Harmonic Forms, Price Inequalities, and Benjamini-Schramm Convergence

Luca F. Di Cerbo
University of Florida
ldicerbo@ufl.edu

Mark Stern*
Duke University
stern@math.duke.edu

Abstract

We study Betti numbers of sequences of Riemannian manifolds which Benjamini-Schramm converge to their universal covers. Using the Price inequalities we developed elsewhere, we derive two distinct convergence results. First, under a negative Ricci curvature assumption and *no* assumption on sign of the sectional curvature, we have a convergence result for *weakly* uniform discrete sequences of closed Riemannian manifolds. In the negative sectional curvature case, we are able to remove the weakly uniform discreteness assumption. This is achieved by combining a refined Thick-Thin decomposition together with a Moser iteration argument for harmonic forms on manifolds with boundary.

*Partially supported by Simons Foundation Grant 3553857

Contents

1 Introduction

Let (M^n, g) be a closed Riemannian manifold. Define the normalized Betti numbers and L^2-Betti numbers respectively as:

$$\tilde{b}_{k,g}(M) := \frac{b_k(M)}{Vol(M)} \quad \text{and} \quad \tilde{b}^{(2)}_{k,g}(M) := \frac{b^{(2)}_k(M)}{Vol(M)}, \tag{1}$$

where $b_k(M)$ denotes the kth Betti number of M and $b^{(2)}_k(M)$ denotes the kth L^2-Betti number. In an influential paper [Lüc94], Lück shows that if M is a closed manifold with residually finite fundamental group, then

$$\tilde{b}^{(2)}_{k,g}(M) = \lim_{l\to\infty} \tilde{b}_{k,g}(M_l),$$

for any tower of coverings $\{M_l\}_l$ of M associated to a cofinal filtration of its fundamental group. The L^2-Betti numbers were originally defined analytically by Atiyah in [Ati76], and Lück's theorem provides a remarkable connection between analysis and topology which has inspired considerable mathematics in the last two or three decades, see for example the bibliography of [Lüc02].

More recently, Abert *et al.* in [ABBGNRS17] and [ABBG18] generalized Luck's approximation theorem in the context of lattices in Lie groups and in the context of finite volume manifolds of negative curvature. To describe this generalization, we first recall a rather weak notion of convergence of Riemannian manifolds, Benjamini-Schramm convergence, which is adapted from graph theory [BS01]. In Riemannian terms, this convergence is given as follows.

Definition 2. Let $(M_l, g_l)_l$ be a sequence of closed Riemannian manifolds which share a common universal Riemannian cover (X, g). Given $x \in M_l$, we denote by $\mathrm{inj}_{g_l}(x)$ the injectivity radius of (M_l, g_l) at x. We define the R-thin part of (M_l, g_l), denoted $(M_l)_{<R}$, by

$$(M_l)_{<R} := \{x \in M_l \,|\, \mathrm{inj}_{g_l}(x) < R\}.$$

Define a relative measure of the thin regions of M_l by

$$\rho(M_l, R) := \frac{Vol_{g_l}((M_l)_{<R})}{Vol_{g_l}(M_l)}. \tag{3}$$

We say that the sequence $(M_l, g_l)_l$ Benjamini-Schramm converges to (X, g), if for any $R > 0$ we have

$$\lim_{l\to\infty} \rho(M_l, R) = 0.$$

Finally, we say that the sequence (M_l, g_l) is *uniformly discrete*, if there exists $\epsilon > 0$ such that for any $l \in \mathbb{N}$:

$$\min_{x\in M_l} \mathrm{inj}_{g_l}(x) \geq \epsilon.$$

We can now state one of the main results in [ABBGNRS17].

Theorem 4 (Corollary 1.4 in [ABBGNRS17])**.** *Let $\{\Gamma_l\}_l$ be a sequence of uniformly discrete, torsion free lattices acting co-compactly on a symmetric space G/K of non-compact type. Let $\{\Gamma_l\backslash G/K\}_l$ be the associated sequence of compact locally symmetric spaces. For any $k \leq \dim(G/K)$, if $\{\Gamma_l\backslash G/K\}_l$ Benjamini-Schramm converges to G/K (equipped with the standard symmetric metric), we have*

$$\lim_{l\to\infty} \tilde{b}_{k,g}(\Gamma_l\backslash G/K) = \beta_k^{(2)}(G/K),$$

where the k-th L^2-Betti number of the symmetric space, $\beta_k^{(2)}(G/K)$, is defined in [ABBGNRS17, 6.24], and satisfies $\tilde{b}_{k,g}^{(2)}(\Gamma\backslash G/K) = \beta_k^{(2)}(G/K)$ for every co-compact torsion free Γ.

When $\{\Gamma_l\}_l$ is a cofinal filtration of a given torsion free lattice acting co-compactly on G/K (*cf.* Theorem 2.1 in [DW78]), the sequence of coverings $\{\Gamma_l\backslash G/K\}_l$ Benjamini-Schramm converges to G/K, and Theorem 4 is a genuine generalization of Lück's original approximation theorem in the case of locally symmetric spaces.

In the real hyperbolic case $G/K = \mathbb{H}^n$, Abert *et al.* in [ABBGNRS17] obtain their strongest result. Remarkably, they are able to remove the uniform discreteness assumption on the lattices.

Theorem 5 (Theorem 1.8. in [ABBGNRS17])**.** *Let $\{\Gamma_l\backslash\mathbb{H}^n\}_l$ be a sequence of compact hyperbolic manifolds of dimension n that Benjamini-Schramm converge to $\mathbb{H}^n$. For any $k \leq n$, we have*

$$\lim_{l\to\infty} \tilde{b}_k(\Gamma_l\backslash\mathbb{H}^n) = \beta_k^{(2)}(\mathbb{H}^n).$$

More recently, in the preprint [ABBG18], four of the seven authors of [ABBGNRS17], extended Lück's approximation theorem to sequences of pinched negatively curved manifolds which Benjamini-Schramm converge to their universal cover.

In this paper, we contribute to this circle of ideas by extending the techniques of [DS17] to prove and *quantify* vanishing of normalized Betti numbers (in certain degrees) along sequences of closed Riemannian manifolds which Benjamini-Schramm converge to their universal covers. Here we consider geometries more general than those considered in [ABBG18]. Our techniques are rather distant from those of [ABBGNRS17] and [ABBG18]. Indeed, we rely on geometric inequalities for harmonic forms on negatively curved Riemannian manifolds which we described in [DS17]. In particular, some of our results do not require *any* direct assumption on the sectional curvature.

The next definition is tailored to our analytical techniques, and it will be used throughout this paper. This definition is related to the notions of convergence considered in [ABBGNRS17] and [ABBG18], but at the same time it contains some new elements.

Definition 6. Let $(M_l, g_l)_l$ be a sequence of closed Riemannian manifolds which share a common universal Riemannian cover (X, g). We say a sequence of manifolds $(M_l, g_l)_l$ is *weakly uniformly discrete* and converges to (X, g) if there exists a sequence $\{R_l\}_l \subset (0, \infty)$ with

$$\lim_{l\to\infty} R_l = \infty,$$

such that

$$\lim_{l\to\infty} \Big(1 + \frac{1}{\mathrm{inj}_{g_l}(M_l)}\Big)^n \rho(M_l, R_l) = 0.$$

Before listing our main results, it is important to state the precise connection between the notion of convergence given in Definition 6 and the usual Benjamini-Schramm convergence (*cf.* [ABBGNRS17] and [ABBG18]).

Remark 7. If a sequence of manifolds $(M_l, g_l)_l$ Benjamini-Schramm converges, then there is always a sequence $\{R_l\}_l$ converging to ∞ such that

$$\lim_{l\to\infty} \rho(M_l, R_l) = 0. \tag{8}$$

Hence every uniformly discrete sequence which Benjamini Schramm converges is weakly uniformly discrete. On the other hand, weakly uniformly discrete sequences may well have injectivity radius that goes to zero along a subsequence, and therefore need not be uniformly discrete in the sense of Definition 2.

We can now state our first result which requires only a negative *Ricci* curvature assumption, and no uniform lower bound on the injectivity radius. On the other hand, we require the weakly uniformly discrete assumption (*cf.* Definition 6).

Theorem 9. *Let (X^n, g) be a simply connected manifold without conjugate points and with $-1 \leq \sec_g \leq 1$. Assume there exists $\delta > 0$ such that*

$$-Ric_g \geq \delta g.$$

Let (M_l, g_l) be a weakly uniformly discrete sequence of closed manifolds converging to (X, g). Then for any $k \in \mathbb{N}$ such that $\delta > 4k^2$, we have

$$\lim_{l \to \infty} \tilde{b}_{k,g}(M_l) = 0.$$

We remark that, in [DS17, Theorem 122], under the same curvature assumptions as in Theorem 9, we proved the following vanishing theorem for L^2-Betti numbers:

$$b_k^{(2)}(M_l) = 0, \quad \text{for all} \quad l.$$

Thus, Theorem 9 asserts the convergence along the Benjamini-Schramm sequence of certain normalized Betti numbers to the corresponding L^2-Betti number.

