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Transition Measures for the Stochastic Burgers Equation

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Dedicated to Percy Deift on the occasion of his sixtieth birthday.

ABSTRACT. We prove that transition measures for the stochastic Burgers equation with rough forcing are equivalent to an explicit non-degenerate Gaussian measure on $\ell_2(\mathbb{Z}/\{0\})$ induced by a natural infinite dimensional Ornstein-Uhlenbeck process. This result can be used to prove that the stochastic Burgers equation with rough forcing is uniquely ergodic.

1. Introduction

We consider the stochastic Burgers equation on $[0, 2\pi]$ given by

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = \Delta u + \frac{dW}{dt} \tag{1.1}$$

and subject to periodic boundary conditions. This equation has attracted much interest in recent years. For example, see [**Sin91, DPD98, GM05, TZ06**] and the references within. Our goal in this paper is modest: We will not concern ourselves with the general theory of the stochastic Burgers equation, but rather explore the mutual absolute continuity of the transition measures induced by the dynamics of (1.1) with respect to appropriate Gaussian measures on a Hilbert space. This work is a continuation of the investigation which began with [**MS05**].

Equation (1.1) is formal notation for the following coupled infinite dimensional system of stochastic differential equations:

$$du_k(t) = \left[-|k|^2 u_k(t) - i\sum_{\ell\in\mathbb{Z}^*} \ell u_\ell(t)u_{k-\ell}(t)\right] dt + \sigma_k dB_k(t),\ k \in \mathbb{Z}/\{0\}. \tag{1.2}$$

Equation (1.2) describes the dynamics of the Fourier coefficients of u with respect to the orthonormal basis for $L^2(0, 2\pi)$ given by $\left\{\frac{1}{\sqrt{2\pi}}e^{ikx}\right\}_{k\in\mathbb{Z}/\{0\}}$ and subject to the

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condition that $\int_0^{2\pi} u(t,x)dx = 0$. Henceforth, we denote $\mathbb{Z}/\{0\}$ by $\mathbb{Z}^*$ and assume that for each k, $\sigma_k \in \mathbb{C}$ and is subject to the following reality condition: $\sigma_k = \overline{\sigma_{-k}}$. We assume that the $\{B_k\}_{k\in\mathbb{Z}^*}$ are independent standard complex Brownian motions subject only to the following reality condition: $B_k = \overline{B_{-k}}$. The estimates in this note imply that (1.2) possesses global solutions in $\ell_2(\mathbb{Z}^*)$. However, our primary interest is the structure of the transition measures induced by (1.2).

We associate to the coupled system (1.2) a natural linear system:

$$dz_k(t) = -|k|^2 z_k(t)dt + \sigma_k dB_k(t),\ k \in \mathbb{Z}^*. \tag{1.3}$$

This is an infinite dimensional Ornstein-Uhlenbeck process which induces a Gaussian measure, μ^z, on $\ell_2(\mathbb{Z}^*)$ with respect to which the Ornstein-Uhlenbeck dynamics is invariant.

We assume that there is an $r > 0$ and positive constants C_1, C_2 such that $\frac{C_1}{|k|^r} < |\sigma_k| < \frac{C_2}{|k|^r}$. Let $P_t^u(x,dy)$ and $P_t^z(x,dy)$ be the time t transition measures on $\ell_2(\mathbb{Z}^*)$ for the u and z process, respectively. Our main theorem follows.

THEOREM 1.1. *If $r > 1$, then $P_t^u(x,dy)$ and $P_t^z(x,dy)$ are mutually absolutely continuous measures on $\ell_2(\mathbb{Z}^*)$ for μ^z almost every x.*

REMARK 1.2. Standard compactness techniques show the existence of an invariant measure for the Burgers system (1.2) (see for example [**DZ**]). The equivalence of the transition measures, when coupled with the existence of an invariant measure for (1.2), can be used to prove unique ergodicity of the stochastic Burgers system (1.2) and shows that the invariant measure for (1.2) is equivalent to μ^z. In this context the Gaussian measure μ^z on $\ell_2(\mathbb{Z}^*)$ plays a role similar to that of Lebesgue measure on $\mathbb{R}^n$.

This paper is organized as follows. In section 2, we define the basic spaces and prove that the relevant dynamics is absorbing. In section 3, we prove probabilistic estimates on the energy norm and the $|\cdot|_{\gamma,\infty}$ norm of solutions to (1.2). In section 4, we prove equivalence of transition measures by introducing an auxiliary process and employing relative entropy estimates.

