

# Aspects of Motives:

## Finite-dimensionality, Chow-Künneth Decompositions and Intersections of Cycles

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Dissertation submitted in partial fulfillment of the requirements for the degree of  
Doctor of Philosophy in the Department of Mathematics  
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ABSTRACT

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# Abstract

This thesis analyzes the Chow motives of 3 types of smooth projective varieties: the desingularized elliptic self fiber product, the Fano surface of lines on a cubic threefold and an ample hypersurface of an Abelian variety. For the desingularized elliptic self fiber product, we use an isotypic decomposition of the motive to deduce the Murre conjectures. We also prove a result about the intersection product. For the Fano surface of lines, we prove the finite-dimensionality of the Chow motive. Finally, we prove that an ample hypersurface on an Abelian variety possesses a Chow-Künneth decomposition for which a motivic version of the Lefschetz hyperplane theorem holds.

To my wife, Jennifer.

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# List of Abbreviations

$f : \mathcal{E} \rightarrow C$	a non-isotrivial elliptic surface with section
$\eta$	generic point of $C$
$\mathcal{E}_\eta$	generic fiber of $f : \mathcal{E} \rightarrow C$
$s : C \rightarrow \mathcal{E}$	section to $f : \mathcal{E} \rightarrow C$
$\mathring{C} \subset C$	the largest open subscheme over which $f$ is smooth
$\Sigma$	$C \setminus \mathring{C}$
$E_{i_\sigma}$	an irreducible component of $f^{-1}(\sigma)$
$f^{-1}(\sigma)$	a degenerate fiber of $f$
$P$	$\mathcal{E}/(-1)_\mathcal{E}$
$p : P \rightarrow C$	induced map over $C$
$g : \mathcal{E} \rightarrow P$	quotient map
$T$	fixed locus on $\mathcal{E}$ for the action of $(-1)_\mathcal{E}$



$\pi_{\mathcal{E}}^+$	$\frac{1}{2}(\Delta_{\mathcal{E}} + {}^t\Gamma_{(-1)_{\mathcal{E}}}) \in \text{Cor}^0(\mathcal{E}, \mathcal{E})$
$\pi_{\mathcal{E}}^-$	$\frac{1}{2}(\Delta_{\mathcal{E}} - {}^t\Gamma_{(-1)_{\mathcal{E}}}) \in \text{Cor}^0(\mathcal{E}, \mathcal{E})$
$\mathfrak{h}(\mathcal{E})^+$	$(\mathcal{E}, \pi_{\mathcal{E}}^+)$
$\mathfrak{h}(\mathcal{E})^-$	$(\mathcal{E}, \pi_{\mathcal{E}}^-)$
$CH^j(\mathcal{E})^{\pm}$	the $\pm$ isotypic subspace for the action of $(-1)_{\mathcal{E}}$
$\epsilon : \mathcal{W} \rightarrow \mathcal{E} \times_C \mathcal{E}$	desingularization of $\mathcal{E} \times_C \mathcal{E}$
$\pi : \mathcal{W} \rightarrow C$	structure map over $C$
$G$	$\mathbb{Z}/2 \times \mathbb{Z}/2$
$g_1$	$(1, 0) \in \mathbb{Z}/2 \times \mathbb{Z}/2$
$g_2$	$(0, 1) \in \mathbb{Z}/2 \times \mathbb{Z}/2$
$g_3$	$(1, 1) \in \mathbb{Z}/2 \times \mathbb{Z}/2$
$\bar{\pi}_i : \mathcal{E} \times_C \mathcal{E} \rightarrow \mathcal{E}$	projection onto $i^{\text{th}}$ factor
$\pi_i : \mathcal{W} \rightarrow \mathcal{E}$	$\mathcal{W} \xrightarrow{\epsilon} \mathcal{E} \times_C \mathcal{E} \xrightarrow{\bar{\pi}_i} \mathcal{E}$
$s_i : \mathcal{E} \rightarrow \mathcal{W}$	section to $\pi_i$ induced by a section $C \rightarrow \mathcal{E}$
$\bar{s}_i : \mathcal{E} \rightarrow \mathcal{E} \times_C \mathcal{E}$	section to $\bar{\pi}_i$ induced by section $s : C \rightarrow \mathcal{E}$
$\pi_{\mathcal{W},1}$	$\frac{1}{4}(\Delta_{\mathcal{W}} - {}^t\Gamma_{g_1}) \circ (\Delta_{\mathcal{W}} + {}^t\Gamma_{g_2})$
$\pi_{\mathcal{W},2}$	$\frac{1}{4}(\Delta_{\mathcal{W}} - {}^t\Gamma_{g_2}) \circ (\Delta_{\mathcal{W}} + {}^t\Gamma_{g_1})$

$\pi_{\mathcal{W},3}$	$\frac{1}{2}(\Delta_{\mathcal{W}} + {}^t\Gamma_{g_3})$
$\mathfrak{h}_{(i)}(\mathcal{W})$	$(\mathcal{W}, \pi_{\mathcal{W},i})$
$U$	strict transform of $T_1 \times_C T_2$ under $\epsilon$
$V$	union of $U$ and an isolated set of points
$\rho : \hat{\mathcal{W}} \rightarrow \mathcal{W}$	blowup of $\mathcal{W}$ along $V$
$\kappa : \mathcal{K} := \hat{\mathcal{W}}/g_3 \rightarrow C$	the structure map over $C$
$q : \hat{\mathcal{W}} \rightarrow \mathcal{K}$	the quotient by $(-1)_{\hat{\mathcal{W}}}$

# Acknowledgements

I would like to offer my sincere thanks to Chad Schoen for serving as my advisor, for his guidance during my graduate study and for his help in editing this thesis. I would like to thank Bruno Kahn for his interest in reading chapters 4 and 5 and for making some valuable suggestions. I would also like to thank Dick Hain, Jayce Getz, and Les Saper for agreeing to serve on my committee.

Countless thanks go to my amazing family for all their love and dedication. My parents, Alicia and Humberto, have been the best role models anyone could ask for and have always taught me the importance of generosity and thoughtfulness. My grandmothers, Aida and Antonia, have spoiled me and have made me feel special my entire life. My grandfather Humberto and my late grandfather Angel, have always been a source of inspiration in teaching me the joys of learning.

A great many thanks also go to my mother-in-law and father-in-law, Julio and Dora, for their unwavering love and support since day one. I would like to extend further thanks to my in-law siblings, Julio Jr. and Milly, for their constant motivation and for making me laugh through tough times.

No amount of words are enough to express my gratitude to my devoted and loving wife, Jennifer, who has stood by me through this long and arduous process. She has done so much more for me than I could ever repay; most importantly, she has made me realize that there are more things to life than work.

Finally, I would like to God for all His blessings and for always giving me the persistence to move forward.

# 1

## Overview of Results

A fundamental (and difficult) problem in algebraic geometry is to characterize the closed subvarieties of a given variety,  $X$ . One approach to this problem, drawing from the notion of homotopy, is to consider classes of subvarieties, where subvarieties within the same class are allowed to vary in a 1-parameter family. More precisely, we let  $Z^p(X)$  be the free Abelian group generated by irreducible codimension  $p$  subvarieties of  $X$  and say that  $\gamma \in Z^p(X)$  is rationally equivalent to 0 if there is some codimension  $p-1$  subvariety  $W \subset X$  and a rational map  $f : W \rightarrow \mathbb{P}^1$  for which  $\gamma = f^{-1}(\infty) - f^{-1}(0)$ . The Chow group of codimension  $p$  cycles on  $X$  is then defined as ([13]):

$$CH^p(X) := Z^p(X)/Z_{rat}^p(X)$$

where  $Z_{rat}^p(X)$  denotes the subgroup generated by cycles rationally equivalent to 0. When  $X$  is smooth and projective and  $H^*$  is a Weil cohomology (such as singular or

$\ell$ -adic cohomology), there is a cycle class map

$$cl^p : CH^p(X) \rightarrow H^{2p}(X) \tag{1.1}$$

that takes a subvariety  $[Y] \in CH^p(X)$  to its fundamental class. The difficulty posed by Chow groups is that often (1.1) is not injective, and the task becomes to understand the kernel (the null-homologous cycles). When  $p = 1$ , the kernel is represented by an Abelian variety known as the Picard group. However, as soon as  $p > 1$ , the task becomes highly nontrivial, as evidenced by the following well-known result.

**Theorem 1.0.1.** *[Mumford, [39] Theorem 3.23] Let  $X$  be a smooth projective surface over  $\mathbb{C}$ . Assume that the kernel (1.1) is representable (with  $H^*$  given by singular cohomology) is represented by an Abelian variety. Then,  $p_g = 0$ .*

**Convention 1.0.1.** *From this point forward, all Chow groups will be taken to have rational coefficients. In other words,  $CH^*$  will mean  $CH^* \otimes \mathbb{Q}$*

The above theorem seems to suggest that not much insight can be gathered from cohomology to understand the Chow group. Nevertheless, Bloch and Beilinson (independently) formulated a set of conjectures about the existence of a well-behaved filtration  $F^\nu$  on the Chow groups  $(\otimes \mathbb{Q}) CH^*$  of a smooth projective variety (see [9] or [8] 5.10). These conjectures are based on the folklore premise there should be a filtration on the Chow group whose graded pieces are determined by cohomology. More precisely, these ask for a filtration that is functorial with respect to pullbacks and push-forwards, that

is compatible with the intersection product and that satisfies

$$F^1CH^j = CH_{hom}^j$$

where  $CH_{hom}^j$  denotes the kernel of (1.1) and  $F^{j+1}CH^j = 0$ . The main difficulties in establishing these conjectures is that they are stated for all smooth projective varieties and there is little indication of what should lie between  $F^1$  and  $F^{j+1}$ .

An equivalent formulation of these conjectures is given by the Murre conjectures (stated in full detail in the sequel), which have the advantage that they may be verified on a case-by-case basis. These conjectures are known to be true for curves, surfaces and certain types of threefolds, an example of which is the elliptic modular threefold in [15]. The first aim of this thesis is to generalize the result of [15] to a slightly broader type of threefold. More precisely, let  $\mathcal{E} \rightarrow C$  be a non-isotrivial, semi-stable elliptic surface with section. Then, we can consider the fiber product  $\mathcal{E} \times_C \mathcal{E}$ . In general, this is a singular threefold, which may be easily desingularized to obtain a smooth threefold  $\mathcal{W} \rightarrow C$  with normal crossings fibers. We will prove the following result.

**Theorem 1.0.2.** *The Murre conjectures hold for  $\mathcal{W}$ .*

With the above assumptions, there is an action of  $\mathbb{Z}/2$  on  $\mathcal{E}$  (acting by inversion on the fibers) and, hence, an action of  $\mathbb{Z}/2 \times \mathbb{Z}/2$  on  $\mathcal{W}$ . The idea of the proof will be to decompose the Chow group of  $\mathcal{W}$  by the action of this group and to express the isotypic components in terms of the Chow groups of varieties dominated by  $\mathcal{W}$ .

An accompanying conjecture to the Bloch-Beilinson conjectures is one posed by

Beauville in [5] asking for a “splitting” of the filtration. A precise formulation can be given for varieties for which  $H^1$  vanishes:

**Conjecture 1.0.1** (Weak Splitting). *For  $X$  a smooth projective variety of dimension  $n$  with  $H^1(X) = 0$ , the group of decomposable cycles*

$$D^n(X) = \text{Im}(CH^1(X)^{\otimes n} \rightarrow CH^n(X))$$

*has rank 1, where  $\cdot$  denotes the intersection product.*

Beauville and Voisin prove this when  $X$  is a K3 surface or an elliptic surface over  $\mathbb{P}^1$  with section (over  $\mathbb{C}$ ) in [7] Theorem 1. The proof uses the fact that every ample divisor is equivalent to a sum of rational curves. Voisin ([43] Corollary 1.10) also proves this for  $n = 4$  in the case that  $X$  is the length 2 Hilbert scheme on a K3 surface or the Fano variety of lines on a cubic fourfold. For special types of elliptic fiber products, we will prove the following stronger version of this conjecture:

**Theorem 1.0.3.** *Let  $\mathcal{E} \rightarrow \mathbb{P}^1$  be a non-isotrivial, semi-stable rational elliptic surface with section and let  $\mathcal{W}$  be the desingularized self-fiber product of  $\mathcal{E}$  as above. Then, the intersection image*

$$CH^2(\mathcal{W}) \cdot CH^1(\mathcal{W}) \subset CH^3(\mathcal{W})$$

*has rank 1.*

In algebraic geometry, the set of morphisms between 2 algebraic varieties can be difficult to understand. A classical technique used to study morphisms  $f : Y \rightarrow X$  is to



view the graph as a subvariety  $\Gamma_f \subset Y \times X$  (or, equivalently, its transpose  ${}^t\Gamma_f \subset X \times Y$ ).

Motivated by this, one can define a category  $M$  of *correspondences* whose objects are smooth projective varieties and whose morphisms are given by *correspondences*:

$$\mathrm{Hom}_M(X, Y) := CH^{d_X}(X \times Y) \otimes \mathbb{Q}. \quad (1.2)$$

Here, a morphism of varieties  $f : Y \rightarrow X$  is identified with  $[{}^t\Gamma_f] \in CH^{d_X}(X \times Y)$ .

Composition on the right hand side of (1.2) is defined in a natural way so that there is

the compatibility  $[{}^t\Gamma_g] \circ [{}^t\Gamma_f] = [{}^t\Gamma_{f \circ g}]$ . The use of Chow groups as the set of morphisms

allows one to view the category of motives as a homotopy category (with  $\mathbb{P}^1$  replacing the unit interval), so that classifying motives becomes the algebraic analogue of classifying

manifolds up to homotopy equivalence. The cycle class map allows one to define a functor

$H^* : M \mapsto \mathrm{Vec}_K$  so that  ${}^t\Gamma_f : X \dashv Y$  corresponds to pull-back  $f^* : H^*(X) \rightarrow H^*(Y)$ . It

is natural to ask when cohomology gives “enough” information about a map on motives.

More precisely, there is the following conjecture:

**Conjecture 1.0.2** (Kimura, [24]). *The kernel of the cycle class map*

$$\mathrm{End}_M(X) \rightarrow \mathrm{End}_K(H^*(X))$$

*is a nilpotent ideal.*

It would be extremely desirable if Kimura’s conjecture were true for all smooth projec-

tive varieties. When the base field is  $\mathbb{C}$  (for instance), this conjecture, together with the

usual Hodge conjecture, would imply that one could check that  $h : X \dashrightarrow Y$  is an isomorphism of motives by checking that the map on cohomology  $H^*(h) : H^*(X) \rightarrow H^*(Y)$  is an isomorphism (see [24]). When a variety satisfies Conjecture 1.0.2, we say that it has finite-dimensional motive.

Let  $X \subset \mathbb{P}^4$  be a smooth degree 3 hypersurface. In this case, the variety parametrizing the lines on  $X$  is a smooth projective surface of general type  $S$ , known as the Fano surface of lines on  $X$ . A result of Bloch states that  $CH^2(S)$  is generated by the intersection of curves on  $S$  ([6], Chapter 1). Using this, we will prove the following result about the map  $i : S \rightarrow A$  from the surface to its Albanese variety  $A := Alb(S)$ :

**Theorem 1.0.4.** *The Albanese map  ${}^t\Gamma_i : A \dashrightarrow S$  is split-surjective and  $S$  has finite-dimensional motive.*

Thus far, all known examples of surfaces satisfying the conjecture have been those for which the Albanese map  $CH_{hom}^2 \rightarrow Alb$  is an isomorphism and those that are dominated by products of curves. Since  $p_g(S) > 0$ , Theorem 1.0.1 shows that  $S$  does not satisfy the first condition and a recent result of [34] is that  $S$  is not dominated by a product of curves, so that the theorem does provide a new example of a surface satisfying the conjecture.

One can define the full category of Chow motives  $\mathcal{M}$  by taking the objects to be triples  $(X, \pi, n)$  where  $\pi \in CH^{dx}(X \times X) = \text{End}_M(X)$  is an idempotent (i.e.,  $\pi^2 = \pi$ ),

$n \in \mathbb{Z}$  and morphisms are:

$$\mathrm{Hom}_{\mathcal{M}}((X, \pi, n), (X', \pi', n')) = \pi' \circ CH^{d_X+n'-n}(X \times X') \circ \pi \quad (1.3)$$

Note that  $M$  is a full subcategory of  $\mathcal{M}$  via the identification of  $X \in M$  with  $\mathfrak{h}(X) = (X, \Delta_X, 0) \in \mathcal{M}$ . For a motive  $N = (X, \pi, n) \in \mathcal{M}$ , we write  $N(m) := (X, \pi, n + m) \in \mathcal{M}$ . The cohomology functor extends to a functor  $H^* : \mathcal{M} \mapsto \mathrm{Vec}_{\mathbb{Q}}$  on the full category of motives. An important problem is the following, known as the Chow-Künneth conjecture.

**Conjecture 1.0.3.** *For  $X$  a smooth projective variety, there exist idempotents  $\pi_{i,X} \in \mathrm{End}_M(X)$  such that:*

$$(a) \quad \pi_{i,X} \circ \pi_{j,X} = 0 \text{ for } i \neq j$$

$$(b) \quad \sum_i \pi_{i,X} = \Delta_X$$

$$(c) \quad H^*(\pi_{j,X}) = \mathrm{Id}|_{H^j(X)}.$$

When  $X$  satisfies this conjecture, we say that  $X$  has a Chow-Künneth decomposition. Indeed, the assumptions of the conjecture would then give us a motivic decomposition of the motive of  $X$

$$\mathfrak{h}(X) \cong \bigoplus_i \mathfrak{h}^i(X)$$

where  $\mathfrak{h}^i(X) = (X, \pi_{i,X})$

**Remark 1.0.1.** *Conjecture 1.0.3 is known to be true when  $X$  is a surface ([27]) or an*

Abelian variety ([12]). In the latter case, the idempotents  $\pi_{j,X}$  may be selected canonically so that

$${}^t\Gamma_{n_X} \circ \pi_{j,X} = n^j \cdot \pi_{j,X}$$

where  $n_X : X \rightarrow X$  is multiplication by  $n$ .

A nontrivial result of Clemens and Griffiths is that the Albanese map  $i : S \rightarrow A$  induces an isomorphism via pull-back  $i^* : H^2(A, \mathbb{Q}) \cong H^2(S, \mathbb{Q})$  ([11], [33] Proposition 4). Using Theorem 1.0.4, we can extend this isomorphism on cohomology to an isomorphism of motives:

**Theorem 1.0.5.** *Let  $\pi_{2,S} \in \text{End}_M(S)$  be any idempotent as in the Conjecture 1.0.3 and  $\pi_{2,A}$  be the canonical idempotent on  $A$ . Then, the Albanese map  $\pi_{2,S} \circ {}^t\Gamma_i \circ \pi_{2,A} : \mathfrak{h}^2(A) \rightarrow \mathfrak{h}^2(S)$  is an isomorphism of motives.*

For the final result of this thesis, we will analyze the motive of an ample divisor on an Abelian variety. For this, let  $A$  be an Abelian variety of dimension  $g$  and  $i : \Theta \hookrightarrow A$  be a smooth ample divisor. Then, the classical Lefschetz hyperplane theorem states that  $i^* : H^j(A) \rightarrow H^j(\Theta)$  is an isomorphism for  $j < g - 1$  (and  $i_* : H^j(\Theta) \xrightarrow{\cong} H^{j+2}(A)$  for  $j > g - 1$ ). When  $j = g - 1$ , the map  $i_* : H^{g-1}(\Theta)(-1) \rightarrow H^{g+1}(A)$  is surjective. There is also the so-called primitive cohomology of  $\Theta$

$$K_\Theta := \text{Ker}(H^{g-1}(\Theta) \xrightarrow{i_*} H^{g+1}(A)).$$

We will extend these results to the motivic setting:

**Theorem 1.0.6.** *There are idempotents  $\pi_{j,\Theta} \in \text{End}_M(\Theta)$  that satisfy Conjecture 1.0.3.*

*Moreover, we have the following isomorphisms:*

$$(i) \pi_{j,\Theta} \circ {}^t\Gamma_i \circ \pi_{j,A} : \mathfrak{h}^j(A) \rightarrow \mathfrak{h}^j(\Theta) \text{ for } j < g - 1$$

$$(ii) \pi_{j+2,\mathcal{A}} \circ \Gamma_i \circ \pi_{j,\Theta} : \mathfrak{h}^j(\Theta)(-1) \rightarrow \mathfrak{h}^{j+2}(A) \text{ for } j > g - 1$$

*Finally, there is a idempotent  $p \in \text{End}_M(\Theta)$  such that the motive  $P = (\Theta, p, 0)$  satisfies  $H^*(P) = K_\Theta$ .*

This thesis is organized as follows. Chapter 2 will be devoted to developing in detail the techniques of Chow motives that will be used in proving the above results. In Chapter 3, we will give more precise statements and proofs of Theorems 1.0.2 and 1.0.3. In Chapter 4, we will treat the Fano surface of lines on a cubic threefold and prove Theorems 1.0.4 and 1.0.5. In Chapter 5, we will prove Theorem 1.0.6.

## 2

# Preliminaries

## 2.1 Chow Motives

Let  $\mathcal{M}_k$  denote the category of Chow motives over  $k$  whose objects are triples  $(X, \pi, m)$ , where  $X$  is a smooth projective variety of dimension  $d$ ,  $\pi \in CH^d(X \times X)$  is an idempotent and  $m \in \mathbb{Z}$ . The morphisms are defined as follows:

$$\begin{aligned} \mathrm{Hom}_{\mathcal{M}_k}((X, \pi, m), (X', \pi', m')) &:= \pi' \circ \mathrm{Cor}^{m'-m}(X, X') \circ \pi \\ &:= \pi' \circ CH^{d+m'-m}(X \times X') \circ \pi \end{aligned}$$

Here, composition is defined in [13] Chapter 16.1. When  $m = 0$ , we will write  $(X, \pi)$  for short.

The category of Chow motives is an additive category and every idempotent possesses a kernel and an image, making it a pseudo-Abelian category. Indeed, given a motive

$M = (X, \pi, m)$  and an idempotent  $\rho \in \text{End}_{\mathcal{M}_k}(M)$ , one can define:

$$\text{Im}(\rho) := (X, \rho, m), \quad \ker(\rho) := \text{Im}(\pi - \rho)$$

so that we have

$$M = \text{Im}(\rho) \oplus \ker(\rho)$$

This provides us with a recipe for obtaining a direct sum decomposition of a motive.

Suppose that  $\pi_1, \pi_2 \in \text{Cor}^0(X, X)$  are idempotents for which  $\pi_1 \circ \pi_2 = 0 = \pi_2 \circ \pi_1$  and  $\pi = \pi_1 + \pi_2$ . Then,  $\rho_1 := \pi \circ \pi_1 \circ \pi$  and  $\rho_2 := \pi \circ \pi_2 \circ \pi$  are idempotents in  $\text{End}_{\mathcal{M}_k}(M)$ . Setting  $M_1 = \text{Im}(\rho_1)$  and  $M_2 = \text{Im}(\rho_2)$ , we obtain

$$M = M_1 \oplus M_2$$

One can iterate this process in order to obtain an  $n$ -fold direct sum decomposition of a motive, as we will see throughout this chapter.