If we assume the sectional curvature to be strictly negative, the techniques developed in [DS17] cover a larger range of Betti numbers. Using this fact, we are able to prove the following (see Theorem 77 for a stronger statement and details of the proof).

Theorem 10. *Let (X^n, g) be a simply connected manifold with*

$$-a^2 \leq \sec_g \leq -1,$$

and $a \geq 1$. Let (M_l, g_l) a sequence of closed Riemannian manifolds BS-converging to (X^n, g). For any $k \in \mathbb{N}$ such that

$$a_{n,k} := (n-1) - 2ka \geq 0,$$

we have

$$\lim_{l \to \infty} \tilde{b}_{k,g}(M_l) = 0.$$

Once again, under the same curvature assumptions as in Theorem 10, we have elsewhere proved the following vanishing theorem for L^2-Betti numbers

$$b_k^{(2)}(X^n) = 0 \quad \Rightarrow \quad b_k^{(2)}(M_l) \quad \text{for all} \quad l;$$

see Section 7 and Proposition 126 in [DS17] (*cf.* also Proposition 4.1 in [DX84] when $a_{n,k} > 0$, and [Dod79] when $a = 1$). Hence Theorem 10 is already a consequence of the preceding references and [ABBG18]. None the less, our convergence result is completely independent of the theory of L^2-Betti numbers, and it follows directly from the Price inequalities for harmonic forms we developed in [DS17]. Indeed all of the analysis can be performed directly on the sequence of compact manifolds, without the need of studying L^2-harmonic forms on the universal Riemannian cover.

Observe that Theorem 10, unlike Theorem 9, does *not* require any uniform discreteness assumption. This greater generality is present in [ABBG18] and Theorem 5 as well, and it depends crucially on the fact that in the negative sectional curvature regime, thanks to the Gromov-Margulis lemma (*cf.* [BGS85]), we understand quite well the topology of regions with small injectivity radius. On the other hand, our proof is substantially different from the approach presented in [ABBG18].

2 Dimension Estimates Revisited

Let (M^n, g) be a closed Riemannian manifold, and denote by $\mathcal{H}_g^k(M)$ the finite dimensional vector space of harmonic k-forms. Define normalized Betti numbers

$$\tilde{b}_{k,g}(M) := \frac{b_k(M)}{Vol_g(M)}. \tag{11}$$

In Section 5 of [DS17] and again in Lemma 14 below, we show

$$\tilde{b}_{k,g}(M) \leq \left(\begin{array}{c} n \\ k \end{array} \right) \max\Big\{ \frac{\|\alpha\|_{L^\infty}^2}{\|\alpha\|_{L^2}^2} : \alpha \in \mathcal{H}_g^k(M) \setminus \{0\} \Big\}. \tag{12}$$

Under various hypotheses on the Ricci curvature or the Riemannian curvature and k, in [DS17] we showed exponential or polynomial bounds in the injectivity radius for the normalized Betti numbers. Those estimates, in conjunction with (12) suffice to establish convergence to zero of sequences of normalized Betti numbers of closed Riemannian manifolds whose injectivity radii diverge, for example, sequences of real hyperbolic manifolds associated to a cofinal filtration of a given torsion free co-compact lattice in $\mathrm{Iso}(\mathbb{H}^n) = PO(n, 1)$, with $k \neq \frac{n}{2}$. On the other hand, for sequences of closed Riemannian manifolds that converge in the Benjamini-Schramm sense, it is not necessarily the case that the injectivity radius goes to infinity (even if the pointed injectivity radius goes to infinity almost everywhere). Thus, we modify the dimension estimate for $\mathcal{H}_g^k(M)$ used in Section 5 of [DS17], in order to obtain vanishing results in this broader context.

Let $K(\cdot, \cdot)$ denote the Schwartz kernel for the L^2 orthogonal projection onto $\mathcal{H}_g^k(M)$. Thus for $x, y \in M$,

$$K(x, y) \in Hom(\Omega^k T_y^* M, \Omega^k T_x^* M).$$

Given an L^2-orthonormal basis $\{\alpha_j\}_{j=1}^l$ for $\mathcal{H}_g^k(M)$, we have

$$K(x, y) = \sum_{i=1}^{l} \alpha_j(x) \langle \cdot, \alpha_j(y) \rangle. \tag{13}$$

Next, we derive a pointwise estimate on the trace of $K(x, x)$.

Lemma 14. *Given $K(\cdot,\cdot)$ as above, we have for any $x \in M$*

$$0 \leq TrK(x,x) \leq \binom{n}{k} \sup_{\alpha \in \mathcal{H}_g^k(M): ||\alpha||_{L^2}^2 = 1} |\alpha(x)|^2.$$

Proof. Fix a point $x \in M$, and let $\{e_i\}_{i=1}^{\binom{n}{k}}$ be a local orthonormal frame for $\Omega^k T^* M$ in a neighborhood of x. Then

$$TrK(x,x) = \sum_{i=1}^{\binom{n}{k}} \langle K(x,x)(e_i), e_i \rangle = \sum_{i=1}^{l} |\alpha_i|_x^2 \geq 0.$$

Next, given a point $p \in M$, there exists a unit eigenvector z of $K(p,p)$ with maximal eigenvalue say λ. Thus, by construction

$$\langle K(p,p)z, z \rangle = \lambda |z|_p^2 = \lambda,$$

with

$$\langle K(p,p)z, z \rangle = \sum_{i=1}^{l} \langle z, \alpha_i(p) \rangle \langle z, \alpha_i(p) \rangle.$$

Thus

$$\begin{aligned}
&\int_M \langle K(x,p)z, K(x,p)z \rangle d\mu_g \\
&= \int_M \Big\langle \sum_{i=1}^{l} \alpha_i(x) \langle z, \alpha_i(p) \rangle, \sum_{j=1}^{l} \alpha_j(x) \langle z, \alpha_j(p) \rangle \Big\rangle d\mu_g \\
&= \sum_{i,j=1}^{l} \langle z, \alpha_i(p) \rangle \langle z, \alpha_j(p) \rangle \int_M \langle \alpha_i(x), \alpha_j(x) \rangle d\mu_g \\
&= \sum_{i,j=1}^{l} \langle z, \alpha_i(p) \rangle \langle z, \alpha_j(p) \rangle \delta_{ij} = \lambda.
\end{aligned}$$

Now, set

$$\alpha(x) := \frac{K(x,p)z}{\sqrt{\lambda}} \in \mathcal{H}_g^k(M),$$

with

$$\|\alpha\|_{L^2} = 1.$$

In sum, we have found an $\alpha \in \mathcal{H}_g^k(M)$ such that

$$||\alpha||_{L^2}^2 = 1, \quad |\alpha(p)|^2 = \lambda.$$

As λ was the largest eigenvalue of $K(p,p)$, we have the estimate

$$TrK(p,p) \leq \binom{n}{k} \lambda \leq \binom{n}{k} \sup_{||\alpha||_{L^2}^2 = 1} |\alpha(p)|^2.$$

Since p is an arbitrary point in M, the proof is complete. □

The following lemma is the usual elliptic regularity for harmonic forms in bounded geometry. One proof is a standard application of Moser iteration. See for example [LS18, Proposition 2.2], where the theorem is proved for hyperbolic manifolds and Proposition 49, where it is proved for manifolds with boundary.

Lemma 15. *Let (M^n, g) be a closed Riemannian manifold with*

$$-a \leq \sec_g \leq 1,$$

and let

$$\operatorname{inj}_g(M) := \min_{p\in M} \operatorname{inj}_g(p) > 0$$

be the global injectivity radius. Given a harmonic k-form $\alpha \in \mathcal{H}^k_g(M)$, for any $p \in M$ and $L < \min(\operatorname{inj}_g(M), 1)$ there exists a strictly positive constant $d(n, a, k, L) := d(n, a, k)(1 + \frac{1}{L})^n$ such that

$$||\alpha||^2_{L^\infty(B_{\frac{L}{2}}(p))} \leq d(n, a, k, L)||\alpha||^2_{L^2(B_L(p))}.$$

Combining Lemma 14 with Lemma 15, we get the key estimate of this section.

Lemma 16. *Given (M^n, g), and $K(\cdot, \cdot)$ as above, there exists a constant $d_0 = d_0(n, a, k, \operatorname{inj}_g(M)) > 0$ such that*

$$0 \leq TrK(x, x) \leq d_0(n, a, k, \operatorname{inj}_g(M)),$$

for any $x \in M$.

Proof. By Lemma 14, we have

$$0 \leq TrK(x, x) \leq \binom{n}{k} \sup_{\alpha\in\mathcal{H}^k_g(M):||\alpha||^2_{L^2}=1} |\alpha(x)|^2.$$

Now apply Lemma 15 to obtain the desired estimate. □

3 Negative Ricci Curvature

In this section, we study manifolds with negative Ricci curvature.

