_i^2 |F_A| \neq 0,$$

where $r_i(\cdot) = \text{dist}(\cdot, p_i)$, denotes the distance to the point p_i .

Remark 4. The second point in the definition above is included in order to require that the Dirac monopoles do have singularities at the points p_1, \dots, p_k . In this way a smooth monopole, such as a flat Abelian connection is not a Dirac monopole.

Back to the setting where (X^3, g) is an AC manifold we shall now construct Dirac monopoles with a specified mass.

Example 3. The standard Dirac monopole on $X = \mathbb{R}^3$, (A_m^D, Φ_m^D) from example 2 is an example of a Dirac monopole. In that case $k = 1$ and $p_1 = 0 \in \mathbb{R}^3$, and recall that

$$\Phi_m^D(x) = m - \frac{1}{2|x|}.$$

We shall now construct finite mass Dirac monopoles on \mathbb{R}^3 in a more systematic way. This will be helpful in illustrating the general construction to come.

As S^1 is Abelian, its adjoint representation is trivial, and so is the adjoint bundle. Therefore, the Higgs field Φ of a Dirac monopole is simply a real valued function. Similarly, the induced connection on the trivial adjoint bundle is the trivial connection and the monopole equation turns into

$$(2.1) \quad F_A = *d\Phi,$$

where the curvature F_A is now simply a closed 2-form. Hence, differentiating the equation above we conclude that Φ is an harmonic function. Moreover, the finite mass requirement implies that Φ must be bounded at infinity. Using separation of variables we find

$$\Phi(x) = m - \frac{k^I}{2|x|},$$

for any $m, k^I \in \mathbb{R}$. Now notice that given Φ one can determine F_A from the monopole equation 2.1. In the case under consideration Φ is defined on $\mathbb{R}^3 \setminus \{0\} \cong \mathbb{R}^+ \times \mathbb{S}^2$ and for $F_A = *d\Phi = k^I d\text{vol}_{\mathbb{S}^2}$ to be the curvature of a connection on a circle bundle over $\mathbb{R}^3 \setminus \{0\}$, the real number k^I must actually be an integer. The standard Dirac monopole on \mathbb{R}^3 from example 2 being the case when $k^I = 1$.

At this point we use the action of \mathbb{R}^3 by translations in order to produce more examples. This gives that for any $p \in \mathbb{R}^3$ and $k^I \in \mathbb{Z}$, $\Phi(x - p) = m - \frac{k^I}{2|x-p|}$.

To finish, we notice that the Abelian monopole equation above is linear and so we can sum solutions in order to produce more solutions. Therefore, for any $(p_1, \dots, p_k) \in (\mathbb{R}^3)^k$ and integers k_i^I we can set

$$\Phi(x) = m - \sum_{i=1}^k \frac{k_i^I}{2|x - p_i|}.$$

The reader may have noticed that for while we have ignored the connection A and focused on Φ . However, Φ determines F_A via equation 2.1, and as the complement of a finite set of points in \mathbb{R}^3 is simply connected, F_A does determine A uniquely up to gauge.

Let $l = b_0(\Sigma)$, then given a total charge $k \in \mathbb{N}$ and mass $m = (m_1, \dots, m_l) \in \mathbb{R}^l \cong H^0(\Sigma, \mathbb{R})$ we shall construct Dirac monopoles with this mass and total charge. The quantity

$$m_0(m) = \frac{1}{l} \sum_{i=1}^{b_0(\Sigma)} m_i,$$

is called the average mass of the monopole.

Definition 6. Let $p = (p_1, \dots, p_k) \in X^k$ and define the current $\delta \in (C_0^\infty(X))^*$, by

$$\delta(f) = \sum_{i=1}^k f(p_i)$$

for a compactly supported $f \in C_0^\infty(X)$.

These points can be repeated and so the current δ has multiplicities giving a vector $\vec{k}^I = (k_1^I, \dots, k_{\#-pts}^I) \in \mathbb{Z}^k$ of charges generating a flux that must then leave X through its ends with some charges $\vec{k}^O \in \mathbb{Z}^l$ with $|\vec{k}^I| = |\vec{k}^O|$. The Laplacian Δ acts $C_0^\infty(X)$ and one can consider its transpose operator, also denoted by Δ acting on $H \in (C_0^\infty)^*$ by $\Delta H(g) = H(\Delta g)$, for all $g \in C_0^\infty(X)$.

Proposition 2. Let $(p_1, \dots, p_k) \in X^k$ and $(m_1, \dots, m_l) \in \mathbb{R}^l$, then there is a current $H^D \in (C_0^\infty(X))^*$ such that $\Delta H^D = \delta$. This can be represented by an integral operator

$$H^D(f) = \int_X f \phi^D,$$

where ϕ^D is an harmonic function on $X \setminus \{p_1, \dots, p_k\}$ and $k^O \in \mathbb{R}^l$ satisfying $|k^O| = |k^I|$ such that

$$(2.2) \quad \phi_D|_{U(p_i)} = -\frac{k_i^I}{2r} + O(r^0)$$

$$(2.3) \quad \phi_D|_{U(\Sigma_i)} = m_i - \frac{k_i^O}{\text{Vol}(\Sigma_i)} \frac{1}{r} + O(r^{-2}).$$

Where $U(p_i)$ and $U(\Sigma_i)$ respectively denote a neighborhood of p_i and the i -th end.

The proof is an exercise in the calculus of variations, which below is hidden by the use of the Riesz representation theorem. Before the proof two lemmas are required. Let V denote the space of functions $f \in L^1_{loc}(X)$, with $\nabla f \in L^2$. Given $g \in L^1_{loc}$, a function $u \in V$ is called a weak solution to $\Delta u = g$ if for all smooth compactly supported ψ

$$\langle du, d\psi \rangle_{L^2} = \int_X g\psi.$$

Recall that if $f : X \rightarrow \mathbb{R}$ is such that $\nabla f \in L^2$, then according to lemma 5 the uniform limit of $f|_{\varphi^{-1}(\{r\} \times \Sigma_i)}$ along any end exists, is constant and will be denoted by $f|_{\Sigma_i}$ on each connected component Σ_i of Σ .

Lemma 7. *Let $g \in L^{\frac{6}{5}}(X)$, then there is a unique weak solution $u_g \in V$ of*

$$\begin{aligned} \Delta u_g &= g \\ u_g|_{\Sigma} &= 0. \end{aligned}$$

Proof. Those $f \in V$ such that $f|_{\Sigma_i}$ vanishes form an Hilbert space H , namely the completion of the smooth compactly supported functions in the inner product $\langle f_1, f_2 \rangle_H = \int_X df_1 \wedge *df_2$, for $f_1, f_2 \in C_0^\infty(X)$. To find a weak solution $u_g \in H$ one proves the linear functional $f \mapsto \int_X gf$ is bounded on H . This follows from

$$\left| \int_X gf \right| \leq \|g\|_{L^{\frac{6}{5}}} \|f\|_{L^6} \leq \|g\|_{L^{\frac{6}{5}}} \|f\|_H,$$

where we used first Hölder's inequality and then the Sobolev inequality from lemma 4. The Riesz representation theorem gives an element $u_g \in H$ such that $\langle u_g, f \rangle_H = \int_X gf$, for all $f \in H$, i.e. u_g is a weak solution to the problem. To prove uniqueness suppose there are two solutions $u, v \in H$. Then $w = u - v$ is an harmonic function in H . Moreover, since $w \in H$ we have $dw \in L^2$ and $w|_{\Sigma} = 0$ and so there is a sequence $R_i \rightarrow \infty$ such that $w|_{r^{-1}(R_i)} = o(R_i^{-1/2})$ and $dw|_{R_i} = o(R_i^{-3/2})$. So one can compute

$$\begin{aligned} 0 &= \int_X w \Delta w = - \int_X (d(w * dw) - dw \wedge *dw) \\ &= \lim_{i \rightarrow \infty} \int_{r^{-1}(R_i)} w * dw + \|w\|_H^2 \\ &= \|w\|_H^2, \end{aligned}$$

and conclude $w \in H$ with $dw = 0$ and so $w = 0$, i.e $u = v$. □

Corollary 2. *Given $m = (m_1, \dots, m_l) \in \mathbb{R}^l$, there is a unique solution $u_m \in V$ of the problem*

$$\begin{aligned} \Delta u_m &= 0 \\ u_m|_{\Sigma_i} &= m_i, \quad i = 1, \dots, l. \end{aligned}$$

Proof. Let $g \in C^\infty(X)$ with $g|_\Sigma = m$ be a smooth extension of m to the whole X , such that $\Delta g \in L^{\frac{6}{5}}$. Then from lemma 7 one concludes that there is a unique solution $\tilde{u} \in H$ to the problem $\Delta \tilde{u} = \Delta g$. Then the solution u_m is given by setting $u_m = g - \tilde{u}$. \square

Corollary 3. *Let $g \in L^{\frac{6}{5}}(X)$ and $m = (m_1, \dots, m_l) \in \mathbb{R}^l$, there is a unique solution $u \in V$ of the problem*

$$\begin{aligned} \Delta u &= g \\ u|_{\Sigma_i} &= m_i, \quad i = 1, \dots, l. \end{aligned}$$

Proof. Let $u_g \in H \subset V$ be the solution to $\Delta u_g = g$, $u_g|_\Sigma = 0$ given by lemma 7 and let $u_m \in V$ be the solution to $\Delta u_m = 0$, $u_m|_\Sigma = m$ given by corollary 2. Then $u = u_g + u_m$ is the desired solution and the uniqueness follows from a similar argument as the one used in the proof of lemma 7. \square

