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COMPLEX CONTACT THREEFOLDS AND THEIR CONTACT CURVES

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§0. Introduction

Let X be a $(2n+1)$-dimensional compact complex manifold. A contact structure on X is a line subbundle L of Ω^1_X such that if θ is a local section of L, then $\theta \wedge (d\theta)^n \neq 0$. This in particular implies the canonical bundle $K_X \cong (n+1)L$. A submanifold $C \subset X$ is called a contact submanifold if all local sections of L vanishes on C. One example of complex contact manifold is a odd-dimensional projective spaces. The present work was initiated by two papers of Bryant. In [Br1] using Penrose's twistor transform $\mathbb{CP}^3 \xrightarrow{T} S^4$, he showed that, among other things, every superminimal Riemann surface can be realized by a smooth contact curves in $\mathbb{CP}^3$. Therefore to study superminimal Riemann surfaces in S^4 is the same as to study contact curves in $\mathbb{CP}^3$. In another paper, Bryant [Br2] studied relationships between manifolds with "exotic" holonomies and certain contact rational curves on a complex contact 3-folds. He showed that the real slice of the moduli space of certain contact rational curves in a complex contact threefold has a so-called "exotic" $\mathcal{G}_3$-structure. Thus he found a missing group from Berger's list of the possible pseudo-Riemannian holonomies which acts irreducibly on each tangent space.

The purpose of this paper is to systematically study complex contact threefolds and contact curves on them. The first main result of this paper is a classification of projective contact threefolds. Using Mori's theory of extremal ray we show that the types of complex *projective* contact threefolds are very limited (Theorem 1.6 and 1.8). The second result is about the moduli space of contact curves on a complex contact threefold (Theorem 2.3). We calculate its Zariski tangent space and the space of obstructions. This result generalizes a theorem of Bryant [Br2], which dealts with only rational contact curves. Finally we study contact curves in $\mathbb{CP}^3$ and obtain a Plücker type formula (Theorem 3.2) for contact curves in $\mathbb{CP}^3$. It was predicted in [Br1] that such a formula should exist.

The paper is organized as follow. There will be three sections. In the beginning of the first section, we will study some general properties of a complex contact threefold. The

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second half will be devoted to the classifications of projective contact threefolds. We will start the second section with an infinitesimal study of the moduli space of contact curves. We will relate its normal bundle to the first prolongation of the dual of the contact bundle. At the end of the second section, we will give some applications of the main theorem in that section. The last section is devoted to studies of contact curves in $\mathbb{CP}^3$. In that section, we have to deal with *singular* contact curves. We obtain a Plücker type formula for contact curves in $\mathbb{CP}^3$. Towards the end of that section we will relate contact geometry in $\mathbb{CP}^3$ to the geometry of the moduli space of rank-two stable vector bundles with $c_1 = 0$ and $c_2 = 1$. We will also pose a question on irreducibility of moduli space of contact curves and speculate how results in the present paper can be generalized to higher-dimensional contact manifolds.

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§1. Compact Complex Contact Threefolds

This section is divided into two parts. In the first part we will study general contact threefolds even including non-projective, non-Kähler ones. The second part deal specifically with projective contact threefolds.

General compact complex contact threefolds.

From now on, unless specifically stated, X will be a contact 3-fold and L is the contact line. Sometimes use a pair (X, L) to mean the same thing. In particular we have $K_X = 2L$. Let us look at some examples of contact threefolds and their contact curves.

Example 1. Let M be a compact complex surface. T_M be its tangent bundle. Then the projectived bundles $\mathbb{P}(T_M)$ a contact structure. A word of caution. By $\mathbb{P}(T_M)$ we mean $T^*_M \setminus \{0\}/\mathbb{C}^*$ instead of $T_M \setminus \{0\}/\mathbb{C}^*$. We can write down its local contact one forms explicitly. Let (x, y) be a local coordinate system on M and λ be a fiber coordinate. Then $dy + \lambda\, dx$ is a local contact one form for $\mathbb{P}(T_M)$. In fact these local contact forms can be glued together to form a contact line bundle, which is the dual of the tautological bundle of $\mathbb{P}(T_M)$. The contact curves are either fibers of the projection of π, or horizontal lifting of curves in M. Since $\Omega^1_M \cong T_M \otimes K_M$, the projectived cotangent bundle $\mathbb{P}(\Omega^1_M) \cong \mathbb{P}(T_M)$. Therefore $\mathbb{P}(\Omega^1_M)$ also have a contact structure. It is easy to see that its contact line bundle is the dual of the tautological line bundle tensored with the pul-back of K_M. The following proposition suggests the contact structure on $\mathbb{P}(T_M)$ is essentially unique.

Proposition 1.1. *For $\mathbb{P}(T_M)$ the moduli space of is $\mathbb{P}\left(GL(T_M)\right)$, where $GL(T_M)$ consists of invertible endmorphisms of T_M. In particular, if T_M is simple (for example if M is $\mathbb{CP}^2$ or a $K3$ surface), then the contact structure is unique.*

Proof. Let $X = \mathbb{P}(T_M)$ and $\mathcal{O}_X(1)$ be the tautological line bundle on X. First of all we have the following standard short exact sequences:

$$0 \longrightarrow \pi^*\Omega^1_M(1) \longrightarrow \Omega^1_X(1) \longrightarrow \Omega^1_{X/M}(1) \longrightarrow 0 \tag{1.1}$$

$$0 \longrightarrow \Omega^1_{X/M}(1) \longrightarrow \pi^* T_M \longrightarrow \mathcal{O}_X(1) \longrightarrow 0 \tag{1.2}$$

Since $H^0(\mathcal{O}_X(1)) = H^0(T_M)$, (1.2) implies that $H^0(\Omega^1_{X/M}(1)) = 0$. Sequence (1.1) in turn gives:

$$H^0(\Omega^1_X(1)) = H^0(\pi^* \Omega^1_M(1)) = End\,(T_M) \tag{1.3}$$

Now it is clear to see that the space of all contact structures is isomorphic to $GL(T_M)$.

□

Example 2. The complex projective three space $\mathbb{CP}^3$ is a contact threefold with infinite many contact structures. They are all equivalent under the actions of the automorphism group of $\mathbb{CP}^3$. Their associated contact line bundles are the same and equal to $\mathcal{O}_{\mathbb{CP}^3}(-2)$. A contact structure is a injective *bundle* (*not sheaf*) homomorphism φ from $\mathcal{O}_{\mathbb{CP}^3}(-2)$ to $\Omega^1_{\mathbb{CP}^3}$. Then it is clear that $N_\varphi = (\operatorname{coker} \varphi)^*(-1)$ is a rank-two vector bundle on $\mathbb{CP}^3$ with $c_1 = 0$ and $c_2 = 1$. In fact N_φ is stable and is a so-called null-correlation bundle. In [OSS], it is shown that any stable rank-two vector bundle on $\mathbb{CP}^3$ with $(c_1, c_2) = (0, 1)$ is a null-correlation bundle. Moreover two such bundles N_φ and $N_{\varphi'}$ are isomorphic if and only if φ and φ' differ by a non-zero constant. Therefore the moduli space of contact structures on $\mathbb{CP}^3$ and the space of rank-two stable vector bundles in $\mathbb{CP}^3$ with $c_1 = 0$ and $c_2 = 1$ are isomorphic. We will come back to this example at the end of section 3.

Among all the contact structures, there is a distinguished one obtained from the twistor map $\mathbb{CP}^3 \xrightarrow{T} S^4$. The contact forms are perpendicular to the tangent directions of the fibers of T. Bryant [Br1] showed that in the affine coordinate chart given by $[1, z_1, z_2, z_3]$, the local contact form is $\omega = dz_1 - z_3 dz_2 + z_2 dz_3$. Curves which are contact with respect to this contact structure are called horizontal curves by Bryant. Their images in S^4 are so-called *superminimal* surfaces in S^4. Therefore to study curves contact with respect to this contact structure is the same as to study superminimal Riemann surfaces in S^4.

By the definition of contact manifold, to check if a complex 3-fold X is contact we need to find a line subbundle $L \subset \Omega^1_X$ such that $\theta \wedge d\theta \neq 0$ for any non-zero local section θ of L. This in particular implies that $K_X = 2L$. However the *non-integrability* condition $\theta \wedge d\,\theta \neq 0$ is difficult to verify in general. Fortunately, the following theorem tells us except for some very special cases the non-integrability is satisfied automatically if $K_X = 2L$.

