Mixing and Enhanced Dissipation in Measure Preserving Dynamical Systems

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Abstract

The movement of particles and energy in a fluid is governed by the advection-diffusion equation. Given an underlying velocity field, a common question in fluid mechanics is to understand the motion described by the advection-diffusion equation. An interesting notion in fluid systems is the concept of mixing, the irreversible thermodynamic process seen by the mixing paint, mixing water of different temperatures, or the behavior of smoke in a smoke-filled rooms. In order to mathematically quantify mixing, we can view fluid systems as a measure preserving dynamical system. This paper will introduce the notion of measure preserving dynamical systems, quantify mixing and enhanced dissipation, and the study long term behavior of solutions to the advection diffusion equation. In particular, we provide an explicit instance of a smooth velocity field that exhibits enhanced dissipation at a rate of $\nu^{\frac{1}{3}}$

1. Introduction

The advection-diffusion equation is the combination of advection and diffusion equations and describes phenomena where particles, energy, or other quantities move in a system due two the processes of advection and diffusion. Diffusion can be thought of as the movement of things from an area of high concentration to an area of low concentration while advection is the transport of things by bulk motion of a fluid. The general advection-diffusion equation is given by

$$\partial_t f + \mathbf{u} \cdot \nabla f - \nu \Delta f = R$$

where advection is described by the second term, diffusion by the third term and R describes the creation and destruction of the things measured. $\Omega \subset \mathbb{R}^2$ is some region. Let $u : \Omega \to \mathbb{R}^2$ be a stationary velocity field that is divergence free. When Ω has a (nontrivial) boundary, we assume $u \cdot n = 0$ on $\partial\Omega$. Let $f : \Omega \times \mathbb{R} \to \mathbb{R}$ be some quantity advected by u. We can think of f as a scalar valued quantity to be measured, such as concentration of some solvent or temperature.

We describe common simplifications to the general equation. When $R = 0$, there are no sources and sinks and thus the velocity field models an incompressible fluid. This gives us the equation of

$$\partial_t f + \mathbf{u} \cdot \nabla f = \nu \Delta f$$

In this paper, we will mostly consider the case of $\nu = 0$, known as the transport equation. We will pivot to the more general case when discussing enhanced dissipation.

1.1 Measure Preserving Dynamical Systems

We consider a set Ω. Let $\mathcal{P}(\Omega)$ denote the power set of Ω. A set $\Sigma \subset \mathcal{P}(\Omega)$ is called *σ-algebra* if it is closed under countable unions and complements. From these properties, it is also closed under countable intersections. A measure is a function $\mu_\Sigma : \Sigma \to \mathbb{R}_+ \cup \{\infty\}$ that is *σ-additive*, or that $\mu_\Sigma(\cup_i A_i) = \sum_i \mu_\Sigma(A_i)$ for disjoint sets A_i. A measure zero set is called a *null set* and its complement is a set of *full measure*. A σ-algebra is *complete* with respect to a measure if it contains every subset of every null set. Given an arbitrary σ-algebra Σ, we can define its *completion* to be the smallest complete σ-algebra containing Σ.

A *measure space* is a triple (Ω, Σ, μ), where Ω is a set, Σ is a complete σ-algebra over Ω, and μ is a measure such that Ω is the countable union of sets with finite measure. Given two measure spaces (Ω, Σ, μ) and (Ω', Σ', ν), we say a function $T : \Omega \to \Omega'$ is a *measurable function* if the pre-image of every measurable set is measurable. A measurable function T is *non-singular* if the pre-image of every null set is a null set and *measure preserving* if $\mu(T^{-1}(A)) = \nu(A)$ for all $A \in \Sigma'$. If (Ω, Σ, μ) is a measure space, and $T : \Omega \to \Omega$ is a non-singular mapping, then T is called a *transformation*. If T is additionally measure preserving, then μ is T*-invariant* and T is called a *measure preserving transformation*. If T is an invertible measure preserving transformation, then its inverse is also measure preserving and non-singular.

A *measure preserving dynamical system* is a tuple (Ω, Σ, μ, T) where (Ω, Σ, μ) is a measure space and T is a measure preserving transformation on Ω. To put the above section into context, we have that $(\mathbb{T}^2, \Sigma, \mu, \Phi)$ is a measure preserving dynamical system. Here, Σ denotes the *Borel σ-algebra* of Ω, the smallest σ-algebra that contains all open sets of $\mathbb{T}^2$ when endowed with the standard topology. By Liouville's Lemma, we know that Φ^{-1} is measure preserving. Since $\det \Phi = 1$, we also have that Φ is measure preserving.

1.2 Functional Analysis Preliminaries

Let (Ω, Σ, μ) and (Ω', Σ', ν) be two measure spaces. We define a following equivalence relation on the set of functions $\{\gamma \,|\, \gamma : \Omega \to \Omega'\}$. We say that two measurable functions $f, g : \Omega \to \Omega'$ are equivalent if they coincide on a full measure set. That is to say, $\mu(\{x \mid f(x) \neq g(x)\}) = 0$. This is clearly a valid equivalence relation as it is reflexive, transitive and symmetric. Then, we can give the following definitions.

Definition 1.1. Let $p \in (0, \infty)$. We define the space $L^p(\Omega, \mu)$ to be the set of equivalence classes of measurable functions $f : \Omega \to \mathbb{R}$ such that

$$\left(\int_\Omega |f(x)|^p \, d\mu \right)^{1/p} < \infty$$

For any $p \geq 1$, the space $L^p(\Omega, \mu)$ is a Banach space with norm given by

$$\|f\|_p^p = \int_\Omega |f(x)|^p \, d\mu$$

The case of $p = 2$ is special in that $L^2(\Omega, \mu)$ is a Hilbert space with inner product given by

$$\langle f, g \rangle^2 = \int_\Omega f \cdot g \, d\mu$$

Another important notion related to L^p spaces is the Fourier Transform. Since we are working with $\Omega = \mathbb{T}^2$, all functions are 2π-periodic. As such, we can define the Fourier series of functions $f : \Omega \to \mathbb{R}$.

Definition 1.2. Given a P-periodic function $u : \Omega \to \mathbb{R}$, we have that

$$u(x) = \sum_{k=-\infty}^{\infty} \hat{u}(k)(x) \cdot e^{\frac{2\pi i k \cdot x}{P}}$$

where $\hat{u}(k)$ denotes the kth Fourier mode of u and is defined by

$$\hat{u}(k) = \frac{1}{2\pi} \int_{-\pi}^{\pi} u(x) \cdot e^{-\frac{2\pi i k \cdot x}{P}} \, dx$$

When Ω is embedded in $\mathbb{R}^n$ for $n > 1$, then k takes on values in $\mathbb{Z}^n$ and the appropriate dot products are taken with the values of x. Taking the Fourier series of a periodic function is essentially leveraging the orthonormal basis of exponential functions on $L^2(\Omega)$.

After defining the Fourier series of periodic functions, we can now consider two important characterizations of Sobolev spaces.

Definition 1.3. Assume that $\Omega \subset \mathbb{R}$. The space $W^{k,p}(\Omega)$ is defined as a subset of $L^p(\Omega)$ such that f and its derivatives up to order k have finite L^p norm. This motivates the natural definition of the norm over $W^{k,p}$ given by

$$\|f\|_{W^{k,p}}^p = \sum_{i=0}^{k} \|D^{(i)} f\|_{L^p}^p$$

It turns out that it suffices to just take the first and last values in the sum, and the induced topologies by the norm are identical. With this norm, $W^{k,p}(\Omega)$ is a Banach space. An important to consider is when $p = 2$ since $W^{k,2} := H^k$ becomes a Hilbert space with the inner product given by

$$\langle f, g \rangle_{H^k} = \sum_{i=0}^{k} \langle D^{(i)} f, D^{(i)} g \rangle_{L^2}$$

Moreover, we have that the H^k norm can be expressed in terms of the Fourier coefficients of f. Specifically,

$$\|f\|_{H^k}^2 = \sum_{n=-\infty}^{\infty} (1 + |n|^2)^k |\hat{f}(n)|^2$$

We also introduce the homogeneous Sobolev norm

$$\|f\|_{\dot{H}^k}^2 = \sum_{n=-\infty}^{\infty} |n|^{2k} |\hat{f}(n)|^2$$

Since we are working on the compact $\mathbb{T}^2$, Poincaré's identity holds and the homogeneous and normal Sobolev norms are identical for mean zero functions. Since the context of this paper meets the required conditions, it is more simple to work with the homogeneous norm.