2. Basic Spaces and Absorbing Balls

Let $\gamma > 0$ and $c = \{c_k\}_{k\in\mathbb{Z}^*}$ be a bi-infinite sequence of complex numbers. Define the $|\cdot|_{\gamma,\infty}$ norm on such a sequence $c = \{c_k\}_{k\in\mathbb{Z}^*}$ by $|c|_{\gamma,\infty} = \sup_{k\in\mathbb{Z}^*} |k|^\gamma |c_k|$. Denote by $\|c\|$ the $\ell_2(\mathbb{Z}^*)$ norm of c, $\left(\sum_{k\in\mathbb{Z}^*} |c_k|^2\right)^{\frac{1}{2}}$. We will use these norms in tandem in order to show that the Burgers solutions remain close to the Ornstein-Uhlenbeck solutions. In particular, if these solutions are sufficiently close in a precise sense described below, then the equivalence of the transition measures will follow from relative entropy and Girsanov type arguments.

Let $z(0) = u(0)$ and $\gamma > 2$. Assume furthermore that $\sup_{t\in[0,T]} |z(t)|_{\gamma,\infty} \le \mathcal{D}$ and $\sup_{t\in[0,T]} \|u\| < \mathcal{E}$. Consider the sequence $\rho = \{\rho_k\}_{k\in\mathbb{Z}^*}$ defined by $\rho_k = u_k - z_k$. The dynamics for ρ satisfy an infinite dimensional system of random ordinary differential equations:

$$\frac{d\rho_k(t)}{dt} = -|k|^2 \rho_k(t) - i \sum_{\ell\in\mathbb{Z}^*} \ell u_\ell(t) u_{k-\ell}(t). \tag{2.1}$$

Define $K_0 = K_0(\gamma, \mathcal{E})$ to be the first positive integer for which $1 > 2\mathcal{E}\mathcal{C}_2|k|^{-\frac{1}{2}}$ for all $|k| > K_0$; here, $\mathcal{C}_2$ is a constant defined in Lemma 5.2. Let $\tilde{\mathcal{D}} = \mathcal{E}K_0^\gamma$ and

$\bar{\mathcal{D}} = \max\{\mathcal{D}, \tilde{\mathcal{D}}\}$. Suppose that for some $t \in [0,T]$ and $k \in \mathbb{Z}^*$, $|\rho_k(t)| = \frac{\bar{\mathcal{D}}}{|k|^\gamma}$, and for all $k' \neq k$, $|\rho_{k'}(t)| \leq \frac{\bar{\mathcal{D}}}{|k'|^\gamma}$. Note that the assumptions on z, u and ρ imply that for all $k' \in \mathbb{Z}^*$, $|u_{k'}| \leq \frac{\mathcal{D}}{|k'|^\gamma} + \frac{\bar{\mathcal{D}}}{|k'|^\gamma} \leq \frac{2\bar{\mathcal{D}}}{|k'|^\gamma}$.

Let $G_k(u) = \sum_{\ell \in \mathbb{Z}^*} |\ell||u_\ell||u_{k-\ell}|$. For $|k| \leq K_0$, the definition of $\bar{\mathcal{D}}$ and the inequality $\sup_{t\in[0,T]} \|u\| \leq \mathcal{E}$ imply that $|u_k(t)| \leq \frac{\bar{\mathcal{D}}}{|k|^\gamma}$. For $|k| \geq K_0$, we have

$$\begin{aligned} |k|^2|\rho_k| &= \frac{\bar{\mathcal{D}}}{|k|^\gamma}|k|^2 > \frac{\bar{\mathcal{D}}}{|k|^\gamma}|k|^2(2\mathcal{E}C_2|k|^{-\frac{1}{2}}) = \frac{\bar{\mathcal{D}}}{|k|^\gamma}(2\mathcal{E}C_2|k|^{\frac{3}{2}}) \geq G_k(u(t)) \\ &\geq \left| i \sum_{\ell\in\mathbb{Z}^*} \ell u_\ell(t) u_{k-\ell}(t) \right| . \end{aligned} \tag{2.2}$$

The strict inequality follows from the fact that $|k| > K_0$ and the definition of K_0 while the second inequality follows from Lemma 5.2. This shows that the ρ vector field "points inward on the boundary" providing an absorbing region in the phase space. We have proven the following proposition.

PROPOSITION 2.1. *Assume that* $\gamma > 2$ *and* $z(0) = u(0)$. *If* $\sup_{t\in[0,T]} |z(t)|_{\gamma,\infty} \leq \mathcal{D}$ *and* $\sup_{t\in[0,T]} \|u\| < \mathcal{E}$ *then* $\sup_{t\in[0,T]} |u|_{\gamma,\infty} \leq 2\bar{\mathcal{D}}$.

3. Probabilistic Control of Norms

The following observation is left as an exercise for the reader.

