On Finsler surfaces of constant flag curvature with a Killing field *

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Abstract

We study two-dimensional Finsler metrics of constant flag curvature and show that such Finsler metrics that admit a Killing field can be written in a normal form that depends on two arbitrary functions of one variable. Furthermore, we find an approach to calculate these functions for spherically symmetric Finsler surfaces of constant flag curvature. In particular, we obtain the normal form of the Funk metric on the unit disk $\mathbb{D}^2$.

Key words and phrases: Finsler metric, constant flag curvature, Killing field, normal form.

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1 Introduction

In Riemannian geometry, one has the concept of sectional curvature. Its analogue in Finsler geometry is called *flag curvature*. A Finsler metric F is said to be of *constant (flag) curvature* if the flag curvature $K =$ constant. One of the fundamental problems in Finsler geometry is to study Finsler metrics of constant (flag) curvature because Finsler metrics of constant flag curvature are the natural generalization of Riemannian metrics of constant sectional curvature. Recently, great progress has been made in studying Finsler metrics of constant curvature. The classification of Randers metrics with constant flag curvature has been completed by D. Bao, C. Robles and Z. Shen [3]. These metrics include the Funk metric on the unit ball and Katok examples [11]. In [19, 18], X. Mo found many new Finsler metrics of constant flag curvature by finding Killing fields of generic Bryant metrics and Mo-Shen-Yang metrics via the navigation problem.

Killing fields on a Finsler manifold M are vector fields induced by local 1-parameter group of isometric transformations of M. They are the natural generalization of Killing fields on a Riemannian manifold and thereby are important in both mathematics and physics.

For instance, Li-Chang-Mo related some Killing fields of Finsler metrics to the symmetry of very special relativity (VSR for short). They find that the isometry group of a class of (α, β)-manifold is the same as the symmetry of VSR [14]. Very special relativity is an interesting approach to investigating the violation of Lorentz invariance that was developed by Cohen-Glashow [7]. Let $\phi(s) = 1 + s$. Then (α, β)-manifold $\left(M, \alpha\phi\left(\frac{\beta}{\alpha}\right)\right)$ becomes a Randers manifold.

The main purpose of this paper is to study Finsler surfaces of constant flag curvature K with a Killing field. By using moving frame theory, we establish normal forms of such Finsler surfaces with $K > 0$, $K = 0$ and $K < 0$ respectively (see (4.16), (5.10) and (6.15) below). In general, the normal form of a class of Finsler metrics clarifies our understanding of such spaces of Finsler metrics [5]. After noting these normal forms, we obtain the following:

Theorem 1.1 *The space of isometry classes of 2-dimensional Finsler metrics of* *constant curvature that admit a Killing field depends on two arbitrary functions of one variable.*

It is worth mentioning that there are many 2-dimensional Finsler metrics with a non-zero Killing field. Let F be a Finsler metric on $\mathbb{B}^2(\mu)$, the ball of radius μ in $\mathbb{R}^2$. F is said to *spherically symmetric* if it satisfies

$$F(Ax,\, Ay) = F(x,\, y)$$

for all $x \in \mathbb{B}^2(\mu)$, $y \in T_x\mathbb{B}^2(\mu)$ and $A \in \mathrm{O}(2)$. Spherically symmetric Finsler metrics F admit the Killing field

$$X = -x^2\frac{\partial}{\partial x^1} + x^1\frac{\partial}{\partial x^2}\,,$$

where $x = (x^1,\, x^2)$ (see (7.3) below). Recently, the study of spherically symmetric Finsler metrics has attracted considerable attention. The classification of projective spherically symmetric Finsler metrics with constant curvature has just been completed recently by Zhou, Mo-Zhu and Li [22, 20, 13]. The following expression for spherically symmetric Finsler metrics had been obtained by Huang-Mo [10, 11] :

$$F = |y|\phi\left(\frac{|x|^2}{2},\, \frac{\langle x,\, y\rangle}{|y|}\right).$$

This expression motivates us to find an approach to calculate two functions of one variable in the normal form of spherically symmetric Finsler surfaces of constant flag curvature (see Section 7). In particular, we will obtain the normal form of the Funk metric on the unit disk $\mathbb{D}^2$.

We will determine the space of Finsler surfaces of constant flag curvature that admit two linearly independent Killing fields in a forthcoming paper.

2 Preliminaries

2.1 The structure equations of a Finsler surface

Let $(M,\, F)$ be an oriented Finsler surface. The function F determines and is determined by the set

$$\Sigma = \{(x,\, y) \in TM \mid F(x,\, y) = 1\,\}$$

which is known as the unit tangent bundle of F. For each $x \in M$, the intersection $\Sigma_x = \Sigma \cap T_xM$ is the *indicatrix*. Define

$$\omega_1 := F_{y^j} dx^j. \tag{2.1}$$

Then ω_1 is a differential form on Σ. The form ω_1 is known in the calculus of variations as the *Hilbert form*. On Σ, there exists a canonical coframing $\omega = (\omega_1, \omega_2, \omega_3)$ satisfying the *structure equations*

$$d\omega_1 = -\omega_2 \wedge \omega_3, \tag{2.2}$$

$$d\omega_2 = -\omega_3 \wedge \omega_1 + I\, \omega_3 \wedge \omega_2, \tag{2.3}$$

$$d\omega_3 = -K\, \omega_1 \wedge \omega_2 - J\, \omega_2 \wedge \omega_3 \tag{2.4}$$

where the functions I, J and K are known as the *main scalar*, the *Landsberg curvature* and the *flag curvature* respectively [4, 16].

Conversely, if Σ is a 3-manifold endowed with a coframing $\omega = (\omega_1, \omega_2, \omega_3)$ that satisfies the structure equations (2.2–4) for some functions I, J, and K on Σ and has the property (which always holds locally) that there exists a smooth submersion $\pi : \Sigma \to M$, where M is a surface, whose fibers are the integral curves of ω_1 and ω_2, then there is a unique immersion $\iota : \Sigma \to TM$ compatible with π that realizes Σ as the unit sphere bundle of a locally defined Finsler structure F on M in such a way that the given coframing is the canonical coframing induced on Σ by the (local) Finsler structure F. In this way, one has a local equivalence between Finsler surfaces and 3-manifolds Σ endowed with a coframing ω that satisfies (2.2–4).

