Web-based Supporting Materials for "Augmented Beta rectangular regression models: A Bayesian perspective" by Jue Wang and Sheng Luo

1 Simulation results for Settings I and II

Table 1: Simulation results when data are simulated from the BEOI regression models (Setting I).

	BEOI regression model				OABR regression model			
	Bias	SD	CP	RMSE	Bias	SD	CP	RMSE
$\omega_0 = -1.5$	−0.002	0.070	0.950	0.070	−0.002	0.069	0.945	0.069
$\omega_1 = -0.5$	0.005	0.102	0.930	0.102	0.006	0.104	0.940	0.104
$\beta_0 = 1.5$	0.001	0.033	0.940	0.033	0.000	0.033	0.940	0.033
$\beta_1 = -0.5$	−0.004	0.044	0.935	0.044	−0.005	0.044	0.940	0.044
$\beta_2 = -0.1$	−0.000	0.006	0.950	0.006	−0.000	0.006	0.955	0.006
$\beta_3 = 0.2$	0.001	0.008	0.925	0.008	0.001	0.008	0.910	0.008
$\eta_0 = 2.5$	−0.004	0.056	0.945	0.056	0.000	0.056	0.930	0.056
$\eta_1 = 0.2$	0.001	0.078	0.940	0.078	−0.002	0.078	0.940	0.078
$\eta_2 = 0.1$	0.000	0.016	0.945	0.016	0.000	0.016	0.945	0.016
$\eta_3 = 0.1$	−0.001	0.022	0.945	0.022	−0.000	0.022	0.940	0.022
$\sigma_1 = 1.2$	0.000	0.056	0.930	0.056	−0.002	0.058	0.925	0.058
$\sigma_2 = 0.6$	−0.002	0.015	0.960	0.015	−0.002	0.015	0.945	0.015
$\sigma_3 = 0.4$	−0.004	0.032	0.935	0.032	−0.009	0.032	0.930	0.034
$\rho_{12} = 0.4$	−0.005	0.043	0.945	0.043	−0.006	0.043	0.945	0.043
$\rho_{13} = 0.1$	−0.004	0.095	0.965	0.095	−0.005	0.098	0.965	0.098
$\rho_{23} = 0.4$	0.003	0.060	0.965	0.060	−0.000	0.059	0.960	0.059
$\alpha = 0$					0.002	0.001		0.002

Table 2: Simulation results when data are simulated from the OABR regression models (Setting II).

	BEOI regression model				OABR regression model			
	Bias	SD	CP	RMSE	Bias	SD	CP	RMSE
$\omega_0 = -1.5$	-0.002	0.069	0.930	0.069	-0.006	0.072	0.935	0.072
$\omega_1 = -0.5$	0.002	0.093	0.940	0.093	0.005	0.096	0.945	0.095
$\beta_0 = 1.5$	0.003	0.041	0.940	0.041	0.001	0.032	0.970	0.032
$\beta_1 = -0.5$	-0.030	0.054	0.905	0.062	-0.006	0.042	0.955	0.043
$\beta_2 = -0.1$	-0.004	0.007	0.865	0.008	-0.000	0.006	0.970	0.006
$\beta_3 = 0.2$	0.010	0.010	0.780	0.014	0.001	0.007	0.955	0.007
$\eta_0 = 2.5$	-0.327	0.070	0.000	0.334	0.012	0.060	0.950	0.061
$\eta_1 = 0.2$	-0.144	0.099	0.640	0.174	-0.014	0.091	0.950	0.092
$\eta_2 = 0.1$	-0.044	0.018	0.220	0.048	-0.002	0.017	0.970	0.017
$\eta_3 = 0.1$	0.001	0.028	0.900	0.028	0.003	0.026	0.950	0.026
$\sigma_1 = 1.2$	0.001	0.054	0.970	0.054	0.002	0.053	0.965	0.053
$\sigma_2 = 0.6$	0.064	0.016	0.015	0.066	-0.001	0.017	0.930	0.017
$\sigma_3 = 0.4$	0.499	0.026	0.000	0.499	-0.015	0.046	0.940	0.048
$\rho_{12} = 0.4$	-0.022	0.045	0.915	0.050	-0.003	0.045	0.930	0.045
$\rho_{13} = 0.1$	0.039	0.059	0.895	0.071	-0.001	0.121	0.935	0.121
$\rho_{23} = 0.4$	0.254	0.026	0.000	0.256	-0.001	0.076	0.935	0.076
$\alpha = 0.2$					0.002	0.012	0.955	0.012

