Gersten-Witt Complex of Hirzebruch Surfaces

by

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Dissertation submitted in partial fulfillment of the requirements for the degree of
Doctor of Philosophy in the Department of Mathematics
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Abstract

The Witt group is classically an abelian group whose elements are represented by isometry classes of anisotropic symmetric bilinear forms over a field. It can be generalized to Witt group of a ring or a scheme X, hence to the Witt sheaf of X.

The Gersten-Witt complex of a scheme X is a cochain complex of Witt groups of residue class fields at all points of X, where the p-th term is the direct sum of Witt groups of residue class field at all codimension p points in X.

Pardon constructed the Gersten-Witt complex of a Gorenstein scheme X, and showed that it is exact when X is the spectrum of a regular local ring which is essentially of finite type over a field of characteristic different from 2. Using this, he defined a flasque resolution of a Witt sheaf. Although the Witt group of a complex surface is a birational invariant, we show that higher cohomologies of the Witt sheaf can distinguish different birational schemes. Specifically, we show that the cohomologies of the Hirzebruch surface H_n are different depending on whether n is even or odd.

Computing cohomologies of the Gersten-Witt complex of a scheme is difficult in practice, because it involves Witt groups of residue fields at all (possibly infinitely many) points of the scheme. However, if the scheme is a complex toric variety, we show that one can construct a quasi-isomorphic cochain complex which has only a finite number of Witt groups of tori, which are orbits of the torus action. Moreover, the Witt group of a complex torus is a vector space of finite dimension. Hence, the

new cochain complex is much simpler than the original Gersten-Witt complex, and its cohomologies can be more easily computed.

Using this quasi-isomorphism, we compute cohomologies of the Witt sheaf on the Hirzebruch surfaces H_n. The Hirzebruch surfaces H_n are projective line bundles over a projective line, one for each integer n. We show that the first and second cohomologies depend on the parity of n.

To my mother

Contents

List of Abbreviations and Symbols

Symbols

$\coprod$	Coproduct.
$\ell_A(M)$	The length of an A-module M.
$\mathcal{O}_X$	The structure sheaf of a scheme X.
$K(X)$	The function field of a scheme X.
$X^{(p)}$	The set of codimension p points in X.
$\kappa(x)$	The residue class field at x.
$\bar{x}$	The closure of the singleton $\{x\}$.
$\mathcal{CM}^p(X)$	The category of coherent sheaves of Cohen-Macaulay $\mathcal{O}_X$-modules of codimension p.
$\mathcal{CM}^p_Y(X)$	The category of coherent sheaves of Cohen-Macaulay $\mathcal{O}_X$-modules of codimension p supported on a closed subscheme $Y \subset X$.
$\mathcal{C}$	The canonical sheaf.
$\mathcal{E}^p$	The p-th term in a minimal injective resolution.
$\mathcal{V}^p$	The kernel of the p-th boundary map $\mathcal{E}^p \to \mathcal{E}^{p+1}$ in a minimal injective resolution.

Abbreviations

CM	Cohen-Macaulay.

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1

Introduction

We will first review some basic definitions of symmetric bilinear forms, and define Witt groups of fields, rings, and schemes. In Section 1.3, we introduce the notion of the Gersten-Witt complex of a scheme. We will see that for certain class of schemes with nice geometric properties, the Gersten-Witt complex is the global section of a flasque resolution of a Witt sheaf on the scheme. We will see how its cohomologies can be computed easily if the scheme is a complex toric variety.

1.1 Symmetric bilinear forms over a field

Let k be a field with $\operatorname{char} k \neq 2$. A *symmetric bilinear form* over k is a map of the form

$$\phi : V \times V \to k,$$

where V is a finite-dimensional k-vector space, such that

$$\begin{aligned} \phi(v, w) &= \phi(w, v), \\ \phi(v + v', w) &= \phi(v, w) + \phi(v', w), \\ \phi(av, w) &= a\phi(v, w), \end{aligned}$$

for every $a \in k$ and $v, v', w \in V$. We will denote the form by (V, ϕ). It is *nonsingular* if the induced map

$$\operatorname{ad}\phi : V \to \operatorname{Hom}(V, k)$$

is bijective, and *anisotropic* if $\phi(v, v) = 0$ implies $v = 0$.

Two bilinear forms (V, ϕ) and (W, ψ) are *isometric* if there is an isomorphism of vector spaces $f : V \xrightarrow{\sim} W$ such that the diagram

$$\begin{array}{ccc} V \times V & \xrightarrow{\ \phi\ } & k \\ {\scriptstyle f\times f}\downarrow\wr & \nearrow_{\psi} & \\ W \times W & & \end{array}$$

commutes.

If $W \subset V$ is a subspace, we define its *orthogonal complement*

$$W^{\perp} := \{v \in V \mid \phi(v, w) \ \ \forall w \in W\}.$$

If $W \subset W^{\perp}$, then W is *totally isotropic*, and if $W = W^{\perp}$, then W is *orthogonal*. There is a dimension equation [7, 1.3],

$$\dim W + \dim W^{\perp} = \dim V. \tag{1.1}$$

A nonsingular form (V, ϕ) is *hyperbolic* if V has an orthogonal subspace, or equivalently, a totally isotropic subspace of half the dimension (by (1.1)).

Every symmetric bilinear form over k can be represented by a symmetric matrix M, and it is nonsingular if and only if $\det M \neq 0$. The forms represented by matrices M and M' are isometric if and only if there is a nonsingular matrix Q such that $M' = QMQ^T$. Every symmetric bilinear form over k can be diagonalized by such a transformation.

The hyperbolic form of dimension 2 is called the *hyperbolic plane*. It is represented by a matrix

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$

which diagonalizes to (recall our assumption that $2 \in A$ is a unit)

$$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} 1 & 1/2 \\ 1 & -1/2 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 1/2 & -1/2 \end{pmatrix}.$$

It can be shown that every hyperbolic space decomposes into a direct sum of hyperbolic planes [7, 3.4(1)].

1.2 Witt groups

1.2.1 Witt group of a field

Let k be as defined above, and let $Q(k)$ be the set of isometry classes of nonsingular symmetric bilinear forms over k. $Q(k)$ is a semigroup, where the addition is defined by the orthogonal sum,

$$[V, \phi] + [W, \psi] = [V \oplus W, \phi \oplus \psi].$$

The Grothendieck group of $Q(k)$ modulo the subgroup generated by hyperbolic forms is called the *Witt group of* k, denoted by $W(k)$. By the diagonalizability, every element of $W(k)$ can be represented by a finite sum of unary forms

$$\langle a_1 \rangle + \langle a_2 \rangle + \cdots + \langle a_r \rangle,$$

where $a_1, \ldots, a_r \in k^\times$. Quotienting by hyperbolic forms allows one to write

$$\langle -a \rangle = -\langle a \rangle \in W(k).$$

Furthermore, it is a consequence of Witt decomposition theorem [7, 4.1] that every element of $W(k)$ can be represented by an anisotropic form.

Note that the isometry relation implies $\langle a^2 \rangle = \langle 1 \rangle$.

Let us look at some examples. The signature map and the dimension map respectivly induce isomorphisms [7, p. 41-42]

$$W(\mathbb{R}) \xrightarrow{\sim} \mathbb{Z}, \qquad W(\mathbb{C}) \xrightarrow{\sim} \mathbb{Z}/2.$$

If $p \in \mathbb{Z}$ is an odd prime, the Witt group of the finite field $\mathbb{F}_p$ is given by[1] [7, p. 45]

$$W(\mathbb{F}_p) = \begin{cases} (\mathbb{Z}/2\mathbb{Z})^2 & \text{if} \quad p \equiv 1 \pmod 4, \\ \mathbb{Z}/4\mathbb{Z} & \text{if} \quad p \equiv 3 \pmod 4. \end{cases}$$

The Witt group of the rationals is given by [7, p. 175]

$$W(\mathbb{Q}) \simeq \mathbb{Z} \oplus \mathbb{Z}/2 \oplus \coprod_{p \neq 2} W(\mathbb{Z}/p\mathbb{Z}). \tag{1.2}$$

We will see that this is derived from the Gersten-Witt complex of $\operatorname{Spec} \mathbb{Z}$ (1.4).

We will come back to this later, but for now it suffices to say that the Gersten-Witt complex is largely an attemp to generalize this isomorphism to schemes.

For future reference, we state a theorem by Springer and Knebusch [8, p. 85]:

Theorem 1.1. *Let A be a discrete valuation ring with maximal ideal $\mathfrak{m}$ and quotient field F, where* $\operatorname{char}(A/\mathfrak{m}) \neq 2$. *If $\pi \in A$ is a generator of $\mathfrak{m}$ and $i = 1$ or 2, there is a unique homomorphism*

$$\partial_i^\pi : W(F) \to W(A/\mathfrak{m})$$

such that

$$\langle \pi^j u \rangle \mapsto \begin{cases} \langle \bar{u} \rangle & \textit{if} \quad j \not\equiv i \pmod 2, \\ 0 & \textit{if} \quad j \equiv i \pmod 2. \end{cases}$$

Note that ∂_2^π depends on the choice of the generator π, while ∂_1^π doesn't. ∂_1^π and ∂_2^π are called the *first and second residue homomorphism*, respectively.

1.2.2 Witt group of a ring

The Witt group can be similarly defined for a ring A in which 2 is a unit. The finite-dimensional k-vector spaces are replaced by finitely generated projective A-modules, the rank of projective modules replacing the dimension of vector spaces.

[1] $W(k)$ admits a ring structure induced by tensor product, but we won't need this in this paper.

The definition of nonsingularity remains the same (i.e., bijectivity of the adjoint). Instead of quotienting the Grothendieck group associated with the semigroup of nonsingular symmetric bilinear forms by hyperbolic forms, we quotient by a larger class of forms called lagrangians[2]: if M is a finitely generated projective A-module, a nonsingular symmetric bilinear form

$$\phi : M \times M \to A$$

is called a *lagrangian* if there is a direct summand $N \subset M$ such that $\phi|_{N \times N} = 0$ and the induced pairing

$$N \times (M/N) \to A$$

is nonsingular (i.e., both adjoints are bijective[3]). The submodule N is called a *sublagrangian*. As in the case of hyperbolic spaces, (M, ϕ) is a lagrangian if and only if M has an orthogonal direct summand, or equivalently, a totally isotropic direct summand with half the rank of M [6, Corollary 2, ii].

The hyperbolic form still plays a role, mainly due to its useful properties (e.g., (1.2) below). Let us first generalize hyperbolic forms in the context of finitely generated projective modules. If M is a finitely generated projective A-module and $M^* := \mathrm{Hom}(M, A)$, the *hyperbolic form* associated with M is defined to be a symmetric bilinear form

$$\phi : (M \oplus M)^* \times (M \oplus M^*) \to A$$

induced by the canonical pairing $M \times M^* \to A$, and requiring that

$$\phi|_{M \times M} = 0, \qquad \phi|_{M^* \times M^*} = 0.$$

[2] Knebusch[6] uses the term "metabolic space" for our lagrangian, "split metabolic space" for our hyperbolic form, "lagrangian" for our sublagrangian, and "sublagrangian" for our totally isotropic space.

[3] In fact, the reflexivity of finitely generated projective modules implies that one adjoint is bijective if and only if the other is.

The reflexivity of finitely generated projective modules ensures its nonsingularity. Note that if $\rho_M : M \to M^{**}$ is the canonical map, (M, ϕ) can be represented by the "matrix"

$$\begin{pmatrix} 0 & \mathrm{id}_{M^*} \\ \rho_M & 0 \end{pmatrix}. \tag{1.3}$$

ρ_M is an isomorphism because M is finitely generated projective. Hence, if M is free, then (1.3) is isometric to a direct sum of the hyperbolic planes,

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

Thus our definition of hyperbolic form agrees with the previous one over fields. Note that the submodule $M \subset M \oplus M^*$ is a sublagrangian, since the canonical pairing

$$M \times M^* \to V$$

is nonsingular by the reflexivity of finitely generated projective modules. Hence, every hyperbolic form is a lagrangian.

We note a useful lemma that we will be needed later:

Lemma 1.2. *If (M, ϕ) is a nonsingular symmetric bilinear form, then the form $(M \oplus M, \phi \oplus (-\phi))$ is isometric to the hyperbolic form associated with M.*

Proof. Let

$$\Delta := \{(m, m) \in M \oplus M \mid m \in M\}, \qquad N := \{(m/2, -m/2) \in M \oplus M \mid m \in M\}.$$

Clearly $\Delta, N \simeq M$, $\Delta \cap N = \{0\}$, and the isomorphism

$$M \oplus M \xrightarrow{\sim} \Delta + N, \qquad (m, m') \mapsto (m + m/2, m' - m'/2)$$

establishes an isometry between $(M \oplus M, \phi \oplus (-\phi)) \simeq (\Delta \oplus N, \phi \oplus (-\phi))$ and the hyperbolic form associated with M. $\square$

Let us look at some examples. The signature map induces an isomorphism $W(\mathbb{Z}) \xrightarrow{\sim} \mathbb{Z}$ [8, p. 23], and there is a split short exact sequence [7, p. 175]

$$0 \to W(\mathbb{Z}) \to W(\mathbb{Q}) \to (\mathbb{Z}/2) \oplus \coprod_{p \neq 2} W(\mathbb{Z}/p\mathbb{Z}) \to 0, \tag{1.4}$$

which gives rise to the isomorphism (1.2). The split exactness of (1.4) is a consequence of the Hasse-Minkowski principle applied to the global field $\mathbb{Q}$. More generally, if we use a Gorenstein ring A of dimension n instead of $\mathbb{Z}$, we can construct a complex of the form

$$0 \to W(A) \to \coprod_{\operatorname{ht}\mathfrak{p}=0} W(\kappa(\mathfrak{p})) \to \cdots \to \coprod_{\operatorname{ht}\mathfrak{p}=n} W(\kappa(\mathfrak{p})) \to 0, \tag{1.5}$$

where $\kappa(\mathfrak{p})$ is the residue class field at $\mathfrak{p} \in \operatorname{Spec} A$. However, (1.5) is not exact in general. As we shall see, (1.5) is the Gersten-Witt complex of $\operatorname{Spec} A$, and Pardon [12, 5.1] proved that it is exact if A is a regular local ring and is of essentially finite type over a field of characteristic different from 2.

1.2.3 Witt group of a scheme

Knebusch[6] defined the Witt group of a scheme $(X, \mathcal{O}_X)$, whose elements are represented by symmetric bilinear forms

$$\phi : \mathcal{M} \times \mathcal{M} \to \mathcal{O}_X,$$

where $\mathcal{M}$ is a locally free sheaf of $\mathcal{O}_X$-modules. The definition of nonsingularity remains the same as in the affine case (i.e., both adjoints are bijective), but the sublagrangian is no longer required to be a split submodule. In fact, sublagrangians always split for affine schemes [6, p. 134], so this definition agrees with the earlier one. The criteria for a sublagrangian is thus an orthogonal submodule, or equivalently, a totally isotropic submodule with half the rank [6, Corollary 2ii]. Moreover, the rank of a totally isotropic submodule cannot exceed half the rank [6, Corollary 2i].

1.3 Gersten-Witt complex

The *Gersten-Witt complex* of a scheme X of dimension n is a complex of the form

$$0 \to \coprod_{x \in X^{(0)}} W(\kappa(x)) \xrightarrow{d^0} \coprod_{x \in X^{(1)}} W(\kappa(x)) \xrightarrow{d^1} \cdots \to \coprod_{x \in X^{(n)}} W(\kappa(x)) \xrightarrow{d^n} 0,$$

where $\kappa(x)$ is the residue class field at $x \in X$, and $X^{(p)} \subset X$ is the subset of codimension p points. A main challenge in constructing such a complex is to define canonical boundary maps d^p. The second residue homomorphism (Theorem 1.1) depends on the choice of the uniformizer, so it cannot be used directly. Several authors constructed the complex using different methods, such as spectral sequences [1], the module of differentials [4, 14], and the canonical sheaf [12]. In the latter, using the analogues of Quillen's localization sequence of K-groups [15, 8.4], Pardon constructed the Gersten-Witt complex of a Gorenstein scheme, and showed that it is acyclic if the scheme is the spectrum of a regular local ring essentially of finite type over a field with characteristic different from 2. This implies the existence of a flasque resolution $\mathcal{W}^\bullet(X)$ of a *Witt sheaf* of a regular scheme X of finite type over k, whose cohomologies furnish us with a new set of invariants for the scheme X.

Computing cohomologies of the Gersten-Witt complex is difficult in practice, because it involves Witt groups of residue class fields at all (possibly infinitely many) points of the scheme. On the other hand, if X is a complex n-dimensional toric variety, then X is filtered as

$$X = X^0 \supset X^1 \supset \cdots \supset X^n \supset X^{n+1} = \varnothing,$$

where $X^p - X^{p+1}$ is a disjoint union of $(n-p)$-tori, $\operatorname{Spec} \mathbb{C}[x_1, 1/x_1, \ldots, x_{n-p}, 1/x_{n-p}]$. Takeda [16] showed the Gersten-Witt complex of K-groups is quasi-isomorphic to a complex of K-groups of coherent sheaves of tori. We will show that the same result holds for Witt groups. The Witt group of the n-torus is known; for example [5, 13],

$$W(\mathbb{C}[x,1/x]) = (\mathbb{Z}/2)^2, \tag{1.6}$$

$$W(\mathbb{C}[x,y,1/x,1/y]) = (\mathbb{Z}/2)^4. \tag{1.7}$$

Hence, the quasi-isomorphic complex would consist of a finite number of Witt groups which are finite-dimensional vector spaces, and its cohomologies are much easier to compute. Using this method, we will compute cohomologies of the Gersten-Witt complex of the toric variety Hirzebruch surface H_n. Specifically, the Gersten-Witt complex of H_n is given by

$$0 \to \coprod_{x \in H_n^{(0)}} W(\kappa(x)) \xrightarrow{d^0} \coprod_{x \in H_n^{(1)}} W(\kappa(x)) \xrightarrow{d^1} \coprod_{x \in H_n^{(2)}} W(\kappa(x)) \xrightarrow{d^2} 0, \tag{1.8}$$

and we will show that it is quasi-isomorphic to a complex of the form

$$0 \to W(H_n - H_n^1) \xrightarrow{d^0_{H_n}} W(H_n^1 - H_n^2) \xrightarrow{d^1_{H_n}} W(H_n^2) \to 0, \tag{1.9}$$

where H_n^p is the closure of codimension p orbits of the torus action,

$$H_n = H_n^0 \supset H_n^1 \supset H_n^2 \supset H_n^3 = \varnothing,$$

and $H_n^p - H_n^{p+1}$ is a finite union of $(n-p)$-tori.

The rest of this paper is organized as follows:

In Chapter 2, we will review Pardon's construction of the Gersten-Witt complex of Gorenstein schemes.

In Chapter 3, we will introduce the Hirzebruch surface H_n, and construct a toric complex which is quasi-isomorphic to the Gersten-Witt complex of H_n.

In Chapter 4, we will prove the quasi-isomorphism.

In Chapter 5, we will compute the boundary maps of the toric complex.

In Chapter 6, we will compute cohomologies of the toric complex, and find that

$$H^0(\mathcal{W}^\bullet(H_n)) = H^0(\mathcal{W}^\bullet(H_n)) = \mathbb{Z}/2 \qquad \forall n \in \mathbb{Z},$$

but

$$H^i(\mathcal{W}^\bullet(H_{\text{even}})) \neq H^i(\mathcal{W}^\bullet(H_{\text{odd}})) \qquad \text{for} \quad i = 1, 2.$$

2

Witt groups with coefficients

To construct a canonical Gersten-Witt complex of a Gorenstein scheme X, Pardon[10] extended the notion of Witt group so that the bilinear forms take values in certain sheaves of $\mathcal{O}_X$-modules. In this section, we will review his construction.

Let X be a scheme of dimension n, and $\mathcal{C}$ a coherent sheaf of $\mathcal{O}_X$-modules with an injective resolution

$$0 \to \mathcal{C} \to \mathcal{E}^0 \xrightarrow{d^0} \mathcal{E}^1 \xrightarrow{d^1} \mathcal{E}^2 \to \cdots \to \mathcal{E}^n \xrightarrow{d^n} 0,$$

where $\mathcal{E}^p = \coprod_{x \in X^{(p)}} i_* E(\kappa(x))$, $E(\kappa(x))$ is the injective hull of the residue class field at x viewed as a constant sheaf on $\bar{x}$, and $i : \bar{x} \hookrightarrow X$ is the inclusion. Such $\mathcal{C}$ is called a *canonical sheaf* for X [2, Chapter 3]. It is unique up to tensor product with a locally free sheaf of rank 1. Not every scheme admits a canonical sheaf, but every regular scheme does, and $\mathcal{O}_X$ is a canonical sheaf in such case. Henceforth, unless otherwise stated, we will assume that X is a regular scheme. Set $\mathcal{V}^p := \ker d^p$.

Definition 2.1. *$\mathcal{CM}^p(X)$ is the category of CM $\mathcal{O}_X$-modules of codimension p.*

$Q(\mathcal{CM}^p(X); \mathcal{C})$ is the category of isometry classes of nonsingular symmetric bi-

linear forms

$$\phi : \mathcal{M} \times \mathcal{M} \to \mathcal{V}^p,$$

where $\mathcal{M} \in \mathcal{CM}^p(X)$*.* $(\mathcal{M}, \phi) \in Q(\mathcal{CM}^p(X); \mathcal{C})$ *is called a lagrangian if there is a submodule* $\mathcal{N} \subset \mathcal{M}$ *such that* $\mathcal{N}, \mathcal{M}/\mathcal{N} \in \mathcal{CM}^p(X)$*,* $\phi|_{\mathcal{N}\times\mathcal{N}} = 0$*, and the induced pairing*

$$\mathcal{N} \times (\mathcal{M}/\mathcal{N}) \to \mathcal{V}^p$$

is nonsingular.

$Q(\mathcal{CM}^p(X); \mathcal{C})$ is a semigroup, where the addition defined by the orthogonal sum. The corresponding Grothendieck group modulo the subgroup generated by lagrangians is denoted by $W(\mathcal{CM}^p(X); \mathcal{C})$.

$\mathcal{CM}^n(X)$ is the category of sheaves of $\mathcal{O}_X$-modules of finite length, and $\mathcal{CM}^0(X)$ is the category of coherent sheaves of locally free $\mathcal{O}_X$-modules. Hence, $W(\mathcal{CM}^0(X); \mathcal{O}_X)$ is the Witt group $W(X)$ defined by Knebusch [6]. Based on this observation, we will use the notation

$$W(X; \mathcal{C}) := W(\mathcal{CM}^0(X); \mathcal{C}),$$

and if $\mathcal{O}_X$ is used as the canonical module, we will suppress it:

$$W(\mathcal{CM}^p(X)) := W(\mathcal{CM}^p(X); \mathcal{O}_X)$$

Now we will construct the so-called *lattice map*,

$$\mathcal{L}^p : \coprod_{x \in X^{(p)}} W(\mathcal{CM}^p(X_x); \mathcal{C}_x) \dashrightarrow W(\mathcal{CM}^{p+1}(X); \mathcal{C}).$$

Definition 2.2. *If* $x \in X$*, let* $i_x : \bar{x} \hookrightarrow X$ *be the inclusion map. Suppose that for each* $x \in X^{(p)}$*, we are given* $[M_x, \phi] \in W(X_x; \mathcal{C}_x)$*. Let*

$$\phi := \coprod_{x \in X^{(p)}} i_{x*}\phi_x, \qquad \mathcal{N} := \coprod_{x \in X^{(p)}} i_{x*}M_x,$$

where the $\mathcal{O}_{X,x}$-module M_x is viewed as a constant sheaf on $\bar{x}$. An $\mathcal{O}_X$-submodule $\mathcal{M} \subset \mathcal{N}$ is called a lattice if $\mathcal{M} \in \mathcal{CM}^p(X)$ and $\mathcal{M}_x = M_x \;\; \forall x \in X^{(p)}$. The lattice is integral with respect to ϕ if $\phi(\mathcal{M} \times \mathcal{M}) \subset \mathcal{V}^p$. If $\mathcal{M}$ is an integral lattice for $(\mathcal{N}, \phi)$, its dual lattice is an $\mathcal{O}_X$-module $\mathcal{M}'$ defined for each affine open subset $U \subset X$ by

$$\mathcal{M}'(U) = \{n \in \mathcal{N}(U) \mid \phi(U)(n, \mathcal{M}(U)) \subset \mathcal{V}^p(U)\}.$$

If $\mathcal{M}$ is an integral lattice for $(\mathcal{N}, \tau)$ and $\mathcal{M}'$ is its dual lattice, there is a well-defined bilinear form

$$\bar{\phi} : \frac{\mathcal{M}'}{\mathcal{M}} \times \frac{\mathcal{M}'}{\mathcal{M}} \dashrightarrow \mathcal{V}^{p+1}$$

given by $\bar{\phi}(\bar{m}'_1, \bar{m}'_2) = d^p(\phi(m'_1, m'_2))$ for each affine open subset $U \subset X$. Pardon[12] proved that there is a well-defined map

$$\mathcal{L}^p : \coprod_{x \in X^{(p)}} W(\mathcal{CM}^p(X_x); \mathcal{C}_x) \dashrightarrow W(\mathcal{CM}^{p+1}(X); \mathcal{C}), \qquad [\mathcal{N}, \phi] \mapsto [\mathcal{M}'/\mathcal{M}, \bar{\phi}].$$

Unfortunately, the integral lattice does not exist in general. To get around this problem, Pardon relaxed the condition $\mathcal{M} \in \mathcal{CM}^p(X)$ to a weaker condition $\mathcal{M} \in \mathcal{S}_2^p(X)$, where $\mathcal{S}_i^p(X)$ is the category of coherent sheaves of $\mathcal{O}_X$-modules $\mathcal{M}$ of codimension p such that

$$\operatorname{depth}_{\mathcal{O}_{X,x}} \mathcal{M}_x \geqslant \min\left\{i, \dim_{\mathcal{O}_{X,x}} \mathcal{M}_x\right\} \qquad \forall x \in X.$$

He then proved the existence of an $\mathcal{S}_2^p$-lattice. There is an inclusion $\mathcal{CM}^p(X) \subset \mathcal{S}_i^p(X)$, which is an equality if $\dim X \leqslant p + i$. Hence, if $\dim X = 2$, then $\mathcal{CM}^p(X) = \mathcal{S}_2^p(X)$, so CM lattice always exists in this case.

