AUTOMORPHIC-TWISTED SUMMATION FORMULAE FOR PAIRS OF QUADRATIC SPACES

by

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Dissertation submitted in partial fulfillment of the requirements for the degree of Doctor of Philosophy in the Department of Mathematics in the Graduate School of Duke University

2023

ABSTRACT

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Abstract

Conjectures of Braverman-Kazhdan, L. Lafforgue, Ngô and Sakellaridis imply that all affine spherical varieties admit generalized Poisson summation formula. In this dissertation we establish a generalized Poisson summation formula for certain spaces of test functions on the zero locus of a quadratic form. The functions are built from the Whittaker coefficients of automorphic representations on GL_n. We also give an expression of the local factors where all the data is unramified.

Contents

List of Symbols

A_1	subgroup of T_G	(3.3.4)
A_2	subgroup of T_G	(3.3.5)
a_y	$\mathrm{diag}(-4\mathcal{Q}'(y_2), I_{n-1}) \in \mathrm{GL}_n$	(2.1.4)
$B_{W_{\rho_{\tau,s}}}$	unramified Bessel function	5.1.11
G	$\{g = (g_1, g_2) \in \mathrm{GL}_2^2(R) : \det g_1 = \det g_2^{-1}\}$	(3.1.1)
G'	SO_4	(3.1.2)
G_1	subgroup of T_G	(3.3.3)
H	SO_{2n+1}	(3.1.1)
ι	embedding map from G to H	(3.2.2)
$I(f, W_{\xi_s})$	global integral	2.1.2
M_1	subgroup of T_G	3.3.1
$M_{\mathrm{SL}_2^2}$	subgroup of maximal torus of $\mathrm{SL}_2 \times \mathrm{SL}_2$	(3.3.8)
M_n	Levi subgroup of Q_n	(3.1.5)
μ	irreducible unramified character of SO_2	5.1
N_1	subgroup of U_2	3.3.6
N_2	subgroup of U_2	3.3.7
N°	unipotent subgroup of H	(3.1.3)
N_n	unipotent radical of Q_n	(3.1.1)
$\overline{N_n}$	opposite unipotent radical of $\overline{Q_n}$	3.1.1
$\mathbb{P}Y'$	quasi-projective subscheme of Y'	2
$\mathcal{Q}$	quadratic form on V_1	4.1
$\mathcal{Q}'$	quadratic form on V_2	4.1
Q_n	standard parabolic subgroup of H	3.1.1
$\overline{Q_n}$	opposite parabolic subgroup of H	3.1.1
w	Weyl group element of H	(3.1.7)
ρ	Weil representation	4.1
τ	irreducible cuspidal representation of GL_n	4.1
T_G	maximal torus of G	3.1
T_H	maximal torus of H	3.1
Θ_f	Theta function	4.1
U_2	maximal unipotent subgroup of G	(2.1.3)
$\overline{U_2}$	opposite of U_2	5.1
V	$V_1 \times V_2$	2
V_i	quadratic space of even dimension	2
W_{ξ_s}	Whittaker function on GL_n when restricted to M_n	4.1.6
$W_{\rho_{\tau,s}}$	local vector in $\mathrm{Ind}_{Q_n}^H(\mathcal{W}(\tau, \psi_0) \otimes \lvert\det\rvert^{s-\frac{1}{2}})$	5.1.1
ξ_s	global smooth holomorphic section in the space $\mathrm{Ind}_{Q_n}^H(\tau \otimes \lvert\det\rvert^{s-\frac{1}{2}})$	4.1
Y	$\{v \in V(R) : Q(v_1) = 2Q'(V_2)\}$	(4.1.3)
Y'	subscheme of Y such that no $y_i = 0$	2

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Chapter 1

Introduction

1.1 The Poisson Summation Conjecture

Let F be a number field and $\mathbb{A}_F$ be its Adele ring. In 1972, generalizing Tate's thesis, Godement and Jacquet [GJ72] build global theory for standard automorphic L-functions for GL_n. The key geometric background is that there is an affine spherical $\mathrm{GL}_n \times \mathrm{GL}_n$-equivariant embedding $\mathrm{GL}_n \hookrightarrow M_n$ of generalizing $\mathbb{G}_m \hookrightarrow \mathbb{G}_a$ in Tate's thesis, where M_n denotes the affine space of $n \times n$ matrices. The main ingredients of the theory are a Schwartz space

$$\mathcal{S}(M_n(\mathbb{A}_F)) = \mathcal{S}(M_n(F_\infty)) \otimes C_c^\infty(M_n(\mathbb{A}_F^\infty)),$$

a Fourier transform $\mathcal{F}_{M_n} : \mathcal{S}(M_n(\mathbb{A}_F)) \to \mathcal{S}(M_n(\mathbb{A}_F))$, equipped with an action $R : \mathcal{S}(M_n(\mathbb{A}_F)) \times \mathrm{GL}_n(\mathbb{A}_F) \times \mathrm{GL}_n(\mathbb{A}_F) \to \mathcal{S}(M_n(\mathbb{A}_F))$, a twisted equivariance property

$$\mathcal{F}_{M_n} \circ R(g_1, g_2) = \frac{|\det g_2|}{|\det g_1|} R(g_1^{-t}, g_2^{-t}) \circ \mathcal{F}_{M_n}$$

for $g_1, g_2 \in \mathrm{GL}_n(\mathbb{A}_F)$, and a Poisson Summation formula

$$\sum_{\gamma \in M_n(F)} f(\gamma) = \sum_{\gamma \in M_n(F)} \mathcal{F}(f)(\gamma)$$

for $f \in \mathcal{S}(M_n(\mathbb{A}_F))$. They formed a zeta integral using the above ingredients and proved the analytic properties of the standard L-functions for GL_n using the Poisson summation formula.

Braverman and Kazhdan in 2000 [BK00] proposed a generalization of Godement-Jacquet theory that conjecturally should give more general Langlands L-Functions $L(s, \pi, \rho)$ for π a cuspidal automorphic representation of reductive groups G and $\rho : \widehat{G} \to \mathrm{GL}_n(\mathbb{C})$. They generalized the embedding $\mathrm{GL}_n \hookrightarrow M_n$ to a $G \times G$-equivariant embedding $G \hookrightarrow X_\rho$, where X_ρ is a normal reductive monoid with unit group G. They conjectured that there should exists a Schwartz space S, a Fourier transform $\mathcal{F}_\rho$ and a Poisson summation formula associated to G and ρ. This program was later refined by L. Lafforgue [Laf14] and Ngô [Ngô14], where L. Lafforgue defined $\mathcal{S}$ using the Plancherel formula of G, and Ngô gave a geometric description for test functions.

Reductive moniods with $G \times G$ action can be viewed as a special case of spherical varieties, which are normal scheme of finite type over F with an action $G \times X \to X$ and X admits an open dense Borel orbit. Conjectures of Sakellaridis [Sak12] suggest that the conjectures for reductive monoids should be generalized to affine spherical varieties. We refer to this conjecture as the Poisson summation conjecture. Specifically, there should exists a conjectural Schwartz space $\mathcal{S}(X)$ which related to the geometric structure of X and a Poisson summation formula for X.

The Poisson summation conjecture implies the meromorphic continuation and functional equations of fairly general Langlands L-functions. By the converse theorem, this implies the existence of many functorial transfers from reductive groups to GL_n. This is a key open problem in the field.

1.2 Poisson Summation Formula for quadratic spaces

In [GL19], such a generalized Poisson summation formula is proved where the underlying scheme is built out of a triple of quadratic spaces. This setting is of particular interest because it is the first case in which the Poisson summation conjecture is known where the underlying affine spherical variety is not a torus bundle over a flag variety. Their method of proof involves replacing the cuspidal representation of $\mathrm{SL}_2^3(\mathbb{A}_F)$ appearing in Garrett's integral representation of the Rankin triple product L-function [Gar87, PSR87] with a θ-function.

This suggests that new summation formulae can be obtained by replacing cusp forms on symplectic groups appearing in known integral representations with θ-functions. In this dissertation, we take another step towards this general program. In more detail, we use the exceptional isogeny $\mathrm{SL}_2 \times \mathrm{SL}_2 \to \mathrm{SO}_4$ to substitute two θ-functions into the Rankin-Selberg integral for $\mathrm{SO}_{2\ell} \times \mathrm{GL}_n$ constructed in [Kap12] in the special case $\ell = 2$. In the next subsection, we state our formula precisely and then give a representation-theoretic interpretation.

Remark. More generally, one may consider substituting restrictions of minimal representations (in the sense of [GS05]) into integral representations of L-functions.

The Rankin-Selberg integral in [Kap12] represents a Langlands-Shahidi L-function, and it is illuminating to consider our procedure from the point of view of the Langlands-Shahidi method. It is well-known that Langlands-Shahidi L-functions can be roughly enumerated by root systems together with a simple root. The Dynkin diagram that remains after deleting the simple root is the Dynkin diagram of a Levi subgroup. Our construction corresponds to the Dynkin diagram D_{n+2} with the unique simple root such that the complement of the root is the Dynkin diagram for $\mathrm{SL}_2 \times \mathrm{SL}_2 \times \mathrm{SL}_n$.

The summation formula we prove in this dissertation enlarges the collection of cases in which we know the Poisson summation conjecture. At present the set of cases is very small ([BK02, GL19, GH20]). This provides crucial test cases to examine for insight into the general picture. Moreover, since the ultimate goal is to study higher rank automorphic L-functions that are currently not understood, incorporating a cusp form of arbitrarily high rank from the outset is a step in the right direction. We also point out that there are methods of building up new summation formulae from old ones modeled on the manner that the summation formula for the basic affine space of [BK99] is built up out of a family of Poisson summation formulae for a two-dimensional symplectic vector space. Thus it is of interest to have various families of summation formulae to serve as building blocks.

1.3 Outline of the Thesis

The structure of this dissertation is as follows. We give an overview of the main results of this dissertation in Chapter 2.

We set up the notation for the various algebraic groups in Chapter 3. We establish and prove our summation formula in Chapter 4. Specifically, in Section 4.1, we establish our summation formula assuming various quantities converge. The main theorem is made rigorous by showing the absolute convergence of the sum of the global integrals in Section 4.2, Section 4.3, and Section 4.4.

In Section 5.1, we give the computation of the local integral when all the data are unramified. We justify in Section 5.2 the absolute convergence of various integrals and the final result in Section 5.1.

We review the results of this thesis and discuss potential future work in Chapter 6.

Chapter 2

Overview of the Results

2.1 Main Results

We give a precise statement of the main results of this dissertation in the section. Let F be a number field and $\mathbb{A}_F$ be its Adele ring. Let d_1, d_2 be two even positive integers, and $V_1 = \mathbb{G}_a^{d_1}, V_2 = \mathbb{G}_a^{d_2}$ be a pair of affine spaces over F equipped with non-degenerate quadratic forms $\mathcal{Q}$ and $\mathcal{Q}'$ respectively. Let $V := V_1 \oplus V_2$.

Let $Y \subset V$ be the closed subscheme whose points in an F-algebra R are

$$Y(R) := \{y = (y_1, y_2) \in V(R) : \mathcal{Q}(y_1) = 2\mathcal{Q}'(y_2)\}. \tag{2.1.1}$$

Below we will use R to denote a "test" F-algebra, sometimes without further comment.

Let

$$V' := \{\gamma = (\gamma_1, \gamma_2) \in V(R) : \gamma_i \neq 0\}.$$

We let $\mathbb{P}Y' \subset \mathbb{P}V$ be the corresponding quasi-projective scheme. This is the scheme attached to the pair of quadratic spaces mentioned above.

Our summation formula will involve functions on Y twisted by Whittaker functions attached to a higher rank cuspidal automorphic representation. Let n be a positive integer. Let τ be a cuspidal automorphic representation of $\mathrm{GL}_n(\mathbb{A}_F)$. Let H

be the split orthogonal group SO_{2n+1}. Let

$$G(R) := \{g = (g_1, g_2) \in \mathrm{GL}_2^2(R) : \det g_1 = \det g_2^{-1}\},$$

and let ξ_s be a smooth holomorphic section from the space

$$\mathrm{Ind}_{Q_n(\mathbb{A})}^{H(\mathbb{A})}(\tau \otimes |\det|^{s-\frac{1}{2}}).$$

Here $Q_n \le H$ is a parabolic subgroup with Levi GL_n. In Section 4.1, following [Sou93, Kap12], we construct a family of inductions of Whittaker functions (lying in $\mathrm{Ind}_{Q_n}^{H}(\mathcal{W}_{\tau,s})$, where $\mathcal{W}_{\tau,s} = \mathcal{W}_\tau \otimes |\det|^{s-\frac{1}{2}}$ and $\mathcal{W}_\tau$ is the Whittaker model for τ)

$$H(\mathbb{A}_F) \times \mathbb{C} \longrightarrow \mathbb{C}$$

$$(h, s) \longmapsto W_{\xi_s}(h, 1).$$

Here W is an indication that this is a Whittaker function on $\mathrm{GL}_n(\mathbb{A}_F)$ for τ when restricted to an appropriate Levi subgroup of H.

Remark. The spaces V, V_1, V_2 have no relationship with the groups G, H and the split $2n+1$-dimensional space H is acting on. We merely require that $n \ge \ell = 2$.

We extend the Weil representation of $\mathrm{SL}_2^2(\mathbb{A}_F)$ on $\mathcal{S}(V(\mathbb{A}_F)) = \mathcal{S}(V_1(\mathbb{A}_F) \times V_2(\mathbb{A}_F))$ to a representation ρ of $G(\mathbb{A}_F)$ on $\mathcal{S}(V_1(\mathbb{A}_F) \times V_2(\mathbb{A}_F) \times \mathbb{A}_F^\times)$ in Section 4.1 via a standard procedure.

We then define for $y \in Y'(\mathbb{A}_F)$ the global integral

$$I(f, W_{\xi_s})(y) = \int_{U_2(\mathbb{A}_F)\backslash G(\mathbb{A}_F)} \rho(g) f(y, 1) \int_{N^\circ(\mathbb{A}_F)} W_{\xi_s}(wu\iota(g), a_y)\psi_1(u)\, du\, dg. \quad (2.1.2)$$

Here

$$U_2 \subset G \tag{2.1.3}$$

is a maximal unipotent subgroup, N° is a certain unipotent subgroup of H (see (3.1.3)), and ι is a embedding map from G to H (see (3.2.2)). Also,

$$a_y = \begin{pmatrix} -4Q'(y_2) & \\ & I_{n-1} \end{pmatrix} \in \mathrm{GL}_n(\mathbb{A}_F) \tag{2.1.4}$$

encodes the value of the quadratic form. We point out that the integral $I(f, W_\xi)(y)$ is only well-defined for $y \in Y(\mathbb{A}_F)$, not for $y \in V(\mathbb{A}_F)$, due to the invariance properties of the Weil representation.

Thus we have a map

$$\mathcal{S}(V(\mathbb{A}_F) \times \mathbb{A}^\times) \otimes \mathrm{Ind}_{Q_n(\mathbb{A}_F)}^{H(\mathbb{A}_F)}(\mathcal{W}_{\tau,s}) \longrightarrow C^\infty(Y'(\mathbb{A}_F)). \tag{2.1.5}$$

We define $\mathcal{S}(Y(\mathbb{A}_F), \tau, s)$ to be the image of (2.1.5). We view this as a Schwartz space of functions on $Y'(\mathbb{A}_F)$ twisted by $\tau \otimes |\det|^{s-\frac{1}{2}}$. One has a restricted direct product decomposition

$$\mathcal{S}(V(\mathbb{A}_F)) \otimes \mathrm{Ind}_{Q_n(\mathbb{A}_F)}^{H(\mathbb{A}_F)}(\mathcal{W}_{\tau,s}) = \otimes_v' \mathcal{S}(V(F_v)) \otimes \mathrm{Ind}_{Q_n(F_v)}^{H(F_v)}(\mathcal{W}_{\tau_v,s}). \tag{2.1.6}$$

Here the restricted direct product is with respect to the basic vectors $\mathbb{1}_{V(\mathcal{O})\times\mathcal{O}^\times} \otimes W_{\rho_{\tau,s}}$, where $\mathcal{O}$ denotes the ring of integers of F, and $W_{\rho_{\tau,s}}$ is the unique normalized spherical vector in $\mathrm{Ind}_{Q_n(F_v)}^{H(F_v)}(\mathcal{W}_{\tau_v,s})$ at unramified places. Thus one has a restricted direct product decomposition

$$\mathcal{S}(Y'(\mathbb{A}_F), \tau, s) = \otimes_v' \mathcal{S}(Y'(F_v), \tau_v, s)$$

with respect to the vectors $I(\mathbb{1}_{V(\mathcal{O})\times\mathcal{O}^\times}, W_{\rho_{\tau,s}})$ at unramified places. We refer to $I(\mathbb{1}_{V(\mathcal{O})\times\mathcal{O}^\times}, W_{\rho_{\tau,s}})$ as the basic function. We give an expression of it in Theorem 2.1.2 below.