2.2 The Bianchi identities

Differentiating (2.3) and using (2.4), (2.2) and (2.3), one deduces

$$(J\, \omega_1 - dI) \wedge \omega_2 \wedge \omega_3 = 0. \tag{2.5}$$

Put

$$dI := I_1\, \omega_1 + I_2\, \omega_2 + I_3\, \omega_3. \tag{2.6}$$

From (2.5) and (2.6), we have $0 = (J - I_1)\, \omega_1 \wedge \omega_2 \wedge \omega_3$. It follows that $I_1 = J$. Thus we obtain the following Bianchi identity

$$dI := J\, \omega_1 + I_2\, \omega_2 + I_3\, \omega_3. \tag{2.7}$$

Differentiating (2.4) and using the structure equations (2.4), (2.2) and (2.3), one obtains

$$-(IK + J_1 + K_3)\,\omega_1 \wedge \omega_2 \wedge \omega_3 = 0 \tag{2.8}$$

where

$$dJ := J_1\,\omega_1 + J_2\,\omega_2 + J_3\,\omega_3, \quad dK := K_1\,\omega_1 + K_2\,\omega_2 + K_3\,\omega_3. \tag{2.9}$$

It follows that $J_1 = -IK - K_3$. Assume that the Finsler surface $(M,\, F)$ has constant flag curvature. Thus $J_1 = -IK$. Substituting this into (2.9) yields

$$dJ := -KI\,\omega_1 + J_2\,\omega_2 + J_3\,\omega_3. \tag{2.10}$$

3 Finsler metrics with a Killing field

Assume that the Finsler surface $(M,\, F)$ has constant flag curvature K and that $(M,\, F)$ admits a non-zero Killing field X. Then its flow φ_t is an isometry on $(M,\, F)$, i.e. $\check{\varphi}_t F = F$ where $\check{\varphi}_t$ is the flow on TM defined by $\check{\varphi}_t(x,\, y) := (\varphi_t(x),\, \varphi_{t*}(y))$. Note that ω_j are globally defined on Σ and φ_t preserves the orientation and Σ. It is easy to see that ${\check{\varphi}_t}^*\omega_j = \omega_j, \quad j = 1,\, 2,\, 3$. Thus, we have

$$\mathcal{L}_{\hat{X}}\omega_j = \lim_{t\to 0} \frac{1}{t}\,(\omega_j - {\check{\varphi}_t}^*\omega_j) = 0, \quad j = 1,\, 2,\, 3 \tag{3.1}$$

where $\hat{X}$ is the natural lift of X to Σ. Equation (3.1) tells us that $\hat{X}$ is a symmetry vector field, that is, a nonzero field whose flow preserves ω_j. It follows that

$$d(\iota_{\hat{X}}\omega_j) + \iota_{\hat{X}}(d\omega_j) = (d \circ \iota_{\hat{X}} + \iota_{\hat{X}} \circ d)\omega_j = \mathcal{L}_{\hat{X}}\omega_j = 0 \tag{3.2}$$

where $\iota_{\hat{X}}$ is the interior product with respect to $\hat{X}$.

Write

$$a_j = \omega_j(\hat{X}) = \iota_{\hat{X}}\omega_j. \tag{3.3}$$

Using (3.2) and (3.3), we have

$$da_j = d(\iota_{\hat{X}}\omega_j) = -\iota_{\hat{X}} d\omega_j, \quad j = 1,\, 2,\, 3. \tag{3.4}$$

Applying the structure equations, we have

$$da_1 = -\iota_{\hat{X}}\,(-\omega_2 \wedge \omega_3) = \left(\iota_{\hat{X}}\omega_2\right)\omega_3 - \left(\iota_{\hat{X}}\omega_3\right)\omega_2 = a_2\omega_3 - a_3\omega_2, \tag{3.5}$$

$$
\begin{aligned}
da_2 &= -\iota_{\hat{X}}\left(-\omega_3\wedge\omega_1 - I\omega_2\wedge\omega_3\right)\\
&= \left(\iota_{\hat{X}}\omega_3\right)\omega_1 - \left(\iota_{\hat{X}}\omega_1\right)\omega_3 + I\iota_{\hat{X}}\left(\omega_2\wedge\omega_3\right)\\
&= a_3\omega_1 - a_1\omega_3 + I da_1,
\end{aligned}
\tag{3.6}
$$

$$
\begin{aligned}
da_3 &= -\iota_{\hat{X}}\left(-K\omega_1\wedge\omega_2 - J\omega_2\wedge\omega_3\right)\\
&= K\left[\left(\iota_{\hat{X}}\omega_1\right)\omega_2 - \left(\iota_{\hat{X}}\omega_2\right)\omega_1\right] + J\iota_{\hat{X}}\left(\omega_2\wedge\omega_3\right)\\
&= K(a_1\omega_2 - a_2\omega_1) + J da_1.
\end{aligned}
\tag{3.7}
$$

By using (3.7), (3.5) and (3.6), we get

$$
\begin{aligned}
\tfrac{1}{2}d\left(Ka_2^2 + a_3^2\right) &= Ka_2 da_2 + a_3 da_3\\
&= Ka_2\left(a_3\omega_1 - a_1\omega_3 + I da_1\right)\\
&\quad + a_3\left[K(a_1\omega_2 - a_2\omega_1) + J da_1\right]\\
&= \left(KIa_2 + Ja_3\right)da_1 - Ka_1\left(a_2\omega_3 - a_3\omega_2\right)\\
&= \left(KIa_2 + Ja_3 - Ka_1\right)da_1.
\end{aligned}
\tag{3.8}
$$

It follows that $Ka_2^2 + a_3^2$ is constant on the level sets of a_1.

Assuming that da_1 is nonvanishing, and that the level sets of a_1 on Σ are connected, one can write

$$
Ka_2^2 + a_3^2 = 2f(a_1) \tag{3.9}
$$

where $f : a_1(\Sigma) \to \mathbb{R}$ is a smooth function Differentiating (3.9) and using (3.8), we have

$$
2\,f'(a_1)\,da_1 = d\big(2f(a_1)\big) = d\left(Ka_2^2 + a_3^2\right) = 2\left(KIa_2 + Ja_3 - Ka_1\right)da_1.
$$

It follows that

$$
KIa_2 + Ja_3 = f'(a_1) + a_1 K \tag{3.10}
$$

on $\{p \in \Sigma \mid da_1(p) \neq 0\}$. (One can also treat the case in which a_1 is constant.)

