Suppression of Chemotactic Blowup by Strong Buoyancy in Stokes-Boussinesq Flow with Cold Boundary

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Abstract

In this paper, we show that the Keller-Segel equation equipped with zero Dirichlet Boundary condition and actively coupled to a Stokes-Boussinesq flow is globally well-posed provided that the coupling is sufficiently large. We will in fact show that the dynamics is quenched after certain time. In particular, such active coupling is blowup-suppressing in the sense that it enforces global regularity for some initial data leading to a finite-time singularity when the flow is absent.

1 Introduction

The Keller-Segel equation is a well known model of chemotaxis [20, 26]. It describes a population of bacteria or slime mold that move in response to attractive external chemical that they themselves secrete. The equation has interesting analytical properties: its solutions can form mass concentration singularities in dimension greater than one (see e.g. [25] where further references can be found). Often, chemotactic processes take place in ambient fluid. One natural question is then how the presence of fluid flow can affect singularity formation. In the case where the ambient flow is passive - prescribed and independent of the bacteria density - it has been shown that presence of the flow can suppress singularity formation. The flows that have been analyzed include some flows with strong mixing properties [21], shear flows [4], hyperbolic splitting flow [15], and some cellular flows [19]. In a similar vein, [9] explored advection induced regularity for the Kuramoto-Sivashinsky equation. The paper [14] studied the phenomena of delayed blow up by transport noise in a more general framework nonlinear PDE that includes the Keller-Segel, Fisher-KPP, and Kuramoto-Sivashinsky equations.

The case where the fluid flow is active - satisfies some fluid equation driven by force exerted the bacteria - is very interesting but harder to analyze. There have been many impressive works that analyzed such coupled systems, usually via buoyancy force; see for example [10, 11, 23, 22, 24, 27, 6, 12, 29, 28] where further references can be found. In some cases results involving global existence of regular solutions (the precise notion of their regularity is different in different papers) have been proved. These results, however, apply either in the settings where the initial data satisfy some smallness assumptions (e.g. [11, 24, 6]) or in the systems where both fluid and chemotaxis equations may not form a singularity if not coupled (e.g. [27, 29, 28]). Recently, in [16] and [30], the authors analyzed Patlak-Keller-Segel equation coupled to the Navier-Stokes equation near Couette flow. Based on ideas of blowup suppression in shear flows and stability of the Couette flow,

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the authors proved that global regularity can be enforced if the amplitude of the Couette flow is dominantly large and if the initial flow is very close to it.

In the recent work of the authors joint with Yao [17], the two dimensional Keller-Segel equation coupled with the incompressible porous media via buoyancy force has been analyzed. It has been proved that in this case, an arbitrary weak coupling constant (i.e, gravity) completely regularizes the system, and the solutions become globally regular for any reasonable initial data. At the heart of the proof is the analysis of potential energy, whose time derivative includes a coercive "main term" $\|\partial_{x_1}\rho\|^2_{H_0^{-1}}$ (where ρ is the bacteria density). Essentially, this H_0^{-1} norm has to become small, and intuitively this implies mixing in the x_1 direction. Hence the solution becomes in some sense quasi-one-dimensional and this arrests singularity formation.

Our goal in this paper is to analyze the Keller-Segel equation in an arbitrary smooth domain in dimensions two and three coupled to the Stokes flow via buoyancy force:

$$\begin{cases} \partial_t\rho + u\cdot\nabla\rho - \Delta\rho + \operatorname{div}(\rho\nabla(-\Delta)^{-1}\rho) = 0, & x\in\Omega,\\ \partial_t u - \Delta u + \nabla p = g\rho e_z,\ \operatorname{div} u = 0, & x\in\Omega,\\ u(0,x) = u_0(x),\ \rho(0,x) = \rho_0(x),\ \rho_0(x)\ge 0, & \\ u|_{\partial\Omega} = 0,\ \rho|_{\partial\Omega} = 0. & \end{cases} \tag{1.1}$$

Here Ω is a smooth, compact domain in $\mathbb{R}^d$, $d=2$ or 3. e_z denotes the unit vector $(0,1)$ when $d=2$ or $(0,0,1)$ when $d=3$. $g\in\mathbb{R}^+$ is the Rayleigh number representing the buoyancy strength. Moreover, the operator $(-\Delta)^{-1}$ denotes the inverse homogeneous Dirichlet Laplacian corresponding to the domain Ω. Throughout this work, we will always work with nonnegative initial density ρ_0, which is biologically relevant. In the case of the Stokes flow, the fluid velocity is more regular, and the equation includes time derivative that complicates matters, partly due to a loss of a "Biot-Savart law" that relates ρ and u directly. We are unable to prove global regularity for all g, and we are not sure if it is true. Our main result is global regularity for strong buoyancy. The proof is completely different from [17]: it relies on softer arguments and the analysis of the large buoyancy limit.

The first part of this paper, corresponding to Section 2, addresses the local well-posedness of strong solutions to (1.1). Before we make precise of the notion of a *strong solution*, we shall first introduce the following useful function spaces: to study the regularity properties of ρ, we consider

$$\begin{aligned} H_0^1 &:= \text{completion of } C_c^\infty(\Omega) \text{ with respect to } H^1 \text{ norm},\\ H_0^{-1} &:= \text{dual space of } H_0^1. \end{aligned}$$

Moreover, we use the traditional notation $W^{k,p}(\Omega)$ to denote Sobolev spaces equipped with norm $\|\cdot\|_{k,p}$ in domain Ω. If $p=2$, we in particular write $H^k(\Omega) = W^{k,2}(\Omega)$ equipped with norm $\|\cdot\|_k$. We will write $W^{k,p}$ (or H^k) instead of $W^{k,p}(\Omega)$ (or $H^k(\Omega)$) for simplicity if there is no confusion over the domain involved. We also say an n-vector field $v=(v_i)_i\in H^k$ if $v_i\in H^k$ for $i=1,\dots,n$.

As we also need to work with Stokes equation, it is standard to introduce the following spaces:

$$C_{c,\sigma}^\infty := \{u\in C_c^\infty(\Omega)\ |\ \operatorname{div} u = 0\},$$

$$H := \text{completion of } C_{c,\sigma}^\infty \text{ with respect to } L^2 \text{ norm},$$

$$V := H\cap H_0^1(\Omega),\ V^* := \text{dual space of } V,$$

where V is equipped with H_0^1 norm, and V^* is equipped with the standard dual norm. We also recall the following useful operators: the Leray projector $\mathbb{P}: L^2\to H$ and the Stokes operator $\mathcal{A} := -\mathbb{P}\Delta : D(\mathcal{A}) = H^2\cap V\to H$. We refer the readers to [7] for a more thorough treatment of such operators. As a common practice in the study of Stokes equation, one may equivalently rewrite the fluid equation as:

$$\partial_t u + \mathcal{A}u = g\mathbb{P}(\rho e_z), \tag{1.2}$$

We will often use this formulation in regularity estimates for the rest of this work.

To make our discussion more precise, we give rigorous definition of a *strong solution* and a *regular solution* to (1.1).

Definition 1.1. *Given initial data $\rho_0 \in H_0^1$, $u_0 \in V$, and some $T > 0$, we say the pair $(\rho(t,x), u(t,x))$ is a strong solution to (1.1) on $[0,T]$ if*

$$\rho \in C([0,T]; H_0^1) \cap L^2((0,T); H^2 \cap H_0^1),\ u \in C([0,T]; V) \cap L^2((0,T); H^2 \cap V),$$
$$\partial_t \rho \in C([0,T]; H_0^{-1}),\ \partial_t u \in C([0,T]; V^*),$$

and (ρ, u) satisfy (1.1) in the distributional sense. Moreover, a solution is regular if it is strong and additionally

$$\rho \in C^\infty((0,T] \times \Omega), \quad u \in C^\infty((0,T] \times \Omega).$$

With this definition, we are able to obtain the following local well-posedness result:

Theorem 1.1. *Given initial data $\rho_0 \in H_0^1$, $u_0 \in V$ with $\rho_0 \geq 0$, there exists a $T_* = T_*(\|\rho_0\|_{L^2}) > 0$ such that there exists a unique strong solution (ρ, u) to problem (1.1) on $[0, T_*]$.*

It will be convenient for us to assume that $T_* \leq 1$ to simplify some estimates.

We will then prove a regularity criterion which allows us to continue a strong solution of (1.1) as long as the $L_t^{\frac{4}{4-d}} L_x^2$ norm of ρ is controlled. More precisely, we have

Theorem 1.2. *Let $\Omega \subset \mathbb{R}^d$, $d = 2, 3$, be a smooth, bounded domain. If the maximal lifespan T_0 of the strong solution (ρ, u) to problem (1.1) is finite, then necessarily*

$$\lim_{t \nearrow T_0} \int_0^t \|\rho\|_{L^2}^{\frac{4}{4-d}} ds = \infty.$$

We remark that a similar result was proved in [21] in the periodic setting for the uncoupled Keller-Segel equation.

In the second part of this work, namely Section 3, we will quantify the quenching effect of the Stokes-Boussinesq flow with strong buoyancy on the Keller-Segel equation equipped with homogeneous Dirichlet boundary condition. To be more precise, we show that the flow can suppress the norm $\|\rho\|_{L^2}$ to be sufficiently small within the time scale of local existence. In particular, we will show the following main result of this work:

Theorem 1.3. *For any smooth, bounded domain $\Omega \subset \mathbb{R}^d$, $d = 2, 3$, and arbitrary initial data $\rho_0 \in H_0^1 \cap L^\infty$, $u_0 \in V$, with $\rho_0 \geq 0$, there exists $g_* = g_*(\|\rho_0\|_{L^\infty}, \|u_0\|_1)$ so that for any $g \geq g_*$, (1.1) admits a strong, global-in-time solution. In particular, ρ is quenched exponentially fast in the sense that*

$$\|\rho(t)\|_{L^2} \leq 2\|\rho_0\|_{L^2} e^{c_0(1-t)}, \tag{1.3}$$

where c_0 is a positive constant that only depends on domain Ω.

We observe that if we fix any smooth passive divergence-free u satisfying the no-flux $u \cdot n = 0$ boundary condition, then one can find smooth initial data ρ_0 such that the solution of the first equation in (1.1) will lead to finite time blow up. The argument proving this is very similar to that of Theorem 8.1 in [21] for the case of $\mathbb{T}^2$; the localization used in that proof makes it insensitive to the boundary condition.

The main step towards showing Theorem 1.3 is to prove that $\|\rho\|_{L^2}$ can be made arbitrarily small within a short time interval in the regime of large g. This step is based on a rather soft compactness/rigidity argument inspired by [8], where quenching phenomena for reaction-diffusion equations with buoyancy were explored. Due to the compactness/rigidity nature of our method, we expect that one can extend Theorem 1.3 to certain unbounded domains that admit Sobolev compactness theorems; for a description of such domains, we refer interested readers to [1]. Extending Theorem 1.3 to more general unbounded domains would need new ideas, and we are not sure such an extension would in general hold true.

In the final part of this paper, namely Section 4, we provide an argument demonstrating that a strong solution to (1.1) is in fact regular. As a consequence of this fact, the main results (i.e. Theorem 1.1, Theorem 1.2, and Theorem 1.3) can all be promoted to regular solutions. The precise statement that we will prove is the following.

Theorem 1.4. *Suppose (ρ, u) is the strong solution to (1.1) with initial data $\rho_0 \in H_0^1, u_0 \in V$ on $[0, T]$ for some $T > 0$. Then (ρ, u) also verifies the following regularity property:*

$$\rho \in C^\infty((0, T]; H_0^1 \cap H^k), \quad u \in C^\infty((0, T]; V \cap H^k)$$

for all $k \in \mathbb{N}$. In particular, (ρ, u) is a regular solution to (1.1).

While such regularization is expected for semilinear parabolic equations, we were unable to locate a convenient reference for a regularity result in the scale of Sobolev spaces H^k with large k in the scenario of (1.1). Thus for the sake of the completeness, we will give explicit *a priori* estimates which lead to this higher regularity in Section 4.

We end this section by declaring several notations and conventions that will be used throughout this work. We will use the expression $f \lesssim g$ to denote the following: there exists some constant C only depending on domain Ω such that $f \leq Cg$. In particular, we will denote a generic constant depending only on Ω by C, and it could change from line to line. Finally, we will use the Einstein summation convention. That is, by default we sum over the repeated indices; e.g. we write $a_i x_i := \sum_i a_i x_i$.

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2 Local Well-Posedness of Strong Solution

In this section, we will establish the local well-posedness of strong solutions to problem (1.1), namely Theorem 1.1. It is well-known that the classical parabolic-elliptic Keller-Segel equation is locally well-posed in domains such as $\mathbb{R}^d$ or $\mathbb{T}^d$, $d = 2, 3$, or in a smooth, bounded domain with Neumann boundary condition on ρ in suitable function spaces (see e.g. [5, 21, 27]). Since we consider a different boundary condition (i.e. Dirichlet boundary condition on ρ), we will give a proof for the sake of completeness.

We first set up an appropriate Galerkin scheme that uses two sets of bases in Subsection 2.1. In Subsection 2.2, we start with a set of lower order *a priori* energy estimates which guarantee spatial regularity of a solution up to H^2. In Subsection 2.3, we will complete the proof of Theorem 1.1 by showing the uniqueness of strong solutions. We will then demonstrate an L^2 regularity criterion (i.e. Theorem 1.2) in Subsection 2.4. In the final subsection, we will show the global well-posedness of strong solution equipped with small ρ_0 measured in L^2 norm. Both the regularity criterion and the small-data global well-posedness results will be instrumental in establishing the global well-posedness of (1.1).

Remark 2.1. *We will only discuss the case when $d = 3$. Then $d = 2$ case follows from similar (and easier) arguments.*

2.1 Galerkin Approximations

Since (1.1) is a system of semilinear parabolic equations in a compact domain, it is convenient to construct a solution to (1.1) by Galerkin approximation. Let $\{v_k\}_k$, $\{\lambda_k\}_k$ be the eigenfunctions and eigenvalues of $-\Delta$. Let $\{w_j\}_j$, $\{\eta_j\}_j$ be the eigenfunctions and eigenvalues of the Stokes operator $\mathcal{A}$. Consider the following approximate system:

$$\begin{cases}\partial_t\rho^{(n)} + \mathbb{Q}_n(u^{(n)}\cdot\nabla\rho^{(n)}) - \Delta\rho^{(n)} + \mathbb{Q}_n(\mathrm{div}(\rho^{(n)}\nabla(-\Delta)^{-1}\rho^{(n)})) = 0,\\ \partial_t u^{(n)} + \mathcal{A}u^{(n)} = g\mathbb{P}_n(\rho^{(n)}e_z),\\ \rho^{(n)}(0) = \mathbb{Q}_n\rho_0,\ u^{(n)}(0) = \mathbb{P}_n u_0,\end{cases} \tag{2.1}$$

where $\mathbb{Q}_n f := (f, v_k)_{L^2}v_k$, $\mathbb{P}_n f := (f, w_j)_{L^2}w_j$. Here $(\cdot,\cdot)_{L^2}$ denotes the standard L^2-inner product. Note that the projection operators $\mathbb{P}_n, \mathbb{Q}_n$ are symmetric with respect to L^2 inner product. Writing the approximated solutions $\rho^{(n)}(t,x) = \rho_k^{(n)}(t)v_k(x)$, $u^{(n)}(t,x) = u_j^{(n)}(t)w_j(x)$ (recall that we are summing over repeated indices), we obtain the following constant-coefficient ODEs in t: for $l = 1,\ldots,n$,

$$\begin{cases}\frac{d}{dt}\rho_l^{(n)} + C_{ljk}^{(n)}u_j^{(n)}\rho_k^{(n)} + \lambda_l\rho_l^{(n)} - D_{ljk}^{(n)}\rho_k^{(n)}\rho_j^{(n)} = 0,\\ \frac{d}{dt}u_l^{(n)} + \eta_l u_l^{(n)} = gC_{kl}\rho_k^{(n)}e_z,\\ \rho_l^{(n)}(0) = (\rho_0, v_l)_{L^2},\ u_l^{(n)}(0) = (u_0, w_l)_{L^2},\end{cases} \tag{2.2}$$

where

$$C_{ljk}^{(n)} := (\mathbb{Q}_n(w_j\cdot\nabla v_k), v_l)_{L^2},\ D_{ljk}^{(n)} := \mathbb{Q}_n(\mathrm{div}(v_k\nabla(-\Delta)^{-1}v_j), v_l)_{L^2},$$
$$C_{kl} := (\mathbb{P}v_k, w_l)_{L^2}.$$

To close the Galerkin approximation argument, we shall prove suitable uniform-in-n energy estimates for $(\rho^{(n)}, u^{(n)})$ and pass to the limit using compactness theorems. For the sake of simplicity, we shall prove such energy estimates in an *a priori* fashion, for sufficiently regular solutions of the original system (1.1). One could verify that all estimates below can be carried over to the approximated solutions $(\rho^{(n)}, u^{(n)})$ in a straightforward manner.