Theorem 1.2. *Let X be a compact complex threefold and $L \subset \Omega^1_X$ such that $K_X = L^2$. If L does not define a contact structure on X, then there is a smooth fibration $X \xrightarrow{\varphi} C$ such that*

1. *C is a smooth curve and $L = \varphi^*(K_C)$.*
2. *φ has connected fibers and they all have trivial canonical bundles.*

If moreover X is Kähler, then X is not simply connected and fibers of φ are either $K3$ surfaces or abelian surfaces.

Proof. Choose a local coordinate cover $\{U_\alpha\}_{\alpha \in \Gamma}$ on X such that $L|_{U_\alpha} = \mathcal{O}_{U_\alpha}(\theta_\alpha)$, where θ_α is a local holomorphic one form. Consider the $\mathcal{O}_{U_\alpha}$-module homomorphism

$\psi : L^2 \to K_X$ defined by $\psi(\theta_\alpha^{\otimes 2}) = \theta_\alpha \wedge d\theta_\alpha$. It is clear that ψ is well-defined. Since $K_X = L^2$, ψ is either nowhere zero or identically zero. If ψ is nowhere zero, then L defines a contact structure. If ψ is identically zero, then $\theta_\alpha \wedge d\theta_\alpha = 0$ for any $\alpha \in \Gamma$. We are going to show that in this case X admits a fibration described above.

For any $\alpha \in \Gamma$ let $(x_\alpha, y_\alpha, z_\alpha)$ be a coordinate system. Set $\theta_\alpha = a_\alpha dx_\alpha + b_\alpha dy_\alpha + c_\alpha dz_\alpha$. Then the fact that $\theta_\alpha \wedge d\theta_\alpha = 0$ implies that a_α, b_α and c_α are constant proportional to one another. Hence $\theta_\alpha = g_\alpha(\lambda_\alpha dx_\alpha + \mu_\alpha dy_\alpha + \sigma_\alpha dz_\alpha)$, where g_α is a holomorphic function on U_α and $\lambda_\alpha, \mu_\alpha$ and σ_α are constant. Since θ_α generates L over U_α, the function g_α has to be invertible in $\mathcal{O}_{U_\alpha}$. Therefore, we may well assume $g_\alpha = 1$. Then we can write $\theta_\alpha = df_\alpha$, where $f_\alpha = \lambda_\alpha x_\alpha + \mu_\alpha y_\alpha + \sigma_\alpha z_\alpha$. Hence we can choose coordinates in such a way such that $\theta_\alpha = dx_\alpha$ for each $\alpha \in \Gamma$.

Let $a_{\alpha\beta}$ be the transition function for L over $U_\alpha \cap U_\beta$. Then On $U_\alpha \cap U_\beta$, we have $\theta_\alpha = a_{\alpha\beta}\theta_\beta$, i.e., $dx_\alpha = a_{\alpha\beta}dx_\beta$. This implies that $\dfrac{\partial x_\alpha}{\partial y_\beta} = \dfrac{\partial x_\alpha}{\partial z_\beta} = 0$, i.e., x_α depends on x_β only. Let $x_\alpha = f_{\alpha\beta}(x_\beta)$ for some one-variable holomorphic function $f_{\alpha\beta}$. It is clear that $f_{\alpha\beta}$ satisfies the co-cycle conditions, i.e., $f_{\alpha\beta} \circ f_{\beta\alpha} = id.$ on $U_\alpha \cap U_\beta$ and $f_{\alpha\beta} \circ f_{\beta\gamma} \circ f_{\gamma\alpha} = id.$ on $U_\alpha \cap U_\beta \cap U_\gamma$. Then $\{f_{\alpha\beta}\}$'s define a complex curve C. It is clear that C is smooth and compact. There is a natural holomorphic map $X \xrightarrow{\varphi} C$ sending $(x_\alpha, y_\alpha, z_\alpha)$ to x_α. Since $a_{\alpha\beta} = \dfrac{df_{\alpha\beta}}{dx_\beta}$, we have $L = \varphi^*(K_C)$. The inclusion $L \subset \Omega^1_X$ fits into the natural short exact sequence:

$$0 \longrightarrow L = \varphi^* K_C \xrightarrow{d\varphi^*} \Omega^1_X \longrightarrow \Omega^1_{X/C} \longrightarrow 0 \tag{1.4}$$

where $\Omega^1_{X/C}$ is the relative cotangent sheaf. The fact that $d\varphi^*$ is a bundle injection implies that $\Omega^1_{X/C}$ is locally free, hence φ is a smooth fibration. By passing to the Stein factorization, we can assume that φ has connected fibers and $\varphi_*(\mathcal{O}_X) = \mathcal{O}_C$. Let $K_{X/C} = \wedge^2 \Omega^1_{X/C}$ be the relative canonical bundle. Then $K_{X/C} = \varphi^*(K_C)$ since $K_X = 2L = 2\varphi^*(K_C)$. Therefore K_F is trivial for any fiber F of φ.

If moreover X is Kähler, then any fiber F is also Kähler. Therefore F is either a $K3$ surface or an Abelian surface. Since $K_{X/C} = \varphi^*(K_C)$, we have $\varphi_* K_{X/C} = K_C$. By a theorem of Fujita-Kawamata [Ka1] $\varphi_* K_{X/C}$ is weakly positive. Hence C is irrational. Leray spectral sequence implies that $h^1(\mathcal{O}_X) = h^0(R^1\varphi_*\mathcal{O}_X) + h^1(\mathcal{O}_C)$. Since C is irrational, $h^1(\mathcal{O}_C) > 0$. Hence $h^1(\mathcal{O}_X) > 0$, therefore $b_1(X) = 2h^1(\mathcal{O}_X) > 0$. Hence X is not simply connected. $\square$

The above theorem implies the following corollary immediately.

Corollary 1.3. *If X is a simply connected Kähler threefold and $L \subset \Omega^1_X$ such that $K_X = 2L$, then L defines a contact structure.*

Theorem 1.4. *Let X be a compact Kähler threefold with $c_1(X) = 0$. Then X has no contact structure.*

Proof. By Beauville's theorem [Be], there is a finite unramified cover $\tilde{X}$ of X such that $\tilde{X} = T \times Y$, where T is a complex torus and Y is simply connected with trivial canonical

bundle. This in particular implies that K_X is a torsion line bundle. Suppose X has a contact structure with a contact line bundle L. We distinguish two cases:

Case I: $\dim Y > 0$. Since K_X is torsion and $K_X = L^2$, we see that L is also torsion. Let m be a non-negative integer such that L^m is trivial. On the one hand, the bundle injection $L \hookrightarrow \Omega^1_X$ implies a bundle injection $\mathcal{O}_X = L^m \hookrightarrow \Omega^1_X$. Hence $h^0(S^m\Omega^1_X) > 0$. Since $\tilde{X}$ is an unramified cover, $h^0(S^m\Omega^1_{\tilde{X}}) > 0$. On the other hand, $h^0(S^m\Omega^1_{\tilde{X}}) = \sum_{p+q=m} h^0(S^p\Omega^1_T)\, h^0(S^q\Omega^1_Y)$. A theorem of Kobayashi [Ko] implies that $h^0(S^q\Omega^1_Y) = 0$ for all $q \geq 0$. Hence $h^0(S^m\Omega^1_{\tilde{X}}) = 0$. This gives a contradiction.

Case II: $\dim Y = 0$. In this case, $\tilde{X}$ is a complex torus. The given contact structure L on X induces a contact structure $\tilde{L}$ on $\tilde{X}$. Since $\Omega^1_{\tilde{X}}$ is trivial, $h^0(\tilde{L}^{-1}) > 0$. However $c_1(\tilde{L}) = 0$. Therefore $\tilde{L}$ has to be trivial. But a trivial line bundle on a torus does not define a contact bundle. This can be shown as follow. Note that $\tilde{L}$ gives rise to a global holomorphic one form θ on $\tilde{X}$. Let $f : \mathbb{C}^3 \longrightarrow \tilde{X}$ be the universal cover, and $\omega = f^*\tilde{\theta}$. Then ω has to be of the form $adx + bdy + cdz$, where a, b, c descend to holomorphic functions on $\tilde{X}$. Hence they must all be constant. Hence $d\omega = 0$. Therefore $\tilde{\theta} \wedge d\tilde{\theta} = 0$. This means that $\tilde{\theta}$ does not define a contact structure.

In either case we get a contradiction. Hence we are done.

□

Next we will give a topological obstruction to existence of a contact structure.

Proposition 1.5. *Let X be a compact complex contact threefold with a contact line bundle $L \subset \Omega^1_X$. Then*

$$\chi_{top.}(X) = 12\chi(\mathcal{O}_X) - \frac{c_1(X)^3}{8}$$

where $c_i(X) = c_i(T_X)$ $i = 1, 2, 3$, and $\chi(\mathcal{O}_X)$ is the holomorphic Euler characteristic, and $\chi_{top.}(X)$ be the topological Euler characteristic.