Definition 17. Let (M, g) be a complete Riemannian manifold. Given any $R > 0$, we define the R-thin part of (M, g) as

$$M_{<R} := \{x \in M \mid \operatorname{inj}_g(x) < R\},$$

where $\operatorname{inj}_g(x)$ is the injectivity radius of (M, g) at the point x. We define the R-thick part, denoted by $M_{\geq R}$, as the complement of the R-thin part.

The proof of [DS17, Theorem 66], implies the following theorem.

Theorem 18. *Let* (M^n, g) *be a compact manifold with* $-1 \leq \sec_g \leq 1$*. Given* $k \in \mathbb{N}$*, assume there exists* $\delta > 4k^2$ *such that*

$$-Ric \geq \delta g.$$

Let ρ *be large enough so that*

$$\frac{\sqrt{\delta}}{2}\coth(\sqrt{\delta}\rho) - k\coth(\rho) \geq \epsilon > 0.$$

There exists $c(n, k, \delta, \epsilon) > 0$ *so that for any* $\alpha \in \mathcal{H}^k_g(M)$ *and* $p \in M$ *with* $\mathrm{inj}_g(p) > \rho + 2$*, we have*

$$\int_{B_\rho(p)} |\alpha|^2 dv \leq c(n, k, \delta, \epsilon) e^{-(\sqrt{\delta}-2k)(\mathrm{inj}_g(p)-\rho-2)} \|\alpha\|^2_{L^2(M,g)}. \tag{19}$$

A corollary of this estimate is the following result for sequences of Riemannian manifolds which Benjamini-Schramm converge to their universal cover.

Theorem 20. *Let* (X^n, g) *be a simply connected manifold without conjugate points with* $-1 \leq \sec_g \leq 1$*. Assume there exists* $\delta > 0$ *such that*

$$-Ric_g \geq \delta g.$$

Let (M_l, g_l) *be a weakly uniformly discrete sequence of closed manifolds converging to* (X, g)*. Then for any* $k \in \mathbb{N}$ *such that* $\delta > 4k^2$*, we have*

$$\lim_{l\to\infty} \frac{b_k(M_l)}{Vol_{g_l}(M_l)} = 0.$$

Proof. Observe that for any $(k, R) \in \mathbb{N} \times (0, \infty)$ such that $\delta > 4k^2$ and $R > max\{\rho + 2, R_0\}$, with ρ as defined in Theorem 18, we have the estimate:

$$\begin{aligned} \frac{b_k(M_l)}{Vol_{g_l}(M_l)} &= \frac{\int_{(M_l)_{<R}} TrK(x,x) d\mu_{g_l}}{Vol_{g_l}(M_l)} + \frac{\int_{(M_l)_{\geq R}} TrK(x,x) d\mu_{g_l}}{Vol_{g_l}(M_l)} \\ &\leq \binom{n}{k} d(n,a,k)\Big(1 + \frac{1}{\mathrm{inj}_{g_l}(M_l)}\Big)^n \rho(M_l, R) + c(n,k,\delta) e^{-(\sqrt{\delta}-2k)(R-\rho-2)}. \end{aligned} \tag{21}$$

Choose $R = R_l$ for some sequence $\{R_j\}_j$ given by the definition of weakly uniformly discrete (Definition 6), and the result follows. □

Remark 22. Theorem 122 in [DS17] implies that, under the curvature assumptions of Theorem 20, we have

$$b^{(2)}_k(M_l) := \dim_{\Gamma_l}(\mathcal{H}^k_2(X)) = 0,$$

for any $k \in \mathbb{N}$ such that $\delta > 4k^2$. Thus, Theorem 20 can alternatively be rephrased by saying that the normalized k-Betti number converge along the sequence to the corresponding k-th L^2-Betti number.

4 Pinched Negative Sectional Curvature

In this section, we study sequences of Riemannian manifolds with negative and pinched sectional curvature which Benjamini-Schramm converge. The starting point is as usual the Price inequality for harmonic forms on manifolds with negative sectional curvature established in [DS17].

4.1 Uniformly Discrete Sequences

We start with sequences of uniformly discrete, negatively curved and pinched manifolds which BS-converge. The key technical point is a Price inequality for harmonic k-forms. For convenience, we assemble in a single statement three results stated distinctly in [DS17].

Theorem 23 (Theorems 87 & 96 and Corollary 108 in [DS17])**.** *Let (M^n, g) be a compact manifold of dimension $n \geq 3$. Assume the sectional curvature is pinched:*

$$-a^2 \leq \sec_g \leq -1$$

with $a \geq 1$. Let k be a non-negative integer such that

$$a_{n,k} := (n-1) - 2ka > 0.$$

For any $\alpha \in \mathcal{H}_g^k(M)$ and for any geodesic ball $B_R(p) \subset M$ with $R > 1 + \frac{\ln(2)}{a_{n,k}}$, there exists a constant $c(n,k) > 0$ so that

$$\int_{B_1(p)} |\alpha|^2 dv \leq c(n,k) e^{-a_{n,k} R} \|\alpha\|^2_{L^2(M,g)}.$$

Finally, if k is a non-negative integer such that

$$a_{n,k} := (n-1) - 2ka = 0,$$

then for any geodesic ball $B_R(p) \subset M$ with $R > 1$, there exists a constant $d(n,k) > 0$ so that

$$\int_{B_1(p)} |\alpha|^2 dv \leq d(n,k)(R-1)^{-1} \|\alpha\|^2_{L^2(M,g)}.$$

As in Section 3, a Price inequality has an immediate consequence for weakly uniformly discrete sequences of Riemannian manifolds which Benjamini-Schramm converge.

Corollary 24. *Let (X^n, g) be a simply connected manifold of dimension $n \geq 3$ with*

$$-a^2 \leq \sec_g \leq -1,$$

and $a \geq 1$. Let (M_l, g_l) be a weakly uniformly discrete sequence of closed manifolds converging to (X, g). For any $k \in \mathbb{N}$ such that

$$a_{n,k} = (n-1) - 2ka \geq 0,$$

we have

$$\lim_{l\to\infty}\frac{b_k(M_l)}{Vol_{g_l}(M_l)}=0.$$

Remark 25. Proposition 126 in [DS17] implies that, under the curvature assumptions of Corollary 24, we have

$$b_k^{(2)}(M_l):=\dim_{\Gamma_l}(\mathcal{H}_2^k(X))=0,$$

for any $k\in\mathbb{N}$ such that $a_{n,k}\geq 0$ (*cf.* also Proposition 4.1 in [DX84] when $a_{n,k}>0$, and [Dod79] when $a=1$). Thus, Corollary 24 can alternatively be rephrased by saying that the normalized k-Betti number converge along the sequence to the corresponding k-th L^2-Betti number.

4.2 Thick-Thin Decomposition Revisited

In this section, we construct an effective Thick–Thin decomposition for closed manifolds with negative sectional curvature. This Thick-Thin decomposition builds upon a construction of Buser, Colbois, and Dodziuk (*cf.* [BCD93]) which we refine for our purposes. These additions are needed for our Moser iteration argument on manifolds with boundary (*cf.* Section 4.3).

Let (M,g) be a compact of dimension $n\geq 3$, with sectional curvatures satisfying

$$-a^2\leq \sec_g\leq -1,$$

for some constant $a\geq 1$. A first geometric consequence of the so-called Gromov-Margulis' Lemma (*cf.* Section 8 in the book [BGS85]) is that there is a positive constant

$$\mu=a^{-1}c_n,$$

$c_n>0$ depending on the dimension only, so that if the set

$$M_\mu:=\{x\in M\mid \mathrm{inj}(x)<\mu\}$$

is not empty, it is then the union of a finite number of disjoint tubes $\{T_{\gamma_i}\}$ around short closed geodesics $\{\gamma_i\}$. For convenience, we will require $\mu<1$. For every tube T_γ, the core geodesic γ has length $l(\gamma)<2\mu$. For every point $p\in\gamma$, and every tangent vector $v\in T_pM$ perpendicular to $\gamma'(0)$, let $\delta_{p,v}(t)$ denote the unit speed geodesic ray emanating from p in the direction of v. We call these rays *radial arcs* and their tangent vector fields the radial vector field $\mathcal{R}$. A Rauch comparison argument gives the following lemma.