2.2 *A priori* Estimates and Existence

Given initial data $\rho_0 \in H_0^1, u \in V$, we first show the following $L_t^\infty L_x^2$ and $L_t^2 H_x^1$ estimates for a strong solution (ρ, u):

Proposition 2.1. *Given initial data $\rho_0 \in H_0^1, u \in V$, we assume (ρ, u) is a strong solution to* (1.1) *on $[0,T]$ for some $T > 0$. Then for $t \in [0,T]$, we have*

$$\frac{d}{dt}\|\rho\|_{L^2}^2 + \|\nabla\rho\|_{L^2}^2 \lesssim \|\rho\|_{L^2}^6, \ \frac{1}{2}\frac{d}{dt}\|u\|_{L^2}^2 + \|\nabla u\|_{L^2}^2 \leq g\|u\|_{L^2}\|\rho\|_{L^2}. \tag{2.3}$$

Moreover, there exists $T_* = T_*(\|\rho_0\|_{L^2}) \in (0,1]$, and a constant $C(\|\rho_0\|_{L^2}, \|u_0\|_{L^2}) > 0$ such that

$$\sup_{t\in[0,T_*]} \|\rho(t)\|_{L^2}^2 + \int_0^{T_*} \|\nabla\rho(t)\|_{L^2}^2 ds \leq 4\|\rho_0\|_{L^2}^2. \tag{2.4}$$

$$\sup_{t\in[0,T_*]} \|u(t)\|_{L^2}^2 + \int_0^{T_*} \|\nabla u(t)\|_{L^2}^2 ds \leq C(\|\rho_0\|_{L^2}, \|u_0\|_{L^2})(g^2+1). \tag{2.5}$$

Proof. First by testing the ρ-equation of (1.1) by ρ and integrating by parts, we have

$$\frac{1}{2}\frac{d}{dt}\|\rho\|_{L^2}^2 + \|\nabla\rho\|_{L^2}^2 = \frac{1}{2}\int_\Omega \rho^3 dx \leq C\|\rho\|_{L^2}^{3/2}\|\nabla\rho\|_{L^2}^{3/2} \leq \frac{1}{2}\|\nabla\rho\|_{L^2}^2 + C\|\rho\|_{L^2}^6,$$

where we used the following standard Gagliardo-Nirenberg inequality in 3D for $f \in H_0^1$ (see [2], for example):

$$\|f\|_{L^3}^3 \leq C\|f\|_{L^2}^{3/2}\|\nabla f\|_{L^2}^{3/2}.$$

After rearranging, we obtain the first inequality of (2.3). Similarly, we test the u-equation of (1.1) by u. After integration by parts, we have

$$\frac{1}{2}\frac{d}{dt}\|u\|_{L^2}^2 + \|\nabla u\|_{L^2}^2 = g\int_\Omega u\cdot(\rho e_z)dx \leq g\|u\|_{L^2}\|\rho\|_{L^2}, \tag{2.6}$$

which proves the second inequality in (2.3). Then, the estimate (2.4) follows immediately from applying Bihari–LaSalle inequality Better to provide a reference since not broadly known to (2.3) and choosing $T_* = T_*(\|\rho_0\|_{L^2}) \leq 1$ sufficiently small. Now integrating (2.6) from 0 to $t \in (0, T_*)$, using (2.4), and taking supremum over t, we have

$$\sup_{t\in[0,T_*]} \|u(t)\|_{L^2} \leq 8g\|\rho_0\|_{L^2}T_* + \|u_0\|_{L^2}. \tag{2.7}$$

Using (2.4) and (2.7) in the integrated in time version of (2.6), we obtain that

$$\int_0^{T_*} \|\nabla u(s)\|_{L^2}^2 ds \leq \|u_0\|_{L^2}^2 + 4gT_*\|\rho_0\|_{L^2}(8g\|\rho_0\|_{L^2}T_* + \|u_0\|_{L^2}) \tag{2.8}$$

The proof of (2.5) is finished after we combine (2.7) and (2.8). □

Remark 2.2. *From now on, any appearance of T_* refers to the time T_* chosen in Proposition 2.1.*

With Proposition 2.1, we will derive the following upgraded temporal and spatial regularity estimates for solution (ρ, u) within the time interval $[0, T_*]$.

Proposition 2.2. *Assuming (ρ, u) to be a strong solution to* (1.1) *with initial data $\rho_0 \in H_0^1, u \in V$, there exists $C(\rho_0, u_0, g) > 0$ such that*

$$\int_0^{T_*} \left(\|\rho(t)\|_2^2 + \|u(t)\|_2^2 + \|\partial_t\rho(t)\|_{L^2}^2 + \|\partial_t u(t)\|_{L^2}^2\right) dt + \sup_{t\in[0,T_*]} (\|\rho(t)\|_1^2 + \|u(t)\|_1^2) \leq C(\rho_0, u_0, g).$$

Proof. Testing the ρ-equation in (1.1) by $-\Delta\rho$ and integrating by parts, we obtain:

$$\frac{1}{2}\frac{d}{dt}\|\nabla\rho\|_{L^2}^2 + \|\Delta\rho\|_{L^2}^2 = \int_\Omega \Delta\rho(u\cdot\nabla\rho) + \int_\Omega \Delta\rho\,\mathrm{div}(\rho\nabla(-\Delta)^{-1}\rho) = I + J.$$

Let us fix $\epsilon > 0$. Using Sobolev embedding, Poincaré inequality, and Young's inequality with ϵ, we can estimate I by:

$$I \le \|\Delta\rho\|_{L^2}\|\nabla\rho\|_{L^2}\|u\|_{L^\infty} \le \epsilon\|\Delta\rho\|_{L^2}^2 + C(\epsilon)\|u\|_2^2\|\nabla\rho\|_{L^2}^2.$$

Moreover, we can write J as:

$$J = \int_\Omega \Delta\rho\left(\nabla\rho\cdot\nabla(-\Delta)^{-1}\rho - \rho^2\right)dx = J_1 + J_2.$$

Using elliptic estimates and Gagliardo-Nirenberg inequality, we can estimate J_1 by:

$$\begin{aligned}
J_1 &\le \|\Delta\rho\|_{L^2}\|\nabla\rho\|_{L^3}\|\nabla(-\Delta)^{-1}\rho\|_{L^6} \lesssim \|\Delta\rho\|_{L^2}\|\nabla\rho\|_{L^3}\|\nabla(-\Delta)^{-1}\rho\|_1 \\
&\lesssim \|\Delta\rho\|_{L^2}\|\nabla\rho\|_{L^2}^{1/2}\|\nabla\rho\|_1^{1/2}\|\rho\|_{L^2} \lesssim \|\rho\|_2^{3/2}\|\nabla\rho\|_{L^2}^{1/2}\|\rho\|_{L^2} \\
&\le \epsilon\|\Delta\rho\|_{L^2}^2 + C(\epsilon)\|\nabla\rho\|_{L^2}^2\|\rho\|_{L^2}^4,
\end{aligned}$$

where we also used Young's inequality in the final step.

We are going to use the following Gagliardo-Nirenberg inequalities: in dimension three,

$$\|\rho\|_{L^4} \lesssim \|\Delta\rho\|_{L^2}^{3/8}\|\rho\|_{L^2}^{5/8};\ \ \|\rho\|_{L^4} \lesssim \|\rho\|_1^{3/4}\|\rho\|_{L^2}^{1/4}.$$

Then we can estimate J_2 as follows:

$$\begin{aligned}
J_2 &\le \|\Delta\rho\|_{L^2}\|\rho\|_{L^4}^2 \le C\|\Delta\rho\|_{L^2}\|\Delta\rho\|_{L^2}^{1/2}\|\rho\|_{L^2}^{5/6}\|\rho\|_1^{1/2}\|\rho\|_{L^2}^{1/6} \\
&= C\|\Delta\rho\|_{L^2}^{3/2}\|\rho\|_1^{1/2}\|\rho\|_{L^2} \le \epsilon\|\Delta\rho\|_{L^2}^2 + C(\epsilon)\|\nabla\rho\|_{L^2}^2\|\rho\|_{L^2}^4,
\end{aligned}$$

Collecting the estimates above and choosing ϵ to be sufficiently small, we obtain the following:

$$\frac{d}{dt}\|\nabla\rho\|_{L^2}^2 + \|\Delta\rho\|_{L^2}^2 \lesssim \left(\|\rho\|_{L^2}^4 + \|u\|_2^2\right)\|\nabla\rho\|_{L^2}^2 \tag{2.9}$$

On the other hand, we test (1.2) by $\mathcal{A}u$. Integrating by parts, we have

$$\frac{1}{2}\frac{d}{dt}\|\nabla u\|_{L^2}^2 + \|\mathcal{A}u\|_{L^2}^2 = g\int_\Omega \mathcal{A}u\cdot\rho e_z \le \frac{1}{2}\|\mathcal{A}u\|_{L^2}^2 + \frac{g^2}{2}\|\rho\|_{L^2}^2$$

Rearranging the above and using Theorem A.1, we conclude that,

$$\frac{d}{dt}\|\nabla u\|_{L^2}^2 + \|u\|_2^2 \le g^2\|\rho\|_{L^2}^2 \le 4g^2\|\rho_0\|_{L^2}^2,\ t\in[0,T_*], \tag{2.10}$$

where the last inequality is due to Proposition 2.1. Integrating (2.10) from 0 to t, $t \le T_*$ and then taking supremum of t on $[0,T_*]$, we obtain

$$\sup_{t\in[0,T_*]}\|\nabla u(t)\|_{L^2}^2 \le 4g^2\|\rho_0\|_{L^2}^2 T_* + \|\nabla u_0\|_{L^2}^2;$$

in addition,

$$\int_0^{T_*} \|u(t)\|_2^2 \leq 4g^2\|\rho_0\|_{L^2}^2 T_* + \|\nabla u_0\|_{L^2}^2. \tag{2.11}$$

It follows that

$$\sup_{t\in[0,T_*]} \|u(t)\|_1^2 + \int_0^{T_*} \|u(t)\|_2^2 dt \leq C(u_0, \rho_0, g).$$

Integrating (2.9) and using (2.11), we have that for all $t \in [0, T_*]$,

$$\|\nabla\rho(t)\|_{L^2}^2 \lesssim \|\rho_0\|_1^2 \exp\left(\int_0^{T_*} (\|\rho\|_{L^2}^4 + \|u\|_2^2 ds\right) \leq \|\rho_0\|_1^2 \exp\left(C(\rho_0, g)T_* + \|u_0\|_1^2\right) < \infty.$$

Similarly to the case of u, we can also use (2.9) to control $\int_0^{T_*} \|\rho(t)\|_2^2 dt$ as well, arriving at

$$\sup_{t\in[0,T_*]} \|\rho(t)\|_1^2 + \int_0^{T_*} \|\rho(t)\|_2^2 \leq C(u_0, \rho_0, g).$$

We have thus showed the spatial regularity of ρ and u.

Finally, we shall obtain regularity estimates for the time derivatives. Using the equation (1.1), we see that

$$\partial_t\rho = -u\cdot\nabla\rho + \Delta\rho - \operatorname{div}(\rho\nabla(-\Delta)^{-1}\rho) \text{ and } \partial_t u = -\mathcal{A}u + g\mathbb{P}(\rho e_z).$$

Using standard Sobolev embeddings and elliptic estimates, we have the following bounds:

$$\begin{aligned}
\int_0^{T_*} \|u\cdot\nabla\rho(t)\|_{L^2}^2 dt &\leq \int_0^{T_*} \|u\|_{L^6}^2 \|\nabla\rho\|_{L^3}^2 dt \lesssim \sup_{t\in[0,T_*]} \|u(t)\|_1^2 \int_0^{T_*} \|\rho(t)\|_2^2 dt,\\
\int_0^{T_*} \|\Delta\rho\|_{L^2}^2 dt &\leq \int_0^{T_*} \|\rho(t)\|_2^2 dt,\\
\int_0^{T_*} \|\operatorname{div}(\rho\nabla(-\Delta)^{-1}\rho)\|_{L^2}^2 dt &\lesssim \int_0^{T_*} \|\rho\|_{L^4}^4 + \|\nabla\rho\cdot\nabla(-\Delta)^{-1}\rho\|_{L^2}^2 dt\\
&\lesssim \sup_{t\in[0,T_*]} \|\rho(t)\|_1^4 T_* + \sup_{t\in[0,T_*]} \|\rho(t)\|_1^2 \int_0^{T_*} \|\rho(t)\|_2^2 dt,\\
\int_0^{T_*} \|\mathcal{A}u\|_{L^2}^2 + g\|\mathbb{P}\rho\|_{L^2}^2 dt &\leq \int_0^{T_*} \|u\|_2^2 + g\|\rho\|_{L^2}^2 dt.
\end{aligned}$$

The above estimates and bounds we proved earlier imply that

$$\int_0^{T_*} \|\partial_t\rho\|_{L^2}^2 dt + \int_0^{T_*} \|\partial_t u\|_{L^2}^2 dt \leq C(u_0, \rho_0, g),$$

and the proof is thus complete. □

With the regularity estimates above, we may construct solutions (ρ, u) from $(\rho^{(n)}, u^{(n)})$. The following standard compactness theorem is useful. We refer interested readers to Theorem IV.5.11 in [3] and Theorem 4 of Chapter 5 in [13] for related proofs.

Theorem 2.1. *Let*

$$E_1 := \{\rho \in L^2((0,T);H^2),\ \partial_t\rho \in L^2((0,T);L^2)\},$$
$$E_2 := \{u \in L^2((0,T);H^2 \cap V),\ \partial_t u \in L^2((0,T);H)\}$$

for some $T > 0$. Then E_1 is continuously embedded in $C([0,T],H^1)$, and E_2 is continuously embedded in $C([0,T],V)$.

With the compactness theorem above, we are ready to show the existence part of Theorem 1.1 as follows.

Corollary 2.1. *Given initial data $\rho_0 \in H_0^1$, $u \in V$, there exists a weak solution (ρ, u) of the system (1.1) satisfying*

$$\rho \in C([0,T_*];H_0^1) \cap L^2((0,T_*);H^2 \cap H_0^1),\ u \in C([0,T_*];V) \cap L^2((0,T_*);H^2 \cap V), \tag{2.12}$$
$$\partial_t\rho \in C([0,T_*];H_0^{-1}),\ \partial_t u \in C([0,T_*];V^*). \tag{2.13}$$

Proof. The uniform bounds in Proposition 2.2 inform us that there exists a subsequence of $\{\rho^{(n)}\}_n, \{u^{(n)}\}_n$, which we still denote by $\rho^{(n)}, u^{(n)}$, and ρ, u, such that

1. $\rho^{(n)} \rightharpoonup \rho$ weak-$*$ in $L^\infty((0,T_*);H_0^1)$, weakly in $L^2((0,T_*);H^2 \cap H_0^1)$; $\partial_t\rho^{(n)} \rightharpoonup \partial_t\rho$ weakly in $L^2((0,T_*);L^2)$,
2. $u^{(n)} \rightharpoonup u$ weak-$*$ in $L^\infty((0,T_*);V)$, weakly in $L^2((0,T_*);H^2 \cap V)$; $\partial_t u^{(n)} \rightharpoonup \partial_t u$ weakly in $L^2((0,T_*);H)$.

It is straightforward to check that the limits ρ and u satisfy (1.1) in the sense of distribution. Invoking Theorem 2.1, we have proved (2.12).

Now, we show $\partial_t u \in C([0,T_*];V^*)$. In view of (1.2), it suffices to show that $-\mathcal{A}u + g\rho e_2 \in C([0,T_*];V^*)$. For simplicity, we show that the most singular term $\mathcal{A}u \in C([0,T_*];V^*)$. But this is indeed true as $u \in C([0,T_*];V)$ and $\mathcal{A}$ is a bounded linear operator from V to V^*. Hence, $\partial_t u \in C([0,T_*];V^*)$.