Proof. (of proposition 2) There is a 1-parameter family of smoothings of the current δ represented by smooth 3-forms $\delta^\epsilon dvol$, such that $\delta(f) = \lim_{\epsilon \rightarrow 0} \int_X \delta^\epsilon f$, for all $f \in C_0^\infty(X)$. The δ^ϵ can be chosen to have uniformly bounded L^1 -norm, i.e. $\|\delta^\epsilon\|_{L^1} = \int_X |\delta^\epsilon| dvol_X = k$, and to be supported on small $\epsilon > 0$ balls around the points p_i . At this point it must be remarked that the original distribution δ does not make sense as an element of H^* , as elements of H need not be bounded. However, it does make sense as a current, i.e. $\delta \in (C_0^\infty(X))^*$ as $\|\delta^\epsilon\|_{L^1} = k$ is bounded independently of ϵ and so

$$|\delta(f)| = \left| \lim_{\epsilon \rightarrow 0} \int_X \delta^\epsilon f \right| \leq k \|f\|_{L^\infty}.$$

The trick now is to understand that for each ϵ the norm $\|\delta^\epsilon\|_{L^{\frac{6}{5}}}$ is still bounded, however not independently of ϵ . Then corollary 3 gives a family of functions $\phi_D^\epsilon \in V$, weakly solving

$$\begin{aligned} \Delta \phi_D^\epsilon &= \delta^\epsilon \\ \phi_D^\epsilon|_{\Sigma_i} &= m_i, \quad i = 1, \dots, l, \end{aligned}$$

and with ϕ_D^ϵ unique for each ϵ . Since the δ^ϵ are smooth, elliptic regularity guarantees that so are the ϕ_D^ϵ . However it must be remarked that the norm $\|\phi_D^\epsilon\|_H$ is not uniformly bounded independently of ϵ . For $f \in C_0^\infty(X)$

$$\delta(f) = \lim_{\epsilon \rightarrow 0} \int_X \delta^\epsilon f = \lim_{\epsilon \rightarrow 0} \langle \phi_D^\epsilon, f \rangle_H,$$

and since for all ϵ we have $\langle \phi_D^\epsilon, f \rangle_H = \int_X \delta^\epsilon f \leq k \|f\|$, the weak limit as $\epsilon \rightarrow 0$ of the ϕ_D^ϵ exists and gives a current ϕ^D weakly solving $\Delta \phi^D = \Delta \delta$. This current is represented by an unbounded function which we still denote by ϕ^D such that the integral $\int_X f \phi^D = \lim_{\epsilon \rightarrow 0} \int_X \phi_D^\epsilon f$ is well defined for all $f \in C_0^\infty(X)$. As the L^1 -norm of the δ^ϵ is bounded independently of ϵ and $\Delta \phi_D^\epsilon = 0$ outside an ϵ -neighborhood of the p_i 's one respectively concludes that $\phi^D \in L^1_{loc}$ and is smooth away from the p_i 's.

Moreover, as the metric is asymptotically conical ϕ_D behaves as 2.3 at each end. Locally on small balls U_{p_i} around each p_i the metric is approximately Euclidean, so that on these 2.2 holds. The last thing to be checked

is the identity $|k^I| = |k^O|$, this follows from integrating

$$\begin{aligned}
0 &= \int_{X \setminus \{p_1, \dots, p_{\#-pts.}\}} \Delta \phi^D dvol = - \int_{X \setminus \{p_1, \dots, p_{\#-pts.}\}} d * d\phi^D \\
&= \lim_{\epsilon \rightarrow 0} \sum_{i=1}^{\#-pts.} \int_{\partial B_\epsilon(p_i)} *d\phi^D - \lim_{r \rightarrow \infty} \sum_{i=1}^{\#-ends.} \int_{\Sigma_i^r} *d\phi^D \\
&= \sum_{i=1}^{\#-pts.} k_i^I - \sum_{i=1}^{\#-ends.} k_i^O,
\end{aligned}$$

where we use Stokes' theorem and the local behavior of ϕ^D both at the singular points p_i and at the ends Σ_i . \square

Remark 5. *The distribution δ can be extended from the smooth compactly supported functions to those f which are smooth and bounded. Then, the second Green's identity gives*

$$\delta(f) = \begin{cases} \int_X d\phi_D \wedge *df, \\ \int_X f \Delta \phi_D + \int_\Sigma f * d\phi_D, \end{cases}$$

where the integrals involving ϕ^D can be interpreted as the corresponding limits as $\epsilon \rightarrow 0$. Since, $*d\phi_D|_{\Sigma_i} = \frac{k_i^O}{Vol(\Sigma_i)} dvol_{\Sigma_i}$,

$$\int_\Sigma f * d\phi_D = \sum_{i=1}^{b_0(\Sigma)} k_i^O \frac{1}{Vol(\Sigma_i)} \int_{\Sigma_i} f,$$

moreover if $df \in L^2$, then f converges to a constant by lemma 5 and the formula above simplifies to $\int_\Sigma f * d\phi_D = \sum_{i=1}^{b_0(\Sigma)} k_i^O f|_{\Sigma_i}$. One still needs to show $\Delta \phi_D = \sum_{i=1}^{\#-pts.} \delta(p_i)$, but this can be achieved by the limit $\lim_{\epsilon \rightarrow 0} \Delta \phi_D^\epsilon$ this can be seen to converge uniformly to zero outside the p_i 's leaving $\sum_{i=1}^{\#-pts.} \delta_{p_i}$. To check that $\Delta \phi_D$ does not get any contribution at ∞ one needs to use the fact that the metric is asymptotically conical and so at the i -th end

$$\phi_D^\epsilon = m_i - \frac{k_i^O}{Vol(\Sigma_i)} \frac{1}{\sqrt{r^2 + \epsilon^2}} + o(r^{-2}),$$

one computes

$$\begin{aligned}
(2.4) \quad \Delta \phi_D^\epsilon &= \frac{\partial^2 \phi_D^\epsilon}{\partial r^2} + \frac{2}{r} \frac{\partial \phi_D^\epsilon}{\partial r} + \dots \\
&= \frac{k_i^O}{Vol(\Sigma_i)} \frac{3\epsilon^2}{(r^2 + \epsilon^2)^{\frac{5}{2}}} + \dots
\end{aligned}$$

and so the integration of $\Delta \phi_D^\epsilon$ times a bounded function is finite and then converges to zero as $\epsilon \rightarrow 0$. Inserting all this information one concludes that δ extends as

$$\delta(f) = \sum_{i=1}^{\#-pts.} k_i^I f(p_i) - \sum_{i=1}^{\#-ends.} k_i^O \frac{1}{Vol(\Sigma_i)} \int_{\Sigma_i} f.$$

The next goal is to construct a line bundle with connection on $U = X \setminus \{p_i\}_{i=1}^{\#-pts.}$

Proposition 3. Let $U = X \setminus \{p_i\}_{i=1}^{\sharp-pt_s}$ and $F_D = *d\phi_D$. Then F_D is a closed 2-form on U and the class $[F_D] \in H^2(U, \mathbb{R}/\mathbb{Z})$ is uniquely determined by a class $[F_X] \in H^2(X, \mathbb{R}/\mathbb{Z})$. Moreover, the F_D is the curvature of a connection on a line bundle over U if and only if the class $[F_X]$ vanishes.

Proof. Let R be one of the following Abelian Groups $\mathbb{Z}, \mathbb{R}, \mathbb{R}/\mathbb{Z}$. Since $H^1(U_{p_i} \setminus \{p_i\}), H^2(U_{p_i})$ and $H^3(X)$ all vanish, the long exact sequence for the pair $U = X \setminus \{p_i\}_{i=1}^{\sharp-pt_s}, \cup_{i=1}^{\sharp-pt_s} U_{p_i}$, gives

$$0 \rightarrow H^2(X, R) \rightarrow H^2(U, R) \rightarrow \bigoplus_{i=1}^{\sharp-pt_s} H^2(U_{p_i} \setminus \{p_i\}, R) \rightarrow 0.$$

The 2-form $F_D = *d\phi_D$ is closed in U , since $dF_D = - * \Delta \phi^D$ which vanishes in U . It represents a class in $H^2(U, \mathbb{R})$ and is the curvature of a connection on a line bundle over U if and only if this class has integer periods, equivalently if $[F_D]$ vanishes in $H^2(U, \mathbb{R}/\mathbb{Z})$. In fact, from the definition of ϕ_D one knows that the image of $[F_D]$ in $H^2(U_{p_i} \setminus \{p_i\}, \mathbb{R}/\mathbb{Z})$ vanishes, since

$$\int_{\partial \bar{U}_{p_i}} F_D = \int_{U_{p_i}} dF_D = \int_{U_{p_i}} \delta = k_i^I \in \mathbb{Z}.$$

So by exactness of the Mayer-Vietoris sequence, the class $[F_D] \in H^2(U, \mathbb{R}/\mathbb{Z})$ is the image of a class in $[F_X] \in H^2(X, \mathbb{R}/\mathbb{Z})$ and for F_D to be the curvature of a connection on a line bundle over U one just needs $[F_X]$ to vanish as well. \square

Definition 7. 1. Define the action of R on $H^0(\Sigma, R) \cong R^{b_0(\Sigma)}$ given by

$$(2.5) \quad c.(m_1, \dots, m_{b_0(\Sigma)}) = (m_1 + c, \dots, m_{b_0(\Sigma)} + c),$$

for $c \in R$ and let $H^0(\Sigma, R)/R$ denote the space of R -equivalence classes.

2. Define the map

$$(2.6) \quad \begin{aligned} b : X^k \times H^0(\Sigma, \mathbb{R})/\mathbb{R} &\rightarrow H^2(X, \mathbb{R}/\mathbb{Z}) \\ (p, [m]) &\mapsto [F_X], \end{aligned}$$

where $[F_X]$ is the class determined via proposition 3. Moreover, fixing the mass m denote by $b_{[m]}$ the map that to $p \in X^k$ assigns $b_{[m]}(p) = b(p, [m]) \in H^2(X, \mathbb{R}/\mathbb{Z})$.

The map b above is indeed well defined, since changing the mass

$$m = (m_1, \dots, m_{b_0(\Sigma)})$$

by an overall constant amounts to add this constant to ϕ_D . Hence $F_D = *d\phi_D$ remains unchanged and proposition 3 can be rephrased as

Corollary 4. Given $(p, [m]) \in X^k \times H^0(\Sigma, \mathbb{R})/\mathbb{R}$, these determine a line bundle L over $U = X \setminus \{p_1, \dots, p_k\}$ with connection ∇_D and curvature $F_D = *d\phi_D$ if and only if $b(p, [m]) = 0$.

