Lattice point methods for combinatorial games

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April 25, 2011

Submitted for Graduation with Distinction
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Abstract. We encode arbitrary finite impartial combinatorial games in terms of lattice points in rational convex polyhedra. Encodings provided by these *lattice games* can be made particularly efficient for octal games, which we generalize to *squarefree games*. These encompass all heap games in a natural setting where the Sprague–Grundy theorem for normal play manifests itself geometrically. We provide polynomial time algorithms for computing strategies for lattice games provided that they have a certain algebraic structure, called an *affine stratification*.

1. Introduction

A combinatorial game is a two-player game where both players have perfect information and moves are deterministic. The games considered here are impartial, meaning that both players have the same available moves from each position (which rules out Chess), and finite, meaning that the game ends in finitely many moves, although we are interested in families of games in which the totality of the available positions is infinite. The quintessential example of such a family of games is Nim, in which a position consists of a finite set of heaps (of beans, say) of given sizes, and a move is accomplished by removing any number of beans from a single heap.

The *normal play* version of Nim, where the last person to move wins, was solved over a century ago [4]. In fact, Sprague and Grundy showed that solving a finite impartial game in normal play can be reduced to solving Nim [14, 7]. In contrast, in *misère play*, the last player to move loses, as in the standard formulation of Dawson's Chess [6]. Misère games are much more complex than normal play games, as observed by Conway [5]. In particular, a polynomial time solution to Dawson's Chess remains elusive, despite recent exciting advances in misère theory by Plambeck and Siegel [13].

Our goal is to provide a setting in which techniques from the theory of lattice points in polyhedra can be brought to bear on questions from misère combinatorial game theory. Our approach is to encode heap-based games (of which Dawson's Chess is an example) as lattice games (Definition 2.9). The lattice encoding allows for generalized misère play, of which normal play and traditional misère play are merely special cases. We show that a natural class of heap-based games, called *squarefree games* (Definition 4.3), exists for which this lattice encoding is efficient.

Any useful "solution" or "strategy" for a game should be a data structure that can be stored efficiently and processed efficiently to compute a winning move, if one exists, from any position. In this sense, the *misère quotients* invented by Plambeck and Siegel [13] constitute excellent data structures for solutions of misère play impartial games when the quotients are finite. However, misère quotients are difficult to compute when they are infinite. In fact, so far all methods for computing misère quotients only halt with certainty on games with finite quotients. Furthermore, even when an infinite misère quotient is given as a finitely presented monoid, say by generators and relations, there remains a need to record the bipartition of the quotient into $\mathcal{P}$-positions and $\mathcal{N}$-positions.

Informally, the lattice games we develop are games played on game boards consisting of lattice points in polyhedra. The idea of the game is that a token is placed on a lattice point, and a set of vectors is given as legal moves, so that during each player's turn, the player picks up the token and moves it by one of the vectors as long as the move lands the token within the game board. Just as Strong games, also known as Maker-Maker games (like Tic-Tac-Toe), and Maker-Breaker games can be encoded on hypergraphs, so can arbitrary finite impartial games be encoded as lattice games ([8, Theorem 5.1]). A game board and rule set uniquely determine the sets of winning and losing positions (Theorem 3.6).

Methods from combinatorial commutative algebra provide clues as to how to express the presence of structure in the sets of winning positions of games. We conjecture that every lattice game has

- a *rational strategy*: the generating function for its winning positions is a ratio of polynomials with integer coefficients (Conjecture 5.7); and
- an *affine stratification*: an expression of its winning positions as a finite disjoint union of finitely generated modules for affine semigroups (Conjecture 6.2).

By Theorem 7.1, a rational strategy can be efficiently computed from an affine stratification, so the second conjecture implies the first, but it is the first that gives an actual data structure exhibiting the two fundamental properties listed above for a successful strategy (Theorem 5.6). However, affine stratifications reflect real, observed phenomena, and we intend affine stratifications to serve as vehicles for producing rational strategies, and they have already been useful as such in examples.

2. Lattice games

Lattice games are played on game boards consisting of lattice points in polyhedra. We first recall a few core notions from Ziegler's book [15]. A *polytope* is the convex hull of a finite set in $\mathbb{R}^d$, while a *polyhedron* is the intersection of finitely many closed affine halfspaces, each one bounded by an affine subspace. Here, an *affine subspace* is a translate of a vector subspace. A polyhedron is *rational* if it has an expression in terms of halfspaces defined by linear inequalities with rational coefficients.

It is intuitively true but not trivial to prove that every bounded polyhedron is in fact a polytope [15, Theorem 1.1]. Every polyhedron Π is a Minkowski sum

$$\Pi = P + C = \{\mathbf{p} + \mathbf{c} \mid \mathbf{p} \in P \text{ and } \mathbf{c} \in C\}$$

of a polytope P and a cone C, where a *cone* is an intersection of closed halfspaces each bounded by a linear hyperplane–that is, passing through the origin. Equivalently, a cone is a subset of $\mathbb{R}^d$ closed under sums and under scaling by nonnegative real numbers. A cone is *pointed* if there exists a linear function which is positive on the cone minus the origin. The *cone generated by* $\mathbf{a}_1, \ldots, \mathbf{a}_n$ is the cone

$$\mathbb{R}_{\geq 0}\{\mathbf{a}_1, \ldots, \mathbf{a}_n\} = \{c_1\mathbf{a}_1 + \cdots + c_n\mathbf{a}_n \mid c_1, \ldots, c_n \geq 0\}.$$

Hence, this cone is rational if and only if there is a choice of $\mathbf{a}_1, \ldots, \mathbf{a}_n \in \mathbb{Q}^d$. The cone C in the expression $\Pi = P + C$ is well defined, and is called the *recession cone* of Π.

On the algebra side of things, we make use of affine semigroups, for which a general reference is [12]. A *semigroup* is a set with an associate binary operation, i.e. a group without an identity element or inverses. If the operation has an identity element, then the semigroup is a *monoid*. An *affine semigroup* is a monoid that is isomorphic to a finitely generated subsemigroup of $\mathbb{Z}^d$ for some d. An affine semigroup is *pointed* if the identity is its only invertible element. This occurs precisely when $\mathbb{R}_{\geq 0}A$ is pointed.

Fix a pointed rational convex polyhedron $\Pi \subset \mathbb{R}^d$ with recession cone C of dimension d. Write $\Lambda = \Pi \cap \mathbb{Z}^d$ for the set of integer points in Π.

Example 2.1. The case of primary interest is $\Pi = C = \mathbb{R}^d_+$, so $\Lambda = \mathbb{N}^d$, in which lattice points with nonnegative coordinates represent positions in the game. The class of *heap games* is subsumed in this context: from an initial finite set of heaps of beans, the players take turns changing a heap—whichever they select—into some number of heaps of smaller sizes. The rules of a heap game specify the allowed smaller sizes. The game of NIM follows this pattern: a player is allowed to remove beans from any single heap, thus either creating one heap of strictly smaller size or deleting the heap entirely. In terms of lattice games, a position $\mathbf{p} = (\pi_1, \ldots, \pi_d) \in \mathbb{N}^d$ represents π_i heaps of size i for $i = 1, \ldots, d$. Octal games, quaternary games, hexadecimal games, and so on, are heap games; we will examine these later (under normal play) in the wider context of *squarefree games*, to be defined and analyzed in Section 4.

Moves in lattice games will require some hypotheses in order to guarantee that positions can reach a suitable neighborhood of the zero game. The geometric condition implying this behavior involves the following notion.

Definition 2.2. Given an extremal ray ρ of a cone C, the *negative tangent cone* of C along ρ is $-T_\rho C = -\bigcap_{H \supset \rho} H_+$ where $H_+ \supseteq C$ is the positive closed halfspace bounded by a supporting hyperplane H for C. Equivalently, $-T_\rho C = \bigcap_{H \supset \rho} -H_+ = \bigcap_{H \supset \rho} H_-$.

Throughout this paper, set $\mathbf{0} = (0, \ldots, 0) \in \mathbb{Z}^d$.

Definition 2.3. A finite subset $\Gamma \subset \mathbb{Z}^d \smallsetminus \{\mathbf{0}\}$ is a *rule set* if

1. there exists a linear function on $\mathbb{R}^d$ that is positive on $C \smallsetminus \{\mathbf{0}\}$ and on Γ; and
2. for each ray ρ of C, some vector $\boldsymbol{\gamma}_\rho \in \Gamma$ lies in the negative tangent cone $-T_\rho C$.

Example 2.4. With notation as in Example 2.1, the positions of the game NIM with heaps of size at most 2 correspond to $\mathbb{N}^2$. Each move either removes a 1-heap, removes a 2-heap, or turns a 2-heap into a 1-heap. Hence the rule set Γ consists of $(1,0)$, $(0,1)$, and $(-1,1)$, respectively. The options of $\mathbf{p} = (\pi_1, \pi_2)$ are the elements of the set $(\mathbf{p} - \Gamma) \cap \mathbb{N}^2$. We verify that Γ is a rule set: for condition 1, the function $\ell : \mathbb{Z}^2 \to \mathbb{Z}$ defined by $\ell(x,y) = x + 2y$ is positive on $\mathbb{N}^2 \smallsetminus \{\mathbf{0}\}$ and on Γ; condition 2 is satisfied by the basis vectors in Γ.

Example 2.5 (Heap games)**.** In the situation of Example 2.1, the rule set of a heap game is, by definition, a finite set of vectors $\boldsymbol{\gamma}$ each having the property that all of the nonzero entries of $\boldsymbol{\gamma}$ are negative, except the last nonzero entry of $\boldsymbol{\gamma}$, which equals 1. Therefore, any linear function $\ell = (\ell_1, \ldots, \ell_d)$ is positive on $\mathbb{N}^d \smallsetminus \{\mathbf{0}\}$ and on Γ as long as ℓ_i is positive and sufficiently bigger than ℓ_{i-1} for each i. The tangent cone axiom is satisfied by definition for heaps of a given size i as long as there is a way to act on a heap of that size; that is, as long as some $\boldsymbol{\gamma} \in \Gamma$ has $\gamma_i = 1$.

Lemma 2.6. *The affine semigroup* $\mathbb{N}\Gamma$ *generated by any rule set* Γ *is pointed.*

Proof. The nonzero vectors in $\mathbb{R}_+\Gamma$ all lie on the same side of the hyperplane given by the vanishing of the linear function. □

Remark 2.7. Condition 1 in Definition 2.3 implies more than Lemma 2.6: it implies also that $\mathbb{N}\Gamma$ and Λ point in the same direction. That is, moves, which are elements of $-\Gamma$, bring positions closer to $\mathbf{0}$.

Lemma 2.8. *Any rule set* Γ *induces a partial order* $\preceq$ *on* Λ *with* $\mathbf{p} \preceq \mathbf{q}$ *if* $\mathbf{q} - \mathbf{p} \in \mathbb{N}\Gamma$.

Proof. This follows immediately from the definitions of poset and Lemma 2.6. □

Definition 2.9 (Lattice games)**.** Given the polyhedral set $\Lambda = \Pi \cap \mathbb{Z}^d$, fix a rule set Γ.