We can also represent L^2 norms using Fourier coefficients. This representation is due to the fact that the Fourier transform is unitary. This is represented by Parseval's Theorem.

Theorem 1.1 (Parseval's Theorem)**.** *If f, g are 2π periodic functions, with Fourier coefficient given by $\hat{f}(k), \hat{g}(k)$, respectively, then*

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) \cdot g(x)\, dx = \sum_{k=-\infty}^{\infty} \hat{f}(k) \hat{g}(k)$$

From this, we can see that

$$\frac{1}{2\pi} \|f\|_{L^2} = \sum_{k=-\infty}^{\infty} |\hat{f}(k)|^2$$

When $\Omega \subset \mathbb{R}^n$ for $n > 1$, the notion of derivatives do not extend nicely since the antiderivative of $D^{(k)} f$ being $D^{(k-1)} f$ need not hold. As such, we define the Sobolev norm using multi-indexes.

Definition 1.4. A multi-index α is a tuple $\alpha = (\alpha_1, \cdots, \alpha_n)$ where all $\alpha_i \in \mathbb{N}_+$ and we define the order of α to be

$$|\alpha| = \sum_{i=1}^{n} \alpha_i$$

With this definition, we can now define Sobolev spaces in high dimensions.

Definition 1.5. Assume $\Omega \subset \mathbb{R}^n$. The space $W^{k,p}(\Omega)$ is the subset of $L^p(\Omega)$ such that the weak derivatives up to order k exist and are in $L^p(\Omega)$. Namely, we require

$$\|D^\alpha(f)\|_{L^p} = \left\| \frac{\partial^{|\alpha|} f}{\partial x_1^{\alpha_1} \cdots \partial x_n^{\alpha_n}} \right\|_{L^p} < \infty$$

for each multi-index α satisfying $|\alpha| \leq k$. The natural norm is then given by

$$\|f\|_{W^{k,p}}^p = \sum_{|\alpha| \leq k} \|D^\alpha f\|_{L^p}^p$$

The analogue of H^k in higher dimensions also follows from this. If $\Omega \subset \mathbb{R}^n$, the space $H^k(\Omega) := W^{k,2}(\Omega)$ can be equipped with the norm

$$||f||_{H^k}^2 = \sum_{k \in \mathbb{Z}^k} (1 + |k|^2)^k |\hat{f(k)}|^2$$

1.3 Mixing

Intuitively, mixing is the irreversible thermodynamic process by which things come together and mix. Real world examples include smoke filling a room, liquids of different temperatures combining in a container, as well as combining paint. There are multiple ways to quantify mixing. We can view mixing as an extension to the property of ergodicity. On the other hand, we can define mixing in terms of weak L^2 convergence, and associated notions such as H^{-1} decay. In this paper, we will discuss these two ways to quantify mixing and provide examples.

1.3.1 Mixing in Ergodic Theory

Quantifying mixing in the frame of ergodic theory is often more straightforward. It simply encapsulates the idea that given any two sets, the measure of their intersection will simply be the product of their measures, which will hold if the two sets spread out evenly through Ω. In this section, we assume that (Ω, Σ, μ, T) is a measure preserving dynamical system. We first define the orbit of a point.

Definition 1.6. The *orbit* of a point $x \in \Omega$ is given by the set $\{T(x, t) \mid t \in \mathbb{R}_+\}$.

The orbit of a point can be thought of as its path defined by T. Intuitively, all points must have dense orbits in order for mixing to occur, since a positive measure set without dense orbits will not "mix". A dynamical system where all points have dense orbits is referred to as *minimal*. Next, we define the notion of ergodicity.

Definition 1.7. The system (Ω, Σ, μ, T) is *ergodic* if $A \in \Sigma$ satisfies $\mu(A \triangle T^{-1}(A)) = 0$, then $\mu(A) \in \{0, 1\}$.

Essentially, this is to say that there exist no invariant sets, or sets that stay constant under the action of T. This notion is important to mixing since if a mapping T is not ergodic, then intuitively any invariant set will not move under T, and thus cannot mix. Lastly, we define weak and strong mixing.

Definition 1.8. Let $A, B \in \Sigma$ be measurable sets. Then, the system (Ω, Σ, μ, T) is *weakly mixing* if

$$\lim_{n \to \infty} \frac{1}{n} \sum_{k=0}^{n-1} |\mu(A \cap T^{-k}(B) - \mu(A)\mu(B)| = 0$$

The system is *strongly mixing* or just *mixing* if

$$\lim_{n \to \infty} \mu(A \cap T^{-n}(B)) = \mu(A)\mu(B)$$

1.3.2 Mixing in Functional Analysis

The description of mixing in terms of L^2 norms is less intuitive. However, the advantage of this perspective is that we can attribute a rate to fluid mixing. In this manner, we can compare the rates of mixing between two mixing dynamical systems. The two characterizations are in fact equivalent under von Neumann's ergodic theorem.

Definition 1.9. We say that the system (Ω, Σ, μ, T) is weakly mixing if given any function $f \in C^2(\Omega, \mu)$, the time average of the sequence given by $(f \circ T^n)_{n \geq 0}$ converges weakly to $\int_\Omega f$. This is to say for any test function $g \in L^2(\Omega, \mu)$ we have

$$\lim_{n \to \infty} \sum_{k=0}^{n-1} \left| \int_\Omega (f \circ T^k) \cdot g \, d\mu - \int_\Omega f \, d\mu \int_\Omega g \, d\mu \right| = 0$$

Similarly, strong mixing can be quantified by removing the time average. The system is weakly mixing if for any $f \in C^2(\Omega, \mu)$, any test function $g \in L^2(\Omega)$ satisfies

$$\lim_{n \to \infty} \int_\Omega (f \circ T^n) \cdot g \, d\mu = \int_\Omega f \, d\mu \int_\Omega g \, d\mu$$

This L^2 formulation does not yet give us the desired rate associated with mixing. In order to evaluate the rate of mixing, we turn to an analysis of Fourier modes. If the initial conditions are mean zero, then the following lemma holds. The rate of decay of the H^{-1} norm can then provide a rate of how a fluid field mixes.

Lemma 1.2. *$\|f \circ T^n\|_{H^{-1}} \to 0$ is equivalent to $f \circ T^{-n} \rightharpoonup 0$*

Proof. The assumption will be that $\Omega \subset \mathbb{R}$, but these results easily extend to $\Omega \subset \mathbb{R}^n$.

Given a system (Ω, Σ, μ, T), assume that Ω is compact and f is mean zero. For notational simplicity, denote $f \circ T^n$ as f^t. Then, assume $\lim_{t\to\infty}\|f^t\|_{H^{-1}} = 0$. We also assume that we have its norm uniformly bounded, $\|f^t\|_{L^2} \le M$. Now, consider some arbitrary test function $g \in L^2(\Omega)$. We have that

$$\begin{aligned}\int_\Omega f^t \cdot g \, d\mu = \sum_{\mathbf{k}=-\infty}^{\infty} \hat{f}^t(k) \cdot \hat{g}(k) &= \sum_{|\mathbf{k}|\le K} k^{-1}\hat{f}^t(k) \cdot k\hat{g}(k) + \sum_{|\mathbf{k}|>K} \hat{f}^t(k) \cdot \hat{g}(k) \\ &\le \|f^t\|_{H^{-1}}\Big(\sum_{|\mathbf{k}|\le K} k^2\hat{g}(k)^2\Big)^{\frac{1}{2}} + \|f^t\|_{L^2}\Big(\sum_{|\mathbf{k}|>K} \hat{g}(k)^2\Big)^{\frac{1}{2}}\end{aligned}$$