Proof. Since $L \subset \Omega^1_X$ is a subbundle (rather that a subsheaf), we conclude that $c_3(\Omega^1_X \otimes L^{-1}) = 0$ by Porteous' formula. This implies that:

$$-c_3 - c_2L - c_1L^2 - L^3 = 0 \tag{1.5}$$

where $c_i = c_i(X)$, for $i = 1, 2, 3$. Using the fact that $c_1 = -K_X = -2L$ and Riemann-Roch formula $\chi(\mathcal{O}_X) = \frac{c_2c_1}{24}$, we get:

$$c_3 = 12\chi(\mathcal{O}_X) - \frac{c_1(X)^3}{8} \tag{1.6}$$

The fact that $c_3 = \chi_{top.}(X)$ implies the proposition immediately.

□

Projective contact threefolds.

In this subsection, we will assume that (X, L) is a complex projective contact threefold. By this we mean that X is a projective complex manifold and $L \subset \Omega^1_X$ defines

a contact structure on X. We will show that the types of these contact threefolds are very limited.

Before we state our next theorem, let us recall that a line bundle is called *nef* if its intersection with every effective curve is non-negative.

Theorem 1.6. *If (X, L) is a projective complex contact threefold and K_X is not nef, then X is either isomorphic to $\mathbb{CP}^3$ or $(X, L) \cong \left(\mathbb{P}(T_M), \mathcal{O}_{\mathbb{P}(T_M)}(-1)\right)$ for some smooth complex projective surface M.*

Proof. Since K_X is not nef, Mori's theory of extremal ray implies that there is a smooth rational curve $C \subset X$ such that $4 \geq -K_X \cdot C > 0$ and C generates an extremal ray $R = \mathbb{R}_+[C]$. If $-K_X \cdot C = 4$, then it is well-known that X is isomorphic to $\mathbb{CP}^3$. Otherwise $-K_X \cdot C = 2$ since $K_X = 2L$. This implies that $L \cdot C = -1$. Consider the restriction of the contact sequence to C:

$$
\begin{array}{ccccccccc}
 & & & & 0 & & & & \\
 & & & & \uparrow & & & & \\
 & & & & N_{C/X} & & & & \\
 & & & & \uparrow & & & & \\
0 & \longrightarrow & L^{\perp}|_C & \longrightarrow & T_X|_C & \longrightarrow & L^*|_C & \longrightarrow & 0 \\
 & & & & \uparrow & & & & \\
 & & & & T_C & & & & \\
 & & & & \uparrow & & & & \\
 & & & & 0 & & & &
\end{array}
\tag{1.7}
$$

Denote by α the composed map $T_C \to L^*|_C$. Since $T_C \cong \mathcal{O}_C(2)$ and $L^*|_C \cong \mathcal{O}_C(1)$, we conclude that α has to be zero, i.e., C is a contact curve. Then by a theorem of Bryant [Br2] or Theorem 2.3 below, $N_{C/X} \cong \mathcal{O}_C \oplus \mathcal{O}_C$. It is shown by Mori [Mo] that X is isomorphic to a conic bundle in this case. Since $L \cdot C = -1$, any deformation of C in X is still reduced and irreducible. Hence X must be a $\mathbb{CP}^1$-bundle over some smooth projective surface M. Therefore we can write X as $\mathbb{P}(E)$ for some rank-two vector bundle on M. Let $X \xrightarrow{\pi} M$ be the natural projection. Note that any fiber of π generates the extremal ray R. Let $\mathcal{O}_X(1)$ be the tautological line bundle of X. It is easy to see that $K_X = -2\mathcal{O}_X(1) + \pi^*\left(K_M + \wedge^2 E\right)$. However $K_X = 2L$. Hence $\pi^*\left(K_M + \wedge^2 E\right) = -2L_0$, where $L_0 = \mathcal{O}_X(1) - L$. Since $L_0 \cdot C = -1 + 1 = 0$, $L_0 = \pi^* L_1$ for some line bundle L_1 on M. Therefore $K_M + \wedge^2 E = -2L_1$, i.e., $\wedge^2 (E \otimes L_1) = \wedge^2 T_M$. Hence if we tensor E by L_1, then we can assume that $\wedge^2 E \cong \wedge^2 T_M$ and $\mathcal{O}_X(-1)$ is the contact line bundle. Then there is a natural bundle injection $\lambda : \mathcal{O}_X(-1) \longrightarrow \Omega^1_X$.

We will show that $E \cong T_M$. We first prove the following claim:

Claim: Let $\Omega^1_{X/M}$ be the relative cotangent bundle. Then

$$H^0\left(\Omega^1_{X/M}(1)\right) = 0, \;\; H^1\left(\Omega^1_{X/M}\right) \cong \mathbb{C}$$

Proof of the Claim. Consider the relative Euler sequence:

$$0 \longrightarrow \Omega^1_{X/M}(1) \longrightarrow \pi^* E \longrightarrow \mathcal{O}_X(1) \longrightarrow 0 \tag{1.8}$$

Since $H^0(\pi^* E) \cong H^0(E) \cong H^0\left(\mathcal{O}_X(1)\right)$, we have $H^0\left(\Omega^1_{X/M}(1)\right) = 0$. Since π is a $\mathbb{CP}^1$-bundle, we have $\pi_* \Omega^1_{X/M} = 0$. Hence Leray spectral sequence for π implies that $H^1\left(\Omega^1_{X/M}\right) \cong H^0\left(R^1\pi_* \Omega^1_{X/M}\right)$. However by the relative duality, $R^1\pi_* \Omega^1_{X/M} \cong (\pi_* \mathcal{O}_X)^* \cong \mathcal{O}_M$. Hence $H^1\left(\Omega^1_{X/M}\right) \cong \mathbb{C}$. Hence the claim is proved.

Now consider the tangential sequence:

$$0 \longrightarrow T_{X/M} \longrightarrow T_X \longrightarrow \pi^* T_M \longrightarrow 0 \tag{1.9}$$

Since $H^0\left(\Omega^1_{X/M}(1)\right) = 0$, the bundle injection (from the contact structure on X) $\lambda : \mathcal{O}_X(-1) \longrightarrow \Omega^1_X$ induces a surjective bundle map $\sigma : \pi^* T_M \longrightarrow \mathcal{O}_X(1)$. Let $\mathcal{N}$ be the kernel of σ. Then $\mathcal{N}$ is a line bundle. Moreover $\mathcal{N} \cong \pi^*\left(\wedge^2 T_M\right) \otimes \mathcal{O}_X(-1)$. However by our assumption, $\wedge^2 E \cong \wedge^2 T_M$. Therefore sequence (1.8) implies that $\mathcal{N} \cong \Omega^1_{X/M}(1)$. Hence we obtained an exact sequence:

$$0 \longrightarrow \Omega^1_{X/M}(1) \longrightarrow \pi^* T_M \longrightarrow \mathcal{O}_X(1) \longrightarrow 0 \tag{1.10}$$

Let e_1, respectively e_2 be the extension class corresponding to (1.8), respectively (1.10). They are elements in $H^1\left(\Omega^1_{X/M}\right) \cong \mathbb{C}$. They are non-zero since their corresponding sequences do not split (because their restrictions to a fiber of π do not split since the normal bundle of the fiber is trivial). Hence they differ only by a non-zero scalar. This last fact implies that $\pi^* E \cong \pi^* T_M$. Since $\pi_* \mathcal{O}_X = \mathcal{O}_M$, we have $E \cong T_M$. Hence we are done.

□

Corollary 1.7. *If X is Fano, i.e., $-K_X$ is ample, then X is isomorphic either to $\mathbb{CP}^3$ or $\mathbb{P}\left(\Omega^1_{\mathbb{CP}^2}\right)$.*

Proof. Suppose that $\mathbb{CP}^3$ or $(X, L) \cong \left(\mathbb{P}(T_M), \mathcal{O}_{\mathbb{P}(T_M)}(-1)\right)$ for some smooth complex projective surface M. Then $-K_X = \mathcal{O}_{\mathbb{P}(T_M)}(-1)$. Hence by our assumption, T_M is ample. This clearly implies that $M \cong \mathbb{CP}^2$ by Mori's proof of Hartshorne conjecture. However as see in Example 1, $\mathbb{P}(T_{\mathbb{CP}^2}) \cong \mathbb{P}\left(\Omega^1_{\mathbb{CP}^2}\right)$. Hence we are done.