To show the needed regularity of $\partial_t\rho$, it suffices to show that $-u\cdot\nabla\rho + \Delta\rho - \mathrm{div}(\rho\nabla(-\Delta)^{-1}\rho) \in C([0,T_*];H_0^{-1})$. A similar argument to how we treat $\mathcal{A}u$ will yield $\Delta\rho \in C([0,T_*];H_0^{-1})$. We then prove strong continuity for the most singular nonlinear term $u\cdot\nabla\rho$, and the term $\mathrm{div}(\rho\nabla(-\Delta)^{-1}\rho)$ will follow from a similar argument. Let $t, s \in [0,T_*]$. Picking $\varphi \in H_0^1$ and integrating by parts, we have

$$\begin{aligned}
&\int_\Omega (u(t,x)\cdot\nabla\rho(t,x) - u(s,x)\cdot\nabla\rho(s,x))\varphi(x)dx \\
&\quad= \int_\Omega (u(t,x)-u(s,x))\cdot\nabla\rho(t,x)\varphi dx + \int_\Omega u(s,\cdot)\cdot\nabla(\rho(t,x)-\rho(s,x))\varphi(x)dx \\
&\quad= \int_\Omega \mathrm{div}((u(t,x)-u(s,x))\rho(t,x))\varphi dx + \int_\Omega \mathrm{div}(u(s,\cdot)(\rho(t,x)-\rho(s,x)))\varphi(x)dx \\
&\quad= -\int_\Omega ((u(t,x)-u(s,x))\rho(t,x))\cdot\nabla\varphi dx - \int_\Omega (u(s,\cdot)(\rho(t,x)-\rho(s,x)))\cdot\nabla\varphi(x)dx.
\end{aligned}$$

The first term on RHS can be estimated by:

$$\begin{aligned}
\int_\Omega ((u(t,x)-u(s,x))\rho(t,x))\cdot\nabla\varphi dx &\le \|u(t,\cdot)-u(s,\cdot)\|_{L^3}\|\rho(t,\cdot)\|_{L^6}\|\varphi\|_1 \\
&\lesssim \|u(t,\cdot)-u(s,\cdot)\|_1\|\rho(t,\cdot)\|_1\|\varphi\|_1 \\
&\le C(\rho_0,u_0,g)\|u(t,\cdot)-u(s,\cdot)\|_1\|\varphi\|_1.
\end{aligned}$$

Note that we used Sobolev embedding in the second inequality and the uniform bound of ρ in $L^\infty((0,T_*);H_0^1)$ norm in the last inequality. Similarly, we can estimate the second term on RHS by:

$$\int_\Omega (u(s,\cdot)(\rho(t,x)-\rho(s,x)))\cdot\nabla\varphi(x)dx \le C(\rho_0,u_0,g)\|\rho(t,\cdot)-\rho(s,\cdot)\|_1\|\varphi\|_1$$

thanks to $u\in L^\infty((0,T);V)$. Combining the two estimates above and using duality, we conclude that

$$\|u(t,\cdot)\cdot\nabla\rho(t,\cdot)-u(s,\cdot)\cdot\nabla\rho(s,\cdot)\|_{H_0^{-1}} \le C(\rho_0,u_0,g)(\|u(t,\cdot)-u(s,\cdot)\|_1+\|\rho(t,\cdot)-\rho(s,\cdot)\|_1)\to 0$$

as $t\to s$ due to $u\in C([0,T_*];V)$ and $\rho\in C([0,T_*];H_0^1)$. This verifies $\partial_t\rho\in C([0,T_*];H_0^{-1})$, and we have proved (2.13). □

2.3 Uniqueness

In this section, we show the uniqueness of regular solutions to problem (1.1).

Proposition 2.3. *Given two solutions (ρ_1,u_1), (ρ_2,u_2) defined on the same time interval $[0,T]$ with the same initial data (ρ_0,u_0), it holds that $\rho_1\equiv\rho_2$, $u_1\equiv u_2$ on $[0,T]$.*

Proof. Consider the differences of the two pairs of solutions: $r=\rho_1-\rho_2$, $w=u_1-u_2$. A straightforward computation yields the following equations satisfied by r,w:

$$\begin{cases}\partial_t r-\Delta r+u_1\cdot\nabla r+w\cdot\nabla\rho_2+\operatorname{div}(r\nabla(-\Delta)^{-1}\rho_1-\rho_2\nabla(-\Delta)^{-1}r)=0,\\ \partial_t w+\mathcal{A}w=g\mathbb{P}(re_z),\end{cases}$$

with boundary conditions $r|_{\partial\Omega}=0$, $w|_{\partial\Omega}=0$ and zero initial condition. Testing the r-equation by r, we obtain

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|r\|_{L^2}^2+\|\nabla r\|_{L^2}^2 &= -\int_\Omega ru_1\cdot\nabla r-\int_\Omega r(w\cdot\nabla\rho_2)+\int_\Omega r\nabla r\cdot\nabla(-\Delta)^{-1}\rho_1\\ &\quad -\int_\Omega \rho_2\nabla r\cdot\nabla(-\Delta)^{-1}r=I_1+I_2+I_3+I_4.\end{aligned}$$

Using incompressibility of u_1, we immediately have $I_1=0$ via integration by parts. Using Hölder inequality and Sobolev embedding, we can estimate I_2 by:

$$I_2\le\|r\|_{L^2}\|w\|_{L^6}\|\nabla\rho_2\|_{L^3}\lesssim\|r\|_{L^2}\|w\|_1\|\rho_2\|_2\le\epsilon\|w\|_1^2+C(\epsilon)\|\rho_2\|_2^2\|r\|_{L^2}^2$$

for any $\epsilon>0$. Using elliptic estimates, Sobolev embedding, and Gagliardo-Nirenberg-Sobolev inequalities, we may estimate I_3 by:

$$\begin{aligned}I_3&\le\|\nabla r\|_{L^2}\|r\|_{L^3}\|\nabla(-\Delta)^{-1}\rho_1\|_{L^6}\lesssim\|\nabla r\|_{L^2}\|r\|_{L^2}^{1/2}\|\nabla r\|_{L^2}^{1/2}\|\rho_1\|_{L^2}\\ &\lesssim\|\rho_1\|_{L^2}\|\nabla r\|_{L^2}^{3/2}\|\rho\|_{L^2}^{1/2}\le\epsilon\|\nabla r\|_{L^2}^2+C(\epsilon)\|\rho_1\|_{L^2}^4\|r\|_{L^2}^2.\end{aligned}$$

Similarly, we can estimate I_4 by

$$I_4\lesssim\|\rho_2\|_{L^\infty}\|r\|_{L^2}\|\nabla r\|_{L^2}\lesssim\|\rho_2\|_2\|r\|_{L^2}\|\nabla r\|_{L^2}\le\epsilon\|\nabla r\|_{L^2}^2+C(\epsilon)\|\rho_2\|_2^2\|r\|_{L^2}^2.$$

On the other hand, we test the w-equation by w:

$$\frac{1}{2}\frac{d}{dt}\|w\|_{L^2}^2+\|\nabla w\|_{L^2}^2=g\int_\Omega w\cdot re_z\le\frac{1}{2}\|w\|_{L^2}^2+\frac{g^2}{2}\|r\|_{L^2}^2.$$

Consider $E(t) := \|w\|_{L^2}^2+\|r\|_{L^2}^2$. Collecting the estimates above and choosing $\epsilon > 0$ to be sufficiently small, we have the following inequality:

$$\frac{dE}{dt} \leq C(\|\rho_2\|_2^2 + \|\rho_1\|_{L^2}^4 + g^2)E(t) =: Cf(t)E(t).$$

Since (ρ_i, u_i) are strong solutions for $i = 1, 2$, we know that $f \in L^1([0, T])$. Since (r, w) assumes zero initial condition, we have $E(0) = 0$. Then an application of Grönwall's inequality on time interval $[0, T]$ implies $E(t) = 0$ for all $t \in [0, T]$ Hence, uniqueness of strong solution is proved. □

2.4 Regularity Criterion

In this section, we aim to prove Theorem 1.2. We first need the following fact on the monotonicity of L^1 norm of cell density ρ:

Lemma 2.1. *Assume Ω to be a smooth domain in either $\mathbb{R}^2$ or $\mathbb{R}^3$. Let (ρ, u) be a strong solution to problem (1.1) on $[0, T]$. Suppose also that ρ_0 is nonnegative. Then for a.e. $t \in [0, T]$, we have*

$$\frac{d}{dt}\|\rho(t)\|_{L^1} \leq 0.$$

Proof. First, we note that by parabolic maximum principle, we must have $\rho(t, x) \geq 0$ in $[0, T] \times \Omega$. Using (1.1), we compute that

$$\begin{aligned}
\frac{d}{dt}\|\rho(t,\cdot)\|_{L^1} &= \frac{d}{dt}\int_\Omega \rho(t,x)dx = \int_\Omega \left(-u\cdot\nabla\rho + \Delta\rho - \mathrm{div}(\rho\nabla(-\Delta)^{-1}\rho)\right)dx \\
&= \int_\Omega \mathrm{div}(\nabla\rho - \rho\nabla(-\Delta)^{-1}\rho)dx = \int_{\partial\Omega} \frac{\partial\rho}{\partial n} - \rho\frac{\partial}{\partial n}(-\Delta)^{-1}\rho dS \\
&= \int_{\partial\Omega} \frac{\partial\rho}{\partial n} dS,
\end{aligned}$$

where $\frac{\partial}{\partial n}$ denotes the outward normal derivative and dS denotes the surface unit. Note that regularity of a strong solution is sufficient to conclude that the right hand side is in $L^2([0, T_*])$, and so $\|\rho(t, \cdot)\|_{L^1}$ is absolutely continuous in time. We also used the incompressibility of u, divergence theorem, and the Dirichlet boundary condition in the derivation above. In view of parabolic maximum principle, we must have

$$\frac{\partial\rho}{\partial n}|_{\partial\Omega} \leq 0.$$

Hence, we conclude that

$$\frac{d}{dt}\|\rho(t,\cdot)\|_{L^1} \leq 0,\ t \in (0, T].$$

□

Now, we are ready to give a proof of the L^2 regularity criterion:

Proof of Theorem 1.2. Assume (ρ, u) is a solution to (1.1) with smooth data (ρ_0, u_0). Let $T_0 > 0$ be its maximal lifespan.

1. $d=2$. Suppose $T_0<\infty$ and

$$\lim_{t\nearrow T_0}\int_0^t \|\rho\|_{L^2}^2 ds = M<\infty.$$

First, we test the u-equation in (1.1) by $\mathcal{A}u$, which yields:

$$\frac{1}{2}\frac{d}{dt}\|\nabla u\|_{L^2}^2+\|\mathcal{A}u\|_{L^2}^2 = g\int_\Omega \mathcal{A}u\cdot\rho e_2 \le \frac{1}{2}\|\mathcal{A}u\|_{L^2}^2+\frac{g^2}{2}\|\rho\|_{L^2}^2,\ t\in[0,T_0).$$

Rearranging the above inequality, using Grönwall inequality, Theorem A.1 and the assumption, we obtain that

$$\sup_{t\in[0,T_0]}\|u\|_1^2+\int_0^{T_0}\|u\|_2^2 ds \le \|u_0\|_1^2+\frac{g^2M}{2}<\infty. \tag{2.14}$$

Testing ρ-equation by $-\Delta\rho$, one obtains that

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|\nabla\rho\|_{L^2}^2+\|\Delta\rho\|_{L^2}^2 &= \int_\Omega \Delta\rho u\cdot\nabla\rho-\int_\Omega\Delta\rho\rho^2+\int_\Omega\Delta\rho\nabla\rho\cdot\nabla(-\Delta)^{-1}\rho\\ &=: Q_1+Q_2+Q_3.\end{aligned}$$

Similarly to the estimate (2.9), we have for any $\epsilon>0$

$$\begin{aligned}Q_1 &\le \|\Delta\rho\|_{L^2}\|\nabla\rho\|_{L^2}\|u\|_{L^\infty}\le\epsilon\|\Delta\rho\|_{L^2}^2+C(\epsilon)\|\nabla\rho\|_{L^2}^2\|u\|_2^2,\\ Q_2 &\le \epsilon\|\Delta\rho\|_{L^2}^2+C(\epsilon)\|\rho\|_{L^4}^4\le\epsilon\|\Delta\rho\|_{L^2}^2+C(\epsilon)\|\rho\|_{L^2}^2\|\nabla\rho\|_{L^2}^2.\end{aligned}$$

The term that we have to treat differently is Q_3. Using Hölder inequality, Sobolev embedding, and an L^p-based elliptic estimate, we have:

$$\begin{aligned}Q_3 &\le \|\Delta\rho\|_{L^2}\|\nabla\rho\|_{L^3}\|\nabla(-\Delta)^{-1}\rho\|_{L^6}\lesssim\|\Delta\rho\|_{L^2}\|\nabla\rho\|_{L^3}\|\nabla(-\Delta)^{-1}\rho\|_{1,\frac{3}{2}}\\ &\lesssim\|\Delta\rho\|_{L^2}\|\nabla\rho\|_{L^3}\|\rho\|_{L^{3/2}}\lesssim\|\Delta\rho\|_{L^2}\|\rho\|_{L^2}^{1/3}\|\nabla^2\rho\|_{L^2}^{2/3}\|\rho\|_{L^1}^{2/3}\|\nabla\rho\|_{L^2}^{1/3}\\ &\lesssim\|\Delta\rho\|_{L^2}^{5/3}\|\rho\|_{L^2}^{1/3}\|\rho\|_{L^1}^{2/3}\|\nabla\rho\|_{L^2}^{1/3}\le\epsilon\|\Delta\rho\|_{L^2}^2+C(\epsilon)\|\rho\|_{L^2}^2\|\rho\|_{L^1}^4\|\nabla\rho\|_{L^2}^2,\end{aligned}$$

where we used the Gagliardo-Nirenberg-Sobolev inequalities

$$\|f\|_{L^{3/2}}\le C\|f\|_{L^1}^{2/3}\|\nabla f\|_{L^2}^{1/3},\ \|\nabla f\|_{L^3}\le C\|f\|_{L^2}^{1/3}\|\nabla^2 f\|_{L^2}^{2/3},$$

in the fourth inequality, and Young's inequality in the last step. By Lemma 2.1, we know that for $t\in[0,T_0)$, $\|\rho(t,\cdot)\|_{L^1}\le\|\rho_0\|_{L^1}$. Then we have

$$Q_3\le\epsilon\|\Delta\rho\|_{L^2}^2+C(\rho_0,\epsilon)\|\rho\|_{L^2}^2\|\nabla\rho\|_{L^2}^2.$$

Choosing $\epsilon>0$ sufficiently small and using the estimates of L_i above, the ρ-estimate can be rearranged as:

$$\frac{d}{dt}\|\nabla\rho\|_{L^2}^2+\|\Delta\rho\|_{L^2}^2\le C(\rho_0)(\|u\|_2^2+\|\rho\|_{L^2}^2)\|\nabla\rho\|_{L^2}^2. \tag{2.15}$$

Using Grönwall inequality, we have:

$$\begin{aligned}\sup_{0\le t\le T_0}\|\nabla\rho(t,\cdot)\|_{L^2}^2+\int_0^{T_0}\|\rho\|_2^2 ds &\lesssim \|\nabla\rho_0\|_{L^2}^2\exp\left(C(\rho_0)\int_0^{T_0}(\|u\|_2^2+\|\rho\|_{L^2}^2)ds\right)\\ &\le C(\rho_0,u_0,M,g,T_0),\end{aligned}$$

where we used the assumption, (2.14), and elliptic estimate. But this implies that one can extend the solution (ρ,u) beyond the supposed lifespan T_0 by Theorem 1.1. This yields a contradiction.

2. $d = 3$. Suppose $T_0 < \infty$ and

$$\lim_{t \nearrow T_0} \int_0^t \|\rho\|_{L^2}^4 ds = M < \infty.$$

Testing the u-equation in (1.1) by Au and deploying estimates similar to the $d = 2$ case, we have

$$\sup_{t \in [0,T_0]} \|\nabla u\|_{L^2}^2 + \int_0^{T_0} \|u\|_2^2 ds \leq \|u_0\|_1^2 + \frac{g^2 \sqrt{MT_0}}{2} < \infty.$$

A derivation identical to (2.9) yields:

$$\frac{d}{dt}\|\nabla \rho\|_{L^2}^2 + \|\Delta \rho\|_{L^2}^2 \lesssim \left(\|\rho\|_{L^2}^4 + \|u\|_2^2\right) \|\nabla \rho\|_{L^2}^2.$$

Applying Grönwall inequality and combining the two estimates above, we have for $t \in [0, T_0]$ that

$$\|\nabla \rho(t, \cdot)\|_{L^2}^2 + \int_0^{T_0} \|\rho\|_2^2 ds \lesssim \|\rho_0\|_1^2 \exp\left(C(\rho_0) \int_0^{T_0} (\|\rho\|_{L^2}^4 + \|u\|_2^2) ds\right) \leq C(\rho_0, u_0, M, g, T_0).$$

And this contradicts the assumption that T_0 is the maximal lifespan in view of Theorem 1.1.

The proof is thus completed. □

2.5 Global Well-Posedness with Small L^2 Data

We conclude Section 2 by making the following observation: a strong solution equipped with initial data (ρ_0, u_0) with $\|\rho_0\|_{L^2}$ sufficiently small survives for all times. This observation is based on an L^2-energy inequality with a higher-than-quadratic nonlinearity, together with the continuation criterion proved in Theorem 1.2. A precise statement of this result is given as follows.

Proposition 2.4. *Let (ρ, u) be a strong solution to (1.1) with initial data $\rho_0 \in H_0^1, u_0 \in V$ and the maximal lifespan $T_0 > 0$. There exists $\epsilon_0 > 0$ sufficiently small, depending only on domain Ω and independent of g, so that if $\|\rho_0\|_{L^2} < \epsilon_0$, $T_0 = \infty$. Moreover, ρ decays exponentially fast i.e.*

$$\|\rho(t)\|_{L^2} \leq \|\rho_0\|_{L^2} e^{-c_0 t}, \tag{2.16}$$

for some constant $c_0 > 0$ depending only on domain Ω.