We are free to choose K, T independently. For any given $\epsilon > 0$, choose K such that $\left(\sum_{|k|>K} \hat{g}(k)^2\right)^{\frac{1}{2}} \le \frac{\epsilon}{2K}$. Then, choose T such that $\|f^t\|_{H^{-1}} \le \frac{\epsilon}{2} \cdot \left(\sum_{|k|\le K} k^2\hat{g}(k)^2\right)^{-\frac{1}{2}}$. This shows that

$$\int_\Omega f^t \cdot g \, d\mu \le \epsilon\Big(\frac{1}{2} + \frac{\|f^t\|_{L^2}}{2K}\Big) \le \epsilon$$

and indeed we have weak convergence to our mean zero function. For the other direction, assume that $f^t \rightharpoonup 0$ and f^t uniformly bounded in L^2 by M. We can take g to be be the orthogonal basis functions given by $b_{\mathbf{k}} := e^{-i\mathbf{k}\cdot\mathbf{x}}$. Then, by our assumption, we have that

$$\int_\Omega f^t b_{\mathbf{k}} = \hat{f}^t(k) \to 0$$

since by the property of harmonics, $\langle e^{-ik\cdot x}, e^{-ij\cdot x}\rangle_{L^2} = 1$ if $j = k$ and 0 otherwise. Moreover, we can also bound the decay of the Fourier coefficients since

$$M^2 \ge |f^t|_{L^2}^2 = \sum_{k=-\infty}^{\infty} |\hat{f}(k)|^2$$

which implies that each $\hat{f}^t(k) \le M$ for all k, t. Then, we can write

$$\begin{aligned}\|f^t\|_{H^{-1}} = \sum_{\mathbf{k}} |k|^{-2}|\hat{f}^t(k)|^2 &= \sum_{|\mathbf{k}|\le K} |k|^{-2}|\hat{f}^t(k)|^2 + \sum_{|\mathbf{k}|>K} |k|^{-2}|\hat{f}^t(k)|^2 \\ &\le \sum_{|\mathbf{k}|\le K} |k|^{-2}|\hat{f}^t(k)|^2 + K^{-2}\sum_{|\mathbf{k}|>K} |\hat{f}^t(k)|^2 \le \sum_{|\mathbf{k}|\le K} |k|^{-2}|\hat{f}^t(k)|^2 + K^{-2}\|f^t\|_{L^2}\end{aligned}$$

Given $\epsilon > 0$, we can choose K such that the latter term $K^{-2}\|f^t\|_{L^2} \le \frac{\epsilon}{2}$. Then, since we have a finite number of terms in the first sum, we take T large enough such that $|\hat{f}^t(K)| \le \frac{1}{K\epsilon}$ which gives the desired bound of $\|f^t\|_{H^{-1}} \le \epsilon$ □

Remark. *Considering the H^{-1} is not necessary in the lemma; this lemma holds for any $H^{-\alpha}$ for any $\alpha > 1$.*

1.4 Enhanced Dissipation

The concept of dissipation is very interconnected with that of mixing. In the general advection diffusion equation, we first consider the case where there is no flow, when $\mathbf{u} = 0$. This is the general heat equation,

$$\partial_t f + \mathbf{u} \cdot \nabla f = \nu\Delta f$$

We provide a simple L^2 estimate on the decay of the norm as $t \to \infty$ using Poincare's Inequality.

Lemma 1.3 (Poincare's Inequality). *Let $1 \leq p < \infty$ and $\Omega \subset \mathbb{R}^n$ a bounded set. Then, there exists a C, dependent only on p, Ω, such that for any $f \in W^{1,p}$ such that f vanishes on the boundary of Ω, we have*

$$\|u\|_{L^p} \leq C\|\nabla u\|_{L^p}$$

Starting with the heat equation, we have

$$\begin{aligned}
\partial_t f &= \nu \Delta f \\
\frac{1}{2}\int_{\mathbb{T}^2} \partial_t(f^2) &= \nu \int_{\mathbb{T}^2} f\Delta f \\
\frac{d}{dt}\Big(\frac{1}{2}\int_{\mathbb{T}^2} f^2\Big) &= \nu\Big(\int_{\partial\mathbb{T}^2} f(\nabla f \cdot n) - \int_{\mathbb{T}^2} |\nabla f|^2\Big) \\
\frac{d}{dt}\Big(\frac{1}{2}\|f\|_{L^2}^2\Big) &= -\nu\|\nabla f\|_{L^2}^2 \leq -C\|f\|_{L^2}^2
\end{aligned}$$

where the last inequality follows from Lemma 1.3. Absorbing all the constant terms, and treating $\|f\|_{L^2}^2$ as a new variable, say x, we have that

$$\frac{dx}{dt} + C'x \leq 0 \implies e^{ct}\Big(\frac{dx}{dt} + C'x\Big) \leq 0 \implies \frac{d}{dt}\Big(xe^{ct}\Big) \leq 0$$

It follows then that if f^t represents the solution f fixed at time t,

$$\|f^t\|_{L^2} \leq \|f^0\|_{L^2} e^{-C''\nu t}$$

This shows that the L^2 norm *dissipates* at a rate of ν^{-1} as $t \to \infty$. We can prove a slightly stronger statement about the decay L^2 norm. The decay of the Fourier modes is decoupled, with the modes at higher frequencies decaying faster. Intuitively, when we re-introduce a fluid flow back into the equation, this rate of decay can increase. We can see one example of this when we having a mixing system, where the Fourier coefficients all go to zero. If we assume that the L^2 norm is conserved, for any finite t, we have that $\|f^t\|_{L^2} = \|f_0\|_{L^2}$. Intuitively, the energy in the lower Fourier coefficients must go toward higher frequencies. We prove the one dimensional version of the statement in the following lemma. It is easily extended to higher dimensions.

Lemma 1.4. *For all $\epsilon > 0$ and $N \in \mathbb{N}$, there exists a $T(N)$ such that for all $t > T$, $\|\mathcal{P}_{>N} f^t\|_{L^2} = (1-\epsilon)\|f^t\|_{L^2}$*

Proof. Let $\epsilon > 0$ be fixed and let N be given. For $k \in \mathbb{N}$, let k_t denote the number such that $t > k_t$ implies $|\hat{f}^t(k)|^2 < \frac{\epsilon}{N}$. Then, let $N' := \max_{k \leq N} k_t$. Choosing any $t > N'$ reveals

$$\|\mathcal{P}_{\leq N} f^t\|_{L^2} = \sum_{k \leq N} |\hat{f}^t(k)|^2 \leq \sum_{k \leq N} \frac{\epsilon}{N} \leq \epsilon$$

from which the desired inequality follows. □

When we consider both a mixing system combined with the inherent L^2 decay of the heat equation, it is intuitive that dissipation should occur at a faster rate since the energy in the Fourier modes goes towards higher frequencies, and dissipation occurs at faster rates at higher frequencies. We characterize the following behavior with the next definition.

Definition 1.10. Let $C \geq 1$ be a real constant, $\delta(\nu) : \mathbb{R}_+ \to \mathbb{R}_+$ a positive real valued function, satisfying

$$\lim_{\nu \to 0} \frac{\nu}{\delta(\nu)} = 0$$

If $u : \Omega \times \mathbb{R} \to \Omega$ is the velocity field in the advection diffusion equation, u is said to be *dissipation enhancing* if for all initial conditions $f_0 \in L^2(\Omega)$, the solution f^t satisfies

$$\Big\|f^t - \int_\Omega f_0\Big\|_{L^2} \leq Ce^{-\delta(\nu)t}\Big\|f_0 - \int_\Omega f_0\Big\|_{L^2}$$

1.5 Problem Formulation

We consider the advection diffusion posed on the two dimensional torus $\mathbb{T}^2$

$$\begin{cases} \partial_t f + u \cdot \nabla f = \nu \Delta f \\ f(x, 0) = f_0(x) \end{cases}$$

where $f_0 : \mathbb{T}^2 \to \mathbb{R}$ are the initial conditions, $u : \mathbb{T}^2 \times \mathbb{R} \to \mathbb{R}^2$ is the velocity field, ν is the diffusivity constant, and $f : \mathbb{T}^2 \times \mathbb{R} \to \mathbb{R}$ is the scalar valued solution map.