2 Zero-one augmented Beta rectangular (ZOABR) model

In this section, we discuss in details how to generalize the proposed OABR regression model to zero-one augmented BR (ZOABR) regression model which accounts for both zero and one boundary values. In model (3), we add a parameter p_{0ij} representing the probability of observing 0 for EQ-VAR from subject i at visit j.

$$f(Y_{ij} = y_{ij}|p_{0ij}, p_{1ij}, \gamma_{ij}, \phi_{ij}, \theta) = \begin{cases} p_{0ij} & \text{if } y_{ij} = 0 \\ p_{1ij} & \text{if } y_{ij} = 1 \\ (1 - p_{0ij} - p_{1ij}) f(Y_{ij} = y_{ij}|\gamma_{ij}, \phi_{ij}, \theta) & \text{if } y_{ij} \in (0, 1). \end{cases}$$

Similar to model (4), we propose ZOABR regression model by regressing the covariates onto p_{0ij},

p_{1ij}, γ_{ij}, and ϕ_{ij} after appropriate transformation by some link functions:

$$\begin{aligned}
&\text{logit}[p_{0ij} = P(y_{ij} = 0|u_{i0})] = \boldsymbol{X}_{i0}\boldsymbol{\omega_0} + u_{i0} \\
&\text{logit}[p_{1ij} = P(y_{ij} = 1|u_{i1})] = \boldsymbol{X}_{i1}\boldsymbol{\omega_1} + u_{i1} \\
&\text{logit}(\gamma_{ij}|u_{i2}) = \boldsymbol{X}_{i2}\boldsymbol{\beta} + u_{i2} \\
&\log(\phi_{ij}|u_{i3}) = \boldsymbol{X}_{i3}\boldsymbol{\eta} + u_{i3},
\end{aligned}$$

where the covariate vectors $\boldsymbol{X}_{i0}$, $\boldsymbol{X}_{i1}$, $\boldsymbol{X}_{i2}$, and $\boldsymbol{X}_{i3}$ can be identical or different. The random effect vector $\boldsymbol{u}_i = (u_{i0}, u_{i1}, u_{i2}, u_{i3})'$ follows a multivariate normal distribution $N_4(0, \boldsymbol{\Sigma})$, where the covariance matrix

$$\boldsymbol{\Sigma} = \left\{ \begin{array}{cccc} \sigma_1^2 & \rho_{12}\sigma_1\sigma_2 & \rho_{13}\sigma_1\sigma_3 & \rho_{14}\sigma_1\sigma_4 \\ \rho_{12}\sigma_1\sigma_2 & \sigma_2^2 & \rho_{23}\sigma_2\sigma_3 & \rho_{24}\sigma_2\sigma_4 \\ \rho_{13}\sigma_1\sigma_3 & \rho_{23}\sigma_2\sigma_3 & \sigma_3^2 & \rho_{34}\sigma_3\sigma_4 \\ \rho_{14}\sigma_1\sigma_4 & \rho_{24}\sigma_2\sigma_4 & \rho_{34}\sigma_3\sigma_4 & \sigma_4^2 \end{array} \right\}. \tag{1}$$

Note that the ZOABR regression model requires an additional constraint: $0 < p_{0ij} + p_{1ij} < 1$. To impose this constraint, one can reparameterize p_{0ij} and p_{1ij} as $p_{0ij} = \nu_{0ij}(1 - \nu_{1ij})$ and $p_{1ij} = \nu_{0ij}\nu_{1ij}$, where $0 < \nu_{0ij}, \nu_{1ij} < 1$, and then regress covariates on ν_{0ij} and ν_{1ij} after transformation by some link function (e.g., logit). Under this reparameterization, $0 < p_{0ij} + p_{1ij} = \nu_{0ij} < 1$. However, the interpretation of ν_{0ij} and ν_{1ij} is not as intuitive as that of p_{0ij} and p_{1ij}. As pointed out by one reviewer, it is more attractive for applied researchers to directly regress on the original p_{0ij} and p_{1ij}. In the Bayesian framework, we can impose the constraint of $0 < p_{0ij} + p_{1ij} < 1$ during the MCMC sampling by skipping the MCMC cycles in which $p_{0ij} + p_{1ij} \geq 1$ for $i = 1, \ldots, I$ and $j = 1, \ldots, J_i$. There are also different frequentist approaches (e.g., Lagrange multipliers) to impose the constraint, but it is out of the scope of this article. To ensure that this constraint can be imposed within the MCMC sampling, we conduct a simulation study similar to simulation setting I, in which data are simulated from the zero-one inflated Beta regression model (BEINF model in the GAMLSS family) and are fitted by the BEINF and ZOABR regression models. The simulation results in Web Table 3 suggest that the ZOABR regression model can correctly estimate all parameters with small bias and the coverage probability being reasonably close to 0.95. The estimate of the shape parameter α is correctly close to zero, suggesting that the overparameterized ZOABR regression model can reduce to the BEINF regression model. To facilitate easy reading and implementation of the ZOABR regression model, the `Stan` codes have been posted in the Web Supplement.