Lemma 26. *Let γ be a geodesic in M satisfying (29). Then*

$$|\nabla\mathcal{R}|(x)\leq a\coth(ad(x,\gamma)).\tag{27}$$

Proof. See for example Chapter 10 in [doC92]. ☐

In every interval $[0, t_0]$ such that $i(\delta_{p,v}(t)) \leq \mu$ for $t \in [0, t_0]$, the function $t \to i(\delta_{p,v}(t))$ is strictly monotonic increasing. Thus, there exists $R_{p,v} > 0$ depending on the initial condition of the geodesic ray such that $i(\delta(R_{p,v})) = \mu$ and $i(\delta_{p,v}(t)) < \mu$ for any $t \in [0, R_{p,v})$. The arc $\delta([0, R_{p,v}])$ is called the maximal arc. Also, different radial arcs are disjoint except possibly for their initial points. Thus, their union T_γ is homeomorphic to $\gamma \times B^{n-1}$ where B^{n-1} is a closed ball inside $\mathbb{R}^{n-1}$, and this explains why they are called tubes. On the other hand, different maximal radial arcs in T_γ may have very different lengths and the boundary of T_γ is not smooth in general. Thus, these object are not great if you want to do calculus on them. We therefore employ a controlled Thick-Thin decomposition due to Buser, Colbois and Dodziuk [BCD93] which we now briefly describe. First, we state the following lemma, observed in [BCD93].

Lemma 28. *There exist constants c_1, c_2 depending only on the dimension n, such that if*

$$l(\gamma) \leq c_1 \exp(-c_2 a)\mu^n a^{n-1}, \tag{29}$$

then $d(x, \gamma) \geq 10$ for every $x \in T_\gamma$ with $\mathrm{inj}(x) = \frac{\mu}{2}$.

Proof. This is a consequence of Lemma 2.4 in [BCD93]. □

From now on, we will regard a geodesic small if and only if its length satisfies the bound (29) of Lemma 28. This means we disregard possibly many small geodesics in the usual Thick-Thin decomposition of M. Thus, we only look at small geodesics which posses fat Margulis tubes around them. This fact plays a role in the constructions that follow.

Next, given a geodesic γ satisfying (29) and $\lambda \in (0, 1)$, we define the following tube around it:

$$U_\gamma^\lambda := \{x \in T_\gamma \mid \mathrm{inj}(x) \leq \lambda\mu\}. \tag{30}$$

Again, there is no a priori reason to believe that the boundary $\partial U_\gamma^\lambda$ is well behaved from a geometric point of view. Thus, we appeal to the following theorem of Buser, Colbois and Dodziuk which ensures the existence of a small deformation of $U_\gamma^{\frac{1}{2}}$ with many nice geometric properties.

Theorem 31 (Theorem 2.14 in [BCD93])**.** *Let γ be a geodesic in M satisfying (29). There exists a smooth hypersurface H_γ contained in $T_\gamma \setminus \gamma$ with the following properties:*

- *The angle θ between the radial vector field $\mathcal{R}$ and the exterior normal of H_γ is less that $\pi/2 - \alpha$ for a constant $\alpha = \alpha(a, n) \in (0, \pi/2)$.*
- *The sectional curvatures of H_γ with respect to the induced metric are bounded in absolute value by a constant depending only on a and n.*

- *H_γ is homeomorphic to $\partial U_\gamma^{\frac{1}{2}}$ by pushing along radial arcs. The distance between $x \in H_\gamma$ and its image $\bar{x} \in \partial U_\gamma^{\frac{1}{2}}$ satisfies $d(x, \bar{x}) \le \mu/50$.*

Next, we explicitly observe the following consequence of Theorem 31. This corollary plays a role in the elliptic estimates presented in Section 4.3.

Corollary 32. *H_γ is locally the graph of a Lipschitz function with Lipschitz constant Λ_H dependent only on a and n.*

Proof. This follows readily from the estimates in [BCD93, p.12] required to prove Theorem 31, in particular from the multiplicative bounds on the gradient of the defining function of H_γ. □

We can now define the tubes in our refined Thick-Thin decomposition. Given a geodesic γ satisfying (29), we consider the tube V_γ around it defined by:

$$H_\gamma = \partial V_\gamma, \text{ and } \gamma \in V_\gamma. \tag{33}$$

In particular, these new tubes always have *smooth* boundaries. We now derive a few lemmas concerning the tubes V_γ that are not directly found in [BCD93]; so, we provide all details of the proof. Let

Lemma 34. *Let γ be a geodesic in M satisfying (29). For any point $x \in H_\gamma$, we have*

$$\frac{26}{50}\mu \ge \operatorname{inj}_g(x) \ge \frac{24}{50}\mu.$$

In particular, $U_\gamma^{\frac{24}{50}} \subset V_\gamma \subset U_\gamma^{\frac{26}{50}}$.

Proof. Given $x \in H_\gamma$, denote by $\bar{x}$ the point of intersection with ∂U_γ of the radial arc from γ to x. We have the standard estimate:

$$d(x, \bar{x}) \ge |\operatorname{inj}_g(x) - \operatorname{inj}_g(\bar{x})|. \tag{35}$$

By definition of U_γ, we have $\operatorname{inj}_g(\bar{x}) = \frac{\mu}{2}$. Thus, by Theorem 31 we have:

$$\frac{\mu}{50} + \frac{\mu}{2} = \frac{26}{50}\mu \ge \operatorname{inj}_g(x) \ge \frac{\mu}{2} - \frac{\mu}{50} = \frac{24}{50}\mu.$$

□

Next, we show that these tubes are uniformly separated.

Lemma 36. *If $\gamma \ne \zeta$ are two distinct closed geodesics in M satisfying (29), then*

$$d(V_\gamma, V_\zeta) > \frac{48}{50}\mu.$$

Proof. Let β be a unit speed geodesic realizing the distance between the compact sets V_γ and V_ζ. There exist $t_1, t_2 \in (0, d(V_\gamma, V_\zeta))$ such that $\beta(t_1) \in \partial T_\gamma$ and $\beta(t_2) \in \partial T_\zeta$. By Lemma 34 we have

$$t_1 \ge \mu - \operatorname{inj}_g(\beta(t_1)) \ge \mu - \frac{26}{50}\mu, \quad (d(V_\gamma, V_\zeta) - t_2) \ge \mu - \frac{26}{50}\mu,$$

so that

$$d(V_\gamma, V_\zeta) \geq \frac{24}{50}\mu + \frac{24}{50}\mu + d(T_\gamma, T_\zeta) > \frac{48}{50}\mu.$$

Note that by the usual Thick-Thin decomposition, the Margulis tubes T_γ and T_ζ are disjoint. □

We also need to know that each V_γ contains a "quantum" of volume. This is essential in studying sequences that BS-converge (*cf.* Lemma 70). In [BCD93], the authors show this is the case for the tubes T_γ. We show that their argument can be extended to the tubes $V_\gamma \subset T_\gamma$.

Lemma 37. *If γ is a closed geodesic satisfying (29), then*

$$Vol(U_\gamma^\lambda) > c_n\Big(\frac{\lambda}{2}\mu\Big)^n, \tag{38}$$

and therefore

$$Vol(V_\gamma) > c_n\Big(\frac{6}{25}\mu\Big)^n.$$

Proof. Let $x \in \partial U_\gamma^{\frac{\lambda}{2}}$, and let y be such that $inj_g(y) = \lambda\mu/2$. We claim that $B_{\frac{1}{2}\lambda\mu}(y)$, the open geodesic ball of radius $\frac{1}{2}\lambda\mu$ centered at y, is entirely contained in U_γ^λ. In fact, for any $z \in B_{\frac{1}{2}\lambda\mu}(y)$ we have

$$\frac{1}{2}\lambda\mu > d(z,y) \geq \Big|\mathrm{inj}_g(z) - \frac{1}{2}\lambda\mu\Big|,$$

which forces $\mathrm{inj}_g(z) < \lambda\mu$. Thus, we conclude that $B_{\frac{1}{2}\lambda\mu}(y) \subset U_\gamma^\lambda$. By volume comparison with Euclidean space we have:

$$Vol(U_\gamma^\lambda) > Vol\Big(B_{\frac{1}{2}\lambda\mu}(y)\Big) \geq c_n\Big(\frac{1}{2}\lambda\mu\Big)^n.$$

Since $V_\gamma \supset U_\gamma^{\frac{24}{50}}$, we have $Vol(V_\gamma) \geq c_n\Big(\frac{6}{25}\lambda\mu\Big)^n$. □

We continue deriving effective estimates for the sizes of tubes in the Buser-Colbois-Dodziuk thick-thin decomposition.