- A *game board* $\mathcal{B}$ is the complement in Λ of a finite Γ-order ideal in Λ called the set of *defeated positions*.
- A *lattice game* is defined by a game board and a rule set.
- A position $\mathbf{p} \in \Lambda$ *has a move to* $\mathbf{q} \in \Lambda$ if $\mathbf{p} - \mathbf{q} \in \Gamma$.
- A move from a position $\mathbf{p}$ to $\mathbf{q}$ is *legal* if $\mathbf{q}$ lies on the game board $\mathcal{B}$.
- The *options* of a position are the positions to which it has legal moves.

Example 2.10. The game board for NIM in Example 2.1 is $\mathcal{B} = \mathbb{N}^d$ in normal play. On the other hand, in misère play, we declare $\mathbf{0}$ to be a defeated position—the only defeated position—so the game board is $\mathcal{B} = \mathbb{N}^d \smallsetminus \{\mathbf{0}\}$.

To analyze the complexity of our algorithms for lattice games, we need to define the input complexity of a lattice game.

Definition 2.11 (**Input complexity of a lattice game**)**.** Let $(\Gamma, \mathcal{B})$ be a lattice game with rule set Γ and game board $\mathcal{B}$. Γ may be represented as a $d \times n$ matrix with entries γ_{ij} for $1 \le i \le d$ and $1 \le j \le n$, where $n = |\Gamma|$. The game board $\mathcal{B}$ may be represented by the m generators of the finite Γ-order ideal $\mathcal{D}$, hence a $d \times m$ matrix with entries a_{ij} for $1 \le i \le d$ and $1 \le j \le m$. The *input complexity* of the lattice game is the number of bits needed to represent these $d(m+n)$ numbers, namely

$$d(m+n) + \sum_{i=1}^{d} \left(\sum_{j=1}^{n} \log_2 |\gamma_{ij}| + \sum_{j=1}^{m} \log_2 |a_{ij}| \right),$$

which is $O(d(n \log \gamma + m \log a))$, where $\gamma = \max\{|\gamma_{ij}|\}$ and $a = \max\{|a_{ij}|\}$.

3. Uniqueness of winning and losing positions

In this section, we show that the sets of winning and losing positions of a lattice game are well-defined. This result, and all of the others in this section, hold in full without the tangent cone axiom for rule sets in Definition 2.3. To begin, here is an algebraic—and seemingly non-recursive—definition of winning and losing positions.

Definition 3.1. If G is a lattice game with game board $\mathcal{B}$ and rule set Γ, then $\mathcal{P}$ is the set of *winning positions* (or *P-positions*) of G, and $\mathcal{N}$ is the set of *losing positions* (or *N-positions*) of G, if $\mathcal{P}$ and $\mathcal{N}$ partition $\mathcal{B}$ and $(\mathcal{P} + \Gamma) \cap \mathcal{B} = \mathcal{N}$.

In a two player game involving players A and B, a standard definition of "winning strategy" is a recursive one that says A has a winning strategy if and only if there exists a move by A such that for all moves by B, A has a winning strategy, with the base case where A has a winning move. The justification for the letters P and N is that P-positions are positions where the **P**revious Player has a winning strategy and N-positions are positions where the **N**ext Player has a winning strategy. Therefore, a player wishes to move to a P-position on his or her turn in order to win.

In game-theoretic terms, the equation $(\mathcal{P} + \Gamma) \cap \mathcal{B} = \mathcal{N}$ says that every position on the game board with a move to a winning position is a losing position. This implies other game-theoretic statements about winning and losing positions, such as the following, which looks similar but is strictly weaker.

Proposition 3.2. *If $\mathcal{B}$ is a game board with winning positions $\mathcal{P}$, losing positions $\mathcal{N}$, and rule set Γ, then $(\mathcal{P} - \Gamma) \cap \mathcal{P} = \varnothing$.*

Proof. Suppose $\mathbf{p} \in (\mathcal{P} - \Gamma) \cap \mathcal{P}$. Then $\mathbf{p} = \mathbf{p}' - \boldsymbol{\gamma}$ for some $\mathbf{p}' \in \mathcal{P}$ and some $\boldsymbol{\gamma} \in \Gamma$. Therefore $\mathbf{p}' = \mathbf{p} + \boldsymbol{\gamma} \in (\mathcal{P} + \Gamma) \cap \mathcal{B} = \mathcal{N}$, a contradiction. □

Proposition 3.2 is weaker than Definition 3.1 because it does not force each losing position to possess a move to some winning position. For example, it is possible to change all but finitely many P-positions to N-positions without violating Proposition 3.2, but Definition 3.1 guarantees the existence of infinitely many P-positions.

Next we explore consequences of the compatibility of the rule set and the game board dictated by the positivity axiom in Definition 2.3.

Lemma 3.3. *Every sequence in Λ decreasing with respect to a rule set Γ is finite.*

Proof. Fix an integer linear function ℓ on $\mathbb{Z}^d$ that is positive on Γ. Then $\ell(\mathbf{q})-\ell(\mathbf{p}) = \ell(\mathbf{q}-\mathbf{p}) \geq 0$ whenever $\mathbf{p} \preceq \mathbf{q}$, with equality if and only if $\mathbf{p} = \mathbf{q}$. Therefore ℓ induces a bijection from each Γ-decreasing sequence in Λ to some decreasing sequence in $\mathbb{Z}$. Therefore, it is enough to show that $\ell(\Lambda)$ is bounded below. But this follows from the fact that ℓ is nonnegative on C. ☐

Definition 3.4. Let $T \subseteq \mathbb{Z}^d$. An element $\mathbf{p} \in T$ is Γ-*minimal* if $(\mathbf{p} - \Gamma) \cap T = \varnothing$. An element $\mathbf{p} \in T$ is $\mathbb{N}\Gamma$-*minimal* (or simply *minimal*) if $(\mathbf{p} - \mathbb{N}\Gamma) \cap T = \varnothing$.

Example 3.5. In misère play, we assume $\Pi = C = \mathbb{R}^d_+$, $\Lambda = \mathbb{N}^d$, and $\mathcal{D} = \{\mathbf{0}\}$. In this case, every position has a Γ-path to $\mathbf{0}$ in Λ if and only if every Γ-minimal element of $\mathcal{B}$ lies in Γ. Indeed, if a Γ-minimal element $\mathbf{p} \in \mathcal{B}$ does not lie in Γ, then $\mathbf{p}-\gamma \notin \mathbb{N}^d$ for every $\gamma \in \Gamma$, and hence $\mathbf{p}$ does not have a Γ-path to $\mathbf{0}$. Conversely, suppose every Γ-minimal element of $\mathcal{B}$ lies in Γ. By Lemma 3.3, every $\mathbf{p} \in \mathcal{B}$ has a Γ-path in $\mathbb{N}^d$ to a Γ-minimal element, and hence to $\mathbf{0}$.

Theorem 3.6. *Given a rule set $\Gamma \subset \mathbb{Z}^d$ and a game board $\mathcal{B}$, there exist unique sets $\mathcal{P}$ and $\mathcal{N}$ of winning and losing positions for $\mathcal{B}$.*

Proof. By Lemma 2.8 and Lemma 3.3, $\mathcal{B}$ has Γ-minimal elements; define $\mathcal{P}_1$ to be the set of these elements. Let $\mathcal{N}_1 = (\mathcal{P}_1+\Gamma)\cap\mathcal{B}$. Inductively, having defined $\mathcal{P}_1, \ldots, \mathcal{P}_{n-1}$ and $\mathcal{N}_1, \ldots, \mathcal{N}_{n-1}$ for some $n \geq 2$, let $\mathcal{P}_n$ consist of the Γ-minimal elements of $\mathcal{B}\smallsetminus\mathcal{P}_{n-1}$, and set $\mathcal{N}_n = (\mathcal{P}_n + \Gamma) \cap \mathcal{B}$. In other words,

- $\mathcal{P}_n$ is the set of all positions $\mathbf{p} \in \mathcal{B}$ for which $(\mathbf{p} - \Gamma) \cap \mathcal{B}$ is contained in $\mathcal{N}_{n-1}$;
- $\mathcal{N}_n$ is the set of all positions $\mathbf{p} \in \mathcal{B}$ such that $\mathbf{p} - \boldsymbol{\gamma} \in \mathcal{P}_n$ for some $\boldsymbol{\gamma} \in \Gamma$.

Note that $\mathcal{N}_{n-2} \subseteq \mathcal{N}_{n-1} \Rightarrow \mathcal{P}_{n-1} \subseteq \mathcal{P}_n \Rightarrow \mathcal{N}_{n-1} \subseteq \mathcal{N}_n$, so it follows by induction on n, starting from $\mathcal{N}_0 = \varnothing$, that these containments all hold.

Lemma 3.7. *Let $\mathcal{P} = \bigcup_{k=1}^{\infty} \mathcal{P}_k$ and $\mathcal{N} = \bigcup_{k=1}^{\infty} \mathcal{N}_k$. Then*

$$\mathcal{P} = \{\mathbf{p} \in \mathcal{B} \mid (\mathbf{p} - \Gamma) \cap \mathcal{B} \subseteq \mathcal{N}\}, \text{ and}$$
$$\mathcal{N} = \{\mathbf{p} \in \mathcal{B} \mid \mathbf{p} - \boldsymbol{\gamma} \in \mathcal{P} \text{ for some } \boldsymbol{\gamma} \in \Gamma\}.$$

Proof. If $\mathbf{p} \in \mathcal{P}$, then $\mathbf{p} \in \mathcal{P}_k$ for some k, so $(\mathbf{p} - \Gamma) \cap \mathcal{B} \subseteq \mathcal{N}_{k-1} \subseteq \mathcal{N}$. On the other hand, if $(\mathbf{p} - \Gamma) \cap \mathcal{B} \subseteq \mathcal{N}$ then $(\mathbf{p} - \Gamma) \cap \mathcal{B} \subseteq \mathcal{N}_{k-1}$ for some k, since Γ is finite, so $\mathbf{p} \in \mathcal{P}_k \subseteq \mathcal{P}$.

If $\mathbf{p} \in \mathcal{N}$ then $\mathbf{p} \in \mathcal{N}_k$ for some k, so $\mathbf{p}-\boldsymbol{\gamma} \in \mathcal{P}_k \subseteq \mathcal{P}$ for some $\boldsymbol{\gamma} \in \Gamma$. On the other hand, if $\mathbf{p}-\boldsymbol{\gamma} \in \mathcal{P}$ for some $\boldsymbol{\gamma} \in \Gamma$, then $\mathbf{p}-\boldsymbol{\gamma} \in \mathcal{P}_k$ for some k, so $\mathbf{p} \in \mathcal{N}_k \subseteq \mathcal{N}$. ☐

For the existence claimed by the theorem, we check that the sets $\mathcal{P}$ and $\mathcal{N}$ from the lemma satisfy the axioms for sets of winning and losing positions, respectively.