Table 3: Simulation results when data are simulated from the BEINF regression model.

	BEINF regression model				ZOABR regression model			
	Bias	SD	CP	RMSE	Bias	SD	CP	RMSE
$\omega_{01} = -2$	-0.001	0.066	0.930	0.065	-0.004	0.063	0.940	0.063
$\omega_{02} = 0.5$	-0.000	0.089	0.920	0.089	0.006	0.086	0.930	0.086
$\omega_{11} = -1.5$	0.006	0.049	0.940	0.049	0.003	0.049	0.960	0.049
$\omega_{12} = -0.5$	-0.001	0.074	0.970	0.074	0.001	0.071	0.980	0.071
$\beta_0 = 1.5$	0.000	0.034	0.960	0.034	-0.001	0.035	0.970	0.035
$\beta_1 = -0.5$	-0.002	0.048	0.950	0.048	-0.008	0.047	0.940	0.047
$\beta_2 = -0.1$	-0.001	0.005	0.980	0.005	-0.001	0.005	0.970	0.005
$\beta_3 = 0.2$	0.001	0.007	0.950	0.007	0.001	0.007	0.960	0.007
$\eta_0 = 2.5$	-0.015	0.063	0.930	0.064	-0.010	0.066	0.950	0.066
$\eta_1 = 0.2$	0.012	0.081	0.940	0.082	0.012	0.086	0.940	0.086
$\eta_2 = 0.1$	0.004	0.016	0.970	0.017	0.003	0.016	0.990	0.017
$\eta_3 = 0.1$	-0.005	0.023	0.950	0.023	-0.005	0.022	0.970	0.022
$\sigma_1 = 0.8$	0.003	0.047	0.970	0.047	-0.004	0.052	0.940	0.051
$\sigma_2 = 0.8$	-0.012	0.050	0.930	0.051	-0.004	0.052	0.950	0.051
$\sigma_3 = 0.6$	-0.000	0.016	0.940	0.016	-0.001	0.015	0.950	0.015
$\sigma_4 = 0.4$	-0.012	0.036	0.940	0.038	-0.012	0.037	0.920	0.039
$\rho_{12} = -0.4$	0.013	0.075	0.950	0.076	0.025	0.074	0.940	0.078
$\rho_{13} = -0.4$	0.005	0.059	0.940	0.059	-0.000	0.053	0.970	0.052
$\rho_{14} = -0.2$	0.004	0.139	0.930	0.138	0.000	0.152	0.920	0.151
$\rho_{23} = 0.4$	0.004	0.057	0.920	0.056	-0.002	0.049	0.950	0.049
$\rho_{24} = 0.2$	0.004	0.135	0.950	0.134	-0.017	0.125	0.950	0.125
$\rho_{34} = 0.2$	0.006	0.067	0.940	0.067	-0.005	0.072	0.930	0.072
$\alpha = 0$					0.002	0.001		0.002

3 Sensitivity analysis results

To ensure that the Bayesian inference of the proposed model is not sensitive to the choices of the prior distributions and hyperparameters, we conduct a sensitivity analysis with various hyperparameters in the simulation setting I, where data are simulated from the BEOI regression model. As described in Section 4.1, the prior distributions of all elements in $\boldsymbol{\omega}$, $\boldsymbol{\beta}$ and $\boldsymbol{\eta}$ are $N(0, \tau^2)$. We use the prior distribution $\sigma_1, \sigma_2, \sigma_3 \sim \text{Inverse-Gamma}(\lambda_1, \lambda_2)$ to ensure positivity. The prior distribution for ρ's is $\rho \sim \text{Uniform}(-1, 1)$. The sensitivity analysis settings are as follows:

- Setting I: $\tau = 5, \lambda_1 = \lambda_2 = 0.0005$
- Setting II: $\tau = 20, \lambda_1 = \lambda_2 = 0.01$
- Setting III: $\tau = 30, \lambda_1 = \lambda_2 = 0.005$
- Setting IV: $\tau = 50, \lambda_1 = \lambda_2 = 0.0001$

Results for part of the regression parameters are summarized in Web Table 4. The very similar estimation results in all four settings suggest that our statistical inference is not sensitive to the choices of the prior distributions and hyperparameters.

Table 4: Sensitivity analysis when data are simulated from the BEOI regression model.

	BEOI regression model				OABR regression model			
	Bias	SD	CP	RMSE	Bias	SD	CP	RMSE
Sensitivity analysis Setting I								
$\omega_0 = -1.5$	-0.003	0.066	0.980	0.066	-0.001	0.065	0.970	0.065
$\omega_1 = -0.5$	0.003	0.098	0.970	0.097	0.005	0.097	0.970	0.096
$\beta_0 = 1.5$	0.002	0.033	0.930	0.032	0.001	0.033	0.940	0.033
$\beta_1 = -0.5$	-0.006	0.045	0.920	0.045	-0.006	0.045	0.930	0.045
$\eta_0 = 2.5$	-0.008	0.053	0.940	0.053	-0.007	0.054	0.950	0.054
$\eta_1 = 0.2$	0.010	0.078	0.950	0.078	0.012	0.079	0.950	0.079
$\alpha = 0$					0.002	0.001		0.002
Sensitivity analysis Setting II								
$\omega_0 = -1.5$	-0.004	0.065	0.980	0.065	-0.001	0.062	0.990	0.062
$\omega_1 = -0.5$	0.003	0.096	0.970	0.096	0.001	0.096	0.980	0.095
$\beta_0 = 1.5$	0.001	0.033	0.950	0.033	0.001	0.033	0.940	0.033
$\beta_1 = -0.5$	-0.006	0.045	0.920	0.045	-0.006	0.045	0.920	0.045
$\eta_0 = 2.5$	-0.008	0.053	0.950	0.054	-0.006	0.054	0.940	0.054
$\eta_1 = 0.2$	0.010	0.079	0.940	0.079	0.010	0.079	0.940	0.080
$\alpha = 0$					0.002	0.001		0.002
Sensitivity analysis Setting III								
$\omega_0 = -1.5$	-0.003	0.067	0.980	0.066	-0.002	0.066	0.960	0.066
$\omega_1 = -0.5$	0.006	0.096	0.970	0.096	0.004	0.098	0.980	0.097
$\beta_0 = 1.5$	0.001	0.033	0.940	0.033	0.002	0.033	0.950	0.033
$\beta_1 = -0.5$	-0.005	0.045	0.930	0.045	-0.007	0.045	0.940	0.045
$\eta_0 = 2.5$	-0.008	0.053	0.940	0.054	-0.006	0.053	0.950	0.053
$\eta_1 = 0.2$	0.010	0.078	0.950	0.079	0.010	0.080	0.950	0.080
$\alpha = 0$					0.002	0.001		0.002
Sensitivity analysis Setting IV								
$\omega_0 = -1.5$	-0.003	0.067	0.970	0.066	-0.003	0.066	0.970	0.066
$\omega_1 = -0.5$	0.004	0.096	0.990	0.096	0.005	0.096	0.970	0.096
$\beta_0 = 1.5$	0.001	0.033	0.950	0.033	0.001	0.033	0.930	0.033
$\beta_1 = -0.5$	-0.006	0.045	0.920	0.045	-0.005	0.045	0.930	0.045
$\eta_0 = 2.5$	-0.009	0.053	0.950	0.053	-0.006	0.054	0.940	0.054
$\eta_1 = 0.2$	0.010	0.078	0.960	0.079	0.010	0.079	0.940	0.080
$\alpha = 0$					0.002	0.001		0.002