Proof. Following the proof of Proposition 2.1, using the energy estimate of ρ, a Gagliardo-Nirenberg-Sobolev inequality, Young's inequality and Poincaré inequality, for $t \in (0, T_0)$ we have that

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}\|\rho\|_{L^2}^2 &\leq -\|\nabla \rho\|_{L^2}^2 + \frac{1}{2}\|\nabla \rho\|_{L^2}^2 + C\|\rho\|_{L^2}^{\frac{12-2d}{4-d}} \\ &\leq -\frac{1}{2C_p}\|\rho\|_{L^2}^2 + C\|\rho\|_{L^2}^{\frac{12-2d}{4-d}} =: f_d(\|\rho\|_{L^2}^2), \end{aligned} \tag{2.17}$$

where C_p denotes the Poincaré constant that only depends on domain Ω. Since $2 < \frac{12-2d}{4-d}$ when $d = 2, 3$, we fix $\epsilon_0 \in (0, 1)$ sufficiently small so that $f_d(s) < -\frac{1}{4C_p}s^2$ for $s \in (0, \epsilon_0]$. Note that such choice of ϵ_0 only depends on domain Ω.

With such choice of ϵ_0, we claim that the solution is global and the L^2 norm never exceeds ϵ_0. Indeed, suppose not and let $\tau \in [0, T_0]$ to be the first time such that $\|\rho(\tau)\|_{L^2} = \epsilon_0$. If $\tau < T_0$, applying (2.17) together with our choice of ϵ_0, we note that

$$\frac{1}{2}\frac{d}{dt}\|\rho\|_{L^2}^2\Big|_{t=\tau} \leq f_d(\epsilon_0^2) < -\frac{1}{4C_p}\epsilon_0^2 < 0-$$

which contradicts the definition of τ. Hence, we must have $\tau = T_0$. But since $\|\rho_0\|_{L^2} < \epsilon_0$, we conclude that

$$\sup_{t\in[0,T_0]} \|\rho(t,\cdot)\|_{L^2} \leq \epsilon_0,$$

which yields

$$\int_0^{T_0} \|\rho(t,\cdot)\|_{L^2}^{\frac{4}{4-d}} dt \leq \epsilon_0^{\frac{4}{4-d}} T_0 < \infty.$$

Then by Theorem 1.2, we must have $T_0 = \infty$, as otherwise we could extend the strong solution beyond T_0, leading to a contradiction.

To prove (2.16), we note from above that $\sup_{t\geq 0} \|\rho(t,\cdot)\|_{L^2} \leq \epsilon_0$. In fact, by our choice of ϵ_0 and (2.17), we have

$$\frac{d}{dt}\|\rho\|_{L^2} \leq -\frac{1}{4C_p}\|\rho\|_{L^2}, \quad t \geq 0.$$

A direct application of Grönwall inequality yields

$$\|\rho(t)\|_{L^2} \leq \|\rho_0\|_{L^2} e^{-\frac{1}{4C_p}t}$$

for all $t \geq 0$. We have thus established (2.16) by setting $c_0 := \frac{1}{4C_p}$ and rearranging the inequality above. □

3 Proof of the Main Theorem: Suppression of Chemotactic Blowup

In this section, our goal is to show Theorem 1.3, namely proving that (1.1) is globally strong in the regime of sufficiently large g. In particular, we will see that the coupling of the Keller-Segel equation to the Stokes flow with sufficiently robust buoyancy term is regularizing, in the sense that the solution $\rho(t,x)$ approaches zero exponentially fast as g is sufficiently large.

3.1 Velocity Control

In this subsection, we remark on two controls on the velocity field u in (1.1) that will be instrumental in our main proof. The first lemma is in fact a standard $H^1_{t,x}$ control of u, where part of which is hidden in our proof of energy estimate in Proposition 4.1. We give a brief derivation here for clarity.

Lemma 3.1. *Let (ρ, u) be a strong solution to problem (1.1) with initial data $\rho_0 \in H_0^1$, $u_0 \in V$. We have*

$$\|u\|_{H^1([0,T_*]\times\Omega)}^2 \leq C(\|\rho_0\|_{L^2}, \|u_0\|_1)(g^2+1). \tag{3.1}$$

Proof. In view of the estimate (2.5) in Proposition 2.1, it suffices to show that

$$\int_0^{T_*} \|\partial_t u(t)\|_{L^2}^2 dt \leq C(\|\rho_0\|_{L^2}, \|u_0\|_1)(g^2+1). \tag{3.2}$$

Testing the u equation in (1.1) by $\partial_t u$, we have

$$\|\partial_t u\|_{L^2}^2 + \frac{1}{2}\frac{d}{dt}\|\nabla u\|_{L^2}^2 = g\int_\Omega \partial_t u\cdot(\rho e_z)dx \le \frac{1}{2}\|\partial_t u\|_{L^2}^2 + \frac{g^2}{2}\|\rho\|_{L^2}^2,$$

where we used incompressibility of u and Cauchy-Schwarz inequality above. Rearranging, integrating in time, and using (2.4) we obtain

$$\begin{aligned}\int_0^t \|\partial_t u(s)\|_{L^2}^2 ds + \|\nabla u(t)\|_{L^2}^2 &\le g^2\int_0^t \|\rho(s)\|_{L^2}^2 ds + \|\nabla u_0\|_{L^2}^2\\ &\le g^2(2T_*\|\rho_0\|_{L^2}^2) + \|u_0\|_1^2\\ &\le C(\|\rho_0\|_{L^2}, \|u_0\|_1)(g^2+1).\end{aligned}$$

By taking supremum of t over $[0,T_*]$, we have arrive at the estimate (3.2). □

The following lemma yields a key additional control over the velocity field by genuinely exploiting the buoyancy forcing structure of the fluid equation in (1.1):

Lemma 3.2. *Let (ρ,u) be a regular solution to problem (1.1) with initial data $\rho_0\in H_0^1$, $u_0\in V$. Then*

$$\sup_{t\in[0,T_*]} \|u(t)\|_{L^2}^2 dt \le C(\|\rho_0\|_{L^2})g + \|u_0\|_{L^2}^2. \tag{3.3}$$

Remark 3.1. *Note that a straightforward L^2 estimate of u only yields a bound $\int_0^{T_*}\|u(t)\|_{L^2}^2 dt \lesssim g^2$. What we display in the lemma is that the structure of buoyancy forcing "gains a g^{-1}".*

Proof. Without loss of generality, assume that Ω contains the origin. Denote $L := \operatorname{diam}(\Omega) > 0$. Multiplying the ρ-equation of (1.1) by $z-L$ (recall that $z = x_d$ when $\Omega\subset\mathbb{R}^d$, $d = 2,3$) and integrating over Ω, we have

$$\frac{d}{dt}\int_\Omega (z-L)\rho dx + \int_\Omega (z-L)(u\cdot\nabla\rho)dx - \int_\Omega (z-L)\Delta\rho dx + \int_\Omega (z-L)\operatorname{div}(\rho\nabla(-\Delta)^{-1}\rho)dx = 0.$$

Moreover using the Dirichlet conditions $\rho|_{\partial\Omega} = 0$ and $u|_{\partial\Omega} = 0$, we note that via integration by parts:

$$\int_\Omega (z-L)(u\cdot\nabla\rho)dx = -\int_\Omega \rho u_z dx + \int_{\partial\Omega}(z-L)\rho u_n dx = -\int_\Omega \rho u_z dx,$$

$$-\int_\Omega (z-L)\Delta\rho dx = \int_\Omega \partial_z\rho dx - \int_{\partial\Omega}(z-L)\frac{\partial\rho}{\partial n}dS,$$

$$\int_\Omega (z-L)\operatorname{div}(\rho\nabla(-\Delta)^{-1}\rho)dx = -\int_\Omega \rho\partial_z(-\Delta)^{-1}\rho dx,$$

where u_n denotes the normal component of u along $\partial\Omega$, and dS denotes the surface measure induced on $\partial\Omega$. Collecting the above computations, we have

$$\int_\Omega \rho u_z dx = \frac{d}{dt}\int_\Omega (z-L)\rho dx + \int_\Omega \partial_z\rho dx - \int_{\partial\Omega}(z-L)\frac{\partial\rho}{\partial n}dS - \int_\Omega \rho\partial_z(-\Delta)^{-1}\rho dx. \tag{3.4}$$

On the other hand, testing the u-equation of (1.1) by u, we also have

$$\frac{1}{2}\frac{d}{dt}\|u\|_{L^2}^2 + \|\nabla u\|_{L^2}^2 = g\int_\Omega \rho u_z. \tag{3.5}$$

From the proof of Lemma 2.1, we also know that $\partial\rho/\partial n \le 0$ on $\partial\Omega$ in $[0,T_*]$. Hence, we have $\int_{\partial\Omega}(z-L)\frac{\partial\rho}{\partial n}dS \ge 0$ by definition of L. Combining this fact with (3.4), (3.5), and integrating on $[0,T_*]$, we have

$$\begin{aligned}\|u(t)\|_{L^2}^2 - \|u_0\|_{L^2}^2 &\le 2g\Big[\int_\Omega (z-L)(\rho(t,x)-\rho_0(x))\,dx + \int_0^t\int_\Omega \partial_z\rho\,dx - \int_0^t\int_\Omega \rho\partial_z(-\Delta)^{-1}\rho\,dx\Big] \\ &\le Cg(\sup_{0\le t\le T_*}\|\rho(t)\|_{L^2} + \sqrt{T_*}\left(\int_0^{T_*}\|\nabla\rho\|_{L^2}^2dt\right)^{1/2} + \int_0^{T_*}\|\rho\|_{L^2}^2dt) \\ &\le C(\|\rho_0\|_{L^2})g,\end{aligned}$$

where we used elliptic estimate in the second inequality, and (2.4) in the final inequality. The proof is therefore completed after integrating in time again. □

3.2 A Key Theorem

In this part, we prove a quantitative characterization of the regularizing effect of the Stokes-Boussinesq flow in (1.1). With a rigidity-type argument inspired by [8], we show that the flow with sufficiently large g can suppress the L^2 energy of ρ to be arbitrarily small within the time scale of local existence, as elucidated in the following theorem:

Theorem 3.1. *Let $\rho_0 \in H_0^1 \cap L^\infty, u_0 \in V$ be initial conditions for the problem* (1.1)*, and consider (ρ,u) to be the corresponding strong solution. For arbitrary $\epsilon > 0$, there exists $g_* = g_*(\|\rho_0\|_{L^\infty}, \|u_0\|_1, \epsilon)$ such that for any $g \ge g_*$,*

$$\inf_{t\in[0,T_*]} \|\rho(t,\cdot)\|_{L^2} \le \epsilon.$$

Proof. We prove by contradiction. Suppose there exists $\epsilon_0 > 0$ such that there is a sequence $\{(\rho_n,u_n,g_n)\}_n$ verifying the following:

1. $g_n \to \infty$;
2. $\{(\rho_n,u_n,g_n)\}_n$ are strong solutions to (1.1) with $\rho=\rho_n, u=u_n, g=g_n$ equipped with initial data $(\rho_{n,0},u_{n,0})$;
3. $\|\rho_{n,0}\|_{L^2} \le \|\rho_0\|_{L^2}$, $\|\rho_{n,0}\|_{L^\infty} \le \|\rho_0\|_{L^\infty}$, and $\|u_{n,0}\|_{L^2} \le \|u_0\|_{L^2}$;
4. for any $t\in[0,T_*]$ and for all n
$$\|\rho_n(t,\cdot)\|_{L^2} > \epsilon_0. \tag{3.6}$$

Note that since the time T_* depends only on $\|\rho_0\|_{L^2}$, we can work on the same time interval for all n. If we are able to derive a contradiction, this would imply the existence of g_* that depends only on $\|u_0\|_1$ and $\|\rho_0\|_\infty$ (note that $\|\rho_0\|_{L^\infty}$ controls $\|\rho_0\|_{L^2}$) - indeed, we are not fixing the initial data for $\{(\rho_n,u_n,g_n)\}_n$ but instead only impose uniform norm bounds on it. Let us consider the normalized velocity $\bar u_n = u_n/g_n$. Now, we divide the proof into the following steps:

- **Step 1: Convergence properties of** (ρ_n,u_n)**.** From (3.1), we have $\|\bar u_n\|_{H^1([0,T_*]\times\Omega)} \le C(\|\rho_0\|_{L^2},\|u_0\|_1)$. Using weak compactness and the Sobolev compact embedding theorem, we obtain that there exists $\bar u_\infty \in H^1([0,T_*]\times\Omega)$ such that

$$\bar u_n \rightharpoonup \bar u_\infty \text{ in } H^1([0,T_*]\times\Omega), \text{ and } \bar u_n \to \bar u_\infty \text{ in } L^2([0,T_*]\times\Omega).$$

In fact, observe that from the estimate (3.3) of Lemma 3.2 it follows that $\|\bar{u}_n\|_{L^2([0,T_*]\times\Omega)} \to 0$ as $n \to \infty$, so $\bar{u}_\infty = 0$. In addition, from the energy estimate (2.4), we may pick a further subsequence, still indexed by n, such that there exists $\rho_\infty \in L^2(0,T_*;H_0^1(\Omega))$ and

$$\rho_n \rightharpoonup \rho_\infty \ \text{ in } L^2(0,T_*;H_0^1(\Omega)).$$

- **Step 2: Derivation of the limiting fluid equation.** Since (ρ_n, u_n) is a strong solution to (1.1) with parameter g_n on $[0,T_*]$, u_n in particular solves the fluid equation in (1.1) in the sense of distributions. That is,

$$-\int_0^{T_*}\int_\Omega (\partial_t\phi)\bar{u}_n dxdt + \int_0^{T_*}\int_\Omega (\mathcal{A}\phi)\bar{u}_n dxdt = \int_0^{T_*}\int_\Omega \rho_n(\phi\cdot e_z)dxdt,$$

for any smooth vector field $\phi \in C_c^\infty([0,T_*]\times\Omega)$ with $\operatorname{div}\phi = 0$. By the convergence properties of ρ_n, u_n as shown in Step 1, and by Lemma 3.2 we find that

$$\rho_\infty e_z = \nabla p_\infty, \ \ (t,x) \in [0,T_*]\times\Omega \tag{3.7}$$

holds in a sense of distributions.

- **Step 3: Nontriviality of ρ_∞.** By maximum principle, we know that ρ_n, and thus ρ_∞, is nonnegative. We would also like to claim that $\rho_\infty \not\equiv 0$. To show this fact, we need the following proposition.

Proposition 3.1. *Let $\Omega \subset \mathbb{R}^d$, $d = 2,3$, be a smooth, bounded domain. Assume (ρ,u) to be the strong solution of problem (1.1) on $[0,T_*]$ with initial condition $\rho_0 \geq 0 \in H_0^1\cap L^\infty$, $u_0 \in V$. If there exists $M > 0$ such that $\sup_{0\leq t\leq T_*}\|\rho(t)\|_{L^2} \leq M$, then we have*

$$\sup_{0\leq t\leq T_*}\|\rho(t)\|_{L^\infty} \leq CM^{\frac{4}{4-d}}.$$

Here C is a constant that only depends on Ω.

A variant of this result has been proved in [21] (Proposition 9.1), in a two dimensional periodic setting. The proof of Proposition 3.1 is similar and for the sake of completeness will be provided in the appendix.

Next, we need the following lemma.

Lemma 3.3. *Let $D \subset \mathbb{R}^d$, $d \in \mathbb{N}$, be a bounded domain, and let $\{f_n\}_n \subset L^2(D)$ be a sequence of nonnegative functions that weakly converges to a function $f \in L^2(D)$. Assume that there exist $M, \epsilon > 0$ such that $\|f_n\|_{L^2} > \epsilon$, $\|f_n\|_{L^\infty} \leq M$ for all n. Then $f \not\equiv 0$.*

Proof. Suppose for the sake of contradiction that $f \equiv 0$. Consider the characteristic function $\phi = \chi_D$. Since D is bounded, $\phi \in L^2(D)$. Then the weak convergence informs us that

$$\lim_{n\to\infty}\int_D f_n = 0.$$

As $f_n \geq 0$ for all n, this is equivalent to $\lim_{n\to\infty}\|f_n\|_{L^1} = 0$. Since $\|f_n\|_{L^\infty} \leq M$, by interpolation we have

$$\|f_n\|_{L^2}^2 \leq \|f_n\|_{L^\infty}\|f_n\|_{L^1} \to 0$$

as $n \to \infty$. But this contradicts the assumption that $\|f_n\|_{L^2} > \epsilon$. □

Observe that from (2.4), we know that $\|\rho_n(t,\cdot)\|_{L^2} \le 4\|\rho_0\|_{L^2}$ for all $t \in [0, T_*]$ and all n. Thus applying Proposition 3.1 to ρ_n we get that $\|\rho_n(t,\cdot)\|_{L^\infty} \le M$ for all $t \in [0, T_*]$, and all n. Here $M = C\|\rho_0\|_{L^2}^{\frac{4}{4-d}}$ is chosen as in Proposition 3.1. Then Lemma 3.3 implies that $\rho_\infty \not\equiv 0$.