□

Remarks: We say that two contact manifolds X_1 and X_2 are *contactly birational* to each other if there is a birational map between them such that contact curves are

mapped to contact curves. It is showed in [Br1] that $\mathbb{CP}^3$ and $\mathbb{P}\left(\Omega^1_{\mathbb{CP}^2}\right)$ are contactly birational to each other. The above Corollary 1.7 answers affirmatively a question of Bryant, who asked if any contact threefold with positive first Chern class is contactly birational to $\mathbb{CP}^3$.

Now we are going to study projective contact threefolds with nef canonical bundle. They are so-called minimal model. We will show that they are either fibered by abelian surfaces, or hyperelliptic surfaces or elliptic curves.

Theorem 1.8. *Let (X, L) be a projective complex contact threefold such that K_X is nef. Then the Kodaira dimension of X is either one or two. If its Kodaira dimension is one, then X admits a fibration onto a smooth curve such that its generic fiber is an abelian surface or, a hyperelliptic surface. If its Kodaira dimension is two, then it admits an elliptic fibrations.*

Proof. Let $\kappa(X)$ be the Kodaira dimension of X. Since K_X is nef, then $\kappa(X)$ is between 0 and 3. If $\kappa(X) = 0$, then $c_1(X) = 0$. We have shown in Theorem 1.4 that this can not happen. Hence $\kappa(X) = 1, 2$, or 3. We are going to distinguish three cases according to $\kappa(X)$.

Case 1: $\kappa(X) = 3$. In this case K_X is big, i.e., $K_X^3 > 0$. Then $L^3 > 0$ also. By a theorem of Tsuji [Ts], T_X (hence Ω^1_X too) is K_X-semistable. Since $L \subset \Omega^1_X$ is a subbundle, we have $L \cdot K_X^2 < \dfrac{K_X \cdot K_X^2}{3}$, i.e., $4L^3 < \dfrac{8}{3}L^3$. This is absurd. Hence $\kappa(X)$ can not be 3.

Case 2: $\kappa(X) = 2$. Then by abundance theorem [Ka2], for a sufficiently large m, $|mK_X|$ is free. Let f be the morphism defined by $|mK_X|$ for some fixed large integer m. Let S be the image of f. Then S is a normal complex surface. By passing to its Stein factorization, we can assume that $f : X \longrightarrow S$ has connected fibers and $f_*\mathcal{O}_X = \mathcal{O}_S$. By the definition of f, we see that there is an ample line bundle H over S such that $mK_X = f^*H$. Let F be a generic fiber of f. Then F is smooth curve and by adjunction formula $K_F = K_X|_F$. Since $mK_X|_F \cong \mathcal{O}_F$ Therefore K_F is torsion. This implies that F is an elliptic curve.

Case 3: $\kappa(X) = 1$. By abundance theorem [Mi] again, we know that $|mK_X|$ is free for sufficiently large m. Let $f : X \longrightarrow C$ be the morphism defined by $|mK_X|$. Then C is a normal projective curve. Hence C is smooth. Let F a general fiber of f. Then F is a smooth projective surface. As we did in the previous case, we can show that $mK_F \cong \mathcal{O}_F$, i.e., K_F is a torsion line bundle. The rest part of the proof is very much similar to the proof of Theorem 1.4 above. Then F either has finite foundmental group, or its universal cover is a complex torus. We claim that F can not have finite foundmental group. This can be shown as follows. The bundle injection $L \hookrightarrow \Omega^1_X$ induces a bundle injection $L|_F \hookrightarrow \Omega^1_F$. Since $2mL|_F \cong mK_X|_F \cong \mathcal{O}_F$, we get a bundle injection $\mathcal{O}_F \hookrightarrow S^{2m}\Omega^1_F$. Hence $H^0\left(S^{2m}\Omega^1_X\right) > 0$. This contradicts a theorem of Kobayashi [Ko]. Therefore F must have infinite foundmental group. Hence F is either an abelian or a hyperelliptic surface.

□

The above theorem gives rise to a natural question: do those manifolds described in the theorem do have a contact structure?. My guess is that to be a contact manifold the

fibration should at least have no singular fibers. Next we show that the trivial abelian fibration over $\mathbb{CP}^1$ does have a contact structure.

Example 3. Let A be any abelian surface. Let $X = \mathbb{CP}^1 \times A$. Let (x, y) be the coordinates on $\mathbb{C}^2$, which is the universal cover of A. Let U_1 and U_2 be the two coordinate cover of $\mathbb{CP}^1$, and z_1 and z_2 be respective coordinates. Then over $U_1 \cap U_2$, $z_1 = 1/z_2$. Consider two local one-forms $\theta_1 = dy + z_1\, dx$ on $U_1 \times A$, and $\theta_2 = z_2\, dy + dx$ on $U_2 \times A$. Then these two local forms define a contact structure on X. In fact the contact line bundle L is just $p_1^*\mathcal{O}_{\mathbb{CP}^1}(-1)$, where $p_1 : X \to \mathbb{CP}^1$ is the first projection.

§2. Contact Curves and Their Moduli Space

Let X be a compact complex contact threefold with a contact line bundle $L \subset \Omega^1_X$. Let C be a smooth contact curve in X. $\mathcal{H}$ be the irreducible component of the Douady space of X the contains $[C]$. Let $\mathcal{H}_c \subset \mathcal{H}$ be the set of all contact curves. The purpose of this section is to study the subspace $\mathcal{H}_c$.

Lemma 2.1. *[Bryant] Let $N_{C/X}$ be the normal bundle of C. Then we have the following short exact sequence:*

$$0 \longrightarrow L^* \otimes K_C \longrightarrow N_{C/X} \longrightarrow L^*|_C \longrightarrow 0 \tag{2.1}$$

Proof. Consider the sequence (1.7). Let $\alpha : T_C \longrightarrow L^*|_C$ be the composed homomorphism. Since C is a contact curve, $\alpha = 0$. This induces a surjective bundle map $N_{C/X} \xrightarrow{\beta} L^*|_C \longrightarrow 0$. Let F be the kernel of β. Then F is a line bundle. By the adjunction formula, we have:

$$\wedge^2 N_{C/X} = K_C \otimes K_X^* = K_C \otimes L^{*2} \tag{2.2}$$

However it is clear that $\wedge^2 N_{C/X} = F \otimes L^*|_C$. Now (2.2) implies that $F = K_C \otimes L^*$. Therefore the lemma is proved. □

Before we continue, let us recall the definition and some properties of the *first prolongation* of a line bundle N on a compact complex manifold, say Y. The first prolongation of N, denoted by $P^1(N)$, is a rank-two bundle obtained via the following exact sequence:

$$0 \longrightarrow \Omega^1_Y(N) \longrightarrow P^1(N) \longrightarrow N \longrightarrow 0 \tag{2.3}$$

where the extension class of (2.3) is $c_1(N) \in H^1(\Omega^1_Y) = \mathrm{Ext}^1_Y(N, \Omega^1_Y(N))$. If $\{g_{\alpha\beta}\}$ are transition functions for N, then the Cech co-cycle $\{d\log(g_{\alpha\beta})\}$ represents $c_1(N)$, hence the extension class for (2.3).

Lemma 2.2.

(1) *Sequence (2.3) splits if and only if $c_1(N) = 0$, i.e., N is flat.*
(2) *Sequence (2.3) always splits cohomologically, i.e., $H^0(P^1(N)) \to H^0(N)$ is always surjective.*

Proof. The first part is true by definition. Let us prove the second part. Choose a local coordinate cover $\{U_\alpha\}$ on Y such that N is trivialized on each U_α. Let $\{g_{\alpha\beta}\}$'s be the transition functions for N under these trivializations. Let $s = \{s_\alpha\}$ be an arbitrary global holomorphic section of N. The second part is equivalent to show that the Cech co-cycle $\{s_\alpha d\log(g_{\alpha\beta})\}$ is a co-boundary with coefficient in $\Omega^1_Y(N)$, i.e., it is zero in $H^1(\Omega^1_Y(N))$. Since s is a global section of N, $s_\alpha = g_{\alpha\beta}s_\beta$. This implies that $s_\alpha d\log(g_{\alpha\beta}) = ds_\alpha - g_{\alpha\beta}ds_\beta$. Hence $\{s_\alpha d\log(g_{\alpha\beta})\}$ is a co-boundary. We are done.

□

We now state our next theorem.