LEMMA 3.1. *If* $\gamma, \epsilon > 0$ *and satisfy* $\limsup_{n\to\infty} |\sigma_n|^2|n|^{2(\gamma+\epsilon)-2} < \infty$ *then for any fixed* T,

$$\lim_{\mathcal{D}\to\infty} \mathbb{P}\left(\sup_{t\in[0,T]} |z(t)|_{\gamma,\infty} < \mathcal{D}\right) = 1. \tag{3.1}$$

In our setting Lemma 3.1 applies for any $\gamma < r + 1$. Next, we need to estimate the probability that $\|u\|$ grows. By a simple martingale estimate analogous to the one used in the proof of Lemma 3.1 of [**MS05**], the following estimate holds.

PROPOSITION 3.2. *Let* $\mathcal{E}^2 = \|u(0)\|^2$ *and* $\mathcal{E}_1 = \sum_{k\in\mathbb{Z}^*} |\sigma_k|^2$. *Assume* $\eta > 1$. *The following estimate holds.*

$$\begin{aligned} &\mathbb{P}\left(\sup_{t\in[0,T]} \|u(t)\|^2 \geq \|u(0)\|^2 + \mathcal{E}_1 T + \left[(\eta-1)\mathcal{E}^2 - \mathcal{E}_1 T\right]\right) \\ &\leq \exp\left\{-\frac{1}{2\sigma_{max}^2}\left[(\eta-1)\mathcal{E}^2 - \mathcal{E}_1 T\right]\right\}, \end{aligned} \tag{3.2}$$

where $\sigma_{max} = \sup_{k\in\mathbb{Z}^*} |\sigma_k|$.

PROOF. By Ito's formula and the Poincaré Inequality,

$$\begin{aligned}
&\left(\|u(t)\|^2 - \|u(0)\|^2\right) \\
= \;& 2\left[-\int_0^t \|\nabla u(s)\|^2 ds + \mathcal{E}_1 t\right] + 4\mathrm{Re}\left\{\int_0^t \sum_{k\in\mathbb{Z}^*} u_k(s)\overline{\sigma_k dB_k(s)}\right\} \\
\leq\; & 2\left[-\int_0^t \|u(s)\|^2 ds + \mathcal{E}_1 t\right] + 4\mathrm{Re}\left\{\int_0^t \sum_{k\in\mathbb{Z}^*} u_k(s)\overline{\sigma_k dB_k(s)}\right\} \\
\leq\; & 2\left[-\frac{1}{|\sigma_{max}|^2}\int_0^t \sum_{k\in\mathbb{Z}^*} |u_k(s)|^2|\sigma_k|^2 ds + \mathcal{E}_1 t\right] + 4\mathrm{Re}\left\{\int_0^t \sum_{k\in\mathbb{Z}^*} u_k(s)\overline{\sigma_k dB_k(s)}\right\}
\end{aligned}$$

Applying the standard L^2 exponential martingale inequality to the first and third term concludes the proof of the proposition. Technically, one needs to localize the local martingale and then apply the inequality, but this is standard fare (See for example [**Mat02, Mat03**]). □

Fixing T and $\epsilon > 0$, assuming that $z(0) = u(0)$ with $\|u(0)\| = \mathcal{E}$, picking $\mathcal{D}$ and γ such that $\mathbb{P}\left(\sup_{t\in[0,T]} |z(t)|_{\gamma,\infty} \leq \mathcal{D}\right) \geq 1 - \frac{\epsilon}{2}$, and picking η sufficiently large that $\mathbb{P}\left(\sup_{t\in[0,T]} \|u(t)\| < \eta\mathcal{E}\right) \geq 1 - \frac{\epsilon}{2}$, Proposition 2.1 implies that

$$\mathbb{P}\left(\sup_{t\in[0,T]} |u(t)|_{\gamma,\infty} \leq 2\bar{\mathcal{D}}\eta\right) \geq 1 - \epsilon, \tag{3.3}$$

where $\bar{\mathcal{D}}$ is defined as in Section 2. The fact that we can pick $\mathcal{D}, \eta$, and γ as prescribed above follows from Lemma 3.1 and Proposition 3.2.

4. Equivalence of Transition Measures

In this section, we consider the diffusion for u on a finite time interval $[0, T]$ and introduce an auxiliary process, $\tilde{u}$, which agrees with u at the times $t = 0$ and $t = T$. $\tilde{u}$ on $[0, T]$ is an Ito process but not a diffusion: In particular, the drift in the $\tilde{u}$ process will be adapted to the appropriate filtration $\{\mathcal{F}_t\}$ and will be better controlled than the drift of the u process.