There is a functor  $\mathfrak{h} : \mathcal{V}_k^{\text{opp}} \mapsto \mathcal{M}_k$  from the category of smooth projective varieties over  $k$  with  $\mathfrak{h}(X) = (X, \Delta_X)$  and with  $\mathfrak{h}(g) = {}^t\Gamma_g \in CH^d(X \times X')$  for any morphism  $g : X' \rightarrow X$ . We observe the following compatibility of composition with the usual push-forward and pull-back of morphisms on Chow groups:

**Lemma 2.1.1** (Liebermann). *Let  $h_X : X' \dashrightarrow X$ ,  $h_Y : Y \dashrightarrow Y'$  be morphisms of smooth projective varieties. Then, for  $\alpha \in CH^*(X \times Y)$ ,  $\beta \in CH^*(X' \times Y')$ , we have*

$$(a) \quad (h_X \times h_Y)_*(\alpha) = \Gamma_{h_Y} \circ \alpha \circ {}^t\Gamma_{h_X}$$

$$(b) (h_X \times h_Y)^*(\beta) = {}^t\Gamma_{h_Y} \circ \beta \circ \Gamma_{h_X}$$

*Proof.* See [13] Proposition 16.1.1. □

One can also define a tensor product in  $\mathcal{M}_k$ . Indeed, if  $M = (X, \pi, m)$  and  $M' = (X', \pi', m')$ , then we set

$$M \otimes M' := (X \times X', \pi \times \pi', m + m')$$

where  $\pi \times \pi' \in CH^{d+d'}(X \times X \times X' \times X') \cong CH^{d+d'}(X \times X' \times X \times X') = Cor^0(X \times X', X \times X')$  under the identification that permutes the inner two factors of  $X \times X \times X' \times X'$ . For  $f \in \text{Hom}_{\mathcal{M}_k}(M, N)$ ,  $f' \in \text{Hom}_{\mathcal{M}_k}(M', N')$  we define  $f \otimes f' := f \times f'$  using this same identification. It is immediate that  $\mathfrak{h} : \mathcal{V}_k^{opp} \mapsto \mathcal{M}_k$  is a tensor functor. Moreover, we have the unit motive,  $\mathbb{1} = (\text{Spec } k, \Delta_k, 0)$ ; the Lefschetz motive,  $\mathbb{L} = (\text{Spec } k, \Delta_k, -1)$ ; and all the tensor products  $\mathbb{L}^n = (\text{Spec } k, \Delta_k, -n)$  for  $n \in \mathbb{Z}$ . We will call any direct sum of motives of the form  $\mathbb{L}^n$  a *sum of Lefschetz motives*. Moreover, we will use the Tate twist notation: for a motive  $M = (X, \pi, m)$ , we set

$$M(n) = (X, \pi, m + n)$$

For  $\Gamma \in Cor^j(X, Y)$  we define an action  $\Gamma_* : CH^i(X) \rightarrow CH^{i+j}(Y)$  by

$$\Gamma_* \alpha := \pi_{2*}(\pi_1^* \alpha \cdot \Gamma)$$

so that  $\Gamma_* \alpha = \Gamma \circ \alpha$  for  $\alpha \in CH^i(X) = Cor^i(\text{Spec } k, X)$ . This allows us to define the



Chow group of a motive  $M = (X, \pi, n)$ :

$$CH^i(M) = p_* CH^{i+n}(X) \cong \text{Hom}_{\mathcal{M}_k}(\mathbb{L}^i, M)$$

Thus, the Chow functor  $CH^i : \mathcal{V}_k^{opp} \rightarrow \text{Vec}_{\mathbb{Q}}$  extends to an additive functor  $CH^i : \mathcal{M}_k \rightarrow \text{Vec}_{\mathbb{Q}}$  via  $\mathfrak{h}$ .

A Weil cohomology theory is a functor  $H^* : \mathcal{V}_k^{opp} \mapsto \text{Vec}_K$  (where  $K$  is a field of characteristic 0) satisfying certain axioms (described in [25] Section 4), which include Poincaré duality, the Künneth theorem, and the existence of a cycle class map, which we will henceforth denote by  $cl : CH^i(X) \rightarrow H^{2i}(X)$ . The primary examples which will concern us are singular cohomology (in case  $k = \mathbb{C}$ ) and  $\ell$ -adic cohomology. Again, this extends to a functor  $H^* : \mathcal{M}_k \mapsto \text{Vec}_K^*$ , and for  $M = (X, \pi, m)$ , we have

$$H^j(M) = \pi_* H^{j+2m}(X)(m),$$

denoting Tate twist.

Finally, there is the extension of scalars functor  $( )_L : \mathcal{M}_k \mapsto \mathcal{M}_L$  for any field extension  $k \subset L$ . One can show that this functor is faithful using a standard argument.

## 2.2 The Murre conjectures in a few basic instances

The Bloch-Beilinson conjectures are a set of fundamental questions about the existence of a well-behaved filtration  $F^\nu$  on the Chow groups  $CH^*$  of smooth projective varieties

(see [9] or [8] 5.10). More precisely, these ask for a filtration that is functorial with respect to pullbacks and push-forwards, that is compatible with the intersection product and that satisfies  $F^1CH^j = CH_{hom}^j$  and  $F^{j+1}CH^j = 0$ . The main difficulties in establishing these conjectures is that they are stated for all smooth projective varieties and there is little indication of what should lie between  $F^1$  and  $F^{j+1}$ . Murre has formulated the following alternative set of conjectures in [29]:

**Conjecture 2.2.1** (Murre). *For  $X$  a smooth projective variety of dimension  $d$  over a field  $k$ ,*

(A) *Conjecture 1.0.3 holds for  $X$ .*

(B) *For a set of idempotents  $\{\pi_{i,X}\}$  as in (A),  $\pi_{i,X*}CH^j(X) = 0$  for  $i \notin \{j, j+1, \dots, 2j\}$ .*

*This defines a descending filtration:*

$$\begin{aligned} F^vCH^j(X) &= \bigcap_{s < v} \ker(\pi_{2j-s,X}|_{CH^j(X)}) \\ &= \bigoplus_{s \geq v} CH^j(\mathfrak{h}^{2j-s}(X)) \end{aligned}$$

*for which  $F^{j+1}CH^j(X) = 0$ .*

(C) *The filtration above is independent of the choice of  $\{\pi_{i,X}\}$ .*

(D)  *$F^1CH^j(X) = CH_{hom}^j(X)$  for all  $j$ .*

A result of Jannsen ([19] Theorem 5.2) is that the Murre conjectures hold for all smooth projective varieties if and only if the Bloch-Beilinson conjectures hold and that,

in fact, the filtrations should agree. Besides the elementary case of curves, the Murre conjectures have been investigated for surfaces ([29] and [30]), products of a surface with 1 or 2 curves ([30] and [23], resp.) uniruled threefolds ([1]), Abelian varieties ([6], [12]), varieties with nef tangent bundles ([17]), and elliptic modular threefolds ([15]). In all these cases, conjectures (A), (B) and (D) are answered directly while for the more subtle (C) only strong evidence of independence is provided. For instance, in handling the case of surfaces, Murre defines natural candidates for  $\pi_{3,X}$  so that

$$F^2CH^2(X) = \ker \pi_{3,X} = \ker (CH_{hom}^2(X) \rightarrow Alb(X)_{\mathbb{Q}})$$

but does not prove that this is the case for all possible choices of  $\pi_{3,X}$ . It should be noted, however, that when  $X$  satisfies conjectures (A), (B), and (D) and has finite-dimensional motive in the sense of [24], then (C) will also hold (Proposition 2.3.1 (e)).

Since we will be interested in the Murre conjectures, we present some basic results in the literature that will be helpful to the reader.

**Example 2.2.1** (Curves). *The simplest case is when  $X$  is a smooth projective (connected) curve. In this case, there is a rather immediate Chow-Künneth decomposition we select  $z \in CH^1(X)$  of degree 1 and set  $\pi_{0,X} = z \times X$ ,  $\pi_{2,X} = X \times z$  and  $\pi_{1,X} = \Delta_X - \pi_{0,X} - \pi_{2,X}$ . It is straightforward to check that this set of correspondences satisfies the conditions of Conjecture (A). One also checks that such a set of idempotents satisfies Conjecture (B). Finally, Conjecture (D) forces the filtration in (C) to be independent.*

The above example illustrates the following useful shortcut. For a smooth projective variety of dimension  $d$ , the idempotents representing extremal degrees of cohomology ( $\pi_{0,X}$  and  $\pi_{2d,X}$ ) can always be taken to have the form  $z \times X$  and  $X \times z$ , respectively. Moreover, the idempotent representing the middle degree ( $\pi_{d,X}$  here) can be taken to be the difference of the diagonal and the sum of the remaining idempotents. This shortcut is particularly useful in handling the case of surfaces when combined with the following result:

**Proposition 2.2.1** ([31] Theorem 6.2.1). *Let  $X$  be a smooth, projective variety of dimension  $d$ . Then, there are idempotents  $\pi_{1,X}, \pi_{2d-1,X} \in \text{End}_{\mathcal{M}_k}(\mathfrak{h}(X))$  such that:*

(a)  $\pi_{1,X}$  and  $\pi_{2d-1,X}$  are mutually orthogonal and  ${}^t\pi_{1,X} = \pi_{2d-1,X}$ .

(b) Setting  $\mathfrak{h}^1(X) = (X, \pi_{1,X}, 0)$ , we have

$$\begin{aligned} H_{et}^*(\mathfrak{h}^1(X)) &= H_{et}^1(\mathfrak{h}^1(X)) = H_{et}^1(X_{\bar{k}}, \mathbb{Q}_\ell) \\ CH^*(\mathfrak{h}^1(X)) &= CH^1(\mathfrak{h}^1(X)) = CH_{hom}^1(X) \end{aligned} \tag{2.1}$$

(c) Setting  $\mathfrak{h}^{2d-1}(X) = (X, \pi_{2d-1,X}, 0)$ , we have

$$\begin{aligned} H_{et}^*(\mathfrak{h}^{2d-1}(X)) &= H_{et}^{2d-1}(\mathfrak{h}^{2d-1}(X)) = H_{et}^{2d-1}(X_{\bar{k}}, \mathbb{Q}_\ell) \\ CH^*(\mathfrak{h}^{2d-1}(X)) &= CH^d(\mathfrak{h}^{2d-1}(X)) = \text{Alb}_X(k) \otimes \mathbb{Q} \end{aligned} \tag{2.2}$$

**Remark 2.2.1.** *It should be noted that while the idempotents in the above result are not uniquely determined, the motives  $\mathfrak{h}^1$  and  $\mathfrak{h}^{2d-1}$  are well-defined up to isomorphism.*

While the Murre conjectures do not make any explicit prediction about the filtration beyond the statement of Conjecture (D), the above result suggests that one should have

$$CH^*(\mathfrak{h}^{2d-1}(X)) = CH^{2d-1}(\mathfrak{h}^1(X)) = Alb_X(k) \otimes \mathbb{Q}$$

and, consequently, that  $F^2CH^d(X) = \mathbb{K}_X := \ker(CH_{num}^d(X) \rightarrow Alb(X) \otimes \mathbb{Q})$ . In fact, Murre (in [29]) does not address the independence of the filtration question (Conjecture (C)) in its most literal form but instead posits that the Albanese kernel is the most natural candidate for  $F^2$ .

The following chart (found in [31] 6.3.1) gives a convenient summary of the Murre conjectures for surfaces:

$M$	$\mathfrak{h}^0(X)$	$\mathfrak{h}^1(X)$	$\mathfrak{h}^2(X)$	$\mathfrak{h}^3(X)$	$\mathfrak{h}^4(X)$
$CH^0(M)$	$CH^0(X)$	0	0	0	0
$CH^1(M)$	0	$Pic^0(X) \otimes \mathbb{Q}$	$NS(X) \otimes \mathbb{Q}$	0	0
$CH^2(M)$	0	0	$\mathbb{K}_X$	$Alb(X) \otimes \mathbb{Q}$	$Num^2(X)$

where  $Num^p$  denotes the group of codimension  $p$  cycles modulo numerical equivalence,

$CH^p / \sim_{num}$ .

## 2.3 Finite-Dimensional Motives

Another important notion for this thesis is that of finite-dimensional motives. To introduce it, recall that  $\mathcal{M}_k$  is a tensor category with tensor product defined as:

$$(X, \pi, m) \otimes (Y, \tau, n) := (X \times Y, \pi \times \tau, m + n) \quad (2.3)$$

Thus, we can define the idempotent

$$\pi_{sym} = \frac{1}{n!} \sum_{g \in \mathfrak{S}_n} \pi^{\otimes n} \circ {}^t\Gamma_g \circ \pi^{\otimes n} \in CH^{dn}(X^{\times n} \times X^{\times n})$$

and its corresponding motive  $\text{Sym}^i M = (X^{\times n}, \pi_{sym}, 0)$ . One can concoct similar idempotents corresponding to the other representations of  $\mathfrak{S}_n$  (including the alternating representation).

**Definition 2.3.1** (Kimura). *We say that a motive  $M$  is oddly finite-dimensional if  $\text{Sym}^n M = 0$  for  $n \gg 0$  and evenly finite-dimensional if  $\wedge^n M = 0$  for  $n \gg 0$ . We say that  $M$  is finite-dimensional if  $M = M_+ \oplus M_-$ , where  $M_+$  is evenly finite-dimensional and  $M_-$  is oddly finite-dimensional.*

We give a quick overview of some of the properties of finite-dimensional motives, the last of which is the proof of Conjecture (C):

**Proposition 2.3.1.** (a)  $M \oplus N$  is finite-dimensional  $\Leftrightarrow M$  and  $N$  are finite-dimensional.

*In this case,  $M \otimes N$  is also finite-dimensional.*

(b) If  $f : M \rightarrow N$  is split-surjective and  $M$  is finite-dimensional, then so is  $N$ .

(c) The motive of a smooth projective curve is finite-dimensional.

(d) If  $M$  is finite-dimensional, then the kernel of the cycle class map

$$\ker(H^*()) : \text{End}_{\mathcal{M}_k}(M) \rightarrow \text{End}_{\mathbb{Q}_\ell}(H^*(M))$$

is a nilpotent ideal.

(e) If  $\mathfrak{h}(X)$  is finite-dimensional and Conjectures (A) and (B) hold for  $X$ , then Conjecture (C) also holds.

*Proof.* For (a), see [24] Corollary 5.11. Then, (b) follows easily from (a). Indeed, if  $g : N \rightarrow M$  is the right inverse, then  $\pi := g \circ f : M \rightarrow M$  is an idempotent. Since  $\mathcal{M}_k$  is pseudo-Abelian, every idempotent has an image and a cokernel. Thus,  $M \cong \text{Im}(\pi) \oplus \text{coker}(\pi)$ . Moreover,  $\text{Im}(\pi) \cong N$  so that by (a),  $N$  is finite-dimensional. Statement (c) follows from [24] Corollary 4.4 and statements (a) and (b). Statement (d) follows from [24] Proposition 7.2. The proof of (e) is found in [19] and uses Lemma 3.1 from [20]. The idea is to let  $\{\pi_{i,X}\}$  and  $\{\pi'_{i,X}\}$  be 2 sets of Chow-Künneth idempotents. Then, from (d), there are nilpotent  $n_i \in \text{End}_{\mathcal{M}_k}(\mathfrak{h}(X))$  for which

$$\pi'_{i,X} = \pi_{i,X} + n_i$$

Squaring both sides, and using the fact that  $\pi_{i,X}$  and  $\pi'_{i,X}$  are idempotents, one obtains:

$$n_i - n_i^2 = \pi_{i,X} \circ n_i + n_i \circ \pi_{i,X}$$

Using this, it is straightforward to check that one has

$$\pi'_{i,X} = (1 - n_i)^{-1} \circ \pi_{i,X} \circ (1 + n_i)$$

where  $(1 - n_i)^{-1} = 1 + n_i + \dots + n_i^{m-1}$  (with  $n_i^m = 0$ ). Since  $(1 + n_i)$  and  $(1 - n_i)$  are both invertible, it follows that

$$\ker(\pi_{i,X}|_{CH^j(X)}) = \ker(\pi'_{i,X}|_{CH^j(X)})$$

so that Conjecture (C) follows. □

As a consequence of (c), (a), the motive of any product of smooth projective curves is finite-dimensional; from (b), so is any variety dominated by a product of curves (such as Abelian varieties). In [14] it is demonstrated that varieties of dimension  $\leq 3$  have finite-dimensional motive if  $CH_0(X)_{hom}$  is representable. (This is true, in particular, if  $X$  is a rationally connected threefold.) Other than this, the following conjecture remains wide open.

**Conjecture 2.3.1** (Kimura, O'Sullivan). *Every motive  $M \in \mathcal{M}_k$  is finite-dimensional.*

A consequence of the above properties is the following trivial proof of Bloch's con-



jecture for surfaces with finite-dimensional motive.

**Proposition 2.3.2.** *Let  $X$  be a surface over  $\mathbb{C}$  with  $p_g(X) = 0$ . Assume that  $\mathfrak{h}^2(X)$  is finite-dimensional. Then, the Albanese kernel  $\mathbb{K}_X = 0$ .*

*Proof.* See [24] Corollary 7.7. □

**Example 2.3.1** (Abelian varieties). *The Beauville decomposition (see [6]) for the Chow ring of an Abelian variety allows one to obtain a canonical Chow-Künneth decomposition. Indeed, for an Abelian variety  $X$  of dimension  $g$ , we have a decomposition*

$$CH^j(X) = \bigoplus_s CH_{(s)}^j(X),$$

where  $n_X^*$  acts as  $n^{2j-s} \cdot \text{id}$  on  $CH_{(s)}^j(X)$ , so that each cycle decomposes into a sum of eigen-components. Applying this to  $\Delta_X \in CH^g(X \times X)$  gives a sum  $\sum_i \pi_{i,X}$ , which by [12] Theorem 3.1 gives the desired Chow-Künneth decomposition. Now, we note that  $\mathfrak{h}(X)$  is finite-dimensional. Indeed, every Abelian variety is dominated by the Jacobian of a curve, which is dominated by a  $n$ -fold self-product of that curve for  $n \gg 0$ . Then, applying Proposition 2.3.1 (a), one concludes that  $\mathfrak{h}(X)$  is finite-dimensional. From (e), it follows that Conjecture (C) holds, provided that (B) holds. The results of [6] indicate that Conjectures (B) and (D) holds modulo the verification of the following 2 conditions:

$$CH_{(s)}^p(X) = 0 \text{ for } s < 0$$

$$CH_{(0)}^p(X) = \text{Num}^p(X) \otimes \mathbb{Q}$$

These conditions are referred to as Beauville's conjectures, which Beauville verifies in [6] for  $p = 1, g - 1, g$  but remain completely open in general.

## 2.4 An Auxiliary Result

In order to prove Theorem 3.1.1, we will need a preliminary result, beginning with the following definition:

**Definition 2.4.1.** *We say that a motive  $M \in \mathcal{M}_k$  possesses a Chow-Künneth decomposition if there exist mutually orthogonal idempotents  $\pi_{i,M} \in \text{End}_{\mathcal{M}_k}(M)$  satisfying the same conditions as in Conjecture (A). Similarly,  $M$  is said to satisfy the Murre conjectures ((B), (C), or (D)) if the idempotents  $\pi_{i,M}$  satisfy the same conditions as in Conjectures ((B), (C), or (D)).*

**Proposition 2.4.1.** *Let  $M, N \in \mathcal{M}_k$ .*

- (a) *If  $M$  and  $N$  possess Chow-Künneth decompositions, then so does  $M \oplus N$ . Similarly, if  $M$  and  $N$  satisfy Murre conjectures (B) and (D), then so does  $M \oplus N$ .*
- (b) *If  $M$  satisfies any of Murre's conjectures, then  $M(1)$  satisfies the same conjectures.*
- (c) *Any sum of Lefschetz motives satisfies the Murre conjectures.*
- (d) *If  $M = (X, \pi)$ ,  $X$  has dimension 3 and  $NS(M) \otimes K \cong H^2(M)$ , then  $M$  possesses a Chow-Künneth decomposition. Moreover, one can arrange it so that the motives  $\mathfrak{h}^2(M) = \text{Im}(\pi_{2,M})$  and  $\mathfrak{h}^4(M) = \text{Im}(\pi_{4,M})$  are sums of Lefschetz motives.*

*Proof.* For (a), we let  $\iota_M : M \hookrightarrow M \oplus N$ ,  $\iota_N : N \hookrightarrow M \oplus N$  with respective left-inverses  $p_M : M \oplus N \hookrightarrow M$  and  $p_N : M \oplus N \hookrightarrow N$ . We observe that  $p_M \circ \iota_N = 0$  and  $p_N \circ \iota_M = 0$ . Then, for any Chow-Künneth decompositions of  $M$  and  $N$ ,  $\{\pi_{i,M}\}$  and  $\{\pi_{i,N}\}$  we set  $\tau_{i,M} = \iota_M \circ \pi_{i,M} \circ p_M$  and  $\tau_{i,N} = \iota_N \circ \pi_{i,N} \circ p_N$  and define:

$$\pi_{i,M \oplus N} := \tau_{i,M} + \tau_{i,N} \text{ for all } i$$

We compute

$$\begin{aligned} \tau_{i,M}^2 &= (\iota_M \circ \pi_{i,M} \circ p_M) \circ (\iota_M \circ \pi_{i,M} \circ p_M) \\ &= (\iota_M \circ \pi_{i,M} \circ \pi_{i,M} \circ p_M) \\ &= (\iota_M \circ \pi_{i,M} \circ p_M) = \tau_{i,M} \end{aligned}$$

and similarly for  $\tau_N$  so that both are idempotents.

Moreover, we compute

$$\begin{aligned} \tau_{i,M} \circ \tau_{i,N} &= (\iota_M \circ \pi_{i,M} \circ p_M) \circ (\iota_N \circ \pi_{i,N} \circ p_N) \\ &= 0 \end{aligned}$$

since  $p_M \circ \iota_N = 0$  (and similarly  $\tau_{i,N} \circ \tau_{i,M} = 0$ ) so that  $\tau_{i,N}$  and  $\tau_{i,M}$  are mutually orthogonal idempotents and so that their sum  $\pi_{i,M \oplus N}$  is an idempotent also. For  $i \neq j$

we compute

$$\begin{aligned}
\pi_{i,M\oplus N} \circ \pi_{j,M\oplus N} &= (\tau_{i,M} + \tau_{i,N}) \circ (\tau_{j,M} + \tau_{j,N}) \\
&= \tau_{i,M} \circ \tau_{j,M} + \tau_{i,N} \circ \tau_{j,M} + \tau_{i,M} \circ \tau_{j,N} + \tau_{i,N} \circ \tau_{j,N} \\
&= \iota_M \circ \pi_{i,M} \circ \pi_{j,M} \circ p_M + \iota_N \circ \pi_{i,N} \circ \pi_{j,N} \circ p_N = 0
\end{aligned}$$

where the second equality follows from  $p_M \circ \iota_N = 0$  and  $p_N \circ \iota_M = 0$  and the third equality follows from  $\pi_{i,M} \circ \pi_{j,M} = 0$  for  $i \neq j$  (and similarly for  $N$ ). Thus,  $\pi_{i,M\oplus N}$  are mutually orthogonal idempotents. Moreover, we compute

$$\begin{aligned}
\sum_i \pi_{i,M\oplus N} &= \sum_i \iota_M \circ \pi_{i,M} \circ p_M + \sum_i \iota_N \circ \pi_{i,N} \circ p_N \\
&= \iota_M \circ p_M + \iota_N \circ p_N = id_{M\oplus N}
\end{aligned}$$

Checking the cohomological condition is straightforward. Now, suppose that  $M$  and  $N$  satisfy Conjectures (B) and (D) and let  $\gamma \in CH^j(M \oplus N)$ . Then, for  $i \notin \{j, j+1, \dots, 2j\}$ , we compute:

$$\begin{aligned}
(\pi_{i,M\oplus N})_* \gamma &= (\tau_{i,M})_* \gamma + (\tau_{i,N})_* \gamma \\
&= (\tau_{i,M})_* \gamma + (\tau_{i,N})_* \gamma \\
&= \iota_{M*} \pi_{i,M*} (p_{M*} \gamma) + \iota_{N*} \pi_{i,N*} (p_{M*} \gamma) = 0
\end{aligned}$$

using Conjecture (B) for  $M$  and  $N$ . Conjecture (D) is also straightforward to check.

For (b) Conjecture (A) is immediate and (B) and (D) follow from

$$CH^j(M(1)) = CH^{j-1}(M), \quad H^i(M(1)) = H^{i-2}(M)$$

Moreover (c) follows from (a) and (b) and the fact that the Murre conjectures hold trivially for the unit motive  $\mathbf{1}$ .