The goal of this thesis is to provide an explicit smooth velocity field, in time and space, that exhibits enhanced dissipation at a rate of $\delta(\nu) = \nu^{\frac{1}{3}}$. This is a novel result since our proposed velocity field would be the first smooth velocity field to obtain this rate. In particular, we consider the velocity field defined by

$$u_{\alpha,M} = \begin{pmatrix} M \sin(y + \alpha t) \\ \sin(x) \end{pmatrix}$$

Our main result is stated in the following theorem.

Theorem 1.5. *When α, M are sufficiently large, the velocity field $u_{\alpha,M}$ exhibits enhanced dissipation at a rate of $\nu^{\frac{1}{3}}$.*

2. Previous Results

In this section, we provide an overview of current known results to frame our contributions. From the context of enhanced dissipation, there exist numerous results for the case of time independent velocity fields. In particular, the cases of shear flows, where $u = (f(y), 0)$, has been extensively studied [2, 8, 6]. In the case of $f(y) = y$, and $f(y) = y^2$, the resulting flows are the well known Couette and Poiseuille flows, respectively. The case of radial flows has also been studied [7]. These time independent flows only exhibit enhanced dissipation of a sharp rate of $\nu^{\frac{1}{3}}$ when the initial datum is assumed to be zero on its streamlines [3]. In general, a time independent flow cannot exhibit enhanced dissipation on all initial data [4].

For systems arising from time dependent flows, there are several explicit examples of flows exhibiting enhanced dissipation on the scale of $|\ln \nu|$ [1, 5]. However, in the case of the [1], the flow was constructed as the solution to stochastic PDEs and are not time-periodic nor uniformly bounded. Recently, there has been an example [5] of an explicit flow that is both time periodic and Lipschitz. However, our theorem seems to be the first instance of a time periodic flow that is smooth with respect to time and space that exhibits enhanced dissipation on all initial data at a rate of $\nu^{\frac{1}{3}}$.

3. Time Independent Velocity Fields

We first prove a few properties of the transport equation in the time independent case, and use those results when we generalize to our proposed velocity field. To solve the transport equation, we employ the characteristic map $\Phi : \Omega \times \mathbb{R} \to \Omega$, mapping $(x_0, t) \mapsto x$, which describes the terminal position x of a particle starting at x_0 after time t, advected by the velocity field . Specifically, we require Φ to satisfy:

$$\frac{d}{dt}\Phi(x, t) = \mathbf{u}(\Phi(x, t))$$
$$\Phi(x, 0) = x$$

implying that the derivative of the position vector is exactly the underlying velocity field, and preserving the initial condition. Then, under this setup, we see that

$$\frac{d}{dt}(f(\Phi(x, t), t)) = \partial_t f(\Phi(x, t), t) + \partial_t \Phi(x, t) = 0$$

which implies that

$$f(\Phi(x,t),t) = f(x,0) = f_0(x)$$

upon substituting x, we see that

$$f(x,t) = f(\Phi^{-1}(x,t),0) = f_0(\Phi^{-t}(x,t))$$

gives the solution f in terms of the initial conditions f_0. We now prove an important property of the characteristic map.

Lemma 3.1 (Liouville Lemma)**.** *Let* **u** *be a velocity field, and* Φ *be its corresponding characteristic map. Let* μ *denote the Lebesgue measure and let* $A \subset \Omega$ *be an arbitrary measurable set. Then, for all* t*, we have*

$$\mu(\Phi^{-1}(A,t)) = \mu(A) \tag{1}$$

In particular, this shows that the characteristic map associated with any velocity field is measure preserving. Thus, we have as an immediate corollary that letting μ be the standard Lebesgue measure and Σ the Borel σ-algebra on $\mathbb{T}^2$, that $(\mathbb{T}^2, \mu, \Sigma, \Phi)$ is a measure preserving dynamical system. We also want to show some properties of the L^2 norm of the solution map. In particular, we show that the norm is preserved for any finite t.

Lemma 3.2. *Let* $f^t := f_0 \circ \Phi^{-t}$ *be the solution map to the transport equation. For any* $t \in \mathbb{R}$*, it follows that*

$$||f_0||_{L^2} = ||f^t||_{L^2}$$

When Ω is simply connected, we can find a stream function ψ to rewrite u. When $u = (\partial_2 v, -\partial_1 v)$ for some $v \in C^2(\Omega, \mathbb{R})$, we see that $\operatorname{div}(u) = 0$ by construction. Taking inspiration from this, we can work backwards to define a stream function ψ to be unique solution to

$$\Delta\psi = \partial_2 u_1 - \partial_1 u_2$$

where the partial derivatives are taken in space. Given such any u, under this construction, we can write $u = \nabla^\perp \psi$.

In order to see the importance of ψ, we first need to understand the behavior of regular points of ψ under the stream function. We define the set R, the regular points of ψ as follows

$$C = \{x \in \Omega \mid \nabla\psi(x) = 0\}$$
$$R = \{x \in \Omega \mid \psi(x) \notin \psi(C)\}$$

The following two results characterizes the behavior of points in R and shows that the other points not in $C \cup R$ are measure zero.

Lemma 3.3. *For every* $x \in R$*, there exists a unique closed* C^1 *Jordan curve* Γ_x *such that for all* $y \in \Gamma_x$*,* $\psi(x) = \psi(y)$*.*

Lemma 3.4. $\mu(\Omega \setminus (C \cup R)) = 0$

With these two results, we can understand ψ as a measure of the energy along streamlines. Since ψ is constant on each streamline, we essentially split R into disjoint streamlines, each associated with a value of ψ. The important fact is that on each streamline, we have that $\frac{d}{dt}\psi = 0$.

4. Proof of Theorem

In order to prove the desired rate of enhanced dissipation, it suffices to consider the simplified transport equation

$$\begin{cases} \partial_t f + u \cdot \nabla f = 0 \\ f(x,0) = f_0(x) \end{cases}$$

4.1 Preliminary Observations

We consider the candidate velocity field

$$u = \begin{pmatrix} M\sin(y+\alpha t) \\ \sin(x) \end{pmatrix}$$

for some fixed $\alpha > 1$. We consider time periodic velocity fields since periodicity still gives a notion of regularity we can use to analyze. By parameterizing the variables x, y as $X = x, Y = y + \alpha t$, we can write

$$X' = x', Y' = y' + \alpha$$

This allows us to write the change of variables

$$u = \begin{pmatrix} M\sin(Y) \\ \alpha + \sin(X) \end{pmatrix}$$

which is now time independent. We calculate the stream function to be

$$\psi(X, Y) = -M\cos(Y) - \alpha X + \cos(X)$$

and it follows that

$$\frac{\partial}{\partial t}\bigg(- M\cos(Y) - \alpha X + \cos(X)\bigg) = 0$$

We note that this is a valid characterization of almost every point in $\mathbb{T}^2$ since $\nabla\psi \neq 0$ anywhere as $\alpha > 1$. By Lemma 3.4, we then have that $\mu(R) = 1$. Redoing the change of variables and setting initial conditions at $t = 0$, we have

$$-M\cos(y+\alpha t) - \alpha x + \cos(x) = -M\cos(y_0) - \alpha x_0 + \cos(x_0)$$

Rearranging and solving for x, we have that

$$\alpha x - \cos(x) = -M\cos(y+\alpha t) + M\cos(y_0) + \alpha x_0 - \cos(x_0)$$

Since $\alpha > 1$, we know that $\alpha x - \cos(x)$ is invertible. Denote $F(x) = \alpha x - \cos(x)$ and the initial conditions as $C_0 := M\cos(y_0) + \alpha x_0 - \cos(x_0)$. It follows that

$$\begin{aligned} x &= F^{-1}(-M\cos(y+\alpha t) + C_0) \\ y' &= \sin(x) = \sin(F^{-1}(-M\cos(y+\alpha t) + C_0)) \end{aligned}$$

From these computations, we have decoupled the two variables and can now provide better characterizations for their behaviors.