4.1. The Auxiliary Process. Let $F_k(u(s)) = -i\sum_{\ell\in\mathbb{Z}^*} \ell u_\ell(s)u_{k-\ell}(s)$ be the nonlinear term in the u process (1.2). Setting

$$\tilde{F}_k(s) = \begin{cases} 0 & \text{if } s < \frac{T}{2} \\ 2e^{-|k|^2(T-s)}F_k(u(2s-T)) & \text{if } s \in \left[\frac{T}{2}, T\right] \end{cases}, \tag{4.1}$$

we define $\tilde{u}$ by

$$d\tilde{u}_k(s) = \left[-|k|^2\tilde{u}_k(s) + \tilde{F}_k(s)\right] ds + \sigma_k dB_k(s) \tag{4.2}$$

$$\tilde{u}_k(0) = u_k(0). \tag{4.3}$$

Notice that

$$\tilde{u}_k(T) = e^{-|k|^2 T}u_k(0) + \int_0^T e^{-|k|^2(T-s)}\tilde{F}_k(s)ds + \int_0^T e^{-|k|^2(T-s)}\sigma_k dB_k(s). \tag{4.4}$$

The first and last terms are identical to the first and last terms in the analogous representation of $u_k(T)$. Next, observe that

$$\begin{aligned}\int_0^T e^{-|k|^2(T-s)}\tilde{F}_k(s)ds &= \int_{\frac{T}{2}}^T e^{-|k|^2(T-s)}\tilde{F}_k(s)ds \\ &= \int_{\frac{T}{2}}^T e^{-|k|^2(T-s)}2e^{-|k|^2(T-s)}F_k(u(2s-T))ds. \end{aligned} \tag{4.5}$$

Setting $\tau = 2s - T$ we arrive at the equality

$$\int_0^T e^{-|k|^2(T-s)}\tilde{F}_k(s)ds = \int_0^T e^{-|k|^2(T-\tau)}F_k(u(\tau))d\tau, \tag{4.6}$$

which shows that $u_k(T) = \tilde{u}_k(T)$. This equality only holds only at time T and the distributions of these processes on the path space, $(C([0,T];\ell_2(\mathbb{Z}^*)),\mathcal{F})$, are different.

4.2. The Truncated Process. Denote by $(C([0,T];\ell_2(\mathbb{Z}^*)),\mathcal{F},\{\mathcal{F}_t\},\mathbb{P}_{z_0}) = (\Omega,\mathcal{F},\{\mathcal{F}_t\},\mu)$ the probability space of continuous $\ell_2(\mathbb{Z}^*)$ valued functions equipped with its Borel σ-algebra $\mathcal{F}$ (induced by the sup norm), the filtration $\mathcal{F}_t$ generated by the finite dimensional cylinder sets determined by times $s \le t$, and the Ornstein-Uhlenbeck measure induced by the z process on $[0,T]$ subject to the initial condition $z(0) = z_0$.

Let $T_{O\to B} : \Omega \to \Omega$ be the map which takes an Ornstein-Uhlenbeck path to the Brownian path which generated it; this is a well defined transformation since the Ornstein-Uhlenbeck process is linear and invertible in the obvious sense. Let $U_N : \Omega \to \Omega$ be the map which takes Brownian paths to the following truncated u process:

$$d\tilde{u}_n^{(N)}(t) = \left[-|k|^2\tilde{u}_k^{(N)}(t) + \tilde{F}_k(t)\mathbb{I}_{\tau_N>t}\right]dt + \sigma_k dB_k(t), \quad \tilde{u}_k^{(N)}(0) = z_k(0), \tag{4.7}$$

where $\mathbb{I}_{\tau_N>t}$ is the indicator function on the event $\{\tau_N > t\}$, $\tau_N \equiv \inf\{t \in [0,T] : \|u(t)\| > N \text{ or } |z(t)|_{\gamma,\infty} > N\} \wedge T$, and $\gamma < r+1$ has been fixed. Let $\mathbb{Q}_N = (U_N \circ T_{O\to B})^*\mathbb{P}_{z_0}$ be the sequence of measures on $(\Omega,\mathcal{F},\{\mathcal{F}_t\})$ induced by the truncated $\tilde{u}^{(N)}$ processes and let $\mathbb{Q}$ be the measure induced on $(\Omega,\mathcal{F},\{\mathcal{F}_t\})$ by the $\tilde{u}$ process. Lemma 3.1 and Proposition 3.2 imply that $\mathbb{P}_{z_0}$ almost surely $\lim_{N\to\infty}\tau_N = T$ and that $\lim_{N\to\infty}\mathbb{Q}_N(A) = \mathbb{Q}(A)$ for every $A \in \mathcal{F}$.