Finally, item (d) is a standard argument, the idea for which is outlined in Appendix C of [31]. □

# 3

## The Motives of Certain Elliptic Threefolds

### 3.1 Statement of Results

Let  $k$  be an algebraically closed field of characteristic 0. The first aim of this chapter is to prove Theorem 1.0.2. To do this, we will need the following set-up. Let

$$f : \mathcal{E} \rightarrow C$$

be a non-isotrivial, semi-stable elliptic surface with section  $s : C \rightarrow \mathcal{E}$ . We then consider the elliptic self-fiber product

$$\mathcal{E} \times_C \mathcal{E}$$

Such a threefold is not smooth, however. Indeed, if  $f$  fails to be smooth at some point  $e_i \in \mathcal{E}$ , then a local computation reveals that  $\mathcal{E} \times_C \mathcal{E}$  is singular at  $(e_1, e_2) \in \mathcal{E} \times_C \mathcal{E}$ . In fact, since we have assumed that the fibers of  $f$  are semi-stable, such points are ordinary double points (see [35] §1). One may obtain a desingularization

$$\epsilon : \mathcal{W} \rightarrow \mathcal{E} \times_C \mathcal{E}$$

by blowing up all such singular points. Then, we can state the following result:

**Theorem 3.1.1.** *With the notation of the preceding paragraph, the desingularized elliptic fiber product  $\mathcal{W}$  satisfies (A), (B), (D) of Murre's conjectures.*

The idea of the proof of Theorem 3.1.1 is to decompose the motive of  $\mathcal{W}$  into isotypic components with respect to a natural group action by  $\mathbb{Z}/2 \times \mathbb{Z}/2$ . This group action may be described on the smooth fibers as that induced by inversion on each factor with respect to the identity  $s_1(c) \times s_2(c)$ .

We will also prove Theorem 1.0.3:

**Theorem 3.1.2.** *Suppose that  $f : \mathcal{E} \rightarrow \mathbb{P}^1$  is a rational (semi-stable) elliptic surface with section  $s : \mathbb{P}^1 \rightarrow \mathcal{E}$ . Further, let  $\mathcal{W}$  be the desingularization of  $\mathcal{E} \times_{\mathbb{P}^1} \mathcal{E}$  described previously. Then, there exists  $\beta \in CH^3(\mathcal{W})$  representing the class of a point on an irreducible rational surface for which*

$$Im(CH^2(\mathcal{W}) \otimes CH^1(\mathcal{W}) \rightarrow CH^3(\mathcal{W})) = \mathbb{Q} \cdot \beta$$

In case  $\mathcal{E}$  is rational, the desingularized self fiber product  $\mathcal{W}$  is simply-connected and the Hodge numbers are  $h^{(2,0)}(\mathcal{W}) = 0$  and  $h^{(3,0)}(\mathcal{W}) = 1$ . The last item of Theorem 3.1.2 is then not surprising, given the stronger result for Calabi-Yau hypersurfaces in  $\mathbb{P}^n$  (see [41] Theorem 3.4). The naïve form of the splitting question would suggest that something similar should hold when  $\mathcal{E}$  is arbitrary. In particular, one would hope to say something about the intersection of CM cycles with divisors.

### 3.2 The Result of Gordon and Murre

As mentioned before, one of the goals of this chapter is to extend the result of Gordon and Murre on the elliptic modular threefolds in [15]. To be more precise, we fix  $N > 0$  and let  $L$  be a field which contains the  $N$ th roots of unity and in which  $2N$  is invertible. Further, let  $M := M_N$  be the modular curve over  $L$  that represents the functor which to an  $L$ -scheme  $S$  associates the set of isomorphism classes of elliptic curves over  $E/S$  with level- $N$  structure, where a level- $N$  structure consists of an isomorphism:

$$\alpha : (\mathbb{Z}/N\mathbb{Z})^2 \times S \xrightarrow{\sim} E[N]/S$$

of group schemes over  $S$ . Further, denote the smooth completion by  $\overline{M}$ . Then, consider the universal elliptic scheme  $f : \mathcal{E}_{\overline{M}} \rightarrow \overline{M}$ . The fiber product

$$\mathcal{E}_{\overline{M}} \times_{\overline{M}} \mathcal{E}_{\overline{M}}$$



has an isolated set of double points for its singular locus, which can be blown up to obtain a smooth threefold,  $\mathcal{W}_{\overline{M}} \rightarrow \overline{M}$ .

**Theorem 3.2.1** (= [15] Theorem 4.2+Theorem 5.1+Theorem 6.2). *With the notation above,*

(1)  $\mathcal{E}_M$  has a Chow-Künneth decomposition and (for some integer  $m$ ) there is a decomposition:

$$\begin{aligned} \mathfrak{h}^0(\mathcal{E}_{\overline{M}}) &\cong \mathbb{1} & \mathfrak{h}^4(\mathcal{E}_{\overline{M}}) &\cong \mathbb{L}^2 \\ \mathfrak{h}^1(\mathcal{E}_{\overline{M}}) &\cong \mathfrak{h}^1(\overline{M}) & \mathfrak{h}^3(\mathcal{E}_{\overline{M}}) &\cong \mathfrak{h}^1(\overline{M})(1) \\ \mathfrak{h}^2(\mathcal{E}_{\overline{M}}) &\cong \mathbb{L}^{\oplus m} \oplus {}^1W \end{aligned}$$

(2)  $\mathcal{W}_M$  has a Chow-Künneth decomposition and (for some integer  $n$ ) there is a decomposition:

$$\begin{aligned} \mathfrak{h}^0(\mathcal{W}_{\overline{M}}) &\cong \mathbb{1} & \mathfrak{h}^6(\mathcal{W}_{\overline{M}}) &\cong \mathbb{L}^3 \\ \mathfrak{h}^1(\mathcal{W}_{\overline{M}}) &\cong \mathfrak{h}^1(\overline{M}) & \mathfrak{h}^5(\mathcal{W}_{\overline{M}}) &\cong \mathfrak{h}^1(\overline{M})(2) \\ \mathfrak{h}^2(\mathcal{W}_{\overline{M}}) &\cong \mathbb{L}^{\oplus m} \oplus {}^1W & \mathfrak{h}^4(\mathcal{W}_{\overline{M}}) &\cong (\mathbb{L}^2)^{\oplus m} \oplus {}^1W(1) \\ \mathfrak{h}^3(\mathcal{W}_{\overline{M}}) &\cong (\mathfrak{h}^1(\overline{M})(1))^{\oplus 3} \oplus {}^2W \end{aligned}$$

Here,  ${}^1W$  and  ${}^2W$  denote the Chow motives for modular forms constructed in [36].

Moreover, Murre conjectures (B) and (D) hold for  $\mathcal{W}_M$ .

### 3.3 Essentials on Elliptic Surfaces

We present here some important facts about elliptic surfaces which we will need in the sequel. Since a great deal of notation is introduced, we have provided an index of notation at the beginning.

#### 3.3.1 The group scheme $\mathring{\mathcal{E}}$

Let  $C$  a smooth connected projective curve and  $f : \mathcal{E} \rightarrow C$  a non-isotrivial elliptic surface; i.e.,  $\mathcal{E}$  is a smooth projective surface for which the generic fiber of  $f$

$$\mathcal{E}_\eta \rightarrow \eta := \text{Spec } k(C)$$

is an elliptic curve defined over  $\eta$  that does not admit a model over  $k$ . We will assume further that  $f : \mathcal{E} \rightarrow C$  possesses a section  $s : C \rightarrow \mathcal{E}$ . Moreover, let  $\mathring{C} \subset C$  be the largest open subscheme over which  $f$  is smooth and let  $\Sigma = C \setminus \mathring{C}$  and for each  $\sigma \in \Sigma$  let  $\{i_\sigma\}$  index the irreducible components  $E_{i_\sigma}$  of the degenerate fiber  $f^{-1}(\sigma)$ . Then, we have the following basic fact:

**Lemma 3.3.1.**  $\mathring{\mathcal{E}}$  has the structure of an Abelian scheme over  $\mathring{C}$  with identity section  $s : \mathring{C} \rightarrow \mathring{\mathcal{E}}$ .

*Proof.* Consider the relative Picard scheme,  $\text{Pic}^0(\mathring{\mathcal{E}}/\mathring{C}) \rightarrow \mathring{C}$ . To prove the lemma, it suffices to show that  $\mathring{\mathcal{E}} \cong \text{Pic}^0(\mathring{\mathcal{E}}/\mathring{C})$  which maps the section  $s$  to the identity section of  $\text{Pic}^0(\mathring{\mathcal{E}}/\mathring{C})$ . By the representability of the relative Picard functor, we have a

commutative diagram:

$$\begin{array}{ccc}
Pic(\mathring{\mathcal{E}} \times_{\mathring{C}} \mathring{\mathcal{E}}) & \longrightarrow & Hom_{\mathring{C}}(\mathring{\mathcal{E}}, Pic^0(\mathring{\mathcal{E}}/\mathring{C})) \\
sp_c \downarrow & & sp_c \downarrow \\
Pic(E_c \times E_c) & \longrightarrow & Hom(E_c, Pic^0(E_c))
\end{array} \tag{3.1}$$

for each  $c \in \mathring{C}$ , where  $sp_c$  denotes the specialization map. Let  $\mathcal{M} \in Pic(\mathring{\mathcal{E}} \times_{\mathring{C}} \mathring{\mathcal{E}})$  be the invertible sheaf corresponding to the diagonal  $\Delta_{\mathring{\mathcal{E}}} \subset \mathring{\mathcal{E}} \times_{\mathring{C}} \mathring{\mathcal{E}}$  and  $\mathcal{L}(s) \in Pic(\mathring{\mathcal{E}})$  be the invertible sheaf corresponding to the image of the section  $s : \mathring{C} \rightarrow \mathring{\mathcal{E}}$ . Then, we consider the invertible sheaf

$$\mathcal{L} = \pi_1^* \mathcal{L}(s) \otimes \pi_2^* \mathcal{L}(s) \otimes \mathcal{M}^{-1}$$

Using (3.1), the corresponding  $C$ -morphism  $\phi : \mathring{\mathcal{E}} \rightarrow Pic^0(\mathring{\mathcal{E}}/\mathring{C})$  is that whose restriction to the fiber over any  $c \in \mathring{C}$  is the morphism  $\phi_c : E_c \rightarrow Pic^0(E_c)$  corresponding to the divisor  $E_c \times c + c \times E_c - \Delta_{E_c}$ . This is the morphism given by  $x \mapsto t_x^* \mathcal{L}(s) \otimes \mathcal{L}(s)^{-1}$  for  $x \in E_c$ , which is an isomorphism mapping  $s(c) \in E_c$  to  $0 \in Pic^0(E_c)$ . Since  $\phi$  is a  $C$ -morphism, it follows that  $\phi$  is an isomorphism as well.  $\square$

### 3.3.2 The quotient by $(-1)_{\mathcal{E}}$

Henceforth, we will make the following assumptions about  $f : \mathcal{E} \rightarrow C$ :

- (1) There are no exceptional curves in any of the fibers (in this case, we say  $\mathcal{E}$  is a *relatively minimal elliptic surface*).
- (2) The degenerate fibers are semi-stable.

We note from Lemma 3.3.1 that  $\mathring{f} : \mathring{\mathcal{E}} \rightarrow \mathring{C}$  possesses an action by  $\mathbb{Z}/2$  which acts by  $-1$  on all the fibers. This extends to an action over  $C$  on the Néron model  $\mathcal{E}' \rightarrow C$ . Since  $\mathcal{E}$  does not possess any exceptional curves on the fibers, an argument as in [3] Chapter 2 shows that this action then extends to  $f : \mathcal{E} \rightarrow C$ .

Now, we consider the following quotient of  $\mathcal{E}$ :

$$p : P := \mathcal{E}/(-1)_{\mathcal{E}} \rightarrow C$$

We would like to show that  $P$  is smooth, which is done by proving:

**Lemma 3.3.2.** *Let  $T$  be the fixed locus for the action of  $(-1)_{\mathcal{E}}$  on  $\mathcal{E}$ .*

- (a) *For any  $\sigma \in \Sigma$ ,  $f^{-1}(\sigma)$  is a set of (reduced) rational curves intersecting in an  $n$ -gon configuration. Moreover, any such curve has self-intersection  $-2$ .*
- (b) *When  $n$  is even,  $T$  does not pass through any of the intersection points in (a). When  $n$  is odd, it passes through exactly 1 intersection point.*
- (c)  *$T$  is a smooth divisor on  $\mathcal{E}$  and, hence,  $P$  is smooth. Moreover,  $T \rightarrow C$  is doubly ramified at all intersection points as in (b) and étale everywhere else.*
- (d) *Any  $g(e) \in g(T)$  at which  $g(T) \rightarrow C$  is doubly ramified is contained in precisely one component of its degenerate fiber.*

*Proof.* (a) follows from the Kodaira classification of degenerate fibers of minimal elliptic surfaces, which can be found in [27]. Since the fibers of  $f$  are semi-stable, a degenerate

fiber of  $\mathcal{E}'$  is isomorphic to

$$\mathbb{G}_m \times \mathbb{Z}/n$$

When  $n$  is even, inversion fixes precisely 4 points of this last group:

$$(1, 0), (-1, 0), (1, \frac{n}{2}), (-1, \frac{n}{2})$$

This means that when  $f^{-1}(\sigma)$  is an even-sided  $n$ -gon (for  $\sigma \in \Sigma \subset C$ ),  $T \cap f^{-1}(\sigma)$  consists of 4 points. Thus,  $T \xrightarrow{f} C$  is 4-to-1 over such  $\sigma$  and, hence, étale over such  $\sigma$ .

On the other hand, when  $n$  is odd, the fixed points of  $\mathbb{G}_m \times \mathbb{Z}/n$  are

$$(1, 0), (-1, 0)$$

These correspond to 2 points on the identity component of  $f^{-1}(\sigma)$  through which  $T$  passes. To find the remaining fixed points on  $f^{-1}(\sigma)$ , we observe that  $(-1)_{\mathcal{E}}$  permutes the intersection points of the components of  $f^{-1}(\sigma)$ . This action fixes precisely one such intersection point, that joining components  $\frac{n-1}{2}, \frac{n+1}{2} \in \mathbb{Z}/n$ . This gives a total of 3 points, establishing (b).

For (c), note that by definition of  $T$ , we have

$$T = \{(-1)_{\mathcal{E}}(e) = e \mid e \in \mathcal{E}\}$$

So, for each  $e \in T$ , the Zariski tangent space is given by

$$T_e T = \{v \in T_e \mathcal{E} \mid d(-1)_{\mathcal{E},e}(v) = v\}$$

Thus, the goal is to show that the rank of this tangent space is 1. The right hand side is precisely the eigenspace of  $d(-1)_{\mathcal{E},e}$  for the eigenvalue 1. Now, we observe that  $d(-1)_{\mathcal{E},e}$  diagonalizes as  $\text{diag}(1, -1)$  for any  $e \in \mathcal{E}$  lying on a smooth fiber of  $f$ . Indeed, the fact that  $(-1)_{\mathcal{E}}$  is a  $C$ -morphism accounts for the eigenvalue 1, and the fact that inversion acts by  $-1$  on the tangent space of a smooth fiber accounts for the eigenvalue  $-1$ . Thus,  $d(-1)_{\mathcal{E},e}$  diagonalizes as  $\text{diag}(1, -1)$  on the open subset of  $\mathcal{E}$  consisting of smooth fibers. Now, linear involutions are diagonalizable and can have only 1 and  $-1$  as eigenvalues. Since the function

$$e \mapsto \text{rank}(\ker(d(-1)_{\mathcal{E},e} \pm id : T_e \mathcal{E} \rightarrow T_e \mathcal{E}))$$

is upper semi-continuous, it follows that the rank of each eigenspace is  $\geq 1$  for all  $e \in \mathcal{E}$ . Since  $\mathcal{E}$  is smooth,  $T_e \mathcal{E}$  has rank 2 everywhere, which forces the rank of  $\ker(d(-1)_{\mathcal{E},e} \pm id : T_e \mathcal{E} \rightarrow T_e \mathcal{E})$  to be 1 at all  $e \in \mathcal{E}$ . Thus,  $T$  is smooth and, so is  $P$ . Also, the above discussion shows that  $T \rightarrow C$  is étale at all points that are not singular points of degenerate fibers. When  $T$  does pass through such a singular point  $e$ , it intersects 2 components of the degenerate fiber of  $e$ , so  $T \rightarrow C$  is ramified at  $e$ . Since the degree of  $T \rightarrow C$  is 4 and  $T$  intersects the fiber of  $e$  in 3 points, it follows that  $T \rightarrow C$  is doubly ramified at  $e$ . This gives (c).

For (d), let  $e \in \mathcal{E}$  be a point at which  $T$  is doubly ramified and suppose  $e$  is the intersection of components  $\frac{n-1}{2}$  and  $\frac{n+1}{2}$ . Since the action of inversion on  $\mathbb{Z}/n$  switches  $\frac{n-1}{2}$  and  $\frac{n+1}{2}$ , it follows that these 2 components are identified in the quotient by  $(-1)_{\mathcal{E}}$ ,  $g : \mathcal{E} \rightarrow P$ . Thus,  $g(e)$  is contained in precisely one component of its degenerate fiber. This gives the lemma.  $\square$

**Remark 3.3.1.** *With a bit of work, one can show that the components of the degenerate fibers of  $P \rightarrow C$  can be successively blown down until all the fibers are irreducible rational curves. The result of this blow down process is a ruled surface over  $C$ .*

Now, we have orthogonal idempotents:

$$\pi_{\mathcal{E}}^+ = \frac{1}{2}(\Delta_{\mathcal{E}} + {}^t\Gamma_{(-1)_{\mathcal{E}}}), \quad \pi_{\mathcal{E}}^- = \frac{1}{2}(\Delta_{\mathcal{E}} - {}^t\Gamma_{(-1)_{\mathcal{E}}}) \in \text{Cor}^0(\mathcal{E}, \mathcal{E})$$

for which  $\pi_{\mathcal{E}}^+ + \pi_{\mathcal{E}}^- = \Delta_{\mathcal{E}}$ . We set

$$\mathfrak{h}(\mathcal{E})^+ := (\mathcal{E}, \pi_{\mathcal{E}}^+), \quad \mathfrak{h}(\mathcal{E})^- := (\mathcal{E}, \pi_{\mathcal{E}}^-)$$

so that there is a decomposition of motives

$$\mathfrak{h}(\mathcal{E}) = \mathfrak{h}(\mathcal{E})^+ \oplus \mathfrak{h}(\mathcal{E})^- \tag{3.2}$$

We observe that

$$\mathfrak{h}(g) \circ {}^t\mathfrak{h}(g) = {}^t\Gamma_g \circ \Gamma_g = (g \times g)^* \Delta_P = \Delta_{\mathcal{E}} + {}^t\Gamma_{(-1)_{\mathcal{E}}} = 2\pi_P$$

using Liebermann's lemma in the second equality. Further,

$${}^t\mathfrak{h}(g) \circ \mathfrak{h}(g) = \Gamma_g \circ {}^t\Gamma_g = (g \times g)_* \Delta_{\mathcal{E}} = \Delta_P$$

so that  $\pi_P \circ \mathfrak{h}(g) : \mathfrak{h}(P) \rightarrow \mathfrak{h}(\mathcal{E})^+$  is an isomorphism whose inverse is  $\frac{1}{2} \cdot {}^t\mathfrak{h}(g) \circ \pi_P$ .

**Remark 3.3.2.** *It is useful to note that*

$$CH^j(\mathfrak{h}(\mathcal{E})^+) = CH^j(\mathcal{E})^+ = g^* CH^j(P), \quad CH^j(\mathfrak{h}(\mathcal{E})^-) = CH^j(\mathcal{E})^-$$

where  $CH^j(\mathcal{E})^\pm$  denotes the subspace of  $CH^j(\mathcal{E})$  on which  $(-1)_{\mathcal{E}}$  acts by  $\pm 1$ .

## 3.4 The Isotypic Decomposition

The role of this section will be to obtain an isotypic decomposition for the motive of  $\mathcal{W}$  and to use this decomposition to give a straightforward proof of Theorem 3.1.1. We recall that  $\epsilon : \mathcal{W} \rightarrow \mathcal{E} \times_C \mathcal{E}$  is the desingularization of the self-fiber product of an elliptic surface  $\mathcal{E}$  as in the previous section. We also let  $\pi : \mathcal{W} \rightarrow C$  denote the structure map over  $C$ .

### 3.4.1 The quotient by $G$

Since there is an action on  $\mathcal{E}$  by  $\mathbb{Z}/2$ , there is an action on  $\mathcal{E} \times_C \mathcal{E}$  on each of the factors, resulting in an action by

$$G := \mathbb{Z}/2 \times \mathbb{Z}/2$$



We would like to extend to  $\mathcal{W}$ . Indeed, for any  $g \in G$  permutes the singular points of  $\mathcal{E} \times_C \mathcal{E}$ , so that there is an equality of ideals

$$(g \circ \epsilon)^{-1}(\mathfrak{J}) = \epsilon^{-1}g^{-1}(\mathfrak{J}) = \epsilon^{-1}(\mathfrak{J})$$

where  $\mathfrak{J}$  is the ideal of the singular points of  $\mathcal{E} \times_C \mathcal{E}$ . So,  $(g \circ \epsilon)^{-1}(\mathfrak{J})$  is an invertible sheaf and, using the universal property of blow-up ([16] Prop. 7.14), we have a commutative diagram:

$$\begin{array}{ccc} \mathcal{W} & \xrightarrow{g} & \mathcal{W} \\ \epsilon \downarrow & & \epsilon \downarrow \\ \mathcal{E} \times_C \mathcal{E} & \xrightarrow{g} & \mathcal{E} \times_C \mathcal{E} \end{array} \quad (3.3)$$

So,  $G = \mathbb{Z}/2 \times \mathbb{Z}/2$  lifts to an action on  $\mathcal{W}$ . We will use the following notation in the sequel:

$$g_1 := (1, 0), \quad g_2 := (0, 1), \quad g_3 := (1, 1) \in G$$

For  $i = 1, 2$ , there are natural maps

$$\pi_i : \mathcal{W} \xrightarrow{\epsilon} \mathcal{E} \times_C \mathcal{E} \xrightarrow{\overline{\pi}_i} \mathcal{E}$$

where  $\overline{\pi}_i : \mathcal{E} \times_C \mathcal{E} \rightarrow \mathcal{E}$  denotes the projection onto the  $i^{\text{th}}$  factor. A section  $s : C \rightarrow \mathcal{E}$  of  $f$  induces sections  $\overline{s}_i : \mathcal{E} \rightarrow \mathcal{E} \times_C \mathcal{E}$  of  $\overline{\pi}_i$ . We would like to lift these to sections of  $\pi_i$ , which is done in the claim below:

**Claim:** The maps  $\pi_i : \mathcal{W} \rightarrow \mathcal{E}$  possess sections:

$$s_i : \mathcal{E} \rightarrow \mathcal{W}$$

*Proof.* There are sections  $\overline{s}_i : \mathcal{E} \rightarrow \mathcal{E} \times_C \mathcal{E}$  of  $\overline{\pi}_i$  induced from the section  $s : C \rightarrow \mathcal{E}$ . Since  $\epsilon : \mathcal{W} \rightarrow \mathcal{E} \times_C \mathcal{E}$  is a birational morphism, we then have rational maps  $s_i : \mathcal{E} \dashrightarrow \mathcal{W}$ . We would like to show that the indeterminacy locus of  $s_i$  is empty, which, for ease of notation, we do only for  $i = 1$ . The indeterminacy locus of  $s_1$  consists of those points  $e \in \mathcal{E}$  for which  $\overline{s}_1(e) = (e, s(f(e))) \in \mathcal{E} \times_C \mathcal{E}$  is singular. This can occur only if  $e$  and  $s(f(e))$  are singular points of a degenerate fiber of  $f$ . This would mean that the image of  $s$  passes through a singular point of a degenerate fiber of  $f$ . However, viewing  $f^\circ : \mathring{\mathcal{E}} \rightarrow \mathring{C}$  as an elliptic scheme with identity section  $\mathring{s} : \mathring{C} \rightarrow \mathring{\mathcal{E}}$ , we see that  $\mathring{s}$  extends to a section  $s : C \rightarrow \mathcal{E}' \subset \mathcal{E}$ , where  $\mathcal{E}'$  is the Néron model as before. So, the image of  $s$  does not pass through any singular points of the degenerate fibers of  $f$ , and the indeterminacy locus is empty.  $\square$

We fix a section  $s : C \rightarrow \mathcal{E}$  of  $f$  giving sections  $s_i$  as in the claim. We then have the following compatibilities of  $\pi_i$  and  $s_i$  with the action of  $G$ :

$$\pi_i \circ g_i = (-1)_{\mathcal{E}} \circ \pi_i, \quad g_i \circ s_i = s_i \circ (-1)_{\mathcal{E}} \tag{3.4}$$

for  $i = 1, 2$ . We also have

$$\pi_1 \circ g_2 = \pi_1, \quad g_2 \circ s_1 = s_1 \tag{3.5}$$

and similarly for  $i = 2$ .