Lemma 3.8. *With $\mathcal{P}$ and $\mathcal{N}$ as in Lemma 3.7, we have $\mathcal{P} \cap \mathcal{N} = \varnothing$.*

Proof. If $\mathbf{p} \in \mathcal{P} \cap \mathcal{N}$ then $\mathbf{p} \in \mathcal{P}_n \cap \mathcal{N}_{n-1}$ for some n, since the unions defining $\mathcal{P}$ and $\mathcal{N}$ are increasing. But $\mathcal{P}_n \subseteq \mathcal{B} \smallsetminus \mathcal{N}_{n-1}$ by definition. □

Lemma 3.9. *With $\mathcal{P}$ and $\mathcal{N}$ as in Lemma 3.7, we have $\mathcal{P} \cup \mathcal{N} = \mathcal{B}$.*

Proof. Suppose $\mathbf{p}$ is Γ-minimal in $\mathcal{B} \smallsetminus (\mathcal{P} \cup \mathcal{N})$. Then $(\mathbf{p} - \Gamma) \cap \mathcal{B} \subseteq \mathcal{P} \cup \mathcal{N}$. Therefore $\mathbf{p}$ must lie in $\mathcal{P}$ or in $\mathcal{N}$ by Lemma 3.7. □

By Lemma 3.7, it follows immediately that $(\mathcal{P} + \Gamma) \cap \mathcal{B} = \mathcal{N}$.

To prove uniqueness, suppose $\mathcal{B} = \mathcal{P} \cup \mathcal{N} = \mathcal{P}' \cup \mathcal{N}'$, where $\mathcal{P}, \mathcal{N}$ and $\mathcal{P}', \mathcal{N}'$ are pairs of winning and losing positions. First, suppose $\mathcal{P} \cap \mathcal{N}' = \varnothing = \mathcal{P}' \cap \mathcal{N}$. The first equality implies $\mathcal{N}' \subseteq \mathcal{N}$ while the second equality implies $\mathcal{N} \subseteq \mathcal{N}'$. Hence $\mathcal{N} = \mathcal{N}'$ and thus $\mathcal{P} = \mathcal{P}'$. Now suppose, by symmetry, that $\mathcal{P} \cap \mathcal{N}' \neq \varnothing$. Let $\mathbf{p} \in \mathcal{P} \cap \mathcal{N}'$. If $\mathbf{p}$ is not Γ-minimal in $\mathcal{B}$, then since $\mathbf{p} \in \mathcal{N}'$, there is some $\boldsymbol{\gamma} \in \Gamma$ such that $\mathbf{p} - \boldsymbol{\gamma} \in \mathcal{P}'$; and since $\mathbf{p} \in \mathcal{P}$, we have $\mathbf{p} - \boldsymbol{\gamma} \in \mathcal{N}$ (note that $\mathbf{p} - \boldsymbol{\gamma} \in \mathcal{B}$ since $\mathcal{P}' \subseteq \mathcal{B}$). Thus $\mathbf{q} = \mathbf{p} - \boldsymbol{\gamma} \in \mathcal{P}' \cap \mathcal{N}$. Continuing in this manner and applying Lemma 3.3, we reduce to the case where $\mathbf{p}$ is Γ-minimal in $\mathcal{B}$. But then $\mathbf{p} \in \mathcal{P}'$, contradicting $\mathbf{p} \in \mathcal{N}'$. Therefore $\mathcal{P} \cap \mathcal{N}' = \varnothing$ and hence $\mathcal{P} = \mathcal{P}'$ and $\mathcal{N} = \mathcal{N}'$. This completes the proof. □

4. Squarefree games

The notion of *octal game* encompasses quite a broad range of examples, but it can sometimes feel contrived, such as when coincidences between the rules and certain heap sizes cause positions with nonempty collections of heaps that nonetheless have no options (see Example 4.4). Lattice games suggest a natural common generalization of octal games, as well as hexadecimal games and indeed arbitrary heap games that, in particular, automatically avoids the no-option phenomenon. The *squarefree games* we define here are precisely those lattice games played on $\mathbb{N}^d$ such that the Sprague–Grundy theorem for normal play finite impartial games is commensurate with the coordinates placed on positions by virtue of the game being on $\mathbb{N}^d$.

In this section, we assume that $\Pi = C = \mathbb{R}^d_+$ and hence $\Lambda = \mathbb{N}^d$. The following notation will come in handy a few times.

Definition 4.1. Given $\mathbf{v} \in \mathbb{R}^d$, let $\mathbf{v}_+$ and $\mathbf{v}_-$ be the nonnegative vectors with disjoint support such that $\mathbf{v} = \mathbf{v}_+ - \mathbf{v}_-$. That is, $\mathbf{v}_+ = \mathbf{v} \wedge \mathbf{0}$ and $\mathbf{v}_- = -(\mathbf{v} \vee \mathbf{0}) = \mathbf{v}_+ - \mathbf{v}$.

Proposition 4.2. *For a rule set Γ, the following are equivalent.*

1. *For each $\gamma \in \Gamma$, and $p, q \in \mathbb{N}^d$, if $p + q - \gamma \in \mathbb{N}^d$ then $p - \gamma \in \mathbb{N}^d$ or $q - \gamma \in \mathbb{N}^d$.*
2. *If $p + q'$ is an option of $p + q$, then q' is an option of q.*
3. *The maximum entry of each $\gamma \in \Gamma$ is at most 1, and there is at most one such entry.*
4. *The positive part γ_+ is a 0-1 vector with at most one 1, for all $\gamma \in \Gamma$.*
5. *Each move takes away at most one heap of one size.*

Proof. $1 \Leftrightarrow 2$: Assume 1 holds. Suppose $p+q'$ is an option of $p+q$, so $p+q' = p+q-\gamma$ for some $\gamma \in \Gamma$, so $q' = q-\gamma$ and hence is an option of q. Conversely, assume 2 holds. Suppose $p+q-\gamma \in \mathbb{N}^d$. Let $q' = q-\gamma \in \mathbb{N}^d$. Then $p+q'$ is an option of $p+q$, so q' is an option of q, hence $q-\gamma \in \mathbb{N}^d$.

$1 \Rightarrow 3$: First we show that the maximum entry is at most 1. Let $M = \max\{\gamma_1, \dots, \gamma_d\}$, and let $p = \lceil \frac{M}{2} \rceil \mathbb{1}$ where $\mathbb{1}$ is the vector with all entries equal to 1. Then the minimum of the entries of $2p-\gamma$ is 1 of odd M and 0 for even M, and hence $2p-\gamma \in \mathbb{N}^d$. However, the minimum of the entries of $p-\gamma$ is $\lceil \frac{M}{2} \rceil - M$ which is negative since $M > 1$. Hence $p-\gamma \notin \mathbb{N}^d$. Now we show that there is at most one entry that is 1. Suppose for some $\gamma = (\gamma_1, \dots, \gamma_d) \in \Gamma$, more than one entry is 1, say $\gamma_i = \gamma_j = 1$ where $i \neq j$. For each $k = 1, \dots, d$, let e_k be the k-th basis vector. Let $p = \sum_{k \neq j} \max\{\gamma_k, 0\} e_k$ and let $q = e_j$. Then $p+q = \gamma \vee \mathbf{0}$ so $p+q-\gamma \in \mathbb{N}^d$, but $(p-\gamma)_j = -1$ and $(q-\gamma)_i = -1$.

$3 \Rightarrow 1$: Suppose 2 holds. Let $\gamma \in \Gamma$ and let $p, q \in \mathbb{N}^d$ with $p+q-\gamma \in \mathbb{N}^d$. For all i such that $p_i + q_i = 0$, it must be that $p_i = q_i = 0$, so we must have $\gamma_i \geq 0$, hence either $(p-\gamma)_i, (q-\gamma)_i \geq 0$. For all j such that $(p+q)_j > 0$, we have either $p_j > 0$ or $q_j > 0$, say $p_j > 0$, so then $(p-\gamma)_j = p_j - \gamma_j \geq 1 - 1 = 0$. Therefore $p-\gamma \in \mathbb{N}^d$.

$3 \Leftrightarrow 4$: This is straightforward.

$4 \Leftrightarrow 5$: In the notation of Examples 2.1 and 2.5, condition 4 is the translation of condition 4 into the language of heaps. □

Definition 4.3. A rule set Γ is *squarefree* if it satisfies any of the equivalent conditions in Proposition 4.2.

Example 4.4. The historically popular *octal games*, invented by Guy and Smith [10], are heap games in which every move consists of selecting a single heap and, depending on the heap's size and the game's rules, either

1. removing the entire heap;
2. removing some beans from the heap, making it smaller; or
3. removing some beans from the heap, splitting it into two smaller heaps.

A problem arises when there is a heap of size k but the rules do not allow the removal of j beans from a heap, for any $j \leq k$. In this case, we may simply ignore heaps of size k (treat them as heaps of size 0), and octal games naturally become a special class of squarefree games.

Definition 4.5. The *normal play* game board in $\mathbb{N}^d$ for a given rule set is the one with no defeated positions: $\mathcal{D} = \varnothing$.

Our main result for normal play squarefree games is the following theorem.

Theorem 4.6. *Let Γ be a squarefree rule set and let $\mathcal{B}$ be the normal play game board $\mathbb{N}^d$ with winning positions $\mathcal{P}$. If $\mathcal{P}_0 = \mathcal{P} \cap \{0,1\}^d$ then*

$$\mathcal{P} = \mathcal{P}_0 + 2\mathbb{N}^d.$$

Furthermore, there is an $O(2^d|\Gamma|)$ algorithm for computing $\mathcal{P}_0$.

To prove Theorem 4.6, we will need the notion of misère congruence.

Definition 4.7. Two positions $p, q \in \mathcal{B}$ are *(misère) congruent*, written $\mathbf{p} \cong \mathbf{q}$, if

$$(\mathbf{p} + C) \cap \mathcal{P} = \mathbf{p} - \mathbf{q} + (\mathbf{q} + C) \cap \mathcal{P}.$$

In other words, $\mathbf{p} + \mathbf{r} \in \mathcal{P} \Leftrightarrow \mathbf{q} + \mathbf{r} \in \mathcal{P}$ for all $\mathbf{r}$ in the recession cone C of $\mathcal{B}$.

It is elementary to verify that congruence is an equivalence relation, and that it is additive, in the sense that $\mathbf{p} \cong \mathbf{q} \Rightarrow \mathbf{p} + \mathbf{r} \cong \mathbf{q} + \mathbf{r}$ for all $\mathbf{r} \in C \cap \mathbb{Z}^d$. Thus, when $\mathcal{B} = \Lambda = C \cap \mathbb{Z}^d$ is a monoid, the quotient of $\mathcal{B}$ modulo congruence is again a monoid, called the *misère quotient*.

Throughout the remainder of this section, we assume that $\mathcal{B}$ is the normal play game board $\mathbb{N}^d$ with winning positions $\mathcal{P}$, losing positions $\mathcal{N}$, and rule set Γ.

Proposition 4.8. *If* $\mathbf{p} \in \mathcal{B}$*, then* $\mathbf{p} \in \mathcal{P} \Leftrightarrow \mathbf{p} \cong \mathbf{0}$.

