Algebraic Data Structures for Decomposing Multipersistence Modules

Joey Li

November 24, 2020

Abstract

Single-parameter persistent homology techniques in topological data analysis have seen increasing usage in recent years. These techniques have found particular success because of the existence of a complete, discrete, efficiently computable invariant to describe persistence modules in the single-parameter case: the barcode. Attempts to develop an equally robust theory of multiparameter persistent homology, however, have been slow to progress because there is no natural multiparameter analogue to the barcode. Relatively little is known about the structure of decompositions of multiparameter persistence (multipersistence) modules or how to classify their indecomposables. In fact, even for the problem of computing decompositions, there currently is no generalization to multiple parameters of the decomposition algorithm from single-parameter persistent homology. In this paper, we define a new algebraic data structure, the QR code, which was first proposed in [1] but was formulated somewhat erroneously. Additionally, we prove a theorem stating that the QR code recovers all the information of the module it encodes. We suggest that this new data structure, which seeks to encode a module using births and deaths rather than births and relations, may be the correct language in which to solve the problem of decomposing arbitrary finitely generated multipersistence modules.

1 Introduction

1.1 Motivation

Single-parameter persistent homology is a technique which applies topological ideas to data analysis problems, and which has been applied to solve problems in a wide range of areas over the past two decades. In applying persistent homology, one constructs an algebraic object, called the persistence module, which encodes topological structure arising from the data. One of the main reasons for the success of single-parameter persistent homology as a computational tool is the existence of a complete discrete invariant to describe a persistence module. In particular, any finitely generated persistence module may be viewed as a $\mathbb{Z}$-graded module over $\Bbbk[t]$, and thus has a direct sum decomposition into indecomposable modules. Furthermore, by the structure theorem for finitely generated modules over PID's, these indecomposable modules may be completely characterized by their lowest and highest grades with nonzero component.[1] In particular, the set of indecomposables for single-parameter persistence modules can be parameterized by pairs of integers, which yields a simple description of a module in terms of a discrete invariant, called the barcode. Moreover, computing a decomposition of a module, hence its barcode, can be done efficiently in the single-parameter setting. When extending to multiparameter persistence however, there is no such decomposition theorem. In fact, not only is there no simple way to list indecomposable multipersistence modules [3], there is currently no general algorithm to decompose multipersistence modules in a way that preserves the intuition of births and deaths from the single-parameter setting [4], which would be important toward understanding not only how to decompose multipersistence modules, but what indecomposable multipersistence modules look like.

This paper seeks to provide a first step toward computing decompositions of multipersistence modules. In [4], the authors propose an algorithm for decomposing multipersistence modules which works in the setting where no two generators or relations appear in the same graded degree. However, the problem of decomposition is still unresolved in the general finitely generated setting. We present a definition of births and deaths in the multiparameter setting which seeks to generalize the notions from the single-parameter setting, and following, define a data structure, the QR code, which captures all the information of a multipersistence module through its births and deaths. We then phrase the problem of decomposing arbitrary multipersistence modules, herein referred to as the decomposition problem, in the language of the QR code, which we believe is the right language in which to develop an interpretable algorithm to solve the decomposition problem.

1.2 Related Work

The main theoretical result in this paper, Definition 3.4 and Theorem 3.5 regarding the QR code, originates from arXiv preprint [1], where it was defined in the far more general $\mathbb{R}^n$-graded case. However, there was an issue in the proof of Theorem 16.5 in that paper due to an issue in the definition of the death module in Definition 15.13. This paper represents an initial attempt to fix that issue and give the correct definition and theorem in the restricted setting of finite-length, $\mathbb{Z}^n$-graded multipersistence modules, which we anticipate will generalize to the arbitrary $\mathbb{R}^n$-graded setting.

1.3 Useful References

To understand the main algebraic result of this paper, a background understanding of persistent homology is not strictly required, though the motivation for the algebraic work in this paper is firmly rooted in computational topology. For example, the definition of a multipersistence module is motivated by abstracting the topological information obtained from multifiltrations on topological spaces, which in practice would correspond to filtrations on a dataset. Similarly, the focus on flat covers and injective hulls is directly motivated by a desire to extend the birth-death viewpoint from single-parameter persistent homology to a multiparameter setting. For a general survey on persistent homology written by two im-

[1] For more information regarding the single-parameter case, see [2].

portant figures in the field, see [5]. For an exposition of the main mathematical framework of multiparameter persistence, see [6]. Finally, for a recent, more informal exposition of the main themes of persistent homology, especially multiparameter persistent homology, see [7]. This last reference is a fairly comprehensive reference on multipersistent homology, and should more than suffice to follow the discussion in this paper.