### 3.4.2 First Decomposition of $\mathfrak{h}(\mathcal{W})$

We can define the following idempotents in  $Cor^0(\mathcal{W}, \mathcal{W})$ :

$$\begin{aligned}\pi_{\mathcal{W},1} &:= \frac{1}{4}(\Delta_{\mathcal{W}} - {}^t\Gamma_{g_1}) \circ (\Delta_{\mathcal{W}} + {}^t\Gamma_{g_2}) \\ \pi_{\mathcal{W},2} &:= \frac{1}{4}(\Delta_{\mathcal{W}} - {}^t\Gamma_{g_2}) \circ (\Delta_{\mathcal{W}} + {}^t\Gamma_{g_1}) \\ \pi_{\mathcal{W},3} &:= \frac{1}{2}(\Delta_{\mathcal{W}} + {}^t\Gamma_{g_3})\end{aligned}$$

We see that  $\pi_{\mathcal{W},1} + \pi_{\mathcal{W},2} + \pi_{\mathcal{W},3} = \Delta_{\mathcal{W}}$  and that  $\pi_{\mathcal{W},i} \circ \pi_{\mathcal{W},j} = 0$  for  $i \neq j$ . Then, we set  $\mathfrak{h}_{(i)}(\mathcal{W}) := (\mathcal{W}, \pi_{\mathcal{W},i})$  so that there is a decomposition of motives:

$$\mathfrak{h}(\mathcal{W}) = \mathfrak{h}_{(1)}(\mathcal{W}) \oplus \mathfrak{h}_{(2)}(\mathcal{W}) \oplus \mathfrak{h}_{(3)}(\mathcal{W})$$

For the proof of the next result, we will need the following identities:

$$\begin{aligned}\pi_{\mathcal{W},1} \circ {}^t\Gamma_{\pi_1} &= \frac{1}{4}(\Delta_{\mathcal{W}} - {}^t\Gamma_{g_1}) \circ (\Delta_{\mathcal{W}} + {}^t\Gamma_{g_2}) \circ {}^t\Gamma_{\pi_1} \\ &= \frac{1}{2}(\Delta_{\mathcal{W}} - {}^t\Gamma_{g_1}) \circ {}^t\Gamma_{\pi_1} \\ &= \frac{1}{2}{}^t\Gamma_{\pi_1} \circ (\Delta_{\mathcal{E}} - {}^t\Gamma_{(-1)_{\mathcal{E}}}) = {}^t\Gamma_{\pi_1} \circ \pi_{\mathcal{E}}^-\end{aligned}\tag{3.6}$$

where the second equality follows from (3.5) and the third follows from (3.4). Similarly, we have

$$\pi_{\mathcal{W},1} \circ \Gamma_{s_1} = \Gamma_{s_1} \circ \pi_{\mathcal{E}}^-\tag{3.7}$$

We also have the corresponding identities for  $i = 2$ .

**Proposition 3.4.1.** *For  $i = 1, 2$ , we have an isomorphism of motives:*

$$\mathfrak{h}_{(i)}(\mathcal{W}) \cong \mathfrak{h}(\mathcal{E})^- \oplus \mathfrak{h}(\mathcal{E})^- \otimes \mathbb{L} \oplus \mathcal{L}_i$$

where  $\mathcal{L}_i$  is a sum of Lefschetz motives.

*Proof.* We prove this result only for  $i = 1$ , as the proof is the same for  $i = 2$ . The idea of the proof will be to construct a decomposition:

$$\mathfrak{h}_{(1)}(\mathcal{W}) = M \oplus M'$$

where  $M$  is isomorphic to  $\mathfrak{h}(\mathcal{E})^- \oplus \mathfrak{h}(\mathcal{E})^- \otimes \mathbb{L}$  and where  $M'$  is a sum of Lefschetz motives.

To show that  $M$  is of this form, we define maps of motives

$$\Phi_1 := \pi_{\mathcal{W},1} \circ ({}^t\Gamma_{\pi_1} + \Gamma_{s_1}) \circ (\pi_{\mathcal{E}}^- \oplus \pi_{\mathcal{E}}^-) : \mathfrak{h}(\mathcal{E})^- \oplus \mathfrak{h}(\mathcal{E})^-(1) \rightarrow \mathfrak{h}_{(1)}(\mathcal{W})$$

$$\Psi_1 := (\pi_{\mathcal{E}}^- \oplus \pi_{\mathcal{E}}^-) \circ (\psi_1 + \Gamma_{\pi_1}) \circ \pi_{\mathcal{W},1} : \mathfrak{h}_{(1)}(\mathcal{W}) \rightarrow \mathfrak{h}(\mathcal{E})^- \oplus \mathfrak{h}(\mathcal{E})^-(1)$$

where

$$\psi_1 := {}^t\Gamma_{s_1} - {}^t\Gamma_{s_1} \circ \Gamma_{s_1} \circ {}^t\Gamma_{\pi_1} \in \text{Cor}^0(\mathcal{W}, \mathcal{E})$$

**Step 1:** We need to establish that

$$\Psi_1 \circ \Phi_1 = \pi_{\mathcal{E}}^- \oplus \pi_{\mathcal{E}}^- \tag{3.8}$$

To this end, we compute

$$\begin{aligned}
\pi_{\mathcal{W},1} \circ ({}^t\Gamma_{\pi_1} + \Gamma_{s_1}) &= \pi_{\mathcal{W},1} \circ {}^t\Gamma_{\pi_1} + \pi_{\mathcal{W},1} \circ \Gamma_{s_1} \\
&= {}^t\Gamma_{\pi_1} \circ \pi_{\mathcal{E}}^- + \Gamma_{s_1} \circ \pi_{\mathcal{E}}^- \\
&= ({}^t\Gamma_{\pi_1} + \Gamma_{s_1}) \circ (\pi_{\mathcal{E}}^- \oplus \pi_{\mathcal{E}}^-)
\end{aligned} \tag{3.9}$$

where the second equality uses (3.6) and (3.7). We also have

$$\begin{aligned}
(\psi_1 \oplus \Gamma_{\pi_1}) \circ ({}^t\Gamma_{\pi_1} + \Gamma_{s_1}) &= (\psi_1 \circ ({}^t\Gamma_{\pi_1} + \Gamma_{s_1})) \oplus (\Gamma_{\pi_1} \circ ({}^t\Gamma_{\pi_1} + \Gamma_{s_1})) \\
&= (\psi_1 \circ ({}^t\Gamma_{\pi_1} + \Gamma_{s_1})) \oplus (\Gamma_{\pi_1} \circ \Gamma_{s_1}) \\
&= (\psi_1 \circ ({}^t\Gamma_{\pi_1} + \Gamma_{s_1})) \oplus \Delta_{\mathcal{E}}
\end{aligned} \tag{3.10}$$

Note that the second equality follows from the statement:

$$\Gamma_{\pi_1} \circ {}^t\Gamma_{\pi_1} = (\pi_1 \times \pi_1)_* \Delta_{\mathcal{W}} = 0$$

This statement in turn uses Liebermann's lemma in the first equality and uses the definition of push-forward and the fact that  $(\pi_1 \times \pi_1)(\Delta_{\mathcal{W}}) = \Delta_{\mathcal{E}}$  has strictly lower dimension than  $\Delta_{\mathcal{W}}$  (see [13] Chapter 1.4) in the second equality.

We can determine the first component of the final line of (3.10):

$$\begin{aligned}
\psi_1 \circ ({}^t\Gamma_{\pi_1} + \Gamma_{s_1}) &= ({}^t\Gamma_{s_1} - {}^t\Gamma_{s_1} \circ \Gamma_{s_1} \circ \Gamma_{\pi_1}) \circ {}^t\Gamma_{\pi_1} + ({}^t\Gamma_{s_1} - {}^t\Gamma_{s_1} \circ \Gamma_{s_1} \circ \Gamma_{\pi_1}) \circ \Gamma_{s_1} \\
&= {}^t\Gamma_{s_1} \circ {}^t\Gamma_{\pi_1} - {}^t\Gamma_{s_1} \circ \Gamma_{s_1} \circ \Gamma_{\pi_1} \circ {}^t\Gamma_{\pi_1} + \Gamma_{s_1} \circ \Gamma_{s_1} - {}^t\Gamma_{s_1} \circ \Gamma_{s_1} \circ \Gamma_{\pi_1} \circ \Gamma_{s_1} \\
&= \Delta_{\mathcal{E}} + {}^t\Gamma_{s_1} \circ \Gamma_{s_1} - {}^t\Gamma_{s_1} \circ \Gamma_{s_1} = \Delta_{\mathcal{E}}
\end{aligned} \tag{3.11}$$

where the third equality follows from the fact that  $\Gamma_{\pi_1} \circ {}^t\Gamma_{\pi_1} = 0$ . Combining (3.10) and (3.11), we obtain

$$(\psi_1 \oplus \Gamma_{\pi_1}) \circ ({}^t\Gamma_{\pi_1} + \Gamma_{s_1}) = \Delta_{\mathcal{E}} \oplus \Delta_{\mathcal{E}} \tag{3.12}$$

Combining the calculations of (3.9) and (3.12), we obtain

$$\begin{aligned}
\Psi_1 \circ \Phi_1 &= (\pi_{\mathcal{E}}^- \oplus \pi_{\mathcal{E}}^-) \circ (\psi_1 \oplus \Gamma_{\pi_1}) \circ \pi_{\mathcal{W},1} \circ ({}^t\Gamma_{\pi_1} + \Gamma_{s_1}) \circ (\pi_{\mathcal{E}}^- \oplus \pi_{\mathcal{E}}^-) \\
&= (\pi_{\mathcal{E}}^- \oplus \pi_{\mathcal{E}}^-) \circ (\psi_1 \oplus \Gamma_{\pi_1}) \circ ({}^t\Gamma_{\pi_1} + \Gamma_{s_1}) \circ (\pi_{\mathcal{E}}^- \oplus \pi_{\mathcal{E}}^-) \circ (\pi_{\mathcal{E}}^- \oplus \pi_{\mathcal{E}}^-) \\
&= (\pi_{\mathcal{E}}^- \oplus \pi_{\mathcal{E}}^-) \circ (\Delta_{\mathcal{E}} \oplus \Delta_{\mathcal{E}}) \circ (\pi_{\mathcal{E}}^- \oplus \pi_{\mathcal{E}}^-) \\
&= \pi_{\mathcal{E}}^- \oplus \pi_{\mathcal{E}}^-
\end{aligned} \tag{3.13}$$

Here, the second equality follows from (3.9) and the third follows from (3.12). This proves (3.8) and the proof of Step 1 is complete.

**Step 2:** The map of motives

$$\Pi_1 := \Phi_1 \circ \Psi_1 : \mathfrak{h}_{(1)}(\mathcal{W}) \rightarrow \mathfrak{h}_{(1)}(\mathcal{W}) \tag{3.14}$$

is an idempotent. Moreover, there is an isomorphism of motives:

$$\Pi_1 \circ \Phi_1 : \mathfrak{h}(\mathcal{E})^- \oplus \mathfrak{h}(\mathcal{E})^- \otimes \mathbb{L} \rightarrow (\mathcal{W}, \Pi_1) \quad (3.15)$$

whose inverse is  $\Psi_1 \circ \Pi_1 : (\mathcal{W}, \Pi_1) \rightarrow \mathfrak{h}(\mathcal{E})^- \oplus \mathfrak{h}(\mathcal{E})^- \otimes \mathbb{L}$ .

For the first statement of this step, we compute

$$\begin{aligned} \Pi_1^2 &= (\Phi_1 \circ \Psi_1) \circ (\Phi_1 \circ \Psi_1) \\ &= \Phi_1 \circ (\pi_{\mathcal{E}}^- \oplus \pi_{\mathcal{E}}^-) \circ \Psi_1 \\ &= \Phi_1 \circ (\pi_{\mathcal{E}}^- \oplus \pi_{\mathcal{E}}^-) \circ (\pi_{\mathcal{E}}^- \oplus \pi_{\mathcal{E}}^-) \circ (\psi_1 \oplus \Gamma_{\pi_1}) \circ \pi_{\mathcal{W},1} \\ &= \Phi_1 \circ (\pi_{\mathcal{E}}^- \oplus \pi_{\mathcal{E}}^-) \circ (\psi_1 \oplus \Gamma_{\pi_1}) \circ \pi_{\mathcal{W},1} \\ &= \Phi_1 \circ \Psi_1 = \Pi_1 \end{aligned}$$

so that  $\Pi_1$  is an idempotent. We can then define the associated motive  $(\mathcal{W}, \Pi_1)$ . To prove the second statement of Step 2, we need to show that

$$\begin{aligned} (\Pi_1 \circ \Phi_1) \circ (\Psi_1 \circ \Pi_1) &= \Pi_1 \\ (\Psi_1 \circ \Pi_1) \circ (\Pi_1 \circ \Phi_1) &= \pi_{\mathcal{E}_1}^- \oplus \pi_{\mathcal{E}_1}^- \end{aligned}$$

Towards the first identity, we have

$$\Pi_1 \circ \Phi_1 \circ \Psi_1 \circ \Pi_1 = \Pi_1 \circ \Pi_1 \circ \Pi_1 = \Pi_1$$

since  $\Pi_1$  is an idempotent. Towards the second identity, we have

$$(\Psi_1 \circ \Pi_1) \circ (\Pi_1 \circ \Phi_1) = \Psi_1 \circ \Pi_1 \circ \Phi_1 = \Psi_1 \circ \Phi_1 \circ \Psi_1 \circ \Phi_1 = (\pi_{\mathcal{E}}^- \oplus \pi_{\mathcal{E}}^-) \circ (\pi_{\mathcal{E}}^- \oplus \pi_{\mathcal{E}}^-) = (\pi_{\mathcal{E}}^- \oplus \pi_{\mathcal{E}}^-)$$

where the third equality follows from (3.13). This concludes the proof of Step 2.

**Step 3:** There is a decomposition of motives:

$$\mathfrak{h}_{(1)}(\mathcal{W}) = (\mathcal{W}, \Pi_1) \oplus (\mathcal{W}, \pi_{\mathcal{W},1} - \Pi_1) \cong \mathfrak{h}(\mathcal{E})^- \oplus \mathfrak{h}(\mathcal{E})^- \otimes \mathbb{L} \oplus (\mathcal{W}, \pi_{\mathcal{W},1} - \Pi_1)$$

To prove this, we need only show that  $\Pi_1$  and  $\pi_{\mathcal{W},1} - \Pi_1$  are mutually orthogonal idempotents since the isomorphism was proved in Step 2. To this end, we compute:

$$\begin{aligned} \Pi_1 \circ \pi_{\mathcal{W},1} &= \Phi_1 \circ \Psi_1 \circ \pi_{\mathcal{W},1} \\ &= \Phi_1 \circ (\pi_{\mathcal{E}}^- \oplus \pi_{\mathcal{E}}^-) \circ (\Psi_1 \oplus \Gamma_{\pi_1}) \circ \pi_{\mathcal{W},1} \circ \pi_{\mathcal{W},1} \\ &= \Phi_1 \circ (\pi_{\mathcal{E}}^- \oplus \pi_{\mathcal{E}}^-) \circ (\Psi_1 \oplus \Gamma_{\pi_1}) \circ \pi_{\mathcal{W},1} \\ &= \Phi_1 \circ \Psi_1 = \Pi_1 \end{aligned}$$

Similarly, one computes  $\pi_{\mathcal{W},1} \circ \Pi_1 = \Pi_1$ . It follows that

$$(\pi_{\mathcal{W},1} - \Pi_1) \circ \Pi_1 = \Pi_1 - \Pi_1^2 = 0$$



and similarly  $\Pi_1 \circ (\pi_{\mathcal{W},1} - \Pi_1) = 0$ . Moreover,

$$\begin{aligned} (\pi_{\mathcal{W},1} - \Pi_1)^2 &= \pi_{\mathcal{W},1}^2 - \pi_{\mathcal{W},1} \circ \Pi_1 - \Pi_1 \circ \pi_{\mathcal{W},1} + \Pi_1^2 \\ &= \pi_{\mathcal{W},1} - 2\Pi_1 + \Pi_1 = \pi_{\mathcal{W},1} - \Pi_1 \end{aligned}$$

This shows that  $\Pi_1$  and  $\pi_{\mathcal{W},1} - \Pi_1$  are mutually orthogonal idempotents, as required.

Now, define

$$\mathcal{L}_1 := (\mathcal{W}, \pi_{\mathcal{W},1} - \Pi_1)$$

**Step 4:** We need to show that  $\mathcal{L}_1$  is a sum of Lefschetz motives. For this, note that it suffices by [39] Corollary 3.5 to show that for every field extension  $k \subset \Omega$ , the Chow group

$$CH^j((\mathcal{L}_1)_\Omega) = (\pi_{\mathcal{W},1} - \Pi_1)_* CH^*(\mathcal{W}_\Omega)$$

is a finite-dimensional  $\mathbb{Q}$  vector space for all  $j$ . This amounts to proving that

$$\Pi_{1*} CH^j(\mathcal{W}_\Omega) \subset \pi_{\mathcal{W},1*} CH^*(\mathcal{W}_\Omega) \text{ has finite index for all } j \quad (3.16)$$

and all field extensions  $k \subset \Omega$ .

To this last end, we first unravel  $\Pi_{1*} CH^j(\mathcal{W}_\Omega)$ . Indeed, we have (by definition):

$$\Pi_1 = \Phi_1 \circ \Psi_1$$

By Step 1,  $\Psi_1$  is split-surjective, so we see that

$$\begin{aligned}\Pi_{1*}CH^j(\mathcal{W}_\Omega) &= \Phi_{1*}(CH^j(\mathfrak{h}(\mathcal{E}_\Omega)^- \oplus \mathfrak{h}(\mathcal{E}_\Omega)^-(1))) \\ &= \Phi_{1*}(CH^j(\mathcal{E}_\Omega)^- \oplus CH^{j-1}(\mathcal{E}_\Omega)^-)\end{aligned}$$

Thus, in order to obtain the statement of (3.16), it suffices to prove that the image of

$$\Phi_{1*} : CH^j(\mathcal{E}_\Omega)^- \oplus CH^{j-1}(\mathcal{E}_\Omega)^- \rightarrow CH^j(\mathcal{W}_\Omega)$$

has finite index in  $\pi_{\mathcal{W},1*}CH^j(\mathcal{W}_\Omega)$ . We observe that this map is given by

$$\Phi_{1*} = \pi_{\mathcal{W},1*} \circ (\pi_1^* + s_{1*}) : CH^j(\mathcal{E}_\Omega)^- \oplus CH^{j-1}(\mathcal{E}_\Omega)^- \rightarrow CH^j(\mathcal{W}_\Omega) \quad (3.17)$$

Then, we have the following lemma, whose proof we postpone:

**Lemma 3.4.1.** *The image of the map*

$$\pi_1^* + s_{1*} : CH^j(\mathcal{E}_\Omega) \oplus CH^{j-1}(\mathcal{E}_\Omega) \rightarrow CH^j(\mathcal{W}_\Omega)^{g_2} \quad (3.18)$$

*has finite index in  $CH^j(\mathcal{W}_\Omega)^{g_2}$ , the invariant subspace of  $CH^j(\mathcal{W}_\Omega)$  for the action of  $g_2$ .*

We can use Lemma 3.4.1 in order to prove that the image of (3.17) has finite index in  $\pi_{\mathcal{W},1*}CH^j(\mathcal{W}_\Omega)$ , completing the proof of Step 4 and, hence, that of the proposition.

To use the lemma, first note that  $\pi_{\mathcal{W},1*}CH^j(\mathcal{W}_\Omega)$  is the anti-invariant subspace of

$CH^j(\mathcal{W}_\Omega)^{g_2}$  for the action of  $g_1$ . Further, using (3.4) we have:

$$(\pi_1^* + s_{1*}) \circ (-1)_{\mathcal{E}}^* = g_1^* \circ (\pi_1^* + s_{1*}),$$

which shows that  $\pi_1^* + s_{1*}$  is equivariant with respect to  $g_1$ . We then apply:

**Lemma 3.4.2.** *Suppose that  $V$  and  $W$  are  $\mathbb{Q}[\mathbb{Z}/2]$ -modules and  $\phi : V \rightarrow W$  is a map of  $\mathbb{Q}[\mathbb{Z}/2]$ -modules for which  $\phi(V) \subset W$  has finite index. Then,  $\phi(V^\pm) \subset W^\pm$  also has finite index.*

*Proof of Lemma 3.4.1.* We will drop the  $\Omega$  notation for convenience. We also make the following definition, which will simplify the notation of the proof.

**Definition 3.4.1.** *We say that  $\gamma \in CH^j(\mathcal{E})$  (or  $CH^j(\mathcal{W})$ ) is supported on the degenerate fibers if it is contained in the subspace:*

$$F_{\mathcal{E}}^j := \text{Im}\left(\bigoplus_{\sigma \in \Sigma} CH^{j-1}(f^{-1}(\sigma)) \rightarrow CH^j(\mathcal{E})\right)$$

$$(\text{or } F_{\mathcal{W}}^j := \text{Im}\left(\bigoplus_{\sigma \in \Sigma} CH^{j-1}(\pi^{-1}(\sigma)) \rightarrow CH^j(\mathcal{W})\right))$$

where the arrows  $CH^{j-1}(f^{-1}(\sigma)) \rightarrow CH^j(\mathcal{E})$  and  $CH^{j-1}(\pi^{-1}(\sigma)) \rightarrow CH^j(\mathcal{W})$  are the push-forwards via the inclusion maps  $f^{-1}(\sigma) \hookrightarrow \mathcal{E}$  and  $\pi^{-1}(\sigma) \hookrightarrow \mathcal{W}$ .

Using the localization sequence, we obtain the following commutative diagram with

rows exact:

$$\begin{array}{ccccc}
F_{\mathcal{E}}^j \oplus F_{\mathcal{E}}^{j-1} & \longrightarrow & CH^j(\mathcal{E}) \oplus CH^{j-1}(\mathcal{E}) & \twoheadrightarrow & CH^j(\mathring{\mathcal{E}}) \oplus CH^{j-1}(\mathring{\mathcal{E}}) \\
\downarrow & & \pi_1^* + s_{1*} \downarrow & & \pi_1^* + s_{1*} \downarrow \\
(F_{\mathcal{W}}^j)^{g_2} & \longrightarrow & CH^j(\mathcal{W})^{g_2} & \twoheadrightarrow & CH^j(\mathring{\mathcal{W}})^{g_2}
\end{array} \tag{3.19}$$

Indeed, certainly the right side of the diagram commutes. The exactness of the lower sequence is obtained by applying the functor  $(\ )^{g_2}$  to the usual localization sequence and noting that  $H^1(\mathbb{Z}/2, \ )$  vanishes for  $\mathbb{Q}$  vector spaces. The exactness of the rows forces the existence of the leftmost vertical arrow. (This circumvents defining a pull-back operation on possibly singular varieties.)