Proof. First we show that $p \in \mathcal{P} \Rightarrow p \cong \mathbf{0}$. What we wish to show is that if $p, q \in \mathcal{P}$, then $p + q \in \mathcal{P}$. We proceed by induction. Clearly this is true if $p = q = \mathbf{0}$. Now assume $p \succ \mathbf{0}$ or $q \succ \mathbf{0}$ and inductively that $\hat{p} \in \mathcal{P} \Rightarrow \hat{p} + q \in \mathcal{P}$ for all $\hat{p} \prec p$ and $p + \hat{q} \in \mathcal{P} \Rightarrow p + \hat{q} \in \mathcal{P}$ for all $\hat{q} \prec q$. Let $\gamma \in \Gamma$ such that $p + q - \gamma \in \mathcal{P}$. Then either $p - \gamma \in \Gamma$ or $q - \gamma \in \Gamma$, since Γ is squarefree. Suppose $p - \gamma \in \Gamma$. Then $p - \gamma \in \mathcal{N}$ so there is $\gamma' \in \Gamma$ such that $p - \gamma - \gamma' \in \mathcal{P}$. By our induction hypothesis, $(p + q - \gamma) - \gamma' = (p - \gamma - \gamma') + q \in \mathcal{P}$, so $p + q - \gamma \in \mathcal{N}$. If $q - \gamma \in \Gamma$, then a similar argument still yields $p + q - \gamma \in \mathcal{N}$. Since γ was arbitrary, $p + q \in \mathcal{P}$.

Now suppose $q \in \mathcal{N}$. Then there is $\gamma \in \Gamma$ such that $q - \gamma \in \mathcal{P}$. Hence $p + q - \gamma \in \mathcal{P}$ by what we just proved, so $p + q \in \mathcal{N}$. Therefore $p \cong \mathbf{0}$.

Conversely, suppose $p \cong \mathbf{0}$. Then $p \in \mathcal{P}$ since $\mathbf{0} \in \mathcal{P}$. □

Proposition 4.9. *If* Γ *is squarefree and* $\mathbf{p} \in \mathcal{B}$*, then* $2\mathbf{p} \cong \mathbf{0}$.

Proof. By Proposition 4.8, it suffices to show that $2\mathbf{p} \in \mathcal{P}$. It is clearly true for $\mathbf{p} = \mathbf{0}$. Now suppose $\mathbf{p} \succ \mathbf{0}$ and $2\hat{\mathbf{p}} \in \mathcal{P}$ for all $\hat{\mathbf{p}} \prec \mathbf{p}$. Let $\boldsymbol{\gamma} \in \Gamma$ such that $2\mathbf{p} - \boldsymbol{\gamma} \in \mathcal{B}$. Since Γ is squarefree, $\mathbf{p} - \boldsymbol{\gamma} \in \mathcal{B}$, hence $2(\mathbf{p} - \boldsymbol{\gamma}) \in \mathcal{B}$. By our induction hypothesis, $2\mathbf{p} - \boldsymbol{\gamma} - \boldsymbol{\gamma} = 2(\mathbf{p} - \boldsymbol{\gamma}) \in \mathcal{P}$, hence $2\mathbf{p} - \boldsymbol{\gamma} \in \mathcal{N}$. Since $\boldsymbol{\gamma}$ was arbitrary, $2\mathbf{p} \in \mathcal{P}$. □

Lemma 4.10. *Let n be a positive integer. If* $\mathbf{p} \in \mathbb{Z}^d$*, then there exist unique* $\mathbf{q} \in \mathbb{Z}^d$ *and* $\mathbf{r} \in \{0, \ldots, n-1\}^d$ *such that* $\mathbf{p} = n\mathbf{q} + \mathbf{r}$.

Proof. Let $1 \le i \le d$. There are unique $q_i \in \mathbb{Z}$ and $r_i \in \{1, \ldots, n-1\}$ with $p_i = nq_i + r_i$ by the division algorithm. Thus $\mathbf{q} = (q_1, \ldots, q_d)$ and $\mathbf{r} = (r_1, \ldots, r_d)$ do the job. □

Now we are ready to prove Theorem 4.6.

Proof of Theorem 4.6. Let $\mathbf{w} \in \mathcal{P}_0$. If $\mathbf{p} \in 2\mathbb{N}^d$, by Propositions 4.8 and 4.9, $\mathbf{w} + \mathbf{p} \cong \mathbf{w} \cong \mathbf{0}$ and hence $\mathbf{w} + \mathbf{p} \in \mathcal{P}$. On the other hand, let $\mathbf{w} \in \mathcal{P}$. By Lemma 4.10, we may write $\mathbf{w} = 2\mathbf{p} + \mathbf{q}$ for some $\mathbf{p} \in \mathcal{B}$ and $\mathbf{q} \in \{0, 1\}^d$. By Propositions 4.8 and 4.9, we have $\mathbf{0} \cong \mathbf{w} \cong 2\mathbf{p} + \mathbf{q} \cong \mathbf{q}$, hence $\mathbf{q} \in \mathcal{P}_0$.

This immediately gives an algorithm for computing $\mathcal{P}_0$. We simply recursively compute whether each position in $\{0,1\}^d$ is in $\mathcal{P}$ or not. We can then use dynamic programming to fill in whether each position in this box is in $\mathcal{P}$, and for each position, we need to look at at most $|\Gamma|$ positions. □

Corollary 4.11. *In normal play, if the rule set is squarefree, then the set of winning positions is a finite disjoint union of translates of an affine semigroup.*

Remark 4.12. Without the squarefree hypothesis, Theorem 4.6 can fail. As an example, consider the rule set $\Gamma = \{(1,0),(0,2)\}$. It is straightforward to check that $(0,0),(0,1) \in \mathcal{P}$ but $(0,2),(0,3) \in \mathcal{N}$. In fact, the set of P-positions is

$$\mathcal{P} = \{(0,0),(0,1),(1,2),(1,3),(2,0),(2,1),(3,2),(3,3)\} + 4\mathbb{N}^2.$$

One might suspect that in general normal play, the P-positions have the form

$$\mathcal{P} = \mathcal{P}_0 + m\mathbb{N}^d.$$

However, more exotic behavior arises for the rule set $\Gamma = \{(1,0),(0,1),(2,2)\}$.

5. Rational strategies

Definition 5.1. If $A \subseteq \mathbb{Z}^d$, then the *generating function* for A is

$$f(A;\mathbf{t}) = \sum_{\mathbf{a}\in A} \mathbf{t}^{\mathbf{a}}.$$

If there exists a finite index set I, nonnegative integers $k(i)$, and rational numbers α_i, along with vectors $\mathbf{p}_i, \mathbf{a}_{ij} \in \mathbb{Z}^d$ and $\mathbf{a}_{ij} \neq \mathbf{0}$ for all i,j such that

$$f(A;\mathbf{t}) = \sum_{i\in I} \alpha_i \frac{\mathbf{t}^{\mathbf{p}_i}}{(1-\mathbf{t}^{\mathbf{a}_{i1}})\cdots(1-\mathbf{t}^{\mathbf{a}_{ik(i)}})},$$

then $f(A;\mathbf{t})$ is a *rational generating function* for A. When computing a rational generating function $f(A;\mathbf{t})$ from some data, for example, the generators $\mathbf{a}_1,\ldots,\mathbf{a}_n$ of an affine semigroup A, then that $f(A;\mathbf{t})$ is a *short rational generating function* if the number $|I|$ of indices is bounded by a polynomial in the input complexity.

Example 5.2. A *rational strategy* for a lattice game is a rational generating function for the set of P-positions [8, Definition 8.1].

Definition 5.3 (**Complexity of short rational generating functions**)**.** Fix a positive integer k. Let $A \subseteq \mathbb{Z}^d$ and suppose that $f(A;\mathbf{t})$ is a rational generating function as in Definition 5.1. If $\mathbf{p}_i = (p_{i1},\ldots,p_{id})$ and $\mathbf{a}_{ij} = (a_{ij1},\ldots,a_{ijd})$ for all i,j, then let $p = \max\{|p_{ij}|\}$ and let $a = \max\{|a_{ijr}|\}$. The *complexity* ι of $f(A;\mathbf{t})$ is

$$|I|(1+d+kd) + \sum_{i\in I}\Big(\sum_{j=1}^{d}\log_2|p_{ij}| + \sum_{j=1}^{k}\sum_{r=1}^{d}\log_2|a_{ijr}|\Big)$$

which is $O(d|I|(\log p + k\log a))$.

Definition 5.4 ([2, Definition 3.2])**.** For Laurent power series

$$f_1(\mathbf{t}) = \sum_{\mathbf{a}\in\mathbb{Z}^d} \beta_{1\mathbf{a}}\mathbf{t}^{\mathbf{a}} \quad\text{and}\quad f_2(\mathbf{t}) = \sum_{\mathbf{a}\in\mathbb{Z}^d} \beta_{2\mathbf{a}}\mathbf{t}^{\mathbf{a}}$$

in $\mathbf{t} \in \mathbb{C}^d$, the *Hadamard product* $f = f_1 \star f_2$ is the power series

$$f(\mathbf{t}) = \sum_{\mathbf{a}\in\mathbb{Z}^d} (\beta_{1\mathbf{a}}\beta_{2\mathbf{a}})\,\mathbf{t}^{\mathbf{a}}.$$

Lemma 5.5. *Fix k and d. Let $A, B \subseteq \mathbb{Z}^d$ lie in the same pointed rational cone C. If $f(A;\mathbf{t})$ and $f(B;\mathbf{t})$ are rational generating functions with $\leq k$ denominator binomials in each, then there is an algorithm for computing $f(A;\mathbf{t}) \star f(B;\mathbf{t})$ as a rational generating function in polynomial time in the complexity of the generating functions.*

Proof. Choose an affine linear function ℓ that is negative on C. Write

$$\begin{aligned} f(A;\mathbf{t}) &= \sum_{i\in I} \alpha_i \frac{\mathbf{t}^{\mathbf{p}_i}}{(1-\mathbf{t}^{\mathbf{a}_{i1}})\cdots(1-\mathbf{t}^{\mathbf{a}_{ik}})} \\ f(B;\mathbf{t}) &= \sum_{j\in J} \beta_i \frac{\mathbf{t}^{\mathbf{q}_i}}{(1-\mathbf{t}^{\mathbf{b}_{j1}})\cdots(1-\mathbf{t}^{\mathbf{b}_{jk}})}, \end{aligned}$$

where $\mathbf{p}_i, \mathbf{q}_i \in \mathbb{Z}^d$, $\mathbf{a}_{ir}, \mathbf{b}_{jr} \in C$ for all i, j, r. Since $\langle \ell, \mathbf{a}_{ir}\rangle < 0$ and $\langle \ell, \mathbf{b}_{jr}\rangle < 0$ for all i, j, r, by Lemma 3.4 of [2] we can compute

$$\frac{\mathbf{t}^{\mathbf{p}_i}}{(1-\mathbf{t}^{\mathbf{a}_{i1}})\cdots(1-\mathbf{t}^{\mathbf{a}_{ik}})} \star \frac{\mathbf{t}^{\mathbf{q}_i}}{(1-\mathbf{t}^{\mathbf{b}_{j1}})\cdots(1-\mathbf{t}^{\mathbf{b}_{jk}})}$$

in polynomial time for each i, j. Since the Hadamard product is bilinear, it follows that we can compute $f(A;\mathbf{t}) \star f(B;\mathbf{t})$ in polynomial time as well. □

The main result of this section is as follows.

Theorem 5.6. *Any rational strategy for a lattice game, presented as a ratio of two polynomials with integer coefficients, produces algorithms for*

- *determining whether a position is a P-position or an N-position, and*
- *computing a legal move to a P-position, given any N-position.*

These algorithms run in polynomial time if the rational strategy is a short rational generating function.