Lemma 4.1. *y can be expressed as the sum of a linear and a periodic function.*

Proof. We perform the change of variables back to (X, Y), we have that

$$Y' - \alpha = \sin(F^{-1}(-M\cos(Y) + C_0))$$

where the constant terms are likewise converted. Isolate Y' and denote the right hand side as

$$g(z) = \sin(F^{-1}(-M\cos(z) + C_0)) + \alpha$$

We note that g is 2π periodic and thus if consider any $s \in (0, 2\pi)$

$$T = \int_0^{2\pi} \frac{dz}{g(z)} = \int_s^{2\pi+s} \frac{dz}{g(z)} = \int_{Y^{-1}(s)}^{Y^{-1}(s+2\pi)} \frac{Y'(t)dt}{g(Y(t))} = Y^{-1}(s+2\pi) - Y^{-1}(s)$$

since we know $g(Y) = Y'$ by definition. If we assume $Y(t) = s$, we have that

$$\begin{aligned} T + Y^{-1}(s) &= Y^{-1}(s+2\pi) \\ Y(T + T^{-1}(s)) &= s + 2\pi \\ Y(T+t) &= Y(t) + 2\pi \end{aligned}$$

which implies by making the change of variables back to (x, y) that

$$y(t+T) - y(t) = 2\pi - \alpha T$$

for all x. This implies that $y = (\frac{2\pi}{T} - \alpha)t + H(t)$ where H is a T periodic function. $\square$

Lemma 4.2. *x is T periodic.*

Proof. Inputting the general equation of y into x, we see that

$$\begin{aligned} x(t) &= F^{-1}(-M\cos(y + \alpha t) + C_0) \\ &= F^{-1}\Big(-M\cos\Big(\frac{2\pi t}{T} + H(t)\Big) + C_0\Big) \\ x(t+T) &= F^{-1}\Big(-M\cos\Big(\frac{2\pi(t+T)}{T} + H(t+T)\Big) + C_0\Big) \\ &= F^{-1}\Big(-M\cos\Big(\frac{2\pi t}{T} + 2\pi + H(t)\Big) + C_0\Big) = x(t) \end{aligned}$$

$\square$

4.2 H^1 Growth of Solution

Showing the H^1 growth of the solution map f^t is an important step in proving the balanced growth criterion, which will be elaborated upon in future sections. Moreover, it is a consequence if u satisfies polynomial mixing.

Theorem 4.3. *Assume that $f_0 \in L^2(T^2, \mu)$ is C^2 and Φ is the flow map corresponding to $u = \begin{pmatrix} M\sin(y+\alpha t) & \sin(x) \end{pmatrix}^T$. Then, there exists $c > 0$ such that*

$$\|f^t\|_{H^1} \geq ct$$

where $f^t : \mathbb{T}^2 \to \mathbb{R}$ is taken to be the solution map $f^t = f_0 \circ \Phi^{-t}$.

Proof. We recall that we can write the solution

$$f((x, y), t) = f_0 \circ \Phi^{-1}((x, y), t)$$

for given initial conditions f_0. By definition, we have that

$$\|f^t\|_{H^1} = \|f^t\|_{L^2} + \|\nabla f^t\|_{L^2}$$

Since the former term stays constant for all finite t, it suffices to consider the latter. By chain rule, we have

$$\nabla f^t = \nabla f_0 \nabla \Phi^{-1}$$

We can consider the individual components of $\nabla\Phi$ and write

$$\nabla\Phi = \begin{pmatrix} \frac{\partial x(t)}{\partial x_0} & \frac{\partial x(t)}{\partial y_0} \\ \frac{\partial y(t)}{\partial x_0} & \frac{\partial y(t)}{\partial y_0} \end{pmatrix}$$

We consider both $x((x_0, y_0), t)$ and $y((x_0, y_0), t)$ and we evaluate both at times $t = nT$. We then have

$$\begin{aligned} x((x_0, y_0), nT) &= F^{-1}(-M\cos(2n\pi + H(0)) + C_0) \\ y((x_0, y_0), nT) &= 2n\pi - n\alpha T + H(0) \end{aligned}$$

The only function whose derivative will grow as time tends towards infinity is that of y. We recall that

$$C_0 = M\cos(y_0) + \alpha x_0 - \cos(x_0)$$

and perform the computations below:

$$\begin{aligned}
\frac{\partial T}{\partial x_0} &= \frac{\partial}{\partial x_0}\int_0^{2\pi} \frac{dz}{\sin(F^{-1}(-M\cos(z)+C_0))+\alpha} \\
&= \int_0^{2\pi} \frac{-K^1 dz}{(\sin(F^{-1}(-M\cos(z)+C_0))+\alpha)^2} \\
K^1 &= \frac{\partial}{\partial x_0}\sin(F^{-1}(-M\cos(z)+C_0))+\alpha \\
&= \cos(F^{-1}(-M\cos(z)+C_0))\cdot K^2 \\
K^2 &= \frac{\partial}{\partial x_0}F^{-1}(-M\cos(z)+C_0)) \\
&= \frac{1}{\alpha+\sin(F^{-1}(-M\cos(z)+C_0))}\cdot K^3 \\
K^3 &= \frac{\partial}{\partial x_0}(-M\cos(z)+C_0) \\
&= \alpha+\sin(x_0)
\end{aligned}$$

which implies that

$$\begin{aligned}
\frac{\partial T}{\partial x_0} &= \int_0^{2\pi} \frac{-\cos(F^{-1}(-M\cos(z)+C_0))(\alpha+\sin(x_0))}{(\sin(F^{-1}(-M\cos(z)+C_0))+\alpha)^3}\,dz \\
\frac{\partial T}{\partial y_0} &= \int_0^{2\pi} \frac{\cos(F^{-1}(-M\cos(z)+C_0))(M\sin(y_0))}{(\sin(F^{-1}(-M\cos(z)+C_0))+\alpha)^3}\,dz
\end{aligned}$$

By plugging these values into the function y, we see that

$$\begin{aligned}
\frac{\partial y}{\partial x_0}((x_0,y_0),nT) &= -n\alpha\int_0^{2\pi} \frac{-\cos(F^{-1}(-M\cos(z)+C_0))(\alpha+\sin(x_0))}{(\sin(F^{-1}(-M\cos(z)+C_0))+\alpha)^3}\,dz + \frac{\partial H}{\partial x_0}(0) \\
\frac{\partial y}{\partial y_0}((x_0,y_0),nT) &= -n\alpha\int_0^{2\pi} \frac{\cos(F^{-1}(-M\cos(z)+C_0))(M\sin(y_0))}{(\sin(F^{-1}(-M\cos(z)+C_0))+\alpha)^3}\,dz + \frac{\partial H}{\partial y_0}(0)
\end{aligned}$$

Since H is periodic, its derivative is bounded. We then consider the full C^1 derivative, given by

$$\nabla f^t = \begin{pmatrix}\partial_1 f_0 & \partial_2 f_0\end{pmatrix}\begin{pmatrix}\frac{\partial}{\partial y_0}y(t) & -\frac{\partial}{\partial y_0}x(t) \\ -\frac{\partial}{\partial x_0}y(t) & \frac{\partial}{\partial x_0}x(t)\end{pmatrix} = \begin{pmatrix}\partial_1 f_0\frac{\partial}{\partial y_0}y(t)-\partial_2 f_0\frac{\partial}{\partial x_0}y(t) & -\partial_1 f_0\frac{\partial}{\partial y_0}x(t)+\partial_2 f_0\frac{\partial}{\partial x_0}x(t)\end{pmatrix}$$

The second term is bounded, and so it suffices to show the growth of the first term. By considering $t = nT$ for integer n, the left hand side can be simplified down to

$$-n\alpha((M\sin(y_0))\partial_1 f_0+(\alpha+\sin(x_0))\partial_2 f_0)\int_0^{2\pi}\frac{\cos(F^{-1}(-M\cos(z)+C_0))}{(\sin(F^{-1}(-M\cos(z)+C_0))+\alpha)^3}\,dz + \partial_1 f_0\frac{\partial H}{\partial y_0}(0) - \partial_2 f_0\frac{\partial H}{\partial x_0}(0)$$