The space $\mathcal{S}(Y'(\mathbb{A}_F), \tau, s)$ comes equipped with a correspondence on the last horizontal line of the following diagram:

$$\begin{array}{ccc} \mathcal{S}(V(\mathbb{A}_F)) \otimes \mathrm{Ind}_{Q_n}^H(\mathcal{W}_{\tau,s}) & \xrightarrow{M(\tau,s)} & \mathcal{S}(V(\mathbb{A}_F)) \otimes \mathrm{Ind}_{Q_n}^H(\mathcal{W}_{\tau^\vee,1-s}) \\ \big\downarrow I & & \big\downarrow I \\ \mathcal{S}(Y'(\mathbb{A}_F), \tau, s) & \dashrightarrow & \mathcal{S}(Y'(\mathbb{A}_F), \tau^\vee, 1-s) \end{array}.$$

Here $M(\tau, s)$ is the usual intertwining operator from $\mathrm{Ind}_{Q_n}^H(\mathcal{W}_{\tau,s})$ to $\mathrm{Ind}_{Q_n}^H(\mathcal{W}_{\tau^\vee,1-s})$.

Let

$$V''(R) := \{(\gamma_1, \gamma_2) \in V(R) : \mathcal{Q}(\gamma_1) = \mathcal{Q}'(\gamma_2) = 0\}. \tag{2.1.7}$$

Our summation formula follows:

Theorem 2.1.1. *For $g \in \mathrm{O}(V_1)(\mathbb{A}_F) \times \mathrm{O}(V_2)(\mathbb{A}_F)$, the sum $\sum_{y\in\mathbb{P}Y'(F)} I(f, W_{\xi_s})(gy)$ admits a meromorphic continuation to the whole s-plane. It satisfies a functional equation*

$$\sum_{y\in\mathbb{P}Y'(F)} I(f, W_{\xi_s})(gy) = \sum_{y\in\mathbb{P}Y'(F)} I(f, M(\tau, s)W_{\xi_s})(gy).$$

Here $f(y, 1) \in \mathcal{S}(V(\mathbb{A}_F) \times \mathbb{A}_F^\times)$ is a Schwartz-Bruhat function such that $\rho(g)f(\gamma, u) = 0$ for all $(g, \gamma, u) \in G(F) \times V''(F) \times F^\times$.

The integral $I(f, W_{\xi_s})(y)$ mixes the arithmetic of the quadratic forms $\mathcal{Q}$ and $\mathcal{Q}'$ and the cuspidal automorphic representation τ. It is Eulerian for each y (see the discussion around (4.1.8)). Ideally, one would like an expression for the unramified local factors in terms of a suitable local model for τ and the point y. We achieve this in

Theorem 2.1.2 below. This is far more difficult than the corresponding calculation in [GL19]. To execute it, we adapt an argument appearing in [Kap12], which ultimately relates the integral to the Bessel model of $\mathrm{Ind}_{Q_n}^{H}(\mathcal{W}_{\tau,s})$ attached to a character on a unipotent subgroup of H and a character on SO_2.

Theorem 2.1.2. *For all the data unramified, $\Re(s)$ large, and $d_2 > d_1$, we have*

$$I(\mathbb{1}_{V(\mathcal{O})\times\mathcal{O}^\times}, W_{\rho_{\tau,s}})(y) = \alpha(y_1, y_2)|4Q'(y_2)|^{-\Re(s)+\frac{1}{2}-\frac{n}{2}} \sum_{k=\mathrm{val}(4\mathcal{Q}'(y_2))}^{\infty} q^{(n-2+\frac{d_2}{2})k} C_{k,s}(y).$$

Here $\alpha(y_1, y_2)$ is as defined in Eq. (5.1.3), *and*

$$C_{k,s}(y) = \int_{iI_F+\sigma_1}\int_{iI_F+\sigma_2} \frac{(1-q^{s_2}+(q-1)q^{-\mathrm{val}(y_2)s_2})\zeta_v(-s_2-\frac{d_1}{2}+2)^2\zeta_v(-s_2)^2(\log q)^2}{4\pi^2\gamma(s-s_1+s_2,\chi'\otimes\tau)}$$
$$\times\, q^{(-s_1+s_2)k} B_{\psi_1',s_1}\begin{pmatrix} -4\mathcal{Q}'(y_2)\varpi^k & & \\ & I_{2n-1} & \\ & & (-4\mathcal{Q}'(y_2)\varpi^k)^{-1}\end{pmatrix} ds_1 ds_2,$$

which is the product of the $-k$-th coefficient in q^{s_1} and the k-th coefficient in q^{s_2} of a product of Laurent series in q^{s_1} and q^{s_2}, where γ represents gamma factor for $\mathrm{GL}_1\times\mathrm{GL}_n$, *and B_{ψ_1',s_1} is the normalized unramified Bessel function defined in 5.1.11.*

Remarks.

- Let b_Z be the basic function at unramified places in the conjectural Schwartz space $\mathcal{S}(Z(\mathbb{A}_F))$. Then the quantity computed in Theorem 2.1.2 should correspond to

$$\int_{\mathrm{GL}_n(F)} b_Z(z, g')\xi_s(1, g')dg.$$

- As mentioned before, the key difference between the results in this paper with the work of Kaplan and Soudry is that we substitute the theta functions on

$G(\mathbb{A})$ for cusp forms in their work. In the unramified setting, we must relate $\rho(g)f(y,1)$ to functions lying in an appropriate Whittaker model in order to apply Kaplan and Soudry's methods.

Let us indicate how our constructions are related to the θ-correspondence and Rankin-Selberg L-functions. Let π_i be a cuspidal automorphic representation of $O(V_i)(F)\backslash O(V_i)(\mathbb{A}_F)$ for $i=1,2$. Assume that π_i is the θ-lift of a cuspidal automorphic representation σ_i of $\mathrm{SL}_2(\mathbb{A}_F)$. Assume moreover that the central character of $\sigma_1 \otimes \sigma_2$ is trivial when restricted to the diagonal copy of $\pm I_2$. Then using the isomorphism

$$\pm(I_2, I_2)\backslash \mathrm{SL}_2 \times \mathrm{SL}_2 \longrightarrow \mathrm{SO}_4$$

the representation $\sigma_1 \otimes \sigma_2$ defines a cuspidal automorphic representation σ of SO_4. Let

$$r : {}^L(\mathrm{SO}_4 \times \mathrm{GL}_n) \longrightarrow \mathbb{G}_a^{4n}$$

be the tensor product of the two standard representations. Finally, ϕ_i is a cusp form in the space of π_i.

Then it follows from [Kap12] that the integral

$$\int_{O(V)(F)\backslash O(V)(\mathbb{A}_F)} \int_{U_2(\mathbb{A}_F)\backslash G(\mathbb{A}_F)} \sum_{y \in \mathbb{P}Y'(F)} \rho(g,h) f(h^{-1}y, 1)\phi_1(h_1)\phi_2(h_2)$$

$$\times \int_{N^\circ(\mathbb{A}_F)} W_{\xi_s}(wu\iota(g), a_y)\psi_1(u)\,du\,dg\,dh$$

is Eulerian, with unramified local factors equal to

$$\frac{L(s, \sigma \times \tau, r)}{L(2s, \pi, \mathrm{Sym}^2)}.$$

2.2 Representation theoretic interpretation

We provide an interpretation of our summation formula from a representation-theoretic perspective.

In (3.1.3) and (4.1.2) we define a unipotent subgroup $N^\circ \subset \mathrm{SO}_{2n+1}$ and a character $\psi_1 : N^\circ(F)\backslash N^\circ(\mathbb{A}_F) \to \mathbb{C}$. In particular, $N^\circ(\mathbb{A}_F)$ acts on

$$\mathrm{Ind}_{Q_n(\mathbb{A}_F)}^{H(\mathbb{A}_F)}(\mathcal{W}_{\tau,s})$$

via ψ_1 (where $\mathcal{W}$ denotes the Whittaker model) and we can consider the coinvariants

$$\mathrm{Ind}_{Q_n(\mathbb{A}_F)}^{H(\mathbb{A}_F)}(\mathcal{W}_{\tau,s})_{N^\circ(\mathbb{A}_F),\psi_1}.$$

There is an embedding $\iota : G \to \mathrm{SO}_{2n+1}$. The image normalizes N° and stabilizes ψ_1, and we can consider the coinvariants

$$(\mathrm{Ind}_{Q_n(\mathbb{A}_F)}^{H(\mathbb{A}_F)}(\mathcal{W}_{\tau,s})_{N^\circ(\mathbb{A}_F),\psi_1} \otimes \mathcal{S}(V(\mathbb{A}_F)))_{G(\mathbb{A}_F)}.$$

The integral $I(f, \xi_s, s)$ may be viewed as a functional

$$(\mathrm{Ind}_{Q_n(\mathbb{A}_F)}^{H(\mathbb{A}_F)}(\mathcal{W}_{\tau,s})_{N^\circ(\mathbb{A}_F),\psi_1} \otimes \mathcal{S}(V(\mathbb{A}_F)))_{G(\mathbb{A}_F)} \longrightarrow C^\infty(Y'(\mathbb{A}_F)). \tag{2.2.1}$$

The functional equation in the main theorem (Theorem 2.1.1) ultimately is a consequence of the existence of the intertwining operator on $\mathrm{Ind}_{Q_n}^{H}(\mathcal{W}_{\tau,s})$ associated with the longest Weyl element and the functional equation of the corresponding Eisenstein series defined in Eq. (4.1.1) with respect to the intertwining operator.

Chapter 3

Preliminaries

3.1 Groups

For this section we let F be a field of characteristic zero. Let $\mathcal{O}$ be the ring of integers of F and ϖ be the uniformizer of $\mathcal{O}$. To define points of F-schemes we let R denote an F-algebra. All algebraic groups we define below are affine algebraic groups over F.

Let

$$J_k := \begin{pmatrix} 0 & & 1 \\ & \iddots & \\ 1 & & 0 \end{pmatrix} \in \mathrm{GL}_k(F)$$

for k a positive integer. Let SO_k be the special orthogonal group with respect to J_k.

We say that a parabolic subgroup of SO_k is standard if it contains the Borel subgroup of upper triangular matrices. Let

$$G(R) := \{(g_1, g_2) \in \mathrm{GL}_2^2(R) : \det g_1 = \det g_2^{-1}\} \quad \text{and} \quad H := \mathrm{SO}_{2n+1}. \tag{3.1.1}$$

Let

$$G' := \mathrm{SO}_4. \tag{3.1.2}$$

We denote by $T_{G'}$ and T_H the corresponding maximal split tori consisting of diagonal matrices.

3.1.1 Subgroups of H

Let Q_H be the standard parabolic subgroup with Levi subgroup whose points in an F-algebra R are

$$M_H(R) :=$$

$$\left\{(x,c)^\wedge = \begin{pmatrix} x & & \\ & c & \\ & & x^* \end{pmatrix} \in H : (x,c) \in \mathrm{GL}_{n-2}(R) \times \mathrm{SO}_5(R), x^* = J_{n-2}({}^t x^{-1})J_{n-2}\right\}.$$

Let N_H be the unipotent subgroup whose points in an F-algebra R are

$$N_H(R) := \left\{ \begin{pmatrix} z & x & y \\ & I_5 & x' \\ & & z^* \end{pmatrix} : x \in M_{(n-2)\times 5}(R), y \in M_{n-2}(R), z \in Z_H(R) \right\}.$$

Here $z^* = J_{n-2}({}^t z^{-1})J_{n-2}$, $x' = -J_5({}^t x)J_{n-2}z^*$, and Z_H is the unipotent radical of the Borel subgroup of upper triangular matrices of GL_{n-2}.

We let Y_H be the subgroup of N_H whose points in an F-algebra R are

$$Y_H(R) :=$$

$$\left\{ \begin{pmatrix} z & 0 & 0 & x & 0 \\ & I_2 & 0 & 0 & x' \\ & & 1 & 0 & 0 \\ & & & I_2 & 0 \\ & & & & z^* \end{pmatrix} : z \in Z_H(R), z^* = J_{n-2}({}^t z^{-1})J_{n-2}, x' = -J_2({}^t x)J_{n-2}z^* \right\},$$

and we denote N° the subgroup of N_H whose points in an F-algebra R are

$$N^\circ(R) = \left\{ \begin{pmatrix} I_{n-2} & x & y & 0 & z \\ & I_2 & 0 & 0 & 0 \\ & & 1 & 0 & y' \\ & & & I_2 & x' \\ & & & & I_{n-2} \end{pmatrix} : x' = -J_2({}^t x)J_{n-2}, y' = -J_1({}^t y)J_{n-2} \right\} \tag{3.1.3}$$

such that N° is isomorphic to $Y_H \backslash N_H$.

For $x \in \mathrm{GL}_n(R)$ let

$$v(x) := \begin{pmatrix} x & & \\ & 1 & \\ & & J_n({}^t x^{-1})J_n \end{pmatrix} \in \mathrm{GL}_{2n+1}(R). \tag{3.1.4}$$

Let Q_n be the standard parabolic subgroup with Levi subgroup M_n whose points in an F-algebra R are

$$M_n(R) := \{v(x) : x \in \mathrm{GL}_n(R)\}. \tag{3.1.5}$$

Let N_n be the unipotent subgroup whose points in an F-algebra R are

$$N_n(R) := \left\{ \begin{pmatrix} z & x & y \\ & 1 & x' \\ & & z^* \end{pmatrix} : z \in Z_n, z^* = J_n({}^t z^{-1})J_n, x' = -J_1({}^t x)J_n z^* \right\}, \tag{3.1.6}$$

where Z_n is the unipotent radical of the Borel subgroup of upper triangular matrices of GL_n.

Accordingly, we denote $\overline{Q}_n \subset H$ as the opposite parabolic subgroup with Levi subgroup M_n, and we let $\overline{N}_n$ be the corresponding unipotent radical of $\overline{Q}_n$.

Let

$$w := \begin{pmatrix} & \frac{1}{2}I_2 & & & \\ & & & & I_{n-2} \\ & & (-1)^{n-2} & & \\ I_{n-2} & & & & \\ & & & 2I_2 & \end{pmatrix} \tag{3.1.7}$$

be a Weyl group element in H.

Let $Q_{G'}$ be a subgroup of H whose points in an F-algebra R are

$$Q_{G'}(R) :=$$

$$\left\{ \begin{pmatrix} a & b & c & -2b & d \\ & 1 & 0 & 0 & -2b' \\ & & 1 & 0 & c' \\ & & & 1 & b' \\ & & & & a^{-1} \end{pmatrix} : a \in R^\times, c \in R, b' = -ba^{-1}, c' = -ca^{-1}, d = \frac{-4b'^2 + c'^2}{2a^{-1}} \right\}.$$

3.2 Embedding of the groups

For the construction of the global integral, we use two embeddings of groups. Here we give the explicit maps we use in our integral.

We have a sporadic isogeny between the algebraic groups $\mathrm{SL}_2 \times \mathrm{SL}_2$ and $G' = \mathrm{SO}_4$.