- **Step 4: Derivation of a contradiction.** Let us consider

$$\psi_n(x) := \int_0^{T_*} \rho_n(t,x)dt, \ \psi_\infty(x) := \int_0^{T_*} \rho_\infty(t,x)dt.$$

In particular, $\psi_\infty \not\equiv 0$ and $\psi_\infty \ge 0$ by Step 3. On the one hand, picking arbitrary $\eta \in L^2(\Omega)$, we have

$$\begin{aligned}\left| \int_\Omega \eta(x)(\psi_n(x) - \psi_\infty(x))dx \right| &= \left| \int_0^{T_*} \int_\Omega \eta(x)(\rho_n(t,x) - \rho_\infty(t,x))dxdt \right| \\ &= \left| \int_0^{T_*} \int_\Omega \eta(x)\chi_{[0,T_*]}(t)(\rho_n(t,x) - \rho_\infty(t,x))dxdt \right|,\end{aligned}$$

which converges to 0 as $\rho_n \rightharpoonup \rho_\infty$ in $L^2([0,T_*] \times \Omega)$. This implies that $\psi_n \rightharpoonup \psi_\infty$ in $L^2(\Omega)$. On the other hand, we note that by Minkowski inequality and Hölder inequality,

$$\|\nabla\psi_n\|_{L^2} \le \int_0^{T_*} \|\nabla\rho_n\|_{L^2} dt \le \sqrt{T_*}\|\nabla\rho_n\|_{L^2([0,T_*]\times\Omega)} \le C(\|\rho_0\|_{L^2}),$$

where we used (2.4) in the last step. Since $\rho_n|_{\partial\Omega} = 0$, we know that $\psi_n \in H_0^1(\Omega)$ with a uniform H^1-norm bound from above. Hence by weak compactness and Sobolev compact embedding theorem, there exists a subsequence, still denoted by ψ_n, and $\tilde{\psi}_\infty \in H_0^1(\Omega)$ such that

$$\psi_n \rightharpoonup \tilde{\psi}_\infty \ \text{ in } H_0^1(\Omega), \ \psi_n \to \tilde{\psi}_\infty \ \text{ in } L^2(\Omega).$$

Indeed, we must have $\tilde{\psi}_\infty = \psi_\infty$ due to the uniqueness of weak limit, and hence $\psi_\infty \in H_0^1(\Omega)$. But now, integrating (3.7) with respect to time, we have

$$\nabla P = \psi_\infty e_z,$$

where $P(x) := \int_0^{T_*} p_\infty(t,x)dt$. But this implies that $\psi_\infty(x) = h(z)$, where h is some single-variable function. Moreover, we know that $\psi_\infty \in H_0^1(\Omega)$. These two facts imply that $\psi_\infty \equiv 0$. However, this contradicts the fact that $\psi_\infty > 0$ - as shown in Lemma 3.3 under assumption (3.6). This completes the proof of the theorem.

□

3.3 Proof of Global Well-Posedness with Large g

From Theorem 3.1, we know that, fixing initial data (ρ_0, u_0), the L^2 norm of ρ is suppressed to a sufficiently small level in the regime of large g. In view of the small-data global well-posedness result stated in Proposition 2.4, the cell density ρ will enter the regime of exponential decay as soon as $\|\rho\|_{L^2}$ touches a sufficiently low level. Combining these two facts will yield Theorem 1.3. We now provide a rigorous proof of the aforementioned idea.

Proof of Theorem 1.3. Let us first choose ϵ_0 as in Proposition 2.4. By Theorem 3.1, there exists $g_* = g_*(\|\rho_0\|_{L^\infty}, \|u_0\|_1)$ such that for every $g \geq g_*$ there exists $\tau_g \in [0, T_*]$ with $\|\rho(\tau_g)\|_{L^2}^2 < \epsilon$. Now we consider the problem (1.1) equipped with initial data $(\rho(\tau_g, x), u(\tau_g, x))$. Applying Proposition 2.4 immediately yields global existence of strong solution (ρ, u) and the exponential decay estimate

$$e^{c_0(t-\tau_g)}\|\rho(t)\|_{L^2} \leq \|\rho(\tau_g)\|_{L^2} \leq 2\|\rho_0\|_{L^2}.$$

We remark that the second inequality above follows from our choice of T_* and the fact that $\tau_g \leq T_*$.

□

4 Promotion to Regular Solution

In this section, we will show that strong solutions are in fact regular solutions. This will be done by establishing the smoothness of a solution (ρ, u) for positive times, namely

$$\rho \in C^\infty((0, T_*] \times \Omega), \ u \in C^\infty((0, T_*] \times \Omega),$$

via energy estimates in arbitrarily high order Sobolev norms. We would like to remark on the following caveat: with Dirichlet boundary condition imposed on both ρ and u, one cannot obtain higher order Sobolev estimates by commuting the differential operator ∂^s with the equation, where ∂^s denotes a general s-th order spatial derivative. The main reason is that when we treat the dissipation term, integration by parts incurs a boundary term that is difficult to control. To remedy this issue, we commute time derivatives ∂_t^k through the equation. It is clear that no boundary terms are generated since ∂_t preserves Dirichlet boundary condition. By applying this strategy, we can improve regularity in time, after which spatial regularity can be upgraded using elliptic estimates.

Again, to obtain the claimed regularity we should proceed by the Galerkin scheme and perform the estimates in Proposition 4.1 for the approximated solutions. Since this step is similar to that in Corollary 2.1, we omit this tedious part and will proceed with only *a priori* estimates as follows.

Proposition 4.1. *Assume (ρ, u) is a regular solution to problem (1.1) with initial condition $\rho_0 \in H_0^1, u_0 \in V$. Then the following bounds hold:*

$$t^k \left(\|\partial_t^l \rho(t, \cdot)\|_{1+k-2l}^2 + \|\partial_t^l u(t, \cdot)\|_{1+k-2l}^2\right) \leq C(\rho_0, u_0, g, k), \tag{4.1}$$

$$t^k \int_t^{T_*} \left(\|\partial_t^l \rho(\tau, \cdot)\|_{2+k-2l}^2 + \|\partial_t^l u(\tau, \cdot)\|_{2+k-2l}^2\right) d\tau \leq C(\rho_0, u_0, g, k), \tag{4.2}$$

for any $t \in (0, T_*]$, $k \in \mathbb{N}$, $0 \leq l \leq \lfloor \frac{k+1}{2} \rfloor$, where $\lfloor \cdot \rfloor$ denotes the floor function.

Proof. We prove the proposition by induction on k. Since the case $k = 0$ is already proved by Proposition 2.2, we now assume that the statement holds up to index $k - 1$. We will discuss two cases based on the parity of k. We also remind the readers that the constant $C(\rho_0, u_0, g, k)$ might change from line to line.

1. k **is odd.** Let us write $S = \frac{k+1}{2}$, and define the s-energy

   $$E_s(\tau) = \|\partial_t^s \rho(\tau, \cdot)\|_{L^2}^2 + \|\partial_t^s u(\tau, \cdot)\|_{L^2}^2$$

   for any $0 \leq s \leq S$. From now on, we fix arbitrary $t \in (0, T_*]$. This case can be detailed into the following steps.

Step 1: show (4.1)**,** (4.2) **with** $l = S$**.** Commuting ∂_t^s with (1.1) for $0 \le s \le S$, we obtain that

$$\partial_t \partial_t^s \rho - \Delta \partial_t^s \rho + \sum_{r=0}^{s} \binom{s}{r} \left[(\partial_t^r u \cdot \nabla) \partial_t^{s-r} \rho + \partial_t^{s-r} \rho \partial_t^r \rho + \nabla \partial_t^{s-r} \rho \cdot \nabla(-\Delta)^{-1} (\partial_t^r \rho) \right] = 0, \tag{4.3a}$$

$$\partial_t \partial_t^s u + \mathcal{A} \partial_t^s u = g \mathbb{P}(\partial_t^s \rho e_z), \tag{4.3b}$$

equipped with boundary conditions $\partial_t^s \rho|_{\partial\Omega} = 0$, $\partial_t^s u|_{\partial\Omega} = 0$. Testing (4.3b) with $s = S$ by $\partial_t^S u$, we obtain that

$$\frac{1}{2} \frac{d}{dt} \|\partial_t^S u\|_{L^2}^2 + \|\nabla \partial_t^S u\|_{L^2}^2 \le \frac{g}{2} \left(\|\partial_t^S u\|_{L^2}^2 + \|\partial_t^S \rho\|_{L^2}^2 \right).$$

Testing (4.3a) with $s = S$ by $\partial_t^S \rho$:

$$\frac{1}{2} \frac{d}{dt} \|\partial_t^S \rho\|_{L^2}^2 + \|\nabla \partial_t^S \rho\|_{L^2}^2 = \sum_{r=0}^{S} \binom{S}{r} (I_r + J_r + K_r), \tag{4.4}$$

where

$$I_r = \int_\Omega (\partial_t^S \rho)(\partial_t^r u \cdot \nabla) \partial_t^{S-r} \rho, \; J_r = \int_\Omega (\partial_t^S \rho) \partial_t^{S-r} \rho (\partial_t^r \rho),$$

$$K_r = \int_\Omega (\partial_t^S \rho) \nabla \partial_t^{S-r} \rho \cdot \nabla(-\Delta)^{-1} (\partial_t^r \rho).$$

To estimate I_r, first note that $I_0 = 0$ by incompressibility and integration by parts. For $1 \le r \le S-1$, we integrate I_r by parts once to obtain:

$$I_r = -\int_\Omega \partial_j \partial_t^S \rho \partial_t^r u_j \partial_t^{S-r} \rho,$$

where we also used the incompressibility of $\partial_t^r u$. Thus, we can estimate:

$$I_r \le \|\nabla \partial_t^S \rho\|_{L^2} \|\partial_t^r u\|_{L^3} \|\partial_t^{S-r} \rho\|_{L^6} \le \delta \|\nabla \partial_t^S \rho\|_{L^2}^2 + C(\delta) \|\partial_t^r u\|_{L^3}^2 \|\partial_t^{S-r} \rho\|_1^2,$$

for some $\delta > 0$. If $r = S$, we instead estimate:

$$I_S \le \|\nabla \partial_t^S \rho\|_{L^2} \|\partial_t^S u\|_{L^2} \|\rho\|_{L^\infty} \le \delta \|\nabla \partial_t^S \rho\|_{L^2}^2 + C(\delta) \|\rho\|_2^2 \|\partial_t^S u\|_{L^2}^2.$$

This concludes the estimates of I_r. To estimate J_r, we note that if $r = 0$ or $r = S$, we have

$$J_r \le \|\partial_t^S \rho\|_{L^2}^2 \|\rho\|_{L^\infty} \lesssim \|\partial_t^S \rho\|_{L^2}^2 \|\rho\|_2$$

If $1 \le r \le S-1$, then we have

$$J_r \le \frac{1}{2} \|\partial_t^S \rho\|_{L^2}^2 + \frac{1}{2} \|\partial_t^r \rho\|_1^2 \|\partial_t^{S-r} \rho\|_1^2.$$

Now we estimate K_r. If $r = 0$, we use the standard elliptic estimate and Young's inequality to obtain:

$$K_0 \le \delta \|\nabla \partial_t^S \rho\|_{L^2}^2 + C(\delta) \|\rho\|_1^2 \|\partial_t^S \rho\|_{L^2}^2,$$

where $\delta > 0$. If $r = S$, we apply elliptic estimates and Sobolev embeddings:

$$K_S \leq \|\nabla\rho\|_{L^3}\|\partial_t^S\rho\|_{L^2}\|\nabla(-\Delta)^{-1}\partial_t^S\rho\|_{L^6} \lesssim \|\nabla\rho\|_1\|\partial_t^S\rho\|_{L^2}^2.$$

If $1 \leq r \leq S-1$, we can estimate

$$K_r \leq \|\nabla\partial_t^{S-r}\rho\|_{L^3}\|\partial_t^S\rho\|_{L^2}\|\nabla(-\Delta)^{-1}\partial_t^r\rho\|_{L^6} \leq \frac{1}{2}\|\partial_t^S\rho\|_{L^2}^2 + C\|\nabla\partial_t^{S-r}\rho\|_1^2\|\partial_t^r\rho\|_{L^2}^2.$$

After choosing $\delta > 0$ to be sufficiently small, the above estimates yield the following differential inequality: for $\tau \in (0, T_*)$,

$$\begin{aligned}\frac{dE_S}{d\tau} + \|\nabla\partial_t^S\rho(\tau,\cdot)\|_{L^2}^2 + \|\nabla\partial_t^S u(\tau,\cdot)\|_{L^2}^2 &\leq C(k)\Big[\left(1 + g + \|\rho\|_2 + \|\rho\|_2^2\right)E_S(\tau) \\ &\quad + \sum_{r=1}^{S-1}\left(\|\nabla\partial_t^{S-r}\rho\|_1^2\|\partial_t^r\rho\|_{L^2}^2 + \|\partial_t^r\rho\|_1^2\|\partial_t^{S-r}\rho\|_1^2 + \|\partial_t^r u\|_{L^3}^2\|\partial_t^{S-r}\rho\|_1^2\right)\Big] \\ &= C(k)\left(F(\tau)E_S(\tau) + \sum_{r=1}^{S-1}G_r(\tau)\right) \end{aligned} \tag{4.5}$$

with

$$F(\tau) = 1 + g + \|\rho(\tau,\cdot)\|_2 + \|\rho(\tau,\cdot)\|_2^2,$$
$$G_r(\tau) = \|\nabla\partial_t^{S-r}\rho\|_1^2\|\partial_t^r\rho\|_{L^2}^2 + \|\partial_t^r\rho\|_1^2\|\partial_t^{S-r}\rho\|_1^2 + \|\partial_t^r u\|_{L^3}^2\|\partial_t^{S-r}\rho\|_1^2.$$

To proceed, we need the following useful lemma:

Lemma 4.1. *There exists $\tau_0 \in [t/2, t]$ such that $E_S(\tau_0) \leq C(\rho_0, u_0, g, k)t^{-k}$.*

Proof. Let us consider (4.3) with $s = S-1$. For any $\tau \in [t/2, t]$, we note that by (4.3b),

$$\|\partial_t^S u(\tau)\|_{L^2}^2 \lesssim \|\mathcal{A}\partial_t^{S-1}u\|_{L^2}^2 + g^2\|\partial_t^{S-1}\rho\|_{L^2}^2 \leq \|\partial_t^{S-1}u\|_2^2 + g^2\|\partial_t^{S-1}\rho\|_{L^2}^2.$$

Integrating over $[t/2, t]$ and using (4.2) at index $k-1$ (which is valid as this is part of the induction hypothesis), we obtain

$$\int_{t/2}^{t}\|\partial_t^S u(\tau)\|_{L^2}^2 d\tau \leq \int_{t/2}^{T_*}\|\partial_t^S u(\tau)\|_{L^2}^2 d\tau \leq C(\rho_0, u_0, g, k)t^{1-k}. \tag{4.6}$$

Similarly, applying Hölder inequality to (4.3a), we have

$$\begin{aligned}\|\partial_t^S\rho\|_{L^2}^2 \lesssim \|\partial_t^{S-1}\rho\|_2^2 + \sum_{r=0}^{S-1}C(k)\Big(&\|\partial_t^r u\|_1^2\|\nabla\partial_t^{S-1-r}\rho\|_1^2 \\ &+ \|\partial_t^{S-1-r}\rho\|_1^2\|\partial_t^r\rho\|_1^2 + \|\nabla\partial_t^{S-1-r}\rho\|_1^2\|\partial_t^r\rho\|_{L^2}^2\Big).\end{aligned} \tag{4.7}$$

Observe that given the induction hypothesis, applying (4.2) with index $k-1$, we have

$$\int_{t/2}^{t}\|\partial_t^{S-1}\rho(\tau)\|_2^2 d\tau \leq C(\rho_0, u_0, g, k)t^{1-k}.$$

Also, for $r = 0, \dots, S-1$,

$$\int_{t/2}^{t} \|\partial_t^r u(\tau)\|_1^2 \|\nabla \partial_t^{S-1-r} \rho(\tau)\|_1^2 d\tau \le C(\rho_0, u_0, g, k) t^{-2r} t^{-2(S-r-1)} = C(\rho_0, u_0, g, k) t^{1-k},$$

where we applied (4.1) with index $2r$ to $\|\partial_t^r u(\tau)\|_1$ and (4.2) with index $2(S-r-1)$ to $\|\nabla \partial_t^{S-1-r} \rho(\tau)\|_1$. In a similar fashion, we can also obtain the following bound:

$$\int_{t/2}^{t} \left[\|\partial_t^{S-1-r} \rho(\tau)\|_1^2 \|\partial_t^r \rho(\tau)\|_1^2 + \|\nabla \partial_t^{S-1-r} \rho(\tau)\|_1^2 \|\partial_t^r \rho(\tau)\|_{L^2}^2 \right] d\tau \le C(\rho_0, u_0, g, k) t^{1-k}.$$

Collecting the estimates above and combining with (4.7), we have

$$\int_{t/2}^{t} \|\partial_t^S \rho(\tau)\|_{L^2}^2 d\tau \le C(\rho_0, u_0, g, k) t^{1-k}. \tag{4.8}$$

Combining (4.6) and (4.8), we have

$$\int_{t/2}^{t} \left(\|\partial_t^S u(\tau)\|_{L^2}^2 + \|\partial_t^S \rho(\tau)\|_{L^2}^2 \right) d\tau \le C(\rho_0, u_0, g, k) t^{1-k}.$$

By mean value theorem, we can find a $\tau_0 \in (t/2, t)$ such that

$$E_S(\tau_0) = \|\partial_t^S u(\tau_0)\|_{L^2}^2 + \|\partial_t^S \rho(\tau_0)\|_{L^2}^2 \le C(\rho_0, u_0, g, k) t^{-k},$$

and this concludes the proof. □

We also need another lemma that treats the terms G_r.