The constant terms at the end do not matter for growth. It suffices to show that there exist choices for (x_0, y_0) such that integral and other factor does not vanish. We consider taking α large. Since $F(x) = \cos x - \alpha x$, intuitively when α is large, we have that $F^{-1}(y) = -y/\alpha$. Let us fix some $\epsilon > 0$ and choose α_1 such that $\frac{M}{\alpha} < \epsilon$. Likewise, choose α_2 such

$$\left|F^{-1}(y)+\frac{y}{\alpha}\right| < \epsilon$$

Take $\alpha = \max(\alpha_1, \alpha_2)$. We first evaluate the term inside the sin and cos. We see that

$$\left|F^{-1}(-M\cos z+C_0)+\frac{-M\cos z+M\cos y_0+\alpha x_0-\cos x_0}{\alpha}\right| < \epsilon$$

which implies that

$$F^{-1}(-M\cos z+C_0) = -x_0+4k_z\epsilon$$

for $k_z \in [0,1)$. We can then write

$$\sin(F^{-1}(-M\cos z + C_0)) = \sin(-x_0 + 4k_z\epsilon) = -\sin(x_0)\cos(4k_z\epsilon) + \cos(x_0)\sin(4k_z\epsilon)$$
$$\cos(F^{-1}(-M\cos z + C_0)) = \cos(-x_0 + 4k_z\epsilon) = \cos(x_0)\cos(4k_z\epsilon) + \sin(x_0)\sin(4k_z\epsilon)$$

Since ϵ is small, we can approximate $\sin(\epsilon) \approx \epsilon$ and $\cos(\epsilon) \approx 1 - \epsilon^2$. We see that the integrand becomes, letting $\epsilon_z = 4k_z\epsilon$

$$\frac{\cos(x_0) - \epsilon_z^2\cos(x_0) + \sin(x_0)\epsilon_z + o(\epsilon_z^3)}{(-\sin(x_0) + \epsilon_z^2\sin(x_0) + \cos(x_0)\epsilon_z + \alpha + o(\epsilon_z^3))^3}$$

Choosing any x_0 such that $\cos(x_0) \neq 0$ allows the integrand to tend towards $\frac{\cos(x_0)}{(\alpha - \sin(x_0))^3}$. Next, we argue that the first factor

$$M\sin(y_0)\partial_1 f_0 + (\alpha + \sin(x_0))\partial_2 f_0$$

is never equivalent to zero. Suppose for contradiction that there exists a $f_0 \in C^2(\mathbb{T}^2)$ such that

$$\nabla f_0(x_0, y_0) \cdot \begin{pmatrix} M\sin(y_0) \\ \alpha + \sin(x_0) \end{pmatrix} = 0$$

for all $(x_0, y_0) \in \mathbb{T}^2$. Then, since the set of vectors orthogonal to ∇f_0 has dimension 1, we can assume that

$$\nabla^\perp f_0 = \begin{pmatrix} \partial_y f_0 \\ -\partial_x f_0 \end{pmatrix} = \kappa \begin{pmatrix} M\sin(y_0) \\ \alpha + \sin(x_0) \end{pmatrix}$$

for some $\kappa \in \mathbb{R}$. A solution to this is given by $f_0 = -M\cos(y_0) - \alpha x + \cos(x_0)$. However, this f_0 is not periodic in 2π, due to the linear term, and thus any constant multiple of f_0 will not be 2π periodic either, leading to a contradiction.

Let us denote $p(x_0, y_0) := M\sin(y_0)\partial_1 f_0 + (\alpha + \sin(x_0))\partial_2 f_0$. By our analysis, $\mu(\{(x_0, y_0) \mid p(x_0, y_0) \neq 0\}) > 0$. So, when we integrate over the partial derivative $\partial_1 f^t$, we evaluate

$$\int_{\mathbb{T}^2} \left| -n\alpha p(x_0, y_0) \int_0^{2\pi} \frac{\cos(F^{-1}(-M\cos(z) + C_0))}{(\sin(F^{-1}(-M\cos(z) + C_0)) + \alpha)^3}\, dz \right|^2 \geq nC$$

for some constant C depending on f_0. □

4.3 H^2 Growth of Solution

To show balanced growth, we also want to give bounds on the H^2 norm. In particular, we want to show it grows no faster than quadratically.

Theorem 4.4. *Assume that $f_0 \in L^2(\mathbb{T}^2, \mu)$ is C^2 and Φ is the flow map corresponding to $u = \begin{pmatrix} M\sin(y + \alpha t) & \sin(x) \end{pmatrix}^T$. Then, there exists $c > 0$ such that*

$$\|f^t\|_{H^1} \leq ct^2$$

where $f^t : \mathbb{T}^2 \to \mathbb{R}$ is taken to be the solution map $f^t = f_0 \circ \Phi^{-t}$.

Proof. For the H^2 norm, we need to consider all the partial second derivatives. As such, the main difficulty is writing out the chain rule terms.

$$\frac{\partial^2 f}{\partial x_0^2} = \partial_1^2(f_0)\Big(\frac{\partial y}{\partial y_0}\Big)^2 - 2(\partial_{1,2}^2 f_0)\Big(\frac{\partial y}{\partial y_0}\frac{\partial y}{\partial x_0}\Big) + \partial_2^2(f_0)\Big(\frac{\partial y}{\partial x_0}\Big)^2 + \partial_1 f_0 \frac{\partial^2 y}{\partial x_0 \partial y_0} - \partial_2 f_0 \frac{\partial^2 y}{\partial x_0^2}$$
$$\frac{\partial^2 f}{\partial y_0^2} = \partial_1^2(f_0)\Big(\frac{\partial x}{\partial y_0}\Big)^2 - 2(\partial_{1,2}^2 f_0)\Big(\frac{\partial x}{\partial y_0}\frac{\partial x}{\partial x_0}\Big) + \partial_2^2(f_0)\Big(\frac{\partial x}{\partial x_0}\Big)^2 - \partial_1 f_0 \frac{\partial^2 x}{\partial y_0^2} + \partial_2 f_0 \frac{\partial^2 x}{\partial x_0 \partial y_0}$$
$$\frac{\partial^2 f}{\partial x_0 \partial y_0} = -\partial_1^2 f_0\Big(\frac{\partial x}{\partial y_0}\frac{\partial y}{\partial y_0}\Big) + \partial_{1,2}^2\Big(\frac{\partial x}{\partial y_0}\frac{\partial y}{\partial x_0} + \frac{\partial x}{\partial x_0}\frac{\partial y}{\partial y_0}\Big) - \partial_2^2 f_0\Big(\frac{\partial x}{\partial x_0}\frac{\partial y}{\partial x_0}\Big) - \partial_1 f_0 \frac{\partial^2 x}{\partial x_0 \partial y_0} + \partial_2 f_0\Big(\frac{\partial^2 x}{\partial x_0^2}\Big)$$

We recall that the only two partial derivatives that grew with respect to time were $\frac{\partial y}{\partial x_0}, \frac{\partial y}{\partial y_0}$. We note that further second derivatives of y will not produce anything higher than linear terms, since the $\frac{\partial T}{\partial x_0}, \frac{\partial T}{\partial y_0}$ are not responsible for the growth of the H^1 norm. Moreover, we see that the only partial derivatives that grow only appear with order at most two, which bounds the growth of the H^2 norm by a quadratic factor. □

4.4 Balanced Growth

With the two lemmas above, we prove a third that uses the growth bounds we established. This lemma simply establishes a bound on how the energy in the lower Fourier modes tends towards higher frequencies. For any initial conditions, we are guaranteed that a fixed proportion of the energy in the L^2 norm wanders away.