Because the flow of $\hat{X}$ preserves the ω_i, it must also preserve I and J. Thus,

$$
\mathcal{L}_{\hat{X}} I = 0 \tag{3.11}
$$

and

$$
\mathcal{L}_{\hat{X}} J = 0. \tag{3.12}
$$

By (2.6) and (3.11), we get

$$\begin{aligned} a_1 J + a_2 I_2 + a_3 I_3 &= \left(\iota_{\hat{X}}\omega_1\right) J + \left(\iota_{\hat{X}}\omega_2\right) I_2 + \left(\iota_{\hat{X}}\omega_3\right) I_3 \\ &= \left(\iota_{\hat{X}} J\right)\omega_1 + J\iota_{\hat{X}}\omega_1 + \left(\iota_{\hat{X}} I_2\right)\omega_2 + I_2\iota_{\hat{X}}\omega_2 \\ &\quad + \left(\iota_{\hat{X}} I_3\right)\omega_3 + I_3\iota_{\hat{X}}\omega_3 \\ &= \iota_{\hat{X}}\left(J\omega_1 + I_2\omega_2 + I_3\omega_3\right) \\ &= \iota_{\hat{X}} dI = (d\circ\iota_{\hat{X}} + \iota_{\hat{X}}\circ d)I = \mathcal{L}_{\hat{X}} I = 0. \end{aligned} \tag{3.13}$$

Similarly, (2.9) and (3.12) imply that $-a_1 KI + a_2 J_2 + a_3 J_3 = 0$. Together with (3.13), (3.7), (3.5) and (3.6), we obtain

$$\begin{aligned} d(a_2 J - a_3 I) &= J\, da_2 + a_2\, dJ - I\, da_3 - a_3\, dI \\ &= J\left(a_3\omega_1 - a_1\omega_3 + I da_1\right) + a_2\left(-KI\omega_1 + J_2\omega_2 + J_3\omega_3\right) \\ &\quad -I\left(Ka_1\omega_2 - Ka_2\omega_1 + J da_1\right) - a_3\left(J\omega_1 + I_2\omega_2 + I_3\omega_3\right) \\ &= (a_2 J_2 - K a_1 I - a_3 I_2)\omega_2 + (a_2 J_3 - a_1 J - a_3 I_3)\omega_3 \\ &= -a_3(J_3 + I_2)\omega_2 + a_2(J_3 + I_2)\omega_3 = (J_3 + I_2) da_1. \end{aligned}$$

It follows that

$$a_2\, J - a_3\, I = g(a_1) \tag{3.14}$$

for some function g, again, assuming that da_1 is nonvanishing and that the level sets of a_1 are connected.

It turns out to be convenient to split the further discussion into cases according to whether $K > 0$, $K = 0$ and $K < 0$. Moreover, by scaling, one can reduce to the cases $K = 1$, $K = 0$ and $K = -1$.

To simplify notation, in the following sections, we shall abbreviate a_1 as a. We will also assume that da is nonvanishing and that the level sets of a are connected.

4 $K = 1$

In this section, we are going to study Finsler surfaces with constant flag curvature $K = 1$. In this case, $a_2^2 + a_3^2$ is a function of a $(:= a_1)$ by (3.9). Without loss of generality, we can assume that $(a_2,\, a_3) \neq (0,\, 0)$. Let

$$a_2^2 + a_3^2 = u(a)^2 \tag{4.1}$$

where u is a positive function on $a(\Sigma) \subset \mathbb{R}$.

Write

$$a_2 = u(a)\,\sin t, \quad a_3 = u(a)\,\cos t \tag{4.2}$$

where $t : \Sigma \to \mathbb{R}$. It follows from (3.10) that

$$a_2 I + a_3 J = u(a)\,u'(a) + a. \tag{4.3}$$

We rewrite (3.14) as follows

$$a_2 J - a_3 I = u(a)^2 v(a) \tag{4.4}$$

where $v(a) := \frac{g(a)}{u(a)^2}$. For notational simplicity in what follows, we will write u, u', or v instead of $u(a)$, $u'(a)$, or $v(a)$.

Solving (4.4) and (4.3) and then using (4.2), we obtain

$$I = \left[u' + \frac{a}{u}\right]\,\sin t - uv\,\cos t, \tag{4.5}$$

$$J = \left[u' + \frac{a}{u}\right]\,\cos t + uv\,\sin t. \tag{4.6}$$

By (4.2), we have

$$da_2 = u'\,\sin t\,da + u\,\cos t\,dt \tag{4.7}$$

and

$$da_3 = u'\,\cos t\,da - u\,\sin t\,dt. \tag{4.8}$$

Plugging (4.2) into (3.5) yields

$$da = u\,(\sin t\,\omega_3 - \cos t\,\omega_2)\,. \tag{4.9}$$

By substituting (4.5) into (3.6) and using (4.7), we obtain

$$\begin{aligned} u'\,\sin t\,da + u\,\cos t\,dt &= da_2 \\ &= a_3\,\omega_1 - a\,\omega_3 + I\,da \\ &= a_3\,\omega_1 - a\,\omega_3 + \left[\left(u' + \frac{a}{u}\right)\,\sin t - uv\,\cos t\right]\,da. \end{aligned}$$

Together with (4.9), we have

$$u\,\cos t\,dt = u\,\cos t\left[\omega_1 - \left(\frac{a}{u}\,\sin t - uv\,\cos t\right)\omega_2 - \left(\frac{a}{u}\,\cos t + uv\,\sin t\right)\omega_3\right].$$

Except where $\cos t = 0$, we get

$$dt = \omega_1 - \Big(\frac{a}{u}\,\sin t - uv\,\cos t\Big)\,\omega_2 - \Big(\frac{a}{u}\,\cos t + uv\,\sin t\Big)\,\omega_3. \qquad (4.10)$$

Similarly, we obtain that (4.10) holds when $\sin t \neq 0$ using (4.6), (3.6), (4.8) and (4.9). Hence (4.10) holds on Σ.