Let $\delta_{p,v}(t)$ be a unit speed radial arc, and let $t = R_\lambda$ be the first time the radial arc intersects the boundary of U_γ^λ. We have the estimate

$$c_n\Big(\frac{\lambda\mu}{2}\Big)^n \leq l(\gamma)a^{-(n-1)}\sinh^{n-1}(aR_\lambda). \tag{39}$$

In order to prove (39), we argue as follows. Let $y \in \delta_{p,v}([0,R_\lambda])$ be a point such that $\mathrm{inj}_g(y) = \frac{1}{2}\lambda\mu$. Then the inclusion $B_{\frac{\lambda\mu}{2}}(y) \subset U_\gamma^\lambda$ follows from the proof of Lemma 37. Next, we claim that

$$B_{\frac{\lambda\mu}{2}}(y) \subset \{z \in U_\gamma^\lambda \mid d(z,\gamma) \leq R_\lambda\}.$$

This follows from the triangle inequality since for any $z \in B_{\frac{\lambda\mu}{2}}(y)$

$$d(z,\gamma) \leq d(\gamma,y) + d(y,z) \leq R_\lambda - \frac{\lambda\mu}{2} + \frac{\lambda\mu}{2} = R_\lambda.$$

Thus, by volume comparison with a space of constant sectional curvature $-a^2$, we obtain the claimed inequality (39). We therefore conclude there exists $k = k(a,n,\lambda) > 0$ such that

$$R_\lambda \geq \frac{\ln(\frac{1}{l(\gamma)})}{(n-1)a} + k(a,n,\lambda). \tag{40}$$

Thus, if we denote by $R_{min,\lambda}(\gamma)$ the length of the shortest radial arc reaching $\partial U_\gamma^\lambda$ we have that

$$\lim_{l(\gamma)\to 0} R_{min,\lambda}(\gamma) \to \infty.$$

We can now show that volume of the tubes V_{γ_k} goes to infinity as $l(\gamma_k) \to 0$. More precisely, we have the following effective estimate.

Lemma 41. *Let (M_k, g_k) be a sequence of closed manifolds of dimension $n \geq 3$ with sectional curvature*

$$-a^2 \leq \sec_g \leq -1$$

and $a \geq 1$. Let $\{\gamma_k\}$ be a sequence of closed geodesics in (M_k, g_k) such that

$$\lim_{k\to\infty} l(\gamma_k) = 0.$$

Then

$$\lim_{k\to\infty} Vol(V_{\gamma_k}) = \infty.$$

In fact, there exists a constant $D(n,a,\mu) > 0$ such that

$$\frac{\ln(\frac{1}{l(\gamma_k)})}{Vol(V_{\gamma_k})} \leq D,$$

for all small geodesics.

Proof. Because of Lemma 34, it suffices to show

$$\lim_{k\to\infty} Vol(U_{\gamma_k}^{\frac{24}{50}}) = \infty. \tag{42}$$

Consider a point $p \in \gamma_k$ and two unit speed radial arcs

$$\delta_1(t) = \delta_{p,v_1}(t), \quad \delta_2(t) = \delta_{p,v_2}(t),$$

such that the angle between the initial velocities v_1, v_2 is greater than or equal to $\frac{2\pi}{N}$ for a fixed integer $N > 2\pi$. By a Rauch comparison argument (see [doC92, Lemma 3.1, p. 259]), the distance between these rays diverges at least as fast as in Euclidean space.

For $i = 1, 2$, define $t = R_i$ to be the first time that $\delta_i(t) \in \partial U^{\frac{12}{50}}_{\gamma_k}$. Without loss of generality, we can assume $R_1 = \min\{R_1, R_2\}$. We therefore have

$$d(\delta_1(R_1), \delta_2(R_1)) \geq \frac{\pi}{N} R_1 \geq \frac{\pi}{N} R_{min, \frac{12}{50}}.$$

We now set

$$p_1 = \delta_1(R_1), \quad p_2 = \delta_2(R_2), \quad p' = \delta_2(R_1).$$

By the triangle inequality, we have

$$d(p_2, p_1) + d(p_1, p) = d(p_2, p_1) + R_1 \geq d(p_2, p) = R_2,$$

and

$$d(p_2, p_1) + d(p_2, p') = d(p_2, p_1) + R_2 - R_1 \geq d(p_1, p').$$

Therefore

$$d(p_1, p_2) \geq \frac{1}{2} d(p_1, p') \geq \frac{\pi}{2N} R_1 \geq \frac{\pi}{2N} R_{min, \frac{12}{50}}.$$

Thus, for $R_{min} > \frac{2N}{\pi} \frac{24\mu}{50}$,

$$B_{p_1}\Big(\frac{12\mu}{50}\Big) \cap B_{p_2}\Big(\frac{12\mu}{50}\Big) = \emptyset.$$

Hence we choose N to be the largest integer such that

$$N < \frac{50\pi}{48\mu}\Big(k(a, n, 12/50) + \ln\Big(\frac{1}{l(\gamma_k)}\Big)\frac{1}{(n-1)a}\Big).$$

For all $i = 1, ..., N$, define R_i as above and set

$$\delta_{p,v_i}(R_i) = \delta_i(R_i) = p_i \in \partial U^{\frac{12}{50}}_{\gamma_k}.$$

Then

$$B_{\frac{12\mu}{50}}(p_i) \cap B_{\frac{12\mu}{50}}(p_j) = \emptyset,$$

for $i \neq j$. Moreover, for all $i = 1, ..., N$ it is easy to see that

$$B_{\frac{12\mu}{50}}(p_i) \subset U^{\frac{24}{50}}_{\gamma_k}. \tag{43}$$

Indeed, for any $q \in B_{\frac{12\mu}{50}}(p_i)$ we have

$$\frac{12\mu}{50} > d(p_i, q) \geq |\mathrm{inj}_g(p_i) - \mathrm{inj}_g(q)| = \Big|\frac{12\mu}{50} - \mathrm{inj}_g(q)\Big| \quad \Rightarrow \quad \mathrm{inj}_g(q) < \frac{24\mu}{50},$$

and $q \in U^{\frac{24}{50}}_{\gamma_k}$. Consequently,

$$Vol(U^{\frac{24}{50}}_{\gamma_k}) \geq \sum_{j=1}^{N} Vol(B_{\frac{12\mu}{50}}(p_j) \geq (N-1)c_n\Big(\frac{12\mu}{50}\Big)^n \geq C(n, a, \mu)\ln\Big(\frac{1}{l(\gamma_k)}\Big),$$

where the first inequality follows from volume comparison with Euclidean space, and were $C(n, a, \mu)$ is a positive constant. The result follows once we set $D(n, a, \mu) = C(n, a, \mu)^{-1}$. □

4.3 Pointwise Bounds for Harmonic Forms on Manifolds with Boundary

Let $M_T := M \setminus \cup_\gamma V_\gamma^\circ$, where the V_γ are the modified tubes defined in (33). We will need elliptic estimates for harmonic forms in M_T satisfying Neumann or Dirichlet boundary conditions on ∂M_T. In order to keep track of both the dependence of the estimates on the geometry of the Margulis tube and on the local injectivity radius, in this section we provide a proof of these estimates.

Proposition 44. *Let (M^n, g) be compact with $-a^2 \leq sec_g \leq -1$. There is a constant $S(a,n)$ depending only on a and n so that for all geodesic balls $B_R(p) \subset M$, with $R \leq 1$, and all $f \in C_c^\infty(B_R(p) \cap M_T)$, one has*

$$S(a,n)\|f\|^2_{L^2_1(B_R(p)\cap M_T)} \geq \|f\|^2_{L^{\frac{2n}{n-2}}(B_R(p)\cap M_T)}. \tag{45}$$

Proof. For $R < \min\{\mathrm{inj}_g(p), 1\}$, there are constants $c_{a,j}$ and $C_{a,j}$, $j = 0, 1$ depending only on a and n so that for any domain $A \subset B_R(p)$,

$$c_{a,0}\|f\|_{L^{\frac{2n}{n-2}}(A,\mathrm{euclidean})} \leq \|f\|_{L^{\frac{2n}{n-2}}(A,g)} \leq C_{a,0}\|f\|_{L^2(A,\mathrm{euclidean})}, \tag{46}$$

and

$$c_{a,1}\|f\|_{L^2_1(A,\mathrm{euclidean})} \leq \|f\|_{L^2_1(A,g)} \leq C_{a,1}\|f\|_{L^2_1(A,\mathrm{euclidean})}. \tag{47}$$

Here $\|f\|_{L^2_1(B_R(p),g)}$ and $\|f\|_{L^2_1(B_R(p),\mathrm{euclidean})}$ denote the usual Sobolev norm

$$\|f\|^2_{L^2(B_R(p))} + \|df\|^2_{L^2(B_R(p))}$$

computed with respect to g and with respect to the Euclidean metric induced by the exponential map. By [Ste70, Theorem 5] and its proof (see also [Jon81]), there exists a bounded extension map

$$E_{T,R,p} : W^{1,2}_0(B_R(p) \cap M_T, \mathrm{euclidean}) \to L^2_1(\mathbb{R}^n, \mathrm{euclidean}),$$

satisfying $E_{T,R,p}f - f = 0$ on $B_R(p) \cap M_T$ and

$$\|E_{T,R,p}f\|_{L^2_1(\mathbb{R}^n,\mathrm{euclidean})} \leq B(\Lambda_H, n)\|f\|_{L^2_1(B_R(p),\mathrm{euclidean})},$$

with bound $B(\Lambda_H, n)$ depending only on dimension and the Lipschitz constant Λ_H for a defining function for ∂M_T (*cf.* Corollary 32). Hence we have

$$\begin{aligned}
\|f\|^2_{L^{\frac{2n}{n-2}}(B_R(p)\cap M_T,g)} &\leq C^2_{a,0}\|f\|^2_{L^{\frac{2n}{n-2}}(B_R(p)\cap M_T,\mathrm{euclidean})} \\
&\leq C^2_{a,0}\frac{(n-1)^2}{(n-2)^2}\|d(E_{T,R,p}f)\|^2_{L^2(\mathbb{R}^n,\mathrm{euclidean})} \\
&\leq C^2_{a,0}\frac{(n-1)^2}{(n-2)^2}B(\Lambda_H,n)^2\|f\|^2_{L^2_1(B_R(p)\cap M_T,\mathrm{euclidean})} \\
&\leq \frac{C^2_{a,0}}{c^2_{a,1}}\frac{(n-1)^2}{(n-2)^2}B(\Lambda_H,n)^2\|f\|^2_{L^2_1(B_R(p)\cap M_T,g)}
\end{aligned} \tag{48}$$

Set

$$S(a,n) := \frac{C_{a,0}^2}{c_{a,1}^2}\frac{(n-1)^2}{(n-2)^2}B(\Lambda_H,n)^2$$

to obtain the desired result. □

Given Proposition 44, we can now prove the main estimate of this section.