Lemma 4.2. *Let τ_0 be chosen as in Lemma 4.1. Then for any $r = 1, \dots, S-1$, we have*

$$\int_{\tau_0}^{T_*} G_r(\tau) d\tau \le C(\rho_0, u_0, g, k) t^{-k}.$$

Proof. We fix $r = 1, \dots, S-1$. By definition of G_r, we can write

$$\begin{aligned} \int_{\tau_0}^{T_*} G_r(\tau) d\tau &= \int_{\tau_0}^{T_*} \left(\|\nabla \partial_t^{S-r} \rho\|_1^2 \|\partial_t^r \rho\|_{L^2}^2 + \|\partial_t^r \rho\|_1^2 \|\partial_t^{S-r} \rho\|_1^2 + \|\partial_t^r u\|_{L^3}^2 \|\partial_t^{S-r} \rho\|_1^2 \right) d\tau \\ &=: \int_{\tau_0}^{T_*} \left(G_r^1(\tau) + G_r^2(\tau) + G_r^3(\tau) \right) d\tau. \end{aligned}$$

Applying (4.1) with index $2r-1$ and (4.2) with index $k-2r+1$ to terms $\|\partial_t^r \rho\|_{L^2}^2$ and $\|\nabla \partial_t^{S-r} \rho\|_1^2$ respectively, we observe that

$$\begin{aligned} \int_{\tau_0}^{T_*} G_r^1(\tau) d\tau &\le C(\rho_0, u_0, g, k) \tau_0^{1-2r} \int_{\tau_0}^{T_*} \|\partial_t^{S-r} \rho(\tau)\|_2^2 d\tau \\ &\le C(\rho_0, u_0, g, k) \tau_0^{-(2r-1)} \tau_0^{-(k-2r+1)} \\ &\le C(\rho_0, u_0, g, k) t^{-k}, \end{aligned}$$

where we used the fact that $\tau_0 > t/2$.

To study the term involving G_r^2, we will apply (4.1) with index $2r$ and (4.2) with index $k-2r$ to terms $\|\partial_t^r \rho\|_1^2$ and $\|\partial_t^{S-r}\rho\|_1^2$ respectively. This yields:

$$\begin{aligned}\int_{\tau_0}^{T_*} G_r^2(\tau)d\tau &\le C(\rho_0,u_0,g,k)\tau_0^{-2r}\int_{\tau_0}^{T_*}\|\partial_t^{S-r}\rho(\tau)\|_1^2 d\tau\\ &\le C(\rho_0,u_0,g,k)\tau_0^{-2r}\tau_0^{-(k-2r)}\\ &\le C(\rho_0,u_0,g,k)t^{-k},\end{aligned}$$

Finally, using Sobolev embedding, Gagliardo-Nirenberg-Sobolev inequality, and Cauchy-Schwarz inequality,

$$\begin{aligned}\int_{\tau_0}^{T_*} G_r^3(\tau)d\tau \le \int_{\tau_0}^{T_*}\|\partial_t^r u\|_{L^3}^2\|\partial_t^{S-r}\rho\|_1^2 d\tau &\lesssim \int_{\tau_0}^{T_*}\|\partial_t^r u\|_{L^2}\|\nabla\partial_t^r u\|_{L^2}\|\partial_t^{S-r}\rho\|_1^2 d\tau\\ &\le C(\rho_0,u_0,g,k)\tau_0^{-\frac{2r-1}{2}}\tau_0^{-r}\int_{\tau_0}^{T_*}\|\partial_t^{S-r}\rho\|_1^2 d\tau\\ &\le C(\rho_0,u_0,g,k)t^{-(k-\frac12)} \le C(\rho_0,u_0,g,k)t^{-k}.\end{aligned}$$

Note that we applied (4.1) with index $2r-1$ to $\|\partial_t^r u\|_{L^2}$, (4.1) with index $2r$ to $\|\nabla\partial_t^r u\|_{L^2}$, and (4.2) with index $k-2r$ to the other $\|\partial_t^{S-r}\rho\|_1^2$. We also used $\tau_0 \le T_* \le 1$ in the final inequality. The proof is thus completed after we combine the estimates above. □

Using induction hypothesis at $k=0$, we have $F \in L^1(0,T_*)$ with the bound $\|F\|_{L^1(0,T_*)} \le C(u_0,\rho_0,g)$. We may thus apply Grönwall inequality to (4.5) on time interval $[\tau_0,t]$, where τ_0 is selected as in Lemma 4.1 above. Using the two lemmas above, we have

$$\begin{aligned}E_S(t) &\le C(k)\left(E_S(\tau_0)+\sum_{r=1}^{S-1}\int_{\tau_0}^t G_r(\tau)d\tau\right)\exp\left(\|F\|_{L^1(0,T_*)}\right)\\ &\le C(\rho_0,u_0,g,k)t^{-k},\end{aligned}\tag{4.9}$$

where we recall that T_* depends only on ρ_0. This verifies (4.1). To verify (4.2), we integrate (4.5) on interval $[t,T_*]$, which yields:

$$\int_t^{T_*}\left(\|\nabla\partial_t^S\rho(\tau)\|_{L^2}^2+\|\nabla\partial_t^S u(\tau)\|_{L^2}^2\right)d\tau \le E_S(t)+C(k)\Bigg(\int_t^{T_*}F(\tau)E_S(\tau)d\tau+\sum_{r=1}^{S-1}\int_t^{T_*}G_r(\tau)d\tau\Bigg).$$

Using (4.9), Lemma 4.2, and the fact that $\frac{t}{2}<\tau_0<t$, we can estimate the above by:

$$\begin{aligned}\int_t^{T_*}(\|\nabla\partial_t^S\rho(\tau)\|_{L^2}^2+\|\nabla\partial_t^S u(\tau)\|_{L^2}^2)d\tau &\le C(\rho_0,u_0,g,k)t^{-k}\\ &\quad + C(\rho_0,u_0,g,k)(t^{-k}\|F\|_{L^1}+t^{-k})\\ &\le C(\rho_0,u_0,g,k)t^{-k}.\end{aligned}$$

This concludes the proof of (4.2) with $l=S$.

Step 2: show (4.1), (4.2) with $l<S$. We will show how we obtain the case when $l=S-1$. Then the rest just follows from another induction on $l=1,\dots,S$ backwards.

We may rewrite the equations (4.3) with $s = S - 1$ as

$$-\Delta\partial_t^{S-1}\rho = -\partial_t^S\rho - \sum_{r=0}^{S-1}\binom{S-1}{r}\Big[\partial_t^r u\cdot\nabla\partial_t^{S-1-r}\rho + \partial_t^{S-1-r}\rho\partial_t^r\rho + \nabla\partial_t^{S-1-r}\rho\cdot\nabla(-\Delta)^{-1}(\partial_t^r\rho)\Big]$$

$$= -\partial_t^S\rho + R_1 \tag{4.10a}$$

$$\mathcal{A}\partial_t^{S-1}u = -\partial_t^S u + g\mathbb{P}(\partial_t^{S-1}\rho e_z) = -\partial_t^S u + R_2 \tag{4.10b}$$

Here, R_1, R_2 are the remainder terms which are essentially of lower order. We will see that these terms can be treated by the induction hypothesis on k. To illustrate this, we show that the following estimates hold:

Lemma 4.3. *For any* $t \in (0, T_*]$,

$$t^{k-\frac{1}{4}}(\|R_1(t)\|_{L^2}^2 + \|R_2(t)\|_{L^2}^2) \le C(\rho_0, u_0, g, k),$$

$$t^{k-\frac{1}{4}}\int_t^{T_*}\left(\|R_1(\tau)\|_1^2 + \|R_2(\tau)\|_1^2\right)d\tau \le C(\rho_0, u_0, g, k).$$

Proof. First, it is straightforward to obtain the following bounds for R_2 by directly imposing the induction hypothesis at index $k-1$:

$$t^{k-1}\|R_2(t)\|_{L^2}^2 + t^{k-1}\int_t^{T_*}\|R_2(t)\|_1^2 dt \le C(\rho_0, u_0, g, k). \tag{4.11}$$

Prior to estimating R_1, we first need an improved bound for $\|u\|_2$: invoking (4.11) with $k = 1$, we have

$$\|R_2(t)\|_{L^2}^2 \le C(\rho_0, u_0, g).$$

Since $S = 1$ when $k = 1$ by definition, we apply the Stokes estimate to (4.10b) with $S = 1$ to see that

$$\|u\|_2^2 \lesssim \|\partial_t u\|_{L^2}^2 + \|R_2\|_{L^2}^2 \le C(\rho_0, u_0, g)(t^{-1} + 1) \le C(\rho_0, u_0, g)t^{-1}, \tag{4.12}$$

where we used Step 1 with $k = 1$ above. Now, we are ready to estimate R_1. We first note that it involves 3 typical terms, namely

$$R_{11}^r := \partial_t^r u\cdot\nabla\partial_t^{S-1-r}\rho,\ R_{12}^r := \partial_t^{S-1-r}\rho\partial_t^r\rho,\ R_{13}^r := \nabla\partial_t^{S-1-r}\rho\cdot\nabla(-\Delta)^{-1}(\partial_t^r\rho),$$

where $0 \le r \le S-1$. We will prove suitable bounds for R_{11}^r, and the rest can be bounded more easily since these terms involve fewer derivatives. If $1 \le r \le S-1$, then by Hölder inequality:

$$\begin{aligned}\|R_{11}^r\|_{L^2}^2 &\le \|\partial_t^r u\|_{L^6}^2\|\nabla\partial_t^{S-1-r}\rho\|_{L^3}^2 \lesssim \|\partial_t^r u\|_1^2\|\partial_t^{S-1-r}\rho\|_1\|\partial_t^{S-1-r}\rho\|_2 \\ &\le C(\rho_0, u_0, g, k)t^{-2r}t^{-\frac{k-2r-1}{2}}t^{-\frac{k-2r}{2}} \\ &\le C(\rho_0, u_0, g, k)t^{-(k-\frac{1}{2})},\end{aligned}$$

where we used (4.1) at indices $2r, k-2r-1, k-2r$ respectively.
If $r = 0$, then we observe that $R_{11}^0 = u\cdot\nabla\partial_t^{S-1}\rho$. We estimate as follows:

$$\begin{aligned}\|R_{11}^0\|_{L^2}^2 &\le \|u\|_{L^\infty}^2\|\partial_t^{S-1}\rho\|_1^2 \le \|u\|_{L^2}^{1/2}\|u\|_2^{3/2}\|\partial_t^{S-1}\rho\|_1^2 \\ &\le C(\rho_0, u_0, g, k)t^{-3/4}t^{-(k-1)} = C(\rho_0, u_0, g, k)t^{-(k-1/4)}\end{aligned}$$

where we used Agmon's inequality in 3D:

$$\|u\|_{L^\infty}^2 \lesssim \|u\|_{L^2}^{1/2}\|u\|_2^{3/2}$$

in the second inequality. We also invoked (4.1) with index 0 to estimate $\|u\|_{L^2}$, (4.1) with index $k-1$ to bound $\|\partial_t^{S-1}\rho\|_1$, and (4.12) to control $\|u\|_2$.

Turning to the second inequality, since $\partial_t^r u = 0$ on $\partial\Omega$, then we can invoke Poincaré inequality to obtain:

$$\begin{aligned}\int_t^{T_*} \|R_{11}^r\|_1^2 d\tau &\lesssim \int_t^{T_*} \|\nabla R_{11}^r\|_{L^2}^2 d\tau \\ &\lesssim \int_t^{T_*} \left(\|\nabla\partial_t^r u\cdot\nabla\partial_t^{S-1-r}\rho\|_{L^2}^2 + \|\partial_t^r u\cdot\nabla^2\partial_t^{S-1-r}\rho\|_{L^2}^2\right) d\tau \\ &=: R_{111}^r + R_{112}^r.\end{aligned}$$

If $1 \le r \le S-1$, using Hölder inequality and Gagliardo-Nirenberg-Sobolev inequalities, we can estimate R_{111}^r by

$$\begin{aligned}R_{111}^r &\le \int_t^{T_*} \|\nabla\partial_t^r u\|_{L^3}^2\|\nabla\partial_t^{S-1-r}\rho\|_{L^6}^2 d\tau \lesssim \int_t^{T_*} \|\nabla\partial_t^r u\|_{L^2}\|\nabla^2\partial_t^r u\|_{L^2}\|\nabla\partial_t^{S-1-r}\rho\|_1^2 d\tau \\ &\le C(\rho_0,u_0,g,k)t^{-r}t^{-(k-2r)}\int_t^{T_*} \|\nabla^2\partial_t^r u\|_{L^2}\|\nabla\partial_t^{S-1-r}\rho\|_1 d\tau \\ &\le C(\rho_0,u_0,g,k)t^{-r}t^{-\frac{k-2r}{2}}t^{-r}t^{-\frac{k-2r-1}{2}} = C(\rho_0,u_0,g,k)t^{-(k-\frac{1}{2})}.\end{aligned}$$

If $r = 0$, then we apply Hölder inequality and a Gagliardo-Nirenberg-Sobolev inequality to estimate that

$$\begin{aligned}R_{111}^0 &\le \int_t^{T_*} \|\nabla u\|_{L^3}^2\|\nabla\partial_t^{S-1}\rho\|_{L^6}^2 d\tau \le \int_t^{T_*} \|u\|_1\|u\|_2\|\partial_t^{S-1}\rho\|_2^2 d\tau \\ &\le C(\rho_0,u_0,g,k)t^{-1/2}\int_t^{T_*} \|\partial_t^{S-1}\rho\|_2^2 d\tau \\ &\le C(\rho_0,u_0,g,k)t^{-1/2}t^{-(k-1)} \le C(\rho_0,u_0,g,k)t^{-(k-1/2)},\end{aligned}$$

where we used the bound (4.12) and (4.2) with index 0 and $k-1$ above.
Now we discuss the bound for R_{112}^r. For $1 \le r \le S-1$, we have

$$\begin{aligned}R_{112}^r &\le \int_t^{T_*} \|\partial_t^r u\|_{L^3}^2\|\nabla^2\partial_t^{S-1-r}\rho\|_{L^6}^2 d\tau \\ &\le \int_t^{T_*} \|\partial_t^r u\|_{L^2}\|\partial_t^r u\|_1\|\partial_t^{S-1-r}\rho\|_3^2 d\tau \\ &\le C(\rho_0,u_0,g,k)t^{-\frac{2r-1}{2}}t^{-r}\int_t^{T_*} \|\partial_t^{S-1-r}\rho\|_3^2 d\tau \\ &\le C(\rho_0,u_0,g,k)t^{-(k-1/2)},\end{aligned}$$

where we used (4.1) with indices $2r-1$ and $2r$ in the third inequality, and (4.2) with index $k-2r$ in the last inequality. If $r = 0$, then we take advantage of Agmon's inequality in 3D again to obtain:

$$\begin{aligned}
R_{112}^0 &\le \int_t^{T_*} \|u\|_{L^\infty}^2 \|\nabla^2 \partial_t^{S-1}\rho\|_{L^2}^2 d\tau \\
&\le \int_t^{T_*} \|u\|_{L^2}^{1/2} \|u\|_2^{3/2} \|\partial_t^{S-1}\rho\|_2^2 d\tau \\
&\le C(\rho_0, u_0, g, k) t^{-3/4} \int_t^{T_*} \|\partial_t^{S-1}\rho\|_2^2 d\tau \\
&\le C(\rho_0, u_0, g, k) t^{-3/4} t^{-(k-1)} \\
&= C(\rho_0, u_0, g, k) t^{-(k-1/4)}.
\end{aligned}$$

Therefore, we arrive at the bound:

$$\int_t^{T_*} \|R_{11}^r\|_1^2 d\tau \le C(\rho_0, u_0, g, k) t^{-(k-1/4)}.$$

Proceeding in a similar fashion, we can acquire similar bounds for the R_{12}^r and R_{13}^r. The proof of the lemma is thus complete after we sum up the estimates above. □

By Step 1, we know that for any $t \in (0, T_*]$,

$$t^k \left(\|\partial_t^S \rho(t)\|_{L^2}^2 + \|\partial_t^S u(t)\|_{L^2}^2 \right) \le C(\rho_0, u_0, g, k),$$

$$t^k \int_t^{T_*} \left(\|\partial_t^S \rho(\tau)\|_1^2 + \|\partial_t^S u(\tau)\|_1^2 \right) d\tau \le C(\rho_0, u_0, g, k).$$

Combining Lemma 4.3 with equations (4.10a), (4.10b), and using elliptic estimates, we conclude that for $t \in (0, T_*]$

$$\|\partial_t^{S-1} \rho(t)\|_2^2 + \|\partial_t^{S-1} u(t)\|_2^2 \le C(\rho_0, u_0, g, k) t^{-k},$$

$$\int_t^{T_*} \left(\|\partial_t^{S-1} \rho(\tau)\|_3^2 + \|\partial_t^{S-1} u(\tau)\|_3^2 \right) d\tau \le C(\rho_0, u_0, g, k) t^{-k},$$

which finishes the case when $l = S - 1$. The rest will follow from an induction in l, and we omit the details here. Hence, we have concluded the case where k is odd.