Proof. Suppose we wish to determine whether $\mathbf{p} \in \mathcal{B}$ is a P-position or an N-position. Let $f(\mathcal{P};\mathbf{t})$ be a rational strategy for the lattice game. By definition, $\mathcal{P}$ and $\mathbf{p}$ both lie in the cone C. It follows from Lemma 5.5 that we can compute $f(\mathcal{P} \cap \mathbf{p};\mathbf{t}) = f(\mathcal{P};\mathbf{t}) \star \mathbf{t}^{\mathbf{p}}$ in $O(\iota^c)$ time, where ι is the complexity of $f(\mathcal{P};\mathbf{t})$ and $\mathbf{t}^{\mathbf{p}}$, and c is some positive integer. We get

$$f(\mathcal{P} \cap \mathbf{p};\mathbf{t}) = \begin{cases} \mathbf{t}^{\mathbf{p}} & \text{if } \mathbf{p} \in \mathcal{P} \\ 0 & \text{if } \mathbf{p} \in \mathcal{N}. \end{cases}$$

Given an N-position $\mathbf{q}$, simply apply this algorithm to all positions $\mathbf{q} - \boldsymbol{\gamma}$ for each legal move $\boldsymbol{\gamma} \in \Gamma$. Since $\mathbf{q} \in \mathcal{N}$, at least one $\mathbf{q} - \boldsymbol{\gamma}$ lies in $\mathcal{P}$, hence this procedure will end in $O(\iota^c|\Gamma|)$ time. □

One of the long-term goals of this project is to solve Dawson's Chess [6]. That is, given any position in Dawson's Chess, we desire efficient algorithms to determine whether the next player to move has a winning strategy, and if so, to find one. This is equivalent to determining whether any given position $\mathbf{p}$ is a P-position or an N-position. By Theorem 5.6, we can do all of this if we have a Dawson's Chess rational strategy for heaps of sufficient size. Alas, it is not known whether rational strategies exist, in general, even for Dawson's Chess.

Conjecture 5.7. *Every lattice game possesses a rational strategy.*

The question, then, is how to find a rational strategy for Dawson's Chess. As a caveat, observe that a fixed lattice game structure only suffices to encode a heap game for heaps of bounded size. Thus, if G_n denotes the lattice game corresponding to Dawson's Chess with heaps of size at most n, then a strategy could consist of a rational strategy for every n, but care must be taken concerning the dependence of the complexity on n. In the next sections, we shall see that affine stratifications serve as data structures from which to extract rational strategies in polynomial time. Thus the problem will be reduced to finding affine stratifications for G_n for all n, and there is hope that some regularity might arise, as n grows, to allow the possibility of computing them in time polynomial in n.

6. Affine stratifications

Definition 6.1 ([8, Definition 8.6])**.** An *affine stratification* of a subset $\mathcal{W} \subseteq \mathbb{Z}^d$ is a partition

$$\mathcal{W} = \biguplus_{i=1}^{r} W_i$$

of $\mathcal{W}$ into a disjoint union of sets W_i, each of which is a finitely generated module for an affine semigroup $A_i \subset \mathbb{Z}^d$; that is, $W_i = F_i + A_i$, where $F_i \subset \mathbb{Z}^d$ is a finite set. An *affine stratification of a lattice game* is an affine stratification of its set of P-positions.

What enters into the complexity is the dimension and a slew of vectors:

- the rule set,
- the generators of the affine semigroups, and
- the module generators for each affine semigroup.

The integer entries of these vectors must be accounted for as part of the complexity, and the number of vectors contributes to the complexity, as well.

Conjecture 6.2. *Every lattice game possesses an affine stratification.*

Example 6.3. Consider again the game of NIM with heaps of size at most 2. An affine stratification for this game is $\mathcal{P} = 2\mathbb{N}^2$; that is, $\mathcal{P}$ consists of all nonnegative integer points with both coordinates even.

Example 6.4. The misère lattice game on $\mathbb{N}^5$ whose rule set forms the columns of

$$\Gamma = \begin{bmatrix} 1 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \end{bmatrix}$$

was one of the motivations for the definitions in [8] because the illustration of the winning positions in this lattice game provided by Plambeck and Siegel [13, Figure 12] possesses an interesting description as an affine stratification. Indeed, for this lattice game, $\mathcal{P} = W_1 \uplus \cdots \uplus W_7$ for modules $W_k = F_k + A_k$ over the afffine semigroups

$$\begin{aligned}
A_1 &= \mathbb{N}\{(2,0,0,0,0),(0,2,0,0,0),(0,0,1,0,0),(0,0,0,2,0),(0,0,0,0,2)\} \\
A_2 &= \mathbb{N}\{(2,0,0,0,0),(0,2,0,0,0),(0,0,0,2,0),(0,0,0,0,2)\} \\
A_3 &= \mathbb{N}\{(2,0,0,0,0),(0,2,0,0,0),(0,0,0,0,2),(0,0,0,2,2)\} \\
A_4 &= \mathbb{N}\{(2,0,0,0,0),(0,2,0,0,0),(0,0,0,2,0),(0,0,0,2,2)\} \\
A_5 &= \mathbb{N}\{(2,0,0,0,0),(0,2,0,0,0)\} \\
A_6 &= \mathbb{N}\{(2,0,0,0,0),(0,2,0,0,0),(0,0,0,2,2)\} \\
A_7 &= \mathbb{N}\{(2,0,0,0,0)\},
\end{aligned}$$

where the finite generating sets F_k consist of the columns of the following:

$$F_1\colon \begin{bmatrix} 0 \\ 0 \\ 3 \\ 0 \\ 0 \end{bmatrix} \qquad F_2\colon \begin{bmatrix} 0 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 & 1 \\ 2 & 2 & 2 & 2 & 2 \\ 0 & 3 & 3 & 3 & 1 \\ 0 & 1 & 3 & 5 & 5 \end{bmatrix} \qquad F_3\colon \begin{bmatrix} 1 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 \\ 2 & 2 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 \\ 9 & 12 & 9 & 8 & 9 & 8 & 10 & 9 & 12 & 13 \end{bmatrix}$$

$$F_4\colon \begin{bmatrix} 1&1&1&1&0&0&0&0&0&0&1&1&1&0&0&0&1&1&0&0&0&0&0&1&1&1&1&0&0&0&1&1&1 \\ 0&0&0&0&1&1&1&0&0&0&0&0&0&1&1&1&1&1&0&0&0&0&0&0&0&0&0&1&1&1&1&1&1 \\ 2&2&2&2&2&2&2&1&1&1&1&1&1&1&1&1&1&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0 \\ 1&1&1&1&0&0&0&0&2&0&1&3&1&2&2&2&3&1&2&4&4&4&4&3&5&5&3&2&4&2&3&3&3 \\ 0&2&4&6&1&5&7&0&4&6&0&4&6&1&5&7&1&5&0&2&4&6&8&0&2&4&6&1&5&7&1&5&7 \end{bmatrix}$$

$$F_5\colon \begin{bmatrix} 1&0&1&1&1&1&0&1&1&1&1&0&0&0&0&0&0&1&1&1&1&1&1&0&0&0&0&0&1&1 \\ 1&0&0&0&0&0&1&1&1&1&1&2&0&0&0&0&0&2&0&0&0&0&0&1&1&1&1&1&3&1 \\ 2&1&1&1&1&1&1&1&1&1&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0 \\ 0&0&0&1&0&1&1&1&2&0&0&0&1&2&0&1&2&0&2&3&0&1&2&1&0&1&0&1&1&2 \\ 3&3&2&3&5&6&4&1&3&4&7&0&2&3&4&5&6&1&2&3&3&4&5&0&2&3&5&6&1&4 \end{bmatrix}$$

$$F_6\colon \begin{bmatrix} 0&0&1&1&0&0&1 \\ 0&0&0&0&1&1&1 \\ 0&0&0&0&0&0&0 \\ 0&1&0&1&0&1&1 \\ 7&8&6&7&8&9&10 \end{bmatrix} \qquad F_7\colon \begin{bmatrix} 0&1&1 \\ 0&0&0 \\ 0&0&0 \\ 0&0&1 \\ 1&0&1 \end{bmatrix}.$$

The choice of $\mathcal{P}$ possessing an affine stratification may seem arbitrary, but actually it does not matter whether we require $\mathcal{P}$ or $\mathcal{N}$ to possess an affine stratification, due to the following result.

Theorem 6.5. *If A and $B \subset A$ both possess affine stratifications, then $A \smallsetminus B$ possesses an affine stratification.*

The plan for proving Theorem 6.5 is to show that removing a translated normal affine semigroup from a normal affine semigroup yields something with an affine stratification, and then to show that intersecting sets with affine stratifications results in a set with an affine stratification.

Lemma 6.6. *Suppose B is the intersection of a polyhedron and a subgroup of $\mathbb{Z}^d$. If A is a normal affine semigroup and $\mathbf{b} + A \subset B$ for some $\mathbf{b} \in B$, then $B \smallsetminus (\mathbf{b} + A)$ has an affine stratification.*

Proof. First we assume that $\mathbf{b} = \mathbf{0}$ and that B is a normal affine semigroup and $\mathbb{R}_{\geq 0}A = \mathbb{R}_{\geq 0}B$. Since $A \subset B$, that means $\mathbb{Z}A$ is a sublattice of $\mathbb{Z}B$ in $\mathbb{Z}^d$, hence B can be written as a finite disjoint union of cosets of A.

Now, suppose B is an arbitrary intersection of a polyhedron Π_B and a lattice L, and $\mathbf{b} \in B$ is arbitrary. We will reduce to the previous case by "carving" away pieces of B that do not lie in $\mathbb{R}_{\geq 0}A$. Suppose $\mathbb{R}_{\geq 0}A$ has a facet (a $(d-1)$-dimensional face) which is not contained in a facet of Π_B. Let H be the bounding hyperplane of this facet and H_- the corresponding negative halfspace (the half that is outside of $\mathbb{R}_{\geq 0}A$). Then $H_- \cap \Pi_B$ is a polyhedron. to reduce the number of facets of $\mathbb{R}_{\geq 0}A$ which do not lie in a facet of Π_B. Thus we have "carved out" a piece $H_- \cap \Pi_B$ of Π_B. By [11, Lemma 2.4], $H_- \cap \Pi_B \cap L$ is a finitely generated module over an affine semigroup. Now replace Π_B with $\Pi_B \smallsetminus H_-$ and repeat. Each time we repeat the argument, we carve out a piece of the original Π_B which has an affine stratification, and furthermore we reduce the number of facets of $\mathbb{R}_{\geq 0}A$ that do not lie in the current Π_B. Eventually we reduce to the case where each facet of $\mathbb{R}_{\geq 0}A$ lie in some facet of Π_B, which is actually the first case above where Π_B is a cone and $\mathbf{b} = \mathbf{0}$. By [11, Corollary 2.8], the union of these pieces possesses an affine stratification.

There is a degenerate case when A is not d-dimensional, but then we may reduce to a lower dimension by carving away $\mathbb{Z}^d \smallsetminus A$. □

Lemma 6.7. *If $\mathcal{W}$ and $\mathcal{W}'$ have affine stratifications, then $\mathcal{W} \cap \mathcal{W}'$ has an affine stratification.*

Proof. By [11, Theorem 2.6], we may write

$$\mathcal{W} = \biguplus_{i=1}^{r} W_i \quad \text{and} \quad \mathcal{W}' = \biguplus_{j=1}^{s} W_i'$$

where each W_i and W_j' is a translate of a normal affine semigroup. Therefore, it suffices to show that the intersection of a translate of a normal affine semigroup with a translate of another normal affine semigroup has an affine stratification, for the union of all of these intersections would then have an affine stratification, by [11, Corollary 2.8].