Remark 6. To make a connection with Hitchin's point of view in [5] one must regard the formal sum of points $\sum_{i=1}^k p_i$ as a sort of divisor on the 3 dimensional Riemannian manifold (X, g) . This divisor can be equivalently regarded as a flat gerbe, since $H^3(X, \mathbb{Z}) = 0$ as X is noncompact. The holonomy of the flat connection represents a class in $H^2(X, \mathbb{R}/\mathbb{Z})$ and then one has the following equivalent statements. 1. The holonomy vanishes. 2. The flat gerbe is trivial. 3 The divisor is linearly trivial. 4 There is a Dirac monopole on U .

Theorem 2. *Let $m \in H^0(\Sigma, \mathbb{R})$, then for each*

$$(p, \alpha) \in b_{[m]}^{-1}(0) \times H^1(X, \mathbb{S}^1)$$

there is a reducible $SU(2)$ bundle $L \oplus L^{-1}$ over $X \setminus \{p_1, \dots, p_k\}$ equipped with a charge k and mass m Dirac monopole (A_D, Φ_D) . In particular, if $b_2 = 0$, then the map b vanishes and there is such a Dirac monopole for all $p \in X^k$.

Moreover, for any two different (p, α) and (p', α') the Dirac monopoles associated with these are not related by any gauge transformation $g \in C^\infty(X, \mathbb{S}^1)$.

Proof. This has 3 steps.

1. From proposition 2 one constructs the function ϕ_D , which determines a 2 form via $F_D = *d\phi_D$. Then, from proposition 3 or equivalently corollary 4 one knows that the two form F_D is the curvature of a connection ∇_D on a line bundle L over $X \setminus \{p_1, \dots, p_k\}$ if and only if $b(p, [m]) = 0$. We also note that the connections ∇_D associated with different tuples of points are obviously gauge inequivalent and may possibly live in different line bundles.
2. A class $\alpha \in H^1(X, \mathbb{S}^1)$ represents a gauge equivalence class of flat connections ∇^F on a torsion line bundle L^F over X . One can now twist this with the Dirac monopole without changing the Higgs field. This gives the line bundle $L \otimes L^F$, which one still denotes by L , equipped with the connection $\nabla = \nabla_D \otimes 1 + 1 \otimes \nabla^F$. The curvature of this connection remains unchanged and so it still satisfies the Bogomolnyi equation $*F_D = d\phi_D$.
3. To finish the proof increase the gauge group to $SU(2)$, either by equipping L with a metric and considering the principal bundle $P = \mathbb{S}^1(L) \times_{U(1)} SU(2)$, or by considering the vector bundle $L \oplus L^{-1}$, associated with P through the standard representation of $SU(2)$ in \mathbb{C}^2 . Then, the adjoint bundle is $\mathfrak{g}_P = \mathbb{R} \oplus L^2$. Define the connection $A_D = \nabla_D \oplus (-\nabla_D)$ on E and Higgs field $\Phi_D = \phi_D \oplus 0$, these do satisfy the monopole equations on U and have Dirac type singularities at the points p_i , for $i = 1, \dots, k$.

□

Remark 7. *The automorphism group of the pair (A_D, Φ_D) constructed above on $X \setminus \cup_{i=1}^k \{p_i\}$ is isomorphic to \mathbb{S}^1 . In fact, given $e^{ia} \in \mathbb{S}^1$, we have $e^{ia\Phi_D} \cdot (A_D, \Phi_D) = (A_D, \Phi_D)$.*

2.1 The Approximate Solution

This section constructs an approximate solution to the monopole equations in proposition 4, whose error term is estimated in lemma 9. Recall the map b from definition in 7 and theorem 2 which proves that if $p = (p_1, \dots, p_k) \in X^k$ and $m \in H^0(\Sigma, \mathbb{R})$ are such that $b(p, [m]) = 0$, then one can construct a reducible Dirac Monopole (A_D, Φ_D) on an $SU(2)$ -bundle $L \oplus L^{-1}$ over $X \setminus \{p_1, \dots, p_k\}$. Before proceeding to the construction of an approximate solution it is helpful to show how one can make this Dirac monopole have very large Higgs field in a big open set in X .

Lemma 8. *Let $[m] \in H^0(\Sigma, \mathbb{R})/\mathbb{R}$ be an \mathbb{R} -equivalence class. Then, there is $\mu \in \mathbb{R}$ (depending on the metric g and the class $[m]$), with the following property. Let ϕ_D the Higgs field constructed in proposition*

2 associated with a representative $m \in [m]$ with average mass $m_0(m) \geq \mu$ and k distinguish points $p_1, \dots, p_k \in X$. Then, there is $\epsilon = \sqrt{\frac{10}{8m_0}}$

$$\phi^D \geq \frac{m_0}{2},$$

on $U_\epsilon = X \setminus \cup_{i=1}^k \overline{B_\epsilon(p_i)}$.

Proof. Let $m_{init} \in H^0(\Sigma, \mathbb{R})$ be a representative in the class $[m]$ with average mass $m_0(m_{init}) = 0$. Then, equation 2.2 in proposition 2 implies that there are $c_i \in \mathbb{R}$ and a positive $\delta < inj(X, g)$ such that on $B_\delta(p_i)$,

$$\phi_D(r) = c_i - \frac{1}{2r} + O(r).$$

One can always add a constant m_0 and construct a new Dirac monopole ϕ_D with mass $m = m_{init} + (m_0, \dots, m_0)$. Then $[m] = [m_{init}]$, but $m_0([m]) = m_0$. Then, by possibly changing the constants c_i by some term linear in δ one can assume that on each $B_\delta(p_i)$

$$\phi_D \geq m_0 + c_i - \frac{1}{2r},$$

and is increasing along the radial coordinate. The goal now is to increase m_0 in order to find $\epsilon < \delta$, such that for $r \in (\epsilon, \delta)$ one has $\phi_D \geq \frac{m_0}{2}$. As the c_i 's do not change with m_0 , one can change m_0 so that $m_0 + c_i \geq \frac{9}{10}m_0$, i.e. $m_0 \geq 10 \max_i(-c_i)$. Then it is enough to solve

$$\frac{9}{10}m_0 - \frac{1}{2r} \geq \frac{m_0}{2},$$

which gives $r \geq \frac{10}{8m_0}$. This quantity is strictly decreasing with increasing m_0 , and so we can arrange for $\sqrt{\frac{10}{8m_0}} \ll \delta$, as δ does not depend on m_0 . Then, set $\epsilon = \sqrt{\frac{10}{8m_0}} > \frac{10}{8m_0}$ for big m_0 . Then, one has

$$\phi_D|_{\partial B_\epsilon} \geq \frac{m_0}{2},$$

so that the problem is reduced to show that $\phi_D \geq \frac{m_0}{2}$ on the whole of $U_\epsilon = X \setminus \cup_i B_\epsilon(p_i)$. This follows from the fact that ϕ_D is harmonic on U_ϵ and so by the maximum principle it has no interior maximum or minimum. These are attained at the ends of X or at the spheres ∂B_ϵ . Since ϵ was chosen smaller than δ , and on each $B_\delta(p_i)$ ϕ_D is increasing with $r = dist(p_i, \cdot)$, the conclusion is that the minimum is attained at the inner boundaries and so $\phi_D \geq \frac{m_0}{2}$ on $X \setminus \cup_i B_\epsilon(p_i)$. \square

Definition 8. Let $m \in H^0(\Sigma, \mathbb{R})$ be a configuration with positive average mass m_0 . Define $X^k(m)$ as the set of $p = (p_1, \dots, p_k) \in X^k$ such that $b(p, [m]) = 0$ and $dist(p_i, p_j) \geq \frac{4}{\sqrt{m_0}}$, for all distinct $i, j \in \{1, \dots, k\}$.

Remark 8. In the case where $b^2(X) = 0$, the map b vanishes identically and $H^0(\Sigma, \mathbb{R}) \cong \mathbb{R}^l$ so $X^k(m)$ is the open subset of X^k where the distinct points are at distance more than $4/\sqrt{m}$ apart. However, in general this is the intersection of this open set $U(m)$ with a submanifold $b_{[m]}^{-1}(0)$ of X^k of dimension $3k - b^2(X)$. Moreover, this submanifold only depends on the equivalence class $[m]$ and so by increasing $m_0(m)$ one can make the open set $U(m)$ as large as one wants and so $X^k(m) = b_{[m]}^{-1}(0) \cap U(m)$ is certainly nonempty.

Proposition 4. Fix an identification $\mathbb{S}^1 \cong (\mathring{M}_1(\mathbb{R}^3))$ with the moduli space of charge 1 centered monopoles on \mathbb{R}^3 . Denote k copies of this by \mathbb{T}^k and let $\check{\mathbb{T}}^{k-1} = \{(e^{i\theta_1}, \dots, e^{i\theta_k}) \in \mathbb{T}^k \mid e^{i(\theta_1 + \dots + \theta_k)} = 1\}$. Then, for all $[m] \in H^0(\Sigma, \mathbb{R})/\mathbb{R}$, there is $\mu_0 > 0$, such that for $m_0(m) > \mu_0$ there is a function

$$(2.7) \quad H : X^k(m) \times H^1(X, \mathbb{S}^1) \times \check{\mathbb{T}}^{k-1} \rightarrow C_{k,m}(X, g),$$

with the property that for each configuration $(A_0, \Phi_0) = H(p, \alpha, \theta)$ there are real numbers $\epsilon_{out}^i = 2\epsilon_{in}^i = O(m_0^{-1/2})$ and $\lambda_i = O(m_0)$ such that

- (A_0, Φ_0) is a configuration on a $SU(2)$ bundle E , such that $E|_{X \setminus \{p_1, \dots, p_k\}} \cong L \oplus L^{-1}$, where $L \oplus L^{-1}$ is the bundle from theorem 2 and $|\Phi_0| > \frac{m_0}{2}$.
- On $X \setminus \cup_{i=1}^{\#} B_{\epsilon_{out}^i}^{pts.}(p_i)$, the configuration (A_0, Φ_0) coincides with a Dirac monopole,
- Using geodesic normal coordinates on each $B_{\epsilon_{in}^i}(p_i)$, (A_0, Φ_0) coincides with a centered charge 1 and mass λ_i BPS monopole on \mathbb{R}^3 .