Pardon's original proof [12, 3.12] contains an error, which prevents its application to non-affine schemes. Here we include a proof which works for any Gorenstein schemes, provided that $p = 0$:

Proposition 2.3. *Let X be a Gorenstein scheme, $x \in X^{(p)}$, and $[N, \psi] \in W(\mathcal{CM}^p(X_x); \mathcal{C}_x)$. If X is affine or $p = 0$, then there exists an integral lattice for $[N, \psi]$.*

Proof. Let $i : \bar{x} \hookrightarrow X$ be the inclusion, and $\mathcal{M} \subset i_*N$ an $\mathcal{O}_X$-submodule such that $\mathcal{M}_x = N$. Then $\mathcal{M} \in \mathcal{S}_1^p(X)$ by [12, 1.19], so the canonical map $\rho : \mathcal{M} \to \mathcal{M}^{**}$ is injective. On the other hand, since $\mathcal{M}_x$ is an $\mathcal{O}_{X,x}$-module of finite length, $\rho_x : \mathcal{M}_x \to \mathcal{M}_x^{**}$ is bijective. Thus, we have a map

$$\mathcal{M}_x^{**} \underset{\sim}{\xrightarrow{\rho_x^{-1}}} \mathcal{M}_x = N.$$

Since taking stalks at x is a left adjoint to i_*, we obtain an injective map $\mathcal{M}^{**} \hookrightarrow i_*N$. By [12, 1.13], $\mathcal{M}^{**} \in \mathcal{CM}^p(X)$, so $\mathcal{M}^{**}$ is a lattice for $[N, \psi]$.

Now consider the composition

$$\theta : \mathcal{M} \times \mathcal{M} \hookrightarrow i_*N \times i_*N \xrightarrow{i_*\psi} i_*\mathcal{E}_x^p \hookrightarrow \mathcal{E}^p \xrightarrow{d^p} \mathcal{E}^{p+1} \xrightarrow{\sim} \bigoplus_{y \in X^{(p+1)}} i_{y*}\mathcal{E}_y^{p+1},$$

where $i_y : \bar{y} \hookrightarrow X$ is the inclusion. If $X = \operatorname{Spec} A$, we can always "clear out denominators" by multiplying $\mathcal{M}$ by some nonzero element $a \in A$, so that $\theta(a\mathcal{M} \times a\mathcal{M}) = 0$, i.e., $i_*\psi(a\mathcal{M} \times a\mathcal{M}) \subset \mathcal{V}^p$. Then $a\mathcal{M} \subset i_*N$ is an integral lattice.

This is not always possible, however, if X is not affine: for example, if $X = \mathbb{P}^1_{\mathbb{C}}$, the only regular functions on X are constant functions, so one can't clear out denominators by multiplying by a regular function. We will show that if $p = 0$, one can construct a subsheaf $\mathcal{D} \subset \mathcal{M}$ using a Weil divisor that cancels out the poles appearing in the image of $i_*\psi$, thus giving $\theta(\mathcal{D} \times \mathcal{D}) = 0$.

So assume that $p = 0$. Then we may assume that $N = K(X)$, viewed as a constant sheaf on X, and that ψ is given by multiplication by some $f \in K(X)$. We may take $\mathcal{M} = \mathcal{O}_X$ as our (non-integral) lattice. Our aim is to find a submodule $\mathcal{D} \subset \mathcal{O}_X$ such that $\mathcal{D} \in \mathcal{CM}^0(X)$ and $\theta(\mathcal{D} \times \mathcal{D}) = 0$. Let

$$(f) := \sum_{y \in X^{(1)}} n_y(f) y \in \operatorname{Div}(X),$$

and

$$(f)^+ := \sum_{y \in X^{(1)}} n_y^+(f) y, \qquad (f)^+ := \sum_{y \in X^{(1)}} n_y^-(f) y,$$

where

$$n_y^+(f) := \begin{cases} n_y(f) & \text{if } n_y(f) \geqslant 0, \\ 0 & \text{if } n_y(f) < 0, \end{cases} \qquad n_y^-(f) := \begin{cases} 0 & \text{if } n_y(f) > 0, \\ n_y(f) & \text{if } n_y(f) \leqslant 0. \end{cases}$$

Define an $\mathcal{O}_X$-module $\mathcal{D}$ for each affine open subset $U \subset X$ by

$$\mathcal{D}(U) := \left\{ g \in K(X) \mid n_y(g) + n_y^-(f) \geqslant 0 \quad \forall y \in U \cap X^{(1)} \right\}.$$

Since

$$\mathcal{O}_X(U) = \left\{ g \in K(X) \mid n_y(g) \geqslant 0 \quad \forall y \in U \cap X^{(1)} \right\},$$

$\mathcal{D}$ is a subsheaf of $\mathcal{O}_X$. Moreover, if $y \in X^{(1)}$ and $\pi_y \subset \mathcal{O}_{X,y}$ is the maximal ideal, then

$$\mathcal{D}_y = \begin{cases} \mathcal{O}_{X,y} & \text{if } y \notin \operatorname{Supp}(f)^-, \\ \pi_y^{|n_y^-(f)|} \mathcal{O}_{X,y} & \text{if } y \in \operatorname{Supp}(f)^-. \end{cases}$$

Hence, if $\eta \in X$ is the generic point, then $\mathcal{D}_\eta = \mathcal{O}_{X,\eta} = K(X)$, and by construction,

$$\theta_y(\mathcal{D}_y \times \mathcal{D}_y) = 0 \in \mathcal{E}_y^1 \qquad \forall y \in X^{(1)}.$$

Hence, $\theta(\mathcal{D} \times \mathcal{D}) = 0$. Finally, since $\mathcal{D}$ is locally free, $\mathcal{D} \in \mathcal{CM}^0(X)$. □

Given $[\mathcal{M}, \phi] \in W(\mathcal{CM}^p(X); \mathcal{C})$, we may localize at $x \in X^{(p)}$ to obtain $[\mathcal{M}_x, \phi_x] \in W(\mathcal{CM}^p(X_x); \mathcal{C}_x)$. Hence, there is a map

$$\mathcal{K}^p : W(\mathcal{CM}^p(X); \mathcal{C}) \to \coprod_{x \in X^{(p)}} W(\mathcal{CM}^p(X_x); \mathcal{C}_x).$$

Pardon[12, 3.9, 3.23] showed that the sequence

$$0 \to W(\mathcal{CM}^p(X); \mathcal{C}) \xrightarrow{\mathcal{K}^p} \coprod_{x \in X^{(p)}} W(\mathcal{CM}^p(X_x); \mathcal{C}_x) \xrightarrow{\mathcal{L}^p} W(\mathcal{CM}^{p+1}(X); \mathcal{C}) \tag{2.1}$$

is exact. Setting $d^p := \mathcal{K}^{p+1} \circ \mathcal{L}^p$, we thus obtain a complex

$$0 \to \coprod_{x \in X^{(0)}} W(\mathcal{CM}^0(X_x); \mathcal{C}_x) \xrightarrow{d^0} \cdots \to \coprod_{x \in X^{(n)}} W(\mathcal{CM}^n(X_x); \mathcal{C}_x) \xrightarrow{d^n} 0. \tag{2.2}$$

Moreover, he showed [12, 5.1] that if X is the spectrum of a regular local ring which is essentially of finite type over a field of characteristic different from 2, then $\mathcal{L}^p$ is surjective and therefore (2.2) is acyclic, with $\ker d^0 = W(\mathcal{S}_1^0(A); \mathcal{C})$. To recover the Gersten-Witt complex from (2.2), he makes use of the dévissage [11, 2.2]

$$\coprod_{x \in X^{(p)}} W(\kappa(x); \Lambda^p N(\mathfrak{m}_x) \otimes_{\mathcal{O}_x} \mathcal{C}_x) \xrightarrow{\sim} \coprod_{x \in X^{(p)}} W(\mathcal{CM}^p(X_x); \mathcal{C}_x), \tag{2.3}$$

where $N(\mathfrak{m}_x) := \mathrm{Hom}_{\mathcal{O}_x}(\mathfrak{m}_x/\mathfrak{m}_x^2, \kappa(x))$. One may always choose an isomorphism

$$\Lambda^p N(\mathfrak{m}_x) \otimes_{\mathcal{O}_X} \mathcal{C}_x \simeq \kappa(x),$$

giving rise to a *non-canonical* isomorphism

$$W(\kappa(x); \Lambda^p N(\mathfrak{m}_x) \otimes_{\mathcal{O}_x} \mathcal{C}_x) \simeq W(\kappa(x)).$$

Thus we obtain the classical Gersten-Witt complex

$$0 \to \coprod_{x \in X^{(0)}} W(\kappa(x)) \xrightarrow{d^0} \coprod_{x \in X^{(1)}} W(\kappa(x)) \xrightarrow{d^1} \cdots \to \coprod_{x \in X^{(n)}} W(\kappa(x)) \xrightarrow{d^n} 0.$$

The local acyclicity of (2.2) implies that we can sheafify it to obtain a flasque resolution. Define a *Witt sheaf* for each affine open subset $U \subset X$ by

$$\mathcal{W}(X; \mathcal{C})(U) := \ker \left(W(\kappa(\eta); \mathcal{C}_\eta) \xrightarrow{d^0} \coprod_{x \in U \cap X^{(1)}} W(\mathcal{CM}^1(X_x); \mathcal{C}_x) \right),$$

where $\eta \in X$ is the generic point. (2.2) then sheafifies to a flasque resolution of

$\mathcal{W}(X;\mathcal{C})$:

$$0 \to i_{\eta *}W(\kappa(\eta);\mathcal{C}_\eta) \xrightarrow{d^0} \coprod_{x\in X^{(1)}} i_{x*}W(\kappa(x);N(\mathfrak{m}_x)\otimes_{\mathcal{O}_{X,x}}\mathcal{C}_x) \xrightarrow{d^1} \cdots$$

$$\cdots \to \coprod_{x\in X^{(n)}} i_{x*}W(\kappa(x);N(\mathfrak{m}_x)\otimes_{\mathcal{O}_{X,x}}\mathcal{C}_x) \xrightarrow{d^n} 0,$$

where $W(\kappa(x);N(\mathfrak{m}_x)\otimes_{\mathcal{O}_{X,x}}\mathcal{C}_x)$ is viewed as a constant sheaf on $\bar{x}$, and $i_x : \bar{x} \hookrightarrow X$ is the inclusion [12, 0.11]. By the Purity Theorem[9], the stalk of $\mathcal{W}(X;\mathcal{C})$ at $x \in X$ is $W(X_x;\mathcal{C}_x)$, so $\mathcal{W}(X;\mathcal{C})(U) = W(U)$.

3

Hirzebruch surfaces

This chapter is organized as follows:

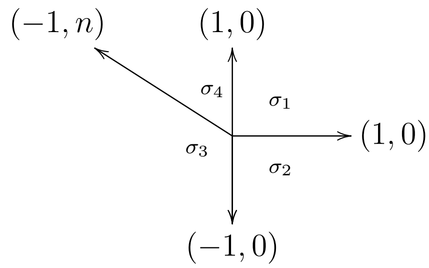
In Section 3.1, we introduce the Hirzebruch surface H_n, and discuss its geometry.

In Section 3.2, we define a new complex of Witt groups on 2-dimensional surfaces supported on codimension p tori. We define these Witt groups in Sections 3.3. In Section 3.4, we define the boundary maps of this complex.

In Section 3.5, we show that the Witt group supported on codimension p tori is isomorphic to the Witt group of the the tori, thus obtaining a complex of Witt groups of tori. Our claim is that this complex is quasi-isomorphic to the Gersten-Witt complex of H_n. We prove the quasi-isomorphism in the next chapter.

3.1 Geometry of Hirzebruch surfaces

The Hirzebruch surface H_n is a $\mathbb{P}^1_{\mathbb{C}}$ bundle $\pi : H_n \to \mathbb{P}^1_{\mathbb{C}}$ obtained by projectivizing the line bundle $\mathcal{O}_{\mathbb{P}^1_{\mathbb{C}}} \oplus \mathcal{O}_{\mathbb{P}^1_{\mathbb{C}}}(-n)$. It can be constructed as a toric variety by a fan depicted below [**?** , p. 7]:

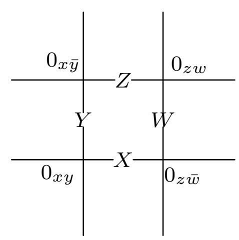


To each cone σ_i corresponds an affine open subset U_{σ_i} :

$$\begin{array}{ccccccc} U_{\sigma_2} & = & \mathrm{Spec}\,\mathbb{C}[x, \bar{y}] & \longleftrightarrow & \mathrm{Spec}\,\mathbb{C}[z, w] & = & U_{\sigma_3} \\ & & \updownarrow & \nwarrow\!\!\!\!\nearrow \;\; \swarrow\!\!\!\!\searrow & \updownarrow & & \\ U_{\sigma_1} & = & \mathrm{Spec}\,\mathbb{C}[x, y] & \longleftrightarrow & \mathrm{Spec}\,\mathbb{C}[z, \bar{w}] & = & U_{\sigma_4} \end{array} \tag{3.1}$$

where $z := 1/x$, $w := 1/x^n y$, $\bar{w} := 1/w$, $\bar{y} := 1/y$. The axes of U_{σ_i} constitute four projective lines in H_n, denoted by X, Y, Z, W:

(3.2)



X and Z are sections of $\pi : H_n \to \mathbb{P}^1_{\mathbb{C}}$. Y and W are fibres over $\mathbb{P}^1_{\mathbb{C}}$, hence have trivial normal bundles, while the normal bundles of X and Z are "twisted" by the integer parameter n. When $n = 0$, there is no twist, and $H_0 \simeq \mathbb{P}^1_{\mathbb{C}} \times \mathbb{P}^1_{\mathbb{C}}$.

Fernández-Carmena [3, 3.4] showed that the Witt group of a smooth complex surface is a birational invariant. Hence,

$$W(H_n) = W(\mathbb{P}^2_{\mathbb{C}}) = \mathbb{Z}/2 \qquad \forall n \in \mathbb{Z}. \tag{3.3}$$

We will adopt the following notations:

$$H^1_n := X \cup Y \cup Z \cup W, \qquad H^2_n := \{0_{xy}, 0_{zw}, 0_{x\bar{y}}, 0_{z\bar{w}}\}.$$

$$T_X := X - \{0_{xy}, 0_{z\bar{w}}\}, \qquad T_Y := Y - \{0_{xy}, 0_{x\bar{y}}\},$$

$$T_Z := Z - \{0_{zw}, 0_{x\bar{y}}\}, \qquad T_W := W - \{0_{zw}, 0_{z\bar{w}}\},$$

$$L := X \cup Z, \qquad N := Y \cup W,$$

$$T_L := T_X \cup T_Z, \qquad T_N := T_Y \cup T_W.$$

Note that $T_L = H_n^1 - N$, $T_N = H_n^1 - L$, and $H_n^2 = L \cap N$.

3.2 A complex of Witt groups supported on tori

Let $H = H_n$ for some $n \in \mathbb{Z}$, and $\eta \in H$ the generic point. The Gersten-Witt complex of H is given by

$$0 \to W(\kappa(\eta)) \xrightarrow{d^0} \coprod_{x \in H^{(1)}} W(\mathcal{CM}^1(H_x)) \xrightarrow{d^1} \coprod_{x \in H^{(2)}} W(\mathcal{CM}^2(H_x)) \to 0. \tag{3.4}$$

In Pardon's construction [12],

$$d^0 : W(\kappa(\eta)) \xrightarrow{\mathcal{L}^0} W(\mathcal{CM}^1(H)) \xrightarrow{\mathcal{K}^1} \coprod_{x \in H^{(1)}} W(\mathcal{CM}^1(H_x)),$$

$$d^1 : \coprod_{x \in H^{(1)}} W(\mathcal{CM}^1(H_x)) \xrightarrow{\mathcal{L}^1} W(\mathcal{CM}^2(H)) \underset{\sim}{\xrightarrow{\mathcal{K}^2}} \coprod_{x \in H^{(2)}} W(\mathcal{CM}^2(H_x)).$$

We claim (Proposition 4.3, Corollary 3.8) that there is a quasi-isomorphic complex

$$0 \to W(H - H^1) \xrightarrow{d_H^0} W(\mathcal{CM}^1_{T_L}(H - N)) \coprod W(\mathcal{CM}^1_{T_N}(H - L)) \xrightarrow{d_H^1} W(\mathcal{CM}^2_{H^2}(H)) \to 0, \tag{3.5}$$

where d_H^0 is the compositioin

$$W(H - H^1) \xrightarrow{\mathcal{L}^0_{H^1}} W(\mathcal{CM}^1_{H^1}(H)) \xrightarrow{\mathcal{K}^1_{H^1}} W(\mathcal{CM}^1_{H^1 - N}(H - N)) \coprod W(\mathcal{CM}^1_{H^1 - L}(H - L)), \tag{3.6}$$

where $\mathcal{K}^1_{H^1} := \mathcal{K}^1_N \coprod \mathcal{K}^1_L$,

$$\mathcal{K}^1_N : W(\mathcal{CM}^1_{H^1}(H)) \;\to\; W(\mathcal{CM}^1_{H^1-N}(H-N)), \tag{3.7}$$

$$\mathcal{K}^1_L : W(\mathcal{CM}^1_{H^1}(H)) \;\to\; W(\mathcal{CM}^1_{H^1-L}(H-L)), \tag{3.8}$$

and $d^1_H := \mathcal{L}^1_{T_L} \coprod \mathcal{L}^1_{T_N}$, where

$$\mathcal{L}^1_{T_L} : W(\mathcal{CM}^1_{T_L}(H-N)) \;\to\; W(\mathcal{CM}^2_{L\cap N}(H)), \tag{3.9}$$

$$\mathcal{L}^1_{T_N} : W(\mathcal{CM}^1_{T_N}(H-L)) \;\to\; W(\mathcal{CM}^2_{L\cap N}(H)). \tag{3.10}$$

Recall that $L \cap N = H^2$. $\mathcal{K}^1_{H^1}$ is an excision map induced by restriction, and it is injective because $\mathcal{K}^1$ is injective [12, 3.9].

In the next couple of sections, we define the Witt groups with support and the lattice maps $\mathcal{L}^0_{H^1}, \mathcal{L}^1_{T_L}, \mathcal{L}^1_{T_N}$.

3.3 Witt groups with support

If Y is a closed subscheme of X, $\mathcal{CM}^p_Y(X)$ will denote the category of coherent sheaves of CM $\mathcal{O}_X$-modules of codimension p supported on Y. Also, if $\mathcal{V}$ is a sheaf of $\mathcal{O}_X$-modules, then $\mathcal{V}_Y$ will denote the sheaf of $\mathcal{O}_X$-modules defined for each affine open subset $U \subset X$ by

$$\mathcal{V}_Y(U) := \bigcup_{i=1}^{\infty} (0 : \mathscr{I}(Y)(U)^i)_{\mathcal{V}(U)},$$

where $\mathscr{I}(Y) \subset \mathcal{O}_X$ is the ideal sheaf of Y.

3.3.1 $W(\mathcal{CM}^1_{H^1}(H))$

Let $Q^1_{H^1}(H)$ be the semigroup of isometry classes of nonsingular symmetric bilinear forms

$$\phi : \mathcal{M} \times \mathcal{M} \to \mathcal{V}^1_{\mathcal{O}_H},$$

where $\mathcal{M} \in \mathcal{CM}^1_{H^1}(H)$. $(\mathcal{M}, \phi) \in Q^1(H)$ is called a *lagrangian* if there is a submodule $\mathcal{N} \subset \mathcal{M}$ such that $\mathcal{N}, \mathcal{M}/\mathcal{N} \in \mathcal{CM}^1_{H^1}(H)$, $\phi|_{\mathcal{N}\times\mathcal{N}} = 0$, and the induced pairing

$$\mathcal{N} \times (\mathcal{M}/\mathcal{N}) \to \mathcal{V}^1_{\mathcal{O}_H}$$

is nonsingular. $W(\mathcal{CM}^1_{H^1}(H))$ is the Grothendieck group of $Q^1_{H^1}(H)$ modulo the subgroup generated by lagrangians. Note that the images of the above bilinear maps lie in $\mathcal{V}^1_{\mathcal{O}_H, H^1}$.

3.3.2 $W(\mathcal{CM}^1_{T_L}(H-N))$

Let $Q^1_{T_L}(H-N)$ be the semigroup of isometry classes of nonsingular symmetric bilinear forms

$$\phi : \mathcal{M} \times \mathcal{M} \to \mathcal{V}^1_{\mathcal{O}_{H-N}},$$

where $\mathcal{M} \in \mathcal{CM}^1_{T_L}(H-N)$. $(\mathcal{M}, \phi) \in Q^1_{T_L}(H-N)$ is called a *lagrangian* if there is a submodule $\mathcal{N} \subset \mathcal{M}$ such that $\mathcal{N}, \mathcal{M}/\mathcal{N} \in \mathcal{CM}^1_{T_L}(H-N)$, $\phi|_{\mathcal{N}\times\mathcal{N}} = 0$, and the induced pairing

$$\mathcal{N} \times (\mathcal{M}/\mathcal{N}) \to \mathcal{V}^1_{\mathcal{O}_{H-N}}$$

is nonsingular. $W(\mathcal{CM}^1_{T_L}(H-N))$ is the Grothendieck group of $Q^1_{T_L}(H-N)$ modulo the subgroup generated by lagrangians. Note that the images of the above bilinear maps lie in $\mathcal{V}^1_{\mathcal{O}_{H-N}, T_L}$.

3.3.3 $W(\mathcal{CM}^1_{T_N}(H-L))$

Let $Q^1_{T_N}(H-L)$ be the semigroup of isometry classes of nonsingular symmetric bilinear forms

$$\phi : \mathcal{M} \times \mathcal{M} \to \mathcal{V}^1_{\mathcal{O}_{H-L}},$$

where $\mathcal{M} \in \mathcal{CM}^1_{T_N}(H-L)$. $(\mathcal{M}, \phi) \in Q^1_{T_N}(H-L)$ is called a *lagrangian* if there is a submodule $\mathcal{N} \subset \mathcal{M}$ such that $\mathcal{N}, \mathcal{M}/\mathcal{N} \in \mathcal{CM}^1_{T_N}(H-L)$, $\phi|_{\mathcal{N}\times\mathcal{N}} = 0$, and the induced pairing

$$\mathcal{N} \times (\mathcal{M}/\mathcal{N}) \to \mathcal{V}^1_{\mathcal{O}_{H-L}}$$

is nonsingular. $W(\mathcal{CM}^1_{T_N}(H-L))$ is the Grothendieck group of $Q^1_{T_N}(H-L)$ modulo the subgroup generated by lagrangians. Note that the images of the above bilinear maps lie in $\mathcal{V}^1_{\mathcal{O}_{H-L},T_N}$.

3.3.4 $\mathcal{CM}^2_{H^2}(H)$

Let $Q^2_{H^2}(H)$ be the semigroup of isometry classes of nonsingular symmetric bilinear forms

$$\phi : \mathcal{M} \times \mathcal{M} \to \mathcal{V}^2_{\mathcal{O}_H},$$

where $\mathcal{M} \in \mathcal{CM}^2_{H^2}(H)$. $(\mathcal{M}, \phi) \in Q^2_{H^2}(H)$ is called a *lagrangian* if there is a submodule $\mathcal{N} \subset \mathcal{M}$ such that $\mathcal{N}, \mathcal{M}/\mathcal{N} \in \mathcal{CM}^2_{H^2}(H)$, $\phi|_{\mathcal{N}\times\mathcal{N}} = 0$, and the induced pairing

$$\mathcal{N} \times (\mathcal{M}/\mathcal{N}) \to \mathcal{V}^2_{\mathcal{O}_H}$$

is nonsingular. $W(\mathcal{CM}^2_{H^2}(H))$ is the Grothendieck group of $Q^2_{H^2}(H)$ modulo the subgroup generated by lagrangians. Note that the images of the above bilinear maps lie in $\mathcal{V}^2_{\mathcal{O}_H,H^2}$.

3.4 Lattice maps with support

We will use $\mathcal{O}_H$ for our canonical sheaf for H. The construction is essentially the same as Pardon's [12]. All we are doing here is to show that his construction works even with the additional support condition.

3.4.1 $\mathcal{L}^0_{H^1} : W(H - H^1) \to W(\mathcal{CM}^1_{H^1}(H))$

Let $[\mathcal{N}, \psi] \in W(H - H^1)$, so that

$$\psi : \mathcal{N} \times \mathcal{N} \to \mathcal{O}_{H-H^1},$$

where $\mathcal{N} \in \mathcal{CM}^0(H - H^1)$. Let $i : H - H^1 \hookrightarrow H$ be the inclusion. An $\mathcal{O}_H$-submodule $\mathcal{M} \subset i_*\mathcal{N}$ is a *lattice* if $\mathcal{M} \in \mathcal{CM}^0(H)$ and $\mathcal{M}|_{H-H^1} = \mathcal{N}$. The lattice is *integral* with respect to ψ if $(i_*\psi)(\mathcal{M} \times \mathcal{M}) \subset \mathcal{O}_H$.

$$\begin{array}{ccc} \mathcal{M} \times \mathcal{M} & \subset & i_*\mathcal{N} \times i_*\mathcal{N} \\ \downarrow & & \downarrow {\scriptstyle i_*\psi} \\ \mathcal{O}_H & \subset & i_*\mathcal{O}_{H-H^1} \end{array}$$

If $\mathcal{M}$ is an integral lattice for $[\mathcal{N}, \psi]$, its *dual lattice* is an $\mathcal{O}_H$-submodule $\mathcal{M}' \subset i_*\mathcal{N}$ defined for each affine open subset $U \subset H$ by

$$\mathcal{M}'(U) = \{n \in i_*\mathcal{N}(U) \mid (i_*\psi)(U)(n, \mathcal{M}(U)) \subset \mathcal{O}_H(U)\}.$$

Then there is an induced symmetric bilinear form

$$\bar{\psi} : \frac{\mathcal{M}'}{\mathcal{M}} \times \frac{\mathcal{M}'}{\mathcal{M}} \dashrightarrow \frac{i_*\mathcal{O}_{H-H^1}}{\mathcal{O}_H}$$

given by $\bar{\psi}(\bar{m}'_1, \bar{m}'_2) = d^0((i_*\psi)(m'_1, m'_2))$ on affine open subsets.