2. k **is even**. Since we have proved the $k = 0$ case, we may write $k = 2S$, $S \ge 1$, and define

$$\tilde{E}_s(t) = \|\nabla \partial_t^s \rho\|_{L^2}^2 + \|\nabla \partial_t^s u\|_{L^2}^2$$

for $0 \le s \le S$. Notice that $\tilde{E}_s(t) \sim \|\partial_t^s \rho\|_1^2 + \|\partial_t^s u\|_1^2$ in view of the Poincaré inequality. The scheme of the proof in this case is the same double induction argument (in forward k and for each k backwards in l), and we will follow the same outline as in the odd case. Considering (4.3) for $s = 1, \dots, S$, we test (4.3a), (4.3b) with $s = S$ by $-\Delta \partial_t^S \rho, \mathcal{A} \partial_t^S u$ respectively, which yields:

$$\begin{aligned}
\frac{1}{2}\frac{d}{dt}\|\nabla \partial_t^S \rho\|_{L^2}^2 + \|\Delta \partial_t^S \rho\|_{L^2}^2 &= \sum_{r=0}^S \binom{S}{r} (\tilde{I}_r + \tilde{J}_r + \tilde{K}_r), \\
\frac{1}{2}\frac{d}{dt}\|\nabla \partial_t^S u\|_{L^2}^2 + \|\mathcal{A} \partial_t^S u\|_{L^2}^2 &= g \int_\Omega \mathcal{A} \partial_t^S u \mathbb{P}(\partial_t^S \rho e_z) \le \frac{g}{2} \tilde{E}_S,
\end{aligned}$$

where for $r = 0, \dots, S$,

$$\tilde{I}_r = \int_\Omega \Delta\partial_t^S \rho (\partial_t^r u \cdot \nabla)\partial_t^{S-r}\rho,\ \tilde{J}_r = \int_\Omega \Delta\partial_t^S \rho \partial_t^r \rho \partial_t^{S-r}\rho,$$

$$\tilde{K}_r = \int_\Omega \Delta\partial_t^S \rho \nabla \partial_t^{S-r}\rho \cdot \nabla(-\Delta)^{-1}\partial_t^r \rho.$$

To estimate $\tilde{I}_r$, we first observe that

$$\tilde{I}_r \leq \|\Delta\partial_t^S\rho\|_{L^2}\|\partial_t^r u\|_{L^6}\|\nabla\partial_t^{S-r}\rho\|_{L^3} \leq \|\Delta\partial_t^S\rho\|_{L^2}\|\partial_t^r u\|_1\|\nabla\partial_t^{S-r}\rho\|_{L^2}^{1/2}\|\nabla^2\partial_t^{S-r}\rho\|_{L^2}^{1/2}.$$

Hence if $r \neq 0$, we may estimate $\tilde{I}_r$ as follows: for any $\epsilon > 0$,

$$\tilde{I}_r \leq \epsilon\|\Delta\partial_t^S\rho\|_{L^2}^2 + C(\epsilon)\|\partial_t^{S-r}\rho\|_1\|\partial_t^{S-r}\rho\|_2\|\partial_t^r u\|_1^2.$$

If $r = 0$, we estimate

$$\tilde{I}_0 = \int_\Omega \Delta\partial_t^S\rho(u\cdot\nabla)\partial_t^S\rho \leq \|\Delta\partial_t^S\rho\|_{L^2}\|u\|_{L^\infty}\|\nabla\partial_t^S\rho\|_{L^2} \leq \epsilon\|\Delta\partial_t^S\rho\|_{L^2}^2 + C(\epsilon)\|u\|_2^2\|\nabla\partial_t^S\rho\|_{L^2}^2.$$

To estimate $\tilde{J}_r$, we have:

$$\tilde{J}_r \leq \epsilon\|\Delta\partial_t^S\rho\|_{L^2}^2 + C(\epsilon)\|\partial_t^r\rho\|_1^2\|\partial_t^{S-r}\rho\|_1^2,$$

where $\epsilon > 0$. Finally, to estimate of $\tilde{K}_r$, we invoke elliptic estimate to obtain

$$\begin{aligned}\tilde{K}_r &\leq \|\Delta\partial_t^S\rho\|_{L^2}\|\partial_t^{S-r}\rho\|_1\|\nabla(-\Delta)^{-1}\partial_t^r\rho\|_{L^\infty} \lesssim \|\Delta\partial_t^S\rho\|_{L^2}\|\partial_t^{S-r}\rho\|_1\|\nabla(-\Delta)^{-1}\partial_t^r\rho\|_2 \\ &\leq \epsilon\|\Delta\partial_t^S\rho\|_{L^2}^2 + C(\epsilon)\|\partial_t^r\rho\|_1^2\|\partial_t^{S-r}\rho\|_1^2,\end{aligned}$$

for any $\epsilon > 0$. Combining the estimates above yields the following energy inequality: for $\tau \in (0, T_*)$,

$$\begin{aligned}\frac{d\tilde{E}_S}{d\tau} + \|\partial_t^S\rho(\tau,\cdot)\|_2^2 + \|\partial_t^S u(\tau,\cdot)\|_2^2 &\leq C(k)\Bigg[\left(g + \|u\|_2^2 + \|\rho\|_2^2\right)\tilde{E}_S(\tau) \\ &\quad + \sum_{r=1}^{S-1}\left(\|\partial_t^{S-r}\rho\|_1\|\partial_t^{S-r}\rho\|_2\|\partial_t^r u\|_1^2 + \|\partial_t^{S-r}\rho\|_1^2\|\partial_t^r\rho\|_1^2\right)\Bigg] \\ &\leq C(k)\left(\tilde{F}(\tau)\tilde{E}_S(\tau) + \sum_{r=1}^{S-1}\tilde{G}_r(\tau)\right), \qquad (4.13)\end{aligned}$$

where $\tilde{F} \in L^1(0, T_*)$ due to the induction hypothesis at $k = 0$. Now, we would like to follow the same plan as that in the odd case. This motivates us to prove lemmas similar to Lemma 4.1, 4.2, and 4.3 adapted to the even case. First, we show the following lemma that parallels Lemma 4.1:

Lemma 4.4. *There exists $\tau_0 \in [t/2, t]$ such that $\tilde{E}_S(\tau_0) \leq C(\rho_0, u_0, g, k)t^{-k}$.*

Proof. We consider (4.3) with $s = S-1$. In view of (4.3b), we have

$$\|\partial_t^S u\|_1^2 \lesssim \|\mathcal{A}\partial_t^{S-1}u\|_1^2 + g^2\|\partial_t^{S-1}\rho\|_1^2 \leq \|\partial_t^{S-1}u\|_3^2 + g^2\|\partial_t^{S-1}\rho\|_1^2,$$

for any $\tau \in [t/2, t]$. Integrating over $[t/2, t]$ and using (4.2) with index $k-1$, we obtain

$$\int_{t/2}^{t} \|\partial_t^S u(\tau)\|_1^2 d\tau \leq \int_{t/2}^{T_*} \|\partial_t^S u(\tau)\|_1^2 d\tau \leq C(\rho_0, u_0, g, k)t^{1-k}. \tag{4.14}$$

To estimate $\|\nabla\partial_t^S\rho\|_{L^2}$, we apply ∇ to both sides of (4.3a) with $s = S-1$, and then use Hölder's inequality:

$$\|\nabla\partial_t^S\rho\|_{L^2}^2 \lesssim \|\partial_t^{S-1}\rho\|_3^2 + \sum_{r=0}^{S-1} C(k)\bigg(\|\nabla(\partial_t^r u\cdot\nabla\partial_t^{S-r-1}\rho)\|_{L^2}^2 + \|\nabla(\partial_t^{S-r-1}\rho\partial_t^r\rho)\|_{L^2}^2 + \|\nabla(\nabla\partial_t^{S-r-1}\rho\cdot\nabla(-\Delta)^{-1}(\partial_t^r\rho))\|_{L^2}^2\bigg). \tag{4.15}$$

To save space, we only consider the most singular term, namely $\|\nabla(\partial_t^r u\cdot\nabla\partial_t^{S-r-1}\rho)\|_{L^2}^2$, and show that

$$\int_{t/2}^{t} \|\nabla(\partial_t^r u\cdot\nabla\partial_t^{S-r-1}\rho)\|_{L^2}^2 d\tau \leq C(\rho_0, u_0, g, k)t^{1-k}. \tag{4.16}$$

The estimates on the rest of the terms follow from a similar argument. To show (4.16), we first compute that

$$\nabla(\partial_t^r u\cdot\nabla\partial_t^{S-r-1}\rho) = \nabla\partial_t^r u\cdot\nabla\partial_t^{S-r-1}\rho + \partial_t^r u\cdot\nabla^2\partial_t^{S-r-1}\rho.$$

The first term can be estimated by

$$\begin{aligned}\|\nabla\partial_t^r u\cdot\nabla\partial_t^{S-r-1}\rho\|_{L^2}^2 &\leq \|\nabla\partial_t^r u\|_{L^4}^2\|\nabla\partial_t^{S-r-1}\rho\|_{L^4}^2\\ &\lesssim \|\nabla\partial_t^r u\|_1^2\|\nabla\partial_t^{S-r-1}\rho\|_1^2\\ &\leq \|\partial_t^r u\|_2^2\|\partial_t^{S-r-1}\rho\|_2^2.\end{aligned}$$

Similarly, we may estimate the second term above by

$$\|\partial_t^r u\cdot\nabla^2\partial_t^{S-r-1}\rho\|_{L^2}^2 \lesssim \|\partial_t^r u\|_1^2\|\partial_t^{S-r-1}\rho\|_3^2$$

Thus for $r = 0, \ldots, S-1$,

$$\begin{aligned}\int_{t/2}^{t} \|\nabla(\partial_t^r u\cdot\nabla\partial_t^{S-r-1}\rho)\|_{L^2}^2 d\tau &\lesssim \int_{t/2}^{t} \left(\|\partial_t^r u\|_2^2\|\partial_t^{S-r-1}\rho\|_2^2 + \|\partial_t^r u\|_1^2\|\partial_t^{S-r-1}\rho\|_3^2\right) d\tau\\ &\leq C(\rho_0, u_0, g, k)(t^{-(2r+1)}t^{-(k-2r-2)} + t^{-2r}t^{-(k-2r-1)})\\ &\leq C(\rho_0, u_0, g, k)t^{-(k-1)}\end{aligned}$$

where we applied (4.1) with index $2r+1$ to $\|\partial_t^r u\|_2$, (4.2) with index $k-2r-2$ to $\|\partial_t^{S-1-r}\rho\|_2$, (4.1) with index $2r$ to $\|\partial_t^r u\|_1$, and (4.2) with index $k-2r-1$ to $\|\partial_t^{S-1-r}\rho\|_3$. In a similar fashion, we can also obtain the following bound:

$$\int_{t/2}^{t} \left[\|\nabla(\partial_t^{S-r-1}\rho\partial_t^r\rho)\|_{L^2}^2 + \|\nabla(\nabla\partial_t^{S-r-1}\rho\cdot\nabla(-\Delta)^{-1}(\partial_t^r\rho))\|_{L^2}^2\right] d\tau \leq C(\rho_0, u_0, g, k)t^{1-k}.$$

Collecting the estimates above and combining with (4.15), we have

$$\int_{t/2}^{t} \|\nabla \partial_t^S \rho(\tau)\|_{L^2}^2 d\tau \le C(\rho_0, u_0, g, k) t^{1-k}. \tag{4.17}$$

Combining (4.14) and (4.17), we have

$$\int_{t/2}^{t} \tilde{E}_S(\tau) d\tau \le C(\rho_0, u_0, g, k) t^{1-k}.$$

By mean value theorem, we can find a $\tau_0 \in (t/2, t)$ such that

$$\tilde{E}_S(\tau_0) \le C(\rho_0, u_0, g, k) t^{-k},$$

and this concludes the proof. □

Then we show a counterpart to Lemma 4.2.

Lemma 4.5. *Let τ_0 be chosen as in Lemma 4.4. Then for any $r = 1, \dots, S-1$, we have*

$$\int_{\tau_0}^{T_*} \tilde{G}_r(\tau) d\tau \le C(\rho_0, u_0, g, k) t^{-(k-\frac{1}{2})}.$$

Proof. Observe that for $r = 1, \dots, S-1$,

$$\tilde{G}_r = \|\partial_t^{S-r}\rho\|_1 \|\partial_t^{S-r}\rho\|_2 \|\partial_t^r u\|_1^2 + \|\partial_t^{S-r}\rho\|_1^2 \|\partial_t^r \rho\|_1^2 =: \tilde{G}_r^1 + \tilde{G}_r^2.$$

To estimate $\tilde{G}_r^2$, apply (4.1) with index $2r$ to $\|\partial_t^r \rho\|_1^2$ and (4.2) with index $k-2r-1$ to $\|\partial_t^{S-r}\rho\|_1^2$:

$$\begin{aligned}
\int_{\tau_0}^{T_*} \tilde{G}_r^2(\tau) d\tau &\le C(\rho_0, u_0, g, k) \tau_0^{-2r} \int_{\tau_0}^{T_*} \|\partial_t^{S-r}\rho\|_1^2 d\tau \\
&\le C(\rho_0, u_0, g, k) \tau_0^{-2r} \tau_0^{-(k-2r-1)} \\
&\le C(\rho_0, u_0, g, k) t^{-(k-1)}.
\end{aligned}$$

To treat the term $\tilde{G}_r^1$, we use the induction hypothesis to obtain that

$$\begin{aligned}
\int_{\tau_0}^{T_*} \tilde{G}_r^1(\tau) d\tau &= \int_{\tau_0}^{T_*} \|\partial_t^{S-r}\rho\|_1 \|\partial_t^r u\|_1 \|\partial_t^{S-r}\rho\|_2 \|\partial_t^r u\|_1 d\tau \\
&\le C(\rho_0, u_0, g, k) \tau_0^{-\frac{k-2r}{2}} \tau_0^{-r} \int_{\tau_0}^{T_*} \|\partial_t^{S-r}\rho\|_2 \|\partial_t^r u\|_1 d\tau \\
&\le C(\rho_0, u_0, g, k) \tau_0^{-\frac{k-2r}{2}} \tau_0^{-r} \tau_0^{-\frac{k-2r}{2}} \tau_0^{-\frac{2r-1}{2}} \\
&\le C(\rho_0, u_0, g, k) t^{-(k-\frac{1}{2})}
\end{aligned}$$

Summing up the two estimates above completes the proof of the lemma. □

Finally, we show a result parallel to Lemma 4.3.

Lemma 4.6. *For any $t \in (0, T_*]$,*

$$t^{k-\frac{1}{4}}(\|R_1(t)\|_1^2 + \|R_2(t)\|_1^2) \leq C(\rho_0, u_0, g, k),$$

$$t^{k-\frac{1}{4}} \int_t^{T_*} \left(\|R_1(\tau)\|_2^2 + \|R_2(\tau)\|_2^2\right) d\tau \leq C(\rho_0, u_0, g, k),$$

where R_1, R_2 are defined as in (4.10).

Proof. First, we note that by applying (4.1) and (4.2) with index $k-2$, we have

$$t^{k-2}\left(\|R_2(t)\|_1^2 + \int_t^{T_*} \|R_2(\tau)\|_2^2 d\tau\right) \leq C(\rho_0, u_0, g, k).$$

Then it suffices for us to show suitable bounds for R_1. Similarly to the proof of Lemma 4.3, we need to control the following typical terms:

$$R_{11}^r := \partial_t^r u \cdot \nabla \partial_t^{S-1-r}\rho,\ R_{12}^r := \partial_t^{S-1-r}\rho \partial_t^r \rho,\ R_{13}^r := \nabla \partial_t^{S-1-r}\rho \cdot \nabla(-\Delta)^{-1}(\partial_t^r \rho),$$

For simplicity, we will only consider in detail the most singular term R_{11}^r, as the estimates for the remaining two terms will follow similarly.