Lemma 4.5 (Balanced Growth)**.** *Let $C > 1$ be a positive constant and f^t be a sequence of $H^2(\mathbb{T}^2)$ functions such that*

$$||f^t||_{L^2}||f^t||_{H^2} \leq C||f^t||_{H^1}^2$$

Then, taking $N(t) = \frac{1}{\sqrt{2C}}\frac{||f^t||_{H^1}}{||f^t||_{L^2}}$, we have

$$||\mathbf{P}_{>N(t)}f^t||_{L^2} \geq \frac{||f^t||_{L^2}}{2C}$$

Proof. We start from the homogeneous $\dot{H}^1$ norm of f^t.

$$||f^t||_{H^1}^2 = \sum_{k\in\mathbb{Z}^2}|k|^2|\hat{f}^t(k)|^2 = \sum_{|k|\leq N}|k|^2|\hat{f}^t(k)|^2 + \sum_{|k|>N}|k|^2|\hat{f}^t(k)|^2$$

We can bound the former sum by N^2 and upper bound the other coefficients by the L^2 norm. For the latter sum, we use Cauchy-Schwarz splitting the product as $|k|^2|\hat{f}^t(k)|^2 = |k|^2|\hat{f}^t(k)|\cdot|\hat{f}^t(k)|$. The next bounds come naturally as definitions of the H^2 and L^2 norms.

$$||f^t||_{H^1}^2 \leq N^2||f^t||_{L^2}^2 + \sqrt{\sum_{|k|>N}|k|^4|\hat{f}^t(k)|^2}\sqrt{\sum_{|k|>N}|\hat{f}^t(k)|^2} \leq N^2||f^t||_{L^2}^2 + ||f^t||_{H^2}||\mathbf{P}_{>N(t)}f^t||_{L^2}$$

The last few steps use the definition of N, as well as the fact that $C > 1$, and combining terms to obtain the desired bound.

$$\begin{aligned} ||f^t||_{H^1}^2 &\leq \frac{1}{2C}||f^t||_{H^1}^2 + \frac{C||f^t||_{H^1}^2}{||f^t||_{L^2}}||\mathbf{P}_{>N(t)}f^t||_{L^2} = ||f^t||_{H^1}^2\Big(\frac{1}{2} + \frac{C}{||f^t||_{L^2}}||\mathbf{P}_{>N(t)}f^t||_{L^2}\Big) \\ \frac{||f^t||_{L^2}}{2C} &\leq ||\mathbf{P}_{>N(t)}f^t||_{L^2} \end{aligned}$$

□

Lemma 4.6 (Enhanced Dissipation)**.** *If $f^t \in H^2(T^2)$ fulfills the balanced growth criterion, and $f^t = f_0 \circ \Phi^{-t}$ is the solution to advection diffusion equation, then the system $(\mathbb{T}^2, \Sigma, \mu, \Phi)$ is dissipation enhancing with rate $\delta(\nu) = \nu^{\frac{1}{3}}$*

From this lemma and our previous results, we see that for any initial data $f_0 \in C^2(\mathbb{T}^2)$, the sequence given by $f_0 \circ \Phi^{-t}$ satisfies Lemma 4.5. Thus, the enhanced dissipation of the velocity field is proved.

5. Future Work

A direction of future work would be to extend these results to show mixing of the system. With this construction, if the system is shown to be mixing, it would be the first instance of a smooth velocity field exhibiting polynomial mixing. An important step in proving mixing would be to show that the defined flow is ergodic. This would allow for a characterization of the set of weak limits in the sequences $\{f^t\}_{t=0}^{\infty}$.

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A. Proofs of Lemmas

We prove the Lemmas 3.1, 3.3, 3.4 We restate the lemmas here for convenience.

Lemma (Liouville Lemma)**.** *Let* $\mathbf{u}$ *be a velocity field, and* Φ *be its corresponding characteristic map. Let* μ *denote the Lebesgue measure and let* $A \subset \Omega$ *be an arbitrary measurable set. Then, for all* t*, we have*

$$\mu(\Phi^{-1}(A,t)) = \mu(A)$$

Proof. When $f_0 \in C^1$, we know that $f = f_0 \circ \Phi^{-1}$ is solution to the transport equation. As such we integrate over the transport equation to get

$$0 = \int_\Omega \partial_t f + \int_\Omega \mathbf{u} \cdot \nabla f = \frac{d}{dt}\int_\Omega f + \int_\Omega \mathrm{div}(\mathbf{u} f)$$

The latter term vanishes by the divergence theorem

$$\int_\Omega \mathrm{div}(\mathbf{u} f) = \int_{\partial\Omega} f\,(\mathbf{u}\cdot n) = 0$$

by our assumptions of boundary conditions. This leaves us with

$$0 = \frac{d}{dt}\int_\Omega f(x,t) = \frac{d}{dt}\int_\Omega f_0(\Phi^{-1}(x,t))$$

In particular, we thus have

$$\int_\Omega f_0(\Phi^{-1}(x,t)) = \int_\Omega f_0(\Phi^{-1}(x,0)) = \int_\Omega f_0(x)$$

Now, let A be any measurable function, and let $\{f_k\}$ be a series of C^1 functions that converge pointwise to $\mathbb{1}_A$ and supported on A. Fix some $\epsilon > 0$. By Egorov's Theorem, there exists some $B \subset A$ with $\mu(B) < \frac{\epsilon}{4}$ such that $\{f_k\}$ converges uniformly to $\mathbb{1}_A$ on $A \setminus B$. It follows then that

$$\lim_{k\to\infty}\int_\Omega f_k(x) = \lim_{k\to\infty}\int_A f_k(x) = \lim_{k\to\infty}\int_{A\setminus B} f_k(x) + \lim_{k\to\infty}\int_B f_k(x) = \int_{A\setminus B}\lim_{k\to\infty} f_k(x) + \lim_{k\to\infty}\int_B f_k(x) = \mu(A\setminus B) + \lim_{k\to\infty}\int_B f_k(x)$$

It follows then that

$$\left|\mu(A) - \lim_{k\to\infty}\int_\Omega f_k(x)\right| = \left|\int_\Omega \lim_{k\to\infty} f_k(x) - \lim_{k\to\infty}\int_\Omega f_k(x)\right| < \frac{\epsilon}{2}$$

since we can simply upperbound the f_k by 1. A similar argument gives

$$\left|\mu(\Phi^{-1}A) - \lim_{k\to\infty}\int_\Omega f_k(\Phi^{-1}(x))\right| = \left|\int_\Omega \lim_{k\to\infty} f_k(\Phi^{-1}(x)) - \lim_{k\to\infty}\int_\Omega f_k(\Phi^{-1}(x))\right| < \frac{\epsilon}{2}$$

Since by our above argument, $\lim_{k\to\infty}\int_\Omega f_k(x) = \lim_{k\to\infty}\int_\Omega f_k(\Phi^{-1}(x)$, we have that $|\mu(A) - \mu(\Phi^{-1}(A))| < \epsilon$. This proves the lemma. □

Lemma. *For every* $x \in R$*, there exists a unique closed* C^1 *Jordan curve* Γ_x *such that for all* $y \in \Gamma_x$*,* $\psi(x) = \psi(y)$*.*

Proof. Let $x \in R$ be arbitrary. Let us define $\gamma_x : \mathbb{R} \to \Omega$ by

$$\partial_t \gamma_x(t) = \nabla^\perp \psi(\gamma_x(t)) \qquad \gamma_x(0) = x$$

which exists since $\nabla^\perp\psi$ is uniformly bounded on a compact set and thus has bounded derivative. We see that on the image of each γ

$$\frac{d}{dt}\psi(\gamma_x(t)) = \partial_t\gamma_x(t)\cdot\nabla\psi(\gamma_x(t)) = 0$$