$\mathcal{M}'/\mathcal{M}$ is supported on H^1 because $\mathcal{M}'$ locally coincides with $\mathcal{M}$ at every point of $H - H^1$ by the nonsingularity of ψ. Moreover, $\mathcal{M}'/\mathcal{M} \in \mathcal{CM}^1(H)$ by [11, 1.2].

Before we prove our main propositions (Proposition 3.4, Proposition 3.5), we need some lemmas:

Lemma 3.1. *In the notation as above,*

$$i_*\mathcal{O}_{H-H^1}/\mathcal{O}_H = \mathcal{V}^1_{\mathcal{O}_H}.$$

Proof. $\mathcal{V}^1_{\mathcal{O}_H} = \frac{K(H)}{\mathcal{O}_H}$, where the function field $K(H)$ of H is viewed as a constant sheaf on H, and for every affine open subset $U \subset H$, $\mathcal{V}^1_{\mathcal{O}_H,H^1}(U) \subset (K(H)/\mathcal{O}_H)(U)$ is the subset of sections with poles along $H^1 \cap U$:

$$\mathcal{V}^1_{\mathcal{O}_H,H^1}(U) = \bigcup_{j=1}^{\infty}(0 : \mathscr{I}(H^1)^j(U))_{(i_*K(H)/\mathcal{O}_H)(U)} = \frac{i_*\mathcal{O}_{H-H^1}}{\mathcal{O}_H}(U) \subset \frac{K(H)}{\mathcal{O}_H}(U).$$

$\square$

Lemma 3.2. *Let* $[\mathcal{M}, \phi] \in W(\mathcal{CM}^{p+1}_{H^1}(X); \mathcal{C})$*, and* $\mathcal{N} \subset \mathcal{M}$ *a totally isotropic submodule such that* $\mathcal{N}, \mathcal{N}^\perp/\mathcal{N} \in \mathcal{CM}^{p+1}(X)$*. If the induced bilinear maps*

$$\alpha : \mathcal{N} \times \mathcal{M}/\mathcal{N}^\perp \to \mathcal{V}^{p+1}, \qquad \beta : \mathcal{N}^\perp/\mathcal{N} \times \mathcal{N}^\perp/\mathcal{N} \to \mathcal{V}^{p+1}$$

are nonsingular and $[\mathcal{N}^\perp/\mathcal{N}, \beta]$ *is a lagrangian, then* $[\mathcal{M}, \phi]$ *is a lagrangian.*

Proof. Let $\mathcal{K}/\mathcal{N} \subset \mathcal{N}^\perp/\mathcal{N}$ be a sublagrangian, where $\mathcal{N} \subset \mathcal{K} \subset \mathcal{N}^\perp$. We will show that $\mathcal{K} \subset \mathcal{M}$ is a sublagrangian. Since $\mathcal{N}, \mathcal{K}/\mathcal{N} \in \mathcal{CM}^{p+1}(X)$, we have $\mathcal{K} \in \mathcal{CM}^{p+1}(X)$ [11, 1.2]. Being a sublagrangian, $\mathcal{K}/\mathcal{N} \subset \mathcal{N}^\perp/\mathcal{N}$ is totally isotropic, so $\mathcal{K} \subset \mathcal{M}$ is totally isotropic, as well. Let

$$\bar{\phi} : \mathcal{K} \times \mathcal{M}/\mathcal{K} \to \mathcal{V}^{p+1}$$

be the induced pairing. We will show that the induced map

$$\operatorname{ad}^\dagger \bar{\phi} : \mathcal{M}/\mathcal{K} \to \mathscr{H}om(\mathcal{K}, \mathcal{V}^{p+1})$$

is bijective. This would imply $\mathcal{M}/\mathcal{K} \in \mathcal{CM}^{p+1}(X)$ [11, 1.6a], and that $\mathcal{K} \subset \mathcal{M}$ is a sublagrangian, finishing the proof.

To this end, note that there is a short exact sequence

$$0 \to \frac{\mathcal{N}^\perp/\mathcal{N}}{\mathcal{K}/\mathcal{N}} \xrightarrow{j} \frac{\mathcal{M}}{\mathcal{K}} \to \frac{\mathcal{M}}{\mathcal{N}^\perp} \to 0.$$

Taking $(-)^* \equiv \mathscr{H}om(-, \mathcal{V}^{p+1})$, we obtain a commutative diagram

$$\begin{array}{ccccccc} 0 \longleftarrow & \mathcal{K}/\mathcal{N} & \longleftarrow & \mathcal{K} & \xleftarrow[\text{inc.}]{i} & \mathcal{N} & \longleftarrow 0 \\ & \downarrow{\scriptstyle \operatorname{ad}\bar{\beta}} & & \downarrow{\scriptstyle \operatorname{ad}\bar{\phi}} & & \downarrow{\scriptstyle \operatorname{ad}\alpha} & \\ & ((\mathcal{N}^\perp/\mathcal{N})/(\mathcal{K}/\mathcal{N}))^* & \xleftarrow{j^*} & (\mathcal{M}/\mathcal{K})^* & \longleftarrow & (\mathcal{M}/\mathcal{N}^\perp)^* & \longleftarrow 0 \end{array}$$

where the rows are exact. Since $\operatorname{ad}\bar{\beta}$ and $\operatorname{ad}\alpha$ are isomorphisms, j^* is surjective, hence $\operatorname{ad}\bar{\phi}$ is bijective. Reflexivity of CM modules [11, 1.6a] then implies that $\operatorname{ad}^\dagger \bar{\phi}$ is also bijective, as desired. $\square$

Lemma 3.3. *Let* $[\mathcal{M}, \phi] \in W(\mathcal{CM}^{p+1}_{H^1}(X); \mathcal{C})$*, and* $\mathcal{N} \subset \mathcal{M}$ *a totally isotropic submodule such that* $\mathcal{N}, \mathcal{N}^\perp/\mathcal{N} \in \mathcal{CM}^{p+1}(X)$*. If the induced bilinear maps*

$$\alpha : \mathcal{N} \times \mathcal{M}/\mathcal{N}^\perp \to \mathcal{V}^{p+1}, \qquad \beta : \mathcal{N}^\perp/\mathcal{N} \times \mathcal{N}^\perp/\mathcal{N} \to \mathcal{V}^{p+1}$$

are nonsingular, then $[\mathcal{M}, \phi] = [\mathcal{N}^\perp/\mathcal{N}, \beta]$ *in* $W(\mathcal{CM}^{p+1}_{H^1}(X); \mathcal{C})$*.*

Proof. We will show that $[\mathcal{M}, \phi] - [\mathcal{N}^\perp/\mathcal{N}, \beta] = [\mathcal{M} \oplus (\mathcal{N}^\perp/\mathcal{N}), \phi \oplus (-\beta)]$ is a lagrangian. Let $\mathcal{N}_1 \equiv \mathcal{N} \oplus 0 \subset \mathcal{M} \oplus (\mathcal{N}^\perp/\mathcal{N})$. Then $\mathcal{N}_1 \in \mathcal{CM}^{p+1}_{H^1}(X)$, and

$$\mathcal{N}_1^\perp = \mathcal{N}^\perp \oplus (\mathcal{N}^\perp/\mathcal{N}) \subset \mathcal{M} \oplus (\mathcal{N}^\perp/\mathcal{N}), \qquad \frac{\mathcal{M} \oplus (\mathcal{N}^\perp/\mathcal{N})}{\mathcal{N}_1^\perp} \simeq \mathcal{M}/\mathcal{N}^\perp.$$

Since $\mathcal{N} \subset \mathcal{N}^\perp$, we have $\mathcal{N}_1 \subset \mathcal{N}_1^\perp$ and

$$\mathcal{N}_1^\perp/\mathcal{N}_1 \simeq (\mathcal{N}^\perp/\mathcal{N}) \oplus (\mathcal{N}^\perp/\mathcal{N}) \in \mathcal{CM}^{p+1}_{H^1}(X).$$

The induced bilinear maps

$$\mathcal{N}_1 \times \frac{\mathcal{M} \oplus (\mathcal{N}^\perp/\mathcal{N})}{\mathcal{N}_1^\perp} \to \mathcal{V}^{p+1}, \qquad \mathcal{N}_1^\perp/\mathcal{N}_1 \times \mathcal{N}_1^\perp/\mathcal{N}_1 \to \mathcal{V}^{p+1},$$

are isomorphic to

$$\alpha : \mathcal{N} \times (\mathcal{M}/\mathcal{N}^\perp) \to \mathcal{V}^{p+1},$$

$$\beta \oplus (-\beta) : (\mathcal{N}^\perp/\mathcal{N}) \oplus (\mathcal{N}^\perp/\mathcal{N}) \times (\mathcal{N}^\perp/\mathcal{N}) \oplus (\mathcal{N}^\perp/\mathcal{N}) \to \mathcal{V}^{p+1},$$

respectively. Note that the latter is a hyperbolic form, hence a lagrangian. Thus, applying Lemma 3.2 with $[\mathcal{M} \oplus (\mathcal{N}^\perp/\mathcal{N}), \phi \oplus (-\beta)]$ in place of $[\mathcal{M}, \phi]$, and $\mathcal{N}_1$ in place of $\mathcal{N}$, it follows that $[\mathcal{M} \oplus (\mathcal{N}^\perp/\mathcal{N}), \phi \oplus (-\beta)]$ is a lagrangian. □

We are now ready to prove one of our main propositions in this chapter:

Proposition 3.4. *There is a well-defined map*

$$\mathcal{L}^0_{H^1} : W(H - H^1) \dashrightarrow W(\mathcal{CM}^1_{H^1}(H)), \qquad [\mathcal{N}, \psi] \mapsto [\mathcal{M}'/\mathcal{M}, \bar{\psi}],$$

where $\mathcal{M}$ *is an integral lattice for* $[\mathcal{N}, \psi]$*.*

Proof. We have seen above that $\mathcal{M}'/\mathcal{M} \in \mathcal{CM}^1_{H^1}(H)$. We must show that 1) $\bar{\psi}$ is nonsingular, 2) $[\mathcal{M}'/\mathcal{M}, \bar{\psi}]$ is independent of the choice of $\mathcal{M}$, and 3) if $[\mathcal{N}, \psi]$ is a lagrangian, so is $[\mathcal{M}'/\mathcal{M}, \bar{\psi}]$.

1) For the nonsingularity of $\bar{\psi}$, we need to show that the induced map

$$\operatorname{ad}\bar{\psi} : \mathcal{M}'/\mathcal{M} \to \mathscr{H}om(\mathcal{M}'/\mathcal{M}, i_*\mathcal{O}_{H-H^1}/\mathcal{O}_H)$$

is bijective. Injectivity is clear from the definition of dual lattice. For surjectivity, let $\beta \in \mathscr{H}om(\mathcal{M}'/\mathcal{M}, i_*\mathcal{O}_{H-H^1}/\mathcal{O}_H)$. The short exact sequence

$$0 \to \mathcal{M} \to \mathcal{M}' \to \mathcal{M}'/\mathcal{M} \to 0$$

induces an exact sequence [11, 1.6b]

$$0 \leftarrow (\mathcal{M}'/\mathcal{M})\hat{} \xleftarrow{\delta} \tilde{M} \leftarrow \tilde{\mathcal{M}}' \leftarrow 0.$$

where $(-)\tilde{} := \mathscr{H}om(-, \mathcal{O}_H)$, $(-)\hat{} := \mathscr{H}om(-, \mathcal{V}^1(H))$. If $\beta = \delta(\alpha)$, there is a commutative diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathcal{M} & \longrightarrow & \mathcal{M}' & \longrightarrow & \mathcal{M}'/\mathcal{M} & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \alpha} & & \downarrow{\scriptstyle \alpha'} & & \downarrow{\scriptstyle \beta} & & \\
0 & \longrightarrow & \mathcal{O}_H & \longrightarrow & i_*\mathcal{O}_{H-H^1} & \longrightarrow & \mathcal{V}^1_{\mathcal{O}_H, H^1} & \longrightarrow & 0
\end{array}$$

where the rows are exact (Lemma 3.1). α' is given on an affine open subset $U \subset H$ by $\alpha'(U) = (i_*\psi)(U)(m, -)$ for some $m \in \mathcal{M}(U)$. Hence,

$$\beta(U) = \bar{\psi}(U)(\bar{m}', -) = \operatorname{ad}\bar{\psi}(U)(\bar{m}')$$

for some $\bar{m}' \in (\mathcal{M}'/\mathcal{M})(U)$, i.e., $\operatorname{ad}\bar{\psi}$ is surjective.

2) Now suppose that $\mathcal{M}_1, \mathcal{M}_2 \subset i_*\mathcal{N}$ are two integral lattices. Then $\mathcal{M}_1 \cap \mathcal{M}_2$ is also an integral lattice, so we may assume $\mathcal{M}_1 \subset \mathcal{M}_2$. Then there are inclusions

$$\mathcal{M}_1 \subset \mathcal{M}_2 \subset \mathcal{M}'_2 \subset \mathcal{M}'_1 \subset i_*\mathcal{N}.$$

Hence, $\mathcal{M}_2/\mathcal{M}_1 \subset \mathcal{M}'_1/\mathcal{M}_1$ is a totally isotropic subspace, and

$$(\mathcal{M}_2/\mathcal{M}_1)^{\perp} = \mathcal{M}'_2/\mathcal{M}_1 \subset \mathcal{M}'_1/\mathcal{M}_1.$$

Hence, there are isomorphisms

$$(\mathcal{M}_2/\mathcal{M}_1)^{\perp}/(\mathcal{M}_2/\mathcal{M}_1) \simeq \mathcal{M}'_2/\mathcal{M}_2, \qquad (\mathcal{M}'_1/\mathcal{M}_1)/(\mathcal{M}_2/\mathcal{M}_1)^{\perp} \simeq \mathcal{M}'_1/\mathcal{M}'_2,$$

and the induced bilinear maps

$$\alpha : \mathcal{M}_2/\mathcal{M}_1 \times (\mathcal{M}'_1/\mathcal{M}_1)/(\mathcal{M}_2/\mathcal{M}_1)^{\perp} \to \mathcal{V}^1(H),$$

$$\beta : (\mathcal{M}_2/\mathcal{M}_1)^{\perp}/(\mathcal{M}_2/\mathcal{M}_1) \times (\mathcal{M}_2/\mathcal{M}_1)^{\perp}/(\mathcal{M}_2/\mathcal{M}_1) \to \mathcal{V}^1(H),$$

are isomorphic to the bilinear forms

$$\alpha' : \mathcal{M}_2/\mathcal{M}_1 \times \mathcal{M}'_1/\mathcal{M}'_2 \to \mathcal{V}^1(H),$$

$$\beta' : \mathcal{M}'_2/\mathcal{M}_2 \times \mathcal{M}'_2/\mathcal{M}_2 \to \mathcal{V}^1(H),$$

respectively. We have shown the nonsingularity of β' in the first part of the proof. Similar proof shows that $\operatorname{ad}\alpha'$ is surjective. The reflexivity of CM modules [11, 1.6] implies that $\mathcal{M}''_1 = \mathcal{M}_1$, which in turn implies injectivity of $\operatorname{ad}\alpha'$. Hence, α' and β' are nonsingular, and therefore α and β are nonsingular. Since $\mathcal{M}_2/\mathcal{M}_1, \mathcal{M}'_2/\mathcal{M}_2 \in \mathcal{CM}^1(H)$ [11, 1.2], the result then follows from applying Lemma 3.3 to $[\mathcal{M}'_1/\mathcal{M}_1, \bar{\psi}] \in W(\mathcal{CM}^1_{H^1}(H))$ and $\mathcal{M}_2/\mathcal{M}_1 \subset \mathcal{M}'_1/\mathcal{M}_1$.

3) Let $[\mathcal{N}, \psi] \in W(H - H^1)$ be a lagrangian. We will show that $[\mathcal{M}'/\mathcal{M}, \bar{\psi}] = 0 \in W(\mathcal{CM}^1_{H^1}(H))$. Since the localization map

$$W(\mathcal{CM}^1_{H^1}(H)) \to \coprod_{x \in H^{(1)}} W(\mathcal{CM}^1(H_x))$$

is injective [12, 3.9], it suffices to show that

$$[\mathcal{M}'_x/\mathcal{M}_x, \bar{\psi}_x] = 0 \in W(\mathcal{CM}^1(H_x)) \qquad \forall x \in H^{(1)}.$$

Let $\mathcal{I} \subset \mathcal{N}$ be a sublagrangian, and define

$$\mathcal{G} := \ker(\mathcal{M} \rightarrowtail i_*\mathcal{N} \to i_*(\mathcal{N}/\mathcal{I})).$$

Since $\mathcal{M}$ is an integral lattice, there is an induced bilinear map

$$i_*\psi : \mathcal{M} \times \mathcal{M} \to \mathcal{O}_H.$$

Since $\mathcal{G} \subset i_*\mathcal{I}$, the submodule $\mathcal{G} \subset \mathcal{M}$ is totally isotropic with respect to $i_*\psi$, and there is an induced pairing

$$\alpha : \mathcal{G} \times \mathcal{M}/\mathcal{G} \to \mathcal{O}_H.$$

Let $(-)^* := \mathscr{H}om(-, \mathcal{O}_H)$. Since $\mathcal{M}' \simeq \mathcal{M}^*$ [12, 3.16], there is a commutative diagram

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathcal{G} & \xrightarrow{j} & \mathcal{M} & \xrightarrow{\pi} & \mathcal{M}/\mathcal{G} & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \operatorname{ad}\alpha} & & \downarrow{\scriptstyle \operatorname{ad} i_*\psi} & & \downarrow{\scriptstyle \operatorname{ad}^\dagger \alpha} & & \\
0 & \longrightarrow & (\mathcal{M}/\mathcal{G})^* & \xrightarrow{\pi^*} & \mathcal{M}^* & \xrightarrow{j^*} & \mathcal{G}^* & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \operatorname{cok}\operatorname{ad}\alpha & \overset{\bar{\pi}^*}{\dashrightarrow} & \mathcal{M}'/\mathcal{M} & \overset{\bar{j}^*}{\dashrightarrow} & \operatorname{cok}\operatorname{ad}^\dagger \alpha & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

where the rows and columns are exact. Since $\mathcal{M} \in \mathcal{CM}^0(H)$ and $\mathcal{M}/\mathcal{G} \rightarrowtail i_*(\mathcal{N}/\mathcal{I}) \in \mathcal{CM}^0(H)$, we have $\mathcal{G}, \mathcal{M}/\mathcal{G} \in \mathcal{S}_1^0(H)$ [12, 1.19]. Hence, $\mathcal{G}_x, \mathcal{M}_x/\mathcal{G}_x \in \mathcal{CM}^0(H_x)$ $\forall x \in H^{(1)}$. By [11, 1.6c], the second and third rows are locally exact at every point of $H^{(1)}$. Let $\mathcal{S} := \operatorname{im}\bar{\pi}^*$. It follows from the commutative diagram that $\bar{\psi}|_{\mathcal{S}\times\mathcal{S}} = 0$, and the local exactness of the third row implies that $\mathcal{S} \subset \mathcal{M}'/\mathcal{M}$ is a sublagrangian at those points (in $H^{(1)}$). $\square$

Since $\mathcal{L}^0_{H^1}$ is defined in the same way as Pardon's lattice map $\mathcal{L}^0$, there is a commutative diagram

$$\begin{array}{ccc} W(\kappa(\eta)) & \xrightarrow{d^0} & \coprod_{x\in H^{(1)}} W(\mathcal{CM}^1(H_x)) \\ \uparrow & & \uparrow \\ W(H-H^1) & \xrightarrow{d^0_H} & W(\mathcal{CM}^1_{T_L}(H-N)) \coprod W(\mathcal{CM}^1_{T_N}(H-L)) \end{array}$$

where $d^0 = \mathcal{K}^1 \circ \mathcal{L}^0$, $d^0_H = \mathcal{K}^1_{H^1} \circ \mathcal{L}^0_{H^1}$, and the vertical maps are induced by inclusion. Note that the bottom right has only two Witt components supported on T_L and T_N, because the image of the lattice map $\mathcal{L}^0_{H^1}$ is supported on H^1.

3.4.2 $\mathcal{L}^1_{T_L} \coprod \mathcal{L}^1_{T_N} : W(\mathcal{CM}^1_{T_L}(H-N) \coprod W(\mathcal{CM}^1_{T_N}(H-L) \to W(\mathcal{CM}^2_{H^2}(H))$

Let $[\mathcal{N}, \psi] \in W(\mathcal{CM}^1_{T_L}(H-N))$, so that

$$\psi : \mathcal{N} \times \mathcal{N} \to \mathcal{V}^1_{\mathcal{O}_{H-N}, T_L},$$

where $\mathcal{N} \in \mathcal{CM}^1_{T_L}(H-N)$. Let $i : H-N \hookrightarrow H$ be the inclusion. An $\mathcal{O}_H$-submodule $\mathcal{M} \subset i_*\mathcal{N}$ is a *lattice* if $\mathcal{M} \in \mathcal{CM}^1_L(H)$ and $\mathcal{M}|_{H-N} = \mathcal{N}$. The lattice is *integral* with respect to ψ if $(i_*\psi)(\mathcal{M}\times\mathcal{M}) \subset \mathcal{V}^1_{\mathcal{O}_H}$.

$$\begin{array}{ccc} \mathcal{M}\times\mathcal{M} & \subset & i_*\mathcal{N}\times i_*\mathcal{N} \\ \downarrow & & \downarrow i_*\psi \\ \mathcal{V}^1_{\mathcal{O}_H} & \subset & i_*\mathcal{V}^1_{\mathcal{O}_{H-N},T_L} \end{array}$$

Since $\mathcal{M} \subset i_*\mathcal{N}$, the image of the left vertical map necessarily lies in $\mathcal{V}^1_{\mathcal{O}_H,L}$.

If $\mathcal{M}$ is an integral lattice for $[\mathcal{N}, \psi]$, its *dual lattice* is an $\mathcal{O}_H$-submodule $\mathcal{M}' \subset i_*\mathcal{N}$ defined for each affine open subset $U \subset H$ by

$$\mathcal{M}'(U) = \left\{ n \in i_*\mathcal{N}(U) \mid (i_*\psi)(U)(n, \mathcal{M}(U)) \subset \mathcal{V}^1_{\mathcal{O}_H,L}(U) \right\}.$$

Applying Bass's theorem [**?** , 2.5] which states that minimal injective resolution is preserved under taking annihilator of a regular element, we obtain an exact sequence

$$0 \to \mathcal{V}^1_{\mathcal{O}_H,L} \to \mathcal{E}^1_{\mathcal{O}_H,L} \xrightarrow{d^1} \mathcal{E}^2_{\mathcal{O}_H,L} \to 0,$$

which is a minimal injective resolution of $\mathcal{V}^1_{\mathcal{O}_H,L}$ as a sheaf of $\mathcal{O}^L_H$-modules, where $\mathcal{O}^L_H$ is the L-adic completion of $\mathcal{O}_H$. Thus, there is an induced map

$$\bar{\psi} : \frac{\mathcal{M}'}{\mathcal{M}} \times \frac{\mathcal{M}'}{\mathcal{M}} \dashrightarrow \mathcal{V}^2_{\mathcal{O}_H}.$$

$\mathcal{M}'$ locally coincides with $\mathcal{M}$ at every point of $H - N$. Also, being submodules of $i_*\mathcal{N}$, $\mathcal{M}'$ and $\mathcal{M}$ are supported on L. Hence, $\mathcal{M}'/\mathcal{M}$ is supported on $N \cap L = H^2$, and the image of $\bar{\psi}$ lies in $\mathcal{V}^2(H)_{H^2}$. Moreover, $\mathcal{M}'/\mathcal{M} \in \mathcal{CM}^2(H)$ by [11, 1.2].

Our next main proposition of this chapter can be proved in the same way as Proposition 3.4:

Proposition 3.5. *There are well-defined maps*

$$\begin{aligned} \mathcal{L}^1_{T_L} : W(\mathcal{CM}^1_{T_L}(H - N)) &\dashrightarrow W(\mathcal{CM}^2_{H^2}(H)), \qquad [\mathcal{N}, \psi] \mapsto [\mathcal{M}'/\mathcal{M}, \bar{\psi}], \\ \mathcal{L}^1_{T_N} : W(\mathcal{CM}^1_{T_N}(H - L)) &\dashrightarrow W(\mathcal{CM}^2_{H^2}(H)), \qquad [\mathcal{N}, \psi] \mapsto [\mathcal{M}'/\mathcal{M}, \bar{\psi}]. \end{aligned}$$

Let $d^1_H := \mathcal{L}^1_{T_L} \coprod \mathcal{L}^1_{T_N}$, so that

$$d^1_H : W(\mathcal{CM}^1_{T_L}(H - N)) \coprod W(\mathcal{CM}^1_{T_N}(H - L)) \to W(\mathcal{CM}^2_{H^2}(H)).$$

Since $\mathcal{L}^1_{T_L}$ and $\mathcal{L}^1_{T_N}$ are defined in the same way as Pardon's lattice map $\mathcal{L}^1$, there is a commutative diagram

$$\begin{array}{ccc} \coprod_{x \in H^{(1)}} W(\mathcal{CM}^1(H_x)) & \xrightarrow{\quad d^1 \quad} & \coprod_{x \in H^{(2)}} W(\mathcal{CM}^2(H_x)) \\ \uparrow & & \uparrow \\ W(\mathcal{CM}^1_{T_L}(H - N)) \coprod W(\mathcal{CM}^1_{T_N}(H - L)) & \xrightarrow{\quad d^1_H \quad} & W(\mathcal{CM}^2_{H^2}(H)) \end{array}$$

where $d^1 = \mathcal{K}^2 \circ \mathcal{L}^1$. Note that the Witt group on the bottom right is supported on four points $0_{xy}, 0_{zw}, 0_{x\bar{y}}, 0_{z\bar{w}} \in H^{(2)}$.