Let

$$\alpha = \frac{\theta}{u} \qquad (4.11)$$

where

$$\theta = \theta_1 + \theta_2 \qquad (4.12)$$

where

$$\theta_1 = (\sin t)\,\omega_2, \quad \theta_2 = (\cos t)\,\omega_3. \qquad (4.13)$$

Using (2.2), (4.5) and (4.10) we get

$$d\theta_1 = (\cos t)\,\omega_1 \wedge \omega_2 - (\sin t)\,\omega_3 \wedge \omega_1 + \Big(\frac{a}{u}\,\cos 2t - u'\sin^2 t + uv\sin 2t\Big)\,\omega_2 \wedge \omega_3.$$

By (2.3), (4.5) and (4.10), we see that

$$d\theta_2 = -(\cos t)\,\omega_1 \wedge \omega_2 + (\sin t)\,\omega_3 \wedge \omega_1 - \Big(\frac{a}{u}\,\cos 2t + u'\cos^2 t + uv\sin 2t\Big)\,\omega_2 \wedge \omega_3.$$

Thus, we have

$$d\theta = -u'\,\omega_2 \wedge \omega_3 \qquad (4.14)$$

from which, together with (4.9), we obtain that the 1-form α is closed. It follows that α is locally an exact differential form, i.e., locally there exists a function b such that $\alpha = db$. Using (4.11), (4.12) and (4.13), we get

$$db = \frac{1}{u(a)}\,(\sin t\,\omega_2 + \cos t\,\omega_3)\,. \qquad (4.15)$$

Taking this together with (4.9) and (4.10), we obtain

$$\begin{aligned} da \wedge db \wedge dt &= u\,(\sin t\,\omega_3 + \cos t\,\omega_2) \wedge \tfrac{1}{u}\,(\sin t\,\omega_2 + \cos t\,\omega_3) \wedge dt \\ &= \omega_3 \wedge \omega_2 \wedge dt \\ &= \omega_3 \wedge \omega_2 \wedge \omega_1 = -dV_\Sigma \neq 0. \end{aligned}$$

Hence $(a,\, b,\, t)$ is a local coordinate system on Σ. Furthermore, we can solve for the ω_i in the form

$$\begin{pmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \end{pmatrix} = \begin{pmatrix} 1 & v & a \\ 0 & -(\cos t)/u & u\,\sin t \\ 0 & (\sin t)/u & u\,\cos t \end{pmatrix} \begin{pmatrix} dt \\ da \\ db \end{pmatrix} \tag{4.16}$$

where we have used (4.9), (4.10) and (4.15). We say that (4.16) is the *normal form* for a Finsler surface of constant flag curvature 1 that admits a Killing field.

Conversely, regarding $u = u(a) > 0$ and $v = v(a)$ as arbitrary functions of a, the above coframing ω_i satisfies the structure equations of an oriented Finsler surface with $K \equiv 1$ and admitting a symmetry vector field. Note that it depends on two arbitrary functions of one variable. Thus, we have shown the following:

Theorem 4.1 *The space of isometry classes of Finsler metrics of constant flag curvature* 1 *that admit a Killing field depends on two arbitrary functions of one variable.*

Now we investigate the geometric meanings of b, t and a. Using (4.2), (4.11), (4.12) and (4.13), one can verify that

$$db\left(\hat{X}\right) = \frac{1}{u}(a_2 \sin t + a_3 \cos t) = 1,$$
$$dt\left(\hat{X}\right) = a - \left(\frac{a}{u}\sin t - uv\cos t\right) a_2 - \left(\frac{a}{u}\cos t - uv\sin t\right) a_3 = 0,$$
$$da\left(\hat{X}\right) = u\,(a_3 \sin t - a_2 \cos t) = 0.$$

By the above formula, we obtain $\hat{X} = \frac{\partial}{\partial b}$. It follows that the b-curves are the integral curves of $\hat{X}$.

Let E be the Reeb vector field of F. Then [8, Proposition 3.2]

$$\omega_1(E) = 1, \quad \omega_2(E) = \omega_3(E) = 0.$$

Together with (4.9), (4.10) and (4.15) we get

$$da(E) = db(E) = 0, \quad dt(E) = 1.$$

It follows that $E = \frac{\partial}{\partial t}$, equivalently, the t-curves are the flows of E. Recall that a curve is a (unit Finslerian speed) geodesic if its canonical lift in Σ is an integral curve of E [4]. Thus the t-curves are the canonical lift of unit geodesics on (M, F).

Finally we are going to discuss the geometric meaning of a. In natural coordinates, we have

$$\hat{X} = v^i \frac{\partial}{\partial x^i} + y^j \frac{\partial v^i}{\partial x^j} \frac{\partial}{\partial y^i} \tag{4.17}$$

where $X = v^i \frac{\partial}{\partial x^i}$; see [17, 18]. Together with (2.1) we have

$$a = a_1 = \omega_1(\hat{X}) = \left(F_{y^k} dx^k\right) \left(v^i \frac{\partial}{\partial x^i} + y^j \frac{\partial v^i}{\partial x^j} \frac{\partial}{\partial y^i}\right) = \omega_1(X) = \iota_X \omega_1.$$

It follows that a is the interior product with respect to the Killing field X of the Hilbert form ω_1 [12, Page 35].

5 $K = 0$

In this section, we are going to investigate Finsler surfaces with a flag curvature $K = 0$. In this case, a_3^2, $a_3 J$ and $a_2\, J - a_3\, I$ are functions of a by using (3.9), (3.10) and (3.14). Let

$$a_3 = u(a) \tag{5.1}$$

where $u(a)$ is a non-zero function on $\Omega := \{p \in \Sigma \mid a_3(p) \neq 0\}$. By using (3.9), (3.10) and (5.1), we have $a_3 J = \left[\frac{u(a)^2}{2}\right]' = u(a)u'(a)$. Together with (5.1) we get

$$J = \frac{u(a)u'(a)}{a_3} = u'(a) \tag{5.2}$$

on Ω. Write

$$a_2 = u(a)t, \quad a_2 J - a_3 I = u(a)^2 v(a) \tag{5.3}$$

where $t : \Sigma \to \mathbb{R}$. It follows from (5.2) that

$$I = \frac{a_2 J - u(a)^2 v(a)}{a_3} = \frac{u(a)tu'(a) - u(a)^2 v(a)}{a_3} = u'(a)t - u(a)v(a). \tag{5.4}$$

Again, for simplicity of notation, let us write u, u' or v instead of $u(a)$, $u'(a)$, or $v(a)$.