Theorem 2.3. *Let $C \subset X$ be a smooth contact curve in a contact threefold. Let $L \subset \Omega^1_X$ be the contact structure. Then the following are true:*

1. *The Zariski tangent space $T_{[C]}\mathcal{H}_c \cong H^0(L^*|_C)$. The space of obstructions is $H^1(L^*|_C)$. In particular, if $h^1(L^*|_C) = 0$, then $\mathcal{H}_c$ is smooth of dimension $h^0(L^*|_C)$.*
2. *The normal bundle $N_{C/X}$ is isomorphic to $P^1(L^*|_C)$, where $P^1(L^*|_C)$ is the first prolongation of $L^*|_C$. In particular, sequence (2.1) splits if and only if $L\cdot C = 0$.*

Proof. As Bryant showed in [Br2] that we can choose a local coordinate cover $\{U_\alpha\}$on X such that on each U_α, L is generated by a local one form θ_α of the form $\theta_\alpha = dy_\alpha - z_\alpha dx_\alpha$, where $(x_\alpha, y_\alpha, z_\alpha)$ are local coordinates and $C \cap U_\alpha = \{y_\alpha = z_\alpha = 0\}$. Denote $C \cap U_\alpha$ by C_α. Let $\left[\dfrac{\partial}{\partial y_\alpha}\right]$ and $\left[\dfrac{\partial}{\partial z_\alpha}\right]$ be the classes of $\dfrac{\partial}{\partial y_\alpha}$ and $\dfrac{\partial}{\partial z_\alpha}$ in $N_{C/X}|_{C_\alpha}$. Then they generate $N_{C/X}|_{C_\alpha}$ over $\mathcal{O}_{C_\alpha}$.

We call an embedded deformation $\{C_t\}$ of C is a *contact* deformation if $\forall t$, C_t is a contact curve with respect to the given contact structure. Given a normal vector field $v_\alpha = a_\alpha\left[\dfrac{\partial}{\partial y_\alpha}\right] + b_\alpha\left[\dfrac{\partial}{\partial z_\alpha}\right]$ with a_α and b_α being in $\mathcal{O}_{U_\alpha}$. Then v_α generates a first-order contact deformation of C_α if and only if $b_\alpha = \dfrac{da_\alpha}{dx_\alpha}$. This is because if v_α generates a one-dimensional deformation $C_{\alpha t}$ of C_α in U_α, then the deformation can be expressed as: $x_\alpha(t) = x_\alpha + c_\alpha t + O(t^2), y_\alpha(t) = a_\alpha t + O(t^2), z_\alpha(t) = b_\alpha t + O(t^2)$. Then $\theta_\alpha|_{C_{\alpha t}} = t\,(\dfrac{da_\alpha}{dx_\alpha} - b_\alpha)dx_\alpha + O(t^2)$. Therefore v_α generates an infinitesimal contact deformation of C_α if and only if $b_\alpha = \dfrac{da_\alpha}{dx_\alpha}$.

Let us denote by $N^0_\alpha = \left\{ a_\alpha\left[\dfrac{\partial}{\partial y_\alpha}\right] + \dfrac{da_\alpha}{dx_\alpha}\left[\dfrac{\partial}{\partial z_\alpha}\right] \;\middle|\; a_\alpha \in \mathcal{O}_{C_\alpha} \right\}$. Then N^0_α is a subset of $N_{C/X}|_{C_\alpha}$, but not a submodule over $\mathcal{O}_{C_\alpha}$. However, we can put a different $\mathcal{O}_{C_\alpha}$-module structure on N^0_α. For any $f_\alpha \in \mathcal{O}_{C_\alpha}$, we define its action on $a_\alpha\left[\dfrac{\partial}{\partial y_\alpha}\right] + \dfrac{da_\alpha}{dx_\alpha}\left[\dfrac{\partial}{\partial z_\alpha}\right]$ by:

$$f_\alpha \circ \left(a_\alpha\left[\frac{\partial}{\partial y_\alpha}\right] + \frac{da_\alpha}{dx_\alpha}\left[\frac{\partial}{\partial z_\alpha}\right]\right) \stackrel{\text{def.}}{=} (f_\alpha a_\alpha)\left[\frac{\partial}{\partial y_\alpha}\right] + \frac{d\,(f_\alpha a_\alpha)}{dx_\alpha}\left[\frac{\partial}{\partial z_\alpha}\right]$$

It is clear that with this $\mathcal{O}_{C_\alpha}$-module structure N^0_α is free and generated by $\left[\dfrac{\partial}{\partial y_\alpha}\right]$.

Next we will show that we can glue these N_α^0's to get a line bundle over C. In fact we will show that this line bundle is isomorphic to $L^*|_C$. First let us prove the following claim.

Claim : Let $\mathcal{I}_\alpha = (y_\alpha, z_\alpha)$ be the ideal sheaf of C_α. After modulating the ideal sheaf $\mathcal{I}_\alpha^2$, on $U_\alpha \cap U_\alpha$ we have:

$$\begin{cases} x_\beta = f_{\beta\alpha}(x_\alpha) \\ y_\beta = a_{\beta\alpha} y_\alpha \\ z_\beta = c_{\beta\alpha} z_\alpha + b_{\beta\alpha} y_\alpha \end{cases}$$

where $f_{\beta\alpha}$, $a_{\beta\alpha}$, $b_{\beta\alpha}$ and $c_{\beta\alpha}$ are holomorphic functions over U_α depending only on x_α. Moreover $b_{\beta\alpha} = \dfrac{da_{\beta\alpha}}{dx_\beta}$ and $c_{\beta\alpha} = a_{\beta\alpha}\dfrac{dx_\alpha}{dx_\beta}$ when restricted to $U_\alpha \cap U_\beta \cap C$.

Proof of the Claim. On $U_\alpha \cap U_\alpha$, we have:

$$dx_\beta = \frac{\partial x_\beta}{\partial x_\alpha} dx_\alpha + \frac{\partial x_\beta}{\partial y_\alpha} dy_\alpha + \frac{\partial x_\beta}{\partial z_\alpha} dz_\alpha \tag{2.4}$$

Note that dx_α generates $\Omega^1_C|_{C_\alpha}$, therefore dx_β and dx_α are proportional to each other on $U_\alpha \cap U_\beta \cap C$. Therefore

$$\frac{\partial x_\beta}{\partial y_\alpha} = \frac{\partial x_\beta}{\partial z_\alpha} = 0 \pmod{\mathcal{I}_\alpha} \tag{2.5}$$

This implies the first equation of the claim.

By the same token, since θ_α and θ_β are proportional, we have $\dfrac{\partial y_\beta}{\partial z_\alpha} - z_\beta \dfrac{\partial x_\beta}{\partial z_\alpha} = 0$. Therefore (2.5) implies that $\dfrac{\partial y_\beta}{\partial z_\alpha} = 0 \,(\mathrm{mod}\, \mathcal{I}_\alpha^2)$. Hence $y_\beta = h_{\alpha\beta}(y_\alpha, x_\alpha) \,(\mathrm{mod}\, \mathcal{I}_\alpha^2)$. Since y_α and y_β are both zero when restricted to $U_\alpha \cap U_\beta \cap C$, we see y_β has to be divisible by y_α. This give the second equation of the claim. The last equation of the claim is obvious since z_β vanishes on $U_\alpha \cap U_\beta \cap C$.

The rest of the claim follows easily from the fact that θ_α and θ_β are proportional to each other on $U_\alpha \cap U_\beta$. Hence the claim is proved.

On the one hand, by the claim, when restricted to $U_\alpha \cap U_\beta \cap$ C, we have:

$$\begin{aligned} \left[\frac{\partial}{\partial y_\alpha}\right] &= \frac{\partial y_\beta}{\partial y_\alpha}\left[\frac{\partial}{\partial y_\beta}\right] + \frac{\partial z_\beta}{\partial y_\alpha}\left[\frac{\partial}{\partial z_\beta}\right] \\ &= a_{\beta\alpha}\left[\frac{\partial}{\partial y_\beta}\right] + \frac{da_{\beta\alpha}}{dx_\beta}\left[\frac{\partial}{\partial z_\beta}\right] \\ &= a_{\beta\alpha} \circ \left[\frac{\partial}{\partial y_\beta}\right] \end{aligned} \tag{2.6}$$

On the other hand, it is clear to see that $\theta_\beta = a_{\beta\alpha}\theta_\alpha \,(\mathrm{mod}\, \mathcal{I}_\alpha)$ on $U_\alpha \cap U_\beta \cap C$. Therefore we conclude that N_α^0's can be glued to get a line bundle on C which is isomorphic to $L^*|_C$. This proves the first part of the theorem.