Stan code for fitting the OABR regression model (4)

```
data{
  int<lower=0> N; // Number of subjects
  int<lower=0> obs; // Number of observations
  int<lower=0> subject[obs]; // Subject ID
  real<lower=0, upper=1> Y[obs];
  int<lower=0> treat[obs];
  real<lower=0> time[obs];
  vector[3] zero;
}
parameters{
  real beta[4];
  real omega[2];
  real eta[4];
  real<lower=0, upper=1> alpha;
  vector[3] U[N];
  real<lower=0> sig1;
  real<lower=0> sig2;
  real<lower=0> sig3;
  real<lower=-1, upper=1> rho12;
  real<lower=-1, upper=1> rho13;
  real<lower=-1, upper=1> rho23;
}
transformed parameters{
  real<lower=0, upper=1> gamma[obs];
  real<lower=0, upper=1> theta[obs];
  real<lower=0> phi[obs];
  real<lower=0, upper=1> mu[obs];
  real<lower=0, upper=1> p[obs];
  cov_matrix[3] Sigma_U;
  // Regress on BR mean, precision phi and probability of observing 1
  for (i in 1:obs) {
    p[i] <- inv_logit(omega[1] + omega[2]*treat[i] + U[subject[i], 1]);
    gamma[i] <- inv_logit(beta[1] + beta[2]*treat[i] + beta[3]*time[i] + beta[4]*treat[i]*time[i] + U[subject[i], 2]);
    phi[i] <- exp(eta[1] + eta[2]*treat[i] + eta[3]*time[i] + eta[4]*treat[i]*time[i] + U[subject[i], 3]);
    theta[i] <- alpha*(1 - 2*fabs(gamma[i]-0.5));
    mu[i] <- (gamma[i] - 0.5*theta[i])/(1-theta[i]);
  }
  // Construct the variance-covariance matrix
  Sigma_U[1,1] <- sig1*sig1;
  Sigma_U[1,2] <- rho12*sig1*sig2;
  Sigma_U[1,3] <- rho13*sig1*sig3;
  Sigma_U[2,1] <- rho12*sig1*sig2;
  Sigma_U[2,2] <- sig2*sig2;
  Sigma_U[2,3] <- rho23*sig2*sig3;
  Sigma_U[3,1] <- rho13*sig1*sig3;
  Sigma_U[3,2] <- rho23*sig2*sig3;
  Sigma_U[3,3] <- sig3*sig3;
}
model{
  real u[obs];
  real v[obs];
  real LL[obs];
  // Construct random effects
  for (i in 1:N) U[i] ~ multi_normal(zero, Sigma_U);
  // Reparameterization of BR distribution
  for (i in 1:obs) {
    u[i] <- mu[i] * phi[i];
    v[i] <- (1-mu[i]) * phi[i];
```

```
  }
  // Construct full conditional log likelihood
  for (i in 1:obs) {
    if (Y[i]==1)
      LL[i] <- log(p[i]); // One-augmented data
    else
      LL[i] <- log((1-p[i])*(theta[i] + (1-theta[i])*exp(beta_log(Y[i], u[i], v[i]))));
  }
  increment_log_prob(LL);
  // prior for regression coefficients
  beta ~ normal(0, 10);
  omega ~ normal(0, 10);
  eta ~ normal(0, 10);
  // prior for shape parameter
  alpha ~ uniform(0, 1);
  // prior for variances of random effects
  sig1 ~ inv_gamma(0.01, 0.01);
  sig2 ~ inv_gamma(0.01, 0.01);
  sig3 ~ inv_gamma(0.01, 0.01);
  rho12 ~ uniform(-1, 1);
  rho13 ~ uniform(-1, 1);
  rho23 ~ uniform(-1, 1);
}
```