Proof. Let $x = (p, \alpha, \theta) \in X^k(m) \times H^1(X, \mathbb{S}^1) \times \check{\mathbb{T}}^{k-1}$. Starting with (p, α) , theorem 2 in the last section constructs a reducible $SU(2)$ Dirac monopole (A_D, Φ_D) on $L \oplus L^{-1}$ over $X \setminus \{p_1, \dots, p_k\}$. From the proof of lemma 8, in a neighborhood of each p_i , one can write

$$\phi_D = -\frac{1}{2r} + c_i + m_0 + O(r),$$

where the c_i 's are constants independent of the average mass m_0 and depend only on the points p_i and the class $[m]$. Moreover there is $\mu > 0$, such that for $m_0 > \max\{1 + 2 \sup_i |c_i|, \mu\}$ and define

$$(2.8) \quad \lambda_i = m_0 + c_i$$

$$(2.9) \quad \epsilon_{in}^i = \lambda_i^{-1/2},$$

and one can check that locally around each p_i for $r > \epsilon_{in}^i$, it holds that $\phi_D > \frac{\lambda_i}{2}$.

The identification $\mathbb{S}^1 \cong \mathring{N}_{1,1}(\mathbb{R}^3)$ gives for each element of $\theta_i \in \mathbb{S}^1$ a framed centered BPS monopole $(A^{BPS}, \Phi^{BPS}, \eta_i)$. As the bundle L has degree one on small spheres around each p_i we can use the framing η_i to identify the trivial \mathbb{C}^2 bundle over $\partial B_{\epsilon_{out}^i} \setminus \{p_i\}$ with the restriction of $L \oplus L^{-1}$. This extends the bundle $L \oplus L^{-1}$ over the points p_i to form the bundle E .

Recall from definition 4 that $|\Phi^{BPS}| \geq \frac{1}{2}$, outside $B_1(0) \subset \mathbb{R}^3$. After scaling these BPS configurations to have mass λ_i , they satisfy $|\Phi_{\lambda_i}^{BPS}| \geq \frac{\lambda_i}{2}$ outside the ball of radius λ_i^{-1} in \mathbb{R}^3 . Let $\epsilon_{out}^i = 2(\lambda_i)^{-1/2}$, then $dist(p_i, p_j) > 2\epsilon_{out}^i$. Then using geodesic normal coordinates centered at the points p_i and $\theta = (\theta_1, \dots, \theta_k) \in \check{\mathbb{T}}^{k-1}$ to identify the bundles we can pullback these scaled monopoles $(A_{\lambda_i}^{BPS}, \Phi_{\lambda_i}^{BPS})$ from a small ball $B_{\epsilon_{out}^i} \subset \mathbb{R}^3$ to the ball $B_{\epsilon_{out}^i}(p_i)$.

Now consider the open cover of X given by the $B_{\epsilon_{out}^i}(p_i)$ and $U_\epsilon = X \setminus \cup_{i=1}^k \overline{B_{\epsilon_{in}^i}(p_i)}$ and let $\{\chi_{out}, \chi_1, \dots, \chi_k\}$ be a partition of unity subordinate to this cover, such that

$$\chi_{in}^i = \begin{cases} 1 & \text{in } B_{\epsilon_{in}^i}(p_i), \\ 0 & \text{in } X \setminus B_{\epsilon_{out}^i}(p_i). \end{cases}, \quad \chi_{out} = \begin{cases} 0 & \text{in } \cup_{i=1}^k B_{\epsilon_{in}^i}(p_i), \\ 1 & \text{in } X \setminus \cup_{i=1}^k B_{\epsilon_{out}^i}(p_i). \end{cases}$$

and

$$\sum_{i=1}^k |\nabla \chi_i| + |\nabla \chi_{out}| \leq c \sup_{i \in \{1, \dots, k\}} (\epsilon_{out}^i - \epsilon_{in}^i)^{-1} \leq c\sqrt{m_0}.$$

Define the approximate solution as the following configuration on E

$$(2.10) \quad \Phi_0 = \chi_{out} \Phi_D + \sum_{i=1}^k \chi_{in}^i \Phi_{\lambda_i}^{BPS}$$

$$(2.11) \quad A_0 = A_D + \sum_{i=1}^k \chi_{in}^i (A_{\lambda_i}^{BPS} - A_D).$$

One must remark that even though the connection A_D does not extend over the points p_i , the connection A_0 does, as on each trivialization over $B_{\epsilon_{in}^i}(p_i)$, the connection A_0 is represented by the connection $A_{\lambda_i}^{BPS}$. \square

Remark 9. Recall that the automorphism group of the pair (A_D, Φ_D) on $X \setminus \cup_{i=1}^k B_{\epsilon_{in}^i}(p_i)$ is isomorphic to \mathbb{S}^1 via $e^{ia} \mapsto e^{ia\Phi_D}$. Hence, given $\theta = (\theta_1, \dots, \theta_k) \in \mathbb{T}^k$, the configuration $(A_0, \Phi_0) = H(p, \alpha, \theta)$ is gauge equivalent via $e^{ia\Phi_0}$ to $H(p, \alpha, \theta + \vec{a})$ where $\theta + \vec{a} = (\theta_1 + a, \dots, \theta_k + a)$.

Lemma 9. There is a constant $c > 0$ independent of m , such that for any approximate solution (A_0, Φ_0) in the image of the map H the quantity $e_0 = *F_{A_0} - \nabla_{A_0} \Phi_0$ vanishes outside the balls $B_{\epsilon_{out}^i}(p_i)$ and its C^0 norm is bounded by

$$|e_0| \leq cm_0.$$

In particular, if the metric g on X is exactly Euclidean on the balls $B_{\epsilon_{out}^i}(p_i)$, then in fact the error e_0 is supported on the annulus $B_{\epsilon_{out}^i}(p_i) \setminus B_{\epsilon_{in}^i}(p_i)$.

Proof. On $X \setminus \cup_{i=1}^k B_{\epsilon_{out}^i}(p_i)$ the configuration (A_0, Φ_0) agrees with the Dirac monopole (A_D, Φ_D) and hence satisfies the monopole equations. So, the error term $e_0 = *F_{A_0} - \nabla_{A_0} \Phi_0$ is supported on the small balls $\cup_{i=1}^k B_{\epsilon_{out}^i}(p_i)$. On each of these balls, say $B_{\epsilon_{out}^i}(p_i)$ the bundle E has a trivialization and $\chi_{in}^i = 1 - \chi_{out}$, these may be used to write

$$\begin{aligned} \Phi_0 &= \Phi_{\lambda_i}^{BPS} + \chi_{out}(\Phi_D - \Phi_{\lambda_i}^{BPS}) \\ A_0 &= A_{\lambda_i}^{BPS} + \chi_{out}(A_D - A_{\lambda_i}^{BPS}). \end{aligned}$$

Since this agrees with $(A_{\lambda_i}^{BPS}, \Phi_{\lambda_i}^{BPS})$ in the ball $B_{\epsilon_{in}^i}(p_i)$, the error term only depends on how far the metric g on these balls differs from the Euclidean one. Computing the curvature and the covariant derivative of the Higgs field

$$\begin{aligned} F_{A_0} &= F_{A_{\lambda_i}^{BPS}} + d_{A_{\lambda_i}^{BPS}} \chi_{out} (A_D - A_{\lambda_i}^{BPS}) + \chi_{out}^2 [(A_D - A_{\lambda_i}^{BPS}) \wedge (A_D - A_{\lambda_i}^{BPS})] \\ &= F_{A_{\lambda_i}^{BPS}} + d\chi_{out} \wedge (A_D - A_{\lambda_i}^{BPS}) + \chi_{out} d_{A_{\lambda_i}^{BPS}} (A_D - A_{\lambda_i}^{BPS}) \\ &\quad + \chi_{out}^2 [(A_D - A_{\lambda_i}^{BPS}) \wedge (A_D - A_{\lambda_i}^{BPS})], \\ \nabla_{A_0} \Phi_0 &= \nabla_{A_{\lambda_i}^{BPS}} \Phi_0 + \chi_{out} [(A_D - A_{\lambda_i}^{BPS}), \Phi_0] \\ &= \nabla_{A_{\lambda_i}^{BPS}} \Phi_{\lambda_i}^{BPS} + d\chi_{out} (\Phi_D - \Phi_{\lambda_i}^{BPS}) + \chi_{out} \nabla_{A_{\lambda_i}^{BPS}} (\Phi_D - \Phi_{\lambda_i}^{BPS}) \\ &\quad + \chi_{out} [(A_D - A_{\lambda_i}^{BPS}), \Phi_0], \end{aligned}$$

one concludes that the error term is given by

$$\begin{aligned}
*F_{A_0} - \nabla_{A_0} \Phi_0 &= *F_{A_{\lambda_i}^{BPS}} - \nabla_{A_{\lambda_i}^{BPS}} \Phi_0 \\
&+ \chi_{out} \left(*d_{A_{\lambda_i}^{BPS}}(A_D - A_{\lambda_i}^{BPS}) - \nabla_{A_{\lambda_i}^{BPS}}(\Phi_D - \Phi_{\lambda_i}^{BPS}) \right) \\
&+ * \left(d\chi_{out} \wedge (A_D - A_{\lambda_i}^{BPS}) \right) - d\chi_{out}(\Phi_D - \Phi_{\lambda_i}^{BPS}) \\
&+ \chi_{out}^2 * \left[(A_D - A_{\lambda_i}^{BPS}) \wedge (A_D - A_{\lambda_i}^{BPS}) \right] \\
&- \chi_{out} \left[(A_D - A_{\lambda_i}^{BPS}), \Phi_0 \right]
\end{aligned}$$