2 Background

This paper takes a minimalist approach, introducing only the algebraic definitions which are strictly necessary to understand the main result and its importance. A more detailed exposition of the full algebraic picture surrounding the data structures we assemble may be found in [1]: see [8] and [9], which are the corrected versions of the relevant parts of [1]. None of the definitions in this section are original, and in particular, most are taken directly from [1].

Definition 2.1. Let Q be a partially ordered set (poset) with partial order $\preceq$. A *module over Q* or *Q-module* is

- a Q-graded vector space $M = \bigoplus_{q \in Q} M_q$ with
- maps $M_q \to M_{q'}$ for all $q \preceq q'$ such that
- $M_q \to M_{q''}$ is equal to the map $M_q \to M_{q'} \to M_{q''}$ for any $q \preceq q' \preceq q''$

A homomorphism $\varphi : M \to N$ of Q-modules is a degree-preserving linear map, that is, a set of linear maps $\varphi_q : M_q \to N_q$ for all $q \in Q$ which commute with the structure homomorphisms $M_q \to M_{q'}$, $N_q \to N_{q'}$.

Remark. More concisely, one may consider the poset Q to be a category with objects corresponding to elements of the poset and morphisms given by the poset relations, and then a Q-module M is a functor $Q \to \mathbf{Vect}$. A homomorphism between Q-modules is then a natural transformation between functors. In general, we will view M as a graded vector space with structure morphisms, but defining M as a functor nicely encapsulates the relations which define a poset module.

Definition 2.2. Let $(P, \preceq) \subset (Q, \preceq)$ be a subposet, and let M be a Q-module. Then the restriction of M to P is the P-module $M|_P$ with $(M|_P)_p = M_p$ for $p \in P$, and with maps $(M|_P)_p \to (M|_P)_{p'}$ induced by the corresponding maps $M_p \to M_{p'}$.

Remark. In the language of functors, the restriction of a Q-module M to a subposet P is the composition of the functor $Q \xrightarrow{M} \mathbf{Vect}$ with the natural inclusion functor $P \xrightarrow{\iota} Q$.

For the purposes of this paper, we will mostly focus on a specific class of poset modules, which arise directly from (multiparameter) persistent homology.

Definition 2.3. A *persistence module* is a module over $(\mathbb{Z}, \leq)$. A *multiparameter persistence (multipersistence) module in n variables* is a module over $(\mathbb{Z}^n, \preceq)$ where $\mathbf{a} \preceq \mathbf{b}$ if $\mathbf{a} = (a_1, \ldots, a_n)$, $\mathbf{b} = (b_1, \ldots, b_n)$ satisfies $a_i \leq b_i$ for each $i = 1, \ldots, n$.

Now crucially, a persistence module can be viewed as a $\mathbb{Z}$-graded modules over $\Bbbk[t]$, where t acts on the vector space at each grade i by the structure morphism $M_i \to M_{i+1}$. This allows for the construction of the barcode, by noting that $\Bbbk[t]$ is a PID and applying the structure theorem for finitely generated modules over PID's to get a decomposition structure.

In the same way, one may view a multipersistence module as a $\mathbb{Z}^n$-graded module over $\Bbbk[t_1, \ldots, t_n]$, where t_i acts on the vector space at each grade $\mathbf{b}$ by the structure morphism $M_{\mathbf{b}} \to M_{\mathbf{b}+\mathbf{e}_i}$. Since $\Bbbk[t_1, \ldots, t_n]$ is not a PID, the problem of decomposing multipersistence modules is more complex.

If we would like to generalize barcodes to multipersistence modules, we must correctly characterize births and deaths in the multiparameter setting.

First, let $\Bbbk[\mathbf{b}+\mathbb{N}^n]$ denote the multipersistence module having a copy of $\Bbbk$ at each degree $\mathbf{v} \succeq \mathbf{b}$ and 0 at all other degrees, with identity morphisms between all copies of $\Bbbk$. Then note that a finitely generated free $\Bbbk[t_1, \ldots, t_n]$-module F can be written as a direct sum $\bigoplus_{\mathbf{b}\in\mathbb{Z}^n} \bigoplus_{i=1}^{n(\mathbf{b})} \Bbbk[\mathbf{b}+\mathbb{N}^n]$ where $n(\mathbf{b})$ is the number of copies of $\Bbbk[\mathbf{b}+\mathbb{N}^n]$. Then any finitely generated $\Bbbk[t_1, \ldots, t_n]$-module has a surjection from a free $\Bbbk[t_1, \ldots, t_n]$-module.