By (5.2) and the first equation of (5.3), we have

$$da_2 = u' t\, da + u\, dt \tag{5.5}$$

and

$$da_3 = u'\, da. \tag{5.6}$$

Substituting (5.1) and the first equation of (5.3) into (3.4), we have

$$da = u\,(t\,\omega_3 - \omega_2). \tag{5.7}$$

By substituting (5.4) into (3.6) and using (5.5), we obtain

$$\begin{aligned} u' t\, da + u\, dt &= da_2 \\ &= a_3\,\omega_1 - a\,\omega_3 + I\, da \\ &= a_3\,\omega_1 - a\,\omega_3 + (u' t - uv)\ da. \end{aligned}$$

Together with (5.7), we have $u\,dt = u\left[\omega_1 + uv\,\omega_2 - \left(\frac{a}{u} + uvt\right)\,\omega_3\right]$. It follows that

$$dt = \omega_1 + uv\,\omega_2 - \left(\frac{a}{u} + uvt\right)\,\omega_3 \tag{5.8}$$

on Ω. By using (2.3), (3.5), (5.1) and (5.2), we have

$$\begin{aligned} d\left(\tfrac{1}{u}\omega_3\right) &= -\tfrac{u'}{u^2}\, da \wedge \omega_3 + \tfrac{1}{u}\, d\omega_3 \\ &= -\tfrac{u'}{u^2}(a_2\omega_3 - a_3\omega_2) \wedge \omega_3 - \tfrac{1}{u} J\,\omega_2 \wedge \omega_3 \\ &= -\tfrac{u'}{u^2}(-u\omega_2 \wedge \omega_3) - \tfrac{u'}{u}\omega_2 \wedge \omega_3 = 0. \end{aligned}$$

We get that the 1-form $\frac{1}{u}\,\omega_3$ is closed. Hence locally there exists a function b such that

$$db = \frac{1}{u}\,\omega_3. \tag{5.9}$$

Taking this together with (5.7) and (5.8) we obtain

$$\begin{aligned} da \wedge db \wedge dt &= u\,(t\,\omega_3 - \omega_2) \wedge \tfrac{1}{u}\,\omega_3 \wedge dt \\ &= -\omega_2 \wedge \omega_3 \wedge \omega_1 = -dV_\Sigma \neq 0. \end{aligned}$$

It follows that $(a,\,b,\,t)$ is a local coordinate system on Σ. Furthermore, just as in the case $K \equiv 1$, we obtain a normal form for $(M,\,F)$ with flag curvature $K \equiv 0$ that admits a Killing field

$$\begin{pmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \end{pmatrix} = \begin{pmatrix} 1 & v & a \\ 0 & -1/u & t\,u(a) \\ 0 & 0 & u \end{pmatrix} \begin{pmatrix} dt \\ da \\ db \end{pmatrix}, \tag{5.10}$$

where $u = u(a) > 0$ and $v = v(a)$ are arbitrary functions of a. Thus, we have the following result:

Theorem 5.1 *The space of isometry classes of Finsler metrics of constant flag curvature* 0 *that admit a Killing field depends on two arbitrary functions of one variable.*

Just as in the previous case of $K \equiv 1$, the geometric meanings of b, t and a are as follows: the b-curves are the integral curves of $\hat{X}$; the t-curves are the canonical lift of unit geodesics on (M, F) and a is the interior product with respect to the Killing field X of the Hilbert form ω_1.

6 $K = -1$

Now let us consider Finsler surfaces of constant flag curvature $K = -1$. In this case, $-a_2^2 + a_3^2$, $-a_2 I + a_3 J$, $a_2 J - a_3 I$ are functions of a by virtue of (3.9), (3.10) and (3.14). By the sign and continuity of $-a_2^2 + a_3^2$, we should investigate the following three subcases:

$$\text{(i)} \ -a_2^2 + a_3^2 > 0, \quad \text{(ii)} \ -a_2^2 + a_3^2 \equiv 0, \quad \text{(iii)} \ -a_2^2 + a_3^2 < 0$$

For brevity, we only discuss the subcase (i), as the others are similar. Moreover, we will continue to assume that $a = a_1 : \Sigma \to \mathbb{R}$ is a submersion with connected fibers.

Let

$$-a_2^2 + a_3^2 = u(a)^2 \tag{6.1}$$

where u is a positive function on $a(\Sigma) \subset \mathbb{R}$. Write

$$a_2 = u(a) \sinh t, \quad a_3 = u(a) \cosh t \tag{6.2}$$

where $t : \Sigma \to \mathbb{R}$. By using (3.9) and (3.10) we have

$$-a_2 I + a_3 J = \left[\frac{1}{2} u(a)^2\right]' + (-1)a = u(a)u'(a) - a. \tag{6.3}$$

By (3.14), we have

$$a_2 J - a_3 I = u(a)^2 v(a) \tag{6.4}$$

where $v(a) := \frac{g(a)}{u(a)^2}$. Again, for simplicity, we will write u, u' or v for $u(a)$, $u'(a)$, or $v(a)$, respectively. Solving (6.3) and (6.4) and using (6.2), we obtain

$$I = \left[u' - \frac{a}{u}\right] \sinh t - uv \cosh t, \tag{6.5}$$

$$J = \left[u' - \frac{a}{u}\right] \cosh t - uv \sinh t. \tag{6.6}$$

By (4.2), we have

$$da_2 = u' \sinh t \, da + u \cosh t \, dt \tag{6.7}$$

and

$$da_3 = u' \cosh t da + u \sinh t \, dt. \tag{6.8}$$

Substituting (6.2) into (3.5) yields

$$da = u \left(\sinh t \, \omega_3 - \cosh t \, \omega_2\right). \tag{6.9}$$

By substituting (6.5) into (3.6) and using (6.7) we obtain

$$\begin{aligned} u' \sinh t \, da + u \cosh t \, dt &= da_2 \\ &= a_3 \, \omega_1 - a \, \omega_3 + I \, da \\ &= a_3 \, \omega_1 - a\omega_3 + \left[(u' - \tfrac{a}{u}) \sinh t - uv \cosh t\right] da. \end{aligned}$$