If $\mathbb{P}$ and $\mathbb{Q}_N$ are mutually absolutely continuous for every N and if $\mathbb{P}(A) = 0$ then

$$\mathbb{Q}(A) = \lim_{N\to\infty}\mathbb{Q}_N(A) = \lim_{N\to\infty} 0 = 0. \tag{4.8}$$

In this case, $\mathbb{Q} \ll \mathbb{P}$. Theorem 5.3 (Girsanov's Theorem) implies that $\mathbb{P}$ and $\mathbb{Q}_N$ are mutually absolutely continuous for every N if the following sum is finite:

$$\begin{aligned}&\sum_{k\in\mathbb{Z}^*}\frac{1}{|\sigma_k|^2}\int_0^T|\tilde{F}_k^{(N)}(s)|^2ds \\ \le\ & 4\sum_{k\in\mathbb{Z}^*}\left[\sup_{t\in[0,T]}|F_k(u(t))\mathbb{I}_{\tau_N>t}|^2\right]\left[\int_0^T\frac{e^{-2|k|^2(T-t)}}{|\sigma_k|^2}dt\right] \\ \le\ & \sup_{t\in[0,T]}\sum_{k\in\mathbb{Z}^*}\frac{2|F_k(u(t))|^2\mathbb{I}_{\tau_N>t}}{|k|^2|\sigma_k|^2} \le \text{poly(N)}\sum_{k\in\mathbb{Z}^*}\frac{|k|^{2r}|k|^2}{|k|^{2\gamma+2}},\end{aligned} \tag{4.9}$$

where poly(N) is a fixed polynomial which is determined from the bounds given in Lemma 5.1 and Proposition 2.1 and the definition of τ_N. The last sum converges if $\gamma > r + \frac{1}{2}$. This is certainly compatible with the condition that $\gamma < r+1$ which is needed in order to control the z process. Thus, if γ has been chosen to satisfy $r+\frac{1}{2} < \gamma < r+1$, then the above argument implies that $\mathbb{P}$ and $\mathbb{Q}_N$ are mutually absolutely continuous for all N. By the preceding remark, we have shown that $\mathbb{Q} \ll \mathbb{P}$.

4.3. Relative Entropy and Removing the Truncation. In Section 4.2 we proved that $\mathbb{Q} \ll \mathbb{P}$. We now show that $\mathbb{P} \ll \mathbb{Q}$ which completes the proof of the equivalence of the z and $\tilde{u}$ processes on the path space, $(C([0,T];\ell_2(\mathbb{Z}^*)),\mathcal{F})$. This immediately implies the equivalence of the measures induced at time T by the z and $\tilde{u}$ processes. Since $\tilde{u}(T) = u(T)$, showing that $\mathbb{P} \ll \mathbb{Q}$ completes the proof of Theorem 1.1.

Using the estimate in Proposition 3.2 and a simple Gaussian bound in the spirit of Lemma 3.1 in order to establish quantitative control for $r+\frac{1}{2} < \gamma < r+1$, one can show that there are constants $c_1, c_2 > 0$ such that $\mathbb{P}(\tau_N < T) \le c_1 e^{-c_2 N}$. This tail bound will play a key role in the relative entropy estimate which follows.

By Lemma 5.4 it suffices to show that the sequence of relative entropies, $\{H(\mathbb{P}|\mathbb{Q}_N)\}$, is uniformly bounded in N: $\sup_N \int \log\left(\frac{d\mathbb{P}}{d\mathbb{Q}_N}\right) d\mathbb{P} < M < \infty$. Since the Radon-Nikodym derivative, $\frac{d\mathbb{P}}{d\mathbb{Q}_N}$, is a local exponential martingale, it suffices to show that

$$\int \left\{ \int_0^T \sum_{k\in\mathbb{Z}^*} \frac{|\tilde{F}_k(s)|^2}{|\sigma_k|^2} ds \right\} d\mathbb{P} < \infty. \tag{4.10}$$

To prove this we use a stopping time argument and Fatou's lemma.

$$\begin{aligned}
&\int \left[\int_0^T \sum_{k\in\mathbb{Z}^*} \frac{|\tilde{F}_k(s)|^2}{|\sigma_k|^2} ds \right] d\mathbb{P} \le \lim_{N\to\infty} \mathbb{E}\left[\sum_{k\in\mathbb{Z}^*} \int_0^T \frac{|\tilde{F}_k^{(N)}(s)|^2}{|\sigma_k|^2} ds \right] \\
=\ & \lim_{N\to\infty} \left\{ \sum_{k\in\mathbb{Z}^*} \left(\mathbb{E}\left[\int_0^T \frac{|\tilde{F}_k(s)|^2}{|\sigma_k|^2} ds \mathbb{I}_{\tau_N = T} \right] + \mathbb{E}\left[\int_0^T \frac{|\tilde{F}_k^{(N)}(s)|^2}{|\sigma_k|^2} ds \mathbb{I}_{\tau_N < T} \right] \right) \right\} \\
\le\ & \lim_{N\to\infty} \sum_{k\in\mathbb{Z}^*} \mathbb{E}\left[\int_0^T \frac{|\tilde{F}_k(s)|^2}{|\sigma_k|^2} ds \mathbb{I}_{\tau_N = T} \right] + \lim_{N\to\infty} \left[\text{poly(N)}\ c_1 e^{-c_2 N} \sum_{k\in\mathbb{Z}^*} \frac{|k|^{2r}|k|^2}{|k|^{2\gamma+2}} \right] \\
=\ & \lim_{N\to\infty} \sum_{k\in\mathbb{Z}^*} \mathbb{E}\left[\int_0^T \frac{|\tilde{F}_k(s)|^2}{|\sigma_k|^2} ds \mathbb{I}_{\tau_N = T} \right] \\
=\ & \lim_{N\to\infty} \sum_{l=1}^N \sum_{k\in\mathbb{Z}^*} \mathbb{E}\left[\int_0^T \frac{|\tilde{F}_k(s)|^2}{|\sigma_k|^2} ds \mathbb{I}_{\tau_l > T \ge \tau_{l-1}} \right] \\
\le\ & \left[\sum_{k\in\mathbb{Z}^*} \frac{|k|^{2r}|k|^2}{|k|^{2\gamma+2}} \right] \lim_{N\to\infty} \sum_{l=1}^N \text{poly(l)}\ e^{-c(l-1)} < \infty.
\end{aligned}$$