We first study $\|R_{11}^r\|_1^2$, and it suffices for us to consider the leading order contribution i.e. $\|\nabla R_{11}^r\|_{L^2}^2$. Recall from the proof of Lemma 4.3 that

$$\|\nabla R_{11}^r\|_{L^2}^2 \lesssim \|\nabla \partial_t^r u \cdot \nabla \partial_t^{S-1-r}\rho\|_{L^2}^2 + \|\partial_t^r u \cdot \nabla^2 \partial_t^{S-1-r}\rho\|_{L^2}^2 =: R_{111}^r + R_{112}^r.$$

To treat R_{111}^r, we see that for any $0 \leq r \leq S-1$, an application of Hölder inequality, Sobolev embedding, and Gagliardo-Nirenberg Sobolev inequality yields:

$$\begin{aligned}
R_{111}^r &\leq \|\nabla \partial_t^r u\|_{L^3}^2 \|\nabla \partial_t^{S-r-1}\rho\|_{L^6}^2 \\
&\lesssim \|\nabla \partial_t^r u\|_{L^2} \|\nabla \partial_t^r u\|_1 \|\nabla \partial_t^{S-r-1}\rho\|_1^2 \\
&\lesssim \|\partial_t^r u\|_1 \|\partial_t^r u\|_2 \|\partial_t^{S-r-1}\rho\|_2^2 \\
&\leq C(\rho_0, u_0, g, k) t^{-r} t^{-\frac{2r+1}{2}} t^{-(k-2r-1)} \\
&\leq C(\rho_0, u_0, g, k) t^{-(k-1/2)},
\end{aligned}$$

where we used (4.1) with indices $2r$, $2r+1$, $k-2r-1$ respectively in the second to the last inequality above. To treat R_{112}^r, we first discuss the case when $1 \leq r \leq S-1$:

$$\begin{aligned}
R_{112}^r &\leq \|\partial_t^r u\|_{L^3}^2 \|\nabla^2 \partial_t^{S-r-1}\rho\|_{L^6}^2 \\
&\leq \|\partial_t^r u\|_{L^2} \|\partial_t^r u\|_1 \|\partial_t^{S-r-1}\rho\|_3^2 \\
&\leq C(\rho_0, u_0, g, k) t^{-\frac{2r-1}{2}} t^{-r} t^{-(k-2r)} \\
&\leq C(\rho_0, u_0, g, k) t^{-(k-1/2)},
\end{aligned}$$

where we used (4.1) with index $2r-1$, $2r$, $k-2r$ respectively in the second to the last inequality above. In the case where $r = 0$, we instead estimate as follows using Agmon's inequality:

$$\begin{aligned}
R_{112}^0 &\leq \|u\|_{L^\infty}^2 \|\nabla^2 \partial_t^{S-1}\rho\|_{L^2}^2 \\
&\lesssim \|u\|_{L^2}^{1/2} \|u\|_2^{3/2} \|\partial_t^{S-1}\rho\|_2^2 \\
&\leq C(\rho_0, u_0, g, k) t^{-\frac{3}{4}} t^{-(k-1)} \\
&\leq C(\rho_0, u_0, g, k) t^{-(k-1/4)},
\end{aligned}$$

where we used (4.1) with index 0, 1, and $k-1$ in the third inequality. Combining the estimates above yields

$$t^{k-1/4}\|R_{11}^r(t)\|_1^2 \le C(\rho_0, u_0, g, k).$$

Now we shall study $\|R_{11}^r\|_2^2$. We still consider the leading order contribution, namely $\|\nabla^2 R_{11}^r\|_{L^2}^2$. A straightforward computation yields:

$$\begin{aligned}\|\nabla^2 R_{11}^r\|_{L^2}^2 &\lesssim \|\nabla^2\partial_t^r u\cdot\nabla\partial_t^{S-1-r}\rho\|_{L^2}^2 + \|\nabla\partial_t^r u\cdot\nabla^2\partial_t^{S-1-r}\rho\|_{L^2}^2 + \|\partial_t^r u\cdot\nabla^3\partial_t^{S-1-r}\rho\|_{L^2}^2\\ &=: \tilde{R}_{111}^r + \tilde{R}_{112}^r + \tilde{R}_{113}^r.\end{aligned}$$

To control $\tilde{R}_{111}^r$, we have for any $t\in(0,T_*]$:

$$\begin{aligned}\tilde{R}_{111}^r &\le \|\nabla^2\partial_t^r u\|_{L^3}^2\|\nabla\partial_t^{S-1-r}\rho\|_{L^6}^2\\ &\lesssim \|\nabla^2\partial_t^r u\|_{L^2}\|\nabla^2\partial_t^r u\|_1\|\nabla\partial_t^{S-1-r}\rho\|_1^2\\ &\lesssim \|\partial_t^r u\|_2\|\partial_t^r u\|_3\|\partial_t^{S-1-r}\rho\|_2^2\\ &\le C(\rho_0,u_0,g,k)t^{-\frac{2r+1}{2}}t^{-\frac{k-2r-1}{2}}\|\partial_t^r u\|_3\|\partial_t^{S-1-r}\rho\|_2\\ &= C(\rho_0,u_0,g,k)t^{-\frac{k}{2}}\|\partial_t^r u\|_3\|\partial_t^{S-1-r}\rho\|_2\end{aligned}$$

where we used (4.1) with indices $2r+1$ and $k-2r-1$ above. Integrating in time, we obtain

$$\begin{aligned}\int_t^{T_*}\tilde{R}_{111}^r d\tau &\le C(\rho_0,u_0,g,k)t^{-\frac{k}{2}}\int_t^{T_*}\|\partial_t^r u\|_3\|\partial_t^{S-1-r}\rho\|_2 d\tau\\ &\le C(\rho_0,u_0,g,k)t^{-\frac{k}{2}}\left(\int_t^{T_*}\|\partial_t^r u\|_3^2 d\tau\right)^{1/2}\left(\int_t^{T_*}\|\partial_t^{S-1-r}\rho\|_2^2 d\tau\right)^{1/2}\\ &\le C(\rho_0,u_0,g,k)t^{-\frac{k}{2}}t^{-\frac{2r+1}{2}}t^{-\frac{k-2r-2}{2}}\\ &\le C(\rho_0,u_0,g,k)t^{-(k-1/2)},\end{aligned}$$

where we used (4.2) with indices $2r+1$ and $k-2r-2$. A similar argument switching the estimates of u and ρ terms yields the same bound for $\tilde{R}_{112}^r$:

$$\int_t^{T_*}\tilde{R}_{112}^r d\tau \le C(\rho_0,u_0,g,k)t^{-(k-1/2)}.$$

To estimate $\tilde{R}_{113}^r$, we first note that for $1\le r\le S-1$,

$$\begin{aligned}\tilde{R}_{113}^r &\le \|\partial_t^r u\|_{L^3}^2\|\nabla^3\partial_t^{S-r-1}\rho\|_{L^6}^2\\ &\le \|\partial_t^r u\|_{L^2}\|\partial_t^r u\|_1\|\partial_t^{S-r-1}\rho\|_4^2\\ &\le C(\rho_0,u_0,g,k)t^{-\frac{2r-1}{2}}t^{-r}\|\partial_t^{S-r-1}\rho\|_4^2\end{aligned}$$

where we used (4.1) with index $2r-1$ and $2r$ respectively in the last inequality above. Integrating in time, we get:

$$\begin{aligned}\int_t^{T_*}\tilde{R}_{113}^r d\tau &\le C(\rho_0,u_0,g,k)t^{-\frac{2r-1}{2}}t^{-r}\int_t^{T_*}\|\partial_t^{S-r-1}\rho\|_4^2 d\tau\\ &\le C(\rho_0,u_0,g,k)t^{-\frac{2r-1}{2}}t^{-r}t^{k-2r}\\ &= C(\rho_0,u_0,g,k)t^{-(k-1/2)},\end{aligned}$$

where we used (4.2) with index $k-2r$ above. In the case where $r=0$, we instead estimate as follows using Agmon's inequality:

$$\begin{aligned}\tilde{R}^0_{113} &\leq \|u\|^2_{L^\infty}\|\nabla^3\partial_t^{S-1}\rho\|^2_{L^2}\\ &\lesssim \|u\|^{1/2}_{L^2}\|u\|^{3/2}_2\|\partial_t^{S-1}\rho\|^2_3\\ &\leq C(\rho_0,u_0,g,k)t^{-\frac{3}{4}}\|\partial_t^{S-1}\rho\|^2_3\end{aligned}$$

where we used (4.1) with indices 0 and 1. Integrating in time yields:

$$\begin{aligned}\int_t^{T_*}\tilde{R}^0_{113}d\tau &\leq C(\rho_0,u_0,g,k)t^{-\frac{3}{4}}\int_t^{T_*}\|\partial_t^{S-1}\rho\|^2_3 d\tau\\ &\leq C(\rho_0,u_0,g,k)t^{-\frac{3}{4}}t^{-(k-1)}\\ &= C(\rho_0,u_0,g,k)t^{-(k-1/4)},\end{aligned}$$

where we used (4.2) with index $k-1$ above. Collecting the estimates above yields

$$t^{k-1/4}\int_t^{T_*}\|\nabla^2 R^r_{11}\|^2_{L^2}\leq C(\rho_0,u_0,g,k).$$

The proof is therefore completed. □

From this point on, a similar argument to the odd case, combining Lemma 4.4, 4.5, and 4.6 above, finishes the proof for the even case. We leave details for the interested reader.

□

Finally, by combining Proposition 4.1 and Sobolev embeddings, we infer Theorem 1.4.

A Appendix

In the appendix, we will remark on one regularity estimate for Stokes operator $\mathcal{A}$ that plays an essential role in our energy estimates. What follows will be a proof of Proposition 3.1 that appeared in the proof of the key result Theorem 3.1.

The regularity result for Stokes operator stated below is standard; proofs can be found for example in [7]:

Theorem A.1. *Let Ω be a bounded C^2 domain. Then there exists a constant C depending only on domain Ω such that for all $u\in D(\mathcal{A})=H^2(\Omega)\cap V$,*

$$\|u\|_2\leq C\|\mathcal{A}u\|_{L^2}.$$

Moreover, there exist constants c_1, C_1 only depending on domain Ω such that

$$c_1\|\nabla u\|_{L^2}\leq\|\mathcal{A}^{1/2}u\|_{L^2}\leq C_1\|\nabla u\|_{L^2}$$

We will now give a proof for Proposition 3.1; its statement is reiterated below.

Proposition A.1. *Let $\Omega \subset \mathbb{R}^d$, $d = 2,3$, be a smooth, bounded domain. Assume (ρ, u) to be the strong solution of problem (1.1) on $[0,T]$ with initial condition $\rho_0 \in H_0^1 \cap L^\infty$ and $\rho_0 \geq 0$. If there exists $M > 0$ such that $\sup_{0\leq t\leq T} \|\rho(t)\|_{L^2} \leq M$, we then have*

$$\sup_{0\leq t\leq T} \|\rho(t)\|_{L^\infty} \leq CM^{\frac{4}{4-d}},$$

where C is a constant only depending on domain Ω.

Proof. In the proof, we consider $t > 0$ and suppress t dependence for the rest of the proof. Let $p \geq 1$ be an integer. We start with the following computation using (1.1):

$$\frac{d}{dt}\|\rho\|_{L^{2p}}^{2p} = 2p\int_\Omega \rho^{2p-1}(-(u\cdot\nabla)\rho - \operatorname{div}(\rho\nabla(-\Delta)^{-1}\rho) + \Delta\rho)\,dx = 2p(I+J+K).$$

Using incompressibility of u, we can compute that

$$I = -\int_\Omega \rho^{2p-1}(u\cdot\nabla)\rho\,dx = -\frac{1}{2p}\int_\Omega u_j\partial_j\rho^{2p} = 0.$$

Integrating by parts, we have

$$J = (2p-1)\int_\Omega \rho^{2p-1}\partial_j\rho\partial_j(-\Delta)^{-1}\rho dx = \frac{2p-1}{2p}\int_\Omega \partial_j(\rho^{2p})\partial_j(-\Delta)^{-1}\rho dx = \frac{2p-1}{2p}\int_\Omega \rho^{2p+1}dx$$

Using chain rule, we also have

$$K = -(2p-1)\int_\Omega \rho^{2p-2}\partial_j\rho\partial_j\rho dx = -\frac{2p-1}{p^2}\int_\Omega |\nabla\rho^p|^2dx.$$

Collecting all computations above, we observe that

$$\frac{d}{dt}\|\rho\|_{L^{2p}}^{2p} = (2p-1)\|\rho\|_{L^{2p+1}}^{2p+1} - \left(4-\frac{2}{p}\right)\|\nabla\rho^p\|_{L^2}^2. \tag{A.1}$$

Now we shall estimate $\|\rho\|_{L^{2^n}}$ inductively on n. The base case $n = 1$ is dealt with by our assumption. Assume for $t \in [0,T]$ we have the bound

$$\|\rho\|_{L^{2^n}} \leq B_n, B_n \geq 1$$

for any $t \in [0,T]$. Define $f = \rho^{2^n}$, and apply $p = 2^n$ to (A.1), we obtain that

$$\frac{d}{dt}\int_\Omega f^2dx \leq -2\|\nabla f\|_{L^2}^2 + 2^{n+1}\|f\|_{L^{2+2^{-n}}}^{2+2^{-n}}. \tag{A.2}$$

Applying a Gagliardo-Nirenberg-Sobolev inequality, we can estimate using Young's inequality that

$$\|f\|_{L^{2+2^{-n}}}^{2+2^{-n}} \lesssim \|\nabla f\|_{L^2}^{d2^{-n-1}}\|f\|_{L^2}^{2+2^{-n}-d2^{-n-1}} \leq d2^{-n-2}\|\nabla f\|_{L^2}^2 + C\|f\|_{L^2}^{\frac{2+2^{-n}-d2^{-n-1}}{1-d2^{-n-2}}}, \tag{A.3}$$

$$\|f\|_{L^2} \lesssim \|\nabla f\|_{L^2}^{\frac{d}{d+2}}\|f\|_{L^1}^{\frac{2}{d+2}}. \tag{A.4}$$

The constants in the above inequalities do not depend on n. Plugging (A.3), (A.4) to (A.2), we obtain

$$\begin{aligned}\frac{d}{dt}\int_\Omega f^2dx &\leq -2\|\nabla f\|_{L^2}^2 + \frac{d}{2}\|\nabla f\|_{L^2}^2 + C_2 2^{n+1}\|f\|_{L^2}^{\frac{2+2^{-n}-d2^{-n-1}}{1-d2^{-n-2}}} \\ &\leq -C_1\|f\|_{L^2}^{\frac{2d+4}{d}}\|f\|_{L^1}^{-\frac{4}{d}} + C_2 2^{n+1}\|f\|_{L^2}^{\frac{2+2^{-n}-d2^{-n-1}}{1-d2^{-n-2}}},\end{aligned} \tag{A.5}$$

where C_1, C_2 are constants only depending on d. Note that given $d = 2, 3$, we have $\frac{2+2^{-n}-d2^{-n-1}}{1-d2^{-n-2}} < \frac{2d+4}{d}$ for $n \geq 1$. Moreover, observe that

$$\|f\|_{L^1} \leq B_n^{2^n} < \infty.$$

Then for each $n \geq 1$, the right hand side of (A.5) becomes negative when $\|f\|_{L^2}$ is sufficiently large. In particular, one can compute that $\|\rho\|_{L^{2^{n+1}}}$ will never reach the value B_{n+1}, where B_{n+1} is defined by the following recursive relation:

$$\log B_{n+1} = \frac{2^{n+2} - d}{2^{n+2} - 2d} \log B_n + \frac{d}{2^n} \left[\log C_3 + (n+1) \log 2\right], \tag{A.6}$$

where C_3 is a constant independent of n. Note that we have

$$\prod_{j=1}^{n} a_j := \prod_{j=1}^{n} \frac{2^{j+2} - d}{2^{j+2} - 2d} = 2^{-n} \prod_{j=1}^{n} \frac{2^{j+2} - d}{2^{j+1} - d} = 2^{-n} \frac{2^{n+2} - d}{4 - d} = \frac{4 - d2^{-n}}{4 - d} \to \frac{4}{4 - d}$$

as $n \to \infty$, where in the second equality we used the telescoping nature of the product. Using the recursive relation (A.6) and $B_1 \leq M$, one might straightforwardly compute that

$$B_{n+1} \leq M^{\prod_{j=1}^{n} a_j} \left[C_3^{d2^{-n} + \sum_{j=0}^{n-1} (\prod_{k=j+1}^{n} a_k) d2^{-j}}\right] \left[2^{d(n+1)2^{-n} + \sum_{j=0}^{n-1} (\prod_{k=j+1}^{n} a_k) d(j+1)2^{-j}}\right]$$

Note that $0 < \prod_{j=1}^{n} a_j \leq \prod_{j=1}^{\infty} a_j \leq \frac{4}{4-d} < 4$ for $d = 2, 3$, we may deduce from the inequality above that

$$\begin{aligned} B_{n+1} &\leq M^{\frac{4}{4-d}} \left[C_3^{4d \sum_{j=0}^{n} 2^{-j}}\right] \left[2^{4d \sum_{j=0}^{n} (j+1)2^{-j}}\right] \\ &\leq C M^{\frac{4}{4-d}}, \end{aligned}$$

where C is a constant depending only on domain Ω and dimension d, since series $\sum_j 2^{-j}$ and $\sum_j (j+1)2^{-j}$ converge. Finally, as Ω is bounded, we have

$$\|\rho\|_{L^\infty} = \lim_{n \to \infty} \|\rho\|_{L^{2^n}} \leq C M^{\frac{4}{4-d}},$$

and the proof of the lemma is complete. □

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