Together with (6.9), we have

$$u \cosh t \, dt = u \cosh t \left[\omega_1 + \left(\frac{a}{u} \sinh t + uv \cosh t\right) \omega_2 - \left(\frac{a}{u} \cosh t + uv \sinh t\right) \omega_3\right]$$

where we have used $1 + \sinh^2 t = \cosh^2 t$. Note that $u \cosh t > 0$. Hence

$$dt = \omega_1 + \left(\frac{a}{u} \sinh t + uv \cosh t\right) \omega_2 - \left(\frac{a}{u} \cosh t + uv \sinh t\right) \omega_3. \tag{6.10}$$

Let

$$\alpha = \frac{\theta}{u} \tag{6.11}$$

where

$$\theta = \theta_2 - \theta_1 \tag{6.12}$$

where

$$\theta_1 = (\sinh t) \, \omega_2, \quad \theta_2 = (\cosh t) \, \omega_3. \tag{6.13}$$

Using (2.3), (6.5) and (6.10) we get

$$d\theta_1 = (\cosh t)\,\omega_1\wedge\omega_2 - (\sinh t)\,\omega_3\wedge\omega_1 + \left(\frac{a}{u}\,\cosh 2t - u'\sinh^2 t + uv\,\sinh 2t\right)\,\omega_2\wedge\omega_3.$$

By (2.4), (4.6) and (6.10), we see that

$$d\theta_2 = (\cosh t)\,\omega_1\wedge\omega_2 - (\sin t)\,\omega_3\wedge\omega_1 + \left(\frac{a}{u}\,\cosh 2t - u'\cosh^2 t + uv\,\sinh 2t\right)\,\omega_2\wedge\omega_3\,.$$

Thus, we obtain (4.14), where θ is defined in (6.12). By using (4.14) and (6.9) we obtain that the 1-form α is closed. It follows that locally there exists a function b such that $\alpha = db$. Using (6.11), (6.12) and (6.13), we get

$$db = \frac{1}{u}\,(\cosh t\,\omega_3 - \sinh t\,\omega_2)\,. \tag{6.14}$$

Together with (6.9) and (6.10) yields

$$\begin{aligned} da\wedge db\wedge dt &= u\,(\sinh t\,\omega_3 - \cosh t\,\omega_2)\wedge \tfrac{1}{u}\,(\cosh t\,\omega_3 - \sinh t\,\omega_2)\wedge dt \\ &= (\cosh^2 t - \sinh^2 t)\omega_3\wedge\omega_2\wedge\omega_1 \\ &= \omega_3\wedge\omega_2\wedge\omega_1 = -dV_\Sigma \neq 0. \end{aligned}$$

Hence $(a,\,b,\,t)$ is a local coordinate system on Σ. Moreover we can solve for the ω_i in the form

$$\begin{pmatrix}\omega_1\\ \omega_2\\ \omega_3\end{pmatrix} = \begin{pmatrix} 1 & v & a \\ 0 & -(\cosh t)/u & u\,\sinh t \\ 0 & -(\sinh t)/u & u\,\cosh t \end{pmatrix}\begin{pmatrix} dt\\ da\\ db\end{pmatrix} \tag{6.15}$$

where we have used (6.9), (6.10) and (6.14). Then (6.15) is the *normal form* for a Finsler surface of constant flag curvature $K \equiv -1$ that admits a Killing field. It depends on two arbitrary functions of one variable, namely $u = u(a) > 0$ and $v = v(a)$. We then have the following:

Theorem 6.1 *The space of isometry classes of Finsler metrics of constant flag curvature* -1 *which admits a Killing field depends on two arbitrary functions of one variable.*

Again, as in the earlier cases the geometric meanings of b, t and a are as follows: the b-curves are the integral curves of $\hat{X}$; the t-curves are the canonical lift of unit geodesics on $(M,\,F)$ and a is the interior product with respect to the Killing field X of the Hilbert form ω_1.

7 Functions $u(a)$ and $v(a)$ for spherically symmetric metrics

The following notations and lemmas will be used in this section. Let $F = |y|\phi\left(\frac{|x|^2}{2}, \frac{\langle x, y\rangle}{|y|}\right)$ be a spherically symmetric Finsler metric on $\mathbb{B}^2(\mu)$. Let

$$r := |y|, \quad t := \frac{|x|^2}{2}, \quad s := \frac{\langle x,\, y\rangle}{|y|}, \tag{7.1}$$

$$r^i := r_i := \frac{y^i}{|y|}, \quad x_i := x^i, \quad s^i := s_i := x_i - sr_i. \tag{7.2}$$

By a straightforward computation one obtains

$$s_{y^i} = \frac{s_i}{r} \tag{7.3}$$

where we have used (7.1) and (7.2).

Lemma 7.1[9] *Let $f = f(r,\, t,\, s)$ be a function on a domain $\mathcal{U} \subset \mathbb{R}^3$.* Then

$$f_{x^i} = (r_i,\, s_i)\begin{pmatrix} f_s + sf_t \\ f_t \end{pmatrix}, \quad f_{y^i} = (r_i,\, s_i)\begin{pmatrix} f_r \\ f_s/r \end{pmatrix}. \tag{7.4}$$

Corollary 7.2[9] *Let $F = |y|\phi\left(\frac{|x|^2}{2}, \frac{\langle x, y\rangle}{|y|}\right)$ be a spherically symmetric Finsler metric on $\mathbb{B}^2(\mu)$. Then*

$$F_{y^i} = \phi r_i + \phi_s s_i. \tag{7.5}$$

and

$$F_0 =: F_{x^i}y^i = r^2 \cdot (\phi_s + s\phi_t). \tag{7.6}$$

The geodesic coefficients G^i can be expressed by (cf [9], [15, Definition 3.3.8])

$$G^i := \frac{r^2}{2}(r^i, s^i)\begin{pmatrix} \bar{u} \\ \bar{v} \end{pmatrix}. \tag{7.7}$$

where

$$\bar{v} := \frac{s\phi_{ts} + \phi_{ss} - \phi_t}{\Delta}, \quad \bar{u} = \frac{1}{\phi}[\phi_s + s\phi_t - (2t - s^2)\phi_s\bar{v}]. \tag{7.8}$$

where

$$\Delta = \phi - s\phi_s + (2t - s^2)\phi_{ss}. \tag{7.9}$$

By a straightforward computation one obtains the following

Lemma 7.3 *Let t and s be functions satisfying (7.1). Then*

$$2t - s^2 = \frac{(x^1y^2 - x^2y^1)^2}{|y|^2}. \tag{7.10}$$

Now we give an approach to calculate the normal forms for known spherically symmetric Finsler metrics of constant flag curvature.