It induces a surjection $G \to G'$ given on points in an F-algebra by

$$\iota_1 : G(R) \longrightarrow G'(R)$$

$$\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}, \begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix} \right) \longmapsto \begin{pmatrix} aa' & -ab' & ba' & bb' \\ -ac' & ad' & -bc' & -bd' \\ ca' & -cb' & da' & db' \\ cc' & -cd' & dc' & dd' \end{pmatrix}. \tag{3.2.1}$$

In the construction in [Kap12, Section 2.1], the embedding of G' in H is given by $\operatorname{diag}(I_{n-2}, G'', I_{n-2})$, where we denote $G'' \subset \mathrm{SO}_5$ as the image of the embedding of G' in $\mathrm{SO}_5 \subset H$. The map $\iota_2 : G' \to \mathrm{SO}_5$ is

$$\begin{pmatrix} a_1 & a_2 & b_1 & b_2 \\ a_3 & a_4 & b_3 & b_4 \\ c_1 & c_2 & d_1 & d_2 \\ c_3 & c_4 & d_3 & d_4 \end{pmatrix} \longmapsto$$

$$\begin{pmatrix} a_1 & \frac{1}{4}a_2 - \frac{1}{2}b_1 & \frac{1}{2}a_2 + b_1 & -\frac{1}{2}a_2 + b_1 & b_2 \\ a_3 - \frac{1}{2}c_1 & \frac{1}{4}a_4 - \frac{1}{2}b_3 + \frac{1}{2} - \frac{1}{8}c_2 + \frac{1}{4}d_1 & \frac{1}{2}a_4 - b_3 - \frac{1}{4}c_2 - \frac{1}{2}d_1 & -\frac{1}{2}a_4 - b_3 + 1\frac{1}{4}c_2 - \frac{1}{2}d_1 & b_4 - \frac{1}{2}d_2 \\ a_3 + \frac{1}{2}c_1 & \frac{1}{4}a_4 - \frac{1}{2}b_3 + \frac{1}{8}c_2 - \frac{1}{4}d_1 & \frac{1}{2}a_4 + b_3 + \frac{1}{4}c_2 + \frac{1}{2}d_1 & -\frac{1}{2}a_4 - b_3 - \frac{1}{4}c_2 + \frac{1}{2}d_1 & b_4 + \frac{1}{2}d_2 \\ -\frac{1}{2}a_3 + \frac{1}{4}c_1 & -\frac{1}{8}a_4 - \frac{1}{4}b_3 + \frac{1}{4} + \frac{1}{16}c_2 - \frac{1}{8}d_1 & -\frac{1}{4}a_4 - \frac{1}{2}b_3 + \frac{1}{8}c_2 + \frac{1}{4}d_1 & \frac{1}{4}a_4 - \frac{1}{2}b_3 + \frac{1}{2} - \frac{1}{8}c_2 + \frac{1}{4}d_1 & -\frac{1}{2}b_4 + \frac{1}{4}d_2 \\ c_3 & \frac{1}{4}c_4 - \frac{1}{2}ds_3 & \frac{1}{2}c_4 + d_3 & -\frac{1}{2}c_4 + d_3 & d_4 \end{pmatrix}.$$

We define the composite map

$$\iota := \iota_2 \circ \iota_1 : G \longrightarrow \mathrm{SO}_5. \tag{3.2.2}$$

3.3 Image of the maps

Using the map from G to $G' = \mathrm{SO}_4$ and G' in SO_5 (which naturally embeds in H), we make the image of subgroups of G in SO_5 precise.

Let M_1 be the subgroup of the maximal torus of G whose points in an F-algebra R are

$$M_1(R) := \left\{ \left(\left(\begin{smallmatrix} 1 & 0 \\ 0 & m^{-1} \end{smallmatrix}\right), \left(\begin{smallmatrix} m & 0 \\ 0 & 1 \end{smallmatrix}\right) \right) : m \in R^\times \right\}. \tag{3.3.1}$$

Lemma 3.3.1. *Let $M_1' = \iota(M_1) \subset \mathrm{SO}_5$. Then*

$$M_1'(R) = \left\{ \begin{pmatrix} m & & \\ & I_3 & \\ & & m^{-1} \end{pmatrix} : m \in R^\times \right\}. \tag{3.3.2}$$

Let G_1 be a subgroup of the maximal torus of G whose points in an F-algebra R are

$$G_1(R) := \left\{ \left(\left(\begin{smallmatrix} 1 & 0 \\ 0 & b^{-1} \end{smallmatrix}\right), \left(\begin{smallmatrix} 1 & 0 \\ 0 & b \end{smallmatrix}\right) \right) : b \in R^\times \right\}. \tag{3.3.3}$$

Lemma 3.3.2. *Let $G_1' = \iota(G_1) < \mathrm{SO}_5$. Then*

$$G_1'(R) = \left\{ \begin{pmatrix} 1 & & & & \\ & \frac{1}{2} + \frac{1}{4}(b + b^{-1}) & \frac{1}{2}(b - b^{-1}) & 2(b - b^{-1}) & \\ & (b - b^{-1}) & \frac{1}{2}(b + b^{-1}) & -\frac{1}{2}(b - b^{-1}) & \\ & \frac{1}{2}(\frac{1}{2} - \frac{1}{4}(b + b^{-1})) & -\frac{1}{4}(b - b^{-1}) & \frac{1}{2} + \frac{1}{4}(b + b^{-1}) & \\ & & & & 1 \end{pmatrix} : b \in R^\times \right\}.$$

Let A_1 be a subgroup of T_G whose points in F-algebra R are

$$A_1(R) := \left\{ \left(\begin{pmatrix} a_1 & 0 \\ 0 & {a_1}^{-1} \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right) : a_1 \in R^\times \right\}. \tag{3.3.4}$$

Lemma 3.3.3. *Let $A'_1 = \iota(A_1)$. Then*

$$A'_1(R) = \left\{ \begin{pmatrix} a_1 & & & & \\ & \frac{1}{2}+\frac{1}{4}(a_1^2+a_1^{-2}) & \frac{1}{2}(a_1^2-a_1^{-2}) & 2(a_1^2-a_1^{-2}) & \\ & (a_1^2-a_1^{-2}) & \frac{1}{2}(a_1^2+a_1^{-2}) & -\frac{1}{2}(a_1^2-a_1^{-2}) & \\ & \frac{1}{2}(\frac{1}{2}-\frac{1}{4}(a_1^2+a_1^{-2})) & -\frac{1}{4}(a_1^2-a_1^{-2}) & \frac{1}{2}+\frac{1}{4}(a_1^2+a_1^{-2}) & \\ & & & & a_1^{-1} \end{pmatrix} : a_1 \in R^\times \right\}.$$

Let A_2 be the subgroup of T_G whose points in F-algebra R are

$$A_2(R) := \left\{ \left(\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} a_2 & 0 \\ 0 & a_2^{-1} \end{pmatrix} \right) : a_2 \in R^\times \right\}. \tag{3.3.5}$$

Lemma 3.3.4. *Let $A'_2 = \iota(A_2)$. Then*

$$A'_2(R) = \left\{ \begin{pmatrix} a_2 & & & & \\ & \frac{1}{2}+\frac{1}{4}(a_2^2+a_2^{-2}) & \frac{1}{2}(-a_2^2+a_2^{-2}) & 2(-a_2^2+a_2^{-2}) & \\ & (-a_2^2+a_2^{-2}) & \frac{1}{2}(a_2^2+a_2^{-2}) & -\frac{1}{2}(a_2^2-a_2^{-2}) & \\ & \frac{1}{2}(\frac{1}{2}-\frac{1}{4}(a_2^2+a_2^{-2})) & -\frac{1}{4}(-a_2^2+a_2^{-2}) & \frac{1}{2}+\frac{1}{4}(a_2^2+a_2^{-2}) & \\ & & & & a_2^{-1} \end{pmatrix} : a_2 \in R^\times \right\}.$$

Note that we have $T_G = A_1 A_2 G_1$.

Let U_2 be the maximal unipotent radical of the Borel subgroup of upper triangular matrices of G. Let N_2 be a subgroup of the U_2 whose points in an F-algebra R are

$$N_1(R) := \left\{ \left(\begin{pmatrix} 1 & \frac{c}{2} \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & -c \\ 0 & 1 \end{pmatrix} \right) : c \in R \right\}. \tag{3.3.6}$$

Lemma 3.3.5. *Let $N_1' = \iota(N_1) < \mathrm{SO}_5$. Then*

$$N_1'(R) = \left\{ \begin{pmatrix} 1 & 0 & c & 0 & -\frac{1}{2}c^2 \\ & 1 & 0 & 0 & 0 \\ & & 1 & 0 & -c \\ & & & 1 & 0 \\ & & & & 1 \end{pmatrix} : c \in R \right\}.$$

Let N_2 be a subgroup of U_2 whose points in an F-algebra R are

$$N_2(R) := \left\{ \left(\begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 2b \\ 0 & 1 \end{pmatrix} \right) : b \in R \right\}. \tag{3.3.7}$$

Lemma 3.3.6. *Let $N_2' = \iota(N_2) < \mathrm{SO}_5$*

$$N_2'(R) = \left\{ \begin{pmatrix} 1 & b & 0 & -2b & 0 \\ & 1 & 0 & 0 & 2b \\ & & 1 & 0 & 0 \\ & & & 1 & -b \\ & & & & 1 \end{pmatrix} : b \in R \right\}.$$

Let $M_{SL_2^2}$ be a subgroup of $\mathrm{SL}_2 \times \mathrm{SL}_2$ whose points in an F-algebra R are

$$M_{SL_2^2}(R) := \left\{ \left(\begin{pmatrix} m & 0 \\ 0 & m^{-1} \end{pmatrix}, \begin{pmatrix} m & 0 \\ 0 & m^{-1} \end{pmatrix} \right) : m \in R^\times \right\}. \tag{3.3.8}$$

Lemma 3.3.7. *Let* $Q_G = M_{SL_2^2} N_1 N_2$. *Then* $\iota(Q_G)(R)$ *is*

$$\left\{ \begin{pmatrix} a^2 & b & c & -2b & d \\ & 1 & 0 & 0 & -2b' \\ & & 1 & 0 & c' \\ & & & 1 & b' \\ & & & & a^{-2} \end{pmatrix} : a \in R^\times, c \in R, b' = -ba^{-1}, c' = -ca^{-1}, d = \frac{-4b'^2 + c'^2}{2a^{-1}} \right\}.$$

3.4 Summary

We have given a Levi decomposition

$$A_1 A_2 G_1 N_1 N_2 = A_1 G_1 M_1 N_1 N_2$$

of the Borel of upper triangular matrices in $G(R) := \{(g_1, g_2) \in \mathrm{GL}_2^2(R) : \det g_1 = \det g_2^{-1}\}$. We let $G' := \mathrm{SO}_4$. Moreover, we have a commutative diagram:

$$\begin{array}{ccc} A_1 A_2 G_1 N_1 N_2 & \hookrightarrow & G \\ & \searrow & \downarrow \iota_1 \\ & & G' := \mathrm{SO}_4 \\ & & \downarrow \iota_2 \\ & & \mathrm{SO}_5 \end{array} \qquad \iota = \iota_2 \circ \iota_1 : G \to \mathrm{SO}_5 .$$

Our main theorems are stated without the use of G', but we require it in the proofs.

3.5 Notations for local fields

Let F be a global field and v a place of F. We denote by $\mathcal{O}$ the ring of integers of F and $\mathcal{O}_v$ the ring of integers of F_v for nonarchimedean v. We denote by ϖ_v a uniformizer for $\mathcal{O}_v$ and $q_v := |\mathcal{O}_v/\varpi_v|$ the residual characteristic. The idelic norm is denoted by $|\cdot|$ and the local norm on F_v (normalized in the usual manner) is denoted by $|\cdot|_v$.

3.6 Measures

We fix a nontrivial character $\psi : F\backslash\mathbb{A}_F \to \mathbb{C}^\times$ and then choose Haar measures on F_v for all places v that are self-dual with respect to ψ_i. This yields a measure on $\mathbb{A}_F$. We let

$$d^\times x_v := \zeta_v(1)\frac{dx_v}{|x|_v}.$$

This is a Haar measure on $F_v^\times$.

For every split reductive group G we fix a maximal compact subgroup $K \leq G(\mathbb{A}_F)$ that is hyperspecial at all finite places and we normalize the Haar measure on $G(F_v)$ so that K_v has measure 1. For the F_v-points of unipotent subgroups we normalize the Haar measure by transporting measures from F_v to the root subgroups in the usual manner.

Chapter 4

Automorphic-Twisted Summation Formula

4.1 The Summation Formula

In this section, we use the Rankin-Selberg integral for $\mathrm{SO}_{2\ell} \times \mathrm{GL}_n$ developed in [Kap12] to deduce the expression of our global integral when we take $\ell = 2$. We state and prove our main theorem of this dissertation in Theorem 4.1.2 assuming the absolute convergence statement.

The main theorem will be made rigorous by showing the absolute convergence of the sum of the global integrals in Section 4.4.

Let F be a number field. We first briefly recall the construction of the Rankin-Selberg integral in [Kap12, Section 3]. Let τ be an irreducible automorphic representation for $\mathrm{GL}_n(\mathbb{A}_F)$.

Let ξ_s be a smooth holomorphic section from the (normalized induction) space

$$\mathrm{Ind}_{Q_n(\mathbb{A}_F)}^{H(\mathbb{A}_F)}(\tau \otimes |\det|^{s-\frac{1}{2}}).$$

We have the Eisenstein series

$$E_{\xi_s}(h) := \sum_{y \in Q_n(F)\backslash H(F)} \xi_s(yh, 1), \tag{4.1.1}$$

where the first variable of ξ_s is on H, and the second variable of ξ_s is on GL_n.

Let ψ be a non-trivial additive character of $F\backslash\mathbb{A}_F$. For $u \in N_H(\mathbb{A}_F)$, let

$$\psi_1(u) := \psi\left(\sum_{i=1}^{n-3} u_{i,i+1} + u_{n-2,n} + \frac{1}{2}u_{n-2,n+2}\right) \tag{4.1.2}$$

be a character of $N_H(\mathbb{A}_F)$, trivial on $N_H(F)$.

Then the ψ_1-coefficient of E_ξ with respect to $N_H(\mathbb{A}_F)$ is

$$E_{\xi_s}^{\psi_1}(h) = \int_{N_H(F)\backslash N_H(\mathbb{A}_F)} E_{\xi_s}(uh)\psi_1(u)du.$$

Let φ be a cusp form on $G(\mathbb{A}_F)$. The Rankin-Selberg integral in this case is

$$I(\varphi, \xi, s) = \int_{G'(F)\backslash G'(\mathbb{A}_F)} \varphi(g)E_{\xi_s}^{\psi_1}(g)dg.$$

This global integral converges absolutely in the whole s-plane except at the poles of the Eisenstein series $E_{\xi_s}(h)$, and the absolute convergence follows from the rapid decay of the cusp form φ and the moderate growth of the Eisenstein series $E_{\xi_s}(h)$.

Let d_1, d_2 be two even positive integers, and $V_1 = \mathbb{G}_a^{d_1}, V_2 = \mathbb{G}_a^{d_2}$ be a pair of affine spaces over F equipped with non-degenerate quadratic forms $\mathcal{Q}$ and $\mathcal{Q}'$ respectively. Let $V := V_1 \oplus V_2$. Let

$$Y(R) := \{y = (y_1, y_2) \in V(R) : \mathcal{Q}(y_1) = 2\mathcal{Q}'(y_2)\}, \tag{4.1.3}$$

and let

$$V' := \{\gamma = (\gamma_1, \gamma_2) \in V(R) : \gamma_i \neq 0\}.$$

We let $\mathbb{P}Y' \subset \mathbb{P}V$ be the corresponding quasi-projective scheme. This is the scheme attached to the pair of quadratic spaces mentioned above.

For a fixed $u \in F^\times$, let ρ_u be the usual Weil representation on $\mathrm{SL}_2^2(\mathbb{A}_F)$ with quadratic forms $u\mathcal{Q}(\gamma)$. Let $f \in \mathcal{S}(V(\mathbb{A}_F) \times \mathbb{A}_F^\times)$.