Now let us prove the second part of the theorem. By the claim, the transition matrix for the normal bundle $N_{C/X}$ over $U_\alpha \cap U_\beta \cap C$ is given by:

$$\begin{aligned} \left[\frac{\partial}{\partial y_\alpha}\right] &= a_{\beta\alpha}\left[\frac{\partial}{\partial y_\beta}\right] + \frac{da_{\beta\alpha}}{dx_\beta}\left[\frac{\partial}{\partial z_\beta}\right] \\ \left[\frac{\partial}{\partial z_\alpha}\right] &= a_{\beta\alpha}\frac{dx_\alpha}{dx_\beta}\left[\frac{\partial}{\partial z_\beta}\right] \end{aligned} \tag{2.7}$$

This implies that the extension class (2.1) is given by $e_{\alpha\beta} = a_{\alpha\beta}\dfrac{da_{\beta\alpha}}{dx_\beta}$. Therefore $e_{\alpha\beta}dx_\beta = a_{\alpha\beta}\, da_{\beta\alpha} = d\log(a_{\beta\alpha})$ gives a class in $H^1(K_C)$, which is the extension class corresponds to the exact sequence (2.1). Hence the normal bundle $N_{C/X} \cong P^1(L^*|_C)$ by Lemma 2.2. Lemmas 2.2 also implies that sequence (2.1) splits if and only if $c_1(L^*|_C) = 0$, i.e., $L \cdot C = 0$. $\square$

Next we will study the moduli space of all curves which are contact with respect to a given contact structure on X. Let $\mathcal{M}_c$ be the moduli space of all contact structures on X. Let $\varphi \in \mathcal{M}_c$ be a contact structure on X and L_φ be the contact line bundle. Since $K_X = 2L_\varphi$, if $\mathrm{Pic}(X)$ has no two torsion, then all the L_φ are the same. To give an element $\varphi \in \mathcal{M}_c$ is the same as to give a bundle injection (which is still denoted by φ) $L_\varphi \xrightarrow{\varphi} \Omega^1_X$. Therefore $\mathcal{M}_c$ is an Zariski open subset of $H^0\left(\Omega^1_X(L^*_\varphi)\right)$.

Let $\mathcal{H}$ be an irreducible component of the Douady space of *smooth* curves in X. Let $\mathcal{C} \subset X \times \mathcal{H}$ be the universal family. The we have two projections, namely, $\mathcal{C} \xrightarrow{p} \mathcal{H}$ and $\mathcal{C} \xrightarrow{q} X$. Let $\omega_{\mathcal{C}/\mathcal{H}}$ be the relative dualizing sheave for p. Given a contact structure φ on X, it induces in a natural way a section s_φ of $\mathcal{E}_\varphi \overset{\text{def.}}{=} p_*\left(\omega_{\mathcal{C}/\mathcal{H}} \otimes q^*L^*_\varphi\right)$. The vanishing locus of s_φ is $\mathcal{H}_\varphi$, the set of curves in $\mathcal{H}$ contact with respect to φ. This proves the first part of the following proposition.

Proposition 2.4. *Let X be a compact complex threefold. Then the following is true:*

1. *Given a contact structure φ on X there is a natural section s_φ of $\mathcal{E}_\varphi$ such that $\mathcal{H}_\varphi$ is the vanishing locus of s_φ.*
2. *If $H^0(L_\varphi|_C) = 0$ and $\mathcal{H}_\varphi$ is non-empty, then the smoothness of $\mathcal{H}_\varphi$ at a point $[C] \in \mathcal{H}_\varphi$ is equivalent to the smoothness of $\mathcal{H}$ at the same point.*
3. *Let $[C] \in \mathcal{H}_\varphi$ be contact curve. If $H^1\left(L^*_\varphi|_C\right) = 0$, then both $\mathcal{H}$ and $\mathcal{H}_\varphi$ is smooth in a neighborood of $[C]$.*
4. *If $\mathcal{H}_\varphi$ is smooth everywhere, then $T_{\mathcal{H}}|_{\mathcal{H}_\varphi} \cong T_{\mathcal{H}_\varphi} \oplus \mathcal{E}_\varphi$, i.e., the restriction of tangent bundle splits. The normal bundle of $\mathcal{H}_\varphi$ in $\mathcal{H}$ is isomorphic to $\mathcal{E}_\varphi$.*

Proof. The first part is proved in the above paragraph. As for the second part, Lemma 2.2 and Theorem 2.3 imply that sequence (2.1) splits cohomologically. Hence the following sequence is exact:

$$0 \longrightarrow H^1\left(L^*_\varphi \otimes K_C\right) \longrightarrow H^1\left(N_{C/X}\right) \longrightarrow H^1\left(L^*_\varphi\right) \longrightarrow 0 \tag{2.8}$$

Since by our assumptions $H^0\left(L|_C\right) = 0$, we have $H^1\left(L^*_\varphi \otimes K_C\right) = 0$ by Riemann-Roch theorem. Hence $H^1\left(N_{C/X}\right) \cong H^1\left(L^*_\varphi\right)$. By local computations as we did in the proof of

Theorem 2.3, we can show that obstructions in $H^1\left(N_{C/X}\right)$ are mapped to obstructions in $H^1\left(L_\varphi^*\right)$. Therefore if $\mathcal{H}_\varphi$ is smooth at $[C]$, then $\mathcal{H}$ is also smooth at $[C]$. Conversely, if $\mathcal{H}$ is smooth at a point $[C]$, then its dimension around that point is $h^0\left(N_{C/X}\right)$. The fact that $\mathcal{H}_\varphi$ is the vanishing locus of the section $s_\varphi \in H^0\left(\mathcal{E}_\varphi\right)$ implies that the dimension of $\mathcal{H}_\varphi$ around the given point $[C]$ is at least $h^0\left(N_{C/X}\right) - h^0\left(L_\varphi^* \otimes K_C\right) = h^0\left(L_\varphi^*\right)$. By Theorem 2.3, $T_{[C]}\mathcal{H}_\varphi = H^0\left(L_\varphi^*\right)$. Therefore $\mathcal{H}_\varphi$ is smooth at $[C]$. Hence the smoothness of $\mathcal{H}_\varphi$ at a point $[C] \in \mathcal{H}_\varphi$ is equivalent to the smoothness of $\mathcal{H}$ at the same point. This proves the part two.

In view of the fact that sequence (2.1) splits cohomologically, the last two parts are clear.

□

Theorem 2.3 and Proposition 2.4 imply that the following corollary, which was proved by Bryant [Br2] using a different method.

Corollary 2.5. *[Bryant] Let (X, L) be a contact threefold. Suppose $C \subset X$ is a smooth contact rational curve in X with $L|_C \cong \mathcal{O}_C(-k-1)$ for some integer $k \geq 0$. Then the moduli space of contact curves that contains C is smooth of dimension $k+2$. Moreover the normal bundle has the decomposition $N_{C/X} \cong \mathcal{O}_C(k) \oplus \mathcal{O}_C(k)$.*

As another consequence of Theorem 2.3 we will show that contact curves in $\mathbb{CP}^3$ can not be complete intersections.

Corollary 2.6. *Let C be a smooth contact curve in $\mathbb{CP}^3$. If C is degenerate, then it is a straight line. If C is non-degenerate, then it can not be a complete intersection.*

Proof. In any case, by Lemma 2.1, we have the following short exact sequence:

$$0 \longrightarrow \omega_C(2) \longrightarrow N_{C/X} \longrightarrow \mathcal{O}_C(2) \longrightarrow 0 \tag{2.9}$$

By Theorem 2.3, the above sequence does not splits.

If C is degenerate, then C is contained in some hyperplane $\mathbb{CP}^2 \subset \mathbb{CP}^3$. Let d be the degree of C. On the one hand, the normal bundle has the decomposition $N_{C/X} \cong \mathcal{O}_C(1) \oplus \mathcal{O}_C(d)$. On the other hand, since $\omega_C \cong \mathcal{O}_C(d-3)$, (2.9) becomes:

$$0 \longrightarrow \mathcal{O}_C(d-1) \longrightarrow N_{C/X} \longrightarrow \mathcal{O}_C(2) \longrightarrow 0 \tag{2.10}$$

Therefore there is a non-trivial homomorphism $v : \mathcal{O}_C(d) \longrightarrow \mathcal{O}_C(2)$. Hence $d \leq 2$. If $d = 2$, then v is an isomorphism. Therefore (2.10) has to split. This is absurd since it does not split. This shows that $d = 1$, i.e., C is a straight line.