3.5 Dévissage

Let X be a scheme, and $x \in X^{(p)}$ a point of codimension p. In its original form, dévissage [11, 2.2] states that there is an isomorphism (see (2.3) for canonical version)

$$W(\kappa(x)) \xrightarrow{\sim} W(\mathcal{CM}^p(X_x)).$$

In order to identify the complex (3.5) with a complex of Witt groups of tori, we need isomorphisms of the form

$$W(T_X) \xrightarrow{\sim} W(\mathcal{CM}^1_{T_X}(H_n - N)). \tag{3.11}$$

There is certainly such a map induced by inclusion. However, unlike $\mathcal{CM}^p(X_x)$, the sheaves in $\mathcal{CM}^1_{T_X}(H_n - N)$ are not of finite length, so dévissage cannot be applied in its original form. In this section, we show that the map is still an isomorphism. First we need a lemma:

Lemma 3.6. *Let*

$$0 \to \mathcal{E}^0 \to \mathcal{E}^1 \to \mathcal{E}^2 \to 0$$

be a minimal injective resolution of $\mathbb{C}[x, y]$. *Then*

$$\mathcal{E}^1_y := \bigcup_{i=1}^{\infty} (0 : y^i)_{\mathcal{E}^1} \simeq \frac{\mathbb{C}(x)[y, 1/y]}{\mathbb{C}(x)[y]}.$$

Proof. Let $\mathcal{V}^p \subset \mathcal{E}^p$ denote the p-th cosyzygy. We have $\mathcal{E}^0 = \mathbb{C}(x, y)$, $\mathcal{V}^1 = \frac{\mathbb{C}(x,y)}{\mathbb{C}[x,y]}$, and

$$\mathcal{V}^1_y := \bigcup_{i=1}^{\infty} (0 : y^i)_{\mathcal{V}^1} \simeq \frac{\mathbb{C}[x, y, 1/y]}{\mathbb{C}[x, y]}.$$

Hence, there is an inclusion $\mathcal{V}_y^1 \hookrightarrow \frac{\mathbb{C}(x)[y,1/y]}{\mathbb{C}(x)[y]}$. We will show that this is an an essential injective extension over $\mathbb{C}[x][[y]]$. This is an equivalent way of saying that it is an injective hull over $\mathbb{C}[x][[y]]$ [**?** , 2.21]. Since $\mathcal{V}_y^1 \subset \mathcal{E}_y^1$ is an injective hull over $\mathbb{C}[x][[y]]$ by Bass [**?** , 2.5], the result follows.

$\frac{\mathbb{C}(x)[y,1/y]}{\mathbb{C}(x)[y]}$ is an injective $\mathbb{C}(x)[[y]]$-module, since it is divisible over PID. This in turn implies that it is injective over $\mathbb{C}[x][[y]]$.

For essentiality, note that every non-zero element of $\frac{\mathbb{C}(x)[y,1/y]}{\mathbb{C}(x)[y]}$ can be represented by a finite sum of the form $\sum_{i \geqslant 1} \frac{f_i}{g_i y^i}$, where $f_i, g_i \in \mathbb{C}[x] \subset \mathbb{C}[x][[y]]$. By multiplying this by $\prod_{i \geqslant} g_i$, one obtains a non-zero element in $\mathcal{V}_y^1 = \frac{\mathbb{C}[x,y,1/y]}{\mathbb{C}[x,y]}$. This implies that $\mathcal{V}_y^1 \hookrightarrow \frac{\mathbb{C}(x)[y,1/y]}{\mathbb{C}(x)[y]}$ is an essential extension. □

Proposition 3.7. *There are isomorphisms*

$$W(T_X) \xrightarrow{\sim} W(\mathcal{CM}^1_{T_X}(H_n - N)), \qquad W(T_Z) \xrightarrow{\sim} W(\mathcal{CM}^1_{T_Z}(H_n - N))$$
$$W(T_Y) \xrightarrow{\sim} W(\mathcal{CM}^1_{T_Y}(H_n - L)), \qquad W(T_W) \xrightarrow{\sim} W(\mathcal{CM}^1_{T_W}(H_n - L))$$

induced by inclusion.

Proof. We will only prove the first isomorphism. The proofs for the other isomorphisms are similar.

T_X can be covered by two affine open subsets, $U = \operatorname{Spec} \mathbb{C}[x, 1/x]$ and $V = \operatorname{Spec} \mathbb{C}[z, 1/z]$, glued together via $x \leftrightarrow 1/z$. If $[\mathcal{M}, \phi] \in W(T_X)$, then

$$\begin{aligned} \phi(U) : \mathcal{M}(U) \times \mathcal{M}(U) &\rightarrow \mathcal{V}^0_{\mathcal{O}_{T_X}}(U) = \mathbb{C}[x, 1/x] \\ \phi(V) : \mathcal{M}(V) \times \mathcal{M}(V) &\rightarrow \mathcal{V}^0_{\mathcal{O}_{T_X}}(V) = \mathbb{C}[z, 1/z] \end{aligned}$$

where $\mathcal{M}(U)$ is a free $\mathbb{C}[x, 1/x]$-module, and $\mathcal{M}(V)$ is a free $\mathbb{C}[z, 1/z]$-module.

On the other hand, viewed as a subset of $H_n - N$, T_X can also be covered by two affine open subsets $U_1 = \operatorname{Spec} \mathbb{C}[x, y, 1/x]$ and $U_4 = \operatorname{Spec} \mathbb{C}[z, \bar{w}, 1/z]$, glued together via $x \leftrightarrow 1/z$ and $y \leftrightarrow z^n\bar{w}$. If $[\mathcal{N}, \psi] \in W(\mathcal{CM}^1_{T_X}(H_n - N))$, then

$$\mathcal{N}(U_1) \times \mathcal{N}(U_1) \overset{\psi(U_1)}{\to} \mathcal{V}^1_{\mathcal{O}_{H_n-N},T_X}(U_1) = \bigcup_{i=1}^{\infty}(0 : y^i)_{\frac{\mathbb{C}(x,y)}{\mathbb{C}[x,y,1/x]}} = \frac{\mathbb{C}[x, y, 1/x, 1/y]}{\mathbb{C}[x, y, 1/x]}$$

$$\mathcal{N}(U_4) \times \mathcal{N}(U_4) \overset{\psi(U_4)}{\to} \mathcal{V}^1_{\mathcal{O}_{H_n-N},T_X}(U_4) = \bigcup_{i=1}^{\infty}(0 : \bar{w}^i)_{\frac{\mathbb{C}(z,\bar{w})}{\mathbb{C}[z,\bar{w},1/z]}} = \frac{\mathbb{C}[z, \bar{w}, 1/z, 1/\bar{w}]}{\mathbb{C}[z, \bar{w}, 1/z]}$$

where $\mathcal{N}(U_1)$ is a CM $\mathbb{C}[x, y, 1/x]$-module of codimension 1 killed by some power of y, and $\mathcal{N}(U_4)$ is a CM $\mathbb{C}[z, \bar{w}, 1/z]$-module of codimension 1 killed by some power of $\bar{w}$.

Let $i : T_X \hookrightarrow H_n - N$ be the inclusion. There is an injection

$$j : i_*\mathcal{V}^0_{\mathcal{O}_{T_X}} \hookrightarrow \mathcal{V}^1_{\mathcal{O}_{H_n-N},T_X},$$

which is given on the affine charts U_1 and U_4 by

$$j(U_1) : \mathbb{C}[x, 1/x] \overset{\frac{1}{y}}{\hookrightarrow} \frac{\mathbb{C}[x, y, 1/x, 1/y]}{\mathbb{C}[x, y, 1/x]}$$

$$j(U_4) : \mathbb{C}[z, 1/z] \overset{\frac{1}{z^n\bar{w}}}{\hookrightarrow} \frac{\mathbb{C}[z, \bar{w}, 1/z, 1/y]}{\mathbb{C}[z, \bar{w}, 1/z]}$$

Thus there is an induced map of Witt groups

$$W(T_X) \dashrightarrow W(\mathcal{CM}^1_{T_X}(H_n - N)), \qquad [\mathcal{M}, \phi] \mapsto [i_*\mathcal{M}, j \circ i_*\phi].$$

We will show that this map is an isomorphism. It suffices to show that this is an isomorphism on one of the affine charts, U_1.

There is a commutative diagram of value groups

$$
\begin{array}{ccccccc}
 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \\
\mathcal{V}^0_{\mathbb{C}[x,1/x]} & = & \mathbb{C}[x,x^{-1}] & \xhookrightarrow{1/y} & \frac{\mathbb{C}[x,y,1/x,1/y]}{\mathbb{C}[x,y,1/x]} & = & \mathcal{V}^1_{\mathbb{C}[x,y,1/x],y} \\
 & & \downarrow & & \downarrow & & \\
\mathcal{E}^0_{\mathbb{C}[x,1/x]} & = & \mathbb{C}(x) & \xhookrightarrow{1/y} & \frac{\mathbb{C}(x)[y,1/y]}{\mathbb{C}(x)[y]} & = & \mathcal{E}^1_{\mathbb{C}(x)[y],y} \\
 & & \downarrow & & \downarrow & & \\
\mathcal{V}^1_{\mathbb{C}[x,1/x]} & = & \frac{\mathbb{C}(x)}{\mathbb{C}[x,1/x]} & \xhookrightarrow{1/y} & \frac{\mathbb{C}(x)[y,1/y]}{\mathbb{C}[x,y,1/x,1/y]+\mathbb{C}(x)[y]} & = & \mathcal{V}^2_{\mathbb{C}[x,y,1/x],y} \\
 & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & &
\end{array}
$$

where the columns are exact. Note that we can make the identification

$$\mathcal{E}^1_{\mathbb{C}(x)[y],y} = \frac{\mathbb{C}(x)[y,1/y]}{\mathbb{C}(x)[y]}$$

by Lemma 3.6. Thus we have an induced commutative diagram of Witt groups

$$
\begin{array}{ccc}
0 & & 0 \\
\downarrow & & \downarrow \\
W(\mathcal{CM}^0(\mathbb{C}[x,1/x]) & \xrightarrow{1/y} & W(\mathcal{CM}^1_y(\mathbb{C}[x,y,1/x])) \\
\downarrow{\scriptstyle \mathcal{K}^0} & & \downarrow{\scriptstyle \mathcal{K}^1_y} \\
W(\mathcal{CM}^0(\mathbb{C}(x))) & \xrightarrow{1/y} & W(\mathcal{CM}^1_y(\mathbb{C}(x)[y])) \\
\downarrow{\scriptstyle \mathcal{L}^0} & & \downarrow{\scriptstyle \mathcal{L}^1_y} \\
W(\mathcal{CM}^1(\mathbb{C}[x,1/x])) & \xrightarrow{1/y} & W(\mathcal{CM}^2_y(\mathbb{C}[x,y,1/x])) \\
\downarrow & & \downarrow \\
0 & & 0
\end{array}
$$

Pardon[12] showed that the first column is exact. His proof easily extends to the second column (shown below), so the second column is also exact. We will show that the second and third horizontal maps are isomorphisms, which implies that the first horizontal map is also an isomorphism.

To this end, note that the modules in $\mathcal{CM}^1_y(\mathbb{C}(x)[y])$ and $\mathcal{CM}^2_y(\mathbb{C}[x,y,1/x])$ are of finite length. Hence, we may apply dévissage [11, 2.2] to reduce the powers of y which annihilate the modules down to 1, thereby representing the forms by the images of the horizontal maps. This proves surjectivity. Injectivity follows from the fact that dévissage preserves lagrangians.

To prove exactness of the second column, first note that surjectivity of $\mathcal{L}^1_y$ follows from surjectivity of $\mathcal{L}^0$ and bijectivity of the bottom horizontal map. Secondly, injectivity of $\mathcal{K}^1_y$ follows from injectivity of $\mathcal{K}^1$ [12, 3.9]. $\mathcal{L}^1_y \circ \mathcal{K}^1_y = 0$ is clear, and the only non-trivial part is to show that $\ker \mathcal{L}^1_y \subset \operatorname{im} \mathcal{K}^1_y$. Let $[N, \psi] \in \ker \mathcal{L}^1_y$, so that

$$[M'/M, \bar{\psi}] + [L_1, \phi_1] = [L_2, \phi_2], \tag{3.12}$$

where M is an integral lattice for $[N, \psi]$, M' is its dual lattice, and $[L_1, \phi_1]$, $[L_2, \phi_2]$ are lagrangians.

First suppose that $[L_1, \phi_1] = 0$, so that $[M'/M, \bar{\psi}]$ is a lagrangian. Let $\bar{K} \subset M'/M$ be a sublagrangian, and $K \subset M'$ its pullback under the quotient map $M' \twoheadrightarrow M'/M$. Since $\bar{K}$ is a sublagrangian, $\bar{K} = \bar{K}^{\perp}$ [6, p. 134], therefore $K = K'$. The isomorphism

$$K' \simeq \operatorname{Hom}(K, \mathcal{V}^1_{\mathbb{C}[x,y,1/x]})$$

then implies that $K \in \mathcal{CM}^1_y(\mathbb{C}[x,y,1/x])$ [12, 1.13], and that

$$\psi|_{K \times K} : K \times K \to \mathcal{V}^1_{\mathbb{C}[x,y,1/x]}$$

is nonsingular. Hence, $[K, \psi|_{K \times K}] \in W(\mathcal{CM}^1_y(\mathbb{C}[x,y,1/x]))$, and clearly

$$\mathcal{K}^1_y([K, \psi_{K \times K}]) = [N, \psi],$$

proving $\ker \mathcal{L}^1_y \subset \operatorname{im} \mathcal{K}^1_y$.

Next, we show that $[L_1, \phi_1] \in \operatorname{im} \mathcal{K}^1_y$, justifying our assumption. By adding $[L_1, -\phi_1]$ to both sides of (3.12), we may assume that $[L_1, \phi_1]$ is a hyperbolic form

(1.2), so that

$$L_1 = T \oplus \tilde{T},$$

where $T \in \mathcal{CM}_y^2(\mathbb{C}[x,y,1/x])$ and $\tilde{T} := \operatorname{Hom}(T, \mathcal{V}^2_{\mathbb{C}[x,y,1/x]})$. By [11, 1.6b], there exists an exact sequence

$$0 \to I \to J \to T \to 0, \tag{3.13}$$

where $J \in \mathcal{CM}_y^1(\mathbb{C}[x,y,1/x])$, and a dual exact sequence

$$0 \to J^* \to I^* \to \tilde{T} \to 0, \tag{3.14}$$

where $J^* := \operatorname{Hom}(J, \mathcal{V}^1_{\mathbb{C}[x,y,1/x]})$, $I^* := \operatorname{Hom}(I, \mathcal{V}^1_{\mathbb{C}[x,y,1/x]})$. Combining (3.13) and (3.14) gives an exact sequence

$$0 \to I \oplus J^* \to J \oplus I^* \to L_1 \to 0.$$

Let $S := I \oplus J^*$, and

$$\sigma : S \times S \to \mathcal{V}^1_{\mathbb{C}[x,y,1/x]}$$

the induced pairing with $\sigma_{|I\times I} = \sigma_{|J^*\times J^*} = 0$. Then $[S,\sigma] \in W(\mathcal{CM}_y^1(\mathbb{C}[x,y,1/x]))$ [11, 1.6a], and $\mathcal{K}_y^1([S,\sigma]) = [L_1, \phi_1]$, as desired. □

Corollary 3.8. *There are canonical isomorphisms*

$$\begin{aligned} W(T_X) \coprod W(T_Z) &\xrightarrow{\sim} W(\mathcal{CM}^1_{T_L}(H_n - N)) \\ W(T_Y) \coprod W(T_W) &\xrightarrow{\sim} W(\mathcal{CM}^1_{T_N}(H_n - L)) \end{aligned}$$

induced by inclusion.

Proof. It is easy to see that if $\mathcal{M} \in \mathcal{CM}^1_{T_L}(H_n - N)$, then $\mathcal{M} = \mathcal{M}_X \oplus \mathcal{M}_Z$, where $\mathcal{M}_X := \cup_{i=1}^{\infty}(0 : \mathscr{I}(X)^i)_{\mathcal{M}}$, $\mathcal{M}_Z := \cup_{i=1}^{\infty}(0 : \mathscr{I}(Z)^i)_{\mathcal{M}}$, and $\mathscr{I}(X), \mathscr{I}(Z) \subset \mathcal{O}_{H_n}$ are the ideal sheaves of X and Z, respectively. Hence, there is an isomorphism

$$\begin{array}{rcl} W(\mathcal{CM}^1_{T_L}(H_n - N)) & \simeq & W(\mathcal{CM}^1_{T_X}(H_n - N)) \coprod W(\mathcal{CM}^1_{T_Z}(H_n - N)), \\ [\mathcal{M}, \phi] & \mapsto & ([\mathcal{M}_X, \phi|_{\mathcal{M}_X}], [\mathcal{M}_Z, \phi|_{\mathcal{M}_Z}]) \\ [\mathcal{M}, \phi] + [\mathcal{N}, \psi] & \leftarrow\!\shortmid & ([\mathcal{M}, \phi], [\mathcal{N}, \psi]) \end{array}$$

and the result follows from Proposition 3.7. □

4

Quasi-isomorphism via toric decomposition

In this chapter, we will prove that the toric complex

$$0 \to W(H_n - H_n^1) \xrightarrow{d^0_{H_n}} W(H_n^1 - H_n^2) \xrightarrow{d^1_{H_n}} W(H_n^2) \to 0 \tag{4.1}$$

is quasi-isomorphic to the Gersten-Witt complex of H_n

$$0 \to \coprod_{x \in H_n^{(0)}} W(\kappa(x)) \xrightarrow{d^0} \coprod_{x \in H_n^{(1)}} W(\kappa(x)) \xrightarrow{d^1} \coprod_{x \in H_n^{(2)}} W(\kappa(x)) \xrightarrow{d^2} 0. \tag{4.2}$$

By Corollary 3.8, this implies that the complex

$$0 \to W(H - H^1) \xrightarrow{d^0_H} W(\mathcal{CM}^1_{T_L}(H-N)) \coprod W(\mathcal{CM}^1_{T_N}(H-L)) \xrightarrow{d^1_H} W(\mathcal{CM}^2_{H^2}(H)) \to 0 \tag{4.3}$$

that we constructed in Chapter 3 is quasi-isomorphic to (4.2).

More generally, let X be a toric variety of dimension n, where

$$X = X^0 \supset X^1 \supset \cdots \supset X^n \supset X^{n+1} = \varnothing$$

is a chain of closures of orbits of the torus action, and $Y^p := X^p - X^{p+1}$ is a finite disjoint union of $(n-p)$-tori. Let

$$R^p(X) := \bigoplus_{x \in X^{(p)}} W(\kappa(x))$$

be the p-th term of the Gersten-Witt complex of X.

In order to prove the quasi-isomorphism, we need the following proposition, due to Pardon:

Proposition 4.1. *Let k be a field with $\operatorname{char} k \neq 2$. Then the Gersten-Witt complex of $\mathbb{A}^n_k$ is acyclic, and $H^0(\mathbb{A}^n_k) = W(\mathbb{A}^n_k)$.*

Proof. We will prove by induction on n. It is trivially true if $n = 0$, so assume that $n \geqslant 1$, and that it is true for $n-1$.

If $\mathfrak{p} \in \operatorname{Spec} k[x_1]$, denote its fibre under the projection

$$\operatorname{Spec} k[x_1, \ldots, x_n] \twoheadrightarrow \operatorname{Spec} k[x_1]$$

by $F_{\mathfrak{p}}$. If $\mathfrak{p} \in \operatorname{Spec} k[x_1]$ is not the generic point, then $\mathfrak{p} = (f)$, where $f \in k[x_1]$ is an irreducible polynomial. Hence,

$$F_{\mathfrak{p}} = \operatorname{Spec}(k[x_1, \ldots, x_n]/(f)) = \operatorname{Spec}((k[x_1]/(f))[x_2, \ldots, x_n]),$$

and $k[x_1]/(f)$ is an algebraic field extension over k. If $\eta \in \operatorname{Spec} k[x_1]$ is the generic point, then $F_\eta = \operatorname{Spec} k(x_1)[x_2, \ldots, x_n]$. Note that $\operatorname{Spec} k[x_1, \ldots, x_n]$ and F_η have the same function field, $k(x_1, \ldots, x_n)$.

Let $A := \operatorname{Spec} k[x_1, \ldots, x_n]$ and $A_1 := \operatorname{Spec} k[x_1]$. There is a commutative diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & W(k[x_1]) & \longrightarrow & W(k(x_1)) & \longrightarrow & \coprod_{\mathfrak{p} \in A_1^{(1)}} W(k[x_1]/\mathfrak{p}) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & W(A) & \longrightarrow & W(F_\eta) & \longrightarrow & \coprod_{\mathfrak{p} \in A_1^{(1)}} W(F_{\mathfrak{p}}) & \longrightarrow & 0
\end{array}$$

where the vertical maps are induced by inclusion. By Karoubi [5], the vertical maps are isomorphisms, and by Pardon [11], the first row is exact. Hence, the second row is also exact.

Now, there is a short exact sequence of Gersten-Witt complexes

$$0 \to \coprod_{\mathfrak{p} \in A_1^{(1)}} R^\bullet(F_\mathfrak{p})[-1] \to R^\bullet(A) \to R^\bullet(F_\eta) \to 0,$$

where $[-1]$ indicates a degree shift by -1. There is a commutative diagram

$$\begin{array}{ccccccccc}
 & & & & & & 0 & & \\
 & & & & & & \downarrow & & \\
 & & 0 & & & & \coprod_{\mathfrak{p} \in A_1^{(1)}} W(K(F_\mathfrak{p})) & \longrightarrow & \dots \\
 & & \downarrow & & & & \downarrow & & \\
0 & \longrightarrow & W(A) & \xrightarrow{\epsilon_A} & W(K(A)) & \xrightarrow{d_A^0} & \coprod_{x \in A^{(1)}} W(\kappa(x)) & \longrightarrow & \dots \\
 & & \downarrow & & \| & & \downarrow & & \\
0 & \longrightarrow & W(F_\eta) & \xrightarrow{\epsilon_{F_\eta}} & W(K(F_\eta)) & \xrightarrow{d_{F_\eta}^0} & \coprod_{x \in F_\eta^{(1)}} W(\kappa(x)) & \longrightarrow & \dots \\
 & & \downarrow & & & & \downarrow & & \\
 & & \coprod_{\mathfrak{p} \in A_1^{(1)}} W(F_\mathfrak{p}) & & & & 0 & & \\
 & & \downarrow & & & & & & \\
 & & 0 & & & & & &
\end{array}$$

where columns are exact, and $K(-)$ denotes the function field. Diagram chasing shows that there is an induced map

$$\coprod_{\mathfrak{p} \in A_1^{(1)}} W(F_\mathfrak{p}) \dashrightarrow \coprod_{\mathfrak{p} \in A_1^{(1)}} W(K(F_\mathfrak{p})).$$

By the induction hypothesis, this map is injective, and the first and third rows in the above diagram are exact. Hence, the second row also is exact. □

The same proof works with Laurent polynomials:

Proposition 4.2. *Let k be a field with* $\operatorname{char} k \neq 2$, *and T a torus (of any dimension) over k. Then the Gersten-Witt complex of T is acyclic, and $H^0(T) = W(T)$.*

We now prove the main proposition of this chapter (see Takeda [16] for K-theoretic analogue):

Proposition 4.3. *The complex $W(Y^\bullet)$ is quasi-isomorphic to $R^\bullet(X)$.*

Proof. The inclusion $X^{p+1} \hookrightarrow X^p$ induces a short exact sequence

$$0 \to R^\bullet(X^{p+1})[-1] \to R^\bullet(X^p) \to R^\bullet(Y^p) \to 0, \tag{4.4}$$

where $[-1]$ indicates a degree shift by -1. Since $Y^p = \coprod_i T_i^p$, $R^\bullet(Y^p)$ is acyclic by Proposition 4.2. Hence, the short exact sequence (4.4) induces an exact sequence

$$0 \to H^0(R^\bullet(X^p)) \xrightarrow{\mathcal{K}^p} H^0(R^\bullet(Y^p)) \overset{\delta^p}{\dashrightarrow} H^0(R^\bullet(X^{p+1})) \xrightarrow{\mathcal{J}^p} H^1(R^\bullet(X^p)) \to 0, \tag{4.5}$$

and isomorphisms

$$H^{k-1}(R^\bullet(X^{p+1})) \xrightarrow{\sim} H^k(R^\bullet(X^p)) \qquad \forall k \geqslant 2.$$

The latter gives rise to a chain of isomorphisms

$$H^1(R^\bullet(X^{p-1})) \xrightarrow{\sim} H^2(R^\bullet(X^{p-2})) \xrightarrow{\sim} \cdots \xrightarrow{\sim} H^p(R^\bullet(X^0)),$$

which are induced by the inclusion $R^\bullet(X^{p-i})[-1] \hookrightarrow R^\bullet(X^{p-i-1})$.

Let $\partial^p := \mathcal{K}^{p+1} \circ \delta^p$. The exactness of the sequence (4.5) implies that there is a complex

$$\cdots \to H^0(R^\bullet(Y^{p-1})) \xrightarrow{\partial^{p-1}} H^0(R^\bullet(Y^p)) \xrightarrow{\partial^p} H^0(R^\bullet(Y^{p+1})) \to \cdots . \tag{4.6}$$

By Proposition 4.1,

$$H^0(R^\bullet(Y^p)) = W(Y^p) = \coprod_i W(T_i^p).$$

Hence, (4.6) gives a complex $W(Y^\bullet)$.