Then, using Lemma 3.3.2 (a), the components of the degenerate fibers of  $f : \mathcal{E} \rightarrow C$  are rational curves. It follows that the components of the degenerate fibers of  $\mathcal{E} \times_C \mathcal{E} \rightarrow C$  are rational surfaces. Hence, so are those of  $\pi : \mathcal{W} \rightarrow C$  since  $\mathcal{W}$  is obtained by blowing up points on  $\mathcal{E} \times_C \mathcal{E}$ . It follows that  $F_{\mathcal{W}}^j$  has finite rank (and, hence, so does  $(F_{\mathcal{W}}^j)^{g_2}$ ). A diagram chase then shows that the image of the middle vertical arrow of (3.19) has finite index if this is also true of the right vertical arrow. We can, in fact, show that this right vertical arrow is surjective using the following claim.  $\square$

**Claim:** The right vertical arrow in (3.19) is surjective:

$$\pi_1^* + s_{1*} : CH^j(\mathring{\mathcal{E}}) \oplus CH^{j-1}(\mathring{\mathcal{E}}) \rightarrow CH^j(\mathring{\mathcal{W}})^{g_2}$$

*Proof of Claim.* Let

$$\mathring{\mathcal{W}}_1 = \mathring{\mathcal{E}} \times_{\mathring{C}} \mathring{P}$$

where  $\mathring{P} = \mathring{\mathcal{E}}/(-1)_{\mathring{\mathcal{E}}}$ . Then, there is a quotient map  $q_1 : \mathring{\mathcal{W}} \rightarrow \mathring{\mathcal{W}}_1$ . Observe that

$$CH^j(\mathring{\mathcal{W}})^{g_2} = q_1^* CH^j(\mathring{\mathcal{W}}_1)$$

So, to prove the claim, it suffices to show that the map:

$$q_{1*} \circ (\pi_1^* + s_{1*}) : CH^j(\mathring{\mathcal{E}}) \oplus CH^{j-1}(\mathring{\mathcal{E}}) \rightarrow CH^j(\mathring{\mathcal{W}}_1) \quad (3.20)$$

is surjective. In order to prove this, we need to analyze the map  $q_{1*} \circ (\pi_1^* + s_{1*})$ . We observe that there is an induced projection  $\pi'_1 : \mathring{\mathcal{W}}_1 \rightarrow \mathring{\mathcal{E}}$  so that the diagram below commutes:

$$\begin{array}{ccc} \mathring{\mathcal{W}} & \xleftarrow{s_1} & \mathring{\mathcal{E}} \\ q_1 \downarrow & \searrow \pi_1 & \\ \mathring{\mathcal{W}}_1 & \xrightarrow{\pi'_1} & \mathring{\mathcal{E}} \\ & \swarrow s'_1 & \end{array} \quad (3.21)$$

where  $s'_1 := q_1 \circ s_1$  is a section of  $\pi'_1 : \mathring{\mathcal{W}}_1 \rightarrow \mathring{\mathcal{E}}$ . Thus, we have  $\pi_1^* = q_1^* \circ (\pi'_1)^*$ ; applying  $q_{1*}$  gives  $q_{1*} \circ \pi_1^* = 2(\pi'_1)^*$ . Thus, the map in (3.20) becomes:

$$2(\pi'_1)^* + s'_{1*} : CH^j(\mathring{\mathcal{E}}) \oplus CH^{j-1}(\mathring{\mathcal{E}}) \rightarrow CH^j(\mathring{\mathcal{W}}_1) \quad (3.22)$$

The task then becomes to show that (3.22) is surjective. To this end, and observe that  $\pi'_1 : \mathring{\mathcal{W}}_1 \rightarrow \mathring{\mathcal{E}}$  is a  $\mathbb{P}^1$ -bundle. Then, by [13] Theorem 3.3, it suffices to show that  $(\pi'_1)^* CH^j(\mathring{\mathcal{E}})$  and  $c_1(\mathcal{O}(1))$  lie in the image of (3.22) for some choice of invertible sheaf

$\mathcal{O}(1)$ . Certainly, this is true for  $(\pi'_1)^*CH^j(\mathring{\mathcal{W}}_1)$ . Also by [13] Theorem 3.3, we have

$$s'_1 = (\pi'_1)^*\gamma + m \cdot c_1(\mathcal{O}(1))$$

for some  $\gamma \in CH^1(\mathring{\mathcal{W}}_1)$  and  $m \in \mathbb{Q}$ . We note that  $m \neq 0$  or otherwise we would have  $s'_1 = (\pi'_1)^*\gamma$ . However,  $\pi'_{1*} \circ (\pi'_1)^* = 0$  while  $\pi'_{1*}(s'_1) = 1 \in CH^0(\mathcal{E})$ . Thus, solving for  $c_1(\mathcal{O}(1))$  and using the projection formula, we have

$$c_1(\mathcal{O}(1)) = \frac{-1}{m}(\pi'_1)^*(\gamma) + \frac{1}{m}s'_1 = 2(\pi'_1)^*\left(\frac{-1}{2m}\gamma\right) + s'_{1*}\left(\frac{1}{m}(s'_1)^*1\right)$$

This shows that  $c_1(\mathcal{O}(1))$  lies in the image of (3.22), as desired. Hence, (3.22) surjects. □

This concludes the proof of Step 4 and the proof of the proposition. □

### 3.4.3 The Kummer construction

Here, we will introduce a smooth projective threefold  $\mathcal{K} \rightarrow C$ , whose generic fiber is the Kummer surface of the generic fiber of  $\pi : \mathcal{W} \rightarrow C$ . We will use  $\mathcal{K}$  in order to obtain a decomposition of the motive of  $\mathcal{W}$ .

With the earlier notation, we have an action by  $g_3 \in G$  which acts by  $(-1)$  on each of the fibers of  $\pi : \mathcal{W} \rightarrow C$ . In order to obtain a quotient by  $g_3$ , we must first alter  $\mathcal{W}$  in a manner described by the following result:

**Proposition 3.4.2.** *There exists a nonsingular subscheme  $V \subset \mathcal{W}$  so that for  $\rho : \hat{\mathcal{W}} \rightarrow \mathcal{W}$ , the blow-up along  $V$ , we have:*

- (1)  $g_3$  lifts to an involution on  $\hat{\mathcal{W}}$ ,  $g_3$ ;
- (2) the fixed locus of  $g_3$  on  $\hat{\mathcal{W}}$  is a smooth divisor.

Thus,  $\mathcal{K} := \hat{\mathcal{W}}/g_3$  is a smooth projective threefold with a morphism  $\kappa : \mathcal{K} \rightarrow C$  whose generic fiber is the Kummer surface of  $\mathcal{E}_\eta \times_\eta \mathcal{E}_\eta$ . Denote the quotient map  $q : \hat{\mathcal{W}} \rightarrow \mathcal{K}$ .

*Proof.* We will take the subscheme  $V$  to be the fixed locus of  $g_3$  on  $\mathcal{W}$ . Since  $g_3$  fixes  $V$ , the universal property of blow-up shows that it lifts to an involution on the blow-up along  $V$ ,  $\rho : \hat{\mathcal{W}} \rightarrow \mathcal{W}$ . So, we first need to prove that  $V$  is smooth. To this end, the fixed locus of  $g_3$  on  $\mathcal{E} \times_C \mathcal{E}$  is precisely  $T \times_C T \subset \mathcal{E} \times_C \mathcal{E}$ , where  $T$  is the branch locus of  $g : \mathcal{E} \rightarrow P$ . Thus, the fixed locus of  $g_3$  must contain the strict transform of  $T \times_C T$  via the desingularization  $\epsilon : \mathcal{W} \rightarrow \mathcal{E} \times_C \mathcal{E}$ , which we denote by  $U$ . We can gain further insight into the fixed locus of  $g_3$  by analyzing the action of  $g_3$  on the exceptional divisors  $\epsilon^{-1}(e_1, e_2)$ , where  $e_i \in f^{-1}(\sigma)$  are singular points of the degenerate fibers through which  $T$  passes. This is done below:

**Lemma 3.4.3.** *The set of fixed points of  $g_3$  on  $\epsilon^{-1}(e_1, e_2)$  consists of 4 distinct points.*

*Proof.* We note that  $g(e_i)$  is contained on a single component of the fiber of  $P \rightarrow C$  over  $f(e_i)$  (Lemma 3.3.2 (d)). Now, let  $\hat{\mathcal{O}}_{P, g(e_i)}$  denote the completion of the local ring

of  $P$  at  $g(e_i)$ . Since  $P$  is smooth (Lemma 3.3.2 (c)), it follows then that

$$\hat{\mathcal{O}}_{P,g(e_i)} \cong k[[t, x]]$$

where  $t$  is a local parameter for  $C$  and  $x = 0$  is the equation defining the component containing  $g(e_i)$ . In  $\hat{\mathcal{O}}_{P,g(e_i)}$ ,  $g(T)$  is defined by  $t - x^2$  since  $g(T)$  is smooth and doubly ramified at  $g(e_i)$  (Lemma 3.3.2 (c)). Also, since  $\mathcal{E}$  is the double cover of  $P$  branched along  $g(T)$ , we have an isomorphism:

$$\hat{\mathcal{O}}_{\mathcal{E},e_i} \cong k[[t, x, y]]/(y^2 + x^2 - t) \cong k[[t, x, y]]/(xy - t)$$

The action of  $(-1)_{\mathcal{E}}$  on  $\hat{\mathcal{O}}_{\mathcal{E},e_i}$  under this identification then switches  $x$  and  $y$  and fixes  $t$ .

It follows that the completion of  $\mathcal{E} \times_C \mathcal{E}$  at  $(e_1, e_2)$  is isomorphic to

$$k[[t, x_1, y_1, x_2, y_2]]/(x_1y_1 - t, x_2y_2 - t) \cong k[[x_1, y_1, x_2, y_2]]/(x_1y_1 - x_2y_2) \quad (3.23)$$

On this ring,  $g_3 = (-1)_{\mathcal{E}} \times_C (-1)_{\mathcal{E}}$  acts by switching  $x_i$  and  $y_i$ . Thus, the action of  $g_3$  on the exceptional divisor  $\epsilon^{-1}((e_1, e_2))$  is given by its action on the projective cone of the associated ring of (3.23). This is the action on the quadric  $x_1y_1 - x_2y_2 = 0$  in  $\mathbb{P}^3$  given by switching  $x_i$  and  $y_i$ . Using the Segre imbedding, this is precisely the involution on  $\mathbb{P}^1 \times \mathbb{P}^1$  defined by

$$([x_1, y_1], [x_2, y_2]) \mapsto ([y_1, x_1], [y_2, x_2]) \quad (3.24)$$



This fixes precisely the 4 points

$$([1, \pm 1], [1, \pm 1])$$

as required. □

Since  $T$  is smooth and  $T \rightarrow C$  has ramification index  $\leq 2$  at all points (Lemma 3.3.2 (c)),  $T \times_C T$  is smooth everywhere except for nodes at  $(e_1, e_2)$ , with the notation above. It follows that the strict transform  $U$  intersects an exceptional divisor  $\epsilon^{-1}(e_1, e_2)$  in 2 distinct points, resolving the singular point at  $(e_1, e_2)$ . These give 2 of the 4 points described in the preceding claim. Thus,  $V$  consists of the smooth subscheme  $U$ , as well as the isolated points of the above lemma. This gives the required smoothness.

What remains is to show that the fixed locus  $g_3$  is  $\rho^{-1}(V)$ . First, we verify that  $g_3$  fixes  $\rho^{-1}(U)$ . Indeed, for every  $c \in \mathring{C}$ ,

$$\rho^{-1}(E_c \times E_c)$$

is the blowup of  $E_c \times E_c$  along the 2-torsion points. Inversion fixes the exceptional divisors  $\rho^{-1}(E_c \times E_c)$ , as in the construction of the Kummer surface of an Abelian surface.

Thus, we are left with the task of showing that  $\rho^{-1}(v)$  is fixed by  $g_3$  (for  $v \in V$  an isolated point). The action of  $g_3$  on this exceptional divisor is determined by the action of  $dg_3$  on  $T_v \epsilon^{-1}(e_1, e_2)$ , which in turn is determined by the differential of (3.24)

on  $\mathbb{P}^1 \times \mathbb{P}^1$ . It is straightforward to calculate that this differential is the identity at every point. Hence, we obtain the proposition.  $\square$

#### 3.4.4 The motive $\mathfrak{h}_{(3)}(\mathcal{W})$

We can now use the Kummer construction to better understand the motive  $\mathfrak{h}_{(3)}(\mathcal{W})$ .

We begin with the result below:

**Lemma 3.4.4.**  *$\mathfrak{h}_{(3)}(\mathcal{W})$  is a summand of  $\mathfrak{h}(\mathcal{K})$ ; that is, there exists  $\mathcal{R} \in \mathcal{M}_k$  for which:*

$$\mathfrak{h}(\mathcal{K}) \cong \mathfrak{h}_{(3)}(\mathcal{W}) \oplus \mathcal{R}$$

*Proof.* We do this in 3 steps:

**Step 1:** Define a map of motives

$$\Phi_3 := \frac{1}{2}\Gamma_q \circ {}^t\Gamma_\rho \circ \pi_{\mathcal{W},3} : \mathfrak{h}_{(3)}(\mathcal{W}) \rightarrow \mathfrak{h}(\mathcal{K})$$

which we claim to be split-injective with left inverse:

$$\Psi_3 := \pi_{\mathcal{W},3} \circ \Gamma_\rho \circ {}^t\Gamma_q : \mathfrak{h}(\mathcal{K}) \rightarrow \mathfrak{h}_{(3)}(\mathcal{W})$$

To verify this, we need to check that  $\Psi_3 \circ \Phi_3 = \pi_{\mathcal{W},3}$ . Indeed, we have

$$\begin{aligned}
\Psi_3 \circ \Phi_3 &= \frac{1}{2} \pi_{\mathcal{W},3} \circ \Gamma_\rho \circ {}^t\Gamma_q \circ \Gamma_q \circ {}^t\Gamma_\rho \circ \pi_{\mathcal{W},3} \\
&= \frac{1}{2} \pi_{\mathcal{W},3} \circ \Gamma_\rho \circ (\Delta_{\hat{\mathcal{W}}} + {}^t\Gamma_{g_3}) \circ {}^t\Gamma_\rho \circ \pi_{\mathcal{W},3} \\
&= \frac{1}{2} \pi_{\mathcal{W},3} \circ (\Delta_{\mathcal{W}} + {}^t\Gamma_{g_3}) \circ \Gamma_\rho \circ {}^t\Gamma_\rho \circ \pi_{\mathcal{W},3} \\
&= \frac{1}{2} \pi_{\mathcal{W},3} \circ (\Delta_{\mathcal{W}} + {}^t\Gamma_{g_3}) \circ \pi_{\mathcal{W},3} \\
&= \pi_{\mathcal{W},3}^3 = \pi_{\mathcal{W},3}
\end{aligned} \tag{3.25}$$

where the third equality follows from the fact that  $\rho \circ g_3 = g_3 \circ \rho$  and the fourth from the fact that

$$\Gamma_\rho \circ {}^t\Gamma_\rho = (\rho \times \rho)_*(\Delta_{\hat{\mathcal{W}}}) = \Delta_{\mathcal{W}}$$

since  $\rho$  is birational. Thus,  $\Phi_3$  is split-injective, which concludes Step 1.

**Step 2:**  $\Pi_3 := \Phi_3 \circ \Psi_3$  is an idempotent. Moreover,  $\Pi'_3 := \Delta_{\mathcal{K}} - \Pi_3$  is also an idempotent which is orthogonal to  $\Pi_3$ .

For the first statement, we note from above that  $\Psi_3 \circ \Phi_3 = \pi_{\mathcal{W},3}$ . We also observe that  $\Phi_3 \circ \pi_{\mathcal{W},3} = \Phi_3$ . It follows then that:

$$\Pi_3^2 = \Phi_3 \circ \Psi_3 \circ \Phi_3 \circ \Psi_3 = \Phi_3 \circ \pi_{\mathcal{W},3} \circ \Psi_3 = \Phi_3 \circ \Psi_3 = \Pi_3$$

The second statement of this step is then straightforward to verify.

Step 2 then gives a decomposition of motives:

$$\mathfrak{h}(\mathcal{K}) = (\mathcal{K}, \Pi_3) \oplus (\mathcal{K}, \Pi'_3)$$

**Step 3:** There is an isomorphism:

$$\Pi_3 \circ \Phi_3 \circ \pi_{\mathcal{W},3} : \mathfrak{h}_{(3)}(\mathcal{W}) \rightarrow (\mathcal{K}, \Pi_3)$$

with inverse  $\pi_{\mathcal{W},3} \circ \Psi_3 \circ \Pi_3 = \Psi_3 \circ \Pi_3$ .

Using 3.25, one readily checks the required identities:

$$\pi_{\mathcal{W},3} \circ \Psi_3 \circ \Pi_3 \circ \Phi_3 \circ \pi_{\mathcal{W},3} = \pi_{\mathcal{W},3}$$

$$\Pi_3 \circ \Phi_3 \circ \pi_{\mathcal{W},3} \circ \Psi_3 \circ \Pi_3 = \Pi_3$$

as was done for (3.15). Let  $\mathcal{R} := (\mathcal{W}, \Pi'_3)$  be the complementary motive. This completes the proof.

□

### 3.5 Proof of Theorem 3.1.1

We must first verify that  $\mathcal{W}$  satisfies the Murre Conjectures (A), (B), and (D). For this we use Proposition 2.4.1 to reduce this to showing that these hold for  $\mathfrak{h}_{(i)}(\mathcal{W})$  for each  $i$ . This can be done in 2 cases.

**Case 1:  $i = 1, 2$**  Using Proposition 3.4.1, there is an isomorphism:

$$\mathfrak{h}_{(i)}(\mathcal{W}) \cong \mathfrak{h}(\mathcal{E})^- \oplus (\mathfrak{h}(\mathcal{E})^- \otimes \mathbb{L}) \oplus \mathcal{L}_i \quad (3.26)$$

Using Proposition 2.4.1 again, we are reduced to proving that  $\mathfrak{h}(\mathcal{E})^-$  satisfies (A), (B) and (D). To do this, we first verify that  $\mathfrak{h}(\mathcal{E})^-$  possesses a Chow-Künneth decomposition (Conjecture (A)). For this, it suffices to show that  $H^j(\mathfrak{h}(\mathcal{E})^-) = 0$  for  $j \neq 2$  (since a motive whose cohomology vanishes in all except 1 degree has a Chow-Künneth decomposition trivially).

**Lemma 3.5.1.**  *$H^j(\mathfrak{h}(\mathcal{E})^-) = 0$  for  $j \neq 2$  for singular or  $\ell$ -adic cohomology.*

*Proof.* Since  $H^j(\mathfrak{h}(\mathcal{E})^-) = H^j(\mathcal{E})^-$ , we need to show that  $H^j(\mathcal{E})^- = 0$  for  $j \neq 2$ . To this end, we note that  $f^* : H^j(C) \rightarrow H^j(\mathcal{E})$  is an isomorphism for  $j = 0, 1$  since there is an isomorphism of Albanese varieties:

$$\text{Alb}(f) : \text{Alb}(\mathcal{E}) \xrightarrow{\cong} \text{Alb}(C) \quad (3.27)$$

Since  $f^*(H^j(C)) \subset H^j(\mathcal{E})^+$ , we obtain the result for  $j = 0, 1$ . By Poincaré duality, the push-forward  $f_{i,*} : H^j(\mathcal{E})(1) \rightarrow H^{j-2}(C)$  is an isomorphism for  $j = 3, 4$ . Since

$$\ker(f_* : H^j(\mathcal{E})(1) \rightarrow H^{j-2}(C)) = H^j(\mathcal{E})^-,$$

it follows that  $H^j(\mathfrak{h}(\mathcal{E})^-) = 0$ . Hence, the lemma. □

We note that verifying Conjectures (B) and (D) on  $\mathfrak{h}(\mathcal{E})^-$  means checking that

- $CH^0(\mathfrak{h}(\mathcal{E})^-) = CH^0(\mathcal{E})^- = 0$
- $CH_{hom}^1(\mathfrak{h}(\mathcal{E})^-) = CH_{hom}^1(\mathcal{E})^- = 0$

since  $\mathfrak{h}(\mathcal{E})^- = \mathfrak{h}^2(\mathcal{E})^-$ . To verify these, note that there is an isotypic decomposition:

$$CH^j(\mathcal{E}) = CH^j(\mathcal{E})^+ \oplus CH^j(\mathcal{E})^- = g^*CH^j(P) \oplus CH^j(\mathcal{E})^-$$

where the second equality uses Remark 3.3.2. Since  $g^*CH^0(P) = CH^0(\mathcal{E})$ , it follows that  $CH^0(\mathcal{E})^- = 0$ , which verifies the first item. For the second item, note that

$$f^* : CH_{hom}^1(C) \xrightarrow[\cong]{\bar{f}^*} CH_{hom}^1(P) \xrightarrow{g^*} CH_{hom}^1(\mathcal{E})$$

is an isomorphism using (3.27). Thus,  $g^*$  is an isomorphism on  $CH_{hom}^1$ , which implies that  $CH_{hom}^1(\mathcal{E})^- = 0$ , as required.

**Case 2:  $i=3$**  We will prove the Chow-Künneth Conjecture for  $\mathfrak{h}_{(3)}(\mathcal{W})$  using the 2 lemmas below:

**Lemma 3.5.2.** *[Vial, [40] Theorem 4.2] Let  $M = (X, \pi) \in \mathcal{M}_k$  with  $\dim(X) = 3$ ,  $k$  algebraically closed and  $H^*$  singular cohomology with  $\mathbb{Q}$  coefficients (when  $k = \mathbb{C}$ ) or  $\ell$ -adic cohomology. Suppose that the intersection product:*

$$NS(M) \otimes N_1(M) \rightarrow \mathbb{Q} \tag{3.28}$$

is nondegenerate for  $N_1(M) = \text{Im}(cl^2 : CH^2(M) \rightarrow H^4(M))$ . Then, there exist a Chow-Künneth decomposition of  $M$  satisfying:

$$(a) \mathfrak{h}^0(M) = \mathbf{1}, \mathfrak{h}^6(M) = \mathbb{L}^3$$

$$(b) \mathfrak{h}^1(M) \otimes \mathbb{L}^2 \cong \mathfrak{h}^5(M)$$

$$(c) CH^*(\mathfrak{h}^1(M)) = CH_{hom}^1(M)$$

$$(d) \mathfrak{h}^{2i}(M) \cong (\mathbb{L}^i)^{\oplus \rho} \text{ for } i = 1, 2 \text{ and } \rho = \dim_{\mathbb{Q}}(NS(M)_{\mathbb{Q}})$$

**Lemma 3.5.3.** *The intersection product (3.28) is nondegenerate when  $M = \mathfrak{h}_{(3)}(\mathcal{W})$ .*

*Proof of Lemma 3.5.3.* We first prove that the cycle class map

$$cl^1 : CH^1(M) \otimes K \rightarrow H^2(M) \tag{3.29}$$

is surjective when  $H^*$  is singular cohomology with  $\mathbb{Q}$  coefficients or  $\ell$ -adic cohomology.

We first do this in the case that  $k = \mathbb{C}$  (and  $H^*$  is singular cohomology) and then extend to char  $k = 0$ . Using the functoriality of the cycle class map, we have

$$\begin{array}{ccc} CH^1(\mathcal{K}) & \xrightarrow{cl^1} & H^2(\mathcal{K}) \\ (\Phi_3 \circ \Psi_3)_* \downarrow & & (\Phi_3 \circ \Psi_3)_* \downarrow \\ CH^1(\mathfrak{h}_{(3)}(\mathcal{W})) & \xrightarrow{cl^1} & H^2(\mathfrak{h}_{(3)}(\mathcal{W})) \end{array} \tag{3.30}$$

so that it suffices to check that the top vertical arrow is surjective; i.e., that

$$NS(\mathcal{K}) = H^2(\mathcal{K})(1)$$

Using the Lefschetz (1, 1) theorem, this is reduced to showing that  $H^{(2,0)}(\mathcal{K}) = 0$ . This follows from the result below since  $\kappa : \mathcal{K} \rightarrow C$  has non-isomorphic fibers.