Definition 2.4. The *minimal free cover* of a finitely generated multipersistence module M is a free $\Bbbk[t_1, \ldots, t_n]$-module F along with a surjection $\varphi : F \to M$ such that the map

$$\mathbb{1} \otimes_{\Bbbk[t_1,\ldots,t_n]} \varphi : \Bbbk \otimes_{\Bbbk[t_1,\ldots,t_n]} F \to \Bbbk \otimes_{\Bbbk[t_1,\ldots,t_n]} M$$

is an isomorphism, where $\Bbbk$ is the $\Bbbk[t_1, \ldots, t_n]$-module with a copy of $\Bbbk$ at grade $\mathbf{0}$ and 0 elsewhere.

One may check, as [6] does, that a minimal free cover is unique up to automorphism. We can relax some parts of the definition above to get a more general definition.

Definition 2.5. A *flat cover* of a module M is a surjection $F \to M$ from a flat module F.

Remark. It is a slight confusion of notation to refer to both a free cover and a flat cover of M as F, but functionally this is not a huge deal, as they will serve the same purpose in all of our discussion. Moreover, in the setting we work in, the distinction between the two is moot – by Serre's problem, free and flat modules over $\Bbbk[t_1, \ldots, t_n]$ are equivalent in the finitely generated case.

Now as noted above, a finitely generated free module F is a direct sums of copies of $\Bbbk[\mathbf{b}+\mathbb{N}^n]$ which naturally correspond to submodules generated by generating elements. Then the images of these elements under a map of a free hull $F \to M$ correspond to generators of M. We will make this notion more precise. Let $I = (t_1, \ldots, t_n) \subset k[t_1, \ldots, t_n]$ be the ideal generated by $t_1, \ldots, t_n$.

Definition 2.6. A *generator* m in degree $\mathbf{b}$ of a module M is an element such that $\overline{m} \in M/IM$ is nonzero.

In particular, a generator is any element which is not mapped to by some element of strictly smaller degree via a structure morphism of M. We can also give another characterization of generators, which turns out to be useful as well.

Definition 2.7. The *extant* submodule of M at degree $\mathbf{v}$ is the submodule $M_{\preceq \mathbf{v}}$ of elements generated in degree at most $\mathbf{v}$. The *elder* submodule of M at degree $\mathbf{v}$ is the submodule $M_{\prec \mathbf{v}}$ of elements generated in degree strictly less than $\mathbf{v}$. A *generator* in degree $\mathbf{v}$ is an element of $M_{\preceq \mathbf{v}}$ which is not in $M_{\prec \mathbf{v}}$.

Remark. This definition, which was correctly formulated in Section 17 of [1], is from the section of [1] which contained the incorrect definition of QR codes, and for which this current paper is a preliminary corrected version.

Now it is not quite right to associate generators with births, as the choice of generator is basis-dependent, in that there might exist multiple values of m which lift from the same value $\overline{m} \in M/IM$. Correspondingly, we define birth as follows.

Definition 2.8. The *top* of a multipersistence module M is the quotient M/IM. A *birth* in M is an element in M/IM.

In summary, births are not equivalent to generators, but generators are lifts of births to the module M. The number of distinct minimal generators in degree $\mathbf{b}$ of M is equal to the (vector space) dimension of $(M/IM)_{\mathbf{b}}$, which is an invariant of the module.

We make one final definition, whose use will become more evident later in section 3:

Definition 2.9. Let $\deg \operatorname{top} M$ be the set of degrees associated to nonzero vector spaces in $\operatorname{top} M = M/IM$. This is an invariant of the module which records the birth degrees of M.

Now we would like to construct deaths in a dual manner. In particular, we take an alternate approach to that in [4] and most other papers in TDA which represent multipersistence modules by free presentations – that is, with generators and relations. Instead, we claim that the dual to a generator is a socle element and the dual to a birth is a death, as we will see shortly.

Definition 2.10. A *socle element* in grade $\mathbf{v}$ of a multipersistence module M is an element $m \in M_{\mathbf{v}}$ such that $t_i m = 0$ for all $i = 1, \ldots, n$. The *socle* of M, $\operatorname{soc}(M)$ is the set of all socle elements of M.