Moreover, given the sequence of $\{\gamma_x(n)\}_{n\in\mathbb{N}}$, there exists some convergent subsequence, denoted $\gamma_x(t_n) \to y \in \Omega$. By continuity, it follows that $\psi(x) = \psi(y)$. To show that Γ_x is closed, it suffices to show that γ is time periodic. For all $s, t \in \mathbb{R}$, we have that

$$\gamma_{\gamma_x(s)}(t) = \gamma_x(s+t)$$

Then, by the implicit function theorem, since $\nabla\psi(y) \neq 0$, there exists a neighborhood such that part of the level set of ψ containing y is given by $\gamma_y(t), t \in (-\epsilon, \epsilon)$, and we can choose an appropriate scale such that $\epsilon < \frac{1}{4}$. By convergence, there exists some large n such that $\gamma_x(t_n) = \gamma_y(\epsilon_n)$ for some $\epsilon_n \in (-\epsilon, \epsilon)$. It follows that there exists $\epsilon_n \in (-\epsilon, \epsilon)$ such that

$$\gamma_x(t_n - \epsilon_n) = \gamma_{\gamma_x(t_n)}(-\epsilon_n) = \gamma_{\gamma_y(\epsilon_n)}(-\epsilon_n) = y$$

for all n sufficiently large. Thus, it follows that x has a periodic of $t_{n+1} - t_n + \epsilon_n - \epsilon_{n+1} > \frac{1}{2} > 0$, which is a non-trivial period. □

Lemma. *Let $\psi \in C^2(\Omega, \mathbb{R})$ be fixed. Let us characterize the points in Ω. Let us define the following sets:*

$$\begin{aligned} C &= \{x \in \Omega \mid \nabla\psi(x) = 0\} \\ R &= \{x \in \Omega \mid \psi(x) \notin \psi(C)\} \\ N &= \Omega \setminus (C \cup R) \end{aligned}$$

Then, $\mu(N) = 0$

We build up to the proof with a series of propositions.

Proposition. *For all $x \in N$, there does not exist an $\epsilon > 0$ such that $D(x, \epsilon) \subset N$.*

Suppose such an $x \in N$ existed. By Sard's Theorem we have that $m(\psi(D(x, \epsilon))) < m(N \cup C) = 0$. Thus we have that the image of the disk has Lebesgue measure zero. Since ψ is C^2, it follows that $\psi(D(x, \epsilon)) = k$ for some $k \in \mathbb{R}$. However, this implies that $\nabla\psi(x) = 0$, which contradicts $x \in N$.

Proposition. *For each $x \in N$, there exists some simple curve $\gamma_x^U : (-\epsilon, \epsilon) \to \Omega$ such that $\gamma_x(0) = x$ and $U \subset \Omega$ is open such that $\gamma_x((-\epsilon, \epsilon)) = U \cap N$.*

For each $x \in N$, we know that $\nabla\psi(x) \neq 0$. Let us consider the function $\psi_x : \Omega \to \mathbb{R}$ given by $\psi_x(y) = \psi(y) - \psi(x)$. By the implicit function theorem, there exists some neighborhood U about $\psi(x)$ such that we can express the level set of $\psi_x = 0$ as a function of x_1 (if $\partial\psi_x/\partial x_2(x) \neq 0$) or of x_2 (if $\partial\psi_x/\partial x_1(x) \neq 0$), one of which is bound to occur since $\nabla\psi_x = \nabla\psi \neq 0$.

Let us induce an equivalence relation on the points in N. We say two points, x, y, are equivalent if the images of γ_x^U and γ_y^V intersect for some open neighborhoods U, V. The equivalence classes induced by this relation are essentially one dimensional curves in Ω. Let us denote the set of these curves by $\{K_\alpha\}_{\alpha \in I}$ for some index set I.

Proposition. *Each K_α is a simple curve in Ω. Moreover, ψ is constant on each K_α.*

Suppose not, and the curve intersected itself at some $y \in \Omega$. We first note that each K_α is contained within a level set of ψ by continuity. Then, if we consider the directional derivatives at y, we note there are two independent directions where the directional derivatives are both zero, implying $\nabla\psi(y) = 0$, a contradiction.

The second property holds by an application of Sard's Theorem.

Proposition. $m(K_\alpha) = 0$ *for all $\alpha \in I$.*

K_α is a one-dimension submanifold embedded in Ω. Each of the maps are C^1 by the implicit function theorem, and thus their composition is also C^1 as the function has a well defined inverse.

In order to ensure that the critical points are separate from the points in N, we construct a sequence of balls with radii converging to zero around each critical point and show each residual set has measure zero, eventually passing to the limit. In particular, let C_δ be the set $\{x \in \Omega \mid d(x, y) < \delta \mid \forall y \in C\}$, and let $N_\delta := N \cap C_\delta^c$. It suffices to show each of the N_δ have measure zero.

Proposition. *On N_δ, each point satisfies $|\nabla\psi| \geq c\delta$ for some positive constant c. Moreover, N_δ is compact.*

Suppose for contradiction that this c does not exist. Then, let $c_1 = 1$ and let $x_1 \in N_\delta$ be the point such that $|\nabla\psi(x_1)| < c_1\delta$. Then, we recursively define $c_n = \min(\frac{1}{n}, |\nabla\psi(x_{n-1})|)$, and $x_n \in N_\delta$ to be the response to the c_n argument. We consider a convergent subsequence of the $\{x_n\}_{n=1}^\infty$, denoted $\{y_n\}$, which converges to some y. Since $|\nabla\psi|$ is continuous, it follows that $|\nabla\psi(y_n)| \to |\nabla\psi(y)|$ as well. Since the sequence is bounded by a convergent sequence in the reals, we have $|\nabla\psi(y)| = 0$, implying $y \in C$. However, this implies that there exists some N such that for all $n > N$, we have $d(y_n, y) < \delta$, contradicting that $\{y_n\} \subset N_\delta$.

To show the set is compact, we consider any convergent sequence $\{x_n\} \subset N_\delta$ that converges to some x. Since $N \cup C$ is closed, we know that $x \in N \cup C$. However, $N_\delta \cap C = \emptyset$, and moreover, x must satisfy $|\nabla\psi(x)| \geq c\delta$, thus $x \in N_\delta$. N_δ, the closed subset of a compact space, is thus compact. Now, we have all the details we need to finish the proof of our desired claim.

Proof. We consider N as the limit of the N_δ. Intuitively, wish to think of N_δ as $\psi(C) \times [0, 1]$. In order to accomplish this, we perform local coordinate transforms leveraging the compactness of N_δ.

Fix some positive $\delta > 0$. At each $x \in N_\delta$, there exists an open neighborhood where ψ can be expressed as a one variable function by the implicit function theorem. Since Ω is simply connected and the curl of the velocity field given by $u = (\partial_y\psi, -\partial_x\psi)$ is zero, it follows that u is a conservative vector field and can be written as $u = (\partial_y\psi, -\partial_x\psi) = (\partial_x\varphi, \partial_y\varphi)$ for some C^2 function $\varphi : \Omega \to \mathbb{R}$. φ can essentially be thought of as flow map of the vector field perpendicular to $\nabla^\perp\psi$. By the implicit function theorem, there also exists some neighborhood where φ can be expressed as a one variable function. For each x, denote the intersubsection of these two neighborhoods as U_x. The union of the U_x cover N_δ, and we choose a finite subcover given by $\{U_n\}_{n=1}^N$.

It suffices to show that $m(U_n) = 0$ for all n. We perform a coordinate transform with x, y to r, θ by the following:

$$r = \psi(x, y) \qquad \theta = \varphi(x, y)$$

and so r is constant on each level set K_α while θ is constant on each level set of φ. By the implicit function theorem theorem, we know that ψ and φ are invertible on U_x, thus we can solve $x = a(r, \theta), y = b(r, \theta)$ for some C^1 functions $a, b : \mathbb{R} \to \mathbb{R}$. Under this coordinate transform, we have that U_x is C^1 diffeomorphic to $\psi(U_x) \times [\alpha, \beta]$ where $[\alpha, \beta]$ is the image U_x under φ. Since $\psi(U_x) \subset \psi(C)$ which has measure zero, it follows that since $\psi(U_x)$ and $\varphi(U_x)$ are both measurable sets, U_x is diffeomorphic to a set of measure zero. Any diffeomorphism is Lipschitz, which stretches the measure of the image by a constant factor, thus $m(U_x) = 0$. It follows that N_δ, covered by finitely many U_x, has measure zero.

Lastly, choose any sequence δ_n positive and converging to zero. We have that $m(N_{\delta_n}) = 0$ for all n, and thus in the limit $m(N) = 0$. □