We construct the Weil representation ρ for $G(\mathbb{A}_F)$ following [YZZ13], which extends the usual Weil representation of $\mathrm{SL}_2^2(\mathbb{A}_F)$ as follows:

$$\rho(g)f(\gamma, u) = \rho_u(g)f(\gamma, u), \quad g \in \mathrm{SL}_2^2(\mathbb{A}_F)$$

$$\rho\left(\left(\begin{pmatrix} 1 & \\ & a \end{pmatrix}, \begin{pmatrix} 1 & \\ & a^{-1} \end{pmatrix}\right)\right)f(\gamma, u) = f(\gamma, a^{-1}u)|a|^{-\frac{\dim V_1}{4}}, \quad a \in \mathbb{A}_F^\times.$$

Here $\mathcal{S}(V(\mathbb{A}_F) \times \mathbb{A}_F^\times)$ is the usual Schwartz space for vector spaces.

Let $f \in \mathcal{S}(V(\mathbb{A}_F) \times \mathbb{A}_F^\times)$ be a Schwartz-Bruhat function such that

$$\rho(g)f(\gamma, u) = 0 \text{ for all } g \in G(\mathbb{A}_F) \text{ and } (\gamma, u) \in V''(F) \times F^\times, \tag{4.1.4}$$

where $V''(R) = \{(\gamma_1, \gamma_2) \in V(R) : \mathcal{Q}(\gamma_1) = \mathcal{Q}'(\gamma_2) = 0\}$.

We let

$$\Theta_f(g) = \sum_{(\gamma,u) \in V(F) \times F^\times} \rho(g)f(\gamma, u)$$

be the theta function on $G(\mathbb{A}_F)$.

Using the formula ofr $I(\varphi, \xi, s)$, we define a global integral as

$$I(\Theta_f, \xi, s) = \int_{G(F)\backslash G(\mathbb{A}_F)} \Theta_f(g) E_{\xi_s}^{\psi_1}(g) dg.$$

Since we take $\rho(g)f(0) = 0$ for all $g \in G(\mathbb{A}_F)$, $\theta_f(g)$ is cuspidal. Thus $I(\Theta_f, \xi, s)$ converges absolutely in the whole s-plane except at the poles of the Eisenstein series

$E^{\psi_1}_{\xi_s}(h)$ similar as the Rankin-Selberg integral $I(\varphi, \xi, s)$.

By the action of Weil representation, we have

$$I(\Theta_f, \xi, s) = \int_{\mathrm{SL}_2^2(F)\backslash G(\mathbb{A}_F)} \sum_{\gamma \in V(F)} \rho(g) f(\gamma, 1) E^{\psi_1}_{\xi_s}(g) dg.$$

We first unfold the Eisenstein series E_{ξ_s} for $\Re(s)$ large.

Lemma 4.1.1. *For $\Re(s)$ large, we have*

$$I(\Theta_f, \xi, s) = \int_{Q_G(F)\backslash G(\mathbb{A}_F)} \sum_{\gamma \in V(F)} \rho(g) f(\gamma, 1) \int_{Y_H(F)\backslash N_H(\mathbb{A})} \xi(w_0 u \iota(g), 1) \psi_1(u) du dg. \tag{4.1.5}$$

Here

$$w_0 = \begin{pmatrix} & & & & I_{n-2} \\ & I_2 & & & \\ & & (-1)^{n-2} & & \\ & & & I_2 & \\ I_{n-2} & & & & \end{pmatrix} \in H.$$

Proof. By the embedding map ι_1 from G to G' (see Eq. (3.2.1)), we have a long exact sequence

$$1 \to \{\pm I_2\} \to \mathrm{SL}_2^2(F) \xrightarrow{\iota_1} G'(F) \xrightarrow{\mathrm{sn}} H^1(F, \pm I_2) \cong F^\times/(F^\times)^2 \to 1,$$

where the map sn denotes the spinor norm.

It follows that

$$\iota_2(G'(F)) = \bigcup_{\epsilon \in F^\times/(F^\times)^2} \begin{pmatrix} I_{n-2} & & & \\ & \epsilon & & \\ & & I_3 & \\ & & & \epsilon^{-1} & \\ & & & & I_{n-2} \end{pmatrix} \iota(\mathrm{SL}_2^2)(F),$$

$$Q_{G'}(F) = \bigcup_{\epsilon \in F^\times/(F^\times)^2} \begin{pmatrix} I_{n-2} & & & \\ & \epsilon & & \\ & & I_3 & \\ & & & \epsilon^{-1} & \\ & & & & I_{n-2} \end{pmatrix} \iota(Q_G)(F).$$

Then

$$Q_G(F)\backslash \mathrm{SL}_2^2(F) \cong \iota(Q_G)(F)\backslash \iota(\mathrm{SL}_2^2)(F) \cong Q_{G'}(F)\backslash \iota_2(G'(F)).$$

By [Kap12, Proof of Proposition 3.1, Page 151-154], after unfolding the Eisenstein series, the only non-vanishing contribution of $E_{\xi_s}^{\psi_1}$ in $I(\Theta_f, \xi, s)$ for $\Re(s)$ large is

$$\sum_{y \in Q_{G'}(F)\backslash \iota_2(G'(F))} \int_{Y_H(F)\backslash N_H(\mathbb{A}_F)} \xi(w_0 y u \iota(g), 1)\psi_1(u) du,$$

which is equivalent to

$$\sum_{y \in Q_G(F)\backslash \mathrm{SL}_2^2(F)} \int_{Y_H(F)\backslash N_H(\mathbb{A}_F)} \xi(w_0 y u \iota(g), 1)\psi_1(u) du.$$

Then we have

$$I_{\Theta_f} = \int_{\mathrm{SL}_2^2(F)\backslash G(\mathbb{A}_F)} \sum_{\gamma\in V(F)} \rho(g) f(\gamma, 1)$$

$$\times \sum_{y\in Q_G(F)\backslash \mathrm{SL}_2^2(F)} \int_{Y^\eta(F)\backslash N_H(\mathbb{A}_F)} \xi(w_0 y u\iota(g), 1)\psi_1(u) du dg$$

$$= \int_{Q_G(F)\backslash G(\mathbb{A}_F)} \sum_{\gamma\in V(F)} \rho(g) f(\gamma, 1) \int_{Y_H(F)\backslash N_H(\mathbb{A}_F)} \xi(w_0 u\iota(g), 1)\psi_1(u) du dg.$$

□

The main theorem of this dissertation is:

Theorem 4.1.2. *Let*

$$I(f, W_{\xi_s})(y) = \int_{U_2(\mathbb{A}_F)\backslash G(\mathbb{A}_F)} \rho(g) f(y, 1) \int_{N^\circ(\mathbb{A}_F)} W_{\xi_s}(wu\iota(g), a_y)\psi_1(u) du dg.$$

Let

$$W_{\xi_s}(wu\iota(g), 1) = \int_{Z_n(F)\backslash Z_n(\mathbb{A})} \xi_s(wu\iota(g), z)\psi_0^{-1}(z) dz, \tag{4.1.6}$$

where w is defined in Eq. (3.1.7) *and f satisfies Eq.* (4.1.4). *For $g \in \mathrm{O}(V_1)(\mathbb{A}_F) \times \mathrm{O}(V_2)(\mathbb{A}_F)$, the sum $\sum_{y\in\mathbb{P}Y'(F)} I(f, W_{\xi_s})(gy)$ admits a meromorphic continuation to the whole s-plane which satisfies a functional equation*

$$\sum_{y\in\mathbb{P}Y'(F)} I(f, W_{\xi_s})(gy) = \sum_{y\in\mathbb{P}Y'(F)} I(f, M(\tau, s)W_{\xi_s})(gy).$$

Here $M(\tau, s)$ is the intertwining operator from $\mathrm{Ind}_{Q_n}^H(\mathcal{W}_{\tau,s})$ to $\mathrm{Ind}_{Q_n}^H(\mathcal{W}_{\tau^\vee}, 1-s)$.

Proof. We use the defining property of the action of the Weil representation on f, and Lemma 3.3.1 through Lemma 3.3.6 to unfold the integral.

Firstly, using Lemma 3.3.1, Lemma 3.3.5, Lemma 3.3.6, and the action of N_1 on f we have

$$\begin{aligned}
I(\Theta_f,\xi,s) &= \int_{Q_G(F)\backslash G(\mathbb{A}_F)} \sum_{\gamma\in V(F)} \rho(g)f(\gamma,1)\int_{Y_H(F)\backslash N_H(\mathbb{A}_F)} \xi_s(w_0u\iota(g),1)\psi_1(u)dudg \\
&= \int_{N_1(\mathbb{A}_F)N_2(F)M_1(F)\backslash G(\mathbb{A}_F)} \int_{N_1(F)\backslash N_1(\mathbb{A}_F)} \sum_{\gamma\in V(F)} \rho(r'g)f(\gamma,1) \\
&\times \int_{Y_H(F)\backslash N_H(\mathbb{A}_F)} \xi_s(w_0ur'\iota(g),1)\psi_1(u)dudr'dg \\
&= \int_{N_1(\mathbb{A})N_2(F)M_1(F)\backslash G(\mathbb{A}_F)} \int_{F\backslash\mathbb{A}_F} \sum_{\gamma\in V(F)} \rho(g)\psi(\frac{c}{2}Q(\gamma_1)-cQ'(\gamma_2))f(\gamma,1) \\
&\times \int_{Y_H(F)\backslash N_H(\mathbb{A}_F)} \xi_s(w_0uc\iota(g),1)\psi_1(u)dudcdg \\
&= \int_{N_1(\mathbb{A}_F)N_2(F)M_1(F)\backslash G(\mathbb{A}_F)} \sum_{\substack{\gamma\in V'(F)\\ Q(\gamma_1)=2Q'(\gamma_2)}} \rho(g)f(\gamma,1) \\
&\times \left(\int_{Y_H(F)\backslash N_H(\mathbb{A}_F)} \xi_s(w_0u\iota(g),1)\psi_1(u)du\right)dg.
\end{aligned}$$

Here the last line holds since by [Kap12, Proof of Propostion 3.1], the function

$$g\mapsto \int_{Y_H(F)\backslash N_H(\mathbb{A}_F)} \xi_s(w_0u\iota(g),1)\psi_1(u)du$$

is invariant on the left for $r'\in N_1(\mathbb{A}_F)$.

Using the action of N_2 on f our integral becomes

$$\int_{N_1(\mathbb{A}_F)N_2(\mathbb{A}_F)M_1(F)\backslash G(\mathbb{A}_F)} \int_{N_2(F)\backslash N_2(\mathbb{A}_F)} \sum_{\substack{\gamma\in V'(F)\\ Q(\gamma_1)=2Q'(\gamma_2)}} \rho(zg)f(\gamma,1)$$

$$\times \int_{Y_H(F)\backslash N_H(\mathbb{A}_F)} \xi_s(w_0 uz\iota(g),1)\psi_1(u)dudzdg$$

$$= \int_{N_1(\mathbb{A}_F)N_2(\mathbb{A}_F)M_1(F)\backslash G(\mathbb{A}_F)} \int_{N_2(F)\backslash N_2(\mathbb{A}_F)} \sum_{\substack{\gamma\in V'(F)\\ Q(\gamma_1)=2Q'(\gamma_2)}} \rho(g)$$

$$\times \psi(bQ(\gamma_1)+2bQ'(\gamma_2))f(\gamma,1) \int_{Y_H(F)\backslash N_H(\mathbb{A}_F)} \xi_s(w_0 uz\iota(g),1)\psi_1(u)dudbdg$$

$$= \int_{N_1(\mathbb{A})N_2(\mathbb{A}_F)M_1(F)\backslash G(\mathbb{A}_F)} \sum_{\substack{\gamma\in V'(F)\\ Q(\gamma_1)=2Q'(\gamma_2)}} \rho(g)f(\gamma,1)$$

$$\times \int_{N_2(F)\backslash N_2(\mathbb{A}_F)} \int_{Y_H(F)\backslash N_H(\mathbb{A})} \xi_s(w_0 uz\iota(g),1)\psi(4bQ'(\gamma_2))\psi_1(u)dudzdg.$$

Using the action of M_1 on f this is

$$\int_{N_1(\mathbb{A}_F)N_2(\mathbb{A}_F)\backslash G(\mathbb{A}_F)} \sum_{\substack{\gamma\in\mathbb{P}V'(F)\\ Q(\gamma_1)=2Q'(\gamma_2)}} \rho(g)f(\gamma,1)$$

$$\times \int_{N_2(F)\backslash N_2(\mathbb{A}_F)} \int_{Y_H(F)\backslash N_H(\mathbb{A}_F)} \xi_s(w_0uz\iota(g),1)\psi_1(u)\psi(4bQ'(\gamma_2))dudzdg$$

$$= \int_{N_1(\mathbb{A}_F)N_2(\mathbb{A}_F)\backslash G(\mathbb{A}_F)} \sum_{y\in\mathbb{P}Y'(F)} \rho(g)f(y,1)$$

$$\times \int_{N_2(F)\backslash N_2(\mathbb{A}_F)} \int_{Y_H(\mathbb{A}_F)\backslash N_H(\mathbb{A}_F)} \int_{Y_H(F)\backslash Y_H(\mathbb{A}_F)} \xi_s(w_0yuz\iota(g),1)$$

$$\times \psi_1(nu)\psi(4b\mathcal{Q}'(y_2))dndudzdg.$$

As in [Kap13, Page 42], for fixed u and g, the function on $N_2(\mathbb{A}_F)$

$$z \mapsto \int_{Y_H(F)\backslash Y_H(\mathbb{A}_F)} \xi_s(w_0nuzg,1)\psi_1(nu)\psi(4b\mathcal{Q}'(y_2))dn$$

is well-defined since the elements of $N_2(\mathbb{A}_F)$ and $Y_H(\mathbb{A}_F)$ commute. Also, since z normalizes $Y_H(\mathbb{A}_F)$ and stabilizes $\psi_1(y)$ and $\xi_s(w_0z,1) = \xi_s(w_0,1)$, the function is left-invariant on $N_2(F)$. The mapping on $N_H(\mathbb{A}_F)$

$$u \mapsto \int_{N_2(F)\backslash N_2(\mathbb{A}_F)} \int_{Y_H(F)\backslash Y_H(\mathbb{A}_F)} \xi_s(w_0nuzg,1)\psi_1(yu)\psi(4b\mathcal{Q}'(y_2))dndz$$

is left-invariant by $Y_H(\mathbb{A}_F)$. Also, z in the integral normalizes $N_H(\mathbb{A}_F)$ and stabilizes

ψ_1. Thus we can interchange uz to zu in the integral. Then we have

$$I(\Theta_f, \xi, s) = \int_{N_2(\mathbb{A}_F)N_2'(\mathbb{A}_F)\backslash G(\mathbb{A}_F)} \sum_{y \in \mathbb{P}Y'(F)} \rho(g) f(y, 1)$$

$$\times \int_{Y_H(\mathbb{A}_F)\backslash N_H(\mathbb{A}_F)} \int_{N_2'(F)\backslash N_2'(\mathbb{A}_F)} \int_{Y_H(F)\backslash Y_H(\mathbb{A}_F)} \xi_s((w_0 n z w_0^{-1}) w_0 u \iota(g), 1)$$

$$\times \psi_1(yu)\psi(4b\mathcal{Q}'(y_2)) dndudzdg.$$

Let

$$w' := \begin{pmatrix} 0 & I_2 \\ I_{n-2} & 0 \end{pmatrix}.$$

As in [Kap13, Page 43], the double integral $\int_{N_2'(F)\backslash N_2'(\mathbb{A}_F)} \int_{Y_H(F)\backslash Y_H(\mathbb{A}_F)}$ can be written as $\int_{\tilde{Z}_n(F)\backslash \tilde{Z}_n(\mathbb{A}_F)}$, where $w'\tilde{Z}_n w'^{-1} = Z_n$.

We note that now the character on the group Z_n is

$$\psi_1'(z) = \psi(-4\mathcal{Q}'(y_2) z_{1,2} + \frac{1}{2} z_{2,3} + \sum_{i=3}^{n-1} z_{i,i+1})$$

for $z \in Z_n(\mathbb{A}_F)$.