Suppose our two translates are $\mathbf{a}_1 + A_1$ and $\mathbf{a}_2 + A_2$. If their intersection is empty, then trivially it has an affine stratification, so we may assume that there is some $\mathbf{a} \in (\mathbf{a}_1 + A_1) \cap (\mathbf{a}_2 + A_2)$. Then $\mathbf{a}_1 - \mathbf{a} + \mathbb{Z}A_1 = \mathbb{Z}A_1$ and $\mathbf{a}_2 - \mathbf{a} + \mathbb{Z}A_2 = \mathbb{Z}A_2$. Therefore

$$\begin{aligned}(\mathbf{a}_1 + \mathbb{Z}A_1) \cap (\mathbf{a}_2 + \mathbb{Z}A_2) &= \mathbf{a} + (\mathbf{a}_1 - \mathbf{a} + \mathbb{Z}A_1) \cap (\mathbf{a}_2 - \mathbf{a} + \mathbb{Z}A_2) \\ &= \mathbf{a} + (\mathbb{Z}A_1 \cap \mathbb{Z}A_2),\end{aligned}$$

i.e., the intersection of the cosets is itself a coset of a lattice. Moreover, the intersection $(\mathbf{a}_1 + \mathbb{R}_{\geq 0}A_1) \cap (\mathbf{a}_2 + \mathbb{R}_{\geq 0}A_2)$ is a polyhedron. By [11, Lemma 2.4], since A_1 and A_2 are normal, we have

$$\begin{aligned}(\mathbf{a}_1 + A_1) \cap (\mathbf{a}_2 + A_2) &= ((\mathbf{a}_1 + \mathbb{R}_{\geq 0}A_1) \cap (\mathbf{a}_1 + \mathbb{Z}A_1)) \cap ((\mathbf{a}_2 + \mathbb{R}_{\geq 0}A_2) \cap (\mathbf{a}_2 + \mathbb{Z}A_2)) \\ &= ((\mathbf{a}_1 + \mathbb{R}_{\geq 0}A_1) \cap (\mathbf{a}_2 + \mathbb{R}_{\geq 0}A_2)) \cap ((\mathbf{a}_1 + \mathbb{Z}A_1) \cap (\mathbf{a}_2 + \mathbb{Z}A_2))\end{aligned}$$

is an intersection of a polyhedron with a coset of a lattice and hence is a finitely generated module over an affine semigroup. In particular, the intersection has an affine stratification. □

Proof of Theorem 6.5 First, assume A is a normal affine semigroup. Suppose

$$B = \biguplus_{i=1}^{r} B_i$$

where each B_i is a translate of a normal affine semigroup. By Lemma 6.6, each $A \setminus B_i$ has an affine stratification. Therefore, by Lemma 6.7, $A \setminus B = A \setminus (\biguplus_{i=1}^{r} B_i) = \bigcap_{i=1}^{r}(A \setminus B_i)$ has an affine stratification. For the general case where A has an affine stratification, each A_i reduces to the previous case, and then we obtain the result by taking the union. □

In what follows, we define the complexity of an affine stratification to be the complexity of the generators and the affine semigroups involved. Roughly speaking, the complexity of an integer k is its binary length (more precisely, $1 + \lceil \log_2 k \rceil$), so the complexity is roughly the sum of the binary lengths of the integer entries of the generators in the finite sets F_i and the coefficients of the vectors generating the affine semigroups A_i; see [1, Section 2] for additional details. To say that an algorithm is *polynomial time when the dimension d is fixed* means that the running time is bounded by $\iota^{\phi(d)}$ for some fixed function ϕ, where ι is the input complexity.

Definition 6.8 (**Complexity of an affine semigroup**)**.** Fix an affine semigroup $A = \mathbb{N}\{\mathbf{a}_1, \ldots, \mathbf{a}_n\}$ in $\mathbb{Z}^d$. Let $\mathbf{a}_i = (a_{1i}, \ldots, a_{di})$. The *complexity* of A is the number of bits needed to represent the dn numbers a_{ij}, which is equal to

$$dn + \sum_{i=1}^{n} \sum_{j=1}^{d} \log_2 |a_{ij}|$$

which is $O(dn \log a)$, where $a = \max\{|a_{ij}|\}$.

Definition 6.9 (**Complexity of an affine stratification**)**.** Let

$$\mathcal{W} = \biguplus_{i=1}^{r} W_i$$

be an affine stratification, where $W_i = F_i + A_i$ for some affine semigroup $A_i \subset \mathbb{Z}^d$ and finite set $F_i \subset \mathbb{Z}^d$. Let $m_i = |F_i|$ and $F_i = \{\mathbf{b}_{i1}, \ldots, \mathbf{b}_{im_i}\}$ where $\mathbf{b}_{ij} = (b_{ij1}, \ldots, b_{ijd})$, and let $A_i = \mathbb{N}\{\mathbf{a}_{i1}, \ldots, \mathbf{a}_{in}\}$, where $\mathbf{a}_{ij} = (a_{ij1}, \ldots, a_{ijd})$. The *complexity* of the affine stratification is the number

$$d\Big(nr + \sum_{i=1}^{r} m_i\Big) + \sum_{i=1}^{r} \sum_{s=1}^{d} \Big(\sum_{j=1}^{n} \log_2 |a_{ijs}| + \sum_{j=1}^{m_i} \log_2 |b_{ijs}| \Big)$$

of bits needed to represent every F_i and A_i. This number is $O(dr(n\log a + m\log b))$, where $a = \max\{a_{ijs}\}$, $b = \max\{b_{ijs}\}$, and $m = \max\{m_i\}$. If ι is an upper bound on the complexity of the affine semigroups A_i, then this becomes $O(r\iota + dmr\log b)$.

Remark 6.10. The existence of affine stratifications as in [8, Conjecture 8.9] is equivalent to the same statement with the extra hypothesis that the rule set generates a saturated (also known as "normal") affine semigroup. There are also a number of ways to characterize the existence of affine stratifications, in general [11, Theorem 2.6], using various combinations of hypotheses such as normality of the affine semigroups involved, or disjointness of the relevant unions. However, some of these freedoms increase complexity in untamed ways, and are therefore unsuitable for efficient algorithmic purposes. Definition 6.1 characterizes the notion of affine stratification in the most efficient terms, where algorithmic computation of rational strategies is concerned; allowing the unions to overlap would make it easier to find affine stratifications, but harder to compute rational strategies from them.

7. Computing rational strategies from affine stratifications

In this section, we prove the following.

Theorem 7.1. *A rational strategy can be algorithmically computed from any affine stratification, in time polynomial in the input complexity of the affine stratification when the dimension d is fixed and the numbers of module generators over the semigroups A_i are uniformly bounded above.*

If Conjecture 6.2 is true, then this theorem implies Conjecture 5.7. The proof of the theorem requires a few intermediate results, the point being simply to keep careful track of the complexities of the constituent elements of affine stratifications.

Lemma 7.2. *Fix $k, d \in \mathbb{N}$. Let $A, B \subseteq \mathbb{Z}^d$ lie in the same pointed rational cone C. If $f(A;\mathbf{t})$ and $f(B;\mathbf{t})$ are short rational generating functions with $\leq k$ binomials in their denominators, then for some $c \in \mathbb{N}$ there is an $O(\iota^c)$ time algorithm for computing the rational function $f(A \cup B;\mathbf{t})$, where ι is an upper bound on the complexity of $f(A;\mathbf{t})$ and $f(B;\mathbf{t})$. If A and B are disjoint, then the complexity of $f(A\cup B;\mathbf{t})$ is bounded by 2ι, and $f(A\cup B;\mathbf{t})$ can be computed in $O(\iota)$ time.*

Proof. This follows from the fact that

$$f(A\cup B;\mathbf{t}) = f(A;\mathbf{t}) + f(B;\mathbf{t}) - f(A\cap B;\mathbf{t})$$

and that $f(A\cap B;\mathbf{t}) = f(A;\mathbf{t}) \star f(B;\mathbf{t})$ can be computed in polynomial time, by Lemma 5.5. □

Corollary 7.3. *Fix k and d. Let $A_1, \ldots, A_m \subseteq \mathbb{Z}^d$ lie in the same pointed rational cone C. If rational generating functions $f(A_1;\mathbf{t}), \ldots, f(A_r;\mathbf{t})$ have $\leq k$ binomials in their denominators, and $A = A_1 \cup \cdots \cup A_m$, then for some $c \in \mathbb{N}$ there is an $O(2^m\iota^c)$*

time algorithm for computing $f(A;\mathbf{t})$ as a rational generating function, where ι is an upper bound on the complexity of $f(A_1;\mathbf{t}),\dots,f(A_r;\mathbf{t})$. If the A_i are pairwise disjoint, then the complexity bound is $O(m\iota)$.

Proof. This follows from Lemma 7.2 and the fact that the number of binomials in the denominators in the rational generating functions may increase by a factor of up to 2 after computing each union. If the A_i are pairwise disjoint, then

$$f(A;\mathbf{t}) = \sum_{i=1}^{m} f(A_i;\mathbf{t})$$

and no intersections need to be computed. □

Lemma 7.4. *Fix n and d. If $A \subseteq \mathbb{Z}^d$ is a pointed affine semigroup generated by n integer vectors and has complexity ι, then for some positive integer c there is an $O(\iota^c)$ time algorithm for computing $f(A;\mathbf{t})$.*

Proof. Let $A = \mathbb{N}\{\mathbf{a}_1,\dots,\mathbf{a}_n\}$. It is algorithmically easy to embed A into $\mathbb{N}^d$: if A has dimension d, then find d linearly independent facets and take their integer inner normal vectors as the columns of the embedding ν; if A has dimension $d' < d$, then find d' linearly independent facets and any $d - d'$ linear integer functions that vanish on A. Use the discussion of [2, Section 7.3] to compute $f(\nu(A);\mathbf{t})$. Then apply ν^{-1} to the exponents in $f(\nu(A);\mathbf{t})$ to get $f(A;\mathbf{t})$. □

Lemma 7.5. *Fix d. Let $W = F + A$, where $A \subseteq \mathbb{Z}^d$ is a pointed affine semigroup with complexity ι and $F \subseteq \mathbb{Z}^d$ is a finite set with $|F| = m$. For some $c \in \mathbb{N}$ there is an $O(2^m\iota^c)$ time algorithm for computing $f(W;\mathbf{t})$ as a rational function.*

Proof. Let $F = \{\mathbf{b}_1,\dots,\mathbf{b}_m\}$. Since F is finite, any linear function that is positive on $A \setminus \{0\}$ is bounded below on W. Therefore, there exists a pointed rational cone that contains each $\mathbf{b}_j + A$. For each j, $f(\mathbf{b}_j + A;\mathbf{t}) = \mathbf{t}^{\mathbf{b}_j} f(A;\mathbf{t})$, each of which has complexity $O(\iota)$ and can be computed in $O(\iota^{c'})$ time, for some $c' > 0$, by Lemma 7.4. Since W is the union of the $\mathbf{b}_j + A$, it follows from Corollary 7.3 that $f(W;\mathbf{t})$ can be computed in $O(2^m\iota^c)$. □

We now return to proving our main theorem.