If C is non-degenerate, we will show that it is not a complete intersection. Suppose C is a complete intersection, we write $C = S_n \cap S_m$, where S_n, resp. S_m is surface of degree n, resp. m. Since C is non-degenerate, both n and m are at least two. Suppose that $n \geq m$. Now sequence (2.9) and the fact that $N_{C/X} \cong \mathcal{O}_C(n) \oplus \mathcal{O}_C(m)$ imply that there is a non-trivial $\mathcal{O}_C$-module homomorphism $u : \mathcal{O}_C(m) \longrightarrow \mathcal{O}_C(2)$. Hence $m \leq 2$. Therefore $m = 2$. This implies that u is in fact an isomorphism. Hence (2.9) has to split. This is a contradiction. Therefore C is not a complete intersection. We are done.

□

§3. Contact Curves in $\mathbb{CP}^3$

In this section we will study contact curves in $\mathbb{CP}^3$. We will give a Plücker type formula for contact curves in $\mathbb{CP}^3$. This formula was predicted in [Br1] by Bryant. At the end of the section, we will relate moduli space of contact line in $\mathbb{CP}^3$ to the set of jumping lines for a null-correlation bundle. In this section we will consider also *singular* contact curves. A singular curve C is *contact* if local contact forms vanish on C_{reg}, the smooth part of C.

Since all contact structures on $\mathbb{CP}^3$ are equivalent under automorphisms of $\mathbb{CP}^3$, we will consider only the distinguished contact structure obtained from the twistor map $\mathbb{CP}^3 \xrightarrow{T} S^4$. In the affine coordinates $[1, z_1, z_2, z_3]$ of $\mathbb{CP}^3$, the local contact form is $\theta = d\,z_1 - z_3\,d\,z_2 + z_2\,d\,z_3$. Bryant [Br1] provided a way to construct all contact curves in $\mathbb{CP}^3$. He showed that all contact curves in $\mathbb{CP}^3$ are "lifts" of curves in $\mathbb{CP}^2$.

First of all, let us make clear what the "lift" means. For a reduced and irreducible curve $D \subset \mathbb{CP}^2$, let D_{reg} be the smooth part of D. Then points in D_{reg} together with their tangent directions form a curve in $\mathbb{P}\left(\Omega^1_{\mathbb{CP}^2}\right)$. This gives a lift of D_{reg} to $\mathbb{P}\left(\Omega^1_{\mathbb{CP}^2}\right)$. Now the Zariski closure of this lift in $\mathbb{P}\left(\Omega^1_{\mathbb{CP}^2}\right)$ is the lift of D to $\mathbb{P}\left(\Omega^1_{\mathbb{CP}^2}\right)$. Sometimes we call this the horizontal lift of D, and we denote it by $\tilde{D}$. It is clear that $\tilde{D}$ is smooth if D has only unramified or simple cuspidal singularities.

Bryant [Br1] defined a birational map $f : \mathbb{P}\left(\Omega^1_{\mathbb{CP}^2}\right) \longrightarrow \mathbb{CP}^3$ sending $(x, y, [\lambda_1, \lambda_2])$ to $\left[\lambda_1, x\lambda_1 - \frac{1}{2}y\lambda_2, y\lambda_1, \frac{1}{2}\lambda_2\right]$, where (x, y) are coordinates on $\mathbb{C}^2 \subset \mathbb{CP}^2$ and $[\lambda_1, \lambda_2]$ are fiber coordinates.

Now we can state the theorem of Bryant [Br1].

Theorem 3.1. *[Bryant] Let C be a contact curve in $\mathbb{CP}^3$. Then C is either a straight line or of the form $f\left(\tilde{D}\right)$, where $\tilde{D} \subset \mathbb{CP}^2$ is the horizontal lift of a reduced and irreducible plane curve D of degree at least two.*

Therefore to study contact curves in $\mathbb{CP}^3$ is the same as to study curves in $\mathbb{CP}^2$ and their lifts to $\mathbb{P}\left(\Omega^1_{\mathbb{CP}^2}\right)$. As we see in the first section, there is a natural contact structure on $\mathbb{P}\left(\Omega^1_{\mathbb{CP}^2}\right)$. The above birational map f is *contact* in the sense that it maps contact curves to contact curves.

Next we offer a more homogeneous way of defining Bryant's map. We can think of $\mathbb{P}\left(\Omega^1_{\mathbb{CP}^2}\right)$ as the universal family of lines in $\mathbb{CP}^2$, or as the flag manifold $F_{1,2}$. Therefore $\mathbb{P}\left(\Omega^1_{\mathbb{CP}^2}\right)$ sits naturally inside $\mathbb{CP}^2 \times \mathbb{CP}^{2*}$, where $\mathbb{CP}^{2*}$ is the dual of $\mathbb{CP}^2$. In fact $\mathbb{P}\left(\Omega^1_{\mathbb{CP}^2}\right)$ is the set of pairs (p, ℓ) such that $p \in \ell$, where p is a point in $\mathbb{CP}^2$ and ℓ is a line (which is thought of as a point in $\mathbb{CP}^{2*}$). Let $[x_i]$, $[y_i]$ $(i = 0, 1, 2)$ be projective coordinates in $\mathbb{CP}^2$ and $\mathbb{CP}^{2*}$. Then $\mathbb{P}\left(\Omega^1_{\mathbb{CP}^2}\right) \subset \mathbb{CP}^2 \times \mathbb{CP}^{2*}$ is defined by the bi-homogeneous polynomial:

$$x_0y_0 + x_1y_1 + x_2y_2 = 0$$

Let $U_2 = \{x_2 \neq 0\} \subset \mathbb{CP}^2$ be an affine open set with affine coordinate $(x, y) = (x_0/x_2, x_1.x_2)$. On U_2, we can identify the fiber coordinate $[\lambda_1, \lambda_2]$ with $[-y_0, y_1]$.

Therefore Bryant's map can be redefined as:

$$f\left([x_0,x_1,x_2],[y_0,y_1,y_3]\right)=[2x_2y_0,2x_0y_0+x_1y_1,2x_1y_0,-x_2y_1] \tag{3.1}$$

Consider two points $p_\infty=[1,0,0]\in\mathbb{CP}^2$ and $p_\infty^*=[0,0,1]\in\mathbb{CP}^{2*}$. Then (p_∞,p_∞^*) is a point in $\mathbb{P}\left(\Omega^1_{\mathbb{CP}^2}\right)$. Let $\ell_\infty\subset\mathbb{CP}^2$ be the line dual to $p_\infty^*\in\mathbb{CP}^{2*}$, and $\ell_\infty^*\subset\mathbb{CP}^{2*}$ be the line dual to $p_\infty\in\mathbb{CP}^2$. Then it is clear that the f is not defined precisely along the curve $\ell_\infty\cup\ell_\infty^*$.

Let $p_1:\mathbb{P}\left(\Omega^1_{\mathbb{CP}^2}\right)\longrightarrow\mathbb{CP}^2$ and $p_2:\mathbb{P}\left(\Omega^1_{\mathbb{CP}^2}\right)\longrightarrow\mathbb{CP}^{2*}$ be the two projections. Note that if $D\subset\mathbb{CP}^2$ is a reduced and irreducible plane curve and $\tilde{D}$ is its lift, then $p_1\left(\tilde{D}\right)=D$, and $p_2\left(\tilde{D}\right)=D^*$, where D^* is the dual curve of D. We say that a plane curve D is *good* if $p_\infty\notin D$ and $p_\infty^*\notin D^*$. The second condition is equivalent to the fact the D doest not tangent to the line ℓ_∞. Also D is good if and only if $D\cap(\ell_\infty\cup\ell_\infty^*)=\emptyset$. If D is good and unramified, then $\tilde{D}$ is smooth, hence $f\left(\tilde{D}\right)$ is also smooth. Note that D^* may be a point. This happens exactly when D is a line in $\mathbb{CP}^2$. To make the following theorem hold for lines also, we understand the degree of D^* as zero if D^* is a point. Now we get the following theorem immediately.

Theorem 3.2. *Let $C\subset\mathbb{CP}^3$ be a contact curve of degree d and geometric genus g, which is obtained from a good plane curve D of degree n. Then*

$$\begin{aligned} d&=n+n^*\\ g&=g(D)\end{aligned} \tag{3.1}$$

where n^* is the degree of the dual curve of D and $g(D)$ is the geometric genus of D.