We use a conjugation by

$$\begin{pmatrix} \frac{1}{2} I_2 & 0 \\ 0 & I_{n-2} \end{pmatrix}$$

to replace the character ψ_1' to a character $\psi_{y,\mathcal{Q}'}$, where $\psi_{y,\mathcal{Q}'}$ is the generic character of Z_n such that

$$\psi_{y,\mathcal{Q}'}(z) = \psi(-4\mathcal{Q}'(y_2)z_{1,2} + z_{2,3} + \cdots + z_{n-1,n}). \tag{4.1.7}$$

Then we have

$$\begin{aligned} I(\Theta_f, \xi, s) &= \int_{N_2(\mathbb{A}_F)N_2'(\mathbb{A}_F)\backslash G(\mathbb{A}_F)} \sum_{y \in \mathbb{P}Y'(F)} \rho(g) f(y,1) \\ &\times \int_{Y_H(\mathbb{A}_F)\backslash N_H(\mathbb{A}_F)} W_{\xi_s}^{\psi_{y,\mathcal{Q}'}}(wu\iota(g), 1)\psi_1(u)dudg \\ &= \sum_{y \in \mathbb{P}Y'(F)} \int_{U_2(\mathbb{A}_F)\backslash G(\mathbb{A}_F)} \rho(g) f(y,1) \\ &\times \int_{Y_H(\mathbb{A}_F)\backslash N_H(\mathbb{A}_F)} W_{\xi_s}^{\psi_{y,\mathcal{Q}'}}(wu\iota(g), 1)\psi_1(u)dudg, \end{aligned}$$

where

$$W_{\xi_s}^{\psi_{y,\mathcal{Q}'}}(wug, 1) = \int_{Z_n(F)\backslash Z_n(\mathbb{A})} \xi_s(wu\iota(g), z)\psi_{y,\mathcal{Q}'}^{-1}(z)dz.$$

We recall that $W_{\xi_s}^{\psi_{y,\mathcal{Q}'}}$ may be factored into a product of local Whittaker functions if ξ_s is a pure tensor, and hence the integral

$$\int_{U_2(\mathbb{A}_F)\backslash G(\mathbb{A}_F)} \rho(g) f(y,1) \int_{Y_H(\mathbb{A}_F)\backslash N_H(\mathbb{A}_F)} W_{\xi_s}^{\psi_{y,\mathcal{Q}'}}(wu\iota(g), 1)\psi_1(u)dudg \tag{4.1.8}$$

is Eulerian. Thus $I(\Theta_f, \xi, s)$ is a sum of Eulerian integrals.

Lemma 4.1.3. *We have*

$$W_{\xi_s}^{\psi_{y,\mathcal{Q}'}}(g, 1) = W_{\xi_s}(g, a_y),$$

where

$$W_{\xi_s}(g, a_y) = \int_{Z_n(F)\backslash Z_n(\mathbb{A})} \xi_s(wu\iota(g), a_y z)\psi_0^{-1}(z)dz.$$

Here ψ_0 *is the standard character on* $Z_n(\mathbb{A})$.

Proof. We have

$$\psi_0(a_y z a_y^{-1}) = \psi_{y,Q'}(z)$$

for $z \in Z_n$.

Thus we have the function

$$g \mapsto W_{\xi_s}(g, a_y)$$

lies in $\mathrm{Ind}_{Q_n}^H(\mathcal{W}_{\tau,s,\psi_{y,Q'}})$ since

$$W_{\xi_s}(g, a_y z') = W_{\xi_s}(g, (a_y z' a_y^{-1})a_y) = \psi_0(a_y z' a_y^{-1})W_{\xi_s}(g, a_y) = \psi_{y,Q'}(z')W_{\xi_s}(g, a_y)$$

for $z' \in Z_n(\mathbb{A})$. □

Thus we have

$$I(\Theta_f, \xi, s) = \sum_{y \in \mathbb{P}Y'(F)} \int_{U_2(\mathbb{A}_F)\backslash G(\mathbb{A}_F)} \rho(g)f(y, 1)$$

$$\times \int_{Y_H(\mathbb{A}_F)\backslash N_H(\mathbb{A}_F)} W_{\xi_s}(wu\iota(g), a_y)\psi_1(u)dudg.$$

The manipulations of the integral will be justified in Section 4.4 by showing the sum converges absolutely for $\Re(s)$ large. Then for $\Re(s)$ large, we have

$$I(\Theta_f, \xi, s) = \sum_{y \in \mathbb{P}Y'(F)} I(f, W_{\xi_s})(y). \tag{4.1.9}$$

Thus we have that $\sum_{y\in\mathbb{P}Y'(F)} I(f, W_{\xi_s})(y)$ admits a meromorphic continuation to all s-plane.

By the functional equation of the Eisenstein series E_{ξ_s}, we obtain the desired functional equation for the sum of the global integral. Also, the poles of our sum of integrals come from the poles of E_{ξ_s} [GPSR87, BG92]. □

4.2 Bounds of the local integrals in the non-Archimedean case

Let F be a non-Archimedean local field of characteristic zero. Let $K_G = G(\mathcal{O})$, and let $K_H = \mathrm{SO}_{2n+1}(\mathcal{O})$. In this section we give bounds for the local factors of the global integral $I(f, W_{\xi_s})$ in the non-Archimedean case.

In Lemma 4.2.1 and Lemma 4.2.2 we first bound the inner integral of our local integral using some techniques from the proof of convergence of non-Archimedean Rankin-Selberg integral for $\mathrm{SO}_{2\ell+1} \times \mathrm{GL}_n$ in [Sou93, Section 4].

We give a bound for our local integral in the general case in Lemma 4.2.3, and then in the unramified case in Lemma 4.2.5.

The local integral is

$$I_s(y) := I(f, W_{\rho_{\tau,s}})(y) = \int_{U_2(F)\backslash G(F)} \rho(g) f(y, 1) \int_{N^\circ(F)} W_{\rho_{\tau,s}}(wug, a_y)\psi_1(u)dudg.$$

For $t = \mathrm{diag}(t_1, \ldots, t_n) \in \mathrm{GL}_n(F)$, let

$$t' := \begin{pmatrix} t & & \\ & 1 & \\ & & w_0 t^{-1} w_0 \end{pmatrix} \in T_H(F), \tag{4.2.1}$$

where $w_0 \in \mathrm{GL}_n(F)$ is the antidiagonal matrix.

For a quasi-character $\eta : F^\times \to \mathbb{C}^\times$ there is a unique real number $\Re(\eta)$ such that $\eta|\cdot|^{-\Re(\eta)}$ is unitary. We say η is **positive** if $\Re(\eta) > 0$.

Lemma 4.2.1. *Let $(n, t', k) \in N_n(F) \times T_H(F) \times K_H$, we have*

$$|W_{\rho_{\tau,s}}(nt'k, a_y)| \leq |\det t|^{\Re(s)+\frac{n-1}{2}} \sum_{j=1}^{l} c_{j,s}\eta_j(a_y t).$$

Here $c_{j,s} \in \mathbb{C}$ and η_j are positive quasi-characters of $T_H(F)$ which depend on τ.

Proof. We use an argument analogous to [Sou93, Lemma 4.4]. Since $W_{\rho_{\tau,s}}$ is K_H-finite, we have

$$W_{\rho_{\tau,s}}(k, t) = \sum_i y_{i,s}(k) W_i(a_y t)$$

where $y_{i,s}(k)$ are matrix coefficients of K_H and $W_i \in W(\tau, \psi^{-1}_{y,\mathcal{Q}'})$. Since each W_i can be majorized by a gauge, there are positive quasicharacters η_j of T_n such that

$$|W_i(a_y t)| \leq \sum_{j=1}^{l} c_{i,j}\eta_j(a_y t),$$

and $|\frac{t_i}{t_{i+1}}|$ for $i = 1, \ldots, n-1$ are bounded (independent of k). We have

$$|W_{\rho_{\tau,s}}(na'k, a_y)| = |\det t|^{\Re(s)-\frac{1}{2}+\frac{n}{2}} |W_{\rho_{\tau,s}}(k, a_y t)|.$$

then the assertion is clear. $\square$

The points of the group $wN^\circ w^{-1}$ in an F-algebra R are

$$wN^\circ(R)w^{-1} =$$

$$\left\{ \begin{pmatrix} 1 & & & & & & \\ & 1 & & & & & \\ & & I_{n-2} & & & & \\ 0 & 0 & v_3' & 1 & & & \\ v_1 & v_2 & T & v_3 & I_{n-2} & & \\ 0 & 0 & v_2' & 0 & & 1 & \\ 0 & 0 & v_1' & 0 & & & 1 \end{pmatrix} : v_1, v_2, v_3 \in R^{n-2}, v_i' = -{}^t v_i J_{n-2}, T \in M_{n-2}(R) \right\}.$$

Let $v \in H(F)$ be a unipotent element of H of the form

$$v = \begin{pmatrix} 1 & & & & & & \\ & 1 & & c & & -\frac{1}{2}c^2 & \\ & & I_{n-2} & & & & \\ & & & 1 & & -c & \\ & & & & I_{n-2} & & \\ & & & & & 1 & \\ & & & & & & 1 \end{pmatrix} \tag{4.2.2}$$

for some $c \in F$.

Lemma 4.2.2. *For $\Re(s)$ large, the integral*

$$\int_{wN^\circ(F)w^{-1}} |W_{\rho_{\tau,s}}(uv, 1)| du$$

converges.

Proof. For $u \in wN^\circ(F)w^{-1}$, we have

$$uv = \begin{pmatrix} 1 & & & & & & \\ & 1 & & c & & -\frac{1}{2}c^2 & \\ & & I_{n-2} & & & & \\ 0 & 0 & v_3' & 1 & & -c & \\ v_1 & v_2 & T & v_2c+v_3 & I_{n-2} & -\frac{1}{2}v_2c^2-v_3c & \\ 0 & 0 & v_2' & 0 & & 1 & \\ 0 & 0 & v_1' & 0 & & & 1 \end{pmatrix}. \tag{4.2.3}$$

We denote the Iwasawa decomposition of uv as $uv = nt'k$ where $(n, t', k) \in$

$N_n(F) \times T_H(F) \times K_H$, and we denote the i-th line of uv as $(uv)_i$.

By Lemma 4.2.1, the integral is majorized by

$$\sum_{j=1}^{\nu} c_{j,s} \int_{wN^{\circ}(F)w^{-1}} [D(uv)]^{\Re(s)+\frac{n-1}{2}} E_j(uv)du, \tag{4.2.4}$$

where

$$D(nt'k) = |\det t|,$$

$$E_j(nt'k) = \eta_j(t).$$

We use techniques following [Sou93, Lemma 1, Section 11.15]. Let $\{e_1, \ldots, e_{2n+1}\}$ be the standard basis of F^{2n+1}. We take the sup-norm on $\wedge^p F^{2n+1}$ according to the basis $\{e_{i_1} \wedge e_{i_2} \wedge \cdots \wedge e_{i_p} | 1 \leq i_1 < \cdots < i_p \leq n\}$. K_H preserves this norm. We have

$$\|v_1 \wedge v_2 \wedge \cdots \wedge v_p\| \leq \|v_1\| \cdot \|v_2\| \cdots \|v_p\|, v_j \in F^{2n+1}.$$

Let $e_{n+1+j} = e_{-n+j-1}$ for $j = 1, \ldots, n$, we have

$$\begin{aligned} |t_{j+1} \cdots t_1| &= \|(e_{-(j+1)}uv) \wedge \cdots (e_{-1}uv)\| \\ &= \|(e_{-(j+1)} + (uv)_{2n+1-j}) \wedge \cdots (e_{-1} + (uv)_{2n+1})\| \\ &\leq \prod_{i=0}^{j} \max\{1, \|(uv)_{2n+1-j}\|\} = \prod_{i=0}^{j} [(uv)_{2n+1-j}]. \end{aligned}$$

Here uv is a matrix, $e_{-j+1}uv$ is a vector, $[(uv)_{2n+1-j}] = \max\{1, \|(uv)_{2n+1-j}\|\}$ and

$\|\cdot\|$ denotes the sup-norm. For

$$h = \begin{pmatrix} h_1 \\ \vdots \\ h_{2n+1} \end{pmatrix} \in H(F),$$

we let

$$\mathcal{L}(h) = \begin{pmatrix} h_{n+2} \\ \vdots \\ h_{2n+1} \end{pmatrix} \tag{4.2.5}$$

be the bottom n rows of h.

Since the coordinates of $\mathcal{L}(uv)$ appear in the coefficients of

$$t_{j+1} \cdots t_1 = e_{-(j+1)}\mathcal{L}(uv) \wedge \cdots e_{-1}\mathcal{L}(uv),$$

we have

$$|t_{j+1} \cdots t_1|^{-1} \geq \max\{1, \|\overline{x}'_{2n+1-j}\|, \ldots, \|\overline{x}'_{2n+1}\|\}.$$

We denote

$$[\mathcal{L}(uv)] = \max\{1, \|\mathcal{L}(uv)\|\}, \tag{4.2.6}$$

where $\|\cdot\|$ is the sup-norm.

Then we have

$$[\mathcal{L}(uv)]^{-2j} \leq |\frac{t_j}{t_{j+1}}| \leq [\mathcal{L}(uv)]^{2j}, j = 1, \ldots, n-1, \tag{4.2.7}$$

and

$$[\mathcal{L}(uv)]^{-n} \leq D(u'v) \leq [\mathcal{L}(uv)]^{-1}. \tag{4.2.8}$$

Since $[\mathcal{L}(uv)] \leq [u][v]$, we have

$$E_j(uv) \leq [\mathcal{L}(uv)]^C \leq [u]^C[v]^C \tag{4.2.9}$$

for some positive constant C which depends only on τ.

By the structure of $\mathcal{L}(uv)$, we have

$$\begin{aligned}
[\mathcal{L}(uv)]^{-1} &= \max\{1, \|v_1\|, \|v_2\|, \|v_3\|, \|T\|, \|-\frac{1}{2}v_2c^2\|, \|v_2c\|, \|v_3c\|\}^{-1} \\
&\leq \max\{1, \|v_1\|, \|v_2\|, \|v_3\|, \|T\|\}^{-1} \\
&= [u]^{-1}.
\end{aligned}$$

Here the middle line holds since if $\max\{1, \|-\frac{1}{2}v_2c^2\|, \|v_2c\|, \|v_3c\|\} = 1$, we have equality, and we have

$$\begin{aligned}
&\max\{1, \|v_1\|, \|v_2\|, \|v_3\|, \|T\|, \|-\frac{1}{2}v_2c^2\|, \|v_2c\|, \|v_3c\|\} \\
&> \max\{1, \|v_1\|, \|v_2\|, \|v_3\|, \|T\|\}
\end{aligned}$$

otherwise.

Thus we have

$$D(uv) \leq [\mathcal{L}(uv)]^{-1} \leq [u]^{-1}. \tag{4.2.10}$$

By Eq. (4.2.9) and Eq. (4.2.10), for $\Re(s) + \frac{n-1}{2} - C > 0$, Eq. (4.2.4) is bounded by

$$\sum_{j=1}^{\nu} c_{j,s}[v]^C \int_{wN^\circ(F)w^{-1}} [u]^{-\Re(s)-\frac{n-1}{2}+C} du \tag{4.2.11}$$

which converges absolutely.

□

Now we proceed to bound our local integral in the general case.

Lemma 4.2.3. *For $\Re(s)$ large enough, we have*

$$I_s(y) \ll \int_{F^\times} \int_{F^\times} f'(a_1y_1, a_2y_2)|a_1|^{\Re(s)+\frac{d_1-n-1}{2}}|a_2|^{\Re(s)+\frac{d_2-n-1}{2}} d^\times a_1 d^\times a_2.$$

Here $f' \in \mathcal{S}(V(F))$ and $\ll$ is synonymous with the big O notation.