Proof of Theorem 7.1. Write

$$\mathcal{P} = \biguplus_{i=1}^{r} W_i$$

where $W_i = F_i + A_i$ for affine semigroups $A_i \subseteq \mathbb{Z}^d$ and finite sets $F_i \subseteq \mathbb{Z}^d$. Let ι be an upper bound on the complexity of each of the A_i. Since the sizes of the F_i are fixed, by Lemma 7.5 we can compute each $f(W_i;\mathbf{t})$ in $O(\iota^c)$ time, for some positive integer c. Since the W_i are pairwise disjoint, by Corollary 7.3 we can compute $f(\mathcal{P};\mathbf{t})$ in $O(r\iota^c)$ time. □

There is little hope that the complexity of calculating affine stratifications—or even merely rational strategies—should be polynomial in the input complexity when certain parameters are not fixed. Indeed, the complexity of the generating function for an affine semigroup fails to be polynomial in the number of its generators [2, Section 7.3]. Thus it makes sense to restrict complexity estimates to lattice games with rule sets of fixed complexity. On the other hand, there is hope that the complexity of an affine stratification should be bounded by the complexity of the rule set. Therefore, once the complexity of the rule set has been fixed, algorithms dealing with affine stratifications could be polynomial.

8. Toward computing misère quotients from affine stratifications

It is still not known if computing misère quotients is decidable. Indeed, all algorithms for computing them as of now are not guaranteed to halt if the quotient is infinite. In this section, we provide an approach to computing the misère quotient of a lattice game provided that we know its affine stratification. Recall the notion of congruence from Definition 4.7.

The following is an algorithm for determining whether two given positions are indistinguishable.

Theorem 8.1. *Given a rational strategy $f(\mathcal{P};\mathbf{t})$, if $\mathbf{p},\mathbf{q}\in\mathcal{B}$, there exists an algorithm for determining the congruence of $\mathbf{p}$ and $\mathbf{q}$ in time polynomial in the input complexity of $f(\mathcal{P};\mathbf{t})$.*

Proof. Let $S_{\mathbf{p}} = (\mathbf{p}+C)\cap\mathcal{P}-\mathbf{p}$ and $S_{\mathbf{q}} = (\mathbf{q}+C)\cap\mathcal{P}-\mathbf{q}$. Since $\mathbf{p}\in C$, we have $\mathbf{p}+C\subseteq C$, and $\mathcal{P}\subseteq C$ by definition, so we may apply Lemma 5.5 to compute $f((\mathbf{p}+C)\cap\mathcal{P};\mathbf{t})$ in polynomial time. Then we can compute $f(S_{\mathbf{p}};\mathbf{t})$ in polynomial time since $f(S_{\mathbf{p}};\mathbf{t}) = \mathbf{t}^{-\mathbf{p}}f((\mathbf{p}+C)\cap\mathcal{P};\mathbf{t})$. Similarly, we compute $f(S_{\mathbf{q}};\mathbf{t})$ in polynomial time. Then $\mathbf{p}$ and $\mathbf{q}$ are indistinguishable if and only if $f(S_{\mathbf{p}};\mathbf{t})-f(S_{\mathbf{q}};\mathbf{t})=0$. □

Corollary 8.2. *Given an affine stratification, there is a polynomial time algorithm for determining the congruence of $\mathbf{p}$ and $\mathbf{q}$.*

Proof. By Theorem 7.1, we can compute a rational strategy for the lattice game in polynomial time. The result then follows from Theorem 8.1. □

This algorithm leads us one step closer to computing misère quotients. Since misère quotients are finitely presented, i.e. can be described by finitely many generators and relations, it suffices to bound how far away from the origin the elements in the relations can get. It turns out that this can be done when the given affine stratification of the lattice game consists solely of one normal affine semigroup.

Fix d. Suppose Π is a polyhedron with recession cone C and integer points $\Lambda = \Pi\cap\mathbb{Z}^d$, and suppose $\mathcal{B}=\Lambda\smallsetminus\mathcal{D}$ is the game board. Fix a rule set Γ and let $\mathcal{P}$ be the resulting set of P-positions.

Theorem 8.3. *Suppose $\mathcal{P}$ is a normal affine semigroup. Then the misère quotient is generated by relations $\mathbf{p} \cong \mathbf{0}$ for some minimal generator $\mathbf{p}$ of $\mathcal{P}$.*

Before we prove this theorem, we establish two key lemmas.

Lemma 8.4. *Suppose $\mathcal{P}$ is an affine semigroup. If $\mathbf{p} \cong \mathbf{q}$, then*

$$(\mathbf{p} + C) \cap \mathbb{R}_{\geq 0}\mathcal{P} - \mathbf{p} = (\mathbf{q} + C) \cap \mathbb{R}_{\geq 0}\mathcal{P} - \mathbf{q}.$$

Proof. Suppose the conclusion is false. Then there exists an extremal ray $\rho = \mathbb{R}_{\geq 0}\mathbf{b}$, where $\mathbf{b}$ has length 1, of C such that

$$(\mathbf{p} + \rho) \cap \mathbb{R}_{\geq 0}\mathcal{P} - \mathbf{p} \neq (\mathbf{q} + \rho) \cap \mathbb{R}_{\geq 0}\mathcal{P} - \mathbf{q}. \tag{1}$$

By [12, Exercise 7.15], $\mathcal{P}$ contains a translate of its saturation $\overline{\mathcal{P}} = \mathbb{Z}\mathcal{P} \cap \mathbb{R}_{\geq 0}\mathcal{P}$. Fix $\mathbf{a} \in \mathcal{P}$ which lies in the interior of C and within the translated saturation. For any $\mathbf{x} \in \mathbb{R}_{\geq 0}\mathcal{P}$, let $\ell_k(\mathbf{x})$ denote the length of the segment $(\mathbf{x} + k\mathbf{a} + \rho) \cap \mathbb{R}_{\geq 0}\mathcal{P}$. By (1), we have $\ell_0(\mathbf{p}) \neq \ell_0(\mathbf{q})$. Without loss of generality, assume $\ell_0(\mathbf{p}) > \ell_0(\mathbf{q})$. Since $\mathbf{a}$ is in the interior of C, ℓ_k is an increasing function of k. Furthermore, by linearity, $\ell_k = \ell_0 + k(\ell_1 - \ell_0)$, thus $\ell_k(\mathbf{p}) - \ell_k(\mathbf{q})$ is an increasing function of k. Therefore, for sufficiently large k, there exists $\mathbf{a}' \in \mathbb{Z}\mathcal{P} \cap \{r\mathbf{b} \mid \ell_k(\mathbf{q}) < r \leq \ell_k(\mathbf{p})\}$. Then $\mathbf{a}' \in (\mathbf{p}+k\mathbf{a}+C) \cap \mathcal{P} - \mathbf{p}$ but $\mathbf{a}' \notin (\mathbf{p}+k\mathbf{a}+C) \cap \mathcal{P} - \mathbf{q}$, which implies $\mathbf{a}'+\mathbf{p} \ncong \mathbf{a}'+\mathbf{q}$, contradicting the hypothesis. □

Lemma 8.5. *Suppose $\mathcal{P}$ is a normal affine semigroup. Then $\mathbf{p} \cong \mathbf{q}$ if and only if*

$$(\mathbf{p} + C) \cap \mathbb{R}_{\geq 0}\mathcal{P} - \mathbf{p} = (\mathbf{q} + C) \cap \mathbb{R}_{\geq 0}\mathcal{P} - \mathbf{q}.$$

Proof. The forward implication follows from Lemma 8.4, so it suffices to show the backward implication. Suppose $(\mathbf{p} + C) \cap \mathbb{R}_{\geq 0}\mathcal{P} - \mathbf{p} = (\mathbf{q} + C) \cap \mathbb{R}_{\geq 0}\mathcal{P} - \mathbf{q}$. By symmetry, it is enough to show that $(\mathbf{p} + C) \cap \mathcal{P} - \mathbf{p} \subseteq (\mathbf{q} + C) \cap \mathcal{P} - \mathbf{q}$. Let $\mathbf{a} \in (\mathbf{p}+C) \cap \mathcal{P} - \mathbf{p}$, so $\mathbf{a}+\mathbf{p} \in (\mathbf{p}+C) \cap \mathcal{P}$. We want to show that $\mathbf{a}+\mathbf{q} \in (\mathbf{q}+C) \cap \mathcal{P}$. Since $\mathbf{a} + \mathbf{p} \in \mathbf{p} + C$, we have $\mathbf{a} \in C$ and therefore $\mathbf{a} + \mathbf{q} \in \mathbf{q} + C$, so it only remains to show that $\mathbf{a} + \mathbf{q} \in \mathcal{P}$. But $\mathbf{a} + \mathbf{q} = (\mathbf{a} + \mathbf{p}) + (\mathbf{q} - \mathbf{p}) \in \mathbb{Z}\mathcal{P}$. Furthermore, $\mathbf{a} \in (\mathbf{p} + C) \cap \mathbb{R}_{\geq 0}\mathcal{P} - \mathbf{p} = (\mathbf{q} + C) \cap \mathbb{R}_{\geq 0}\mathcal{P} - \mathbf{q}$, so $\mathbf{a} + \mathbf{q} \in (\mathbf{q} + C) \cap \mathbb{R}_{\geq 0}\mathcal{P}$. Since $\mathcal{P}$ is normal, $\mathbf{a} + \mathbf{q} \in \mathbb{Z}\mathcal{P} \cap \mathbb{R}_{\geq 0}\mathcal{P} = \mathcal{P}$. □

We have two cases when $\mathcal{P}$ is a normal affine semigroup. In one case, the real cone generated by $\mathcal{P}$ "fills up" the ambient cone, i.e., $\mathbb{R}_{\geq 0}\mathcal{P} = C$, whereas in the other case it does not. We now see that the former case is a trivial case.

Proposition 8.6. *Suppose $\mathcal{P}$ is a normal affine semigroup. If $\mathbb{R}_{\geq 0}\mathcal{P} = C$ and if $\mathbf{p} \in \mathcal{P}$, then $\mathbf{p} \cong \mathbf{0}$.*

Proof. We wish to show that, for $\mathbf{p} \in \mathcal{P}$, we have $\mathbf{p}+\mathbf{p}' \in \mathcal{P}$ if and only if $\mathbf{p}' \in \mathcal{P}$. The "if" direction follows immediately from the definition of a semigroup. Now suppose $\mathbf{p} + \mathbf{p}' \in \mathcal{P}$. Then $\mathbf{p}' = (\mathbf{p} + \mathbf{p}') - \mathbf{p} \in \mathbb{Z}\mathcal{P}$ and $\mathbf{p}' \in C$, hence $\mathbf{p}' \in \mathbb{Z}\mathcal{P} \cap C = \mathcal{P}$, since $\mathcal{P}$ is normal. □

The following results focus on the interesting case where $\mathbb{R}_{\geq 0}\mathcal{P} \subsetneq C$.