The first of these terms would vanish if the metric g is Euclidean in these balls. The terms $|A_D - A_{\lambda_i}^{BPS}|$ and $|\Phi_D - \Phi_{\lambda_i}^{BPS}|$ always appear multiplied by χ_{out} or $d\chi_{out}$ which are supported outside $B_{\epsilon_{in}}(p_i)$. So, one starts by estimating these on the annulus $B_{\epsilon_{out}}(p_i) \setminus B_{\epsilon_{in}}(p_i)$, where lemma 6 gives

$$\Phi_D = \Phi_{\lambda_i}^{BPS} + O(\lambda_i e^{-\lambda_i r}), \quad A_D = A_{\lambda_i}^{BPS} + O(\lambda_i e^{-\lambda_i r}),$$

and since on these annulus $r \in (\epsilon_{in}^i, \epsilon_{out}^i)$ and $\lambda_i = O(m_0)$, while $\epsilon_{out}^i = O(m_0^{-1/2})$

$$|A_D - A_{\lambda_i}^{BPS}| + |\Phi_D - \Phi_{\lambda_i}^{BPS}| = O(m_0 e^{-\sqrt{m_0}}).$$

and this gives

$$\begin{aligned}
|* \left(d\chi_{out} \wedge (A_D - A_{\lambda_i}^{BPS}) \right)| + |d\chi_{out}(\Phi_D - \Phi_{\lambda_i}^{BPS})| &= O(m_0^{3/2} e^{-\sqrt{m_0}}) \\
|\chi_{out}^2 (A_D - A_{\lambda_i}^{BPS})^2| + |\chi_{out} \left[(A_D - A_{\lambda_i}^{BPS}), \Phi_0 \right]| &= O(m_0 e^{-\sqrt{m_0}}) \\
\chi_{out}^2 \left(|* d_{A_{\lambda_i}^{BPS}}(A_D - A_{\lambda_i}^{BPS})| + |\nabla_{A_{\lambda_i}^{BPS}}(\Phi_D - \Phi_{\lambda_i}^{BPS})| \right) &= O(m_0 e^{-\sqrt{m_0}}).
\end{aligned}$$

Back to evaluate the first term $*F_{A_{\lambda_i}^{BPS}} - \nabla_{A_{\lambda_i}^{BPS}} \Phi_0$, one needs to compare the metric g with the Euclidean metric g_0 on the balls $B_{\epsilon_{out}^i}(p_i)$. Let the $\{x_i\}_{i=1,2,3}$ denote geodesic normal coordinates centered at the point p_i and recall that these are used to pullback the configuration $(A_{\lambda_i}^{BPS}, \Phi_{\lambda_i}^{BPS})$ from \mathbb{R}^3 to this small ball. In these coordinates $g = g_0 + x_i x_j \gamma^{ij} + O(x^3)$ for some symmetric 2-tensor $\gamma = x_i x_j \gamma^{ij}$. After a short computation expanding the formula $\omega \wedge * \omega = |\omega|_g^2 \text{dvol}_g$ for any k -form ω , one concludes that

$$*_g \omega = \left(1 + \frac{\gamma(\omega, \omega)}{|\omega|_{g_0}^2} - \frac{1}{6} Ric^{ij} x_i x_j \right) *_0 \omega + O(x^3).$$

Applying this to $*F_{A_{\lambda_i}^{BPS}}$ gives

$$\begin{aligned}
*F_{A_{\lambda_i}^{BPS}} - \nabla_{A_{\lambda_i}^{BPS}} \Phi_{\lambda_i}^{BPS} &= *_0 F_{A_{\lambda_i}^{BPS}} - \nabla_{A_{\lambda_i}^{BPS}} \Phi_{\lambda_i}^{BPS} + \\
&+ \left(\frac{\gamma(F_{A_{\lambda_i}^{BPS}}, F_{A_{\lambda_i}^{BPS}})}{|F_{A_{\lambda_i}^{BPS}}|_{g_0}^2} - \frac{1}{6} Ric^{ij} x_i x_j \right) *_0 F_{A_{\lambda_i}^{BPS}},
\end{aligned}$$

the first term vanishes because $(A_{\lambda_i}^{BPS}, \Phi_{\lambda_i}^{BPS})$ is a monopole for the metric g_0 . Regarding the second term it is $O((\epsilon_{out}^i)^2 |F_{A_{\lambda_i}^{BPS}}|_{g_0})$, then since $\epsilon_{out}^i = O(m_0^{-1/2})$ and $|F_{A_{\lambda_i}^{BPS}}|_{g_0} = O(m_0^2)$, which gives

$$|*F_{A_{\lambda_i}} - \nabla_{A_{\lambda_i}} \Phi_{\lambda_i}|_g = O(m_0).$$

Summing all these terms for big m_0 (all the previous ones were lower order compared to this one), gives the result in the statement. \square

This lemma points out one other "problem", there is no hope in controlling the error term in the C^0 norm provided by the metric g . This is related to the fact that there is no mass independent lower bound on the first eigenvalue of $d_2 d_2^*$ for the large mass BPS monopole. The idea to overcome this issue is to rescale all data inside the balls $B_{\epsilon_{out}^i}(p_i)$ and work there with the metric $g_{m_0^{-1}} = m_0^2 \exp_{m_0^{-2}}^* g$. Then one can use the scaling identity

$$(2.12) \quad |*_\epsilon F_{A_{\epsilon\lambda_i}} - \nabla_{A_{\epsilon\lambda_i}} \Phi_{\epsilon\lambda_i}|_{g_\epsilon} = \epsilon^2 |*_F_{A_{\lambda_i}} - \nabla_{A_{\lambda_i}} \Phi_{\lambda_i}|_g,$$

for all $\epsilon > 0$. Applying this with $\epsilon = m_0^{-1}$ gives the following result

Corollary 5. *There is $c \in \mathbb{R}^+$, such that $e_0 = *_F_{A_0} - \nabla_{A_0} \Phi_0$ vanishes on $X \setminus \cup_{i=1}^k B_{\epsilon_{out}^i}(p_i)$ and one the interior of these balls*

$$|(e_0)_{m_0^{-1}}|_{g_{m_0^{-1}}} = m_0^{-2} |e_0|_g \leq cm_0^{-1}.$$

Based on this, one can define function spaces where not only the error term will be small but the first eigenvalue of $d_2 d_2^*$ is bounded from below.

3 Proof of the Main Theorem

Our goal is to apply a version of the Banach space Implicit Function theorem to solve the monopole equation. In order to do that we divide our proof into 3 major steps: 1. In subsection 3.1 we introduce suitable function spaces. 2. In subsection 3.2 we prove that using these function spaces the linearisation of the monopole equation d_2 is surjective. 3. Finally in subsection 3.3 we use the result from the second step in order to find a solution to the monopole equation. This also requires the spaces defined in the first step to give the initial approximate solution a small error term. In fact, we shall be able to control the error term in the approximate solutions in a uniform manner with respect to the mass of the approximate solution.

3.1 Function Spaces

This section introduces function spaces specially adapted to solve the monopole equation. To proceed with the definition of these Function spaces some preparation is needed. Let (A_0, Φ_0) be the approximate solution constructed in section 2.1. Then on $U = X \setminus \cup_{i=1}^k B_{\epsilon_{out}^i}(p_i)$ the adjoint bundle \mathfrak{g}_E splits as $\mathfrak{g}_E = \mathbb{R} \oplus L^2$ and one writes a section $f = f^\parallel \oplus f^\perp$ according to this splitting. Let $\beta \in \mathbb{R}$ and $K \subset X$ containing $\cup_{i=1}^k B_1(p_i)$ and for each $n \in \mathbb{N}_0$, define $W_n : X \rightarrow \mathbb{R}$ to be smooth weight functions, interpolating between the values

$$W_n = \begin{cases} m_0^{-2+n}, & \text{in } \cup_{i=1}^k B_{\epsilon_{out}^i}(p_i), \\ 1, & \text{at } \cup_{i=1}^k \partial B_1(p_i), \\ \rho^{n-\beta-\frac{3}{2}} & \text{in } X \setminus K, \end{cases}$$

Define for smooth compactly supported f , the norms

$$(3.1) \quad \begin{aligned} \|f\|_{H_{0,\beta}}^2 &= \sum_{i=1}^k \int_{B_1(p_i)} |W_0 f|_g^2 dvol_g \\ &+ \int_{X \setminus \cup_{i=1}^k B_1(p_i)} \left(|W_0 f^\parallel|_g^2 + |f^\perp|_g^2 \right) dvol_g. \end{aligned}$$

$$(3.2) \quad \begin{aligned} \|f\|_{H_{n,\beta}}^2 &= \sum_{j=0}^n \sum_{i=1}^k \int_{B_1(p_i)} |W_j \nabla_{ABPS}^j f|_g^2 dvol_g \\ &+ \sum_{j=0}^k \int_{X \setminus \cup_{i=1}^k B_1(p_i)} \left(|W_j \nabla_{A_0}^j f^\parallel|_g^2 + |ad_{\Phi_0}^{n-j} \left(\nabla_{A_0}^j f^\perp \right)|_g^2 \right) dvol_g. \end{aligned}$$

Definition 9. For $n \in \mathbb{N}_0$ and $\beta \in \mathbb{R}$, let $H_{n,\beta}$ be the Hilbert space completion of $C_0^\infty(X, (\Lambda^0 \oplus \Lambda^1) \otimes \mathfrak{g}_E)$ in the inner product obtained from polarizing the norms 3.1 and 3.2.