Proof. We apply the Iwasawa decomposition of $U_2(F)\backslash G(F)$ with respect to the usual Borel subgroup of $G(F)$. Since $W_{\rho_{\tau,s}}$ is smooth, it suffices to bound

$$\int_{T_G(F)} \int_{K_G} |\rho(ak)f(y,1)|\delta_{B_G}^{-1}(a) \int_{N^\circ(F)} |W_{\rho_{\tau,s}}(wu\iota(ak), a_y)|dudadk.$$

By the defining property of the Weil representation, the above integral is

$$\int_{G_1}\int_{A_1(F)}\int_{A_2(F)}\int_{K_G} |\rho(xa_1a_2k)f(y,1)|\delta_{B_G}^{-1}(a_1a_2)$$

$$\times \int_{N^\circ(F)} |W_{\rho_{\tau,s}}|(wu\iota(xa_1a_2k),a_y)duda_1da_2dk$$

$$= \int_{F^\times}\int_{F^\times}\int_{F^\times}\int_{K_G} |\rho(k)f(a_1y_1,a_2y_2,b)||a_1|^{\frac{d_1}{2}}|a_2|^{\frac{d_2}{2}}|a_1a_2|^{-2}$$

$$\times \int_{N^\circ(F)} |W_{s,\phi_Q}|(wu\iota\left(\begin{pmatrix} a_1 & \\ & a_1^{-1}b^{-1}\end{pmatrix}, \begin{pmatrix} a_2 & \\ & a_2^{-1}b\end{pmatrix}\right)\iota(k),a_y)dud^\times bd^\times a_1d^\times a_2dk.$$

Here $\iota\left(\begin{pmatrix} a_1 & \\ & a_1^{-1}b^{-1}\end{pmatrix}, \begin{pmatrix} a_2 & \\ & a_2^{-1}b\end{pmatrix}\right)$ is

$$\begin{pmatrix} I_{n-1} &&&&&& \\ & a_1a_2 &&&&& \\ && \frac{1}{2}+\frac{1}{4}(b^{-1}a_1a_2^{-1}+ba_1^{-1}a_2) & \frac{1}{2}(b^{-1}a_1a_2^{-1}-ba_1^{-1}a_2) & 2(b^{-1}a_1a_2^{-1}-ba_1^{-1}a_2) && \\ && (b^{-1}a_1a_2^{-1}-ba_1^{-1}a_2) & \frac{1}{2}(b^{-1}a_1a_2^{-1}+ba_1^{-1}a_2) & -\frac{1}{2}(b^{-1}a_1a_2^{-1}-ba_1^{-1}a_2) && \\ && \frac{1}{2}(\frac{1}{2}-\frac{1}{4}(b^{-1}a_1a_2^{-1}+ba_1^{-1}a_2)) & -\frac{1}{4}(b^{-1}a_1a_2^{-1}-ba_1^{-1}a_2) & \frac{1}{2}+\frac{1}{4}(b^{-1}a_1a_2^{-1}+ba_1^{-1}a_2) && \\ &&&&& (a_1a_2)^{-1} & \\ &&&&&& I_{n-1}\end{pmatrix}.$$

Apply the Iwasawa decomposition to $\mathrm{SO}_3(F)$ we have for $b\in F^\times$,

$$\begin{pmatrix} \frac{1}{2}+\frac{1}{4}(b+b^{-1}) & \frac{1}{2}(b-b^{-1}) & 2(b-b^{-1}) \\ (b-b^{-1}) & \frac{1}{2}(b+b^{-1}) & -\frac{1}{2}(b-b^{-1}) \\ \frac{1}{2}(\frac{1}{2}-\frac{1}{4}(b+b^{-1})) & -\frac{1}{4}(b-b^{-1}) & \frac{1}{2}+\frac{1}{4}(b+b^{-1})\end{pmatrix} = \begin{pmatrix} 1 & c & -\frac{1}{2}c^2 \\ 0 & 1 & -c \\ 0 & 0 & 1\end{pmatrix}\begin{pmatrix} \lfloor b\rceil & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \lfloor b\rceil^{-1}\end{pmatrix}k'.$$

Here $k'\in\mathrm{SO}_3(\mathcal{O})\subset K_H$, and

$$\lfloor b\rceil = \begin{cases} b \text{ if } |b|\le 1 \\ b^{-1} \text{ if } |b|>1\end{cases}, \tag{4.2.12}$$

$$c = \begin{cases} -2 & \text{if } \lfloor b \rceil = b \\ 2 & \text{if } \lfloor b \rceil = b^{-1} \end{cases}.$$

We also notice that

$$\begin{pmatrix} 1 & c & -\frac{1}{2}c^2 \\ 0 & 1 & -c \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \lfloor b^2 \rceil & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \lfloor b \rceil^{-1} \end{pmatrix} = \begin{pmatrix} \lfloor b \rceil & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \lfloor b \rceil^{-1} \end{pmatrix} \begin{pmatrix} 1 & c\lfloor b \rceil^{-1} & -\frac{1}{2}c^2 \lfloor b \rceil^{-2} \\ 0 & 1 & -c\lfloor b \rceil^{-1} \\ 0 & 0 & 1 \end{pmatrix}.$$

Thus our local integral is majorized by

$$\begin{aligned} &\int_{F^\times} \int_{F^\times} \int_{F^\times} \int_{K_G} |\rho(k) f(a_1 y_1, a_2 y_2, b)| |a_1|^{\frac{d_1}{2}} |a_2|^{\frac{d_2}{2}} |a_1 a_2|^{-2} \\ &\times \int_{N^\circ(F)} |W_{\rho_{\tau,s}}(wu \operatorname{diag}(I_{n-2}, a_1 a_2, I_3, (a_1 a_2)^{-1}, I_{n-2}) t n' k'' \iota(k), a_y)| \\ &\times du d^\times b d^\times a_1 d^\times a_2 dk. \end{aligned} \tag{4.2.13}$$

Here

$$t = \begin{pmatrix} I_{n-1} & & & & \\ & \lfloor b^{-1} a_1 a_2^{-1} \rceil & & & \\ & & 1 & & \\ & & & \lfloor b a_1^{-1} a_2 \rceil & \\ & & & & I_{n-1} \end{pmatrix}, \tag{4.2.14}$$

$$n' = \begin{pmatrix} I_{n-1} & & & & \\ & 1 & c\lfloor b a_1^{-1} a_2 \rceil & -\frac{1}{2}c^2 \lfloor b a_1^{-1} a_2 \rceil^2 & \\ & & 1 & -c\lfloor b a_1^{-1} a_2 \rceil & \\ & & & 1 & \\ & & & & I_{n-1} \end{pmatrix}, \tag{4.2.15}$$

$$k'' = \begin{pmatrix} I_{n-1} & & \\ & k' & \\ & & I_{n-1} \end{pmatrix}. \tag{4.2.16}$$

We have

$$w \begin{pmatrix} I_{n-2} &&&&&& \\ & a_1a_2 &&&&& \\ && \lfloor b^{-1}a_1a_2^{-1} \rfloor &&&& \\ &&& 1 &&& \\ &&&& \lfloor ba_1^{-1}a_2 \rfloor && \\ &&&&& (a_1a_2)^{-1} & \\ &&&&&& I_{n-2} \end{pmatrix} w^{-1}$$

$$= \begin{pmatrix} a_1a_2 &&&&&& \\ & \lfloor b^{-1}a_1a_2^{-1} \rfloor &&&&& \\ && I_{n-2} &&&& \\ &&& 1 &&& \\ &&&& I_{n-2} && \\ &&&&& \lfloor ba_1^{-1}a_2 \rfloor & \\ &&&&&& (a_1a_2)^{-1} \end{pmatrix}.$$

Then by the property of $W_{\rho_{\tau,s}}$ Eq. (4.2.13) is

$$\int_{F^\times}\int_{F^\times}\int_{F^\times}\int_{K_G} |\rho(k)f(a_1y_1, a_2y_2, b)||a_1|^{\frac{d_1}{2}}|a_2|^{\frac{d_2}{2}}|a_1a_2|^{\Re(s)-\frac{n+1}{2}}|\lfloor b^{-1}a_1a_2^{-1} \rfloor|^{\Re(s)-\frac{n-3}{2}}$$

$$\times \int_{N^\circ(F)} |W_{\rho_{\tau,s}}(wun'k''\iota(k), a_y \mathrm{diag}(a_1a_2, \lfloor b^{-1}a_1a_2^{-1} \rfloor, I_{n-2}))| du d^\times b d^\times a_1 d^\times a_2 dk.$$

This is

$$\int_{F^\times}\int_{F^\times}\int_{F^\times}\int_{K_G} |\rho(k)f(a_1y_1, a_2y_2, b)||a_1|^{\frac{d_1}{2}}|a_2|^{\frac{d_2}{2}}|a_1a_2|^{\Re(s)-\frac{n+1}{2}}|\lfloor b^{-1}a_1a_2^{-1} \rfloor|^{\Re(s)-\frac{n-3}{2}}$$

$$\times \int_{N^\circ(F)} |W_{\rho_{\tau,s}}((wuw^{-1})(wn'w^{-1})wk''\iota(k), a_y \mathrm{diag}(a_1a_2, \lfloor b^{-1}a_1a_2^{-1} \rfloor, I_{n-2}))|$$

$$\times du d^\times b d^\times a_1 d^\times a_2 dk.$$

Since

$$wn'w^{-1} = \begin{pmatrix} 1 &&&&&& \\ & 1 && -\frac{1}{2}c\lfloor ba_1^{-1}a_2 \rfloor^{-1} && -\frac{1}{4}c^2\lfloor ba_1^{-1}a_2 \rfloor^2 & \\ && I_{n-2} &&&& \\ &&& 1 && \frac{1}{2}c\lfloor ba_1^{-1}a_2 \rfloor & \\ &&&& I_{n-2} && \\ &&&&& 1 & \\ &&&&&& 1 \end{pmatrix},$$

we have

$$[wn'w^{-1}] \ll |\lfloor b^{-1}a_1a_2^{-1}\rfloor|^{-2}.$$

Then the above integral is majorized by

$$\int_{F^\times}\int_{F^\times}\int_{F^\times}\int_{K_G} |\rho(k)f(a_1y_1,a_2y_2,b)||a_1|^{\frac{d_1}{2}}|a_2|^{\frac{d_2}{2}}|a_1a_2|^{\Re(s)-\frac{n+1}{2}}|\lfloor b^{-1}a_1a_2^{-1}\rfloor|^{\Re(s)-\frac{n-3}{2}}$$

$$\times \int_{wN^\circ(F)w^{-1}} |W_{\rho_{\tau,s}}(u(wn'w^{-1})wk''\iota(k), a_y \mathrm{diag}(a_1a_2, \lfloor b^{-1}a_1a_2^{-1}\rfloor, I_{n-2}))|$$

$$\times\, du d^\times b d^\times a_1 d^\times a_2 dk.$$

Since $wk''k \in K_H$, we apply Lemma 4.2.1 and Lemma 4.2.2. Then the local integral is majorized by

$$\sum_{j=1}^{\nu} c_{j,s} \int_{F^\times}\int_{F^\times}\int_{F^\times}\int_{K_G} |\rho(k)f(a_1y_1,a_2y_2,b)||a_1|^{\frac{d_1}{2}}|a_2|^{\frac{d_2}{2}}|a_1a_2|^{\Re(s)-\frac{n+1}{2}}$$

$$\times|\lfloor b^{-1}a_1a_2^{-1}\rfloor|^{\Re(s)-\frac{n-3}{2}-2C}\eta_j(\mathrm{diag}(-4\mathcal{Q}'(y_2)a_1a_2, \lfloor b^{-1}a_1a_2^{-1}\rfloor, I_{n-2})) \tag{4.2.17}$$

$$\times \left(\int_{wN^\circ(F)w^{-1}} [u]^{-\Re(s)-\frac{n-1}{2}+C} du\right) d^\times b d^\times a_1 d^\times a_2 dk.$$

Here η_j are positive quasi-characters that depend only on τ.

Also, for $u \in wN^\circ(F)w^{-1}$, if we denote the Iwasawa decomposition of $u(wn'w^{-1})$ by $u(wn'w^{-1}) = nt'k$ (using notations as in Lemma 4.2.1), then

$$a_y \mathrm{diag}(a_1a_2, \lfloor b^{-1}a_1a_2^{-1}\rfloor, I_{n-2})t$$

lies in the support of a gauge on $\mathrm{GL}_n(F)$. Also, there are constants c_1 that depends only on τ such that

$$\left|\frac{-4Q'(y_2)a_1a_2}{||\lfloor b^{-1}a_1a_2^{-1}\rfloor||}\frac{t_1}{t_2}\right| \leq c_1.$$

Thus by Lemma 4.2.2 we have

$$|-4Q'(y_2)a_1a_2| \leq c_1[u]||\lfloor b^{-1}a_1a_2^{-1}\rfloor||^{-2}. \tag{4.2.18}$$

Therefore,

$$|\eta_j|(a_y\text{diag}(a_1a_2, \lfloor b^{-1}a_1a_2^{-1}\rfloor, I_{n-2})) \leq c_1[u]^{c_2}||\lfloor b^{-1}a_1a_2^{-1}\rfloor||^{-2c_2-c_3}$$

for some positive integers c_2, c_3 that depend only on τ for $j = 1, \ldots, \nu$.

Thus the integral is majorized by

$$\sum_{j=1}^{\nu} c_{j,s} \int_{F^\times}\int_{F^\times}\int_{F^\times}\int_{K_G} |\rho(k)f(a_1y_1, a_2y_2, b)||a_1|^{\frac{d_1}{2}}|a_2|^{\frac{d_2}{2}}|a_1a_2|^{\Re(s)-\frac{n+1}{2}}$$

$$\times ||\lfloor b^{-1}a_1a_2^{-1}\rfloor||^{\Re(s)-\frac{n-3}{2}-2C-4c_2-c_3}\left(\int_{wN^\circ(F)w^{-1}}[u]^{-\Re(s)-\frac{n-1}{2}+C+2c_2}du\right)$$

$$\times d^\times b d^\times a_1 d^\times a_2 dk.$$

Then by definition of the symbol $\lfloor\cdot\rfloor$ (see Eq. (4.2.12)) and $[\cdot]$ (as in Eq. (4.2.6)), for $\Re(s)$ large, the above sum of integrals is majorized by a constant times

$$\int_{F^\times}\int_{F^\times}\int_{K_G} |\rho(k)\tilde{f}(a_1y_1, a_2y_2)||a_1|^{\frac{d_1}{2}}|a_2|^{\frac{d_2}{2}}|a_1a_2|^{\Re(s)-\frac{n+1}{2}}d^\times a_1 d^\times a_2 dk,$$

where $\tilde{f} \in \mathcal{S}(V(F))$.

Let $\tilde{\tilde{f}}(v) = \int_{K_G}\rho(k)f(v)$ for $v \in V(F)$, we have $\tilde{\tilde{f}} \in \mathcal{S}(V(F))$. The integral is equal to

$$\int_{F^\times}\int_{F^\times}|\tilde{\tilde{f}}(a_1y_1,a_2y_2)||a_1|^{\frac{d_1}{2}}|a_2|^{\frac{d_2}{2}}|a_1a_2|^{\Re(s)-\frac{n+1}{2}}d^\times a_1 d^\times a_2,$$

which converges for $\Re(s)$ large enough. □

Now we give a bound for the local integral in the unramified case. Suppose the data are unramified as in Section 5.1. Suppose F is such that $q \geq n$, and $|Q'(y_2)| = 1$. Let $f = \mathbb{1}_{V(\mathcal{O})\times\mathcal{O}^\times}$.

In the unramified case, $\rho(g)f(y,1)$ and $W_{\rho_{\tau,s}}$ are right invariant by K_G and K_H, so after applying the Iwasawa decomposition, we have that

$$I_s(y) = \int_{M_1(F)}\int_{G_1(F)}\rho(mx)f(y)\delta_{B_G}^{-1}(mx)\int_{N^\circ(F)}W_{\rho_{\tau,s}}(wu\iota(mx),a_y)\psi(u)dudmdx.$$

Lemma 4.2.4. *Let $a = (a_1,\dots,a_n) \in \mathrm{GL}_n(F)$ be such that*

$$|a_1| = q^{-k_{y_1}} \geq |a_2| = q^{-k_{y_2}} \geq \cdots \geq |a_n| = q^{-k_n},$$

where $k_n \leq 0$ (i.e. the positive cone). Let W_τ be the unramified Whittaker function for τ. There exists a positive integer c_0 which depends on τ such that

$$|W_\tau(a)| \leq \delta_{B_{\mathrm{GL}_n}}^{\frac{1}{2}}(a)q^{-k_n n c_0}|\mathrm{det}a|^{-c_0}.$$

Proof. The result follows from arguments in [JPSS79, Section 2.4] for $k_n = 0$ by twisting the corresponding rational representation of $\mathrm{GL}_n(\mathbb{C})$ in the explicit formula of $W_\tau(a)$ (see [CS80a]). □

Lemma 4.2.5. *For $\Re(s)$ large enough, we have*

$$|I_s(y)| \leq \zeta_v(\Re(s) + c_\tau)$$
$$\times \int_{F^\times}\int_{F^\times} |\mathbb{1}_{V(\mathcal{O}_F)}(a_1y_1, a_2y_2)||a_1|^{\Re(s)+\frac{d_1-n-1}{2}-c_0}|a_2|^{\Re(s)+\frac{d_2-n-1}{2}-c_0}d^\times a_1 d^\times a_2,$$

where $c_0 > 0$, c_τ are integers depend only on τ.