There is a commutative diagram

$$\begin{array}{ccccccc}
 & & H^0(R^\bullet(Y^{p-1})) & & & & 0 \\
 & & \downarrow{\scriptstyle \delta^{p-1}} & \searrow{\scriptstyle \partial^{p-1}} & & & \downarrow \\
0 & \longrightarrow & H^0(R^\bullet(X^p)) & \xrightarrow{\mathcal{K}^p} & H^0(R^\bullet(Y^p)) & \xrightarrow{\delta^p} & H^0(R^\bullet(X^{p+1})) \\
 & & \downarrow{\scriptstyle \mathcal{J}^{p-1}} & & & \searrow{\scriptstyle \partial^p} & \downarrow{\scriptstyle \mathcal{K}^{p+1}} \\
 & & H^1(R^\bullet(X^{p-1})) & & & & H^0(R^\bullet(Y^{p+1})) \\
 & & \downarrow & & & & \\
 & & 0 & & & &
\end{array}$$

where the rows and columns are exact. Hence, there is a commutative diagram

$$\begin{array}{ccc}
H^0(R^\bullet(X^p)) & \underset{\sim}{\xrightarrow{\mathcal{K}^p}} & \ker \partial^p \\
\cup & & \cup \\
\operatorname{im} \delta^{p-1} & \xrightarrow{\sim} & \operatorname{im} \partial^{p-1}
\end{array}$$

which gives rise to isomorphisms

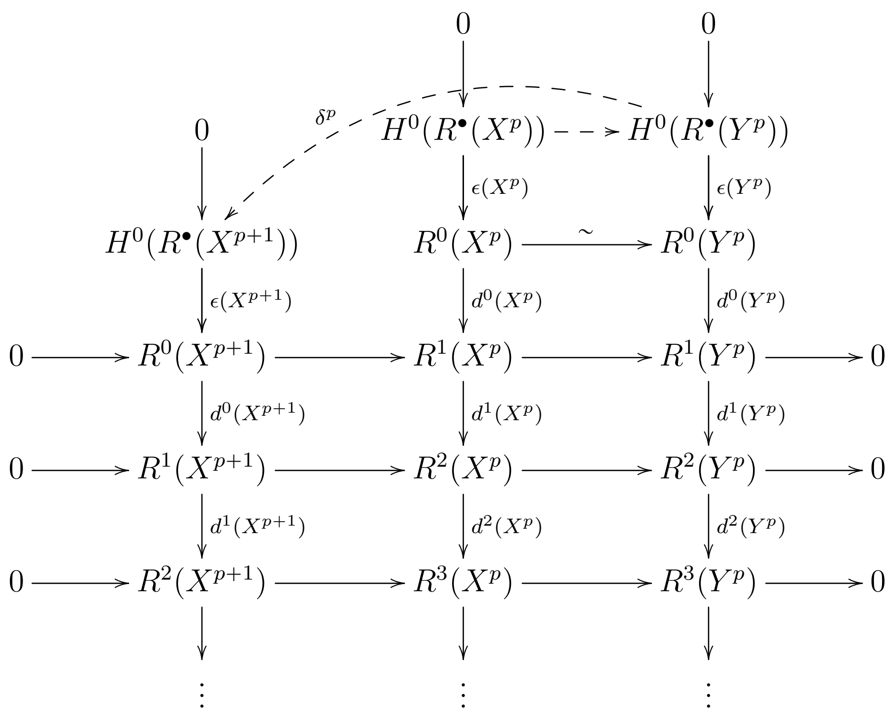
$$H^p(R^\bullet(X^0)) \overset{\sim}{\leftarrow} H^1(R^\bullet(X^{p-1})) \underset{\sim}{\overset{\mathcal{J}^{p-1}}{\leftarrow}} \frac{H^0(R^\bullet(X^p))}{\operatorname{im} \delta^{p-1}} \underset{\sim}{\overset{\mathcal{K}^p}{\rightarrow}} \frac{\ker \partial^p}{\operatorname{im} \partial^{p-1}} = H^p(W(Y^\bullet)),$$

where the first two isomorphisms are induced by inclusion.

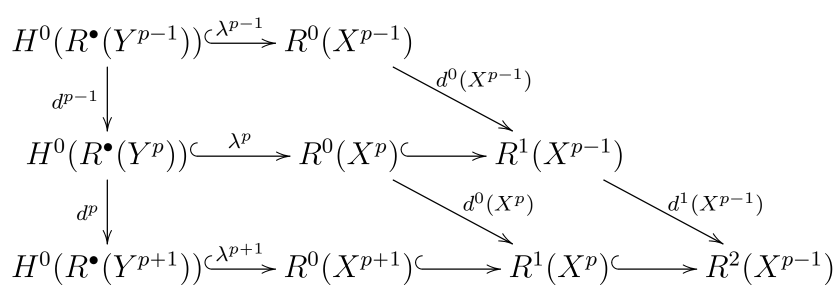
Now we will show that this isomorphism is induced by a chain map

$$R^\bullet(X^0) \dashleftarrow W(Y^\bullet).$$

By the short exact sequence (4.4), there is a commutative diagram



where the rows and columns are exact. By the isomorphism $R^0(X^p) \xrightarrow{\sim} R^0(Y^p)$, there is a map $\lambda^p : H^0(R^\bullet(Y^p)) \dashrightarrow R^0(X^p)$, and a commutative diagram



Hence, there are inclusions of chains

$$W(Y^\bullet) \overset{\lambda^\bullet}{\hookrightarrow} R^\bullet(X^{p-1})[-p+1] \hookrightarrow R^\bullet(X^{p-2})[-p+2] \hookrightarrow \cdots \hookrightarrow R^\bullet(X^0),$$

which induces the isomorphism $H^p(W(Y^\bullet)) \xrightarrow{\sim} H^p(R^\bullet(X^0))$. $\square$

5

Computations

In this chapter, we will compute the boundary maps of the toric complex (3.5). For the sake of simplicity (and without loss of generality), we will assume that $n \geqslant 0$.

From our choice of affine coordinates (3.1), we have $H_n - H_n^1 = \operatorname{Spec} \mathbb{C}[x, y, 1/x, 1/y]$. Then $W(H_n - H_n^1)$ is a $\mathbb{Z}/2$-vector space of dimension 4, generated by the unary forms $\langle 1\rangle, \langle x\rangle, \langle y\rangle, \langle xy\rangle$ [5, 3.11]. On the other hand, by Corollary 3.8,

$$W(\mathcal{CM}^1_{T_L}(H_n - N)) \coprod W(\mathcal{CM}^1_{T_N}(H_n - L)) \tag{5.1}$$

is generated by 8 basis elements corresponding to the basis elements of

$$W(T_X), \quad W(T_Y), \quad W(T_Z), \quad W(T_W),$$

each of which is generated by two basis elements [5, 3.9]. Hence, $d^0_{H_n}$ can be represented by an 8-by-4 matrix. On the other hand, H_n^2 consists of four points (refer to the picture (3.2)), and since the Witt group of a point is $W(\mathbb{C}) = \mathbb{Z}/2$, $W(\mathcal{CM}^2_{H_n^2}(H_n))$ is a $\mathbb{Z}/2$-vector space of dimension 4, so $d^1_{H_n}$ can be represented by a 4-by-8 matrix.

Proposition 5.1. *With above choice of basis, the matrix representation for* $d^0_{H_n}$ *is* *given by*

$$d^0_{H_{even}} = \begin{array}{c} \langle 1_x\rangle \\ \langle x\rangle \\ \langle 1_y\rangle \\ \langle y\rangle \\ \langle 1_z\rangle \\ \langle z\rangle \\ \langle 1_w\rangle \\ \langle w\rangle \end{array} \overset{\begin{array}{cccc} \langle 1\rangle & \langle x\rangle & \langle y\rangle & \langle xy\rangle \end{array}}{\begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}}, \quad d^0_{H_{odd}} = \begin{array}{c} \langle 1_x\rangle \\ \langle x\rangle \\ \langle 1_y\rangle \\ \langle y\rangle \\ \langle 1_z\rangle \\ \langle z\rangle \\ \langle 1_w\rangle \\ \langle w\rangle \end{array} \overset{\begin{array}{cccc} \langle 1\rangle & \langle x\rangle & \langle y\rangle & \langle xy\rangle \end{array}}{\begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}}.$$

Proof. Let us first determine the entries in the first column. To do this, we need to see where the form $(\mathcal{O}_{H_n - H_n^1}, \langle 1\rangle)$ is sent to by the composition $\mathcal{K}^1_{H_n^1} \circ \mathcal{L}^0_{H_n^1}$. To apply the lattice map $\mathcal{L}^0_{H_n^1}$, we need to find an integral lattice for $(\mathcal{O}_{H_n - H_n^1}, \langle 1\rangle)$. We claim that $\mathcal{O}_{H_n}$ is an integral lattice for $(\mathcal{O}_{H_n - H_n^1}, \langle 1\rangle)$. To see this, we check that the image of the bilinear form

$$\mathcal{O}_{H_n}(U_{\sigma_i}) \times \mathcal{O}_{H_n}(U_{\sigma_i}) \xrightarrow{\langle 1\rangle} (j_*\mathcal{O}_{H_n - H_n^1})(U_{\sigma_i})$$

lies in $\mathcal{V}^0_{\mathcal{O}_{H_n}}(U_{\sigma_i})$ for every i, where $j : H_n - H_n^1 \hookrightarrow H_n$ is the inclusion (see (3.1) for the definition of U_{σ_i}.) For example, on U_{σ_1}, the form $(\mathcal{O}_{H_n - H_n^1}, \langle 1\rangle)$ is given by

$$\mathbb{C}[x, y, 1/x, 1/y] \times \mathbb{C}[x, y, 1/x, 1/y] \xrightarrow{1} \mathbb{C}[x, y, 1/x, 1/y],$$

and the image of the bilinear form

$$\mathcal{O}_{H_n}(U_{\sigma_1}) \times \mathcal{O}_{H_n}(U_{\sigma_1}) = \mathbb{C}[x, y] \times \mathbb{C}[x, y] \xrightarrow{1} \mathbb{C}[x, y, 1/x, 1/y]$$

lies in $\mathcal{V}^0_{\mathcal{O}_{H_n}}(U_{\sigma_1}) = \mathcal{O}_{H_n}(U_{\sigma_1}) = \mathbb{C}[x, y]$.

To find its dual lattice, note that

$$\mathcal{O}'_{H_n}(U_{\sigma_1}) := \{f \in \mathbb{C}[x, y, 1/x, 1/y] \mid f \cdot \mathbb{C}[x, y] \subset \mathbb{C}[x, y]\} = \mathbb{C}[x, y] = \mathcal{O}_{H_n}(U_{\sigma_1}).$$

We can similarly check that $\mathcal{O}'_{H_n}(U_{\sigma_i}) = \mathcal{O}_{H_n}(U_{\sigma_i})$ for every i, so $\mathcal{O}_{H_n}$ is self-dual, resulting in $\mathcal{L}^0_{H^1_n}(\langle 1\rangle) = 0$. Hence, $d^0_{H_n}(\langle 1\rangle) = \mathcal{K}^1_{H^1_n} \circ \mathcal{L}^0_{H^1_n}(\langle 1\rangle) = 0$, and the entries in the first column are all 0.

Now let us determine the entries in the second column.

The form $(\mathcal{O}_{H_n - H^1_n}, \langle x\rangle)$ is given on the affine charts by

$$\begin{aligned}
U_{\sigma_1} &: \mathbb{C}[x,y,1/x,1/y] \times \mathbb{C}[x,y,1/x,1/y] \xrightarrow{x} \mathbb{C}[x,y,1/x,1/y] \\
U_{\sigma_2} &: \mathbb{C}[x,\bar{y},1/x,1/\bar{y}] \times \mathbb{C}[x,\bar{y},1/x,1/\bar{y}] \xrightarrow{x} \mathbb{C}[x,\bar{y},1/x,1/\bar{y}] \\
U_{\sigma_3} &: \mathbb{C}[z,w,1/z,1/w] \times \mathbb{C}[z,w,,1/z,1/w] \xrightarrow{1/z} \mathbb{C}[z,w,1/z,1/w] \\
U_{\sigma_4} &: \mathbb{C}[z,\bar{w},1/z,1/\bar{w}] \times \mathbb{C}[z,\bar{w},1/z,1/\bar{w}] \xrightarrow{1/z} \mathbb{C}[z,\bar{w},1/z,1/\bar{w}]
\end{aligned}$$

This time, $\mathcal{O}_{H_n}$ is not an integral lattice because the image of the bilinear form

$$\mathcal{O}_{H_n}(U_{\sigma_3}) \times \mathcal{O}_{H_n}(U_{\sigma_3}) = \mathbb{C}[z,w] \times \mathbb{C}[z,w] \xrightarrow{1/z} \mathbb{C}[z,w,1/z,1/w]$$

does not lie in $\mathcal{V}^0_{\mathcal{O}_{H_n}}(U_{\sigma_3}) = \mathcal{O}_H(U_{\sigma_3}) = \mathbb{C}[z,w]$. On the other hand,

$$\mathcal{O}_{H_n}(-W)(U_{\sigma_3}) = z \cdot \mathbb{C}[z,w],$$

and the image of the bilinear form

$$z \cdot \mathbb{C}[z,w] \times z \cdot \mathbb{C}[z,w] \xrightarrow{1/z} \mathbb{C}[z,w,1/z,1/w]$$

does lie in $\mathbb{C}[z,w]$. We can similarly check that the image of the bilinar form

$$\mathcal{O}_{H_n}(-W)(U_{\sigma_i}) \times \mathcal{O}_{H_n}(-W)(U_{\sigma_i}) \xrightarrow{\langle x\rangle} (j_*\mathcal{O}_{H_n - H^1_n})(U_{\sigma_i})$$

lies in $\mathcal{V}^0_{\mathcal{O}_{H_n}}(U_{\sigma_i})$ for every i. Hence, $\mathcal{O}_{H_n}(-W)$ is an integral lattice for $(\mathcal{O}_{H_n - H^1_n}, \langle x\rangle)$. To find its dual lattice, note that

$$\begin{aligned}
\mathcal{O}_{H_n}(-W)'(U_{\sigma_1}) &:= \{f \in \mathbb{C}[x,y,1/x,1/y] \mid f \cdot x \cdot \mathbb{C}[x,y] \subset \mathbb{C}[x,y]\} \\
&= \frac{1}{x} \cdot \mathbb{C}[x,y] \\
&= \mathcal{O}_{H_n}(Y)(U_{\sigma_1}),
\end{aligned}$$

We can similarly check on the other affine open subsets to conclude that $\mathcal{O}_{H_n}(-W)' = \mathcal{O}_{H_n}(Y)$. Hence, $\mathcal{L}^0_{H_n^1}(\langle x \rangle)$ is given by

$$\frac{\mathcal{O}_{H_n}(Y)}{\mathcal{O}_{H_n}(-W)} \times \frac{\mathcal{O}_{H_n}(Y)}{\mathcal{O}_{H_n}(-W)} \xrightarrow{\langle x \rangle} \frac{i_* \mathcal{O}_{H_n - H_n^1}}{\mathcal{O}_{H_n}}. \tag{5.2}$$

Now we apply the map

$$\mathcal{K}^1_{H_n^1} : W(\mathcal{CM}^1_{H_n^1}(H_n)) \to W(\mathcal{CM}^1_{H_n^1 - N}(H_n - N)) \coprod W(\mathcal{CM}^1_{H_n^1 - L}(H_n - L))$$

by restricting the domain from H_n to $H_n - N$ and $H_n - L$. (Recall $L := X \cup Z$, $N := Y \cup W$.) Restricting (5.2) to $H_n - N$ gives zero because

$$\left. \frac{\mathcal{O}_{H_n}(Y)}{\mathcal{O}_{H_n}(-W)} \right|_{H_n - N} = \frac{\mathcal{O}_{H_n - N}}{\mathcal{O}_{H_n - N}} = 0.$$

By Corollary 3.8, this implies that $\mathcal{K}^1_{H_n^1} \circ \mathcal{L}^0_{H_n^1}(\langle x \rangle)$ has no component in the subspace generated by $\langle 1_x \rangle, \langle x \rangle, \langle 1_z \rangle, \langle z \rangle$, hence the corresponding rows in the second column are 0.

On the other hand, restricting (5.2) to $H_n - L$ gives

$$\frac{\mathcal{O}_{H_n - L}(T_Y)}{\mathcal{O}_{H_n - L}(-T_W)} \times \frac{\mathcal{O}_{H_n - L}(T_Y)}{\mathcal{O}_{H_n - L}(-T_W)} \xrightarrow{\langle x \rangle} \frac{i_* \mathcal{O}_{H_n - H_n^1}}{\mathcal{O}_{H_n - L}}. \tag{5.3}$$

On $(H_n - L) \cap U_{\sigma_1} = \operatorname{Spec} \mathbb{C}[x, y, 1/y]$, (5.3) gives a commutative diagram

$$\begin{array}{ccc} \frac{\frac{1}{x} \cdot \mathbb{C}[x,y,1/y]}{\mathbb{C}[x,y,1/y]} \times \frac{\frac{1}{x} \cdot \mathbb{C}[x,y,1/y]}{\mathbb{C}[x,y,1/y]} & \xrightarrow{x} & \frac{\mathbb{C}[x,y,1/x,1/y]}{\mathbb{C}[x,y,1/y]} \\ {\scriptstyle 1/x \times 1/x} \uparrow \wr & & \uparrow {\scriptstyle 1/x} \\ \mathbb{C}[y, 1/y] \times \mathbb{C}[y, 1/y] & \xrightarrow{1} & \mathbb{C}[y, 1/y] \end{array} \tag{5.4}$$

while on $(H_n - L) \cap U_{\sigma_3} = \operatorname{Spec} \mathbb{C}[z, w, 1/w]$, it gives a commutative diagram

$$\begin{array}{ccc} \frac{\mathbb{C}[z,w,1/w]}{z \cdot \mathbb{C}[z,w,1/w]} \times \frac{\mathbb{C}[z,w,1/w]}{z \cdot \mathbb{C}[z,w,1/w]} & \xrightarrow{1/z} & \frac{\mathbb{C}[z,w,1/z,1/w]}{\mathbb{C}[z,w,1/w]} \\ {\scriptstyle 1 \times 1} \uparrow \wr & & \uparrow {\scriptstyle 1/z} \\ \mathbb{C}[w, 1/w] \times \mathbb{C}[w, 1/w] & \xrightarrow{1} & \mathbb{C}[w, 1/w] \end{array}$$

We therefore conclude that $\mathcal{K}^1_L \circ \mathcal{L}^0_{H_n}(\langle x\rangle) = \langle 1_y\rangle + \langle 1_w\rangle$. Hence, in the second column, the rows corresponding to $\langle 1_y\rangle$ and $\langle 1_w\rangle$ are 1, and the rows corresponding to $\langle y\rangle$ nad $\langle w\rangle$ are 0.

So far our results didn't depend on n. Now we will see that the third and fourth columns do depend on n.

To determine the third column, consider the form $(\mathcal{O}_{H_n - H^1_n}, \langle y\rangle)$, given on the affine charts by

$$\begin{array}{rcccc} U_{\sigma_1} & : & \mathbb{C}[x,y,1/x,1/y]\times\mathbb{C}[x,y,1/x,1/y] & \stackrel{y}{\to} & \mathbb{C}[x,y,1/x,1/y] \\ U_{\sigma_2} & : & \mathbb{C}[x,\bar{y},1/x,1/\bar{y}]\times\mathbb{C}[x,\bar{y},1/x,1/\bar{y}] & \stackrel{1/\bar{y}}{\to} & \mathbb{C}[x,\bar{y},1/x,1/\bar{y}] \\ U_{\sigma_3} & : & \mathbb{C}[z,w,1/z,1/w]\times\mathbb{C}[z,w,,1/z,1/w] & \stackrel{z^n/w}{\to} & \mathbb{C}[z,w,1/z,1/w] \\ U_{\sigma_4} & : & \mathbb{C}[z,\bar{w},1/z,1/\bar{w}]\times\mathbb{C}[z,\bar{w},1/z,1/\bar{w}] & \stackrel{z^n\bar{w}}{\to} & \mathbb{C}[z,\bar{w},1/z,1/\bar{w}] \end{array}$$

We use the same argument as above to conclude that $\mathcal{O}_{H_n}(-Z)$ is an integral lattice for $(\mathcal{O}_{H_n - H^1_n}, \langle y\rangle)$. For example, $\mathcal{O}_{H_n}(-Z)(U_{\sigma_3}) = w\cdot\mathbb{C}[z,w]$, and the image of the bilinear map

$$w\cdot\mathbb{C}[z,w]\times w\cdot\mathbb{C}[z,w] \stackrel{z^n/w}{\to} \mathbb{C}[z,w,1/z,1/w]$$

lies in $\mathcal{V}^0_{\mathcal{O}_{H_n}}(U_{\sigma_3}) = \mathbb{C}[z,w]$. To find its dual lattice, note that

$$\begin{aligned} \mathcal{O}_{H_n}(-Z)'(U_{\sigma_3}) &:= \left\{ f\in\mathbb{C}[z,w,1/z,1/w] \mid f\cdot\frac{z^n}{w}\cdot w\cdot\mathbb{C}[z,w]\subset\mathbb{C}[z,w]\right\} \\ &= \frac{1}{z^n}\cdot\mathbb{C}[z,w] \\ &= \mathcal{O}_{H_n}(X+nW)(U_{\sigma_3}). \end{aligned}$$

We can similarly check on the other affine open subsets to conclude that $\mathcal{O}_{H_n}(-Z)' = \mathcal{O}_{H_n}(X+nW)$. Hence, $\mathcal{L}^0_{H^1_n}(\langle y\rangle)$ is given by

$$\frac{\mathcal{O}_{H_n}(X+nW)}{\mathcal{O}_{H_n}(-Z)}\times\frac{\mathcal{O}_{H_n}(X+nW)}{\mathcal{O}_{H_n}(-Z)} \stackrel{\langle y\rangle}{\to} \frac{i_*\mathcal{O}_{H_n - H^1_n}}{\mathcal{O}_{H_n}}. \tag{5.5}$$

Now we apply the map

$$\mathcal{K}^1_{H^1_n} : W(\mathcal{CM}^1_{H^1_n}(H_n)) \to W(\mathcal{CM}^1_{H^1_n - N}(H_n - N)) \coprod W(\mathcal{CM}^1_{H^1_n - L}(H_n - L))$$

by restricting the domains to $H_n - N$ and $H_n - L$. Restricting (5.5) to $H_n - L$ gives

$$\frac{\mathcal{O}_{H_n-L}(nT_W)}{\mathcal{O}_{H_n-L}} \times \frac{\mathcal{O}_{H_n-L}(nT_W)}{\mathcal{O}_{H_n-L}} \xrightarrow{\langle y\rangle} \frac{i_*\mathcal{O}_{H_n-H^1_n}}{\mathcal{O}_{H_n-L}}, \tag{5.6}$$

while restricting it to $H_n - N$ gives

$$\frac{\mathcal{O}_{H_n-N}(T_X)}{\mathcal{O}_{H_n-N}(-T_Z)} \times \frac{\mathcal{O}_{H_n-N}(T_X)}{\mathcal{O}_{H_n-N}(-T_Z)} \xrightarrow{\langle y\rangle} \frac{i_*\mathcal{O}_{H_n-H^1_n}}{\mathcal{O}_{H_n-N}}. \tag{5.7}$$

On $(H_n - L) \cap U_{\sigma_1} = \operatorname{Spec}\mathbb{C}[x, y, 1/y]$, (5.6) is zero because

$$\left.\frac{\mathcal{O}_{H_n-L}(nT_W)}{\mathcal{O}_{H_n-L}}\right|_{(H_n-L)\cap U_{\sigma_1}} = \frac{\mathcal{O}_{(H_n-L)\cap U_{\sigma_1}}}{\mathcal{O}_{(H_n-L)\cap U_{\sigma_1}}} = 0,$$

while on $(H_n - L) \cap U_{\sigma_3} = \operatorname{Spec}\mathbb{C}[z, w, 1/w]$, it becomes

$$\frac{\frac{1}{z^n}\cdot\mathbb{C}[z,w,1/w]}{\mathbb{C}[z,w,1/w]} \times \frac{\frac{1}{z^n}\cdot\mathbb{C}[z,w,1/w]}{\mathbb{C}[z,w,1/w]} \xrightarrow{z^n/w} \frac{\mathbb{C}[z,w,1/z,1/w]}{\mathbb{C}[z,w,1/w]}. \tag{5.8}$$

If n is even, then

$$\frac{\frac{1}{z^{n/2}}\cdot\mathbb{C}[z,w,1/w]}{z\cdot\mathbb{C}[z,w,1/w]} \subset \frac{\frac{1}{z^n}\cdot\mathbb{C}[z,w,1/w]}{z\cdot\mathbb{C}[z,w,1/w]}$$

is a totally isotropic subspace of (5.8) of half the rank, i.e., a sublagrangian. Hence, the Witt class of (5.8) is zero, i.e., $\mathcal{K}^1_L \circ \mathcal{L}^0_{H^1_{\text{even}}}(\langle y\rangle) = 0$. This implies that $d^0_{H\text{even}}(\langle y\rangle)$ has no component in the subspace spanned by $\langle 1_y\rangle, \langle y\rangle, \langle 1_w\rangle, \langle w\rangle$, and the corresponding rows in the third column are zero.

On the other hand, if n is odd, then

$$M := \frac{\frac{1}{z^{(n-1)/2}}\cdot\mathbb{C}[z,w,1/w]}{\mathbb{C}[z,w,1/w]} \subset \frac{\frac{1}{z^n}\cdot\mathbb{C}[z,w,1/w]}{\mathbb{C}[z,w,1/w]}$$

is a totally isotropic subspace of (5.8), and

$$M^\perp = \frac{\frac{1}{z^{(n+1)/2}} \cdot \mathbb{C}[z,w,1/w]}{\mathbb{C}[z,w,1/w]},$$

so $M^\perp/M \simeq \mathbb{C}[w,1/w]$, and there is a commutative diagram

$$\begin{array}{ccc} \frac{\frac{1}{z^{(n+1)/2}}\cdot\mathbb{C}[z,w,1/w]}{\frac{1}{z^{(n-1)/2}}\cdot\mathbb{C}[z,w,1/w]} \times \frac{\frac{1}{z^{(n+1)/2}}\cdot\mathbb{C}[z,w,1/w]}{\frac{1}{z^{(n-1)/2}}\cdot\mathbb{C}[z,w,1/w]} & \xrightarrow{z^n/w} & \frac{\mathbb{C}[z,w,1/z,1/w]}{\mathbb{C}[z,w,1/w]} \\ {\scriptstyle 1/z^{(n+1)/2}\times 1/z^{(n+1)/2}}\uparrow\wr & & \uparrow{\scriptstyle 1/z} \\ \mathbb{C}[w,1/w]\times\mathbb{C}[w,1/w] & \xrightarrow{1/w} & \mathbb{C}[w,1/w] \end{array} \tag{5.9}$$

By Lemma A.7(2) and Lemma 3.3, $M \mapsto M^\perp/M$ does not change the Witt class, so we conclude that $\mathcal{K}^1_L \circ \mathcal{L}^0_{H^1_{odd}}(\langle y\rangle) = \langle 1/w\rangle = \langle w\rangle$. Hence, in the third column, the row corresponding to $\langle w\rangle$ is 1, while the rows corresponding to $\langle 1_y\rangle, \langle y\rangle, \langle 1_w\rangle$ are 0.