In particular, $\operatorname{soc}(M)$ is naturally a submodule of M, and has a nice description via $\operatorname{soc}(M) = \bigoplus_{\mathbf{v} \in \mathbb{Z}^n} \operatorname{Hom}_{\Bbbk[t_1,\ldots,t_n]}(\Bbbk[\mathbf{v}], M)$ where $\Bbbk[\mathbf{v}]$ is the module with a copy of $\Bbbk$ at degree $\mathbf{v}$ and zeros everywhere else.

Now a socle element is any element which is mapped to zero under any structure morphism, while a generator is an element which is not mapped to under any structure morphism. It is in this sense that socle elements are dual to generators, and thus, should represent deaths.

As free modules encoded generators for the module M, we now define injective hulls to encode cogenerators for M, which loosely correspond to socle elements.

Definition 2.11. An *injective hull* is an injective module E along with an injection $M \hookrightarrow E$. An injective hull is *minimal* if $\operatorname{soc}(M) \to \operatorname{soc}(E)$ is an isomorphism.

Again, as with the top, we define the following object with the socle.

Definition 2.12. Let $\deg \operatorname{soc} M$ be the set of degrees associated to nonzero vector spaces in $\operatorname{soc} M$. This is an invariant of the module, and records the birth degrees of M.

Finally, toward the goal of creating a data structure that encodes a multipersistence module via its births and deaths, we define the flange presentation, first defined in [8].

Definition 2.13 ([8, Def. 5.12])**.** A *flange presentation* of M is a homomorphism $\varphi : F \to E$ from a flat module F to an injective module E such that $\varphi(F) \cong M$. A flange presentation is *minimal* if $F \twoheadrightarrow M$ is a minimal free hull and $M \hookrightarrow E$ is a minimal injective hull.

Remark. The term flange comes from a portmanteau of "flat" and "injective" (flainj).

Remark. We define the flange presentation using flat modules rather than free modules in order that we may generalize it more readily in future applications. However, in this paper, we only consider finitely generated modules, and in this context, flat modules are free modules as remarked above.

In the setting where M is of finite length, F is given by a set of generators and E is given by a set of cogenerators, and thus a flange presentation is constructed to contain all the information about where any given generator dies along each socle element.

3 The QR Code

In this section, we define the QR code, which seeks to encode a module succinctly by its relevant maps between births and deaths, and prove that it indeed captures all the data of a multipersistence module. Throughout this section, let M be a finite-length $\mathbb{Z}^n$-graded module over $\Bbbk[t_1, \ldots, t_n]$. We expect these results to hold for finitely generated $\mathbb{Z}^n$-graded modules as well, which is the main case we are concerned with from a computational perspective.

Definition 3.1. The *birth poset* $\mathcal{B}_M$ of M is the subposet of $\mathbb{Z}^n$ with only the set of grades in $\deg \operatorname{top} M$. The *birth module* of M is the graded vector space $\operatorname{Birth} M = \bigoplus_{\mathbf{v} \in \deg \operatorname{top} M} M_{\mathbf{v}}$, which has a poset module structure over $\mathcal{B}_M$. To be precise, $\operatorname{Birth} M$ is the restriction of the $\mathbb{Z}^n$-module M to the subposet $\mathcal{B}_M \subset \mathbb{Z}^n$.

Definition 3.2. The *death poset* $\mathcal{D}_M$ is the subposet of $\mathbb{Z}^n$ with only the set of grades in $\deg \operatorname{soc} M$. The *death module* of M is the graded vector space $\operatorname{Death} M = \prod_{\mathbf{v} \in \deg \operatorname{soc} M} M_{\mathbf{v}}$, which has a poset module structure over $\mathcal{D}_M$. To be precise, $\operatorname{Death} M$ is the restriction of the $\mathbb{Z}^n$-module M to the subposet $\mathcal{D}_M \subset \mathbb{Z}^n$.

Definition 3.3. For any degree $\mathbf{a} \in \deg \operatorname{soc} M$, define the *headstone map* $\partial_{\mathbf{a}} : \operatorname{Birth} M \to \operatorname{Death}_{\mathbf{a}} M$ by letting $(\partial_{\mathbf{a}})|_{\operatorname{Birth}_{\mathbf{b}} M}$ be the natural map $M_{\mathbf{b}} \to M_{\mathbf{a}}$ induced by the $\Bbbk[t_1, \ldots, t_n]$ module structure of M for all birth degrees $\mathbf{b} \preceq \mathbf{a}$.