Proof. Let $U=\mathbb{P}\left(\Omega^1_{\mathbb{CP}^2}\right)\setminus(\ell_\infty\cup\ell_\infty^*)$. By the definition of the Bryant's map, it is clear that $f^*\mathcal{O}_{\mathbb{CP}^3}(1)|_U\cong p_1^*\mathcal{O}_{\mathbb{CP}^2}(1)\otimes p_2^*\mathcal{O}_{\mathbb{CP}^{2*}}(1)|_U$. Since D is *good*, $\tilde{D}$ is contained in U. Therefore the degree of $C\subset\mathbb{CP}^3$ is the same as $p_1^*\mathcal{O}_{\mathbb{CP}^2}(1)\cdot\tilde{D}+p_2^*\mathcal{O}_{\mathbb{CP}^{2*}}(1)\cdot\tilde{D}=n+n^*$, where n^* is the degree of the dual of D. Since C and D are obviously isomorphic to each other, $g(C)=g(D)$. Hence we are done.

□

In particular, if D has only traditional singularities (see [GH] for definitions), then the following corollary is an immediate consequence of the above theorem and the classical Plücker formula.

Corollary 3.3. *Let C be a contact curve in $\mathbb{CP}^3$ obtained from a good plane curve D of degree n with traditional singularities. Then*

$$\begin{aligned} d&=n^2-2\delta-3\kappa\\ g&=\frac{1}{2}(n-1(n-2)-\delta-\kappa\end{aligned} \tag{3.2}$$

where κ, resp. δ is the number of cusps, resp. nodes of D.

Remark: The requirement that D is *good* simplifies the situation a lot, but it is not essential. By studying the resolution of the map f carefully, we should be able to handle the case of non-good curves. In that case, the degree of C will be less than $n + n^*$ due to the the singularities of the map f.

Before we go on, let us recall that a *complex involution* on a complex manifold Y is an automorphism σ of Y such that $\sigma^2 = id.$.

Let $\mathcal{H}_c(d, g)$ be the space of contact curves (even including singular ones) in $\mathbb{CP}^3$ of degree d and geometric genus g. The above theorem shows that a plane curve and its dual give two contact curves with the same degree and genus. In this way, we get a natural complex involution on $\mathcal{H}_c(d, g)$. That is:

Corollary 3.4. *There is a natural complex involution on $\mathcal{H}_c(d, g)$.*

Proof. There is a natural involution (denoted by σ_0) on $\mathbb{P}\left(\Omega^1_{\mathbb{CP}^2}\right)$ sending $[x_0, x_1, x_2]$ to $[y_0, y_1, y_2]$. Let $\tilde{D}$ be the lift of D. Then it is clear that $\sigma_0(D) = \tilde{D}^*$, the lift of D^* from $\mathbb{CP}^{2*}$. Let $C^* = f\left(\tilde{D}^*\right)$. Then C^* is also a contact curve $\mathbb{CP}^3$. We thus define $\sigma(C) = C^*$. Then σ is an complex involution. Hence we are done.

□

We next present an example, in which the contact geometry is related to the moduli space of instanton bundles on $\mathbb{CP}^3$. We will study the moduli space $\mathcal{H}_c(1, 0)$, the space of contact lines in $\mathbb{CP}^3$.

Example 4. Recall that the moduli space of contact structures on $\mathbb{CP}^3$ is isomorphic to $\mathcal{M}(0, 1)$, the moduli space of rank-two stable bundles with $c_1 = 0$ and $c_2 = 1$ (see Example 2.). As it is shown in [Ha] that $\mathcal{M}(0, 1)$ is isomorphic to the space of all non-singular anti-symmetric 4×4 complex matrices. Hence it can be identified with $\mathbb{CP}^5 - G(1, 3)$. Note that $G(1, 3)$ is the space of all straight lines in $\mathbb{CP}^3$. It is isomorphic to $\mathbb{Q}^4$, the smooth hyperquadric in $\mathbb{CP}^5$. As noted in [Ha] $\mathbb{Q}^4$ induces a duality between non-singular hyperplane sections of $\mathbb{Q}^4$ and points in $\mathbb{CP}^5 - G(1, 3)$. Given a point $\varphi \in \mathbb{CP}^5 - G(1, 3)$, draw all the lines through φ and tangent to $\mathbb{Q}^4$. The points where these lines tangent to $\mathbb{Q}^4$ actually lie on an unique linear subspace $\mathbb{CP}^4 \subset \mathbb{CP}^5$. This linear subspace gives a hyperplane section $H_\varphi \subset \mathbb{Q}^4$. Barth [Ba] showed that H_φ can be identified with the jumping set Z_φ for N_φ, the stable bundle corresponds to φ in Example 2. However we will show that Z_φ can be identified with $\mathcal{H}_\varphi \subset G(1, 3)$, the space of contact lines with respect to φ.

Proposition 3.5. *The set of contact lines $\mathcal{H}_\varphi$ can be identified with the set of jumping lines of Z_φ. In particular, $\mathcal{H}_\varphi$ is isomorphic to a nonsingular quadrics $\mathbb{Q}^3 \subset \mathbb{CP}^4$, hence it has a complex involution and a real structure.*

Proof. Let C is line and is contact with respect to the given contact structure φ. Then the sequence (1.7) for $\mathbb{CP}^3$ implies that we have the following short exact sequence:

$$0 \longrightarrow \mathcal{O}_C(2) \longrightarrow L^\perp|_C \longrightarrow \mathcal{O}_C \longrightarrow 0 \tag{3.3}$$

where $L = \mathcal{O}_{\mathbb{CP}^3}(-2)$ is the contact bundle of $\mathbb{CP}^3$, and $L^\perp$ is the bundle of vector fields perpendicular to local sections of L. Since $H^1(\mathcal{O}_C(2)) = 0$, the above sequence splits. Hence $L^\perp|_C \cong \mathcal{O}_C \oplus \mathcal{O}_C(2)$. The definition of N_φ implies that $N_\varphi = L^\perp|_C \otimes \mathcal{O}_{\mathbb{CP}^3}(-1)$. Hence $N_\varphi \cong \mathcal{O}_C(-1) \oplus \mathcal{O}_C(1)$. Therefore C is a jumping line for N_φ. So $\mathcal{H}_\varphi \subset Z_\varphi$. However $\dim \mathcal{H}_\varphi = h^0(\mathcal{O}_C(2)) = 3$, which is the dimension of Z_φ. Since Z_φ is smooth and irreducible, we get $\mathcal{H}_\varphi = Z_\varphi$. Hence we are done.

□

We close this section by posing a question. Fix a pair of integers (d, g) such that $d \geq g + 3$. Ein [Ei] showed that the Hilbert scheme $\mathcal{H}(d, g)$ of smooth curves in $\mathbb{CP}^3$ with degree d and genus g is irreducible. Given a contact structure φ on $\mathbb{CP}^3$, we have a closed subscheme $\mathcal{H}_\varphi(d, g)$ of $\mathcal{H}(d, g)$ consisting of *smooth* contact curves. Then it is natural to ask:

Question 1. *If $\mathcal{H}_\varphi(d, g)$ is non-empty, is it irreducible?.*

By Proposition 2.2, $\mathcal{H}_\varphi(d, g)$ is smooth. Then if it is connected then it is irreducible. Hence the above question is equivalent to asking if $\mathcal{H}_\varphi(d, g)$ is connected. We remarked in the previous section that $\mathcal{H}_\varphi(d, g)$ is a vanishing locus of a section s_φ of a vector bundle $\mathcal{E}_\varphi$ over $\mathcal{H}(d, g)$. One may hope to use Fulton-Lazarsfeld's connectivity theorem [FL]. But there are two esential difficulties that make this approach impossible. First of all, $\mathcal{H}(d, g)$ is not projective. Secondly, most importantly, the vector bundle $\mathcal{E}_\varphi$ is *not* ample. We have to take some other approach.

Remarks on higher-dimensional contact manifolds

Suppose that $n \geq 2$ and X is a $(2n + 1)$-dimensional complex contact manifold with a contact line bundle L. Then in this situation, we can study contact n-dimensional submanifolds in X. Some of the general results in this paper can be generalized to this situation without suitable modifications. For example, Theorem 1.2, 1.4 and 2.3 can be generalized properly. However, Theorem 1.6 and Theorem 1.8 can not. This is because in higher dimensions we don't have a good understanding of extremal rays and pluri-canonical systems as we do in three-dimensional case. For example, we don't even know if a projective manifold X having an extremal ray with length $\dim(X) + 1$ is isomorphic to a complex projective space. In fact, this is an open conjecture in Mori's theory. However if X is Fano, then X is isomorphic to the projectivized tangent bundle of a projective space provided that Picard number of X is at least two. This is not difficulty to prove using claims of Wisniewski [Wi] (at the end of the paper) and techniques of this paper. We would like to ask the following question:

Question 2. *Let X be a contact $(2n+1)$-dimensional Fano manifold with Picard number $\rho(X) = 1$. Is it true that $X \cong \mathbb{CP}^{2n+1}$?.*

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