Now on $(H_n - N)\cap U_{\sigma_1} = \operatorname{Spec}\mathbb{C}[x,y,1/x]$, (5.7) gives a commutative diagram

$$\begin{array}{ccc} \frac{\frac{1}{y}\cdot\mathbb{C}[x,y,1/x]}{\mathbb{C}[x,y,1/x]} \times \frac{\frac{1}{y}\cdot\mathbb{C}[x,y,1/x]}{\mathbb{C}[x,y,1/x]} & \xrightarrow{y} & \frac{\mathbb{C}[x,y,1/x,1/y]}{\mathbb{C}[x,y,1/x]} \\ {\scriptstyle 1/y\times 1/y}\uparrow\wr & & \uparrow{\scriptstyle 1/y} \\ \mathbb{C}[x,1/x]\times\mathbb{C}[x,1/x] & \xrightarrow{1} & \mathbb{C}[x,1/x] \end{array} \tag{5.10}$$

while on $(H_n - N)\cap U_{\sigma_3} = \operatorname{Spec}\mathbb{C}[z,w,1/z]$, it gives a commutative diagram

$$\begin{array}{ccc} \frac{\mathbb{C}[z,w,1/z]}{w\cdot\mathbb{C}[z,w,1/z]} \times \frac{\mathbb{C}[z,w,1/z]}{w\cdot\mathbb{C}[z,w,1/z]} & \xrightarrow{z^n/w} & \frac{\mathbb{C}[z,w,1/z,1/w]}{\mathbb{C}[z,w,1/z]} \\ {\scriptstyle 1\times 1}\uparrow\wr & & \uparrow{\scriptstyle 1/w} \\ \mathbb{C}[z,1/z]\times\mathbb{C}[z,1/z] & \xrightarrow{z^n} & \mathbb{C}[z,1/z] \end{array} \tag{5.11}$$

Hence, we conclude that $\mathcal{K}^1_N \circ \mathcal{L}^0_{H^1_n}(\langle y\rangle) = \langle 1_x\rangle + \langle z^n\rangle$. This implies that on the third column, the rows corresponding to $\langle 1_x\rangle$ and $\langle z^n\rangle$ are 1, while the rows corresponding to $\langle x\rangle$ and $\langle z^{n+1}\rangle$ are 0.

Finally, let us determine the entries in the fourth column. The form $(\mathcal{O}_{H_n - H_n^1}, \langle xy\rangle)$ is given on the affine charts by

$$\begin{array}{rcccl} U_{\sigma_1} & : & \mathbb{C}[x,y,1/x,1/y] \times \mathbb{C}[x,y,1/x,1/y] & \stackrel{xy}{\to} & \mathbb{C}[x,y,1/x,1/y] \\ U_{\sigma_2} & : & \mathbb{C}[x,\bar{y},1/x,1/\bar{y}] \times \mathbb{C}[x,\bar{y},1/x,1/\bar{y}] & \stackrel{x/\bar{y}}{\to} & \mathbb{C}[x,\bar{y},1/x,1/\bar{y}] \\ U_{\sigma_3} & : & \mathbb{C}[z,w,1/z,1/w] \times \mathbb{C}[z,w,,1/z,1/w] & \stackrel{z^{n-1}/w}{\to} & \mathbb{C}[z,w,1/z,1/w] \\ U_{\sigma_4} & : & \mathbb{C}[z,\bar{w},1/z,1/\bar{w}] \times \mathbb{C}[z,\bar{w},1/z,1/\bar{w}] & \stackrel{z^{n-1}\bar{w}}{\to} & \mathbb{C}[z,\bar{w},1/z,1/\bar{w}] \end{array}$$

Using the same argument as above, we conclude that $\mathcal{O}_{H_n}(-Z-W)$ is an integral lattice. For example, $\mathcal{O}_{H_n}(-Z-W)(U_{\sigma_3}) = zw\cdot\mathbb{C}[z,w]$, and the image of the bilinear form

$$zw \cdot \mathbb{C}[z,w] \times zw \cdot \mathbb{C}[z,w] \stackrel{z^{n-1}/w}{\to} \mathbb{C}[z,w,1/z,1/w]$$

lies in $\mathbb{C}[z,w]$. To find its dual lattice, note that

$$\begin{aligned} \mathcal{O}_{H_n}(-Z-W)'(U_{\sigma_3}) &:= \left\{ f \in \mathbb{C}[z,w,1/z,1/w] \mid f \cdot \frac{z^{n-1}}{w} \cdot zw \cdot \mathbb{C}[z,w] \subset \mathbb{C}[z,w] \right\} \\ &= \frac{1}{z^n} \cdot \mathbb{C}[z,w] \\ &= \mathcal{O}_{H_n}(X+Y+nW)(U_{\sigma_3}). \end{aligned}$$

We can similarly check on the other affine open subsets to conclude that $\mathcal{O}_{H_n}(-Z-W)' = \mathcal{O}_{H_n}(X+Y+nW)$. Hence, $\mathcal{L}^0_{H_n^1}(\langle xy\rangle)$ is given by

$$\frac{\mathcal{O}_{H_n}(X+Y+nW)}{\mathcal{O}_{H_n}(-Z-W)} \times \frac{\mathcal{O}_{H_n}(X+Y+nW)}{\mathcal{O}_{H_n}(-Z-W)} \stackrel{\langle xy\rangle}{\to} \frac{i_*\mathcal{O}_{H_n - H_n^1}}{\mathcal{O}_{H_n}}. \tag{5.12}$$

Now we apply the map

$$\mathcal{K}^1_{H_n^1} : W(\mathcal{CM}^1_{H_n^1}(H_n)) \to W(\mathcal{CM}^1_{H_n^1 - N}(H_n - N)) \coprod W(\mathcal{CM}^1_{H_n^1 - L}(H_n - L))$$

by restricting the domains to $H_n - N$ and $H_n - L$.

Restricting (5.12) to $H_n - N$ gives

$$\frac{\mathcal{O}_{H_n-N}(T_X)}{\mathcal{O}_{H_n-N}(-T_Z)} \times \frac{\mathcal{O}_{H_n-N}(T_X)}{\mathcal{O}_{H_n-N}(-T_Z)} \xrightarrow{xy} \frac{i_*\mathcal{O}_{H_n-H_n^1}}{\mathcal{O}_{H_n-N}}, \tag{5.13}$$

while restricting to $H_n - L$ gives

$$\frac{\mathcal{O}_{H_n-L}(T_Y+nT_W)}{\mathcal{O}_{H_n-L}(-T_W)} \times \frac{\mathcal{O}_{H_n-L}(T_Y+nT_W)}{\mathcal{O}_{H_n-L}(-T_W)} \xrightarrow{xy} \frac{i_*\mathcal{O}_{H_n-H_n^1}}{\mathcal{O}_{H_n-L}}. \tag{5.14}$$

On $(H_n - N) \cap U_{\sigma_1} = \operatorname{Spec}\mathbb{C}[x, y, 1/x]$, (5.13) gives a commutative diagram

$$\begin{array}{ccc} \frac{\frac{1}{y}\cdot\mathbb{C}[x,y,1/x]}{\mathbb{C}[x,y,1/x]} \times \frac{\frac{1}{y}\cdot\mathbb{C}[x,y,1/x]}{\mathbb{C}[x,y,1/x]} & \xrightarrow{xy} & \frac{\mathbb{C}[x,y,1/x,1/y]}{\mathbb{C}[x,y,1/x]} \\ {\scriptstyle 1/y\times 1/y}\uparrow\wr & & \uparrow{\scriptstyle 1/y} \\ \mathbb{C}[x,1/x]\times\mathbb{C}[x,1/x] & \xrightarrow{x} & \mathbb{C}[x,1/x] \end{array}$$

while on $(H_n - N) \cap U_{\sigma_3} = \operatorname{Spec}\mathbb{C}[z, w, 1/z]$, it gives a commutative diagram

$$\begin{array}{ccc} \frac{\mathbb{C}[z,w,1/z]}{w\cdot\mathbb{C}[z,w,1/z]} \times \frac{\mathbb{C}[z,w,1/z]}{w\cdot\mathbb{C}[z,w,1/z]} & \xrightarrow{z^{n-1}/w} & \frac{\mathbb{C}[z,w,1/z,1/w]}{\mathbb{C}[z,w,1/z]} \\ {\scriptstyle 1\times 1}\uparrow\wr & & \uparrow{\scriptstyle 1/w} \\ \mathbb{C}[z,1/z]\times\mathbb{C}[z,1/z] & \xrightarrow{z^{n-1}} & \mathbb{C}[z,1/z] \end{array}$$

Hence, we conclude that $\mathcal{K}_N^1 \circ \mathcal{L}_{H_n^1}^0(\langle xy\rangle) = \langle x\rangle + \langle z^{n-1}\rangle$. This implies that in the fourth column, the rows corresponding to $\langle x\rangle$ and $\langle z^{n-1}\rangle$ are 1, while the rows corresponding to $\langle 1_x\rangle$ and $\langle z^n\rangle$ are 0.

On the other hand, on $(H_n - L) \cap U_{\sigma_1} = \operatorname{Spec}\mathbb{C}[x, y, 1/y]$, (5.14) gives a commutative diagram

$$\begin{array}{ccc} \frac{\frac{1}{x}\cdot\mathbb{C}[x,y,1/y]}{\mathbb{C}[x,y,1/y]} \times \frac{\frac{1}{x}\cdot\mathbb{C}[x,y,1/y]}{\mathbb{C}[x,y,1/y]} & \xrightarrow{xy} & \frac{\mathbb{C}[x,y,1/x,1/y]}{\mathbb{C}[x,y,1/y]} \\ {\scriptstyle 1/x\times 1/x}\uparrow\wr & & \uparrow{\scriptstyle 1/x} \\ \mathbb{C}[y,1/y]\times\mathbb{C}[y,1/y] & \xrightarrow{y} & \mathbb{C}[y,1/y] \end{array} \tag{5.15}$$

while on $(H_n - L) \cap U_{\sigma_3} = \operatorname{Spec} \mathbb{C}[z, w, 1/w]$, it becomes

$$\frac{\frac{1}{z^n} \cdot \mathbb{C}[z, w, 1/w]}{z \cdot \mathbb{C}[z, w, 1/w]} \times \frac{\frac{1}{z^n} \cdot \mathbb{C}[z, w, 1/w]}{z \cdot \mathbb{C}[z, w, 1/w]} \overset{z^{n-1}/w}{\longrightarrow} \frac{\mathbb{C}[z, w, 1/z, 1/w]}{\mathbb{C}[z, w, 1/w]}. \tag{5.16}$$

If n is odd, then

$$\frac{\frac{1}{z^{(n-1)/2}} \cdot \mathbb{C}[z, w, 1/w]}{z \cdot \mathbb{C}[z, w, 1/x]} \subset \frac{\frac{1}{z^n} \cdot \mathbb{C}[z, w, 1/w]}{z \cdot \mathbb{C}[z, w, 1/x]}$$

is a totally isotropic subspace of (5.16) of half the rank, i.e., a sublagrangian. Hence, the Witt class of (5.16) is zero, and together with (5.15), we conclude that

$$\mathcal{K}_L^1 \circ \mathcal{L}^0_{H^1_{\text{odd}}}(\langle xy \rangle) = \langle y \rangle.$$

This implies that in the fourth column, the row corresponding to $\langle y \rangle$ is 1, while the rows corresponding to $\langle 1_y \rangle, \langle 1_w \rangle, \langle w \rangle$ are 0.

On the other hand, if n is even, then

$$M := \frac{\frac{1}{z^{n/2-1}} \cdot \mathbb{C}[z, w, 1/w]}{z \cdot \mathbb{C}[z, w, 1/w]} \subset \frac{\frac{1}{z^n} \cdot \mathbb{C}[z, w, 1/w]}{z \cdot \mathbb{C}[z, w, 1/w]}$$

is a totally isotropic subspace of (5.16), and

$$M^{\perp} = \frac{\frac{1}{z^{n/2}} \cdot \mathbb{C}[z, w, 1/w]}{z \cdot \mathbb{C}[z, w, 1/w]},$$

so $M^{\perp}/M \simeq \mathbb{C}[w, 1/w]$, and there is a commutative diagram

$$\begin{array}{ccc} \frac{\frac{1}{z^{n/2}} \cdot \mathbb{C}[z,w,1/w]}{\frac{1}{z^{n/2-1}} \cdot \mathbb{C}[z,w,1/w]} \times \frac{\frac{1}{z^{n/2}} \cdot \mathbb{C}[z,w,1/w]}{\frac{1}{z^{n/2-1}} \cdot \mathbb{C}[z,w,1/w]} & \overset{z^{n-1}/w}{\longrightarrow} & \frac{\mathbb{C}[z,w,1/z,1/w]}{\mathbb{C}[z,w,1/w]} \\ {\scriptstyle 1/z^{n/2} \times 1/z^{n/2}} \uparrow \wr & & \uparrow {\scriptstyle 1/z} \\ \mathbb{C}[w, 1/w] \times \mathbb{C}[w, 1/w] & \overset{1/w}{\longrightarrow} & \mathbb{C}[w, 1/w] \end{array} \tag{5.17}$$

As we noted earlier, $M \mapsto M^{\perp}/M$ does not change the Witt class. Hence, together with (5.15), we conclude that

$$\mathcal{K}_L^1 \circ \mathcal{L}^1_{H^1_{\text{even}}}(\langle xy \rangle) = \langle y \rangle + \langle 1/w \rangle = \langle y \rangle + \langle w \rangle.$$

This implies that on the fourth column, the rows corresponding to $\langle y\rangle$ and $\langle w\rangle$ are 1, while the rows corresponding to $\langle 1_y\rangle$ and $\langle 1_w\rangle$ are 0. This completes the proof. $\square$

Proposition 5.2. *With the same choice of basis as in Proposition 5.1, the map* $d^1_{H_n}$ *is represented by the matrix*

$$d^1_{H_{even}} = \begin{array}{c} \\ \langle 0_{xy}\rangle \\ \langle 0_{z\bar{w}}\rangle \\ \langle 0_{x\bar{y}}\rangle \\ \langle 0_{zw}\rangle \end{array} \begin{array}{c} \begin{array}{cccccccc} \langle 1_x\rangle & \langle x\rangle & \langle 1_y\rangle & \langle y\rangle & \langle 1_z\rangle & \langle z\rangle & \langle 1_w\rangle & \langle w\rangle \end{array} \\ \begin{pmatrix} 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 \end{pmatrix} \end{array},$$

$$d^1_{H_{odd}} = \begin{array}{c} \\ \langle 0_{xy}\rangle \\ \langle 0_{z\bar{w}}\rangle \\ \langle 0_{x\bar{y}}\rangle \\ \langle 0_{zw}\rangle \end{array} \begin{array}{c} \begin{array}{cccccccc} \langle 1_x\rangle & \langle x\rangle & \langle 1_y\rangle & \langle y\rangle & \langle 1_z\rangle & \langle z\rangle & \langle 1_w\rangle & \langle w\rangle \end{array} \\ \begin{pmatrix} 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 \end{pmatrix} \end{array}.$$

Proof. Recall that $d^1_{H_n} := \mathcal{L}^1_{T_L} \coprod \mathcal{L}^1_{T_N}$, where

$$\begin{aligned} \mathcal{L}^1_{T_L} : W(\mathcal{CM}^1_{T_L}(H_n - N)) &\rightarrow W(\mathcal{CM}^2_{H_n^2}(H_n)), \\ \mathcal{L}^1_{T_N} : W(\mathcal{CM}^1_{T_N}(H_n - L)) &\rightarrow W(\mathcal{CM}^2_{H_n^2}(H_n)). \end{aligned}$$

$W(\mathcal{CM}^1_{T_L}(H_n - N))$ is generated by $\langle 1_x\rangle, \langle x\rangle, \langle 1_z\rangle, \langle z\rangle$, and $W(\mathcal{CM}^1_{T_N}(H_n - L))$ is generated by $\langle 1_y\rangle, \langle y\rangle, \langle 1_w\rangle, \langle w\rangle$ (Corollary 3.8). There is a commutative diagram

$$\begin{array}{ccc} W(\mathcal{CM}^1_{T_L}(H_n - N)) & \xrightarrow{\mathcal{L}^1_{T_L}} & W(\mathcal{CM}^2_{H^2}(H_n)) \\ \uparrow & & \uparrow \\ W(\mathcal{CM}^1_{T_X}(U_{\sigma_1} - Y)) & \xrightarrow{\mathcal{L}^1_{\sigma_1,y}} & W(\mathcal{CM}^2_{0_{xy}}(U_{\sigma_1})) \end{array}$$

where the vertical maps are induced by inclusion. Hence, we can compute $\mathcal{L}^1_{T_L}$ in terms of the affine lattice maps $\mathcal{L}^1_{\sigma_1,y}, \mathcal{L}^1_{\sigma_2,\bar{y}}, \mathcal{L}^1_{\sigma_3,w}, \mathcal{L}^1_{\sigma_4,\bar{w}}$, and $\mathcal{L}^1_{T_N}$ in terms of

$\mathcal{L}^1_{\sigma_1,x}, \mathcal{L}^1_{\sigma_2,x}, \mathcal{L}^1_{\sigma_3,z}, \mathcal{L}^1_{\sigma_4,z}$. The integral lattice and dual lattice for $\mathcal{L}^1_{\sigma_1,y}$ are computed using the value group

$$\mathcal{V}^1_{\mathcal{O}_{U_{\sigma_1}},y} = \frac{\mathbb{C}[x,y,1/y]}{\mathbb{C}[x,y]} \simeq \frac{\mathbb{C}[x,y,1/y]+\mathbb{C}[x,y,1/x]}{\mathbb{C}[x,y,1/x]} \subset \frac{\mathbb{C}[x,y,1/x,1/y]}{\mathbb{C}[x,y,1/x]}. \tag{5.18}$$

We saw in the computation of $\mathcal{L}^0_{H^1_n}(\langle y\rangle)$ that on $(H_n - N)\cap U_{\sigma_1} = \operatorname{Spec}\mathbb{C}[x,y,1/x]$, $\langle 1_x\rangle$ is given by (5.10)

$$\frac{\frac{1}{y}\cdot\mathbb{C}[x,y,1/x]}{\mathbb{C}[x,y,1/x]} \times \frac{\frac{1}{y}\cdot\mathbb{C}[x,y,1/x]}{\mathbb{C}[x,y,1/x]} \xrightarrow{y} \frac{\mathbb{C}[x,y,1/x,1/y]}{\mathbb{C}[x,y,1/x]}. \tag{5.19}$$

Let $M := \frac{\frac{1}{y}\cdot\mathbb{C}[x,y]+\mathbb{C}[x,y,1/x]}{\mathbb{C}[x,y,1/x]} \subset \frac{\frac{1}{y}\cdot\mathbb{C}[x,y,1/x]}{\mathbb{C}[x,y,1/x]}$. Since the image of the bilinear form

$$\frac{\frac{1}{y}\cdot\mathbb{C}[x,y]+\mathbb{C}[x,y,1/x]}{\mathbb{C}[x,y,1/x]} \times \frac{\frac{1}{y}\cdot\mathbb{C}[x,y]+\mathbb{C}[x,y,1/x]}{\mathbb{C}[x,y,1/x]} \xrightarrow{y} \frac{\mathbb{C}[x,y,1/x,1/y]}{\mathbb{C}[x,y,1/x]}$$

lies in $\mathcal{V}^1_{\mathcal{O}_{U_{\sigma_1}},y}$ (5.18), M is an integral lattice for (5.19). Moreover,

$$M' = \left\{ f \in \frac{\frac{1}{y}\cdot\mathbb{C}[x,y,1/x]}{\mathbb{C}[x,y,1/x]} \mid f\cdot\frac{1}{y}\cdot y \in \frac{\frac{1}{y}\cdot\mathbb{C}[x,y]+\mathbb{C}[x,y,1/x]}{\mathbb{C}[x,y,1/x]} \right\} = M,$$

so M is self-dual. Hence, $\mathcal{L}^1_{\sigma_1,y}(\langle 1_x\rangle) = 0$. This implies that $\mathcal{L}^1_{T_L}(\langle 1_x\rangle)$ has no $\langle 0_{xy}\rangle$ component, so the corresponding row in the first column is 0.

On $(H_n - N)\cap U_{\sigma_4} = \operatorname{Spec}[z,\bar{w},1/z]$, $\langle 1_x\rangle$ is represented by (see 5.19)

$$\frac{\frac{1}{\bar{w}}\cdot\mathbb{C}[z,\bar{w},1/z]}{\mathbb{C}[z,\bar{w},1/z]} \times \frac{\frac{1}{\bar{w}}\cdot\mathbb{C}[z,\bar{w},1/z]}{\mathbb{C}[z,\bar{w},1/z]} \xrightarrow{z^n\bar{w}} \frac{\mathbb{C}[z,\bar{w},1/z,1/\bar{w}]}{\mathbb{C}[z,\bar{w},1/z]}. \tag{5.20}$$

Using the the value group

$$\mathcal{V}^1_{\mathcal{O}_{U_{\sigma_4}},\bar{w}} = \frac{\mathbb{C}[z,\bar{w},1/\bar{w}]}{\mathbb{C}[z,\bar{w}]} \simeq \frac{\mathbb{C}[z,\bar{w},1/\bar{w}]+\mathbb{C}[z,\bar{w},1/z]}{\mathbb{C}[z,\bar{w},1/z]} \subset \frac{\mathbb{C}[z,\bar{w},1/z,1/\bar{w}]}{\mathbb{C}[z,\bar{w},1/z]}, \tag{5.21}$$

one finds that $M := \frac{\frac{1}{\bar{w}}\cdot\mathbb{C}[z,\bar{w}]+\mathbb{C}[z,\bar{w},1/z]}{\mathbb{C}[z,\bar{w},1/z]}$ is an integral lattice, and $M' = \frac{\frac{1}{z^n\bar{w}}\cdot\mathbb{C}[z,\bar{w}]+\mathbb{C}[z,\bar{w},1/z]}{\mathbb{C}[z,\bar{w},1/z]}$ is its dual lattice. Since the rank homomorphism induces

$$W(\mathcal{CM}^2_{0_{xy}}(U_{\sigma_1})) = W(\mathbb{C}) = \mathbb{Z}/2,$$

$\dim_{\mathbb{C}}(M'/M) = n$ implies that $\mathcal{L}^1_{\sigma_4,\bar{w}}(\langle 1_x\rangle)$ is zero if and only if n is even. This implies that $\mathcal{L}^1_{T_L}(\langle 1_x\rangle)$ has no $\langle 0_{z\bar{w}}\rangle$ component if and only if n is even, so the corresponding row in the first column is zero if and only if n is even.

Since $\langle 1_x\rangle$ is supported on T_X (Corollary 3.8), the rest of the entries in the first column are zero.

The entries in the second column are obtained in the same way. On $(H_n - N) \cap U_{\sigma_1} = \operatorname{Spec}\mathbb{C}[x, y, 1/x]$, $\langle x\rangle$ is represented by (see 5.19)

$$\frac{\frac{1}{y}\cdot\mathbb{C}[x,y,1/x]}{\mathbb{C}[x,y,1/x]} \times \frac{\frac{1}{y}\cdot\mathbb{C}[x,y,1/x]}{\mathbb{C}[x,y,1/x]} \xrightarrow{xy} \frac{\mathbb{C}[x,y,1/x,1/y]}{\mathbb{C}[x,y,1/x]}. \tag{5.22}$$

One finds that $M = \frac{\frac{1}{y}\cdot\mathbb{C}[x,y]+\mathbb{C}[x,y,1/x]}{\mathbb{C}[x,y,1/x]}$ is an integral lattice, but and that its dual lattice is $M' = \frac{\frac{1}{xy}\cdot\mathbb{C}[x,y]+\mathbb{C}[x,y,1/x]}{\mathbb{C}[x,y,1/x]}$. Then $\dim_{\mathbb{C}} M'/M = 1$, so that $\mathcal{L}^1_{\sigma_1,y}(\langle x\rangle)$ is non-zero. This implies that $\mathcal{L}^1_{T_L}(\langle x\rangle)$ has a non-zero $\langle 0_{xy}\rangle$ component, so the corresponding row in the second column is 1.

On $(H_n - N) \cap U_{\sigma_4} = \operatorname{Spec}[z, \bar{w}, 1/z]$, $\langle x\rangle$ is represented by (see 5.22)

$$\frac{\frac{1}{\bar{w}}\cdot\mathbb{C}[z,\bar{w},1/z]}{\mathbb{C}[z,\bar{w},1/z]} \times \frac{\frac{1}{\bar{w}}\cdot\mathbb{C}[z,\bar{w},1/z]}{\mathbb{C}[z,\bar{w},1/z]} \xrightarrow{z^{n-1}\bar{w}} \frac{\mathbb{C}[z,\bar{w},1/z,1/\bar{w}]}{\mathbb{C}[z,\bar{w},1/z]}. \tag{5.23}$$

Using the value group $\mathcal{V}^1_{\mathcal{O}_{U_{\sigma_4}},\bar{w}}$ (5.21), one finds that $M := \frac{\frac{z}{\bar{w}}\cdot\mathbb{C}[z,\bar{w}]+\mathbb{C}[z,\bar{w},1/z]}{\mathbb{C}[z,\bar{w},1/z]}$ is an integral lattice, and $M' = \frac{\frac{1}{z^n\bar{w}}\cdot\mathbb{C}[z,\bar{w}]+\mathbb{C}[z,\bar{w},1/z]}{\mathbb{C}[z,\bar{w},1/z]}$ is its dual lattice. Then $\dim_{\mathbb{C}} M'/M = n+1$, so $\mathcal{L}^1_{\sigma_4,\bar{w}}(\langle x\rangle)$ is zero if and only if n is odd. This implies that $\mathcal{L}^1_{T_L}(\langle x\rangle)$ has no $\langle 0_{z\bar{w}}\rangle$ component if and only if n is odd, so the corresponding row in the second column is 0 if and only if n is odd.