Proof. As in Lemma 4.2.3, and since in the unramified case $|2| = 1$, $\rho(k)f(y) = f(y)$ for $k \in K_G$, we have

$$|I_s(y)| \leq$$
$$\int_{F^\times}\int_{F^\times}\int_{F^\times} |f(a_1y_1, a_2y_2, b)||a_1|^{\frac{d_1}{2}}|a_2|^{\frac{d_2}{2}}|a_1a_2|^{\Re(s)-\frac{n+1}{2}}|\lfloor b^{-1}a_1a_2^{-1}\rfloor|^{\Re(s)-\frac{n-3}{2}}$$
$$\times \int_{wN^\circ(F)w^{-1}} |W_{\rho_{\tau,s}}|(u(wn'w^{-1}), a_y\mathrm{diag}(a_1a_2, \lfloor b^{-1}a_1a_2^{-1}\rfloor, I_{n-2}))du d^\times b d^\times a_1 d^\times a_2.$$

Using the notation as in Lemma 4.2.1 and Lemma 4.2.2, we write $u(wn'w^{-1}) = nt'k$, by Eq. (4.2.7) and Eq. (4.2.8) we have

$$|t_n| \leq [\mathcal{B}(u(wn'w^{-1}))]^{n-2} \leq ([u]|\lfloor b^{-1}a_1a_2^{-1}\rfloor|^{-2})^{n-2}.$$

Also, by the property of $W_{\rho_{\tau,s}}$, we have

$$|W_{\rho_{\tau,s}}|(u(wn'w^{-1}), a_y\mathrm{diag}(a_1a_2, \lfloor b^{-1}a_1a_2^{-1}\rfloor, I_{n-2}))$$
$$=|D(u(wn'w^{-1}))|^{\Re(s)+\frac{n-1}{2}}W_\tau(a_y\mathrm{diag}(a_1a_2, \lfloor b^{-1}a_1a_2^{-1}\rfloor, I_{n-2})t).$$

Thus by Lemma 4.2.4 and Eq. (4.2.8), for $\Re(s)$ large we have

$$
\begin{aligned}
&|W_{\rho_{\tau,s}}|(u(wn'w^{-1}), a_y\mathrm{diag}(a_1a_2, \lfloor b^{-1}a_1a_2^{-1}\rfloor, I_{n-2}))\\
\le&[u]^{-\Re(s)-n-\frac{1}{2}}([u]|\lfloor b^{-1}a_1a_2^{-1}\rfloor|^{-2})^{(n-2)nc_0}|\det(a_y\mathrm{diag}(a_1a_2, \lfloor b^{-1}a_1a_2^{-1}\rfloor, I_{n-2})t)|^{-c_0}\\
=&[u]^{-\Re(s)-n-\frac{1}{2}}([u]|\lfloor b^{-1}a_1a_2^{-1}\rfloor|^{-2})^{(n-2)nc_0}|\det(\mathrm{diag}(a_1a_2, \lfloor b^{-1}a_1a_2^{-1}\rfloor, I_{n-2})t)|^{-c_0}.
\end{aligned}
$$

By Eq. (4.2.8) again, we have

$$
|\det t|^{-c_0} \le ([u]|\lfloor b^{-1}a_1a_2^{-1}\rfloor|^{-2})^{c_0n}.
$$

Thus, we obtain

$$
\begin{aligned}
&|W_{\rho_{\tau,s}}|(u(wn'w^{-1}), \mathrm{diag}(a_1a_2, \lfloor b^{-1}a_1a_2^{-1}\rfloor, I_{n-2}))\\
\le&|a_1a_2|^{-c_0}[u]^{-\Re(s)-n-\frac{1}{2}+c_0n^2-c_0n}|\lfloor b^{-1}a_1a_2^{-1}\rfloor|^{-c_0(2n^2-2n+1)}.
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
|I_s(y)| \le& \int_{F^\times}\int_{F^\times}\int_{F^\times} |f(a_1y_1, a_2y_2, b)||a_1|^{\frac{d_1}{2}}|a_2|^{\frac{d_2}{2}}|a_1a_2|^{\Re(s)-\frac{n-1}{2}-c_0}\\
&\times |\lfloor b^{-1}a_1a_2^{-1}\rfloor|^{\Re(s)-\frac{n-3}{2}-c_0(2n^2-2n+1)}\int_{wN^\circ(F)w^{-1}}[u]^{-\Re(s)-n-\frac{1}{2}+c_0n^2-c_0n}dud^\times a_1d^\times a_2.
\end{aligned}
$$

Then, for $\Re(s)$ large, we have

$$|I_s(y)| \leq \int_{F^\times}\int_{F^\times}\int_{F^\times} |\mathbb{1}_{V(\mathcal{O}_F)\times\mathcal{O}_F^\times}(a_1y_1,a_2y_2,b)||a_1|^{\frac{d_1}{2}}|a_2|^{\frac{d_2}{2}}|a_1a_2|^{\Re(s)-\frac{n-1}{2}-c_0}$$

$$\times \int_{wN^\circ(F)w^{-1}} [u]^{-\Re(s)-n-\frac{1}{2}+c_0n^2-c_0n} du d^\times b d^\times a_1 d^\times a_2$$

$$= \int_{F^\times}\int_{F^\times} |\mathbb{1}_{V(\mathcal{O}_F)}(a_1y_1,a_2y_2)||a_1|^{\frac{d_1}{2}}|a_2|^{\frac{d_2}{2}}|a_1a_2|^{\Re(s)-\frac{n-1}{2}-c_0}$$

$$\times \int_{wN^\circ(F)w^{-1}} [u]^{-\Re(s)-n-\frac{1}{2}+c_0n^2-c_0n} du d^\times a_1 d^\times a_2.$$

We have

$$\int_{wN^\circ(F)w^{-1}} [u]^{-\Re(s)-n-\frac{1}{2}+2c_0n(n-2)-c_0} du \leq 1 + \sum_{k=1}^{\infty}\int_{[u]=q^k} q^{-(\Re(s)+n+\frac{1}{2}-2c_0n(n-2)+c_0)k} du$$

$$\leq 1 + \frac{q^{-\Re(s)+(c_0+1)n^2-(c_0+2)n-c_0-\frac{5}{2}}}{1-q^{-\Re(s)+(c_0+1)n^2-(c_0+2)n-c_0-\frac{5}{2}}}$$

$$= \zeta_v(\Re(s)-(c_0+1)n^2+(c_0+2)n+c_0+\tfrac{5}{2}).$$

Therefore, we obtain

$$|I_s(y)| \leq \zeta_v(\Re(s)-(c_0+1)n^2+(c_0+2)n+c_0+\tfrac{5}{2})$$

$$\times \int_{F^\times}\int_{F^\times} |\mathbb{1}_{V(\mathcal{O}_F)}(a_1y_1,a_2y_2)||a_1|^{\frac{d_1}{2}}|a_2|^{\frac{d_2}{2}}|a_1a_2|^{\Re(s)-\frac{n-1}{2}-c_0} d^\times a_1 d^\times a_2$$

$$= \zeta_v(\Re(s)+c_\tau)$$

$$\times \int_{F^\times}\int_{F^\times} |\mathbb{1}_{V(\mathcal{O}_F)}(a_1y_1,a_2y_2)||a_1|^{\frac{d_1}{2}}|a_2|^{\frac{d_2}{2}}|a_1a_2|^{\Re(s)-\frac{n-1}{2}-c_0} d^\times a_1 d^\times a_2,$$

where c_τ is some integer depends only on τ. □

4.3 Bounds of the local integrals in the Archimedean case

In this section we give bounds for the local integrals in the Archimedean case. Let F be an Archimedean local field. Let K_G be the maximal compact subgroup of $G(F)$, and let K_H be the maximal compact subgroup of $H(F)$.

The local integral in the Archimedean case is

$$I_s(y) := I(f, W_{\rho_{\tau,s}}) = \int_{U_2(F)\backslash G(F)} \rho(g) f(y,1) \int_{N^\circ(F)} W_{\rho_{\tau,s}}(wu\iota(g), a_y)\psi_1(u) du dg.$$

As in the non-Archimedean case, in Lemma 4.3.1 and Lemma 4.3.2 we first use techniques similar to those employed in the proof of absolute convergence of the Archimedean local integral for the Rankin-Selberg integral for $\mathrm{SO}_{2\ell+1}\times\mathrm{GL}_n$ in [Sou93, Section 5] to bound our inner integral.

For $t \in \mathrm{GL}_n(F)$ let

$$t' := \begin{pmatrix} t & & \\ & 1 & \\ & & w_0 t^{-1} w_0 \end{pmatrix}, \tag{4.3.1}$$

where $w_0 \in \mathrm{GL}_n(F)$ is the antidiagonal matrix.

Lemma 4.3.1. *Let $(n, t', k) \in N_n(F) \times T_H(F) \times K_H$, there is a positive integer N and $c_s \in \mathbb{R}_{>0}$ depending on $W_{\rho_{\tau,s}}$ such that*

$$|W_{\rho_{\tau,s}}(nt'k, 1)| \le c_s |\det t|^{\Re(s)+\frac{n-1}{2}} \|t\|^N$$

for $t = \operatorname{diag}(t_1, t_2, \ldots, t_{n-1}, 1)$, *where*

$$\|t\|^2 = 1 + \sum_{i=1}^{n-1} |t_i|^2 + \sum_{i=1}^{n-1} |t_i|^{-2}.$$

Proof. We argue analogously as in [Sou93, Lemma 5.2]. One has

$$W_{\rho_{\tau,s}}(nt'k, 1) = |\det t|^{s+\frac{n-1}{2}} W_{\rho_{\tau,s}}(k, t).$$

By the property of $W_{\rho_{\tau,s}}$, there is a continuous seminorm p on the space τ and a positive integer N such that

$$|W_{\rho_{\tau,s}}(k, t)| \leq \|t\|^N p(W_{\rho_{\tau,s}}(k, 1)).$$

Since $p(W_{\rho_{\tau,s}}(k, 1))$ is bounded on K_H, we deduce the lemma. □

Lemma 4.3.2. *For* $v \in H(F)$ *as defined in Eq.* (4.2.2) *and* $\operatorname{Re}(s)$ *large,*

$$\int_{wN^\circ(F)w^{-1}} |W_{\rho_{\tau,s}}(uv, 1)| du$$

converges.

Proof. As in Lemma 4.2.2, for $u \in wN^\circ(F)w^{-1}$ and v a unipotent element of the form Eq. (4.2.2), we denote the Iwasawa decomposition of uv as $uv = nt'k$ where $(n, t', k) \in N_n(F) \times T_H(F) \times K_H$, and we denote the i-th line of uv as $(uv)_i$.

By Lemma 4.3.1, there is a positive integer N such that for $\xi_{\tau,s}$ there is $c_s \in \mathbb{C}$

for $\xi_{\tau,s}$ such that

$$|W_{\rho_{\tau,s}}(nt'k,1)| \leq c_s|\det t|^{\Re(s)+\frac{n-1}{2}}|w_\tau(t_n)|\|t\|^N. \tag{4.3.2}$$

Here $t = \operatorname{diag}(t_1t_2\cdot\ldots\cdot t_n, t_2\cdot\ldots\cdot t_n,\ldots,t_{n-1}t_n,t_n)$, $\|t\| = \|t_n^{-1}t\|$, and w_τ is the central character of τ. We assume N is even, then $\|a\|^N$ is a sum of positive quasicharacters.

As in the non-Archimedean case, we denote $\mathcal{L}(uv) = \begin{pmatrix} (uv)_{n+2} \\ \vdots \\ (uv)_{2n+1} \end{pmatrix}$.

Using the technique as in [Sou93, Section 7.3, Lemma 3] we get

$$(1+\|\mathcal{L}(uv)\|^2)^{-\frac{n}{2}} \leq \det(t) \leq (1+\|\mathcal{L}(uv)\|^2)^{-\frac{1}{2}}.$$

Here $\|\mathcal{L}(uv)\|$ denotes the standard norm on $M_{n\times(2n+1)}(F)$, and

$$\max\{\tfrac{t_j}{t_{j+1}},\tfrac{t_{j+1}}{t_j}\} \leq (1+\|\mathcal{L}(uv)\|^2)^n, j=1,\ldots,n-1. \tag{4.3.3}$$

Similar to the non-Archimedean case, we have

$$\begin{aligned}(1+\|\mathcal{L}(uv)\|)^{-\frac{1}{2}} &= (1+\sup\{\|v_1\|,\|v_2\|,\|v_3\|,\|T\|,\|-\frac{1}{2}v_2c^2\|,\|v_3c\|\})^{-\frac{1}{2}} \\ &\leq (1+\sup\{\|v_1\|,\|v_2\|,\|v_3\|,\|T\|\})^{-\frac{1}{2}} \\ &= (1+\|u\|)^{-\frac{1}{2}},\end{aligned}$$

where $\|\cdot\|$ denotes the standard matrix norms.

By Eq. (4.3.2) we have

$$|W_{\rho_{\tau,s}}|(uv,1) \leq \sum_j c_s(1+\|u\|)^{-\frac{\Re(s)}{2}-\frac{n-2}{4}}|\chi_j(t)|. \tag{4.3.4}$$

Here the sum is finite and the χ_j are positive quasi-characters which depend on τ.

By Eq. (4.3.3), we have

$$|\chi_j(t)| \leq (1 + \|\mathcal{L}(uv)\|^2)^C \leq (1 + \|u\|^2)^C (1 + \|v\|^4)^C$$

for some positive constant C which depends on τ.

Thus we have

$$\int_{wN^\circ(F)w^{-1}} |W_{\rho_{\tau,s}}(uv, 1)| du \leq \sum_j c_s (1 + \|v\|^4)^C \int_{wN^\circ(F)w^{-1}} (1 + \|u\|)^{-\frac{\Re(s)}{2} - \frac{n-2}{4} + C} du. \tag{4.3.5}$$

The sum over j is finite. Since by definition $\|\cdot\|$ is positive, the integral converges for $\Re(s)$ large.

□

We now proceed to bound our local integral $I_s(y)$.

Lemma 4.3.3. *For $y = (y_1, y_2) \in \mathbb{P}Y'(F)$, we have that*

$$I_s(y) \ll \int_{F^\times} \int_{F^\times} f'(a_1 y_1, a_2 y_2) |a_1|^{\Re(s) + \frac{d_1 - n - 1}{2} - c_0} |a_2|^{\Re(s) + \frac{d_2 - n - 1}{2} - c_0} d^\times a_1 d^\times a_2$$

for $\Re(s)$ large enough where $f' \in \mathcal{S}(V(F))$ is nonnegative and $c_0 \in \mathbb{R}_{\geq 0}$ depends only on τ as in Lemma 4.2.5.