**Proposition 3.5.1** (Tankeev, [38] Proposition 5.3). *Let  $\mathcal{K}$  be a smooth projective three-fold over  $\mathbb{C}$  with a non-isotrivial morphism  $\kappa : \mathcal{K} \rightarrow C$  onto a smooth projective curve whose general fiber is a K3 surface. Then,  $H^{(2,0)}(\mathcal{K}) = 0$ .*

To extend this to  $\ell$ -adic cohomology, note that this follows immediately for  $k = \mathbb{C}$  using the comparison isomorphism between singular and étale cohomology. Since the Néron-Severi group and  $\ell$ -adic cohomology are stable under algebraically closed extension, the result also extends to any  $k$  of characteristic 0. (The reader will recall that the base field is assumed to be algebraically closed and of characteristic 0.) This proves the surjectivity of (3.29). Because of this, the lemma reduces to proving:

$$(\ , \ ) : H^2(\mathfrak{h}_{(3)}(\mathcal{W})) \otimes H^4(\mathfrak{h}_{(3)}(\mathcal{W})) \rightarrow K$$

is non-degenerate for singular or  $\ell$ -adic cohomology. This is exactly the statement of Lemma 3.5.4 given below. □

**Lemma 3.5.4.** *The cup product*

$$(\ , \ ) : H^2(\mathfrak{h}_{(3)}(\mathcal{W})) \otimes H^4(\mathfrak{h}_{(3)}(\mathcal{W})) \rightarrow K \tag{3.31}$$

*is nondegenerate for singular or  $\ell$ -adic cohomology.*



*Proof.* By Poincaré duality, we have nondegeneracy of the cup product:

$$(\ , \ ) : H^2(\mathcal{W}) \otimes H^4(\mathcal{W}) \rightarrow K \quad (3.32)$$

To prove the lemma, we need to show that

$$(\ , \ ) : H^2(\mathcal{W})^{g_3} \otimes H^4(\mathcal{W})^{g_3} \rightarrow K \quad (3.33)$$

is nondegenerate. So, let  $\alpha \in H^2(\mathcal{W})^{g_3}$  be a nonzero class. Such classes exist since, for instance,  $NS(\mathcal{W})^{g_3}$  contains the class of a smooth fiber of  $\mathcal{W} \rightarrow C$ . We need to find  $\beta \in H^4(\mathcal{W})^{g_3}$  for which  $(\alpha, \beta) \neq 0$ . Indeed, by the nondegeneracy of (3.32), there exists some  $\gamma \in H^4(\mathcal{W})$ . We set

$$\beta = 2\pi_{\mathcal{W}, 3*}\gamma = \gamma + g_3^*\gamma$$

and we compute

$$(\alpha, \beta) = (\alpha, \gamma + g_3^*\gamma) = (\alpha, \gamma) + (\alpha, g_3^*\gamma) = (\alpha, \gamma) + (g_3^*\alpha, \gamma) = 2(\alpha, \gamma) \neq 0$$

where the third equality follows from the fact that  $(g_3^* \ , \ ) = ( \ , \ g_3^* )$  ( $g_3$  is an involution) and the fourth follows from the fact that  $\alpha \in H^2(\mathcal{W})^{g_3}$ . It follows that (3.33) is nondegenerate, which proves the lemma.  $\square$

Using Lemma 3.5.2 conditions (a), (b) and (d), we have the following:

$$\mathfrak{h}_{(3)}^0(\mathcal{W}) \cong \mathbb{1}, \quad \mathfrak{h}_{(3)}^6(\mathcal{W}) \cong \mathbb{L}^3,$$

$$\mathfrak{h}_{(3)}^2(\mathcal{W}) \cong \mathbb{L}^{\oplus m}, \quad \mathfrak{h}_{(3)}^4(\mathcal{W}) \cong (\mathbb{L}^2)^{\oplus m}, \quad \mathfrak{h}_{(3)}^5(\mathcal{W}) \cong \mathfrak{h}_{(3)}^1(\mathcal{W}) \otimes \mathbb{L}^2$$

where  $m = \dim_{\mathbb{Q}}(NS(M))$ . As a consequence, we see that  $CH^1(\mathfrak{h}_{(3)}^2(\mathcal{W}))$  has rank  $m$ .

Since  $CH^1(\mathfrak{h}_{(3)}^1(\mathcal{W})) = CH_{hom}^1(\mathfrak{h}_{(3)}(\mathcal{W}))$  from condition (c) of the lemma and since

$$CH^1(\mathfrak{h}_{(3)}^1(\mathcal{W})) \cap CH^1(\mathfrak{h}_{(3)}^2(\mathcal{W})) = 0$$

it follows that

$$CH^1(\mathfrak{h}_{(3)}^2(\mathcal{W})) \cong NS(\mathfrak{h}_{(3)}(\mathcal{W})) \tag{3.34}$$

so that

$$CH^1(\mathfrak{h}_{(3)}^2(\mathcal{W})) = CH^1(\mathfrak{h}_{(3)}^1(\mathcal{W})) \oplus CH^1(\mathfrak{h}_{(3)}^2(\mathcal{W})) \tag{3.35}$$

From condition (c), we also see that

$$CH^j(\mathfrak{h}_{(3)}^1(\mathcal{W})) = 0 \text{ for } j \neq 1 \tag{3.36}$$

To finish the proof, we need to prove Murre Conjectures (B) and (D) hold for  $\mathfrak{h}_{(3)}(\mathcal{W})$ .

This amounts to checking the following:

- (I)  $CH_{hom}^i(\mathfrak{h}_{(3)}^{2i}(\mathcal{W})) = 0$  for  $i = 1, 2, 3$
- (II)  $CH^1(\mathfrak{h}_{(3)}^j(\mathcal{W})) = 0$  for  $j = 0, 3, 4, 5, 6$
- (III)  $CH^2(\mathfrak{h}_{(3)}^j(\mathcal{W})) = 0$  for  $j = 0, 1, 5, 6$
- (IV)  $CH^3(\mathfrak{h}_{(3)}^j(\mathcal{W})) = 0$  for  $j = 0, 1, 2$

From (3.5),  $\mathfrak{h}_{(3)}^{2i}(\mathcal{W})$  is a sum of Lefschetz motives, so (I) follows immediately. From (3.35), we see that:

$$CH^1(\mathfrak{h}_{(3)}(\mathcal{W})) = CH^1(\mathfrak{h}_{(3)}^1(\mathcal{W})) \oplus CH^1(\mathfrak{h}_{(3)}^2(\mathcal{W}))$$

so that (II) follows. We can also easily obtain (III) and (IV) when  $j$  is even. Indeed, we observe that  $\mathfrak{h}_{(3)}^{2i}(\mathcal{W})$  is a direct sum of the motive  $\mathbb{L}^i$ , which means that  $CH^2(\mathfrak{h}_{(3)}^{2i}(\mathcal{W})) = 0$  if  $i \neq 2$  and  $CH^3(\mathfrak{h}_{(3)}^{2i}(\mathcal{W})) = 0$  if  $i \neq 3$ . We also have (III) and (IV) for  $j = 1$  using (3.36). Finally, we have

$$CH^2(\mathfrak{h}_{(3)}^5(\mathcal{W})) = CH^2(\mathfrak{h}_{(3)}^1(\mathcal{W}) \otimes \mathbb{L}^2) = CH^0(\mathfrak{h}_{(3)}^1(\mathcal{W})) = 0$$

where the first equality follows from (3.5) and the second from (3.36). This completes the verification of the Murre Conjectures for  $\mathfrak{h}_{(3)}(\mathcal{W})$ .

### 3.6 Proof of Theorem 3.1.2

We recall for the reader that  $C = \mathbb{P}^1$  in this theorem and that  $\mathcal{E} \xrightarrow{f} \mathbb{P}^1$  is a rational (semistable) elliptic surface with section. Let  $\mathcal{W}$  be the desingularization of  $\mathcal{E} \times_{\mathbb{P}^1} \mathcal{E}$  obtained by blowing up the double points. We would like to show that the image of

$$CH^1(\mathcal{W}) \otimes CH^2(\mathcal{W}) \rightarrow CH^3(\mathcal{W})$$

has rank 1. We follow the proof of [7] Theorem 1 and begin with the following lemma:

**Lemma 3.6.1.** *There exists a well-defined  $\beta \in CH^3(\mathcal{W})$  that is represented by a point on an irreducible rational surface.*

*Proof.* Let  $\beta, \beta'$  be 2 points on irreducible rational surfaces  $R$  and  $R'$  (respectively). We would like to show that  $\beta \sim_{rat} \beta'$ . Now, we can assume that both  $R$  and  $R'$  are birational onto their images under the blowup

$$\epsilon : \mathcal{W} \rightarrow \mathcal{E} \times_{\mathbb{P}^1} \mathcal{E}$$

Indeed, if say  $R$  is an exceptional divisor, note that the fibers of  $\mathcal{W} \rightarrow \mathbb{P}^1$  are connected, so there is some other rational surface  $D$  on the same fiber as  $R$  such that  $D \cap R \neq \emptyset$ . Since  $CH^2$  of a rational surface has rank 1, we can move  $\beta$  to  $D \cap R$  and, hence, to  $D$ . Thus, we can assume that both  $R$  and  $R'$  are birational onto their images under  $\epsilon$ . Since

$$\epsilon_* : CH^3(\mathcal{W}) \rightarrow CH^3(\mathcal{E} \times_{\mathbb{P}^1} \mathcal{E})$$

is an isomorphism, it suffices to show that  $\epsilon_*\beta \sim_{rat} \epsilon_*\beta'$ . However,  $R$  and  $R'$  are birational onto  $\epsilon(R)$  and  $\epsilon(R')$ , so we are reduced to proving the lemma for  $\mathcal{E} \times_{\mathbb{P}^1} \mathcal{E}$ ; that is, we need to show that there is a well-defined  $\alpha \in CH^3(\mathcal{E} \times_{\mathbb{P}^1} \mathcal{E})$  which is represented by a point on a rational surface of  $\mathcal{E} \times_{\mathbb{P}^1} \mathcal{E}$ .

So, let  $\alpha$  and  $\alpha'$  be 2 points on irreducible rational surfaces  $R$  and  $R'$  (respectively).

We would like to show that  $\alpha \sim_{rat} \alpha'$ . This can be done in 2 cases.

**Case 1:**  $R \cap R' \neq \emptyset$ .  $CH^2$  has rank 1 for a rational surface. We can then move  $\beta$  to

$R \cap R'$  (and, hence, to  $R'$ ). So,  $\alpha \sim \alpha'$ .

**Case 2:**  $R \cap R' = \emptyset$ . We need to find some sequence of rational surfaces connecting  $R$  and  $R'$ . This case will then be complete by Case 1. For this, we have the following claim, whose proof we postpone.

**Claim:** There exists an ample divisor  $H$  that is a sum of rational surfaces.

Let  $H$  be an ample divisor on  $\mathcal{W}$  as in the claim. Then, observe that  $H \cap R \neq \emptyset$ ,  $H \cap R' \neq \emptyset$ . Since  $H$  is connected, this gives a chain of rational surfaces:

$$R_0 = R, R_1, \dots, R_n = R'$$

such that  $R_i \cap R_{i+1} \neq \emptyset$ , as required.

*Proof of claim.* From Remark 2.2 of [7], there is an ample divisor  $h$  on  $\mathcal{E}$  which is a sum of rational curves. In fact, one can take

$$h = m \cdot s + \sum_i n_i F_i \tag{3.37}$$

where  $s$  is the image of a section  $s : \mathbb{P}^1 \rightarrow \mathcal{E}$  and the sum ranges over all components  $F_i$  of all singular fibers. (Note that  $m, n_i > 0$ .) We can then take

$$H = \bar{\pi}_1^*(h) + \bar{\pi}_2^*(h)$$

where  $\bar{\pi}_i : \mathcal{E} \times_{\mathbb{P}^1} \mathcal{E} \rightarrow \mathcal{E}$  denotes the projection onto the  $i^{\text{th}}$  factor. Let  $j : \mathcal{E} \times_{\mathbb{P}^1} \mathcal{E} \rightarrow \mathcal{E} \times \mathcal{E}$  be the inclusion. Then,

$$H = j^*(p_1^*(h) + p_2^*(h))$$

where  $p_i : \mathcal{E} \times \mathcal{E} \rightarrow \mathcal{E}$  denotes the projection onto the  $i^{\text{th}}$  factor. Since  $p_1^*(h) + p_2^*(h)$  is ample, so is  $H$ . Finally, we need to check that  $\bar{\pi}_1^*(h)$  is a sum of rational surfaces (and similarly for  $\bar{\pi}_2^*(h)$ ). Indeed, we see from (3.37) that

$$\bar{\pi}_1^*(h) = m \cdot (s \times_{\mathbb{P}^1} \mathcal{E}) + \sum_i n_i F_i \times f^{-1}(f(F_i))$$

By assumption,  $s \times_{\mathbb{P}^1} \mathcal{E} \cong \mathcal{E}$  is rational. Moreover,  $F_i \times f^{-1}(f(F_i))$  is a sum of rational surfaces. This gives the claim.  $\square$

What remains is then to show that

$$\text{Im}(CH^1(\mathcal{W}) \otimes CH^2(\mathcal{W}) \rightarrow CH^3(\mathcal{W})) = \mathbb{Q} \cdot \beta \quad (3.38)$$

To this end, we will need the following lemma:

**Lemma 3.6.2.**  *$CH^1(\mathcal{W})$  is generated by rational surfaces.*

*Proof.* The localization sequence gives:

$$0 \rightarrow F_{\mathcal{W}}^1 \rightarrow CH^1(\mathcal{W}) \rightarrow CH^1(\mathcal{W}_\eta) \rightarrow 0$$

where  $\eta = \text{Spec}(k(t))$  is the generic point. Since all the components of the singular

fibers of  $\mathcal{W} \rightarrow \mathbb{P}^1$  are rational surfaces,  $F_{\mathcal{W}}^1$  is generated by rational surfaces. So what remains is to show that there exists a subset of rational surfaces

$$\mathcal{S} = \{S_i\} \subset CH^1(\mathcal{W}) \quad (3.39)$$

whose image under

$$CH^1(\mathcal{W}) \rightarrow CH^1(\mathcal{W}_\eta)$$

generates  $CH^1(\mathcal{W}_\eta) = CH^1(\mathcal{E}_\eta \times_\eta \mathcal{E}_\eta)$ . Then, we note that the representability of the Picard functor gives an exact sequence:

$$0 \rightarrow \pi_1^* CH^1(\mathcal{E}_\eta) \oplus \pi_1^* CH^1(\mathcal{E}_\eta) \rightarrow CH^1(\mathcal{E}_\eta \times_\eta \mathcal{E}_\eta) \rightarrow Hom_\eta(\mathcal{E}_\eta, \mathcal{E}_\eta) \otimes \mathbb{Q} \rightarrow 0 \quad (3.40)$$

where we identify  $\mathcal{E}_\eta \cong Pic^0(\mathcal{E}_\eta)$  since  $\mathcal{E}_\eta(\eta) \neq \emptyset$ . We note that in (3.40),

$$\Gamma_{\phi_\eta} \in CH^1(\mathcal{E}_\eta \times_\eta \mathcal{E}_\eta)$$

maps to  $\phi_\eta \in Hom_\eta(\mathcal{E}_\eta, \mathcal{E}_\eta)$ . Thus, we can generate  $CH^1(\mathcal{E}_\eta \times_\eta \mathcal{E}_\eta)$  using  $\pi_1^* CH^1(\mathcal{E}_\eta)$ ,  $\pi_1^* CH^1(\mathcal{E}_\eta)$ , and  $\{\Gamma_{\phi_\eta}\}$ . We also observe that there is an isomorphism:

$$Hom_{\mathbb{P}^1}(\mathring{\mathcal{E}}, \mathring{\mathcal{E}}) \xrightarrow{\cong} Hom_\eta(\mathcal{E}_\eta, \mathcal{E}_\eta)$$

where  $\mathring{\mathbb{P}}^1 \subset \mathbb{P}^1$  denotes the maximal subscheme over which  $f : \mathcal{E} \rightarrow \mathbb{P}^1$  is smooth. This isomorphism follows from the fact that  $f : \mathring{\mathcal{E}} \rightarrow \mathring{\mathbb{P}}^1$  is the Néron model of  $\mathcal{E}_\eta$  by Cor. 1.4

of [2]. So, for any  $\phi_\eta \in \text{Hom}_\eta(\mathcal{E}_\eta, \mathcal{E}_\eta)$ , there exists a unique lift  $\phi \in \text{Hom}_{\mathbb{P}^1}(\mathring{\mathcal{E}}, \mathring{\mathcal{E}})$ . We see that  $\Gamma_\phi \in CH^1(\mathring{\mathcal{E}} \times_{\mathbb{P}^1} \mathring{\mathcal{E}})$  is a rational surface and, hence, so is  $\overline{\Gamma_\phi} \in CH^1(\mathcal{E} \times_{\mathbb{P}^1} \mathcal{E})$ . Thus, we select a generating set for  $\{\gamma_i\} \subset CH^1(\mathcal{E})$  consisting of rational curves (which exists by Remark 2.2 of [7]). Then, we can set

$$\mathcal{S} := \{\pi_1^* \gamma_i, \pi_2^* \gamma_i, \overline{\Gamma_\phi}\}$$

This gives  $\mathcal{S}$  as in (3.39) and proves the claim. □

In order to obtain (3.38), it suffices by the previous claim to prove that

$$R \cdot \gamma \in \mathbb{Q} \cdot \beta$$

for  $R$  a rational surface and  $\gamma \in CH^2(\mathcal{W})$ . Let  $j : R \hookrightarrow \mathcal{W}$  be the inclusion. Then, the projection formula gives:

$$R \cdot \gamma = j_* j^* \gamma$$

Since  $R$  is rational  $j^* \gamma \sim_{rat} m \cdot p$  for some  $m \in \mathbb{Q}$  and  $p$  a point on  $R$ . But

$$j_*(p) = \beta \in CH^3(\mathcal{W})$$

using Lemma 3.6.1. This completes the proof. □



## 4

# The Motive of the Fano Surface

Let  $k$  be a field with  $\text{char } k \neq 2$  and let  $X \subset \mathbb{P}^4$  be a smooth cubic threefold. We denote by  $S(X)$  (or just  $S$ ) the Fano variety of lines in  $X$ . This is known to be a smooth, connected projective surface of general type. It turns out that this surface possesses a great many remarkable properties. It is known, for instance, that the Albanese map is an imbedding  $i : S \hookrightarrow A := \text{Alb}(S)$  and that the pull-back  $H^2(A) \xrightarrow{i^*} H^2(S)$  is an isomorphism. In this chapter, we will prove a motivic version of this isomorphism. More precisely, we have the following result:

**Theorem 4.0.1.** *The morphism  ${}^t\Gamma_i : \mathfrak{h}(A) \rightarrow \mathfrak{h}(S)$  is split-surjective.*

This will show that the motive of the Fano surface is *finite-dimensional in the sense of Kimura*. The primary known examples of surfaces with finite-dimensional motive are those for which either:

- (i) The Chow group of nullhomologous 0-cycles is representable.

(ii) There exists a dominant rational map from a product of smooth projective curves.

Since  $p_g(S) > 0$ , Mumford's Theorem ([39] Theorem 3.13) shows that  $CH_{hom}^2(S)$  is not representable (when  $k = \mathbb{C}$ ). Moreover, a recent result in [34] shows that  $S$  is not dominated by a product of curves. To the author's knowledge, the Fano surface is the first example of a surface with finite-dimensional motive for which neither (i) nor (ii) holds. One reason for interest in finite-dimensionality is that if  $M \in \mathcal{M}_k$  is finite-dimensional, then any morphism  $f : M \rightarrow M$  that induces an isomorphism on cohomology is actually an isomorphism of motives. This will be important to proving:

**Theorem 4.0.2.** *The pull-back along the Albanese imbedding  $i : S \hookrightarrow A$  induces an isomorphism  $\mathfrak{h}^2(A) \xrightarrow{\cong} \mathfrak{h}^2(S)$  in  $\mathcal{M}_k$ .*

## 4.1 A General Principle

The following variant of the Manin principle will facilitate the proof of Theorem 4.0.1, as pointed out to the author by B. Kahn.

**Proposition 4.1.1.** *Let  $X$  and  $Y$  be smooth projective connected varieties over a field  $k$  with a morphism  $f : Y \rightarrow X$  such that  $f_{\Omega}^* : CH^i(X_{\Omega}) \rightarrow CH^i(Y_{\Omega})$  is surjective for all  $i$  and every algebraically closed extension  $k \subset \Omega$ . Then,  $\mathfrak{h}(f) : \mathfrak{h}(X) \dashv \mathfrak{h}(Y)$  is split-surjective.*

This principle has appeared in the literature in various guises (see, for instance, [19] Theorems 3.5 and 3.6). However, since this exact statement is difficult to find, we give a proof below, following [21]. We subdivide the proof into three claims:

**Claim:**  $f_L^* : CH^i(X_L) \rightarrow CH^i(Y_L)$  is surjective for all  $i$  and every extension  $k \subset L$ .

*Proof of Claim.* Let  $\beta \in CH^i(Y_L)$ . We need to find  $\alpha \in CH^i(X_L)$  for which  $f_L^*(\alpha) = \beta$ . By assumption,  $f_{\bar{L}}^* : CH^i(X_{\bar{L}}) \rightarrow CH^i(Y_{\bar{L}})$  is surjective, so there is some  $\bar{\alpha} \in CH^i(X_{\bar{L}})$  such that  $f_{\bar{L}}^*(\bar{\alpha}) = \beta_{\bar{L}}$ . Let  $L' \supset L$  be some finite Galois extension for which there is  $\alpha' \in CH^i(X_{L'})$  with  $\alpha'_{L'} = \bar{\alpha}$ . Indeed, there is some finite extension for which this holds; since Chow groups are unchanged by passing to a purely inseparable extension, we can assume that  $L'$  is a separable extension. This allows us to pass to a Galois extension. Then, let  $G = Gal(L'/L)$  and define  $\text{Tr}(\alpha') = \frac{1}{|G|} \sum_{g \in G} g^* \alpha'$ . By [13] Example 1.7.2.6, this yields  $\alpha \in CH^i(X_L)$  such that  $\alpha_{L'} = \text{Tr}(\alpha')$ . We compute:

$$(f_L^*(\alpha))_{L'} = f_{L'}^*(\alpha_{L'}) = f_{L'}^*(\text{Tr}(\alpha')) = \text{Tr}(f_{L'}^*(\alpha')) = \text{Tr}(\beta_{L'}) = \beta_{L'}$$

Since  $CH^i(X_L) \rightarrow CH^i(X_{L'})$  is injective, it follows that  $f_L^*(\alpha) = \beta$ , as desired.  $\square$

**Claim:**  $(f \times id_Z)^* : CH^i(X \times Z) \rightarrow CH^i(Y \times Z)$  is surjective for all  $i$  and all projective varieties  $Z$  over  $k$ .

*Proof of Claim.* One easily reduces to the case that  $Z$  is connected. We first observe that the pull-back

$$(f \times id_Z)^* : CH^i(X \times Z) \rightarrow CH^i(Y \times Z)$$

is defined even when  $Z$  is singular. Indeed,  $f$  is a local complete intersection morphism since the source and the target are smooth, and one readily checks that so is the (fiber)

product

$$f \times id_Z : X \times Z \rightarrow Y \times Z.$$

Thus, using [13] Chapter 6.6, we can define  $(f \times id_Z)^*$ . We then observe the following commutative diagram with rows exact:

$$\begin{array}{ccccccc} \varinjlim_W CH^{i-c}(X \times W) & \xrightarrow{(id_X \times j_W)^*} & CH^i(X \times Z) & \longrightarrow & CH^i(X_{k(Z)}) & \longrightarrow & 0 \\ (f \times id_W)^* \downarrow & & (f \times id_Z)^* \downarrow & & (f_{k(Z)})^* \downarrow & & \\ \varinjlim_W CH^{i-c}(Y \times W) & \xrightarrow{(id_Y \times j_W)^*} & CH^i(Y \times Z) & \longrightarrow & CH^i(Y_{k(Z)}) & \longrightarrow & 0 \end{array} \quad (4.1)$$

where  $k(Z)$  is the function field of  $Z$ , the limit ranges over subvarieties  $j_W : W \hookrightarrow Z$  with  $\dim(W) \leq \dim(Z)$  and  $c = \dim(Z) - \dim(W)$ . The localization sequence implies that the rows are exact. For the commutativity of the left diagram, note first that there is a Cartesian diagram:

$$\begin{array}{ccc} Y \times W & \xrightarrow{id_Y \times j_W} & Y \times Z \\ f \times id_W \downarrow & & f \times id_Z \downarrow \\ X \times W & \xrightarrow{id_X \times j_W} & X \times Z \end{array} \quad (4.2)$$

with the left and right vertical arrows of the same relative codimension. Then, using [13] Theorem 6.2, one obtains the desired commutativity.