Since $\langle x \rangle$ is supported on T_X, the rest of the entries in the second column are zero.

We move on to the third column.

We saw in the computation of $\mathcal{L}^0_{H^1_n}(\langle x \rangle)$ that on $(H_n - L) \cap U_{\sigma_1} = \operatorname{Spec} \mathbb{C}[x, y, 1/y]$, $\langle 1_y \rangle$ is given by (5.4)

$$\frac{\frac{1}{x} \cdot \mathbb{C}[x, y, 1/y]}{\mathbb{C}[x, y, 1/y]} \times \frac{\frac{1}{x} \cdot \mathbb{C}[x, y, 1/y]}{\mathbb{C}[x, y, 1/y]} \xrightarrow{x} \frac{\mathbb{C}[x, y, 1/x, 1/y]}{\mathbb{C}[x, y, 1/y]}. \tag{5.24}$$

Using the value group

$$\mathcal{V}^1_{\mathcal{O}_{U_{\sigma_1}}, x} = \frac{\mathbb{C}[x, y, 1/x]}{\mathbb{C}[x, y]} \simeq \frac{\mathbb{C}[x, y, 1/x] + \mathbb{C}[x, y, 1/y]}{\mathbb{C}[x, y, 1/y]} \subset \frac{\mathbb{C}[x, y, 1/x, 1/y]}{\mathbb{C}[x, y, 1/y]}, \tag{5.25}$$

one finds that $\frac{\frac{1}{x} \cdot \mathbb{C}[x,y] + \mathbb{C}[x,y,1/y]}{\mathbb{C}[x,y,1/y]}$ is a self-dual integral lattice, so $\mathcal{L}^1_{\sigma_1, x}(\langle 1_y \rangle) = 0$. This implies that $\mathcal{L}^1_{T_N}(\langle 1_y \rangle)$ has no $\langle 0_{xy} \rangle$ component, so the corresponding row in the third column is 0.

On $(H_n - L) \cap U_{\sigma_2} = \operatorname{Spec} \mathbb{C}[x, \bar{y}, 1/\bar{y}]$, $\langle 1_y \rangle$ is given by (see (5.24))

$$\frac{\frac{1}{x} \cdot \mathbb{C}[x, \bar{y}, 1/\bar{y}]}{\mathbb{C}[x, y, 1/y]} \times \frac{\frac{1}{x} \cdot \mathbb{C}[x, \bar{y}, 1/\bar{y}]}{\mathbb{C}[x, \bar{y}, 1/\bar{y}]} \xrightarrow{x} \frac{\mathbb{C}[x, \bar{y}, 1/x, 1/\bar{y}]}{\mathbb{C}[x, \bar{y}, 1/\bar{y}]}. \tag{5.26}$$

Using the value group

$$\mathcal{V}^1_{\mathcal{O}_{U_{\sigma_2}}, x} = \frac{\mathbb{C}[x, \bar{y}, 1/x]}{\mathbb{C}[x, \bar{y}]} \simeq \frac{\mathbb{C}[x, \bar{y}, 1/x] + \mathbb{C}[x, \bar{y}, 1/\bar{y}]}{\mathbb{C}[x, \bar{y}, 1/\bar{y}]} \subset \frac{\mathbb{C}[x, \bar{y}, 1/x, 1/\bar{y}]}{\mathbb{C}[x, \bar{y}, 1/\bar{y}]}, \tag{5.27}$$

one finds that $\frac{\frac{1}{x} \cdot \mathbb{C}[x,\bar{y}] + \mathbb{C}[x,\bar{y},1/\bar{y}]}{\mathbb{C}[x,\bar{y},1/\bar{y}]}$ is a self-dual integral lattice, so $\mathcal{L}^1_{\sigma_2, x}(\langle 1_y \rangle) = 0$. This implies that $\mathcal{L}^1_{T_N}(\langle 1_y \rangle)$ has no $\langle 0_{x\bar{y}} \rangle$ component, so the corresponding row in the third column is 0.

Since $\langle 1_y \rangle$ is supported on T_Y, the rest of the entries in the third column are zero.

We move on to the fourth column.

On $(H_n - L) \cap U_{\sigma_1}$, $\langle y \rangle$ is represented by (see (5.24))

$$\frac{\frac{1}{x}\cdot \mathbb{C}[x,y,1/y]}{\mathbb{C}[x,y,1/y]} \times \frac{\frac{1}{x}\cdot \mathbb{C}[x,y,1/y]}{\mathbb{C}[x,y,1/y]} \xrightarrow{xy} \frac{\mathbb{C}[x,y,1/x,1/y]}{\mathbb{C}[x,y,1/y]}. \tag{5.28}$$

Using the value group $\mathcal{V}^1_{\mathcal{O}_{U_{\sigma_1}},x}$ (5.25), one finds that $M := \frac{\frac{1}{x}\cdot\mathbb{C}[x,y]+\mathbb{C}[x,y,1/y]}{\mathbb{C}[x,y,1/y]}$ is an integral lattice, and that $M' := \frac{\frac{1}{xy}\cdot\mathbb{C}[x,y]+\mathbb{C}[x,y,1/y]}{\mathbb{C}[x,y,1/y]}$ is its dual lattice. Since $\dim_{\mathbb{C}} M'/M = 1$, we conclude that $\mathcal{L}^1_{\sigma_2,x}(\langle y \rangle) \neq 0$. This implies that $\mathcal{L}^1_{T_N}(\langle y \rangle)$ has a nonzero $\langle 0_{xy} \rangle$ component, so the corresponding row in the fourth column is 1.

On $(H_n - L) \cap U_{\sigma_2}$, $\langle y \rangle$ is represented by (see 5.26)

$$\frac{\frac{1}{x}\cdot \mathbb{C}[x,\bar{y},1/\bar{y}]}{\mathbb{C}[x,y,1/y]} \times \frac{\frac{1}{x}\cdot \mathbb{C}[x,\bar{y},1/\bar{y}]}{\mathbb{C}[x,\bar{y},1/\bar{y}]} \xrightarrow{x/\bar{y}} \frac{\mathbb{C}[x,\bar{y},1/x,1/\bar{y}]}{\mathbb{C}[x,\bar{y},1/\bar{y}]} \tag{5.29}$$

Using the value group $\mathcal{V}^1_{\mathcal{O}_{U_{\sigma_2}},x}$ (5.27), one finds that $M := \frac{\frac{\bar{y}}{x}\cdot\mathbb{C}[x,\bar{y}]+\mathbb{C}[x,\bar{y},1/\bar{y}]}{\mathbb{C}[x,\bar{y},1/\bar{y}]}$ is an integral lattice, and that $M' := \frac{\frac{1}{x}\cdot\mathbb{C}[x,\bar{y}]+\mathbb{C}[x,\bar{y},1/\bar{y}]}{\mathbb{C}[x,\bar{y},1/\bar{y}]}$ is its dual lattice. Then $\dim_{\mathbb{C}} M'/M = 1$, so $\mathcal{L}^1_{\sigma_2,x}(\langle y \rangle) \neq 0$. This implies that $\mathcal{L}^1_{T_N}(\langle y \rangle)$ has a nonzero $\langle 0_{x\bar{y}} \rangle$ component, so the corresponding row in the fourth column is 1.

Since $\langle y \rangle$ is supported on T_Y, the rest of the entries in the fourth column are zero.

We move on to the fifth column.

We saw in the computation of $\mathcal{L}^0_{H^1_n}(\langle y \rangle)$ that on $(H_n - N) \cap U_{\sigma_3} = \mathbb{C}[z,w,1/z]$, $\langle 1_z \rangle$ is represented by (5.11)

$$\frac{\mathbb{C}[z,w,1/z]}{w\cdot\mathbb{C}[z,w,1/z]} \times \frac{\mathbb{C}[z,w,1/z]}{w\cdot\mathbb{C}[z,w,1/z]} \xrightarrow{1/w} \frac{\mathbb{C}[z,w,1/z,1/w]}{\mathbb{C}[z,w,1/z]}. \tag{5.30}$$

Using the value group

$$\mathcal{V}^1_{\mathcal{O}_{U_{\sigma_3}},w} = \frac{\mathbb{C}[z,w,1/w]}{\mathbb{C}[z,w]} \simeq \frac{\mathbb{C}[z,w,1/w]+\mathbb{C}[z,w,1/z]}{\mathbb{C}[z,w,1/z]} \subset \frac{\mathbb{C}[z,w,1/z,1/w]}{\mathbb{C}[z,w,1/z]}, \tag{5.31}$$

one finds that $\frac{\mathbb{C}[z,w]+w\cdot\mathbb{C}[z,w,1/x]}{w\cdot\mathbb{C}[z,w,1/x]}$ is a self-dual integral lattice, so $\mathcal{L}^1_{\sigma_3,w}(\langle 1_z\rangle) = 0$. This implies that $\mathcal{L}^1_{T_L}(\langle 1_z\rangle)$ has no $\langle 0_{zw}\rangle$ component, so the corresponding row in the fifth column is 0.

On $(H_n - N) \cap U_{\sigma_2} = \mathbb{C}[x, \bar{y}, 1/x]$, $\langle 1_z\rangle$ is represented by (see 5.30)

$$\frac{\mathbb{C}[x,\bar{y},1/x]}{\bar{y}\cdot\mathbb{C}[x,\bar{y},1/x]} \times \frac{\mathbb{C}[x,\bar{y},1/x]}{\bar{y}\cdot\mathbb{C}[x,\bar{y},1/x]} \overset{x^n/\bar{y}}{\to} \frac{\mathbb{C}[x,\bar{y},1/x,1/\bar{y}]}{\mathbb{C}[x,\bar{y},1/x]}, \tag{5.32}$$

and using the value group $\mathcal{V}^1_{\mathcal{O}_{U_{\sigma_2}},x}$ (5.27), one finds that $M := \frac{\mathbb{C}[x,\bar{y}]+\bar{y}\cdot\mathbb{C}[x,\bar{y},1/x]}{\bar{y}\cdot\mathbb{C}[x,\bar{y},1/x]}$ is an integral lattice, and that $M' := \frac{\frac{1}{x^n}\cdot\mathbb{C}[x,\bar{y}]+\bar{y}\cdot\mathbb{C}[x,\bar{y},1/x]}{\bar{y}\cdot\mathbb{C}[x,\bar{y},1/x]}$ is its dual lattice. Hence, $\dim_{\mathbb{C}} M'/M = n$, so $\mathcal{L}^1_{\sigma_2,x}(\langle 1_z\rangle) = 0$ if and only if n is even. This implies that $\mathcal{L}^1_{T_L}(\langle 1_z\rangle)$ has no $\langle 0_{x\bar{y}}\rangle$ component if and only if n is even, so the corresponding row in the fifth column is 0 if and only if n is even.

Since $\langle 1_z\rangle$ is supported on T_Z, the rest of the entries in the fifth column are zero.

We move on to the sixth column.

On $(H_n - N) \cap U_{\sigma_3} = \mathbb{C}[z, w, 1/z]$, $\langle z\rangle$ is represented by (see 5.30)

$$\frac{\mathbb{C}[z,w,1/z]}{w\cdot\mathbb{C}[z,w,1/z]} \times \frac{\mathbb{C}[z,w,1/z]}{w\cdot\mathbb{C}[z,w,1/z]} \overset{z/w}{\to} \frac{\mathbb{C}[z,w,1/z,1/w]}{\mathbb{C}[z,w,1/z]}. \tag{5.33}$$

Using the value group $\mathcal{V}^1_{\mathcal{O}_{U_{\sigma_3}},w}$ (5.31), one finds that $M := \frac{\mathbb{C}[z,w]+w\cdot\mathbb{C}[z,w,1/z]}{w\cdot\mathbb{C}[z,w,1/z]}$ is an integral lattice, and that $M' := \frac{\frac{1}{z}\cdot\mathbb{C}[z,w]+w\cdot\mathbb{C}[z,w,1/z]}{w\cdot\mathbb{C}[z,w,1/z]}$ is its dual lattice. Hence, $\dim_{\mathbb{C}} M'/M = 1$, so $\mathcal{L}^1_{\sigma_3,w}(\langle z\rangle) \neq 0$. This implies that $\mathcal{L}^1_{T_L}(\langle z\rangle)$ has a non-zero $\langle 0_{zw}\rangle$ component, so the corresponding row in the sixth column is 1.

On $(H_n - N) \cap U_{\sigma_2} = \mathbb{C}[x, \bar{y}, 1/x]$, $\langle z\rangle$ is represented by (see 5.32)

$$\frac{\mathbb{C}[x,\bar{y},1/x]}{\bar{y}\cdot\mathbb{C}[x,\bar{y},1/x]} \times \frac{\mathbb{C}[x,\bar{y},1/x]}{\bar{y}\cdot\mathbb{C}[x,\bar{y},1/x]} \overset{x^{n-1}/\bar{y}}{\to} \frac{\mathbb{C}[x,\bar{y},1/x,1/\bar{y}]}{\mathbb{C}[x,\bar{y},1/x]}. \tag{5.34}$$

Using the value group

$$\mathcal{V}^1_{\mathcal{O}_{U_{\sigma_2}},\bar{y}} = \frac{\mathbb{C}[x,\bar{y},1/\bar{y}]}{\mathbb{C}[x,\bar{y}]} \simeq \frac{\mathbb{C}[x,\bar{y},1/\bar{y}]+\mathbb{C}[x,\bar{y},1/x]}{\mathbb{C}[x,\bar{y},1/x]} \subset \frac{\mathbb{C}[x,\bar{y},1/x,1/\bar{y}]}{\mathbb{C}[x,\bar{y},1/x]}, \tag{5.35}$$

one finds that $M = \frac{\mathbb{C}[x,\bar{y}]+\bar{y}\cdot\mathbb{C}[x,\bar{y},1/x]}{\bar{y}\cdot\mathbb{C}[x,\bar{y},1/x]}$ is an integral lattice, and that $M' = \frac{\frac{1}{x^{n-1}}\cdot\mathbb{C}[x,\bar{y}]+\bar{y}\cdot\mathbb{C}[x,\bar{y},1/x]}{\bar{y}\cdot\mathbb{C}[x,\bar{y},1/x]}$ is its dual lattice. Then $\dim_{\mathbb{C}} M'/M = n-1$, so $\mathcal{L}^1_{\sigma_2,\bar{y}}(\langle z\rangle) \neq 0$ if and only if n is even. This implies that $\mathcal{L}^1_{T_L}(\langle z\rangle)$ has a non-zero $\langle 0_{x\bar{y}}\rangle$ component if and only if n is even, so the corresponding row in the sixth column is 1 if and only if n is even.

Since $\langle z\rangle$ is supported on T_Z, the rest of the entries in the sixth column are zero.

For the seventh and eight columns, we consider different parities of n separately at the outset:

(1) When n is even : We saw in the computation of $\mathcal{L}^0_{H^1_n}(\langle xy\rangle)$ that on $(H_n - L)\cap U_{\sigma_3} = \operatorname{Spec}\mathbb{C}[z,w,1/w]$, $\langle 1_w\rangle$ is given by (see (5.16) and (5.17))

$$\frac{\frac{1}{z^n}\cdot\mathbb{C}[z,w,1/w]}{z\cdot\mathbb{C}[z,w,1/w]} \times \frac{\frac{1}{z^n}\cdot\mathbb{C}[z,w,1/w]}{z\cdot\mathbb{C}[z,w,1/w]} \xrightarrow{z^{n-1}/w^2} \frac{\mathbb{C}[z,w,1/z,1/w]}{\mathbb{C}[z,w,1/w]}. \tag{5.36}$$

Using the value group

$$\mathcal{V}^1_{\mathcal{O}_{U_{\sigma_3}},z} = \frac{\mathbb{C}[z,w,1/z]}{\mathbb{C}[z,w]} \simeq \frac{\mathbb{C}[z,y,1/z]+\mathbb{C}[z,y,1/w]}{\mathbb{C}[z,w,1/w]} \subset \frac{\mathbb{C}[z,w,1/z,1/w]}{\mathbb{C}[z,w,1/w]}, \tag{5.37}$$

one finds that $\frac{\frac{w}{z^{n/2}}\cdot\mathbb{C}[z,w]+z\cdot\mathbb{C}[z,w,1/w]}{z\cdot\mathbb{C}[z,w,1/w]}$ is a self-dual integral lattice, so $\mathcal{L}^1_{\sigma_3,z}(\langle w\rangle) = 0$. This implies that $\mathcal{L}^1_{T_N}(\langle w\rangle)$ does not have $\langle 0_{zw}\rangle$ component, so the corresponding row in the seventh column is 0.

On $(H_n - L)\cap U_{\sigma_4} = \operatorname{Spec}\mathbb{C}[z,\bar{w},1/\bar{w}]$, $\langle 1_w\rangle$ is given by (see 5.36)

$$\frac{\frac{1}{z^n}\cdot\mathbb{C}[z,\bar{w},1/\bar{w}]}{z\cdot\mathbb{C}[z,\bar{w},1/\bar{w}]} \times \frac{\frac{1}{z^n}\cdot\mathbb{C}[z,\bar{w},1/\bar{w}]}{z\cdot\mathbb{C}[z,\bar{w},1/\bar{w}]} \xrightarrow{z^{n-1}\bar{w}^2} \frac{\mathbb{C}[z,\bar{w},1/z,1/\bar{w}]}{\mathbb{C}[z,\bar{w},1/\bar{w}]}. \tag{5.38}$$

Using the value group

$$\mathcal{V}^1_{\mathcal{O}_{U_{\sigma_4}},z} = \frac{\mathbb{C}[z,\bar{w},1/z]}{\mathbb{C}[z,\bar{w}]} \simeq \frac{\mathbb{C}[z,\bar{w},1/z]+\mathbb{C}[z,\bar{w},1/\bar{w}]}{\mathbb{C}[z,\bar{w},1/\bar{w}]} \subset \frac{\mathbb{C}[z,\bar{w},1/z,1/\bar{w}]}{\mathbb{C}[z,\bar{w},1/\bar{w}]}, \tag{5.39}$$

one finds that $\frac{\frac{1}{z^{n/2}\bar{w}}\cdot\mathbb{C}[z,\bar{w}]+z\cdot\mathbb{C}[z,\bar{w},1/\bar{w}]}{z\cdot\mathbb{C}[z,\bar{w},1/\bar{w}]}$ is a self-dual integral lattice, so $\mathcal{L}^1_{\sigma_4,z}(\langle 1_w\rangle) = 0$. This implies that $\mathcal{L}^1_{T_N}(\langle 1_w\rangle)$ has no $\langle 0_{z\bar{w}}\rangle$ component, so the corresponding row in the seventh column is 0.

Since $\langle 1_w \rangle$ is supported on T_W, the rest of the entries in the seventh column are zero.

We move on to the eighth column.

On $(H_n - L) \cap U_{\sigma_3} = \mathbb{C}[z, w, 1/w]$, $\langle w \rangle$ is represented by (see 5.36)

$$\frac{\frac{1}{z^n} \cdot \mathbb{C}[z, w, 1/w]}{z \cdot \mathbb{C}[z, w, 1/w]} \times \frac{\frac{1}{z^n} \cdot \mathbb{C}[z, w, 1/w]}{z \cdot \mathbb{C}[z, w, 1/w]} \overset{z^{n-1}/w}{\longrightarrow} \frac{\mathbb{C}[z, w, 1/z, 1/w]}{\mathbb{C}[z, w, 1/w]}. \tag{5.40}$$

Using the value group $\mathcal{V}^1_{\mathcal{O}_{U_{\sigma_3}},z}$ (5.37), one finds that $M := \frac{\frac{w}{z^{n/2}} \cdot \mathbb{C}[z,w] + z \cdot \mathbb{C}[z,w,1/w]}{z \cdot \mathbb{C}[z,w,1/w]}$ is an integral lattice, and that $M' := \frac{\frac{1}{z^{n/2}} \cdot \mathbb{C}[z,w] + z \cdot \mathbb{C}[z,w,1/w]}{z \cdot \mathbb{C}[z,w,1/w]}$ is its dual lattice. Then $\dim_{\mathbb{C}} M'/M = 1$, so $\mathcal{L}^1_{\sigma_4,z}(\langle w \rangle) \neq 0$. This implies that $\mathcal{L}^1_{T_N}(\langle w \rangle)$ has a non-zero $\langle 0_{zw} \rangle$ component, so the corresponding row in the eighth column is 1.

On $(H_n - L) \cap U_{\sigma_4} = \operatorname{Spec} \mathbb{C}[z, \bar{w}, 1/\bar{w}]$, $\langle w \rangle$ is represented by (see 5.38)

$$\frac{\frac{1}{z^n} \cdot \mathbb{C}[z, \bar{w}, 1/\bar{w}]}{z \cdot \mathbb{C}[z, \bar{w}, 1/\bar{w}]} \times \frac{\frac{1}{z^n} \cdot \mathbb{C}[z, \bar{w}, 1/\bar{w}]}{z \cdot \mathbb{C}[z, \bar{w}, 1/\bar{w}]} \overset{z^{n-1}\bar{w}}{\longrightarrow} \frac{\mathbb{C}[z, \bar{w}, 1/z, 1/\bar{w}]}{\mathbb{C}[z, \bar{w}, 1/\bar{w}]}. \tag{5.41}$$

Using the value group $\mathcal{V}^1_{\mathcal{O}_{U_{\sigma_4}},z}$ (5.39), one finds that $M := \frac{\frac{1}{z^{n/2}} \cdot \mathbb{C}[z,\bar{w}] + z \cdot \mathbb{C}[z,\bar{w},1/\bar{w}]}{z \cdot \mathbb{C}[z,\bar{w},1/\bar{w}]}$ is an integral lattice, and that $M' := \frac{\frac{1}{z^{n/2}\bar{w}} \cdot \mathbb{C}[z,\bar{w}] + z \cdot \mathbb{C}[z,\bar{w},1/\bar{w}]}{z \cdot \mathbb{C}[z,\bar{w},1/\bar{w}]}$ is its dual lattice. Hence, $\dim_{\mathbb{C}} M'/M = 1$, so $\mathcal{L}^1_{\sigma_4,z}(\langle w \rangle) \neq 0$. This implies that $\mathcal{L}^1_{T_N}(\langle w \rangle)$ has a non-zero $\langle 0_{z\bar{w}} \rangle$ component, so the corresponding row in the eigth column is 1.

Since $\langle w \rangle$ is supported on T_W, the rest of the entries in the eighth column are zero.

(2) When n is odd : We saw in the computation of $\mathcal{L}^0_{H^1_n}(\langle y \rangle)$ that on $(H_n - L) \cap U_{\sigma_3} = \mathbb{C}[z, w, 1/w]$, $\langle 1_w \rangle$ is represented by (see (5.8) and (5.9))

$$\frac{\frac{1}{z^n} \cdot \mathbb{C}[z, w, 1/w]}{\mathbb{C}[z, w, 1/w]} \times \frac{\frac{1}{z^n} \cdot \mathbb{C}[z, w, 1/w]}{\mathbb{C}[z, w, 1/w]} \overset{z^n/w^2}{\longrightarrow} \frac{\mathbb{C}[z, w, 1/z, 1/w]}{\mathbb{C}[z, w, 1/w]}. \tag{5.42}$$

Using the value group $\mathcal{V}^1_{\mathcal{O}_{U_{\sigma_3}},z}$ (5.37), one finds that $\frac{\frac{w}{z^{(n+1)/2}}\cdot\mathbb{C}[z,w]+\mathbb{C}[z,w,1/w]}{\mathbb{C}[z,w,1/w]}$ is a self-dual integral lattice, so $\mathcal{L}^1_{\sigma_3,z}(\langle w\rangle) = 0$. This implies that $\mathcal{L}^1_{T_N}(\langle w\rangle)$ has no $\langle 0_{zw}\rangle$ component, so the corresponding row in the seventh column is 0.

On $(H_n - L)\cap U_{\sigma_4} = \operatorname{Spec}\mathbb{C}[z,\bar{w},1/\bar{w}]$, $\langle 1_w\rangle$ is given by (see (5.42))

$$\frac{\frac{1}{z^n}\cdot\mathbb{C}[z,\bar{w},1/\bar{w}]}{\mathbb{C}[z,\bar{w},1/\bar{w}]}\times\frac{\frac{1}{z^n}\cdot\mathbb{C}[z,\bar{w},1/\bar{w}]}{\mathbb{C}[z,\bar{w},1/\bar{w}]}\overset{z^n\bar{w}^2}{\to}\frac{\mathbb{C}[z,\bar{w},1/z,1/\bar{w}]}{\mathbb{C}[z,\bar{w},1/\bar{w}]}. \tag{5.43}$$

Using the value group $\mathcal{V}^1_{\mathcal{O}_{U_{\sigma_4}},z}$ (5.39), one finds that $\frac{\frac{1}{z^{(n+1)/2}\bar{w}}\cdot\mathbb{C}[z,\bar{w}]+\mathbb{C}[z,\bar{w},1/\bar{w}]}{\mathbb{C}[z,\bar{w},1/\bar{w}]}$ is a self-dual integral lattice, so $\mathcal{L}^1_{\sigma_4,z}(\langle 1_w\rangle) = 0$. This implies that $\mathcal{L}^1_{T_N}(\langle 1_w\rangle)$ has no $\langle 0_{z\bar{w}}\rangle$ component, so the corresponding row in the seventh column is 0.