The headstone map plays well with the poset module structure by constructon, since it relies on the fact that M is a functor. Additionally, note the headstone map is additive, since the maps $M_{\mathbf{b}} \to M_{\mathbf{a}}$ induced by the $\Bbbk[t_1, \ldots, t_n]$ module structure of M are additive.

Definition 3.4. The *QR code* of a $\mathbb{Z}^n$-graded module M over $\Bbbk[t_1, \ldots, t_n]$ is the map

$$\partial = \prod_{\mathbf{a} \in \deg \operatorname{soc} M} \partial_{\mathbf{a}} : \operatorname{Birth} M \to \operatorname{Death} M$$

induced by the universal property of the product.

Note this is not quite a morphism of modules over posets, because the underlying posets of Birth M and Death M are not the same, but we do generally have

$$\text{Birth}_{\mathbf{b}}M \xrightarrow{\partial_{\mathbf{a}}} \text{Death}_{\mathbf{a}}M \to \text{Death}_{\mathbf{a}'}M$$

is the same map as

$$\text{Birth}_{\mathbf{b}}M \xrightarrow{\partial_{\mathbf{a}'}} \text{Death}_{\mathbf{a}'}M$$

for all degrees $\mathbf{b} \preceq \mathbf{a} \preceq \mathbf{a}'$.

Now we claim that the QR code records all the relevant information of the module M.

Theorem 3.5. *Let M be a finite-length $\mathbb{Z}^n$-module.*

1. *We can recover M functorially from its QR code as the image of a natural homomorphism*
$$\bigoplus_{\mathbf{b}\in\mathcal{B}_M} \Bbbk[\mathbf{b}+\mathbb{N}^n] \otimes_{\Bbbk} \text{Birth}_{\mathbf{b}}M \xrightarrow{\widehat{\partial}} \prod_{\mathbf{a}\in\mathcal{D}_M} \Bbbk[\mathbf{a}-\mathbb{N}^n] \otimes_{\Bbbk} \text{Death}_{\mathbf{a}}M.$$

2. *We can naturally recover a minimal flange presentation of M from the flange presentation above.*

Proof. For any fixed $\mathbf{b} \in \mathcal{B}_M$ and $\mathbf{a} \in \mathcal{D}_M$, there is a natural map

$$\Bbbk[\mathbf{b}+\mathbb{N}^n] \otimes_{\Bbbk} \text{Birth}_{\mathbf{b}}M \xrightarrow{i\otimes\partial_{\mathbf{a}}} \Bbbk[\mathbf{a}-\mathbb{N}^n] \otimes_{\Bbbk} \text{Death}_{\mathbf{a}}M$$

which is simply given by $i \otimes \partial_{\mathbf{a}}$ where i is the natural map $\Bbbk[\mathbf{b}+\mathbb{N}^n] \to \Bbbk[\mathbf{a}-\mathbb{N}^n]$ given by $1 \in \Bbbk$ for each degree contained in both sets $\mathbf{b}+\mathbb{N}^n$ and $\mathbf{a}-\mathbb{N}^n$ and 0 else. This map is in fact a homomorphism since our death functor $\partial_{\mathbf{a}}$ is additive. Then this naturally extends to the map $\widehat{\partial}$ by universality of the direct sum on the left of equation **??** and the direct product on the right of equation **??**.

Now we claim $\widehat{\partial}$ factors through M, and in fact has image M. First, note any map $\Bbbk[\mathbf{b}+\mathbb{N}^n] \otimes_{\Bbbk} \text{Birth}_{\mathbf{b}}M \xrightarrow{i\otimes\partial_{\mathbf{a}}} \Bbbk[\mathbf{a}-\mathbb{N}^n] \otimes_{\Bbbk} \text{Death}_{\mathbf{a}}M$ factors through M. In particular, we may define the map $f_{\mathbf{b}} : \Bbbk[\mathbf{b}+\mathbb{N}^n] \otimes_{\Bbbk} \text{Birth}_{\mathbf{b}}M \to M$ such that in any degree $\mathbf{v} \succeq \mathbf{b}$, the copy of $m \in \text{Birth}_{\mathbf{b}}M$ sitting in degree $\mathbf{v}$ in $\Bbbk[\mathbf{b}+\mathbb{N}^n] \otimes_{\Bbbk} \text{Birth}_{\mathbf{b}}M$ goes to the image of m in degree $\mathbf{v}$ under the natural graded structure of M. Likewise, for m in degree $\mathbf{v}$, let $g_{\mathbf{a}} : M \to \Bbbk[\mathbf{a}-\mathbb{N}^n] \otimes_{\Bbbk} \text{Death}_{\mathbf{a}}M$ take m to the element in $(\Bbbk[\mathbf{a}-\mathbb{N}^n] \otimes_{\Bbbk} \text{Death}_{\mathbf{a}}M)_{\mathbf{v}}$ corresponding to the natural image in $\text{Death}_{\mathbf{a}}M$ under the $\Bbbk[t_1,\ldots,t_n]$-module structure of M. Then the composite $g_{\mathbf{a}} \circ f_{\mathbf{b}}$ is equal to $(\partial_{\mathbf{a}})|_{\text{Birth}_{\mathbf{b}}M}$ by construction. Now since a map $f_{\mathbf{b}}$ exists for every birth degree $\mathbf{b}$ and a map $g_{\mathbf{a}}$ exists for every death degree $\mathbf{a}$, universality gives us maps