The claim then follows by an induction argument on  $n = \dim(Z)$ . The case  $n = 0$  follows from the previous claim. Assume then that it holds for  $n - 1$ . Then, we note that the rightmost vertical arrow in (4.1) is surjective by the previous claim. The leftmost arrow is surjective by the inductive hypothesis. A diagram chase then shows that the statement is true for  $n$ . Hence, the claim.  $\square$

**Claim:**  ${}^t\Gamma_f$  possesses a right-inverse.

*Proof of Claim.* Taking  $Z = Y$  and  $i = \dim(Y)$  in the above claim, we obtain that

$$CH^i(Y \times X) \xrightarrow{(id_Y \times f)^*} CH^i(Y \times Y)$$

is surjective. So, there is some  $\gamma \in CH^i(X \times Y)$  for which  $(id_Y \times f)^*\gamma = \Delta_Y$ . Applying Liebermann's lemma then gives

$${}^t\Gamma_f \circ \gamma = (id_Y \times f)^*\gamma = \Delta_Y.$$

This is the desired result. □

## 4.2 Proof of Theorem 4.0.1

Assume that  $S$  has a  $k$ -rational point and let  $i : S \rightarrow A$  be the corresponding Albanese morphism. To prove the theorem, we need to show that the correspondence  ${}^t\Gamma_i \in \text{Cor}^0(A \times S)$  possesses a right-inverse. The general principle then shows that it suffices to prove that the pull-back  $i_\Omega^* : CH^j(A_\Omega) \rightarrow CH^j(S_\Omega)$  is surjective for  $j = 1, 2$  and all algebraically closed extensions  $k \subset \Omega$ .

For a smooth cubic threefold  $X$  over  $k$ ,  $S$  is the Hilbert scheme of  $X$  with Hilbert polynomial  $t + 1$ . We recall the following base change compatibility for Hilbert schemes

(see, for instance, [32]); i.e., for an extension  $k \subset \Omega$ , we have

$$S(X_\Omega) \cong S(X)_\Omega$$

Thus,  $S_\Omega$  is the Fano surface of lines of  $X_\Omega$ . Moreover, since the Albanese is compatible with base extension, we view  $i_\Omega : S_\Omega \rightarrow A_\Omega$  as an Albanese map.

**Proposition 4.2.1.** *For all algebraically closed extensions  $k \subset \Omega$ , the pull-back  $i_\Omega^* : CH^1(A_\Omega) \rightarrow CH^1(S_\Omega)$  is an isomorphism.*

*Proof.* Since  $i_\Omega : S_\Omega \rightarrow A_\Omega$  is an Albanese map, we have  $Pic^0(A_\Omega) \xrightarrow{i_\Omega^*} Pic^0(S_\Omega)$  is an isomorphism, and so it suffices to show that we have an isomorphism  $i_\Omega^* : NS(A_\Omega)_\mathbb{Q} \rightarrow NS(S_\Omega)_\mathbb{Q}$  of Néron-Severi groups. Now, let  $\ell \neq \text{char } k$ . Then,  $H^2(A_\Omega, \mathbb{Q}_\ell(1)) \xrightarrow{i_\Omega^*} H^2(S_\Omega, \mathbb{Q}_\ell(1))$  is an isomorphism by [33] Proposition 4. Note that  $S$  and  $A$  may be defined over some finitely generated field  $k_0$ . Upon passing to a large enough extension of  $k_0$ , we may also assume that  $C$  possesses a model over  $k_0$  and that  $NS(C_{k_0} \times A_{k_0})_\mathbb{Q} \cong NS(C \times A)_\mathbb{Q}$  (and, similarly, for  $C \times S$ ). Note that this is possible because the Néron-Severi group is finitely generated. Thus, we need to show that

$$NS(C_{k_0} \times A_{k_0})_\mathbb{Q} \xrightarrow[\cong]{(id_C \times i)^*} NS(C_{k_0} \times S_{k_0})_\mathbb{Q}$$

For this, let  $G := Gal(k/k_0)$  be the absolute Galois group and  $\ell \neq \text{char } k$ . The Künneth

theorem on cohomology then gives a ( $G$ -module) isomorphism:

$$H^2(C \times A, \mathbb{Q}_\ell(1)) \xrightarrow[\cong]{(id_C \times i)^*} H^2(C \times S, \mathbb{Q}_\ell(1)) \quad (4.3)$$

The result then follows from applying the functor  $H^0(G, -)$  to (4.3) and noting that the Tate conjecture holds for the left-hand side (by Faltings' theorem and the fact that  $k_0$  is a finitely generated field).  $\square$

We have the following important result of Bloch:

**Proposition 4.2.2** (Bloch, [9] Proposition 1.7). *Let  $S$  be the Fano surface of lines of a smooth cubic threefold in  $\mathbb{P}^4$  over an algebraically closed field of characteristic  $\neq 2$ . Then, the intersection product  $CH^1(S) \otimes CH^1(S) \rightarrow CH^2(S)$  is surjective.*

*Proof.* It suffices to show that  $CH_{hom}^1(S) \otimes CH^1(S) \rightarrow CH_{hom}^2(S)$  is surjective. We can prove a stronger statement, in fact. Let  $\Gamma \in CH^2(S \times X)$  be the cylinder correspondence; i.e., the cycle whose fiber over  $s \in S$  is the corresponding line  $\ell_s \in X$ . Further, let  $CH^1(S)^I \subset CH^1(S)$  denote the subgroup of incidence divisors. This is the group generated by the set  $\{D_s := {}^t\Gamma_*\ell_s \mid s \in S\}$ . As noted earlier for the general  $s \in S$ ,  $D_s$  agrees with the preceding definition. We show that the intersection product

$$CH^1(S)^I \otimes CH_{hom}^1(S)^I \rightarrow CH_{hom}^2(S) \quad (4.4)$$

is surjective. To this end, it suffices to show that  $2 \cdot (r - s)$  lies in the image of (4.4), for  $r$  and  $s$  on some open subvariety of  $S$ . Working with the general  $r, s$ , it is possible to

find some  $t \in S \setminus \{r, s\}$  such that  $t \in D_r \cap D_s$  since  $D_r \cdot D_s$  has degree 5. We can then write  $2 \cdot (r - s) = 2 \cdot (r - t) + 2 \cdot (t - s)$  and assume that  $\ell_r \cap \ell_s \neq \emptyset$  in  $X$ . Upon passing to some smaller open subvariety, we can assume that  $t \notin D_t$ . Then, let  $a = \ell_r \cap \ell_t$  and  $b = \ell_s \cap \ell_t$ . Now, a basic geometric fact is that exactly 6 distinct lines pass through the general  $x \in X$ ; thus, we have

$${}^t\Gamma_*[x] = s_1 + \dots + s_6 \in CH^2(S)$$

for distinct  $s_i \in S$ . In other words,  ${}^t\Gamma_*[x]$  is the sum over the 6 distinct lines passing through  $x$ . This way, we have

$${}^t\Gamma_*[a] + [s] = D_r \cdot D_t + [r] + [t]$$

$${}^t\Gamma_*[b] + [r] = D_s \cdot D_t + [s] + [t]$$

It is an exercise to the reader to show  $X$  is rationally connected (no matter what  $k$  is).

This means that  $CH^3(X) = \mathbb{Q}$ . Thus, it follows that  ${}^t\Gamma_*[a] \equiv {}^t\Gamma_*[b] \Rightarrow (D_s - D_r) \cdot D_t = 2 \cdot (r - s)$ , as desired.  $\square$

**Proposition 4.2.3.** *For all algebraically closed extensions  $k \subset \Omega$ , the pullback  $CH^2(A_\Omega) \xrightarrow{i_\Omega^*} CH^2(S_\Omega)$  is surjective.*

*Proof.* Since pull-back commutes with intersection product on Chow groups, we have



the following diagram:

$$\begin{array}{ccc}
CH^1(A_\Omega) \otimes CH^1(A_\Omega) & \longrightarrow & CH^2(A_\Omega) \\
i_\Omega^* \times i_\Omega^* \downarrow & & i_\Omega^* \downarrow \\
CH^1(S_\Omega) \otimes CH^1(S_\Omega) & \longrightarrow & CH^2(S_\Omega)
\end{array} \tag{4.5}$$

From Proposition 4.2.1, the left vertical arrow is surjective. Since  $\Omega$  is algebraically closed and  $S_\Omega$  is the Fano surface of lines of  $X_\Omega$ , we can apply Proposition 4.2.2 to deduce that the bottom horizontal arrow is also surjective.. So, the right vertical arrow is surjective, as desired.  $\square$

### 4.3 Proof of Theorem 4.0.2

Fix  $\pi_{2,S}$  a Chow-Künneth idempotent. We define

$$\mathfrak{h}^2(i) = \pi_{2,S} \circ {}^t\Gamma_i \circ \pi_{2,A} \in \text{Hom}_{\mathcal{M}_k}(\mathfrak{h}^2(A), \mathfrak{h}^2(S))$$

The goal is to show that  $\mathfrak{h}^2(i)$  is an isomorphism of motives. This means that we need to find some  $\psi \in \text{Hom}_{\mathcal{M}_k}(\mathfrak{h}^2(S), \mathfrak{h}^2(A))$  for which

$$\psi \circ \mathfrak{h}^2(i) = \pi_{2,A} \in \text{End}_{\mathcal{M}_k}(\mathfrak{h}^2(A)), \quad \mathfrak{h}^2(i) \circ \psi = \pi_{2,S} \in \text{End}_{\mathcal{M}_k}(\mathfrak{h}^2(S)) \tag{4.6}$$

Since  $\mathfrak{h}^2(A)$  and  $\mathfrak{h}^2(S)$  are evenly finite-dimensional, it suffices by Theorem 2.3.1 (d) to find some such  $\psi$  for which these equalities hold cohomologically. Moreover,  $i^* = \mathfrak{h}^2(i)_* :$

$H^2(A, \mathbb{Q}_\ell) \rightarrow H^2(S, \mathbb{Q}_\ell)$  is an isomorphism for  $\ell \neq \text{char } k$ . Thus, if we can find some

$$\gamma \in CH^2(S \times A) \tag{4.7}$$

for which  $\gamma_* : H^2(S, \mathbb{Q}_\ell) \rightarrow H^2(A, \mathbb{Q}_\ell)$  is the inverse of  $i^*$ , we can set  $\psi := \pi_{2,A} \circ \gamma \circ \pi_{2,S}$ , and this gives the desired correspondence in (4.6).

From [4], the image of the map  $S \times S \rightarrow A$  defined by  $(x, y) \mapsto i(x) - i(y)$  is an ample divisor  $\Theta$ . Moreover, we have

$$\frac{1}{3!} \cdot \Theta \wedge \Theta \wedge \Theta = [S] \in H^6(A, \mathbb{Q}_\ell(3))$$

The Hard Lefschetz theorem then implies that  $\wedge[S] : H^2(A, \mathbb{Q}_\ell) \rightarrow H^8(A, \mathbb{Q}_\ell)(3)$  is an isomorphism. Also, since the Lefschetz standard conjecture is true for Abelian varieties ([25] Proposition 4.3), it follows that there is a correspondence  $\phi \in CH^2(A \times A)$  such that  $\phi_* : H^8(A, \mathbb{Q}_\ell) \rightarrow H^2(A, \mathbb{Q}_\ell)(-3)$  is the inverse of  $\wedge[S]$ . The projection formula then shows that  $i_* \circ i^* = \wedge[S]$  so that  $\phi_* \circ i_* \circ i^*$  is the identity on  $H^2(A, \mathbb{Q}_\ell)$ . Since  $i^* : H^2(A, \mathbb{Q}_\ell) \rightarrow H^2(S, \mathbb{Q}_\ell)$  is an isomorphism, it follows that  $\gamma = \phi \circ \Gamma_i$  is the desired inverse.

# 5

## The Motive a Theta Divisor

Let  $k$  be an algebraically closed field and let  $A$  be an Abelian variety of dimension  $g$  with  $i : \Theta \hookrightarrow A$  a smooth ample divisor. The first goal of this chapter is then to prove the following:

**Theorem 5.0.1.**  $\Theta$  satisfies conjecture 1.0.3.

The Lefschetz hyperplane theorem gives isomorphisms  $i^* : H^j(A) \rightarrow H^j(\Theta)$  for  $j < g - 1$  and  $i_* : H^j(\Theta) \rightarrow H^{j+2}(A)$  for  $j > g - 1$ . The proof of Theorem 5.0.1 gives a particular set of idempotents  $\pi_{j,\Theta}$  and we set  $\mathfrak{h}^j(\Theta) = (\Theta, \pi_{j,\Theta}, 0)$ . We also set  $\mathfrak{h}^j(A) = (A, \pi_{j,A}, 0)$ , where  $\pi_{j,A}$  are the canonical idempotents constructed in [12]. We are then able to prove the following motivic version of the Lefschetz hyperplane theorem:

**Theorem 5.0.2.** (a) The pull-back  $\mathfrak{h}^j(i) := \pi_{j,\Theta} \circ {}^t\Gamma_i \circ \pi_{j,A} : \mathfrak{h}^j(A) \rightarrow \mathfrak{h}^j(\Theta)$  is an isomorphism for  $j < g - 1$ .

(b) The push-forward  ${}^t\mathfrak{h}^j(i) := \pi_{j+2,A} \circ \Gamma_i \circ \pi_{j,\Theta} : \mathfrak{h}^j(\Theta) \rightarrow \mathfrak{h}^{j+2}(A)(1)$  is an isomor-

phism for  $j > g - 1$ .

(c)  $\mathfrak{h}^{g-1}(i)$  is split-injective and  ${}^t\mathfrak{h}^{g-1}(i)$  is split-surjective.

(d) There is an idempotent  $p \in CH^{g-1}(\Theta \times \Theta)$  which is orthogonal to  $\pi_{j,\Theta}$  for  $j \neq g - 1$  and for which the motive  $P := (\Theta, p, 0)$  satisfies  $H^*(P) = K_\Theta := \ker(i_* : H^{g-1}(\Theta) \rightarrow H^{g+1}(A))$ .

We can specialize to the case that  $k = \mathbb{C}$  and  $H^*$  is singular cohomology with  $\mathbb{Q}$ -coefficients. The primitive cohomology of  $\Theta$ ,

$$K_\Theta = \ker(i_* : H^{g-1}(\Theta, \mathbb{Q}) \rightarrow H^{g+1}(A, \mathbb{Q})(1)),$$

is the only Hodge substructure of  $H^*$  not coming from  $A$ . So, one should expect to encounter difficulty in analyzing the motive  $P$ . The simplest nontrivial case is when  $A$  is a principally polarized Abelian fourfold and  $[\Theta] \in CH^1(A)$  is its principal polarization. In this case,  $\Theta$  is generally a smooth divisor and  $H^*(P) = K_\Theta$  has Hodge level 1. Conjecturally, a motive over  $\mathbb{C}$  whose singular cohomology has Hodge level 1 should correspond to an Abelian variety ([24] Remark 7.12). We have the following partial result:

**Proposition 5.0.1.** *There exists an Abelian variety  $J$  such that  $\mathfrak{h}^1(J)(-1) \cong P \Leftrightarrow$  we have  $p_{L*}CH_0(\Theta_L) = 0$  for all field extensions  $\mathbb{C} \subset L$ .*

## 5.1 Preliminaries

Let  $i : \Theta \hookrightarrow A$  be a smooth ample divisor and let  $\mathfrak{h}^j(A) = (A, \pi_{j,A}, 0)$  be a summand in the canonical Chow-Künneth decomposition of  $A$ . Then, we define the *Lefschetz operator*:

$$L_\Theta := \Delta_*(\Theta) \in CH^{g+1}(A \times A).$$

The most essential result for the proofs of theorems 5.0.1 and 5.0.2 is the following in [26], a motivic version of the Hard Lefschetz theorem:

**Theorem 5.1.1** (Künnemann). *Assume that  $[\Theta] = (-1)_A^*[\Theta] \in CH^1(A)$ .*

- (a)  $(L_\Theta)_*\alpha = \alpha \cup [\Theta]$  for  $\alpha \in H^*(A)$
- (b) The operator  $\pi_{2g-j,A} \circ L_\Theta^{g-j} \circ \pi_{j,A} : \mathfrak{h}^j(A)(g-j) \rightarrow \mathfrak{h}^{2g-j}(A)$  is an isomorphism of motives for  $j \leq g$ . That is, there exists a correspondence  $\Lambda_\Theta \in CH^{g-1}(A \times A)$  such that the following relations hold for  $j \leq g$ :

$$\begin{aligned} \pi_{j,A} \circ \Lambda_\Theta^{g-j} \circ L_\Theta^{g-j} \circ \pi_{j,A} &= \pi_{j,A} \\ \pi_{2g-j,A} \circ L_\Theta^{g-j} \circ \Lambda_\Theta^{g-j} \circ \pi_{2g-j,A} &= \pi_{2g-j,A} \end{aligned} \tag{5.1}$$

- (c) Set  $\pi_{j,A} = 0$  for all  $j \notin \{0, 1, \dots, 2g\}$ . Then, we have  $L_\Theta \circ \pi_{j,A} = \pi_{j+2,A} \circ L_\Theta$  and  $\Lambda_\Theta \circ \pi_{j,A} = \pi_{j-2,A} \circ \Lambda_\Theta$ .

*Proof.* See [26] Theorem 4.1. It should be noted that (b) holds more generally for Abelian schemes. It is a technical result that uses properties of the Fourier transform

for Chow groups of Abelian schemes.  $\square$

By Theorem 5.1.1 (a) and the projection formula, we have  $(L_\Theta)_* = \cup[\Theta] = i_* \circ i^*$ .

The result below shows that this is true on the level of correspondences:

**Lemma 5.1.1.**  $L_\Theta = \Gamma_i \circ {}^t\Gamma_i \in CH^{g+1}(A \times A)$ .

*Proof.* From the obvious commutative diagram:

$$\begin{array}{ccc} \Theta & \xrightarrow{\Delta_\Theta} & \Theta \times \Theta \\ i \downarrow & & i \times i \downarrow \\ A & \xrightarrow{\Delta_A} & A \times A \end{array} \quad (5.2)$$

we have  $L_\Theta = (\Delta_A)_*(\Theta) = (\Delta_A)_*(i_*1) = (i \times i)_*(\Delta_\Theta) = \Gamma_i \circ \Delta_\Theta \circ {}^t\Gamma_i = \Gamma_i \circ {}^t\Gamma_i$ , where the penultimate step follows from Lemma 2.1.1.  $\square$

## 5.2 Proofs of Theorems 5.0.1 and 5.0.2

Since  $k$  is algebraically closed, it's possible to find some  $a \in A(k)$  such that  $t_a^*[\Theta] \in CH^1(A)$  is invariant under  $(-1)^*$ . So, we can assume that  $(-1)_A^*[\Theta] = [\Theta]$ , so that the results of the previous section are applicable.

*Proof of Theorem 5.0.1.* For the proof, we will need to exhibit correspondences  $\pi_{j,\Theta} \in$

$CH^{g-1}(\Theta \times \Theta)$  which satisfy conjecture 1.0.3. These are given as follows:

$$\begin{aligned}
\pi_{j,\Theta} &= {}^t\Gamma_i \circ \pi_{j,A} \circ \Lambda_{\Theta}^{g-j} \circ L_{\Theta}^{g-j-1} \circ \Gamma_i \text{ for } j < g-1, \\
\pi_{j,\Theta} &= {}^t\Gamma_i \circ L_{\Theta}^{j-g+1} \circ \Lambda_{\Theta}^{j-g+2} \circ \pi_{j+2,A} \circ \Gamma_i \text{ for } j > g-1, \\
\pi_{g-1,\Theta} &= \Delta_{\Theta} - \sum_{j \neq g-1} \pi_{j,\Theta}.
\end{aligned} \tag{5.3}$$

Since  $\sum \pi_{j,\Theta} = \Delta_{\Theta}$  holds by definition, it suffices to check conditions (a) and (c) of conjecture 1.0.3. For  $j < g-1$ , we have

$$\begin{aligned}
\pi_{j,\Theta}^2 &= {}^t\Gamma_i \circ \pi_{j,A} \circ \Lambda_{\Theta}^{g-j} \circ L_{\Theta}^{g-j-1} \circ \Gamma_i \circ {}^t\Gamma_i \circ \pi_{j,A} \circ \Lambda_{\Theta}^{g-j} \circ L_{\Theta}^{g-j-1} \circ \Gamma_i \\
&= {}^t\Gamma_i \circ \pi_{j,A} \circ \Lambda_{\Theta}^{g-j} \circ L_{\Theta}^{g-j} \circ \pi_{j,A} \circ \Lambda_{\Theta}^{g-j} \circ L_{\Theta}^{g-j-1} \circ \Gamma_i \\
&= {}^t\Gamma_i \circ \pi_{j,A} \circ \Lambda_{\Theta}^{g-j} \circ L_{\Theta}^{g-j-1} \circ \Gamma_i = \pi_{j,\Theta}
\end{aligned}$$

Here, the second equality holds by Lemma 5.1.1, the third holds by Theorem 5.1.1 (b).

Similarly, for  $j > g-1$  we have:

$$\begin{aligned}
\pi_{j,\Theta}^2 &= {}^t\Gamma_i \circ \pi_{j,A} \circ L_{\Theta}^{j-g+1} \circ \Lambda_{\Theta}^{j-g+2} \circ \pi_{j+2,A} \circ \Gamma_i \circ {}^t\Gamma_i \circ L_{\Theta}^{j-g+1} \circ \Lambda_{\Theta}^{j-g+2} \circ \pi_{j+2,A} \circ \Gamma_i \\
&= {}^t\Gamma_i \circ L_{\Theta}^{j-g+1} \circ \Lambda_{\Theta}^{j-g+2} \circ \pi_{j+2,A} \circ L_{\Theta} \circ L_{\Theta}^{j-g+1} \circ \Lambda_{\Theta}^{j-g+2} \circ \pi_{j+2,A} \circ \Gamma_i \\
&= {}^t\Gamma_i \circ L_{\Theta}^{j-g+1} \circ \Lambda_{\Theta}^{j-g+2} \circ \pi_{j+2,A} \circ L_{\Theta}^{j-g+2} \circ \Lambda_{\Theta}^{j-g+2} \circ \pi_{j+2,A} \circ \Gamma_i \\
&= {}^t\Gamma_i \circ L_{\Theta}^{j-g+1} \circ \Lambda_{\Theta}^{j-g+2} \circ \pi_{j+2,A} \circ \Gamma_i = \pi_{j,\Theta}
\end{aligned}$$

Thus,  $\pi_{j,\Theta}^2 = \pi_{j,\Theta}$  for  $j \neq g-1$ . Before proving the same for  $j = g-1$ , we show that the orthogonality condition of (a) (in conjecture 1.0.3) holds; that is,  $\pi_{j,\Theta} \circ \pi_{j',\Theta} = 0$  for

$j \neq j'$  and  $j, j' \neq g - 1$ . We do this for the case of  $j \neq j' < g - 1$ :

$$\begin{aligned}
\pi_{j,\Theta} \circ \pi_{j',\Theta} &= {}^t\Gamma_i \circ \pi_{j,A} \circ \Lambda_{\Theta}^{g-j} \circ L_{\Theta}^{g-j-1} \circ \Gamma_i \circ {}^t\Gamma_i \circ \pi_{j',A} \circ \Lambda_{\Theta}^{g-j'} \circ L_{\Theta}^{g-j'-1} \circ \Gamma_i \\
&= {}^t\Gamma_i \circ \pi_{j,A} \circ \Lambda_{\Theta}^{g-j} \circ L_{\Theta}^{g-j} \circ \pi_{j',A} \circ \Lambda_{\Theta}^{g-j'} \circ L_{\Theta}^{g-j'-1} \circ \Gamma_i \\
&= {}^t\Gamma_i \circ \Lambda_{\Theta}^{g-j} \circ L_{\Theta}^{g-j} \circ \pi_{j,A} \circ \pi_{j',A} \circ \Lambda_{\Theta}^{g-j'} \circ L_{\Theta}^{g-j'-1} \circ \Gamma_i \\
&= 0
\end{aligned} \tag{5.4}$$

Again, the second equality holds by Lemma 5.1.1 and the last equality follows from orthogonality of  $\pi_{j,A}$ . The third equality holds because we have

$$\pi_{j,A} \circ \Lambda_{\Theta}^{g-j} \circ L_{\Theta}^{g-j} = \Lambda_{\Theta}^{g-j} \circ L_{\Theta}^{g-j} \circ \pi_{j,A}$$

which follows by repeated application of Theorem 5.1.1 (c). The remaining cases of orthogonality ( $j \neq j'$  and  $j, j' \neq g - 1$ ) are identical to (5.4).