Proposition 49. *Let h be a strongly harmonic form on M_T satisfying Dirichlet or Neumann boundary conditions on ∂M_T. Then there exist $c_n \in (0,\infty)$ independent of M and $C_G > 0$ depending on the second fundamental form of H_γ, the Margulis constant of M, and the Riemann curvature tensor of M such that*

$$\|h\|^2_{L^\infty(B_L(p))} \le c_n S(a,n)^{\frac{n}{2}}\left(\frac{1}{L^2}+C_G\right)^{\frac{n}{2}}\|h\|^2_{L^2(B_{2L}(p))}. \tag{50}$$

In particular, choosing $L=\frac{\mu}{4}$, we have

$$|h|^2(p) \le r(a,n)\|h\|^2_{L^2(B_{\frac{\mu}{2}}(p))}. \tag{51}$$

where $r(a,n) := c_n S(a,n)^{\frac{n}{2}}\left(\frac{4}{\mu^2}+C_G\right)^{\frac{n}{2}}$.

Proof. By Lemma 36, the connected components of ∂M_T are uniformly apart. Without loss of generality, we assume h satisfies Dirichlet boundary conditions: the pullback to the boundary of h and d^*h vanishes. Let ψ be a smooth function compactly supported in M_T (but not necessarily compactly supported in M_T^0). Then we have

$$\begin{aligned} 0 &= \int_{M_T}\langle \Delta h, \psi^2 h\rangle dv \\ &= \int_{M_T}(\langle \nabla h, \nabla(\psi^2 h)\rangle + \langle R^{riem}h,\psi^2 h\rangle)dv - \int_{\partial M_T}\langle \nabla_\nu h, \psi^2 h\rangle d\sigma, \end{aligned} \tag{52}$$

where ν is an outward pointing unit normal and R^{riem} denotes the curvature operator given in a local orthonormal frame $\{e_i\}_i$ and dual coframe $\{\omega^i\}_i$ by $R^{riem} = -e(\omega^i)e^*(\omega^j)R^{riem}(e_i,e_j)$, with $e(\omega^p)$ denoting exterior multiplication on the left by ω^p, $e^*(\omega^p)$ its adjoint, and $R^{riem}(\cdot,\cdot)$ the Riemannian curvature 2 form. Since h is strongly harmonic, we have

$$dh = \sum_{j\ge 1} e(\omega^i)\nabla_{e_i}h = 0, \quad d^*h = -\sum_{j\ge 1}e^*(\omega^i)\nabla_{e_i}h = 0,$$

so that if $e_1=\nu$ on ∂M_T, we obtain the identities

$$\nabla_\nu h = -\sum_{j>1}\langle (e^*(\omega^1)e(\omega^j) - e^*(\omega^j)e(\omega^1))\nabla_j h. \tag{53}$$

Since h satisfies Dirichlet boundary conditions, we have $e(\omega^1)h=0$ on ∂M_T. Thus, we have

$$-\langle \nabla_\nu h, \psi^2 h\rangle = -\sum_{j>1}\langle (e^*(\omega^j)[e(\omega^1),\nabla_j]h, \psi^2 h\rangle = \langle IIh, \psi^2 h\rangle, \tag{54}$$

where II denotes the second fundamental form operator

$$II = II_{jk} e^*(\omega^j) e(\omega^k).$$

From (52) and (54), we obtain a Bochner formula for harmonic forms satisfying Dirichlet boundary conditions:

$$0 = \int_{M_T} (\langle \nabla h, \nabla(\psi^2 h)\rangle + \langle R^{riem} h, \psi^2 h\rangle) dv + \int_{\partial M_T} \langle IIh, \psi^2 h\rangle d\sigma. \tag{55}$$

This equality now allows us to proceed with the usual proof of Moser iteration with one additional modification required for controlling the boundary term. We repeat the standard argument and add in the contribution from the boundary. We will follow the treatment in [CLS16].

Given $p \in M_T$, let $\eta : \mathbb{R} \to [0,1]$ be a smooth function identically 1 on $(-\infty, L]$ and supported on $(-\infty, 2L]$, with $|d\eta| \leq \frac{2}{L}$. Set $\eta_k(x) = \eta(2^k(d(x,p) - L))$. Observe that the function $\eta_k(t)$ is equal to one on $(-\infty, L(1+2^{-k})]$, and it is supported on $(-\infty, L(1+2^{1-k})]$. Let χ_k denote the characteristic function of $B_k := B(p, L(1+2^{-k}))$. Then

$$\chi_k \leq \eta_k \leq \chi_{k-1}, \tag{56}$$

and

$$|d\eta_k| \leq \frac{2^{k+1}}{L} \chi_{k-1}. \tag{57}$$

Now we choose $\psi = \eta_k |h|^{p_k - 1}$ in (55) with p_k to be chosen later to get

$$\begin{aligned} \int_{M_T} |\nabla(\eta_k |h|^{p_k-1} h)|^2 dv = \int_{M_T} (|d(\eta_k |h|^{p_k-1})|h||^2 - \langle R^{riem} h, \eta_k^2 |h|^{2p_k-2} h\rangle) dv \\ - \int_{\partial M_T} \langle IIh, \eta_k^2 |h|^{2p_k-2} h\rangle d\sigma. \end{aligned} \tag{58}$$

Expanding and rearranging terms yields

$$\begin{aligned} \int_{M_T} |\nabla(\eta_k |h|^{p_k-1} h)|^2 dv = \int_{M_T} \Big| \Big(\frac{p_k - 1}{p_k}\Big) d(\eta_k |h|^{p_k}) + \frac{1}{p_k} d(\eta_k) |h|^{p_k} \Big|^2 dv \\ - \int_{M_T} \langle R^{riem} h, \eta_k^2 |h|^{2p_k-2} h\rangle dv \\ - \int_{\partial M_T} \langle IIh, \eta_k^2 |h|^{2p_k-2} h\rangle d\sigma. \end{aligned} \tag{59}$$

Observe that after some manipulation

$$\begin{aligned} &\int_{M_T} \Big| \Big(\frac{p_k - 1}{p_k}\Big) d(\eta_k |h|^{p_k}) + \frac{1}{p_k} d(\eta_k) |h|^{p_k} \Big|^2 dv \\ &\quad \leq \Big(1 - \frac{1}{p_k}\Big) \int_{M_T} |d(\eta_k |h|^{p_k})|^2 dv + \frac{1}{p_k} \int_{M_T} |d(\eta_k)|h|^{p_k}|^2 dv. \end{aligned}$$

Now Kato's inequality implies the pointwise bound

$$|\nabla(\eta_k |h|^{p_k-1}h)| \geq |d(\eta_k |h|^{p_k})|,$$

so that

$$\begin{aligned}\frac{1}{p_k}\int_{M_T} |d(\eta_k|h|^{p_k})|^2 dv \leq \frac{1}{p_k}\int_{M_T} |d(\eta_k)|h|^{p_k}|^2 dv + C_R \int_{M_T} \eta_k^2 |h|^{2p_k} dv \\ + C_{II}\int_{\partial M_T} \eta_k^2 |h|^{2p_k} d\sigma, \end{aligned} \tag{60}$$

where $C_{II} := \|II\|_{L^\infty(\partial M_T)}$, and $C_R := \|R^{riem}\|_{L^\infty(M_T)}$. Let ϕ be a C^1 cutoff function compactly supported in $[0, \frac{24\mu}{50})$ satisfying

$$\phi(t) = 1, \text{ for } t \leq \frac{12\mu}{50}, \text{ and } |d\phi(t)| \leq \frac{100}{12\mu}.$$