Remark 10. 1. Recall the error e_0 of (A_0, Φ_0) is supported in the small balls $B_{\epsilon_{out}^i}(p_i)$ and to make it small in the C^0 norm we used an $\epsilon = m_0^{-1}$ -rescaled norm $g_\epsilon = \epsilon^{-2} \exp_\epsilon^* g$, i.e. for $f \in \Gamma(X, \Lambda^l \otimes \mathfrak{g}_E)$ one needed to use $|f_\epsilon|_{g_\epsilon} = \epsilon^2 |f|_g$, where $f_\epsilon = \exp_\epsilon^* f$ if $l = 1$ and $f_\epsilon = \epsilon \exp_\epsilon^* f$ if $l = 0$. Indeed, the functions W_n are defined so that $W_0 = m_0^{-2}$, on $\cup_{i=1}^k B_{\epsilon_{out}^i}(p_i)$.

2. Outside a big compact set K the function spaces to be defined will coincide with the usual L^2 spaces for the f^\perp component and the Lockhart-McOwen weighted spaces in the f^\parallel component. These are suitable to order to find a bounded right inverse to the operator d_2 .

3.2 The Linear Equation

The goal of this section is to solve the linear equation associated with the linearized monopole equation. Namely, one needs to find a range for $\beta \in \mathbb{R}$ such that if $g \in H_{\beta-1}$, then one can solve

$$d_2 f = g$$

for $f \in H_{1,\beta}$. In other words one needs to construct a right inverse Q to the operator $d_2 : H_{1,\beta} \rightarrow H_{\beta-1}$. This whole section is occupied with the proof of

Proposition 5. Let (X, g) be asymptotically conical with $b^2(X) = 0$. Then there is $\mu > 0$, such that if $m_0(m) > \mu$, the operator

$$d_2 : H_{1,\beta} \rightarrow H_{0,\beta-1},$$

associated with any approximate solution (A_0, Φ_0) constructed in proposition 2.1 is surjective for all $\beta > -1$. In particular, there is a uniformly bounded right inverse $Q : H_{\beta-1} \rightarrow H_{1,\beta}$.

To prove this proposition we need to show that the cokernel of d_2 in $H_{0,\beta-1}$ vanishes. Using the L^2 inner product one can identify $H_{\beta-1}^* \cong H_{0,-\beta-2}$ and the cokernel of d_2 in $H_{0,\beta-1}$ with the kernel of d_2^* in $H_{0,-\beta-2}$. The goal is to find a range of β for which this vanishes. To prove this it enough to prove an inequality of the form

$$(3.3) \quad \|d_2^* f\|_{H_{0,\alpha-1}} \geq c \|f\|_{H_{0,\alpha}},$$

for some $\alpha \in \mathbb{R}, c > 0$ and all $f \in H_{1,\alpha}$. Then, we set $\beta = -\alpha - 2$ and construct $Q : H_{0,\beta-1} \rightarrow H_{1,\beta}$ to be such that $u = Qf$ is the unique solution of $d_2 u = f$ such that u is $H_{1,\beta}$ -orthogonal to the kernel of d_2 in $H_{1,\beta}$. To proceed with the proof, let χ be a bump function such that

$$\chi = \begin{cases} 0 & \text{in } \cup_{i=1}^k B_{\epsilon_{in}^i}(p_i), \\ 1 & \text{in } U = X \setminus \overline{\cup_{i=1}^k B_{\epsilon_{out}^i}(p_i)}, \end{cases}$$

and vanishing with all derivatives on $\partial B_{\epsilon_{in}^i}(p_i)$. One can further suppose that there is a constant $C > 0$, such that $|d\chi|_g \leq C m_0^{1/2}$ on each annulus where $d\chi$ is supported. Hence its L^2 norm is bounded by $\|d\chi\|_{L^2} \leq C m_0^{-1/2}$, as the volume of the set containing the support of $d\chi$ is of order $(\epsilon_{out}^i)^3 = O(m_0^{-3/2})$. Then, for any section $f \in \Omega^1(X, \mathfrak{g}_P)$ one can write $f = (1 - \chi)f + \chi f$, where the first term is supported inside the balls $B_{\epsilon_{out}^i}(p_i)$ and the second one is supported on U , the complement to their closure.

Lemma 10. *There is $\mu > 0$ such that if $m_0(m) > \mu$ and $\alpha < -1$, there is a constant $c > 0$, such that*

$$\|d_2^*(\chi f)\|_{H_{0,\alpha-1}} \geq c \|\chi f\|_{H_{0,\alpha}},$$

for all $f \in H_{1,\alpha}$.

Proof. Recall that in the complement U , the approximate solution (A_0, Φ_0) gives a splitting $\mathfrak{g}_P \cong \mathbb{R} \oplus L^2$, which is preserved by the operator d_2 . Moreover, this is an orthogonal decomposition with respect to the Ad -invariant metric on \mathfrak{g}_P . So

$$\|d_2^*(\chi f)\|_{H_{0,\alpha-1}}^2 = \|d_2^*(\chi f^\parallel)\|_{H_{0,\alpha-1}}^2 + \|d_2^*(\chi f^\perp)\|_{H_{0,\alpha-1}}^2,$$

and it is enough to separately prove the inequality for each of these terms.

1. The longitudinal component χf^\parallel is a one form, so $\|\chi f^\parallel\|_{H_{n,\alpha}} = \|\chi f^\parallel\|_{L_{n,\alpha}^2}$ and

$$d_2^*(\chi f^\parallel) = (*d(\chi f^\parallel), -d^*(\chi f^\parallel)).$$

Then using corollary 1 guarantees that $\|d_2^*(\chi f^\parallel)\|_{H_{0,\alpha-1}}^2 \geq c \|\chi f^\parallel\|_{H_{1,\alpha}}^2$, for all $\alpha < -1$ and some c not depending of f .

2. For the transverse component χf^\perp , $\|\chi f^\perp\|_{H_{0,\alpha}} = \|\chi f^\perp\|_{L^2}$ for all $\alpha \in \mathbb{R}$. So we need to consider

$$(3.4) \quad d_2^*(\chi f^\perp) = \left(*d_{A_0}(\chi f^\perp) + [\chi f^\perp, \Phi_0], -\nabla_{A_0} \chi f^\perp \right),$$

in usual L^2 spaces. Notice that χf^\perp is supported in U , where the approximate solution (A_0, Φ_0) coincides with the Dirac monopole (A_D, Φ_D) (see the second bullet in 4). Using the decomposition $\mathfrak{g}_P = \mathbb{R} \oplus L^2$, we have $\Phi_D = \phi_D \oplus 0$, and $|\phi_D| > \frac{m_0}{2}$. Then, integrating by parts together with the Weitzenböck formula 1.4, gives

$$(3.5) \quad \begin{aligned} \|d_2^*(\chi f^\perp)\|_{L^2}^2 &= \langle \chi f^\perp, d_2 d_2^*(\chi f^\perp) \rangle_{L^2} \\ &= \langle \chi f^\perp, \nabla_D^* \nabla_D(\chi f^\perp) + \phi_D^2 \chi f^\perp + Ric(\chi f^\perp) \rangle_{L^2} \\ &= \|\nabla_D(\chi f^\perp)\|_{L^2}^2 + \|\phi_D \chi f^\perp\|_{L^2}^2 + \langle \chi f^\perp, \chi Ric(f^\perp) \rangle_{L^2}. \end{aligned}$$

Recall, from lemma 8 that, ϕ_D is harmonic on U and is bounded from below by its values at $\partial\bar{U}$, these being greater or equal to $\frac{m_0}{2}$. Then, $\phi_D \geq \frac{m_0}{2}$ on U and

$$\begin{aligned} \|d_2^*(\chi f^\perp)\|_{L^2}^2 &\geq \|\nabla_D(\chi f^\perp)\|_{L^2}^2 + \left(\frac{m_0^2}{4} - \sup_X |Ric|\right) \|\chi f^\perp\|_{L^2}^2 \\ &\geq \|\nabla_D(\chi f^\perp)\|_{L^2}^2 + \frac{\frac{m_0^2}{4} - \sup_X |Ric|}{(m_0 + \max_i(m_i))^2} \|\chi \phi_D f^\perp\|_{L^2}^2 \\ &\geq c_2 \|\chi f^\perp\|_{H_{1,\alpha}} \end{aligned}$$

for any $\alpha \in \mathbb{R}$ and one can take $c_2 = \min\left\{1, \frac{\frac{m_0^2}{4} - \sup_X |Ric|}{(m_0 + \max_i(m_i))^2}\right\} > 0$ for $m_0 > 2\sqrt{\|Ric\|_{L^\infty}}$.

Sum these two inequalities to get

$$\|d_2^*(\chi f)\|_{H_{0,\alpha-1}} \geq c \|\chi f\|_{H_{1,\alpha}},$$

with $c = \min\{c_1, c_2\}$. For the two inequalities to be true and the statement to hold one needs $\alpha < -1$ due to the component χf^\parallel , and $m_0 > \mu > 2\sqrt{\|Ric\|_{L^\infty}}$ for the component χf^\perp . \square

We shall now use lemma 10 to prove the inequality 3.3. This is done as follows. One minimizes the functional

$$J(f) = \|d_2^* f\|_{H_{0,\alpha-1}}^2,$$

subject to the constraint that $\|f\|_{H_{0,\alpha}} = 1$. The minimizer f satisfies an elliptic equation of the kind

$$d_2 d_2^* f + g_1 d_2^* f + g_2 f = 0,$$

where g_1, g_2 are smooth functions. A standard iteration argument in elliptic PDE theory gives an L^∞ bound on the minimizer f , over compact sets. So that one can write $\|f\|_{L^\infty(K)} < C$, for $C > 0$ some constant and K some compact set containing $\cup_{i=1}^k B_{\epsilon_{out}^i}(p_i)$. Denote by $\sigma_{d_2^*}$ the symbol of the operator d_2^* , then

$$\begin{aligned} \|d_2^*(\chi f)\|_{0,H_{\alpha-1}}^2 &= \|\sigma_{d_2^*}(d\chi)f + \chi d_2^* f\|_{H_{0,\alpha-1}}^2 \\ &\leq \|\sigma_{d_2^*}(d\chi)f\|_{H_{0,\alpha-1}}^2 + \|\chi d_2^* f\|_{H_{0,\alpha-1}}^2 \\ &\leq C_2 \|d\chi\|_{H_{0,\alpha-1}}^2 + \|d_2^* f\|_{H_{0,\alpha-1}}^2. \end{aligned}$$

where in the first term one used the bound $\|f\|_{L^\infty} < C$ on f and $C_2 > 0$ denotes some possibly larger constant in order to also bound $|\sigma_{d_2^*}|$. In the last term it was used that $\|\chi d_2^* f\|_{H_{0,\alpha-1}}^2 \leq \|d_2^* f\|_{H_{0,\alpha-1}}^2$.