What remains for the verification of condition (a) is to show that:

(i)  $\pi_{g-1,\Theta}^2 = \pi_{g-1,\Theta}$

(ii)  $\pi_{g-1,\Theta} \circ \pi_{j,\Theta} = 0 = \pi_{j,\Theta} \circ \pi_{g-1,\Theta}$  for  $j \neq g - 1$

For (i), let  $\pi = \sum_{k \neq g-1} \pi_{k,\Theta}$ . Since the summands are mutually orthogonal idempotents by the preceding verifications, it follows that  $\pi^2 = \pi$ . Since  $\pi_{g-1,\Theta} = \Delta_{\Theta} - \pi$  by definition, we have

$$\pi_{g-1,\Theta}^2 = (\Delta_{\Theta} - \pi)^2 = \Delta_{\Theta} + \pi^2 - 2\pi = \Delta_{\Theta} - \pi = \pi_{g-1,\Theta}$$



For (ii), let  $j \neq g - 1$  and note that

$$\begin{aligned}\pi_{g-1,\Theta} \circ \pi_{j,\Theta} &= (\Delta_{\Theta} - \pi) \circ \pi_{j,\Theta} = \pi_{j,\Theta} - \sum_{k \neq g-1} \pi_{k,\Theta} \circ \pi_{j,\Theta} \\ &= \pi_{j,\Theta} - \pi_{j,\Theta} = 0\end{aligned}$$

where the third equality holds since  $\pi_{k,\Theta} \circ \pi_{j,\Theta} = 0$  for  $j \neq k$ . Similarly, one has  $0 = \pi_{j,\Theta} \circ \pi_{g-1,\Theta}$ . This completes the verification of item (a) in conjecture 1.0.3.

Finally, we prove (c) in conjecture 1.0.3. It suffices to show that  $\pi_{j,\Theta}$  acts as the identity on  $H^j(\Theta)$  and trivially on  $H^{j'}(\Theta)$  for  $j \neq j'$  and any Weil cohomology  $H^*$ . One easily reduces this to the case that  $j \neq g - 1$ . We will verify this for  $j < g - 1$ . Since  $\pi_{j,A}$  acts as 0 on  $H^{j'}(A)$  for  $j \neq j'$ , we need only show that  $\pi_{j,\Theta}$  acts as the identity on  $H^j(\Theta)$ . To this end, let  $\phi := \Lambda_{\Theta}^{g-j} \circ L_{\Theta}^{g-j-1} \circ \Gamma_i$  so that

$$\pi_{j,\Theta} = {}^t\Gamma_i \circ \pi_{j,A} \circ \phi$$

Since  $H^*$  is a Weil cohomology,  ${}^t\Gamma_{i*} = i^* : H^j(A) \rightarrow H^j(\Theta)$  is an isomorphism (see [25]). Moreover, by Hard Lefschetz,  $(\phi \circ {}^t\Gamma_i)_* = (\Lambda_{\Theta}^{g-j})_* \circ (L_{\Theta}^{g-j})_*$  is the identity on  $H^j(A)$ . Thus,  $i^*$  and  $\phi_*$  are inverses, from which it follows that  $(\pi_{j,\Theta})_*$  is the identity on  $H^j(\Theta)$  for  $j < g - 1$ . The case of  $j > g - 1$  is nearly identical, only that one uses the fact that  $i_*$  is an isomorphism.  $\square$

*Proof of Theorem 5.0.2.* The statements of (a) and (b) are that  $\mathfrak{h}^j(i)$  and  ${}^t\mathfrak{h}^j(i)$  are isomorphisms for  $j < g - 1$  and  $j > g - 1$ , respectively. To show this, we need to

construct their inverse isomorphisms:

$$\phi_j := \pi_{j,A} \circ \Lambda_{\Theta}^{g-j} \circ L_{\Theta}^{g-j-1} \circ \Gamma_i \circ \pi_{j,\Theta} \text{ for } j < g-1$$

$$\phi_j := \pi_{j,\Theta} \circ {}^t\Gamma_i \circ L_{\Theta}^{j-g+1} \circ \Lambda_{\Theta}^{j-g+2} \circ \pi_{j+2,A} \text{ for } j > g-1$$

Then, for  $j < g-1$ , we have

$$\begin{aligned} \phi_j \circ \mathfrak{h}^j(i) &= \pi_{j,A} \circ \Lambda_{\Theta}^{g-j} \circ L_{\Theta}^{g-j-1} \circ \Gamma_i \circ \pi_{j,\Theta} \circ {}^t\Gamma_i \circ \pi_{j,A} \\ &= \pi_{j,A} \circ \Lambda_{\Theta}^{g-j} \circ L_{\Theta}^{g-j-1} \circ \Gamma_i \circ {}^t\Gamma_i \circ \pi_{j,A} \circ \Lambda_{\Theta}^{g-j} \circ L_{\Theta}^{g-j-1} \circ \Gamma_i \circ {}^t\Gamma_i \circ \pi_{j,A} \\ &= \pi_{j,A} \circ \Lambda_{\Theta}^{g-j} \circ L_{\Theta}^{g-j} \circ \pi_{j,A} \circ \Lambda_{\Theta}^{g-j} \circ L_{\Theta}^{g-j} \circ \pi_{j,A} \\ &= \pi_{j,A} \end{aligned}$$

where the third and fourth equalities hold by Theorem 5.1.1 (b). Similarly, we have

$$\begin{aligned} \mathfrak{h}^j(i) \circ \phi_j &= \pi_{j,\Theta} \circ {}^t\Gamma_i \circ \pi_{j,A} \circ \Lambda_{\Theta}^{g-j} \circ L_{\Theta}^{g-j-1} \circ \Gamma_i \circ \pi_{j,\Theta} \\ &= \pi_{j,\Theta}^3 = \pi_{j,\Theta} \end{aligned}$$

We conclude that  $\mathfrak{h}^j(i)$  and  $\phi_j$  are inverses for  $j < g-1$ , proving (a). For (b), we have

$$\begin{aligned} {}^t\mathfrak{h}^j(i) \circ \phi_j &= \pi_{j+2,A} \circ \Gamma_i \circ \pi_{j,\Theta} \circ {}^t\Gamma_i \circ L_{\Theta}^{j-g+1} \circ \Lambda_{\Theta}^{j-g+2} \circ \pi_{j+2,A} \\ &= \pi_{j+2,A} \circ \Gamma_i \circ {}^t\Gamma_i \circ L_{\Theta}^{j-g+1} \circ \Lambda_{\Theta}^{j-g+2} \circ \pi_{j+2,A} \circ \Gamma_i \circ {}^t\Gamma_i \circ L_{\Theta}^{j-g+1} \circ \Lambda_{\Theta}^{j-g+2} \circ \pi_{j+2,A} \\ &= \pi_{j+2,A} \circ L_{\Theta}^{j-g+2} \circ \Lambda_{\Theta}^{j-g+2} \circ \pi_{j+2,A} \circ L_{\Theta}^{j-g+2} \circ \Lambda_{\Theta}^{j-g+2} \circ \pi_{j+2,A} \\ &= \pi_{j,A} \end{aligned}$$

Similarly, we have

$$\begin{aligned}\phi_j \circ {}^t\mathfrak{h}^j(i) &= \pi_{j,\Theta} \circ {}^t\Gamma_i \circ L_{\Theta}^{j-g+1} \circ \Lambda_{\Theta}^{j-g+2} \circ \pi_{j+2,A} \circ \Gamma_i \circ \pi_{j,\Theta} \\ &= \pi_{j,\Theta}^3 = \pi_{j,\Theta}\end{aligned}$$

So,  ${}^t\mathfrak{h}^j(i)$  and  $\phi_j$  are inverses for  $j > g - 1$ . For (c), we need to show that  $\mathfrak{h}^{g-1}(i)$  and  ${}^t\mathfrak{h}^{g-1}(i)$  are split-injective and split-surjective, respectively. Their left and right inverses will be:

$$\begin{aligned}\phi_{g-1} &= \pi_{g-1,A} \circ \Lambda_{\Theta} \circ \Gamma_i \circ \pi_{g-1,\Theta} \\ \psi_{g-1} &= \pi_{g-1,\Theta} \circ {}^t\Gamma_i \circ \Lambda_{\Theta} \circ \pi_{g+1,A}.\end{aligned}\tag{5.5}$$

To this end, we begin by noting that for  $j < g - 1$ :

$$\begin{aligned}\pi_{j,\Theta} \circ {}^t\Gamma_i &= {}^t\Gamma_i \circ \pi_{j,A} \circ \Lambda_{\Theta}^{j-g} \circ L_{\Theta}^{g-j} \\ &= {}^t\Gamma_i \circ \pi_{j,A}\end{aligned}\tag{5.6}$$

Similarly, we have  $\Gamma_i \circ \pi_{j,\Theta} = \pi_{j+2,A} \circ \Gamma_i$  for  $j > g - 1$ . So, we write  $\pi = \sum_{j \neq g-1} \pi_{j,\Theta}$  as

before and obtain:

$$\begin{aligned}
\Gamma_i \circ \pi \circ {}^t\Gamma_i \circ \pi_{g-1,A} &= \sum_{j < g-1} \Gamma_i \circ \pi_{j,\Theta} \circ {}^t\Gamma_i \circ \pi_{g-1,A} + \sum_{j > g-1} \Gamma_i \circ \pi_{j,\Theta} \circ {}^t\Gamma_i \circ \pi_{g-1,A} \\
&= \sum_{j < g-1} \Gamma_i \circ {}^t\Gamma_i \circ \pi_{j,A} \circ \pi_{g-1,A} + \sum_{j > g-1} \pi_{j+2,A} \circ \Gamma_i \circ {}^t\Gamma_i \circ \pi_{g-1,A} \\
&= \sum_{j < g-1} L_\Theta \circ \pi_{j,A} \circ \pi_{g-1,A} + \sum_{j > g-1} \pi_{j+2,A} \circ L_\Theta \circ \pi_{g-1,A} \\
&= \sum_{j > g-1} L_\Theta \circ \pi_{j,A} \circ \pi_{g-1,A} = 0
\end{aligned} \tag{5.7}$$

where the third equality holds by the mutual orthogonality of  $\pi_{j,A}$  and the fourth holds because  $L_\Theta \circ \pi_{j,A} = \pi_{j+2,A} \circ L_\Theta$ . Thus, we have:

$$\begin{aligned}
\phi_{g-1} \circ \mathfrak{h}^{g-1}(i) &= \pi_{g-1,A} \circ \Lambda_\Theta \circ \Gamma_i \circ \pi_{g-1,\Theta} \circ {}^t\Gamma_i \circ \pi_{g-1,A} \\
&= \pi_{g-1,A} \circ \Lambda_\Theta \circ \Gamma_i \circ (\Delta_\Theta - \pi) \circ {}^t\Gamma_i \circ \pi_{g-1,A} \\
&= \pi_{g-1,A} \circ \Lambda_\Theta \circ \Gamma_i \circ {}^t\Gamma_i \circ \pi_{g-1,A} - \pi_{g-1,A} \circ \Lambda_\Theta \circ \Gamma_i \circ \pi \circ {}^t\Gamma_i \circ \pi_{g-1,A} \\
&= \pi_{g-1,A} \circ \Lambda_\Theta \circ L_\Theta \circ \pi_{g-1,A} = \pi_{g-1,A}
\end{aligned}$$

Here, the second term on the third line vanishes by (5.7). So,  $\mathfrak{h}^{g-1}(i)$  is split-injective. A similar calculation shows that  ${}^t\mathfrak{h}^{g-1}(i)$  is split-surjective with right inverse  $\psi_j$ . The completes the proof of (c).

Finally, for (d) we define:

$$\pi'_{g-1,\Theta} := {}^t\Gamma_i \circ \pi_{g-1,A} \circ \Lambda_\Theta \circ \Gamma_i \in CH^{g-1}(\Theta \times \Theta)$$

As in the proof of Theorem 5.0.1, one can show that  $\pi'_{g-1,\Theta}$  is an idempotent, is orthogonal to  $\pi_{j,\Theta}$  for  $j \neq g-1$ . It follows that

$$\pi'_{g-1,\Theta} \circ \pi_{g-1,\Theta} = \pi'_{g-1,\Theta} - \sum_{j \neq g-1} \pi'_{g-1,\Theta} \circ \pi_{j,\Theta} = \pi'_{g-1,\Theta}$$

Similarly, one has  $\pi_{g-1,\Theta} \circ \pi'_{g-1,\Theta} = \pi'_{g-1,\Theta}$ . Write  $\mathfrak{h}_1^{g-1}(\Theta) = (\Theta, \pi'_{g-1,\Theta}, 0)$  for the corresponding motive and define:

$$p := \pi_{g-1,\Theta} - \pi'_{g-1,\Theta} \in CH^{g-1}(\Theta \times \Theta)$$

We have

$$\begin{aligned} p^2 &= (\pi_{g-1,\Theta} - \pi'_{g-1,\Theta})^2 = \pi_{g-1,\Theta}^2 + (\pi'_{g-1,\Theta})^2 - 2\pi_{g-1,\Theta} \circ \pi'_{g-1,\Theta} \\ &= \pi_{g-1,\Theta} + \pi'_{g-1,\Theta} - 2\pi'_{g-1,\Theta} = \pi_{g-1,\Theta} - \pi'_{g-1,\Theta} = p \end{aligned}$$

so that  $p$  is an idempotent. Write  $P := (\Theta, p, 0)$  for the corresponding motive. We also have

$$p \circ \pi'_{g-1,\Theta} = (\pi_{g-1,\Theta} - \pi'_{g-1,\Theta}) \circ \pi'_{g-1,\Theta} = \pi'_{g-1,\Theta} - \pi'_{g-1,\Theta} = 0$$

so that  $p$  and  $\pi'_{g-1,\Theta}$  are orthogonal. This gives a decomposition of motives:

$$\mathfrak{h}^{g-1}(\Theta) = P \oplus \mathfrak{h}_1^{g-1}(\Theta) \tag{5.8}$$

The same argument for Theorem 5.0.1 (c) shows that  $H^*(\mathfrak{h}_1^{g-1}(\Theta)) = i^*H^{g-1}(\Theta)$ . Thus,

applying  $H^*$  to (5.8), it follows that  $H^*(P) = K_\Theta$ .  $\square$

### 5.3 The complementary motive $P$

Now, let  $k = \mathbb{C}$  and  $H^*$  be singular cohomology with  $\mathbb{Q}$ -coefficients. We consider the case of  $A$  a principally polarized Abelian variety, whose principal polarization is the class of  $i : \Theta \rightarrow A$ . Since we are interested in the motive  $P$ , we need  $\Theta$  to be nonsingular. The simplest nontrivial case is that of  $g = 4$ , where a well-known result of Mumford in [28] is that  $\Theta$  is generally nonsingular. Now, let  $K_{\Theta, \mathbb{Q}} := \ker(i_* : H^{g-1}(\Theta, \mathbb{Q}) \rightarrow H^{g+1}(A, \mathbb{Q})(1))$  be the primitive cohomology. Then, we have the following:

**Lemma 5.3.1.**  *$K_\Theta$  is a rational Hodge structure of level 1 and dimension 10.*

*Proof.* Since  $H^3(\Theta)$  and  $H^3(A)$  both have Hodge level 3, we need to show that  $i^* : H^{3,0}(A) \rightarrow H^{3,0}(\Theta)$  is an isomorphism. Since this map is already injective, it will suffice to show that  $h^{3,0}(\Theta) = h^{3,0}(A) = 4$ . By adjunction,  $\omega_\Theta \cong \mathcal{O}_\Theta(\Theta)$ , so  $h^0(\Theta, \mathcal{O}_\Theta(\Theta)) = h^{3,0}(\Theta)$ . We can use the long exact sequence to compute  $h^0(\Theta, \mathcal{O}_\Theta(\Theta))$ :

$$0 \rightarrow H^0(A, \mathcal{O}_A) \rightarrow H^0(A, \mathcal{O}_A(\Theta)) \xrightarrow{res} H^0(\Theta, \mathcal{O}_\Theta(\Theta)) \rightarrow H^1(A, \mathcal{O}_A) \rightarrow H^1(A, \mathcal{O}_A(\Theta)) = 0$$

Since  $\Theta$  is a principal polarization,  $h^0(A, \mathcal{O}_A(\Theta)) = 1$  so that the restriction arrow is 0. Moreover,  $h^1(A, \mathcal{O}_A) = 4$ , so it follows that  $h^{3,0}(\Theta) = 4 = h^{3,0}(A)$ . Thus,  $i^* : H^{3,0}(A) \rightarrow H^{3,0}(\Theta)$  is an isomorphism and  $K_\Theta$  has Hodge level 1. To determine the dimension of  $K_\Theta$ , we first compute  $\chi(\Theta) = c_3(T\Theta)$ . Applying the Chern polynomial to the adjunction

sequence in this case, one obtains that  $c_3(T\Theta) = -c_1(\mathcal{O}(\Theta))^4 = -4! = -24$ . Using the Lefschetz hyperplane theorem, one also computes that  $\chi(\Theta) = 42 - h^3(\Theta)$ , so that  $h^3(\Theta) = 66$ . Since  $h^3(A) = \binom{8}{3} = 56$ , it follows that  $K_\Theta$  has dimension 10.  $\square$

Thus,  $H^*(P, \mathbb{Q})$  has Hodge level 1 when  $g = 4$ . Now, consider the intermediate Jacobian of  $K_\Theta$ :

$$J(K_\Theta) = K_{\Theta, \mathbb{C}} / (F^2 K_{\Theta, \mathbb{C}} \oplus K_{\Theta, \mathbb{Z}})$$

This is a principally polarized Abelian variety of dimension 5, and we have an isomorphism of rational Hodge structures  $H^1(J(K_\Theta), \mathbb{Q})(-1) \cong H^3(P, \mathbb{Q})$ . The generalized Hodge conjecture predicts that this isomorphism arises from a correspondence  $\Gamma \subset J(K_\Theta) \times \Theta$ . The existence of  $\Gamma$  was proved in [18]. One may take this a step further and ask whether  $\mathfrak{h}^1(J(K_\Theta))(-1)$  and  $P$  are isomorphic as motives. Proposition 5.0.1 provides a partial answer to this; i.e., we have  $\mathfrak{h}^1(J(K_\Theta))(-1) \cong P$  if  $p$  acts trivially on  $CH_0(\Theta_L)$  for all field extensions  $\mathbb{C} \subset L$  (and conversely). We will need the following definition for the proof:

**Definition 5.3.1.** *We say that  $M = (X, \pi, 0) \in \mathcal{M}_k$  has representable Chow group in codimension  $i$  if there exists a smooth complete (possibly reducible) curve  $C$  and  $\Gamma \in CH^i(C \times X)$  such that  $CH_{alg}^i(M_L) = \pi_{L*} CH_{alg}^i(X_L)$  lies in  $\Gamma_{L*} CH_{alg}^1(C_L) \subset CH_{alg}^i(X_L)$  for every field extension  $k \subset L$ .*

*Proof of Proposition 5.0.1.* Suppose that we have some Abelian variety  $J$  for which

$\mathfrak{h}^1(J)(-1) \cong P$ . Then, applying  $CH^3(-)_L$  to both sides we obtain

$$p_{L*}CH_0(\Theta_L) = p_{L*}CH^3(\Theta_L) \cong CH^3(\mathfrak{h}^1(J)_L(-1)) = CH^2(\mathfrak{h}^1(J)_L)$$

From [12] Theorem 2.19, we have  $CH^2(\mathfrak{h}^1(J)_L) = 0$  so that  $p_{L*}CH_0(\Theta_L) = 0$ . For the converse, suppose that  $p_{L*}CH_0(\Theta_L) = 0$  for all field extensions  $k \subset L$ . Then we have:

**Lemma 5.3.2.**  *$P$  has representable Chow group in codimension 2.*

*Proof of Lemma.* We use the same argument as in [10]. There is a localization sequence:

$$\lim_{D \subset \Theta} CH^2(\Theta \times D) \xrightarrow{(id_{\Theta} \times j_D)_*} CH^3(\Theta \times \Theta) \xrightarrow{(id_{\Theta} \times K)^*} CH^3(\Theta_K) \longrightarrow 0 \quad (5.9)$$

where the limit runs over all subvarieties  $D$  of codimension 1 and  $K = \mathbb{C}(\Theta)$  is the function field of  $\Theta$ . We have  $(id_{\Theta} \times K)^* \Delta_{\Theta} = \eta_K$ , the generic point of  $\Theta$ . From Lemma 5.1.1, we have  $p = p \circ \Delta_{\Theta} = (p \times id_{\Theta})_* \Delta_{\Theta}$  so that

$$(id_{\Theta} \times K)^*(p) = (id_{\Theta} \times K)^*(p \times id_{\Theta})_* \Delta_{\Theta} = p_{K*}(id_{\Theta} \times K)^* \Delta_{\Theta} = p_{K*}(\eta_K)$$

Since  $p_{K*}(\eta_K) = 0$  by assumption, the exactness of (5.9) gives some subvariety  $D$  and  $\alpha \in CH^2(\Theta \times D)$  for which  $p = (id_{\Theta} \times j_D)_* \alpha$ . After desingularizing, we can assume that  $D$  is smooth (although  $j_D$  may no longer be an inclusion). By Lemma 5.1.1, we have

$$p = (id_{\Theta} \times j_D)_* \alpha = \Gamma_{j_D} \circ \alpha$$



Thus  $p_{L*}CH_{alg}^2(\Theta_L) \subset j_{D*}CH_{alg}^1(D_L)$ . By the representability of the Picard functor, this means there is some smooth complete  $C$  and some  $\Gamma \in CH^1(C \times D)$  such that  $\Gamma_{L*}CH_{alg}^1(C_L) = CH_{alg}^1(D_L)$  for all field extension  $\mathbb{C} \subset L$ . This proves the lemma.  $\square$

Thus, we see that the Chow group of  $P$  is representable in every codimension. By [39] Theorem 3.4, it follows that the motive of  $P$  decomposes as

$$\bigoplus \mathbb{1}(i)^{\oplus n_i} \oplus \mathfrak{h}^1(J_i)(-i)$$

for integers  $n_i$  and Abelian varieties  $J_i$ . Since the cohomology of  $P$  is 0 in all but degree 3, this means that  $P \cong \mathfrak{h}^1(J)(-1)$  for some Abelian variety  $J$ . Since we have  $H^1(J(K_\Theta), \mathbb{Q})(-1) \cong H^3(P, \mathbb{Q}) \cong H^1(J, \mathbb{Q})(-1)$  (as rational Hodge structures), it follows that  $J$  and  $J(K_\Theta)$  are isogenous.  $\square$

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