This concludes the proof of $\mathbb{P} \ll \mathbb{Q}$ and, by the above discussion, the proof of Theorem 1.1.

5. Technical Lemmas

5.1. Bounds on Nonlinearity. We prove several technical lemmas which provide bounds for the Burgers nonlinearity when the solutions satisfy certain conditions. Given a sequence $a = \{a_k\}_{k\in\mathbb{Z}^*} \in \ell_2(\mathbb{Z}^*)$ we set $G_k(a) = \sum_{\ell\in\mathbb{Z}^*} |\ell||a_\ell||a_{k-\ell}|$.

LEMMA 5.1. *If $\gamma > 2$ and $|a|_{\gamma,\infty} = \mathcal{D}$, then $G_k(a) \le \frac{\mathcal{C}_1\mathcal{D}^2}{|k|^\gamma}|k|$, where $\mathcal{C}_1$ depends only on γ.*

PROOF. We begin by breaking the sum $\sum_{\ell\in\mathbb{Z}^*}$ in the definition of G_k into three sums:

$$\sum_{\ell\in\mathbb{Z}^*} = \sum_1 + \sum_2 + \sum_3 = \sum_{|\ell|\le\frac{|k|}{2}} + \sum_{\frac{|k|}{2}<|\ell|\le 2|k|} + \sum_{|\ell|>2|k|} . \tag{5.1}$$

We estimate each of these sums separately. We begin with $\sum_1$:

$$\begin{aligned}\sum_1 \le\ & \left(\frac{2}{|k|}\right)^\gamma \sum_{|\ell|\le\frac{|k|}{2}} |\ell||a_\ell| \left(\frac{|k|}{2}\right)^\gamma |a_{k-\ell}| \le \left(\frac{2}{|k|}\right)^\gamma \sum_{|\ell|\le\frac{|k|}{2}} |\ell||a_\ell||k-\ell|^\gamma|a_{k-\ell}| \\ \le\ & \frac{2^\gamma\mathcal{D}}{|k|^\gamma}\sum_{|\ell|\le\frac{|k|}{2}} |\ell||a_\ell| \le \frac{2^\gamma\mathcal{D}^2}{|k|^\gamma}\sum_{|\ell|\le\frac{|k|}{2}} |\ell|^{1-\gamma} \le \frac{2^\gamma\mathcal{D}^2}{|k|^\gamma}c_1,\end{aligned}$$

where c_1 is a constant which only depends on γ. Next, we estimate $\sum_2$:

$$\begin{aligned}\sum_2 \le\ & \frac{2^\gamma\mathcal{D}}{|k|^\gamma}\sum_{\frac{|k|}{2}<|\ell|\le 2|k|} |\ell||a_{k-\ell}| \le \frac{2^\gamma\mathcal{D}}{|k|^\gamma}|k| \sum_{\frac{|k|}{2}<|\ell|\le 2|k|} \frac{|\ell|}{|k|}|a_{k-\ell}| \\ \le\ & \frac{2^{\gamma+1}\mathcal{D}}{|k|^\gamma}|k| \sum_{\frac{|k|}{2}<|\ell|\le 2|k|} |a_{\ell-k}| \le \frac{2^{\gamma+1}\mathcal{D}}{|k|^\gamma}|k| \sum_{|\ell|\le 3|k|} |a_\ell| \\ \le\ & \frac{2^{\gamma+1}\mathcal{D}^2}{|k|^\gamma}|k| \sum_{|\ell|\le 3|k|} |\ell|^{-\gamma} \le \frac{2^{\gamma+1}\mathcal{D}^2}{|k|^\gamma}c_2|k|,\end{aligned}$$

where c_2 depends only on γ. Finally, we estimate $\sum_3$:

$$\begin{aligned}\sum_3 \le\ & \sum_{|\ell|>2|k|} |\ell|^{1-\gamma}|\ell|^\gamma|a_\ell||a_{k-\ell}| \le \mathcal{D}\sum_{|\ell|>2|k|} |\ell|^{1-\gamma}|a_{k-\ell}| \\ =\ & \mathcal{D}\sum_{|\ell|>2|k|} |\ell|^{1-\gamma}|k-\ell|^{-\gamma}||k-\ell|^\gamma|a_{k-\ell}| \le \mathcal{D}^2 \sum_{|\ell|>2|k|} |\ell|^{1-\gamma}|k-\ell|^{-\gamma} \\ =\ & \frac{\mathcal{D}^2}{|k|^\gamma}\sum_{|\ell|>2|k|} |\ell|^{1-\gamma}\left|\frac{k}{k-\ell}\right|^\gamma \le \frac{\mathcal{D}^2}{|k|^\gamma}\sum_{|\ell|>2|k|} |\ell|^{1-\gamma} \le \frac{\mathcal{D}^2}{|k|^\gamma}c_3|k|^{2-\gamma},\end{aligned}$$

where c_3 depends only on γ. This concludes the proof that $G_k(a) \le \frac{\mathcal{C}_1\mathcal{D}^2}{|k|^\gamma}|k|$. □

LEMMA 5.2. *If $\gamma > 2$, $\|a\| = \mathcal{E}$, and $|a|_{\gamma,\infty} = \mathcal{D}$, then $G_k(a) \le \frac{\mathcal{C}_2\mathcal{D}}{|k|^\gamma}\mathcal{E}|k|^{\frac{3}{2}}$ where $\mathcal{C}_2$ depends only on γ.*

PROOF. Exactly as in the proof of Lemma 5.1, we break the sum, $\sum_{\ell\in\mathbb{Z}*}$, into three parts, $\sum_1, \sum_2, \sum_3$, and estimate each part separately. We begin by estimating $\sum_1$:

$$\sum_1 \le \frac{2^\gamma \mathcal{D}}{|k|^\gamma} \sum_{|\ell|\le \frac{|k|}{2}} |\ell||a_\ell| \le \frac{2^\gamma \mathcal{D}}{|k|^\gamma}\mathcal{E}\left[\sum_{|\ell|\le\frac{|k|}{2}} |\ell|^2\right]^{\frac{1}{2}} \le \frac{2^\gamma\mathcal{D}}{|k|^\gamma}\tilde{c}_1\mathcal{E}|k|^{\frac{3}{2}},$$

where $\tilde{c}_1$ depends only on γ. Next, we estimate $\sum_2$:

$$\sum_2 \le \frac{2^{\gamma+1}\mathcal{D}}{|k|^\gamma}|k| \sum_{\frac{|k|}{2}<|\ell|\le 2|k|} |a_{k-\ell}| \le \frac{2^{\gamma+1}\mathcal{D}}{|k|^\gamma}|k|\mathcal{E}\left[\sum_{\frac{|k|}{2}<|\ell|\le 2|k|} 1\right]^{\frac{1}{2}} \le \frac{2^{\gamma+1}\mathcal{D}}{|k|^\gamma}\tilde{c}_2\mathcal{E}|k|^{\frac{3}{2}},$$

where $\tilde{c}_2$ depends only on γ. Finally, we estimate $\sum_3$:

$$\sum_3 \le \mathcal{D}\sum_{|\ell|>2|k|} |\ell|^{1-\gamma}|a_{k-\ell}| \le \mathcal{D}\tilde{c}_3\mathcal{E}\left(|k|^{3-2\gamma}\right)^{\frac{1}{2}} = \frac{\mathcal{D}}{|k|^\gamma}\tilde{c}_3\mathcal{E}|k|^{\frac{3}{2}},$$

where $\tilde{c}_3$ depends only on γ. This completes the proof that $G_k(a) \le \frac{C_2\mathcal{D}}{|k|^\gamma}\mathcal{E}|k|^{\frac{3}{2}}$. □

5.2. A Version of Girsanov's Theorem. Suppose that we have stochastic processes $X_i(t)$, $i = 1, 2$ on the path space $C([0,T],\mathbb{X})$ where $\mathbb{X}$ is some separable Hilbert space and $T \in (0,\infty]$. Furthermore, assume that X_i satisfies the equation

$$\begin{aligned} dX_i(t) &= f_i(t, X_i[0,t])dt + g dW(t),\ t \in [0,T] \\ X_i(0) &= x_0. \end{aligned} \tag{5.2}$$

For fixed t, the functions f_1 and f_2 map the space $C_{[0,t]} = C([0,t],\mathbb{X})$ to $\mathbb{X}$. By $X[0,t]$ we mean the segment of the trajectory on $[0,t]$. $W(t)$ is a cylindrical Brownian motion over a separable Hilbert space $\mathbb{Y}$ and g is a fixed invertible Hilbert-Schmidt operator from $\mathbb{Y} \to \mathbb{X}$. For any $\mathcal{B} \subset C_{[0,T]}$, define measures $P^{(i)}_{[0,T]}(\cdot;\mathcal{B})$ on the path space as:

$$P^{(i)}_{[0,T]}(A;\mathcal{B}) = \mathbb{P}\{X_i[0,T] \in A \cap \mathcal{B}\},\ \text{for } A \subset C_{[0,T]}.$$

Define also $D(t,\cdot) = f_1(t,\cdot) - f_2(t,\cdot)$.