On $(H_n - L)\cap U_{\sigma_3} = \operatorname{Spec}\mathbb{C}[z,w,1/w]$, $\langle w\rangle$ is given by (see (5.42))

$$\frac{\frac{1}{z^n}\cdot\mathbb{C}[z,w,1/w]}{\mathbb{C}[z,w,1/w]}\times\frac{\frac{1}{z^n}\cdot\mathbb{C}[z,w,1/w]}{\mathbb{C}[z,w,1/w]}\overset{z^n/w}{\to}\frac{\mathbb{C}[z,w,1/z,1/w]}{\mathbb{C}[z,w,1/w]}. \tag{5.44}$$

Using the value group $\mathcal{V}^1_{\mathcal{O}_{U_{\sigma_3}},z}$ (5.37), one finds that $M := \frac{\frac{w}{z^{(n+1)/2}}\cdot\mathbb{C}[z,w]+\mathbb{C}[z,w,1/w]}{\mathbb{C}[z,w,1/w]}$ is an integral lattice, and that $M' := \frac{\frac{1}{z^{(n+1)/2}}\cdot\mathbb{C}[z,w]+\mathbb{C}[z,w,1/w]}{\mathbb{C}[z,w,1/w]}$ is its dual lattice. Hence, $\dim_{\mathbb{C}} M'/M = 1$, so $\mathcal{L}^1_{\sigma_4,z}(\langle w\rangle)\neq 0$. This implies that $\mathcal{L}^1_{T_N}(\langle w\rangle)$ has a non-zero $\langle 0_{zw}\rangle$ component, so the corresponding row in the eighth column is 1.

On $(H_n - L)\cap U_{\sigma_4} = \operatorname{Spec}\mathbb{C}[z,\bar{w},1/\bar{w}]$, $\langle w\rangle$ is given by (see (5.43))

$$\frac{\frac{1}{z^n}\cdot\mathbb{C}[z,\bar{w},1/\bar{w}]}{\mathbb{C}[z,\bar{w},1/\bar{w}]}\times\frac{\frac{1}{z^n}\cdot\mathbb{C}[z,\bar{w},1/\bar{w}]}{\mathbb{C}[z,\bar{w},1/\bar{w}]}\overset{z^n\bar{w}}{\to}\frac{\mathbb{C}[z,\bar{w},1/z,1/\bar{w}]}{\mathbb{C}[z,\bar{w},1/\bar{w}]}. \tag{5.45}$$

Using the value group $\mathcal{V}^1_{\mathcal{O}_{U_{\sigma_4}},z}$ (5.39), one finds that $M := \frac{\frac{1}{z^{(n+1)/2}}\cdot\mathbb{C}[z,\bar{w}]+\mathbb{C}[z,\bar{w},1/\bar{w}]}{\mathbb{C}[z,\bar{w},1/\bar{w}]}$ is an integral lattice, and that $M' := \frac{\frac{1}{z^{(n+1)/2}\bar{w}}\cdot\mathbb{C}[z,\bar{w}]+\mathbb{C}[z,\bar{w},1/\bar{w}]}{\mathbb{C}[z,\bar{w},1/\bar{w}]}$ is its dual lattice. Then $\dim_{\mathbb{C}} M'/M = 1$, so $\mathcal{L}^1_{\sigma_4,z}(\langle w\rangle)\neq 0$. This implies that $\mathcal{L}^1_{T_N}(\langle w\rangle)$ has a non-zero $\langle 0_{z\bar{w}}\rangle$ component, so the corresponding row in the eighth column is 1.

Since $\langle w \rangle$ is supported on T_W, the rest of the entries in the eighth column are zero. □

Remark 5.3. *The matrix representation for* $d^1_{H^1_n}$ *can also be deduced from Schmid's result [14] that the canonical map*

$$W(\mathbb{P}^1; \mathcal{O}_{\mathbb{P}^1}(n)) \to \coprod_{x \in \mathbb{P}^{1(1)}} W(x)$$

is given by the second residue homomorphism (Theorem 1.1) at $x \neq \infty$, *and by the first (resp. second) residue homomorphism at* $x = \infty$ *if* n *is odd (resp. even).*

6

Cohomologies

Now that we have seen the quasi-isomorphism between the toric complex (4.3) and the Gersten-Witt complex (4.2) of H_n, we compute cohomologies using the former. We will see that they are cohomologies of the Witt sheaf $U \mapsto W(U)$ on H_n.

We first verify that $d^1_{H_n} \circ d^0_{H_n} = 0$. Next, to compute the cohomologies, we put the matrices into Smith normal forms[1] :

$$d^0_{H_{\text{even}}} \simeq \begin{array}{c} \\ \langle 1_x \rangle \\ \langle x \rangle \\ \langle 1_y \rangle \\ \langle y \rangle \\ \langle 1_z \rangle \\ \langle z \rangle \\ \langle 1_w \rangle \\ \langle w \rangle \end{array} \begin{array}{c} \begin{array}{cccc} \langle 1 \rangle & \langle x \rangle & \langle y \rangle & \langle xy \rangle \end{array} \\ \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \end{array} , \quad d^0_{H_{\text{odd}}} \simeq \begin{array}{c} \\ \langle 1_x \rangle \\ \langle x \rangle \\ \langle 1_y \rangle \\ \langle y \rangle \\ \langle 1_z \rangle \\ \langle z \rangle \\ \langle 1_w \rangle \\ \langle w \rangle \end{array} \begin{array}{c} \begin{array}{cccc} \langle 1 \rangle & \langle x \rangle & \langle y \rangle & \langle xy \rangle \end{array} \\ \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \end{array} .$$

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[1] They are computed using a Mathematica package *IntegerSmithNormalForm* from `http://library.wolfram.com/infocenter/MathSource/682`.

$$d^1_{H_{\text{even}}} \simeq \begin{array}{c} \\ \langle 0_{xy}\rangle \\ \langle 0_{z\bar{w}}\rangle \\ \langle 0_{x\bar{y}}\rangle \\ \langle 0_{zw}\rangle \end{array} \begin{array}{c} \begin{array}{cccccccc} \langle 1_x\rangle & \langle x\rangle & \langle 1_y\rangle & \langle y\rangle & \langle 1_z\rangle & \langle z\rangle & \langle 1_w\rangle & \langle w\rangle \end{array} \\ \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \end{array},$$

$$d^1_{H_{\text{odd}}} \simeq \begin{array}{c} \\ \langle 0_{xy}\rangle \\ \langle 0_{z\bar{w}}\rangle \\ \langle 0_{x\bar{y}}\rangle \\ \langle 0_{zw}\rangle \end{array} \begin{array}{c} \begin{array}{cccccccc} \langle 1_x\rangle & \langle x\rangle & \langle 1_y\rangle & \langle y\rangle & \langle 1_z\rangle & \langle z\rangle & \langle 1_w\rangle & \langle w\rangle \end{array} \\ \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \end{pmatrix} \end{array}.$$

Therefore we have

$$\dim_{\mathbb{C}} \ker d^0_{H_n} = 1, \qquad \dim_{\mathbb{C}} \operatorname{im} d^0_{H_n} = 3,$$

$$\dim_{\mathbb{C}} \ker d^1_{H_{\text{even}}} = 5, \qquad \dim_{\mathbb{C}} \ker d^1_{H_{\text{odd}}} = 4,$$

$$\dim_{\mathbb{C}} \operatorname{im} d^1_{H_{\text{even}}} = 3, \qquad \dim_{\mathbb{C}} \operatorname{im} d^1_{H_{\text{odd}}} = 4,$$

from which we conclude that

$$H^0(\mathcal{W}^\bullet(H_{\text{even}})) = \mathbb{Z}/2, \qquad H^1(\mathcal{W}^\bullet(H_{\text{even}})) = (\mathbb{Z}/2)^2, \qquad H^2(\mathcal{W}^\bullet(H_{\text{even}})) = \mathbb{Z}/2,$$

$$H^0(\mathcal{W}^\bullet(H_{\text{odd}})) = \mathbb{Z}/2, \qquad H^1(\mathcal{W}^\bullet(H_{\text{odd}})) = \mathbb{Z}/2, \qquad H^2(\mathcal{W}^\bullet(H_{\text{odd}})) = 0.$$

Note that

$$\ker d^0_{H_n} = H^0(\mathcal{W}^\bullet(H_n)) = \mathbb{Z}/2 \qquad \forall n \in \mathbb{Z}. \tag{6.1}$$

Fernández-Carmena [3, 3.4] showed that the Witt group of a complex surface is a birational invariant, so that

$$W(H_n) = W(\mathbb{P}^2_{\mathbb{C}}) = \mathbb{Z}/2 \qquad \forall n \in \mathbb{Z}.$$

On the other hand, since the rank is a local invariant, the localization map

$$W(H_n) \to W(H_{n,\eta}) = \Gamma(H_n, \mathcal{W}^0(H_n)),$$

where $\eta \in H_n$ is the generic point, is an injection. By (6.1), we have

$$W(H_n) = H^0(\mathcal{W}^\bullet(H_n)).$$

By the Purity Theorem [9], $\mathcal{W}^\bullet$ is a resolution of the sheaf $U \mapsto W(U)$ on H_n.

Appendix A

Technical lemmas

We first note a useful lemma, which is easy to prove:

Lemma A.1. *Let M, N, V be A-modules, and*

$$\phi : M \times N \to V$$

an A-bilinear pairing. If

$$\operatorname{ad}\phi : M \to \operatorname{Hom}_A(N, V), \qquad \operatorname{ad}^\dagger \phi : N \to \operatorname{Hom}_A(M, V)$$

are the adjoints, then

$$\operatorname{ad}\phi = \operatorname{Hom}(\operatorname{ad}^\dagger \phi, V) \circ \rho_M, \qquad \operatorname{ad}^\dagger \phi = \operatorname{Hom}(\operatorname{ad}\phi, V) \circ \rho_N, \tag{A.1}$$

where

$$\begin{aligned} \operatorname{Hom}(\operatorname{ad}\phi, V) : \operatorname{Hom}_A(M, V) &\leftarrow \operatorname{Hom}_A(\operatorname{Hom}_A(N, V), V), \\ \operatorname{Hom}(\operatorname{ad}^\dagger \phi, V) : \operatorname{Hom}_A(N, V) &\leftarrow \operatorname{Hom}_A(\operatorname{Hom}_A(M, V), V), \end{aligned}$$

and

$$\begin{aligned} \rho_M : M &\to \operatorname{Hom}_A(\operatorname{Hom}_A(M, V), V), \\ \rho_N : N &\to \operatorname{Hom}_A(\operatorname{Hom}_A(N, V), V) \end{aligned}$$

are the canonical maps.

Note that if ρ_M and ρ_N are isomorphisms, then $\operatorname{ad}\phi$ is bijective if and only if $\operatorname{ad}^\dagger \phi$ is bijective.

Now let $A := \mathbb{C}[x,y]$, $V_y^1 := \frac{\mathbb{C}[x,y,1/y]}{\mathbb{C}[x,y]}$, and denote $(-)^* := \operatorname{Hom}_A(-, V_y^1)$. Pardon proved the following :

1. If $M \in \mathcal{CM}_y^1(A)$, then $M^* \in \mathcal{CM}_y^1(A)$ [12, 1.13].

2. $M \in \mathcal{CM}_y^1(A)$, then the canonical map

$$\rho_M : M \to M^{**}$$

   is bijective [12, 1.17].

3. If

$$0 \to M' \to M \to M'' \to 0$$

   is an exact sequence in $\mathcal{CM}_y^1(A)$, then the induced sequence

$$0 \leftarrow (M')^* \leftarrow M^* \leftarrow (M'')^* \leftarrow 0$$

   is exact [11, 1.6c].

4. If $M \in \mathcal{CM}_y^1(A)$ and $N \subset M$ is a submodule, then $N \in \mathcal{CM}_y^1(A)$ [12, 1.19].

Lemma A.2. *Let $M, N \in \mathcal{CM}_y^1(A)$, and*

$$\phi : M \times N \to V_y^1$$

a bilinear pairing. Then the following are equivalent:

1. ϕ is nonsingular.

2. $\operatorname{ad}\phi$ *is bijective.*

3. $\mathrm{ad}^\dagger \phi$ *is bijective.*

4. $\mathrm{ad}\,\phi$ *and* $\mathrm{ad}^\dagger \phi$ *are injective.*

Proof. (1)⇔(2)⇔(3) follows from bijectivity of $\rho_M : M \to M^{**}$. (1)⇒(4) is obvious.

(1)⇐(4) : Suppose that $\mathrm{ad}\,\phi$ and $\mathrm{ad}^\dagger \phi$ are injective. Applying $\mathrm{Hom}_A(-, V_y^1)$ to the short exact sequence

$$0 \to M \stackrel{\mathrm{ad}\,\phi}{\to} N^* \to N^*/M \to 0$$

gives an injection

$$\mathcal{CM}_y^1(A) \ni N = N^{**} \hookleftarrow (N^*/M)^*.$$

Hence, $(N^*/M)^* \in \mathcal{CM}_y^1(A)$. Then $N^*/M = (N^*/M)^{**} \in \mathcal{CM}_y^1(A)$, so there is an exact sequence

$$0 \leftarrow M^* \stackrel{(\mathrm{ad}\,\phi)^*}{\leftarrow} N^{**} \leftarrow (N^*/M)^* \leftarrow 0,$$

so $(\mathrm{ad}\,\phi)^*$ is surjective. Since $\mathrm{ad}^\dagger \phi = (\mathrm{ad}\,\phi)^* \circ \rho_N$ and ρ_N is bijective, $\mathrm{ad}^\dagger \phi$ is surjective. A similar argument shows that $\mathrm{ad}\,\phi$ is surjecive. □

Lemma A.3. *Let* $M \in \mathcal{CM}_y^1(A)$*,* $N \subset M$ *a submodule such that* $M/N \in \mathcal{CM}_y^1(A)$*, and*

$$\phi : M \times M \to V_y^1$$

a nonsingular symmetric A-bilinear form. Then $M/N^\perp \in \mathcal{CM}_y^1(A)$*, and* $N = N^{\perp\perp}$*. Moreover, the induced pairings*

$$\alpha : N \times M/N^\perp \to V_y^1, \qquad \beta : N^\perp \times M/N \to V_y^1$$

are nonsingular.

Proof. Let

$$\mathrm{ad}\,\beta : N^\perp \to \mathrm{Hom}_A(M/N, V_y^1), \qquad \mathrm{ad}^\dagger \beta : M/N \to \mathrm{Hom}_A(N^\perp, V_y^1)$$

be the adjoints of β. There is a commutative diagram

$$\begin{array}{ccc} M & \xrightarrow[\sim]{\operatorname{ad}\phi} & \operatorname{Hom}_A(M, V_y^1) \\ \uparrow & & \uparrow \\ N^\perp & \overset{\operatorname{ad}\beta}{\dashrightarrow} & \operatorname{Hom}_A(M/N, V_y^1) \end{array}$$

Bijectivity of $\operatorname{ad}\phi$ implies that $\operatorname{ad}\beta$ is bijective, so

$$\operatorname{Hom}(\operatorname{ad}\beta, V_y^1) : (N^\perp)^* \leftarrow (M/N)^{**}$$

is bijective. By (A.1), $\operatorname{ad}^\dagger\beta$ is then bijective. In particular, injectivity of $\operatorname{ad}^\dagger\beta$ implies that $N^{\perp\perp} \subset N$. Since $N \subset N^{\perp\perp}$, we have $N = N^{\perp\perp}$.

Now

$$\operatorname{ad}^\dagger\alpha : M/N^\perp \to \operatorname{Hom}_A(N, V_y^1)$$

is clearly injective, so $M/N^\perp \in \mathcal{CM}_y^1(A)$. Hence, we can apply the same argument as above with N replaced by $N^\perp$ to conclude that $\operatorname{ad}\alpha$ is bijective. $\square$

The modules in $\mathcal{CM}_y^1(A)$ do not necessarily have finite length. However, if $M \in \mathcal{CM}_y^1(A)$ is $\mathbb{C}[x]$-torsion-free, then one can define a notion similar to length; $y^{k-1}M/y^kM$ is a free module of finite rank over $\mathbb{C}[x]$, so there is a finite chain of submodules

$$M = M^0 \supset M^1 \supset M^2 \supset \cdots \supset M^n = 0,$$

where $M^i/M^{i-1} \simeq \mathbb{C}[x]$. We then define $\ell_y(M) := n$. Note the following:

- ℓ_y is an additive function on $\mathbb{C}[x]$-torsion-free modules in $\mathcal{CM}_y^1(A)$.
- $M^i/M^{i-1} \in \mathcal{CM}_y^1(A)$, and it is $\mathbb{C}[x]$-torsion-free.
- If $M \in \mathcal{CM}_y^1(A)$ is $\mathbb{C}[x]$-torsion-free, then so is M^*.

Lemma A.4. *Let $M \in \mathcal{CM}^1_y(A)$ be $\mathbb{C}[x]$-torsion-free. If $N \subset M$ is a submodule, then $M/N \in \mathcal{CM}^1_y(A)$.*

Proof. It is clear that M/N is $\mathbb{C}[x]$-torsion-free. Hence, $\operatorname{depth}_A(M/N) \geqslant 1$, so $M/N \notin \mathcal{CM}^1_y(A)$ implies that M/N has dimension 2, i.e., it is of finite length. Then it is killed by a product of maximal ideals of the form $(x-a, y-b) \subset \mathbb{C}[x,y]$, contradicting the $\mathbb{C}[x]$-torsion-freeness. □

Lemma A.5. *If $M \in \mathcal{CM}^1_y(A)$ is $\mathbb{C}[x]$-torsion-free, then $\ell_y(M) = \ell_y(M^*)$.*

Proof. Let $\ell_y(M) = n$, so that there is a chain of submodules

$$M = M^0 \supset M^1 \supset M^2 \supset \cdots \supset M^n = 0$$

such that $M^i/M^{i-1} \simeq \mathbb{C}[x]$. We prove by induction on n. There is a short exact sequence of modules in $\mathcal{CM}^1_y(A)$:

$$0 \leftarrow M/M^1 \leftarrow M \leftarrow M^1 \leftarrow 0.$$

Taking $\operatorname{Hom}_A(-, V^1_y)$ gives an exact sequence [11, 1.6c]

$$0 \to (M/M^1)^* \to M^* \to (M^1)^* \to 0.$$

Note that as a $\mathbb{C}[x,y]$-module, $M/M^1 = \mathbb{C}[x]$ is killed by y. Hence, the image of any homomorphism $M/M^1 \to V^1_y$ lies in $(0:y)_{V^1_y} = \frac{\frac{1}{y}\cdot\mathbb{C}[x,y]}{\mathbb{C}[x,y]} \simeq \mathbb{C}[x]$. Hence, $(M/M^1)^* \simeq \mathbb{C}[x]$, so $\ell_y((M/M^1)^*) = 1$. Then by the additivity of ℓ_y and the induction hypothesis,

$$\ell_y(M^*) = 1 + \ell_y((M^1)^*) = 1 + (n-1) = n = \ell_y(M).$$

□

Lemma A.6. *Let $M \in \mathcal{CM}_y^1(A)$ be $\mathbb{C}[x]$-torsion-free, and*

$$\phi : M \times M \to V_y^1$$

a nonsingular symmetric bilinear form. If $N \subset M$ is a subspace, then

$$\ell_y(N) + \ell_y(N^\perp) = \ell_y(M).$$

Proof. By Lemma A.3, the induced pairing

$$N \times M/N^\perp \to V_y^1$$

is nonsingular, so $N \simeq (M/N^\perp)^*$. Hence, by Lemma A.5,

$$\ell_y(N) = \ell_y((M/N^\perp)^*) = \ell_y(M/N^\perp) = \ell_y(M) - \ell_y(N^\perp).$$

□

Lemma A.7. *Let $M \in \mathcal{CM}_y^1(A)$ be $\mathbb{C}[x]$-torsion-free,*

$$\phi : M \times M \to V_y^1$$

a nonsingular symmetric bilinear form, and $N \subset M$ is a totally isotropic subspace.

1. *$2 \cdot \ell_y(N) \leqslant \ell_y(M)$, and equality holds if and only if N is orthogonal, i.e., $N = N^\perp$.*

2. *The induced bilinear form*

$$\bar{\phi} : \frac{N^\perp}{N} \times \frac{N^\perp}{N} \to V_y^1$$

is nonsingular. Moreover, if M has an orthogonal submodule, so does $N^\perp/N$.

Proof. (1) : Since $N \subset N^{\perp}$, $\ell_y(N) \leqslant \ell_y(N^{\perp})$. Hence, by Lemma A.6,

$$2 \cdot \ell_y(N) \leqslant \ell_y(N) + \ell_y(N^{\perp}) = \ell_y(M).$$

So $2 \cdot \ell_y(N) = \ell_y(M)$ if and only if $\ell_y(N) = \ell_y(N^{\perp})$, i.e., if $N = N^{\perp}$.

(2) : Since $N^{\perp}/N \in \mathcal{CM}_y^1(A)$ by Lemma A.4, to prove nonsingularity of $\bar{\phi}$, it suffices to prove injectivity of $\operatorname{ad}\bar{\phi}$. But this is clear from $N^{\perp\perp} = N$.

Now suppose that $K \subset M$ is an orthogonal submodule. Let Λ be the set of totally isotropic submodules containing N, and $S \in \Lambda$ a maximal element. We will show that $S \subset M$ is an orthogonal submodule. This implies that $S/N \subset N^{\perp}/N$ is an orthogonal submodule, completing the proof.

First assume that $S \cap K = 0$. Since S is totally isotropic, $2 \cdot \ell_y(S) \leqslant \ell_y(M)$ by the first part of the proof. We will show that this is in fact an equality. Suppose, by way of contradiction, that $2 \cdot \ell_y(S) < \ell_y(M)$. Since K is orthogonal, $\ell_y(MWittclass) = 2 \cdot \ell_y(K)$ by the first part of the proof. Hence, $\ell_y(S) < \ell_y(K)$, and by Lemma A.6,

$$\ell_y(S^{\perp}) = \ell_y(M) - \ell_y(S) > \ell_y(M) - \ell_y(K).$$

Hence, $\ell_y(M) < \ell_y(S^{\perp}) + \ell_y(K)$, and this implies that $S^{\perp} \cap K \neq 0$, so there exists a non-zero element $m \in S^{\perp} \cap K$. Since $S \cap K = 0$ by assumption, $m \notin S$. Since $m \in K$ and $K = K^{\perp}$, $\psi(m, m) = 0$. Hence, $S + (m) \subset M$ is a totally isotropic submodule strictly containing S, contradicting the maximality of S. Hence, $2 \cdot \ell_y(S) = \ell_y(M)$, so $S \subset M$ is an orthogonal submodule by the first part of the proof.

Now for the general case, let $J := S \cap K$. Then $J \subset M$ is a totally isotropic submodule, so there is an induced bilinear form

$$\bar{\psi} : J^{\perp}/J \times J^{\perp}/J \to V_y^1.$$

We have shown that this is nonsingular. Note that

$$J \subset S \subset S^{\perp} \subset J^{\perp}, \qquad J \subset K = K^{\perp} \subset J^{\perp}.$$

$S/J \subset J^\perp/J$ is maximal among totally isotropic submodules of $J^\perp/J$. Moreover,

$$(S/J) \cap (K/J) = 0 \subset J^\perp/J,$$

so by the previous case, $S/J \subset J^\perp/J$ is an orthogonal submodule. Hence, by Lemma A.3, the induced pairings

$$\alpha : J \times M/J^\perp \to V_y^1, \qquad \beta : S/J \times \frac{J^\perp/J}{S/J} \to V_y^1$$

are nonsingular. We will show that the pairing

$$\gamma : S \times M/S \to V_y^1$$

is nonsingular, which implies that $S = S^\perp$.

There is an exact sequence

$$0 \to \frac{J^\perp/J}{S/J} \to \frac{M}{S} \to \frac{M}{J^\perp} \to 0.$$

Taking $\mathrm{Hom}_A(-, V_y^1)$ gives a commutative diagram

$$\begin{array}{ccccccccc} 0 & \leftarrow & S/J & \leftarrow & S & \leftarrow & J & \leftarrow & 0 \\ & & \downarrow{\scriptstyle \mathrm{ad}\,\beta}\,\wr & & \downarrow{\scriptstyle \mathrm{ad}\,\gamma} & & \wr\,\downarrow{\scriptstyle \mathrm{ad}\,\alpha} & & \\ 0 & \leftarrow & ((J^\perp/J)/(S/J))^* & \leftarrow & (M/S)^* & \leftarrow & (M/J^\perp)^* & \leftarrow & 0 \end{array}$$

where the rows are exact. Hence, $\mathrm{ad}\,\gamma$ is bijective, which implies that

$$\mathrm{ad}^\dagger \gamma : M/S \to \mathrm{Hom}_A(S, V_y^1)$$

is bijective. In particular, injectivity of $\mathrm{ad}^\dagger \gamma$ implies that $S = S^\perp$. □

Remark A.8. *From the proof of Proposition 3.7, we know that there is an isomorphism*

$$1/y : W(\mathbb{C}[x]) \xrightarrow{\sim} W(\mathcal{CM}_y^1(A)). \tag{A.2}$$

Hence, every element of $W(\mathcal{CM}^1_y(A))$ *can be represented by a* $\mathbb{C}[x]$*-torsion-free module. Lemma A.7(2) and Lemma 3.3 then suggests a way to obtain an inverse map of (A.2); let* $[N, \psi] \in W(\mathcal{CM}^1_y(A))$*, where* N *is* $\mathbb{C}[x]$*-torsion-free, and* $y^k N = 0$ *for some* $k \geqslant 1$*. If* $k = 1$*, then* N *is a* $\mathbb{C}[x]$*-module, and the image of* ψ *lies in the image of the embedding*

$$\mathcal{V}^0_{\mathbb{C}[x]} = \mathbb{C}[x] \overset{1/y}{\hookrightarrow} \frac{\mathbb{C}[x, y, 1/y]}{\mathbb{C}[x, y]} = \mathcal{V}^1_{\mathbb{C}[x,y],y}.$$

Hence, $[N, y \cdot \psi] \in W(\mathbb{C}[x])$*. If* $k \geqslant 2$*, then* $2k - 2 \geqslant k$*, so* $y^{k-1}N \subset N$ *is a totally isotropic submodule, and there is an induced form*

$$\bar{\psi} : \frac{(y^{k-1}N)^\perp}{y^{k-1}N} \times \frac{(y^{k-1}N)^\perp}{y^{k-1}N} \to \mathcal{V}^1_{\mathbb{C}[x,y],y}.$$

By Lemma A.7(2) and Lemma 3.3, $[N, \psi] = [\frac{(y^{k-1}N)^\perp}{y^{k-1}N}, \bar{\psi}]$*. Note that now we have* $y^{k-1} \cdot \frac{(y^{k-1}N)^\perp}{y^{k-1}N} = 0$*. Hence, by repeating this procedure, we will end up with the* $k = 1$ *case above.*

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Biography

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