Step 1 First of all, let us calculate $a = a(t, s)$. Most of known spherically symmetric Finsler metrics of constant flag curvature are projectively flat. Hence ϕ satisfies the following projectively flat equation [10]: $s\phi_{ts} + \phi_{ss} - \phi_t = 0$. It follows that $\phi - s\phi_s = h(2t - s^2)$. where h is a function. In this case, $a = a(z)$ where $z = 2t - s^2$ (see (7.15) below).

Let $F = |y|\phi\left(\frac{|x|^2}{2}, \frac{\langle x, y\rangle}{|y|}\right)$ be a spherically symmetric Finsler metric on $\mathbb{B}^2(\mu)$. We can express $x = (x^1, x^2)$ in the polar coordinate system,

$$x^1 = \rho\cos\theta, \qquad x^2 = \rho\sin\theta. \tag{7.11}$$

By a straightforward computation one obtains

$$X := \frac{\partial}{\partial\theta} = -\rho\sin\theta\frac{\partial}{\partial x^1} + \rho\cos\theta\frac{\partial}{\partial x^2} = -x^2\frac{\partial}{\partial x^1} + x^1\frac{\partial}{\partial x^2}. \tag{7.12}$$

From (7.11) we have

$$F = \sqrt{p^2 + q^2\rho^2}\phi\left(\frac{\rho}{2}, \frac{p\rho}{\sqrt{p^2 + q^2\rho^2}}\right)$$

It follows that

$$X(F) = 0. \tag{7.13}$$

(7.13) tells us that $\frac{\partial}{\partial\theta}$ is a Killing field of F [17] and all spherically symmetric Finsler surfaces admit a non-zero Killing field $\frac{\partial}{\partial\theta}$. Let X be the vector field (7.2) on $\mathbb{B}^2(\mu)$.

By using (4.17) and (7.12), we have

$$\hat{X} = -x^2\frac{\partial}{\partial x^1} + x^1\frac{\partial}{\partial x^2} - y^2\frac{\partial}{\partial y^1} + y^1\frac{\partial}{\partial y^2}. \tag{7.14}$$

From (2.1), (7.1), (7.2), (7.5), (7.10), (3.3) and (7.14), we obtain

$$\begin{aligned} a_1 &= \left(F_{y^1}dx^1 + F_{y^2}dx^2\right)\left(-x^2\tfrac{\partial}{\partial x^1} + x^1\tfrac{\partial}{\partial x^2} - y^2\tfrac{\partial}{\partial y^1} + y^1\tfrac{\partial}{\partial y^2}\right) \\ &= -F_{y^1}x^2 + F_{y^2}x^1 \\ &= -\left(\phi r_1 + \phi_s s_1\right)x^2 + \left(\phi r_2 + \phi_s s_2\right)x^1 = (\phi - s\phi_s)\tfrac{\sqrt{2t-s^2}}{|y|}. \end{aligned} \tag{7.15}$$

Step 2 We now calculate $f(z)$ and $g(z)$ for fixed constant K (cf (3.9) and (3.14)). A direct calculation yields [21, Proposition 3.2]

$$D := \det(g_{ij}) = \phi^3\Delta \tag{7.16}$$

where Δ is given in (3.9). Note that $\{\omega_1,\ \omega_2\}$ is the Berwald frame on the Finsler surface $\mathbb{B}^2(\mu)$. It follows that $\omega_2 = \frac{\sqrt{D}}{F}\left(-y^2dx^1 + y^1dx^2\right)$ [1]. Together with (3.3), (7.14) and (7.16), we have

$$a_2 = \frac{\sqrt{D}}{F}(y^2x^2 + y^1x^1) = \frac{\sqrt{\phi^3\Delta}}{\phi|y|}\langle x,\ y\rangle = s\phi^{\frac{1}{2}}\Delta^{\frac{1}{2}}. \tag{7.17}$$

We express the geodesic coefficients G^i by (cf [6])

$$G^i = Py^i + Q^i \tag{7.18}$$

where

$$P = \frac{F_0}{2F} = \frac{r}{2\phi}\left(\phi_s + s\phi_t\right) = \frac{r}{2}(\bar{u} - s\bar{v}), \tag{7.19}$$

$$Q = \frac{r^2}{2}\bar{v}x^i \tag{7.20}$$

where we have used (7.2) and (7.7). By (7.18), we have the connection coefficients

$$N^i_j := \frac{\partial G^i}{\partial y^j} = P_{y^j}y^i + P\delta^i_j + x^i\left(\frac{r^2}{2}\bar{v}\right)_{y^j}. \tag{7.21}$$

On the other hand, we have (cf [2], Page 93)

$$\omega_3 = \frac{\sqrt{D}}{F^2}\left[y^1(dy^2 + N^2_j dx^j) - y^2(dy^1 + N^1_j dx^j)\right].$$