In this setting, we have the following result which is a variation on Lemma B.1 from [**Mat02**] and follows quickly from Girsanov's Theorem.

THEOREM 5.3. *Assume there exists a constant $D_* \in (0,\infty)$ such that*

$$\exp\left\{\frac{1}{2}\int_0^T \left|g^{-1}D\big(t, X_i[0,t]\big)\right|^2_{\mathbb{Y}} dt\right\}\mathbb{I}_{\mathcal{B}}(X_i[0,t]) < D_* \tag{5.3}$$

almost surely for $i = 1, 2$. Then the measures $P^{(1)}_{[0,T]}(\cdot;\mathcal{B})$ and $P^{(2)}_{[0,T]}(\cdot;\mathcal{B})$ are equivalent.

5.3. Basic Relative Entropy Inequality. The following lemma provides a condition which, if satisfied, implies the absolute continuity of a fixed probability measure with respect to a probability measure which is the limit measure of sequence of measures. It can be thought of a way of establishing a type of uniform integrability in the present setting.

LEMMA 5.4. *Let $(\Omega, \mathcal{F}, \mu)$ be a probability space and let $(W, \mathcal{W})$ be a Polish space equipped with its Borel σ-algebra. Let $T : \Omega \to W$ and $T_n : \Omega \to W$, $n = 1, 2, \cdots$, be measurable transformations. Let $\mathbb{P} = T^*\mu$ and $\mathbb{Q}_n = T_n^*\mu$ be their respective push forward measures on W. Assume that there is a probability measure $\mathbb{Q}$ on W such that for any measurable $A \in \mathcal{W}$, $\mathbb{Q}_n(A) \to \mathbb{Q}(A)$. If $\mathbb{P} \ll \mathbb{Q}_n$, $\mathbb{Q}_n \ll \mathbb{P}$, and $\limsup_{n\to\infty} H(\mathbb{P}|\mathbb{Q}_n) = \limsup_{n\to\infty} \left|\int \frac{d\mathbb{P}}{d\mathbb{Q}_n} \log \frac{d\mathbb{P}}{d\mathbb{Q}_n} d\mathbb{Q}_n\right| < M < \infty$, then $\mathbb{P} \ll \mathbb{Q}$.*

PROOF. Denote by $H(\mu|\nu) = \int \frac{d\mu}{d\nu} log \frac{d\mu}{d\nu} d\nu$ the relative entropy of the probability measure μ with respect to ν (when it exists). We begin by proving a well-known relative entropy inequality. If μ and ν are mutually absolutely continuous, $f \in L^1(\mu)$, and $H(\mu|\nu) < \infty$, then

$$\int f d\mu \le H(\mu|\nu) + \log\left(\int e^f d\nu\right).$$

This inequality follows from the simple calculation:

$$\begin{aligned}\int f d\mu - \log\left(\int e^f d\nu\right) &= \int f d\mu - \log\left(\int e^f \frac{d\nu}{d\mu} d\mu\right) \\ &\le \int f d\mu - \int \log\left(e^f \frac{d\nu}{d\mu}\right) d\mu = \int \log \frac{d\mu}{d\nu} d\mu = H(\mu|\nu).\end{aligned}$$

In particular, for any $c > 0$ the inequality becomes

$$\int f d\mu \le \frac{1}{c} H(\mu|\nu) + \frac{1}{c} \log\left(\int e^{cf} d\nu\right).$$

Letting $f = \chi_A$, the characteristic function of a set $A \in \mathcal{W}$, this inequality specialized to $\mathbb{P}$ and $\mathbb{Q}_n$ becomes

$$\mathbb{P}(A) \le \frac{1}{c} H(\mathbb{P}|\mathbb{Q}_n) + \frac{1}{c} \log((e^c - 1)\mathbb{Q}_n(A) + 1).$$

Fix $c > 0$. If $\mathbb{Q}(A) = 0$, then $\mathbb{Q}_n(A) \to 0$ by assumption. Since $\limsup H(\mathbb{P}|\mathbb{Q}_n) < M < \infty$, as $n \to \infty$ the right hand side is bounded by $\frac{2M}{c}$. $\mathbb{P}(A) = 0$ since c is arbitrary. Thus, $\mathbb{P} \ll \mathbb{Q}$. □

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