Then $\phi(d(x, H))$ is Lipschitz with Lipschitz constant $\frac{25}{3\mu}$, and by construction, it is identically equal to one on ∂M_T. Now, let $\mathcal{R}$ be the radial vector field as defined in Theorem 31. Also, let θ be the angle between $\mathcal{R}$ and the unit normal ν on ∂M_T. We then have

$$\langle \nu, \mathcal{R}\rangle = \cos\theta \geq \cos(\pi/2 - \alpha) = \sin\alpha,$$

where $\alpha \in (0, \pi/2)$ is as in the first statement of Theorem 31. Thus, by applying the divergence theorem to $\mathcal{R}$

$$\begin{aligned}
& C_{II}\int_{\partial M_T} \eta_k^2 |h|^{2p_k} d\sigma \\
&\leq \frac{C_{II}}{\sin(\alpha)}\int_{\partial M_T} \eta_k^2 |h|^{2p_k}\langle \nu, \mathcal{R}\rangle d\sigma \\
&= \frac{C_{II}}{\sin(\alpha)}\int_{M_T} d(\eta_k^2 |h|^{2p_k}\phi(d(x,H)) i_{\mathcal{R}} dv) \\
&\leq \frac{C_{II}}{\sin(\alpha)}\|d(\eta_k|h|^{p_k})\|_{L^2}\|\phi\eta_k|h|^{p_k}\|_{L^2(M_T)} \\
&\quad + \frac{C_{II}}{\sin(\alpha)}\Big(\frac{25}{3\mu} + a\coth(10a)\Big)^2 \|\eta_k|h|^{p_k}\|^2_{L^2(M_T)} \\
&\leq \frac{1}{2p_k}\|d(\eta_k|h|^{p_k})\|^2_{L^2(M_T)} + \frac{C_{II}}{\sin(\alpha)}\Big(\frac{100}{\mu^2} + \frac{p_k C_{II}}{2\sin(\alpha)}\Big)\|\eta_k|h|^{p_k}\|^2_{L^2(M_T)},
\end{aligned} \tag{61}$$

where we have used Lemmas 26 and 28 to bound the covariant derivative of $i_{\mathcal{R}}$ above. Inserting this inequality back into (60) gives

$$\|\eta_k|h|^{p_k}\|^2_{L^2_1(M_T)} \leq 2\Big(\frac{4^{k+1}}{L^2} + \frac{100 p_k C_{II}}{\mu^2\sin(\alpha)} + \frac{p_k^2 C_{II}^2}{2\sin^2(\alpha)} + C_R + \frac{1}{2}\Big)\|h\|^{2p_k}_{L^{2p_k}(B_{k-1})}, \tag{62}$$

Now we follow the usual Moser iteration proof as in [CLS16]. By Proposition 44, we have

$$\|\eta_k|h|^{p_k}\|^2_{L^{\frac{2n}{n-2}}(M_T)} \leq S(a,n)\|d(\eta_k|h|^{p_k})\|_{L^2_1(M_T)},$$

so that

$$\|\eta_k|h|^{p_k}\|^2_{L^{\frac{2n}{n-2}}(M_T)} \leq 2S(a,n)\Big(\frac{4^{k+1}}{L^2}+\frac{100p_kC_{II}}{\mu^2\sin(\alpha)}+\frac{p_k^2C_{II}^2}{2\sin^2(\alpha)}+C_R+\frac{1}{2}\Big)\|h\|^{2p_k}_{L^{2p_k}(B_{k-1})}. \tag{63}$$

Set

$$C_G := \frac{100C_{II}}{\mu^2\sin(\alpha)}+\frac{C_{II}^2}{2\sin^2(\alpha)}+C_R+\frac{1}{2}. \tag{64}$$

For $p_k \geq 1$,

$$\|h\|^{2p_k}_{L^{\frac{2np_k}{n-2}}(B_k)} \leq 2S(a,n)p_k^2 4^{k+1}\Big(\frac{1}{L^2}+C_G\Big)\|h\|^{2p_k}_{L^{2p_k}(B_{k-1})}. \tag{65}$$

Choose now $p_k = (\frac{n}{n-2})^k$. Then taking p_k roots of (65) and iterating yields

$$\|h\|^2_{L^{2p_{k+1}}(B_k)} \leq \prod_{j=0}^{k} S(a,n)^{\frac{1}{p_j}} 4^{\frac{j+2}{p_j}} p_j^{\frac{2}{p_j}}\Big(\frac{1}{L^2}+C_G\Big)^{\frac{1}{p_j}}\|h\|^2_{L^{2p_j}(B_{j-1})}. \tag{66}$$

Taking the limit as $k \to \infty$ and setting

$$c_n := \prod_{k=1}^{\infty} 4^{\frac{k+2}{p_k}} \tag{67}$$

yields

$$\|h\|^2_{L^\infty(B_L(p))} \leq c_n S(a,n)^{\frac{n}{2}}\Big(\frac{1}{L^2}+C_G\Big)^{\frac{n}{2}}\|h\|^2_{L^2(B_{2L}(p))}. \tag{68}$$

If a harmonic form h satisfies Dirichlet boundary conditions, then $*h$ satisfies Neumann boundary conditions, and $|*h| = |h|$. Hence the Dirichlet estimate implies the Neumann estimate. □

4.4 Non Uniformly Discrete Sequences

Let (X^n, g) be a simply connected manifold of dimension $n \geq 3$. Assume the sectional curvature is pinched:

$$-a^2 \leq \sec_g \leq -1$$

with $a \geq 1$. Let (M_l, g_l) be a sequence of closed manifolds BS-converging to (X, g), which we now assume *not* to be uniformly discrete. For any element (M_l, g_l) in the sequence, consider a small geodesic γ such that

$$l(\gamma) \leq c_1 \exp(-c_2 a)\mu^n a^{n-1}$$

as in (29) of Lemma 28. Remove from M_l the union of the modified tubes V^l_γ to get a manifold with boundary

$$M_{l,T} := M_l \setminus \cup_\gamma V^l_\gamma, \quad \partial M_{l,T} = \cup_\gamma H^l_\gamma.$$

Denote by $N_T(M_l)$ the number of disjoint tubes V^l_γ in M_l. From the long exact sequence in cohomology,

$$... \to H^k(M_{l,T}, \partial M_{l,T}; \mathbb{R}) \to H^k(M_l; \mathbb{R}) \to H^k(\cup_\gamma V^l_\gamma; \mathbb{R}) \to H^{k+1}(M_{l,T}, \partial M_{l,T}; \mathbb{R}) \to ...,$$

we obtain the inequality

$$b_k(M_l) \leq \dim_{\mathbb{R}} H^k(M_{l,T}, \partial M_{l,T}; \mathbb{R}) + N_T(M_l). \tag{69}$$

The next lemma shows that we can control $N_T(M_l)$ in terms of the total volume.

Lemma 70. *Let (X^n, g) be a simply connected manifold of dimension $n \geq 3$ with*

$$-a^2 \leq \sec_g \leq -1,$$

and $a \geq 1$. If (M_l, g_l) is a sequence of closed manifolds BS-converging to (X, g), then

$$\frac{N_T(M_l)}{Vol_{g_l}(M_l)} < \frac{\rho(M_l, \mu)}{c_n \left(\frac{21}{200}\mu\right)^n}. \tag{71}$$

Proof. By Lemma 37, every Margulis tube V^l_γ satisfies

$$Vol(V^l_\gamma) > c_n \left(\frac{21}{200}\mu\right)^n := \epsilon_0$$

where c_n is a positive constant depending on the dimension only, and $\mu(n, a) > 0$ is the usual Margulis constant for a negatively curved a-pinched n-manifold. By Lemma 34, if γ is a short geodesic satisfying Equation (29), then

$$V^l_\gamma \subset (M_l)_{<\mu}.$$

Thus, we have

$$\frac{N_T(M_l) \cdot \epsilon_0}{Vol(M_l)} \leq \frac{\sum_\gamma Vol(V^l_\gamma)}{Vol(M_l)} < \rho(M_l, \mu).$$

□

Lemma 72. *Let (X^n, g) be a simply connected manifold of dimension $n \geq 3$ with*

$$-a^2 \leq \sec_g \leq -1,$$

and $a \geq 1$. Let (M_l, g_l) be a sequence of closed manifolds BS-converging to (X, g). For any $k \in \mathbb{N}$ such that

$$a_{n,k} = (n-1) - 2ka > 0,$$

we have for all $R >> 1$,

$$\frac{\dim_{\mathbb{R}} H^k(M_{l,T}, \partial M_{l,T}; \mathbb{R})}{Vol_{g_l}(M_l)} \leq \left(\begin{array}{c} n \\ k \end{array} \right) r(a,n)\rho(M_l, R) + c(n,k)e^{-a_{n,k}(R-1)}.$$

If $a_{n,k} = 0$, we have for all $R >> 1$,

$$\frac{\dim_{\mathbb{R}} H^k(M_{l,T}, \partial M_{l,T}; \mathbb{R})}{Vol_{g_l}(M_l)} \leq \left(\begin{array}{c} n \\ k \end{array} \right) r(a,n)\rho(M_l, R) + d(n,k)(R-2)^{-1}.$$