Now notice that $|d\chi|$ is supported on the union of the annuli $B_{\epsilon_{out}^i}(p_i) \setminus B_{\epsilon_{in}^i}(p_i)$, where $|d\chi| \leq C m_0^{1/2}$.

However, in this region the spaces $H_{0,\alpha-1}$ use the rescaled norm $|(d\chi)_{m_0^{-1}}|_{g_{m_0^{-1}}} \leq C m_0^{-3/2}$, see equation 3.1 and the definitions preceding it. Moreover, having into account the volume of the annuli one obtains $\|d\chi\|_{H_{0,\alpha-1}}^2 \leq C m_0^{-3}$, so that

$$\|d_2^* f\|_{H_{0,\alpha-1}}^2 \geq \|d_2^*(\chi f)\|_{H_{0,\alpha-1}}^2 - C_3 m_0^{-3},$$

for some $C_3 > 0$. Insert here the intermediate inequality from proposition 10

$$\begin{aligned} \|d_2^* f\|_{H_{\alpha-1}}^2 &\geq c \|\chi f\|_{H_{1,\alpha}}^2 - C_3 m_0^{-3} \\ &\geq c \|f\|_{H_{0,\alpha}}^2 - c \sum_{i=1}^k \|f\|_{H_{0,\alpha}(B_{\epsilon_{out}^i}(p_i))}^2 - C_3 m_0^{-3}. \end{aligned}$$

The first term is c as f satisfies the constraint that $\|f\|_{H_{0,\alpha}} = 1$. The last two ones can be made arbitrarily small using the L^∞ bound on f over compact sets and by letting m_0 get big. Hence one obtains the inequality in equation 3.3, then the discussion above it proves the main proposition 5 of this section.

3.3 The Nonlinear Terms

Equipped with the surjectivity of the operator d_2 , proved in proposition 5, we will now apply the Implicit Function Theorem to solve the monopole equation $*F_{A_0+a} - \nabla_{A_0+a}(\Phi_0 + \phi) = 0$. This can be written in the form

$$e_0 + d_2(a, \phi) + N((a, \phi), (a, \phi)) = 0,$$

where e_0 is the error term of the approximate solution (A_0, Φ_0) , d_2 the linearisation of the monopole equation at (A_0, Φ_0) and N is the nonlinear term $N((a, \phi), (b, \psi)) = *\frac{1}{2}[a \wedge b] - \frac{1}{2}[a, \psi] - \frac{1}{2}[b, \phi]$. We shall look for a solutions (a, ϕ) in the image of the right inverse to d_2 , i.e. $(a, \phi) = Qu$ for some u a 1-form with values in \mathfrak{g}_P , such that $u \in H_{0,\beta-1}$, with $\beta > -1$. Then, the equation is

$$(3.6) \quad u + N(Qu, Qu) = -e_0,$$

which is solved using the following particular version of the contraction mapping theorem

Lemma 11. *Let B be a Banach space and $q : B \rightarrow B$ a smooth map such that for all $u, v \in B$*

$$\|q(u) - q(v)\| \leq k(\|u\| + \|v\|)\|u - v\|,$$

for some fixed constant k (i.e. independent of u and v). Then, if $\|v\| \leq \frac{1}{10k}$ there is a unique solution u to the equation

$$(3.7) \quad u + q(u) = v,$$

which satisfies the bound $\|u\| \leq 2\|v\|$.

In this lemma one interprets 3.7 as a fixed point equation. Under those hypothesis, the contraction mapping principle applies and one obtains the estimate by keeping track of the norms in the iterations, see [6]. This will be used to prove

Proposition 6. *Let $k \in \mathbb{N}$ and (X, g) be asymptotically conical with $b^2(X) = 0$. Then, there is $\mu > 0$, such that for all $m \in H^0(\Sigma, \mathbb{R})$ with $m \geq \mu$ and $(p, \alpha, \theta) \in X^k(m) \times H^1(X, \mathbb{S}^1) \times \check{\mathbb{T}}^{k-1}$. There is a unique charge k and mass m monopole (A, Φ) , which can be written as $(A, \Phi) = (A_0, \Phi_0) + Qu$ and satisfies*

$$\|(A, \Phi) - (A_0, \Phi_0)\|_{H_{1,-\frac{1}{2}}} \leq Cm^{-\frac{7}{4}},$$

where $(A_0, \Phi_0) = H(p, \alpha, \theta)$ is the approximate solution from proposition 4 and C denotes a constant independent of m .

Lemma 12. *Let $\beta = -1/2 \notin D(d + d^*)$. Then, there is $C > 0$, such that for all $f, g \in H_{1,\beta}$*

$$\|N(f, g)\|_{H_{0,\beta-1}} \leq C\|f\|_{H_{1,\beta}}\|g\|_{H_{1,\beta}}.$$

Proof. That $\beta = -\frac{1}{2} \notin D(d + d^*)$ follows from the first item in lemma 3 and the fact that the Laplacian Δ_Σ has positive eigenvalues. To prove the inequality, let χ be the bump function 3.2 from last section and write $N(f, g)$ as the sum of two components, one supported inside the balls $B_\epsilon(p_i)$ and the other one outside these. We shall prove the inequality separately for each of these two components.

1. If f, g are supported inside the ball $B_{\epsilon_{out}}^i(p_i)$, then using both the Hölder and Sobolev inequalities

$$\begin{aligned}\|N(f, g)\|_{H_{0, \beta-1}} &= \|m_0^{-2}fg\|_{L^2(B_\epsilon(p_i))} \leq C\|m_0^{-1}f\|_{L^4(B_\epsilon(p_i))}\|m_0^{-1}g\|_{L^4(B_\epsilon(p_i))} \\ &\leq C'\|m_0^{-1}f\|_{L_1^2(B_\epsilon(p_i))}\|m_0^{-1}g\|_{L_1^2(B_\epsilon(p_i))} \\ &\leq C''\|f\|_{H_{1, \beta}}\|g\|_{H_{1, \beta}}.\end{aligned}$$

Moreover the constant $C'' > 0$ is independent of m_0 .

2. For f, g supported on the big open set $U = X \setminus \cup_{i=1}^k \overline{B_\epsilon(p_i)}$, we can write $f = f^\parallel + f^\perp$, and similarly for g . Then

$$N(f, g) = \left(N(f^\parallel, g^\perp) + N(f^\perp, g^\parallel)\right) + N(f^\perp, g^\perp),$$

where the first two terms have values in \mathfrak{g}_P^\perp , while the third has values in \mathfrak{g}_P^\parallel . We shall now bound these separately. For $\beta \leq -\frac{1}{2}$ the Hölder and Sobolev inequalities give

$$\begin{aligned}\|N(f^\perp, g^\parallel)\|_{H_{0, \beta-1}} &\leq C\|\phi_D N(f^\perp, g^\parallel)\|_{L^2} \\ &\leq C\|\rho^{\beta+\frac{3}{2}-\beta-\frac{3}{3}-\frac{3}{6}}\phi_D N(f^\perp, g^\parallel)\|_{L^2} \\ &\leq C'\|\rho^{\beta+\frac{3}{2}-1}\phi_D f^\perp\|_{L^3}\|\rho^{-\beta-\frac{3}{6}}g^\parallel\|_{L^6} \\ &\leq C'\|\phi_D f^\perp\|_{L^3}\|g^\parallel\|_{L_{0, \beta}^6} \\ &\leq C''\|\phi_D f^\perp\|_{L_1^2}\|g^\parallel\|_{L_{1, \beta}^2},\end{aligned}$$

where it was used that $\beta + \frac{3}{2} - 1 = \beta + \frac{1}{2} \leq 0$, for $\beta \leq -\frac{1}{2}$ and that $\|g^\parallel\|_{L_{0, \beta}^6} = \|\rho^{-\beta-\frac{3}{6}}g^\parallel\|_{L^6}$, by definition. Since the weighted H -norm uses standard L_1^2 spaces along the components in \mathfrak{g}_P^\perp and weighted norms along the \mathfrak{g}_P^\parallel one can bound the last item above as

$$\|N(f^\perp, g^\parallel)\|_{H_{0, \beta-1}} \leq C''\|f^\perp\|_{H_{1, \beta}}\|g^\parallel\|_{H_{1, \beta}}.$$

The term $N(f^\parallel, g^\perp)$ follows by a similar computation. To evaluate the other term, which lies in \mathfrak{g}_P^\parallel note that for $\beta \geq -\frac{1}{2}$

$$\begin{aligned}\|N(f^\perp, g^\perp)\|_{H_{0, \beta-1}} &\leq C\|\rho^{-\beta+1-\frac{3}{2}}N(f^\perp, g^\perp)\|_{L^2} \\ &\leq C\|N(f^\perp, g^\perp)\|_{L^2},\end{aligned}$$

hence one can once again use the Hölder and Sobolev inequalities to bound the above quantity by $\frac{1}{m_0^2}\|\phi_D f^\perp\|_{L_1^2}\|\phi_D g^\perp\|_{L_1^2} \leq C\|f\|_{H_{1, \beta}}\|g\|_{H_{1, \beta}}$, for $m_0 > \sqrt{C}$. □