Together with (3.3), (7.14) and (7.21), we get

$$\begin{aligned} a_3 &= \frac{\sqrt{D}}{F^2}\left(|y|^2 + x^2y^2N_1^1 + x^1y^1N_2^2 - x^2y^1N_1^2 - x^1y^2N_2^1\right) \\ &= \frac{\sqrt{D}}{F^2}\left(|y|^2 + P\langle x,\, y\rangle + (I)\right). \end{aligned} \tag{7.22}$$

where

$$\begin{aligned} (I) : &= x^1x^2y^2\left(\frac{r^2}{2}\bar{v}\right)_{y^1} + x^1x^2y^1\left(\frac{r^2}{2}\bar{v}\right)_{y^2} \\ &\quad -(x^1)^2y^2\left(\frac{r^2}{2}\bar{v}\right)_{y^2} - (x^2)^2y^1\left(\frac{r^2}{2}\bar{v}\right)_{y^1} \\ &= -\bar{v}(x^1y^2 - x^2y^1)^2 + \frac{r^2}{2}\left(\bar{v}_{y^1}x^2 - \bar{v}_{y^2}x^1\right)(x^1y^2 - x^2y^1) \\ &= -\frac{1}{2}(2\bar{v} - s\bar{v}_s)(x^1y^2 - x^2y^1)^2 = -\frac{r^2}{2}(2\bar{v} - s\bar{v}_s)(2t - s^2) \end{aligned} \tag{7.23}$$

where we have used (7.10) and

$$\bar{v}_{y^j} = \frac{\bar{v}_s}{r}\left(x^j - s\frac{y^j}{r}\right), \qquad \bar{v}_{y^1}x^2 - \bar{v}_{y^2}x^1 = \frac{s\bar{v}_s}{r^2}(x^1y^2 - x^2y^1). \tag{7.24}$$

Plugging (7.19) and (7.23) into (7.22) yields

$$\begin{aligned} a_3 &= \frac{\sqrt{D}}{F^2}\left[r^2 + \frac{sr^2}{2}(\bar{u} - s\bar{v}) - \frac{r^2}{2}(2\bar{v} - s\bar{v}_s)(2t - s^2)\right] \\ &= \frac{\Delta^{\frac{1}{2}}}{2\phi^{\frac{1}{2}}}\left[2 + s(\bar{u} - s\bar{v}) - (2\bar{v} - s\bar{v}_s)(2t - s^2)\right] \end{aligned} \tag{7.25}$$

where we have used (7.16). The main scalar I of F is given by

$$\begin{aligned} I &= \left[\frac{F}{D}\left(\frac{\partial}{\partial y^1}\log\sqrt{D}\right)dx^1 + \frac{F}{D}\left(\frac{\partial}{\partial y^2}\log\sqrt{D}\right)dx^2\right]\left(-F_{y^2}\frac{\partial}{\partial x^1} + F_{y^1}\frac{\partial}{\partial x^2}\right) \\ &= \frac{F}{2D^{\frac{3}{2}}}(F_{y^1}D_{y^2} - F_{y^2}D_{y^1}) \\ &= \frac{\phi^2D_s}{2D^{\frac{3}{2}}}(r_2s_1 - r_1s_2) = \sqrt{2t - s^2}\frac{\phi^2D_s}{2D^{\frac{3}{2}}} \end{aligned} \tag{7.26}$$

where we have made use of (7.5) and the following fact:

$$D_{y^i} = D_s\frac{s_i}{r}.$$

Together with (7.25), we have

$$a_3I = \frac{\Delta^{\frac{1}{2}}}{2\phi^{\frac{1}{2}}}T\sqrt{2t - s^2}\frac{\phi^2D_s}{2D^{\frac{3}{2}}} = \frac{\Psi}{4\Delta\phi}\sqrt{2t - s^2}T \tag{7.27}$$

where

$$T := 2 + s(\bar{u} - s\bar{v}) - (2\bar{v} - s\bar{v}_s)(2t - s^2)$$

and

$$\Psi := 3\phi_s \Delta + \phi \Delta_s = \frac{D_s}{\phi^2}.$$

By Lemma 7.1, we have

$$I_{x^j} y^j = r(I_s + sI_t). \tag{7.28}$$

Note that I is homogeneous of degree zero with respect to y. Hence $y^j \frac{\partial I}{\partial y^j} = 0$. Together with (2.7), (7.21) and (7.28), we get

$$J = e_1(I) = \frac{1}{\phi}(I_s + sI_t) - \frac{(II)}{F} \tag{7.29}$$

where

$$e_1 := \frac{1}{F} y^j \left(\frac{\partial}{\partial x^j} - N^k_j \frac{\partial}{\partial y^k} \right)$$

and

$$\begin{aligned} (II): &= \tfrac{x^k y^j}{2} (r^2 \bar{v})_{y^j} \tfrac{\partial I}{\partial y^k} \\ &= r^2 x^k \left(\bar{v} + \tfrac{y^j}{2} \bar{v}_{y^j} \right) \tfrac{\partial I}{\partial y^k} = r \bar{v} I_s (2t - s^2) \end{aligned} \tag{7.30}$$

where we have used

$$y^j \bar{v}_{y^j} = 0, \qquad \frac{\partial I}{\partial y^j} = I_s \frac{s_j}{r}.$$

Substituting (7.30) into (7.29) yields

$$J = \frac{1}{\phi} \Box I \tag{7.31}$$

where

$$\Box := s \frac{\partial}{\partial t} + \left[1 - (2t - s^2) \bar{v} \right] \frac{\partial}{\partial s}.$$

By a straightforward computation one obtains

$$\Box \left(\sqrt{2t - s^2} \right) = s \sqrt{2t - s^2} \bar{v}. \tag{7.32}$$

Substituting (7.26) into (7.31) and using (7.17) and (7.32) we have

$$\begin{aligned} a_2 J &= s \phi^{\frac{1}{2}} \Delta^{\frac{1}{2}} \cdot \tfrac{1}{2\phi} \Box \left(\tfrac{\sqrt{2t - s^2} \Psi}{\phi^{\frac{1}{2}} \Delta^{\frac{3}{2}}} \right) \\ &= \tfrac{s \sqrt{2t - s^2}}{4 \phi \Delta^2} \left[2\Delta (\Box \Psi + s \Psi \bar{v}) - \Psi (\Delta \Box \log \phi + 3 \Box \Delta) \right]. \end{aligned} \tag{7.33}$$

By using (7.17),(7.25), (7.27) and (7.33), we obtain

$$f = f(z), \qquad g = g(z).$$

Step 3 Let $u(a)^2 := f(z(a))$ and $v(a) = \frac{g(z(a))}{f(z(a))}$. Substituting these these into (4.16), (5.10) and (6.15), we obtain the normal forms for known spherically symmetric Finsler metrics of constant flag curvature.

For example, for the Funk metric on the unit disk $\mathbb{D}^2$, we find $u(a) = \sqrt{1+4a^2}$ and $v(a) = \frac{-3a}{1+4a^2}$.

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