Proof. For any $l \in \mathbb{N}$, denote by

$$i : \partial M_{l,T} \to M_{l,T}$$

the injection map. Recall that the relative cohomology $H^k(M_{l,T}, \partial M_{l,T}; \mathbb{R})$ is isomorphic to the space of harmonic forms satisfying Dirichlet boundary conditions:

$$H^k(M_{l,T}, \partial M_{l,T}; \mathbb{R}) \simeq \{\alpha \in C^\infty(\Omega^k T^*(M_{l,T} \backslash \partial M_{l,T})) \mid \Delta_k \alpha = 0,\ i^*\alpha = 0\}.$$

Let $\{\alpha_i\}_i$ be an L^2 orthonormal basis of harmonic forms satisfying Dirichlet boundary conditions, and consider the function

$$TrK(x,x) = \sum_i |\alpha_i(x)|^2$$

which satisfies

$$\int_{M_{l,T}} TrK(x,x)dv = \dim_{\mathbb{R}} H^k(M_{l,T}, \partial M_{l,T}; \mathbb{R}).$$

By Proposition 49, for any unit norm harmonic k-form α satisfying Dirichlet boundary conditions in $M_{l,T}$, there exists a constant $r(a,n) > 0$ such that

$$|\alpha(p)|^2 \leq r(a,n), \tag{73}$$

for any $p \in M_{l,T}$. On the other hand if we take a point in $p \in M_{l,T}$ with large injectivity radius, we can apply Theorem 23 to obtain much stronger bounds. Consider $p \in M_l$ with $inj_g(p) = R >> 1$. We observe that any such point lies in $M_{l,T}$ and must be quite distant from $\partial M_{l,T} \subset M_l$. Indeed, by Lemma 34, as any point $q \in \partial M_{l,T} = \cup_\gamma H^l_\gamma$ satisfies $\mathrm{inj}_{g_l}(q) \leq \frac{26}{50}\mu$, we have

$$d_{g_l}(p,q) \geq \mathrm{inj}_{g_l}(p) - \frac{26}{50}\mu > R - \mu > R - 1, \tag{74}$$

as we have assumed $\mu < 1$. Thus if $p \in M_l$ has $\mathrm{inj}_{g_l}(p) \geq R >> 1$, then $\bar{B}_{R-1}(p) \subset M^0_{l,T}$. Set

$$(M_{l,T})_{<R} := (M_l)_{<R} \cap M_{l,T}$$

and similarly $(M_{l,T})_{\geq R}$ as its complement in $M_{l,T}$. The Price inequality given in Theorem 23 then tells us that for $R >> 1$,

$$\int_{(M_{l,T})_{\geq R}} TrK(x,x)dv \leq \begin{cases} c(n,k)e^{-a_{n,k}(R-1)}Vol((M_{l,T})_{\geq R}) & \text{if} \quad a_{n,k} > 0, \\ d(n,k)(R-2)^{-1}Vol((M_{l,T})_{\geq R}) & \text{if} \quad a_{n,k} = 0. \end{cases} \tag{75}$$

Moreover, by (51)

$$\int_{(M_{l,T})_{<R}} TrK(x,x)dv \leq \begin{pmatrix} n \\ k \end{pmatrix} r(a,n)Vol((M_{l,T})_{<R}). \tag{76}$$

Combining (75) and (76) together with the inequalities

$$\frac{Vol_{g_l}((M_{l,T})_{<R})}{Vol_{g_l}(M_l)} \leq \rho(M_l, R),$$

we immediately obtain the desired inequalities. □

We now have the main theorem of this section.

Theorem 77. *Let (X^n, g) be a simply connected manifold of dimension $n \geq 3$ with*

$$-a^2 \leq \sec_g \leq -1,$$

and $a \geq 1$. Let (M_l, g_l) be a sequence of closed manifolds BS-converging to (X,g). For any $k \in \mathbb{N}$ such that

$$a_{n,k} = (n-1) - 2ka > 0,$$

we have for all $R >> 1$,

$$\frac{b_k(M_l)}{Vol_{g_l}(M_l)} \leq (\epsilon_0^{-1} + \begin{pmatrix} n \\ k \end{pmatrix} r(a,n))\rho(M_l,R) + c(n,k)e^{-a_{n,k}(R-1)}.$$

If $a_{n,k} = 0$, we have for all $R >> 1$,

$$\frac{b_k(M_l)}{Vol_{g_l}(M_l)} \leq (\epsilon_0^{-1} + \begin{pmatrix} n \\ k \end{pmatrix} r(a,n))\rho(M_l,R) + d(n,k)(R-2)^{-1}.$$

Consequently

$$\lim_{l\to\infty} \frac{b_k(M_l)}{Vol_{g_l}(M_l)} = 0.$$

Proof. The proof follows from (69), Lemma 70, and Lemma 72, and the monotonicity of in R of $\rho(M,R)$. ☐

References

[ABBGNRS17] M. Abert, N. Bergeron, I. Biringer, T. Gelander, N. Nikolov, J. Raimbault, I. Samet, On the growth of L^2-invariants for sequences of lattices in Lie groups. *Ann. of Math. (2)* **185** (2017), no. 3, 711-790.

[ABBG18] M. Abert, N. Bergeron, I. Biringer, T. Gelander, Convergence of normalized Betti numbers in nonpositive curvature. *arXiv:1811.02520v2* [math.GT].

[Ati76] M. F. Atiyah, Elliptic operators, discrete groups and von Neumann algebras. *Colloque "Analyse et Topologie" en l'Honneur de Henri Cartan (Orsay, 1974)*, pp. 43-72, Astérisque, No. 32-32, *Soc. Math. France, Paris*, 1976.

[Aub82] T. Aubin, *Nonlinear Analysis on Manifolds. Monge-Ampére Equations*, Grundlehren Math. Wiss. 252, Springer-Verlag, New York, 1982.

[BGS85] W. Ballmann, M. Gromov, V. Schroeder, Manifolds of nonpositive curvature. Progress in Mathematics, 61. *Birkhäuser Boston, Inc., Boston, MA,* 1985

[BS01] I. Benjamini, O. Schramm, Recurrence of distributional limits of finite planar graphs. *Electron. J. Prob.* **6** (2001), no. 23, 13 pp.

[BCD93] P. Buser, B. Colbois, J. Dodziuk, Tubes and Eigenvalues for Negatively Curves Manifolds. *J. Geom. Anal.* **3** (1993), no. 1, 1-26.

[CLS16] S. Cherkis, A. Larraín-Hubach, M. Stern, Instantons on Multi-Taub-NUT Spaces I: Asymptotic Form and Index Theorem, *arXiv:1608.00018v1* [math.DG].

[DW78] D. DeGeorge and N. Wallach, Limit formulas for multiplicities in $L^2(G/\Gamma)$, I. *Annals of Math.* **107** (1978), no. 1, 133-150.

[DW79] D. DeGeorge and N. Wallach, Limit formulas for multiplicities in $L^2(G/\Gamma)$, II. *Annals of Math.* **109** (1979), no. 3, 477-495.

[DS17] L. F. Di Cerbo, M. Stern, Price Inequalities and Betti number growth on manifolds without conjugate points. *arXiv:1704.06354v2* [math.DG]. To appear in *Comm. Anal. Geom.*

[doC92] M. P. do Carmo, Riemannian Geometry. Mathematics: Theory & Applications. *Birkhuser Boston, Inc., Boston, MA,* 1992.

[Dod79] J. Dodziuk, L^2 Harmonic forms on rotationally symmetric riemannian manifolds. *Proc. Amer. Math. Soc.* **77** (1979), no. 3, 395-400.

[DX84] H. Donnelly, F. Xavier, On the Differential Form Spectrum of Negatively Curved Riemannian Manifolds. *Amer. J. Math.* **106** (1984), no. 1, 169-185.

[Jon81] P. Jones, Quasiconformal mappings and extendability of functions in sobolev spaces. *Acta Math.* **147** (1981), no. 1-2, 71-88.

[LS18] M. Lipnowski, M. Stern, Geometry of the Smallest 1-form Laplacian Eigenvalue on Hyperbolic Manifolds, *Geom. Funct. Anal.* **28** (2018), no. 6, 1717-1755.

[Lüc94] W. Lück, Approximating L^2-invariants by their finite-dimensional analogues. *Geom. Funct. Anal.* **4** (1994), no.4, 455-481.

[Lüc02] W. Lück, L^2-invariants: theory and applications to geometry and K-theory. Results in Mathematics and Related Areas. 3rd Series. A Series of Modern Surveys in Mathematics, 44. *Springer-Verlag, Berlin,* 2002.

[Ste70] E. M. Stein, Singular integrals and differentiability properties of functions. Princeton Mathematical Series, No. 30 *Princeton University Press, Princeton, N. J.* 1970.