Proposition 8.7. *Suppose $\mathcal{P}$ is a normal affine semigroup, and let $\mathbb{R}_{\geq 0}\mathcal{P} \subsetneq C$. If $\mathbf{p} \cong \mathbf{q} \in \mathcal{P}$, then one of the rays $\mathbb{R}_{\geq 0}(\mathbf{p}-\mathbf{q})$ or $\mathbb{R}_{\geq 0}(\mathbf{q}-\mathbf{p})$ is contained in a proper face F of $\mathbb{R}_{\geq 0}\mathcal{P}$. Moreover, if ρ_C is a ray of C, then F is contained in the proper face through which $\mathbf{p}+\rho_C$ exits the cone $\mathbb{R}_{\geq 0}\mathcal{P}$.*

Proof. By Lemma 8.4, we have

$$(\mathbf{p}+C) \cap \mathbb{R}_{\geq 0}\mathcal{P} - \mathbf{p} = (\mathbf{q}+C) \cap \mathbb{R}_{\geq 0}\mathcal{P} - \mathbf{q}. \tag{2}$$

Let $\rho = \mathbb{R}_{\geq 0}(\mathbf{p}-\mathbf{q})$. Suppose neither ρ nor $-\rho$ lie in a proper face of $\mathbb{R}_{\geq 0}\mathcal{P}$. Then either both lie outside of $\mathbb{R}_{\geq 0}\mathcal{P}$ or one lies in the interior of $\mathbb{R}_{\geq 0}\mathcal{P}$. If both lie outside, then $\mathbf{p}+\rho$ exits $\mathbb{R}_{\geq 0}\mathcal{P}$ through at least one of its proper faces F'. Since $\mathbb{R}_{\geq 0}\mathcal{P} \subsetneq C$, there exists a ray ρ_C of C such that $\mathbf{p}+\rho_C$ also exits $\mathbb{R}_{\geq 0}\mathcal{P}$ through F'. But then since ρ points out of the cone $\mathbb{R}_{\geq 0}\mathcal{P}$, the lengths of $(\mathbf{p}+\rho_C) \cap \mathbb{R}_{\geq 0}\mathcal{P}$ and $(\mathbf{q}+\rho_C) \cap \mathbb{R}_{\geq 0}\mathcal{P}$ are not equal, contradicting (2). Now suppose, without loss of generality, that ρ lies in the interior of $\mathbb{R}_{\geq 0}\mathcal{P}$. Again, since $\mathbb{R}_{\geq 0}\mathcal{P} \subsetneq C$, there exists a ray ρ'_C of C such that $\mathbf{q}+\rho'_C$ exits $\mathbb{R}_{\geq 0}\mathcal{P}$ through some proper face F''. Since ρ points into the interior of $\mathbb{R}_{\geq 0}\mathcal{P}$, we have that $(\mathbf{p}+\rho'_C) \cap \mathbb{R}_{\geq 0}\mathcal{P}$ is longer than $(\mathbf{q}+\rho'_C) \cap \mathbb{R}_{\geq 0}\mathcal{P}$, again contradicting (2). ☐

Corollary 8.8. *Suppose $\mathcal{P}$ is a normal affine semigroup, and let $\mathbb{R}_{\geq 0}\mathcal{P} \subsetneq C$. If $\mathbf{p} \cong \mathbf{q} \in \mathcal{P}$, then either $\mathbf{q} \in \mathbf{p}+\mathbb{R}_{\geq 0}\mathcal{P}$ or $\mathbf{p} \in \mathbf{q}+\mathbb{R}_{\geq 0}\mathcal{P}$.*

Lemma 8.9. *Suppose $\mathcal{P}$ is a normal affine semigroup. If $\mathbf{p} \cong \mathbf{q} \in \mathcal{P}$ and $\mathbf{a} \in \mathcal{P}$ such that $\mathbf{p}-\mathbf{a}$ and $\mathbf{q}-\mathbf{a}$ lie in $\mathbb{R}_{\geq 0}\mathcal{P}$, then $\mathbf{p}-\mathbf{a} \cong \mathbf{q}-\mathbf{a}$.*

Proof. By Lemma 8.5, it suffices to show that

$$(\mathbf{p}-\mathbf{a}+C) \cap \mathbb{R}_{\geq 0}\mathcal{P} - \mathbf{p} = (\mathbf{q}-\mathbf{a}+C) \cap \mathbb{R}_{\geq 0}\mathcal{P} - \mathbf{q}.$$

Equivalently, we wish to show that $\mathbf{p}-\mathbf{a}+\mathbf{b} \in \mathbb{R}_{\geq 0}\mathcal{P}$ if and only if $\mathbf{q}-\mathbf{a}+\mathbf{b} \in \mathbb{R}_{\geq 0}\mathcal{P}$. By Corollary 8.8, assume without loss of generality that $\mathbf{q} \in \mathbf{p}+\mathbb{R}_{\geq 0}\mathcal{P}$. In particular, $\mathbf{q}-\mathbf{p} \in \mathbb{R}_{\geq 0}\mathcal{P}$. If $\mathbf{p}-\mathbf{a}+\mathbf{b} \in \mathbb{R}_{\geq 0}\mathcal{P}$, then $\mathbf{q}-\mathbf{a}+\mathbf{b} = (\mathbf{p}-\mathbf{a}+\mathbf{b})+(\mathbf{q}-\mathbf{p}) \in \mathbb{R}_{\geq 0}\mathcal{P}$ since cones are closed under addition. Conversely, suppose $\mathbf{p}-\mathbf{a}+\mathbf{b} \notin \mathbb{R}_{\geq 0}\mathcal{P}$. Then the ray with base point $\mathbf{p}-\mathbf{a}$ passing through $\mathbf{p}-\mathbf{a}+\mathbf{b}$ exits $\mathbb{R}_{\geq 0}\mathcal{P}$ through some proper face F. Thus $\mathbf{q}-\mathbf{p} \in F$, by Proposition 8.7. Thus $\mathbf{q}-\mathbf{a}+\mathbf{b} = (\mathbf{p}-\mathbf{a}+\mathbf{b})+(\mathbf{q}-\mathbf{p}) \in \mathbb{R}_{\geq 0}$. ☐

Proposition 8.10. *Suppose $\mathcal{P}$ is a normal affine semigroup and $\mathbf{p}+\mathbf{q} \cong \mathbf{0}$ for some $\mathbf{p}, \mathbf{q} \in \mathcal{P}$. Then $\mathbf{p} \cong \mathbf{q} \cong \mathbf{0}$.*

Proof. In the case where $\mathbb{R}_{\geq 0}\mathcal{P} = C$, this follows trivially from Proposition 8.6 since $\mathbf{p} \cong \mathbf{0}$ for all $\mathbf{p} \in \mathcal{P}$, so we may assume $\mathbb{R}_{\geq 0}\mathcal{P} \subsetneq C$. By Proposition 8.7, $\mathbf{p}+\mathbf{q}$ lies in a proper face F of $\mathcal{P}$. It follows from [12, Definition 7.8] and [12, Lemma 7.12] that $\mathbf{p} \in F$ and $\mathbf{q} \in F$. By symmetry, it suffices to show that $\mathbf{p} \cong \mathbf{0}$ and by Lemma 8.5, it suffices to show that $\mathbf{a} \in \mathbb{R}_{\geq 0}\mathcal{P}$ if and only if $\mathbf{p}+\mathbf{a} \in \mathbb{R}_{\geq 0}\mathcal{P}$. If $\mathbf{a} \in \mathbb{R}_{\geq 0}\mathcal{P}$, then

$\mathbf{p} + \mathbf{a} \in \mathbb{R}_{\geq 0}\mathcal{P}$ by closure under addition. Conversely, suppose $\mathbf{a} \notin \mathbb{R}_{\geq 0}\mathcal{P}$. Then $\mathbf{p} + \mathbf{a} \notin \mathbb{R}_{\geq 0}\mathcal{P}$ since $\mathbf{p} \in F$. □

Corollary 8.11. *Suppose $\mathcal{P} = \mathbb{N}\{\mathbf{a}_1, \ldots, \mathbf{a}_n\}$ is a normal affine semigroup and $\mathbf{p} \cong \mathbf{0}$. If $\mathbf{p} = c_{i_1}\mathbf{a}_{i_1} + \cdots + c_{i_k}\mathbf{a}_{i_k}$, then $\mathbf{a}_{i_j} \cong \mathbf{0}$ for all $j = 1, \ldots, k$.*

Proof. Follows by induction on Proposition 8.10. □

We now have enough intermediate results to prove the main theorem.

Proof of Theorem 8.3 If $\mathbb{R}_{\geq 0}\mathcal{P} = C$, this is trivial since $\mathbf{p} \cong \mathbf{0}$ for all $\mathbf{p} \in \mathcal{P}$. Suppose we have a relation of the form $\mathbf{p}_1 \cong \mathbf{p}_2$, neither necessarily in $\mathcal{P}$. Then there exists $\mathbf{a} \in C$ such that $\mathbf{p}_1 + \mathbf{a}, \mathbf{p}_2 + \mathbf{a} \in \mathcal{P}$. Moreover, $\mathbf{p}_1 + \mathbf{a} \cong \mathbf{p}_2 + \mathbf{a}$. By Corollary 8.8, assume without loss of generality that $\mathbf{p}_2 + \mathbf{a} \in \mathbf{p}_1 + \mathbf{a} + \mathbb{R}_{\geq 0}\mathcal{P}$, so $\mathbf{p}_2 - \mathbf{p}_1 \in \mathbb{R}_{\geq 0}\mathcal{P}$. Then it follows from Lemma 8.9 that $\mathbf{p}_2 - \mathbf{p}_1 = (\mathbf{p}_2 + \mathbf{a}) - (\mathbf{p}_1 + \mathbf{a}) \cong \mathbf{0}$. So, the original relation reduces to a relation of the form $\mathbf{p} \cong \mathbf{0}$ for some $\mathbf{p} \in \mathcal{P}$. But if $\mathcal{P} = \mathbb{N}\{\mathbf{a}_1, \ldots, \mathbf{a}_n\}$, where the $\mathbf{a}_1, \ldots, \mathbf{a}_n$ are minimal generators, then we may write $\mathbf{p} = c_{i_1}\mathbf{a}_{i_1} + \cdots + c_{i_k}\mathbf{a}_{i_k}$, and hence $\mathbf{a}_{i_k} \cong \mathbf{0}$. □

Corollary 8.12. *There is an algorithm for computing the misère quotient of the monoid $C \cap \mathbb{Z}^d$ if $\mathcal{P}$ is a normal affine semigroup.*

Proof. Compute the Hilbert basis $\{\mathbf{a}_1, \ldots, \mathbf{a}_n\}$ of $\mathcal{P}$, then check if $\mathbf{a}_i \cong \mathbf{0}$ for each $i = 1, \ldots, n$. □

It may seem as though this case is far too special, since in our definition of affine stratification, the affine semigroups are not necessarily normal. However, by [11, Theorem 2.6], we may assume that each affine semigroup A given in the affine stratification is normal.

Acknowledgements

I thank Professor Ezra Miller for providing me with such an interesting problem to work on for the past two years. I am deeply grateful to him for the guidance, support, and mentorship he has provided. Moreover, I thank him for carefully reading this manuscript and for his insightful comments and suggestions. I also thank Professor David Kraines for his support and direction of the PRUV program, which initially gave me the opportunity to start research on this project, and for introducing me to Ezra in the first place. Many of the results and proofs found in this thesis are taken nearly or completely verbatim from papers by myself and Ezra ([8] and [9]).

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