Stan code for fitting the ZOABR regression model

```
data{
  int<lower=0> N; // Number of subjects
  int<lower=0> obs; // Number of observations
  int<lower=0> subject[obs]; // Subject ID
  real<lower=0, upper=1> Y[obs];
  int<lower=0> treat[obs];
  real<lower=0> time[obs];
  vector[4] zero;
}
parameters{
  real beta[4];
  real omega0[2];
  real omega1[2];
  real eta[4];
  real<lower=0, upper=1> alpha;
  vector[4] U[N];
  real<lower=0> sig1;
  real<lower=0> sig2;
  real<lower=0> sig3;
  real<lower=0> sig4;
  real<lower=-1, upper=1> rho12;
  real<lower=-1, upper=1> rho13;
  real<lower=-1, upper=1> rho14;
  real<lower=-1, upper=1> rho23;
  real<lower=-1, upper=1> rho24;
  real<lower=-1, upper=1> rho34;
}
transformed parameters{
  real<lower=0, upper=1> gamma[obs];
  real<lower=0, upper=1> theta[obs];
  real<lower=0> phi[obs];
  real<lower=0, upper=1> mu[obs];
  real<lower=0, upper=1> p0[obs];
  real<lower=0, upper=1-p0[obs]> p1[obs];
  cov_matrix[4] Sigma_U;
  // Regress on BR mean, precision phi and probability of observing 1
  for (i in 1:obs) {
    p0[i] <- inv_logit(omega0[1] + omega0[2]*treat[i] + U[subject[i], 1]);
    p1[i] <- inv_logit(omega1[1] + omega1[2]*treat[i] + U[subject[i], 2]);
    gamma[i] <- inv_logit(beta[1] + beta[2]*treat[i] + beta[3]*time[i] + beta[4]*treat[i]*time[i] + U[subject[i], 3]);
    phi[i] <- exp(eta[1] + eta[2]*treat[i] + eta[3]*time[i] + eta[4]*treat[i]*time[i] + U[subject[i], 4]);
    theta[i] <- alpha*(1 - 2*fabs(gamma[i]-0.5));
    mu[i] <- (gamma[i] - 0.5*theta[i])/(1-theta[i]);
  }
  // Construct the variance-covariance matrix
  Sigma_U[1,1] <- sig1*sig1;
  Sigma_U[1,2] <- rho12*sig1*sig2;
  Sigma_U[1,3] <- rho13*sig1*sig3;
  Sigma_U[1,4] <- rho14*sig1*sig4;
  Sigma_U[2,1] <- rho12*sig1*sig2;
  Sigma_U[2,2] <- sig2*sig2;
  Sigma_U[2,3] <- rho23*sig2*sig3;
  Sigma_U[2,4] <- rho24*sig2*sig4;
  Sigma_U[3,1] <- rho13*sig1*sig3;
  Sigma_U[3,2] <- rho23*sig2*sig3;
  Sigma_U[3,3] <- sig3*sig3;
  Sigma_U[3,4] <- rho34*sig3*sig4;
  Sigma_U[4,1] <- rho14*sig1*sig4;
```

```
  Sigma_U[4,2] <- rho24*sig2*sig4;
  Sigma_U[4,3] <- rho34*sig3*sig4;
  Sigma_U[4,4] <- sig4*sig4;
}
model{
  real u[obs];
  real v[obs];
  real LL[obs];
  // Construct random effects
  for (i in 1:N) U[i] ~ multi_normal(zero, Sigma_U);
  // Reparameterization of BR distribution
  for (i in 1:obs) {
    u[i] <- mu[i] * phi[i];
    v[i] <- (1-mu[i]) * phi[i];
  }
  // Construct full conditional log likelihood
  for (i in 1:obs) {
    if (Y[i]==0)
      LL[i] <- log(p0[i]); // Zero augmented
    else {
      if (Y[i]==1)
        LL[i] <- log(p1[i]); // One augmented
      else
        LL[i] <- log((1-p0[i]-p1[i])*(theta[i] + (1-theta[i])*exp(beta_log(Y[i], u[i], v[i]))));
    }
  }
  increment_log_prob(LL);
  // prior for regression coefficients
  beta ~ normal(0, 10);
  omega0 ~ normal(0, 10);
  omega1 ~ normal(0, 10);
  eta ~ normal(0, 10);
  // prior for shape parameter
  alpha ~ uniform(0, 1);
  // prior for variances of random effects
  sig1 ~ inv_gamma(0.01, 0.01);
  sig2 ~ inv_gamma(0.01, 0.01);
  sig3 ~ inv_gamma(0.01, 0.01);
  sig4 ~ inv_gamma(0.01, 0.01);
  rho12 ~ uniform(-1, 1);
  rho13 ~ uniform(-1, 1);
  rho14 ~ uniform(-1, 1);
  rho23 ~ uniform(-1, 1);
  rho24 ~ uniform(-1, 1);
  rho34 ~ uniform(-1, 1);
}
```