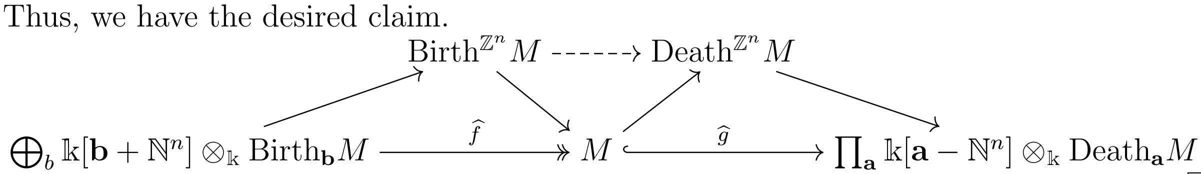
$$\bigoplus_{\mathbf{b}\in\mathcal{B}_M} \Bbbk[\mathbf{b}+\mathbb{N}^n] \otimes_{\Bbbk} \text{Birth}_{\mathbf{b}}M \xrightarrow{\widehat{f}} M \xrightarrow{\widehat{g}} \prod_{\mathbf{a}\in\mathcal{D}_M} \Bbbk[\mathbf{a}-\mathbb{N}^n] \otimes \text{Death}_{\mathbf{a}}M$$

induced by $\{f_{\mathbf{b}} : \mathbf{b} \in \mathcal{B}_M\}$ and $\{g_{\mathbf{a}} : \mathbf{a} \in \mathcal{D}_M\}$. Then we claim $\widehat{g}\widehat{f} = \widehat{\partial}$, but this is true by construction since we chose $f_{\mathbf{b}}$ and $g_{\mathbf{a}}$ such that $g_{\mathbf{a}}f_{\mathbf{b}} = (\partial_{\mathbf{a}})|_{\text{Birth}_{\mathbf{b}}M}$ for each $\mathbf{a}$ and $\mathbf{b}$. Finally, we have to check that $\widehat{f}$ is surjective and $\widehat{g}$ is injective. To see that $\widehat{f}$ is surjective, simply note that by construction, $\widehat{f}$ is surjective on all birth degrees, and thus it follows that it must be surjective on the whole module. Likewise, to see that $\widehat{g}$ is injective, note that if any element in M of degree $\mathbf{v}$ maps to 0 in $\Bbbk[\mathbf{a} - \mathbb{N}^n] \otimes \mathrm{Death}_{\mathbf{a}} M$ for all $\mathbf{a} \succeq \mathbf{v}$, then $\mathbf{v}$ must map to 0 in $\mathrm{Death}_{\mathbf{a}} M$ for all $\mathbf{a} \succeq \mathbf{v}$, which implies it must in fact be zero. Thus, we have shown the claim.

Next, let us show how to recover a minimal flange presentation from this homomorphism. Let $\mathrm{Birth}^{\mathbb{Z}^n} M$ be the $\mathbb{Z}^n$ module defined by $\bigoplus_{\mathbf{v} \in \mathbb{Z}^n} \varinjlim_{\mathbf{b} \preceq \mathbf{v}} \mathrm{Birth}_{\mathbf{b}} M$. This is universal among $\mathbb{Z}^n$ modules with a map from $\mathrm{Birth}\, M$, and so $\widehat{f}$ factors through $\mathrm{Birth}^{\mathbb{Z}^n} M$ by this universal property. In particular, $\mathrm{Birth}^{\mathbb{Z}^n} M$ is a minimal free hull of M. It clearly surjects onto M because $\widehat{f}$ is a surjection, and it must be free since M is finite length, hence finitely generated, and we can find generators which will generate $\mathrm{Birth}^{\mathbb{Z}^n} M$. Minimality comes from the universality of $\mathrm{Birth}^{\mathbb{Z}^n} M$.