Proof. (of proposition 6) We start by showing that $q(u) = N(Qu, Qu)$ satisfies the hypothesis stated in lemma 11, for $B = H_{0, \beta-1}$ and $\beta = -1/2$. This is done by computing

$$\begin{aligned}\|N(Qu, Qu) - N(Qv, Qv)\|_{H_{0, -\frac{3}{2}}} &= \|N(Q(u+v), Q(u-v))\|_{H_{0, -\frac{3}{2}}} \\ &\leq C\|Q(u+v)\|_{H_{1, -\frac{1}{2}}}\|Q(u-v)\|_{H_{1, -\frac{1}{2}}} \\ &\leq C\left(\|u\|_{H_{0, -\frac{3}{2}}} + \|v\|_{H_{0, -\frac{3}{2}}}\right)\|u-v\|_{H_{0, -\frac{3}{2}}},\end{aligned}$$

where the constant C may possibly change from one line to the other. The only hypothesis of lemma 11 which remains to be checked is that it is possible to make $\|e_0\|_{H_{0,-\frac{3}{2}}} \leq \frac{1}{10C}$. This is an immediate consequence of corollary 5. This states the existence of an approximate monopole $H(p, \alpha, \theta)$, whose error term e_0 satisfies $W_0|e_0|_g \leq cm_0^{-1}$, for some constant $c > 0$ not depending on m_0 . Moreover, $|e_0|$ is supported on $\cup_{i=1}^k B_{\epsilon_{out}^i}(p_i)$

$$\|e_0\|_{H_{0,-\frac{3}{2}}}^2 = \int_X |W_0 e_0|^2 \leq ck m_0^{-2} \text{Vol}(B_{\epsilon_{out}^i}(p_i)) \leq ck m_0^{-7/2}$$

and so for big m_0 one obtains a solution u to equation 3.6. In fact from lemma 11, there is a constant c' such that this is the unique solution u to equation 3.6 with $\|u\|_{H_{0,-\frac{3}{2}}} \leq c' m_0^{-7/4}$. Then by setting $(A, \Phi) = (A_0 + a, \Phi_0 + \phi)$, where $(a, \phi) = Qu$, one obtains a monopole which is the unique one of this form satisfying

$$\|(A, \Phi) - (A_0, \Phi_0)\|_{H_{1,-\frac{1}{2}}} \leq C m_0^{-7/4}.$$

□

4 Uniqueness in a Gauge Orbit

Proposition 6 can be used to make the following

Definition 10. Let $k \in \mathbb{N}$ and (X, g) be asymptotically conical with $b^2(X) = 0$ and $m \in H^0(\Sigma, \mathbb{R}) \cong \mathbb{R}$, be such that $m \geq \mu$, so that proposition 6 applies and defines a map

$$(4.1) \quad \tilde{h} : X^k(m) \times H^1(X, \mathbb{S}^1) \times \check{\mathbb{T}}^{k-1} \rightarrow M_{k,m}.$$

The image of the map \tilde{h} above corresponds to monopoles (A, Φ) , which can be written as $(A_0, \Phi_0) + Qu$, where $\|Qu\|_{H_{1,-1/2}} \leq C m^{-7/4}$ and (A_0, Φ_0) is an approximate solution in the image of the map H from proposition 4. The goal of this section is to show that given (A, Φ) in the image of \tilde{h} , all gauge equivalence classes of monopoles close to (A, Φ) come from this construction, i.e. to show

Theorem 3. The map \tilde{h} descends to a local diffeomorphism

$$(4.2) \quad h : X^k(m) \times H^1(X, \mathbb{S}^1) \times \check{\mathbb{T}}^{k-1} \rightarrow \mathcal{M}_{k,m}.$$

The rest of this section is dedicated to prove this theorem. We shall now introduce some notation, let $P = X^k(m) \times H^1(X, \mathbb{S}^1) \times \check{\mathbb{T}}^{k-1}$ denote the parameter space, $x \in P$ be a point and $(A, \Phi) = \tilde{h}(x)$, then $[(A, \Phi)] = h(x) \in \mathcal{M}_{k,m}$ is the corresponding gauge equivalence class. Moreover, $(A, \Phi) = (A_0, \Phi_0) + Qu$, where $(A_0, \Phi_0) = H(x)$ and $\|Qu\|_{H_{1,-1/2}} \leq C m^{-7/4}$.

Invoking the implicit function theorem, we may take a model for $\mathcal{M}_{k,m}$ as a slice in $C_{k,m}$. This identifies a neighborhood of $h(x)$ in $\mathcal{M}_{k,m}$ with a neighborhood of zero in $\ker(d_1^* \oplus d_2)_{-1/2} \subset T_{\tilde{h}(x)} C_{k,m}$. Since $d_1^* \oplus d_2$ is Fredholm [7], we can fix an orthonormal basis of $\ker(d_1^* \oplus d_2)_{-1/2}$ and define a projection

$$\pi_h : T_{\tilde{h}(x)} C_{k,m} \rightarrow \ker(d_1^* \oplus d_2)_{-1/2}.$$

Then it is enough to show that there are neighborhoods U of $x \in P$ and V of $0 \in \ker(d_1^* \oplus d_2)_{-1/2}$, such that

$$\pi_h \circ \tilde{h} : U \rightarrow V$$

is a diffeomorphism.

Lemma 13. *There is $\mu > 0$ and $R > 0$ such that for all $m \geq \mu$ the map*

$$\pi_h \circ H : P \rightarrow \ker(d_1^* \oplus d_2)_{-1/2},$$

maps an open neighborhood U' of $x \in P$ onto B_R a ball of radius R in $\ker(d_1^ \oplus d_2)_{-1/2} \subset C_{k,m}$, equipped with the metric $H_{1,-1/2}$, i.e.*

$$B_R = \{u \in \ker(d_1^* \oplus d_2)_{-1/2} \mid \|u\|_{H_{1,-1/2}} < R\}.$$

Proof. Notice that given $x \in P$, the map $dH_x : TP \rightarrow T_{H(x)}C_{k,m}$ is injective and we can define the norm

$$\|v\|_P = \|dH_x(v)\|_{H_{1,-\frac{1}{2}}},$$

where $v \in T_xP$ and $dH_x(v) = \frac{d}{dt}\Big|_{t=0} H(\exp_x(tv))$. It follows from [7] that $d_1^* \oplus d_2$ has index $i = 4k + b^1(X) - 1$ (our weight $\beta = -\frac{1}{2}$ corresponds in that reference to the weight $\gamma = -\beta - \frac{3}{2} = -1$). Moreover, the same proof as that of proposition 5 gives that $d_1^* \oplus d_2$ is surjective and so we have $\dim(\ker(d_1^* \oplus d_2)) = \dim(T_xP)$. Hence, in order to show that $\pi_h \circ H$ is an isomorphism it is enough to show that it is injective. In fact we shall prove an inequality of the form

$$(4.3) \quad \|\pi_h \circ dH_x(v) - dH_x(v)\|_{H_{1,\beta}} \leq c(m)\|v\|_P,$$

with $c(m) > 0$ converging to zero as $m \rightarrow +\infty$ and all $v \in T_xP$. To prove this inequality notice that the operator $d_1^* \oplus d_2$ has closed image, $\pi_h(dH_x(v))$ is in its kernel and $\pi_h dH_x(v) - dH_x(v)$ is orthogonal to it. So, there is a constant $c > 0$ such that for all v

$$\|d_1^* \oplus d_2(\pi_h dH_x(v) - dH_x(v))\|_{H_{0,\beta-1}} \geq c\|\pi_h dH_x(v) - dH_x(v)\|_{H_{1,\beta}},$$

and the left hand side is equal to $\|d_1^* \oplus d_2(dH_x(v))\|_{H_{0,\beta-1}}$ which by construction of the map H is no greater than $c(m)\|dH_x(v)\|_{H_{1,\beta}} = c(m)\|v\|_P$, with $c(m) \rightarrow 0$ as $m \rightarrow +\infty$. Hence, for large enough m , the map $\pi_h \circ dH$ is a degree 1 map of a sphere of radius R_1 in T_xP into a topological sphere in the annulus of in $\ker(d_1^* \oplus d_2)$ whose inner and outer radius are $R_1 - c(m)$ and $R_1 + c(m)$ respectively. Hence, by the implicit function theorem, for large m , $\pi_h \circ H$ is a local diffeomorphism and maps an m independent neighborhood U' of x onto a neighborhood of $H(x)$. Moreover, one can use equation 4.3 to prove that there is an inequality

$$\|\pi_h dH_x(v)\|_{H_{1,\beta}} \geq (1 - c(m))\|v\|_P$$

and so the derivative $\pi_h \circ dH_x$ is uniformly bounded by below for large m . This implies that for large m , $\pi_h \circ H(U')$ is a neighborhood of $\pi_h \circ H(x)$ with a fixed size, in particular it contains a ball of a fixed radius B_R around $\pi_h(H(x))$. \square

Recalling the discussion preceding the statement of the previous lemma, we shall now go on to prove that there are U and V such that

$$\pi_h \circ \tilde{h} : U \rightarrow V$$

is a diffeomorphism. We write $\tilde{h}(x) = H(x) + Qu$ with Qu given by proposition 6. Hence, it follows from elliptic regularity that we can obtain $H_{k,-\frac{1}{2}}$ bounds in on Qu for all k , Moreover, these are such that converge to 0 as $m \rightarrow \infty$, in particular

$$\|Qu\|_{H_{1,-\frac{1}{2}}} \leq Cm^{-\frac{7}{4}}.$$

Hence $\pi_h(\tilde{h}(x)) = \pi_h(H(x)) + \pi_h(Qu)$ and $\|\pi_h(Qu)\|_{H_{1,-\frac{1}{2}}}$ is very small. It follows that the equation $\pi_h(\tilde{h}(x)) = y$, for y contained in a ball of radius $R' < R$ always has a unique solution provided that $Cm^{-\frac{7}{4}} < R - R'$. This can always be achieved by taking m to be large enough, namely

$$m > \mu = \left(\frac{C}{R - R'} \right)^{\frac{7}{4}}.$$

Hence we can take $V = B_{R'}$ and $U \subset U'$ to be its inverse image via $\pi_h \circ \tilde{h}$. This finishes the proof of theorem 3.

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