Dually, let $\mathrm{Death}^{\mathbb{Z}^n} M$ be the $\mathbb{Z}^n$ module defined by $\bigoplus_{\mathbf{v} \in \mathbb{Z}^n} \varprojlim_{\mathbf{v} \preceq \mathbf{a}} \mathrm{Death}_{\mathbf{a}} M$. This is universal among $\mathbb{Z}^n$ modules with a map to $\mathrm{Death}\, M$. Then $\widehat{g}$ factors through $\mathrm{Death}^{\mathbb{Z}^n} M$, again by universality. Then $\mathrm{Death}^{\mathbb{Z}^n} M$ corresponds to a minimal injective hull of M. In particular, M injects into $\mathrm{Death}^{\mathbb{Z}^n} M$ since $\widehat{g}$ is an injection and $\mathrm{Death}^{\mathbb{Z}^n} M$ is an essential extension since the map on socle degrees from M to $\mathrm{Death}^{\mathbb{Z}^n} M$ is an isomorphism. Since M is finite length, it is finitely cogenerated and we can pick a set of cogenerators which cogenerate $\mathrm{Death}^{\mathbb{Z}^n} M$, hence it is injective. Minimality comes from the universality of $\mathrm{Death}^{\mathbb{Z}^n} M$

Then drawing out the diagram below, we get an induced map $\mathrm{Birth}^{\mathbb{Z}^n} M \to \mathrm{Death}^{\mathbb{Z}^n} M$. This is in fact a minimal flange presentation since we have argued above that $\mathrm{Birth}^{\mathbb{Z}^n} M$ is a minimal free hull of M and $\mathrm{Death}^{\mathbb{Z}^n} M$ is an minimal injective hull of M, and we have $\mathrm{Birth}^{\mathbb{Z}^n} M \twoheadrightarrow M \hookrightarrow \mathrm{Death}^{\mathbb{Z}^n} M$ in the diagram, where these maps are induced naturally from $\widehat{\partial}$. Thus, we have the desired claim.



□

4 The Decomposition Problem

Our main computational goal is to give an algorithm to compute a total decomposition as a direct sum of indecomposable modules, of an arbitrary finitely generated multipersistence module M. Such a decomposition always exists and is unique up to isomorphism on summands, by the Krull-Schmidt theorem [10]. In [4], the authors give an algorithm to do this decomposition in the restricted setting where M has no two generators or relations in the same degree. The essential theoretical result behind their algorithm is that a total decomposition of the module formed by starting with all generators of M and considering all relations in degree $\prec \mathbf{v}$ can be extended to a total decoposition of the module formed by considering all relations in degree $\preceq \mathbf{v}$. In this way, one can iteratively add in relations of higher degree until one has recovered the module with all relations, along with a total decomposition of it.

The key step in their algorithm is the method to compute a total decomposition of the module formed by considering all relations of degree $\preceq \mathbf{v}$ given a total decomposition of the module formed by considering all relations of degree $\prec \mathbf{v}$. This is also the step which relies on the assumption that there are no two relations in the same degree, as having more than one relation in the same degree turns out to significantly complicate the picture, changing the problem from an essentially discrete question of including a given relation or not to a continuous question of choosing the correct basis elements in the given degree in order to maintain a total decomposition.

Now in our language, a decomposition of a multipersistence module is equivalent to finding a direct sum decomposition of the QR code, since M is functorially recovered from the QR code. In particular, a direct sum decomposition $\operatorname{Birth} M = \bigoplus_{i=1}^{n} N_i$ along with a direct sum decomposition $\operatorname{Death} M = \bigoplus_{i=1}^{n} P_i$ such that $\partial(N_i) \subset P_i$ for each $i = 1, \dots, n$ induces a direct sum decomposition of M by Theorem 3.5.

Our approach will be analogous to that of [4], although we will instead consider a filtration on the birth degrees rather than the degrees of relations.

Definition 4.1. A *downset* D of a poset $(Q, \preceq)$ is a subposet of Q such that if $d \in D$, then $b \in D$ for all $b \in Q$ such that $b \preceq d$.

Definition 4.2. For M a Q-module and D a downset of Q, define $M|_D$ to be the restriction (Definition 2.2) of M to a D-module.

Then we can compute a decomposition of M iteratively. In particular, we may let $D \subset \mathcal{B}_M$ be some downset and suppose we have some valid direct sum decomposition of $M|_D$. Then for $\mathbf{v} \in \mathcal{B}_M \smallsetminus D$ some minimal element so that $D \cup \{\mathbf{v}\}$ is a downset, we wish to extend our decomposition of $M|_D$ to a decomposition of $M|_{D\cup\{\mathbf{v}\}}$. Equivalently, given some pair of decompositions of $(\operatorname{Birth} M)|_D$ and $\operatorname{Death} M$ which are compatible, we would like to extend this to a pair of decompositions $(\operatorname{Birth} M)|_{D\cup\{\mathbf{v}\}}$ and $\operatorname{Death} M$ which are compatible.

Precisely, assume we have $(\operatorname{Birth} M)|_D = K_1 \oplus K_2$ and $\operatorname{Death} M = L_1 \oplus L_2$ such that $\partial(K_1) \subset L_1$ and $\partial(K_2) \subset L_2$. We wish to extend this to a decomposition $(\operatorname{Birth} M)|_{D\cup\{\mathbf{v}\}} = K_1' \oplus K_2'$ and $\operatorname{Death} M = L_1' \oplus L_2'$ such that $\partial(K_1') \subset L_1'$ and $\partial(K_2') \subset L_2'$. In practice, we may wish to keep total decompositions at each step, ie direct sums of indecomposable summands, but it suffices to simply be able to check whether any module has a decomposition into two summands and then recurse.

Note part of this decomposition of $(\operatorname{Birth} M)|_{D\cup\{\mathbf{v}\}}$ is already induced by the decomposition of $(\operatorname{Birth} M)|_D$. In particular, there are two components in $(\operatorname{Birth} M)_{\mathbf{v}}$ which are the images of K_1 and K_2 in degree $\mathbf{v}$ under structure morphisms. Then extending to a decomposition $(\operatorname{Birth} M)_{D\cup\{\mathbf{v}\}}$ corresponds to giving a choice of basis for the remaining piece of $(\operatorname{Birth} M)_{\mathbf{v}}$ which is compatible with the existing $\mathcal{D}_M$-module decomposition of $\operatorname{Death} M$. The exact details of this procedure are left to a future study.

5 Future Directions

Due to time constraints, we did not develop the new definition of the QR code in the arbitrary $\mathbb{Z}^n$-graded or $\mathbb{R}^n$-graded setting, but we plan to do this later. Additionally, a goal from the theoretical standpoint would be to define a functorial bar code from the QR code, which would not be a map $\operatorname{Birth} M \to \operatorname{Death} M$ but would be a map from $\operatorname{top}(M)$ to $\operatorname{soc}(M)$.

Additionally, an important next step in this work would be to fully give the algorithm to decompose arbitrary finitely generated multipersistence modules. Herein, we have developed some new language to formulate the problem more precisely, and it is our expectation that this will yield a way to decompose these modules in the general setting.

References

[1] Ezra Miller. Data structures for real multiparameter persistence modules, 2020. 1, 2, 3, 5

[2] Afra Zomorodian and Gunnar Carlsson. Computing persistent homology. *Discrete & Computational Geometry*, 33(2):249–274, 2005. 2

[3] Mickaël Buchet and Emerson G Escolar. Every 1d persistence module is a restriction of some indecomposable 2d persistence module. *arXiv preprint arXiv:1902.07405*, 2019. 2

[4] Tamal K. Dey and Cheng Xin. Generalized persistence algorithm for decomposing multi-parameter persistence modules, 2019. 2, 5, 8, 9

[5] Herbert Edelsbrunner and John Harer. Persistent homology-a survey. *Contemporary mathematics*, 453:257–282, 2008. 3

[6] Gunnar Carlsson and Afra Zomorodian. The theory of multidimensional persistence. *Discrete & Computational Geometry*, 42(1):71–93, 2009. 3, 4

[7] Michael Lesnick. Lecture notes for math 840: Multiparameter persistence. 2019. 3

[8] Ezra Miller. Homological algebra of modules over posets, 2020. 3, 6

[9] Ezra Miller. Essential graded algebra over polynomial rings with real exponents, 2020. 3

[10] Magnus Bakke Botnan and William Crawley-Boevey. Decomposition of persistence modules, 2019. 8