Proof. Applying the Iwasawa decomposition of $G(F)$ with respect to the standard Borel subgroup, we have

$$I_s(y) = \int_{T_G(F)} \int_{K_G} \rho(ak) f(y, 1) \delta_{B_G}^{-1}(a) \int_{N^\circ(F)} W_{\rho_{\tau,s}}(wu\iota(ak), a_y) \psi_1(u) du d^\times a_1 d^\times a_2 dk.$$

By the action of Weil representation it suffices to bound

$$\int_{F^\times}\int_{F^\times}\int_{F^\times}\int_{K_G} |\rho(k)f(a_1y_1,a_2y_2,b)||a_1|^{\frac{d_1}{2}}|a_2|^{\frac{d_2}{2}}|a_1a_2|^{-2}$$

$$\times\int_{N^\circ(F)} |W_{\rho_{\tau,s}}|(wu\iota\left(\begin{pmatrix} a_1 & \\ & a_1^{-1}b^{-1}\end{pmatrix},\begin{pmatrix} a_2 & \\ & a_2^{-1}b\end{pmatrix}\right)\iota(k),a_y)dud^\times bd^\times a_1d^\times a_2dk.$$

Here $\iota\left(\begin{pmatrix} a_1 & \\ & a_1^{-1}b^{-1}\end{pmatrix},\begin{pmatrix} a_2 & \\ & a_2^{-1}b\end{pmatrix}\right)$ is

$$\begin{pmatrix} I_{n-1} &&&&&& \\ & a_1a_2 &&&&& \\ && \frac12+\frac14(b^{-1}a_1a_2^{-1}+ba_1^{-1}a_2) & \frac12(b^{-1}a_1a_2^{-1}-ba_1^{-1}a_2) & 2(b^{-1}a_1a_2^{-1}-ba_1^{-1}a_2) && \\ && (b^{-1}a_1a_2^{-1}-ba_1^{-1}a_2) & \frac12(b^{-1}a_1a_2^{-1}+ba_1^{-1}a_2) & -\frac12(b^{-1}a_1a_2^{-1}-ba_1^{-1}a_2) && \\ && \frac12(\frac12-\frac14(b^{-1}a_1a_2^{-1}+ba_1^{-1}a_2)) & -\frac14(b^{-1}a_1a_2^{-1}-ba_1^{-1}a_2) & \frac12+\frac14(b^{-1}a_1a_2^{-1}+ba_1^{-1}a_2) && \\ &&&&& (a_1a_2)^{-1} & \\ &&&&&& I_{n-1}\end{pmatrix}.$$

As in the non-Archimedean case, applying the Iwasawa decomposition of $\mathrm{SO}_3(F)$ we have for $b\in F^\times$

$$\begin{pmatrix} \frac12+\frac14(b+b^{-1}) & \frac12(b-b^{-1}) & 2(b-b^{-1}) \\ (b-b^{-1}) & \frac12(b+b^{-1}) & -\frac12(b-b^{-1}) \\ \frac12(\frac12-\frac14(b+b^{-1})) & -\frac14(b-b^{-1}) & \frac12+\frac14(b+b^{-1})\end{pmatrix} = \begin{pmatrix} 1 & c & -\frac12c^2 \\ 0 & 1 & -c \\ 0 & 0 & 1\end{pmatrix}\begin{pmatrix} a & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & a^{-1}\end{pmatrix}k',$$

where $k'\in K_{\mathrm{SO}_3}\subset K_H$.

For $F=\mathbb{R}$, let $K'=\mathrm{diag}(\mathrm{SO}(2,\mathbb{R}),1)$ and S be such that ${}^tSJ'S=J_3$, where $J'=\mathrm{diag}(-I_2,1)$. We have $K_{\mathrm{SO}_3}={}^tSK'S$. Let $\|\cdot\|$ denote the Euclidean vector norm. Using the above decomposition, we have

$$\|(\tfrac12(\tfrac12-\tfrac14(b+b^{-1})),-\tfrac14(b-b^{-1}),\tfrac12+\tfrac14(b+b^{-1}))\|=\|(0,0,a^{-1})k\|,$$

$$\|((b-b^{-1}),\tfrac12(b+b^{-1}),-\tfrac12(b-b^{-1}))\|=\|(0,1,-ca^{-1})k'\|.$$

Since the action of K_{SO_3} preserve the Euclidean norm, we get

$$|a^{-1}| = \|(0,0,a^{-1})\| = \|(\tfrac{1}{2}(\tfrac{1}{2} - \tfrac{1}{4}(b+b^{-1})), -\tfrac{1}{4}(b-b^{-1}), \tfrac{1}{2} + \tfrac{1}{4}(b+b^{-1}))\|,$$

$$|ca^{-1}| \le \|(0,1,-ca^{-1})\| = \|((b-b^{-1}), \tfrac{1}{2}(b+b^{-1}), -\tfrac{1}{2}(b-b^{-1}))\|.$$

For $F = \mathbb{C}$, $K_{\mathrm{SO}_3} \cong \mathrm{SO}(3,\mathbb{R})$. Similar as in the real case, the action of K_{SO_3} preserve $\|\cdot\|$. We have

$$|a^{-1}| = \|(0,0,a^{-1})\|^2 = \|(\tfrac{1}{2}(\tfrac{1}{2} - \tfrac{1}{4}(b+b^{-1})), -\tfrac{1}{4}(b-b^{-1}), \tfrac{1}{2} + \tfrac{1}{4}(b+b^{-1}))\|^2,$$

$$|ca^{-1}| \le \|((b-b^{-1}), \tfrac{1}{2}(b+b^{-1}), -\tfrac{1}{2}(b-b^{-1}))\|^2.$$

Therefore in both cases, we have

$$|a| \ll \max(|b|,|b^{-1}|)^{-1} = \min(|b|,|b^{-1}|),$$

$$|ca^{-1}| \ll \max(|b|,|b^{-1}|).$$

We denote $\lfloor b \rfloor = b$ if $|b| \le 1$ and $\lfloor b \rfloor = b^{-1}$ otherwise. Since

$$\begin{pmatrix} 1 & c & -\frac{1}{2}c^2 \\ 0 & 1 & -c \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} a & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & a^{-1} \end{pmatrix} = \begin{pmatrix} a & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & a^{-1} \end{pmatrix} \begin{pmatrix} 1 & ca^{-1} & -\frac{1}{2}c^2a^{-2} \\ 0 & 1 & -ca^{-1} \\ 0 & 0 & 1 \end{pmatrix},$$

the local integral is majorized by

$$\int_{F^\times}\int_{F^\times}\int_{F^\times}\int_{K_G}|\rho(k)f(a_1y_1,a_2y_2,b)||a_1|^{\frac{d_1}{2}-2}|a_2|^{\frac{d_2}{2}-2}|a_1a_2|^{\Re(s)-\frac{n+1}{2}}$$

$$\times\,|\lfloor b^{-1}a_1a_2^{-1}\rfloor|^{\Re(s)-\frac{n-3}{2}}\int_{N^\circ(F)}|W_{\rho_{\tau,s}}|(wunk''\iota(k),a_y\mathrm{diag}(a_1a_2,\lfloor b^{-1}a_1a_2^{-1}\rfloor,I_{n-2}))$$

$$\times\,dud^\times bd^\times a_1d^\times a_2dk.$$

This is

$$\int_{F^\times}\int_{F^\times}\int_{F^\times}\int_{K_G}|\rho(k)f(a_1y_1,a_2y_2,b)||a_1|^{\frac{d_1}{2}-2}|a_2|^{\frac{d_2}{2}-2}$$

$$\times\,|a_1a_2|^{\Re(s)-\frac{n+1}{2}}|\lfloor b^{-1}a_1a_2^{-1}\rfloor|^{\Re(s)-\frac{n-3}{2}}$$

$$\times\int_{wN^\circ(F)w^{-1}}|W_{\rho_{\tau,s}}|(u(wnw^{-1})wk''\iota(k),a_y\mathrm{diag}(a_1a_2,\lfloor b^{-1}a_1a_2^{-1}\rfloor,I_{n-2}))$$

$$\times\,dud^\times bd^\times a_1d^\times a_2dk,$$

where

$$n=\begin{pmatrix}I_{n-1}&&&&\\&1&ca^{-1}&-\frac{1}{2}c^2a^{-2}&\\&&1&-ca^{-1}&\\&&&1&\\&&&&I_{n-1}\end{pmatrix},\quad k''=\begin{pmatrix}I_{n-1}&&\\&k'&\\&&I_{n-1}\end{pmatrix}.$$

Since

$$wnw^{-1}=\begin{pmatrix}1&&&&&&\\&1&&-\frac{1}{2}ca^{-1}&&-\frac{1}{4}c^2a^{-2}&\\&&I_{n-2}&&&&\\&&&1&&\frac{1}{2}ca^{-1}&\\&&&&I_{n-2}&&\\&&&&&1&\\&&&&&&1\end{pmatrix},$$

by Eq. (4.3.4) and Eq. (4.3.5) in Lemma 4.3.2, the local integral is majorized by

$$\sum_j c_s \int_{F^\times}\int_{F^\times}\int_{F^\times}\int_{K_G} |\rho(k)f(a_1y_1,a_2y_2,b)||a_1|^{\frac{d_1}{2}-2}|a_2|^{\frac{d_2}{2}-2}|a_1a_2|^{\Re(s)-\frac{n+1}{2}}$$

$$\times |\lfloor b^{-1}a_1a_2^{-1}\rfloor|^{\Re(s)-\frac{n-3}{2}}|\chi_j|(a_y\mathrm{diag}(a_1a_2,\lfloor b^{-1}a_1a_2^{-1}\rfloor,I_{n-2}))(1+\|-\frac{1}{2}ca^{-1}\|^4)^C$$

$$\times\left(\int_{wN^\circ(F)w^{-1}}(1+\|u\|)^{-\frac{\Re(s)}{2}-\frac{n-2}{4}+C}du\right)d^\times b d^\times a_1 d^\times a_2 dk.$$

Since $|ca^{-1}|\ll|\lfloor b\rfloor|^{-1}$, the above sum is majorized by

$$\sum_j c_s \int_{F^\times}\int_{F^\times}\int_{F^\times}\int_{K_G} |\rho(k)f(a_1y_1,a_2y_2,b)||a_1|^{\frac{d_1}{2}-2}|a_2|^{\frac{d_2}{2}-2}|a_1a_2|^{\Re(s)-\frac{n+1}{2}}$$

$$\times |\lfloor b^{-1}a_1a_2^{-1}\rfloor|^{\Re(s)-\frac{n-3}{2}-C'}|\chi_1^j(-4Q'(y_2)a_1a_2)||\chi_2^j(\lfloor b^{-1}a_1a_2^{-1}\rfloor)|$$

$$\times\left(\int_{wN^\circ(F)w^{-1}}(1+\|u\|)^{-\frac{\Re(s)}{2}-\frac{n-2}{4}+C}du\right)d^\times b d^\times a_1 d^\times a_2 dk.$$

Here χ_1^j, χ_2^j are positive quasi-characters, and C' is some positive integer depends only on τ.

Thus for $\Re(s)$ large, our sum is majorized by

$$\sum_j c_s \int_{F^\times}\int_{F^\times}\int_{F^\times}\int_{K_G} |\rho(k)f(a_1y_1,a_2y_2,b)||a_1|^{\frac{d_1}{2}-2}|a_2|^{\frac{d_2}{2}-2}$$

$$\times|a_1a_2|^{\Re(s)-\frac{n+1}{2}}|-4Q'(y_2)|^{c_j}|a_1a_2|^{c_j}d^\times b d^\times a_1 d^\times a_2 dk,$$

where c_j is some integer depends on τ. This is majorized by

$$\sum_j c_s \int_{F^\times}\int_{F^\times}\int_{F^\times}\int_{K_G} |\rho(k)f(a_1y_1, a_2y_2, b)||a_1|^{\frac{d_1}{2}-2}|a_2|^{\frac{d_2}{2}-2}$$

$$\times|a_1a_2|^{\Re(s)-\frac{n+1}{2}}|a_1y_1|^{c_j}|a_2y_2|^{c_j}d^\times b d^\times a_1 d^\times a_2 dk.$$

Then for $\Re(s)$ large, the local integral is majorized by a constant times a finite sum of integrals of the form

$$\int_{F^\times}\int_{F^\times}\int_{K_G} |\rho(k)\tilde{f}(a_1y_1, a_2y_2)||a_1|^{\frac{d_1}{2}-2}|a_2|^{\frac{d_2}{2}-2}|a_1a_2|^{\Re(s)-\frac{n+1}{2}}d^\times a_1 d^\times a_2 dk$$

$$= \int_{F^\times}\int_{F^\times} |\tilde{\tilde{f}}(a_1y_1, a_2y_2)||a_1|^{\Re(s)+\frac{d_1-n-1}{2}}|a_2|^{\Re(s)+\frac{d_2-n-1}{2}}d^\times a_1 d^\times a_2,$$

where $\tilde{f} \in \mathcal{S}(V(F))$ and $\tilde{\tilde{f}}(v) = \int_{K_G}\rho(k)f(v)$ for $v \in V(F)$. Then we deduce the lemma. $\square$

4.4 Absolute Convergence

In this section we handle the absolute convergence of the sum of the global integral in the main theorem (Theorem 4.1.2), which will make the proof of Theorem 4.1.2 rigorous. The proof follows from the absolute convergence of the local integrals in the Archimedean case and the non-Archimedean case.

Lemma 4.4.1. *The sum of the global integral*

$$\sum_{y\in\mathbb{P}Y'(F)} I(f, W_{\xi_s})(y)$$

$$= \sum_{y\in\mathbb{P}Y'(F)} \int_{U_2(\mathbb{A}_F)\backslash G(\mathbb{A}_F)} \rho(g)f(y,1) \int_{N^\circ(\mathbb{A}_F)} W_{\xi_s}(wu\iota(g),1)\psi(u)dudg$$

converges absolutely for $\Re(s)$ *large enough.*

Proof. Let $y = (y_1, y_2) \in \mathbb{P}Y'(F)$. Let S be a finite set of places of F which includes the infinite places and all the finite places such that $q_v < n$, $f^S = \mathbb{1}_{V(\widehat{\mathcal{O}}^S)\times\prod_{v\notin S}\mathcal{O}_v^\times}$, τ_v unramified, $|\mathcal{Q}'(y_2)|_v = 1$ and $\rho_v(k)f_v = f_v$ for $k \in G(\mathcal{O}_v)$ for $v \notin S$.

Using the results and notations of Lemma 4.2.3, Lemma 4.2.5 and Lemma 4.3.3, for $\Re(s)$ large we have

$$I(f, W_{\xi_s})(y)$$

$$= \prod_{v|\infty} I_v(f_v, W_{\rho_{\tau,s}})(y) \prod_{v \in \mathrm{S}-\infty} I_v(f_v, W_{\rho_{\tau,s}})(y) \prod_{v \notin \mathrm{S}} I_v(f_v, W_{\rho_{\tau,s}})(y)$$

$$\ll \prod_{v|\infty} \int_{F_v^\times} \int_{F_v^\times} |f'_v(a_1y_1, a_2y_2)||a_1|_v^{\Re(s)+\frac{d_1-n-1}{2}-c_0}|a_2|_v^{\Re(s)+\frac{d_2-n-1}{2}-c_0} d^\times a_1 d^\times a_2$$

$$\times \prod_{v \in \mathrm{S}-\infty} \int_{F^\times} \int_{F^\times} |f'_v(a_1y_1, a_2y_2)||a_1|_v^{\Re(s)+\frac{d_1-n-1}{2}-c_0}|a_2|_v^{\Re(s)+\frac{d_2-n-1}{2}-c_0} d^\times a_1 d^\times a_2$$

$$\times \prod_{v \notin \mathrm{S}} \zeta_v(\Re(s)+c_\tau) \int_{F^\times} \int_{F^\times} |\mathbb{1}_{V(\mathcal{O}_F)}(a_1y_1, a_2y_2)||a_1|_v^{\Re(s)+\frac{d_1-n-1}{2}-c_0}$$

$$\times |a_2|_v^{\Re(s)+\frac{d_2-n-1}{2}-c_0} d^\times a_1 d^\times a_2$$

$$= \zeta_F(\Re(s)+c_\tau) \int_{\mathbb{A}_F^\times} \int_{\mathbb{A}_F^\times} |f'(a_1y_1, a_2y_2)||a_1|^{\Re(s)+\frac{d_1-n-1}{2}-c_0}$$

$$\times |a_2|^{\Re(s)+\frac{d_2-n-1}{2}-c_0} d^\times a_1 d^\times a_2.$$

Here $f' \in \mathcal{S}(V(\mathbb{A}_F))$, and $c_0 > 0$ is a constant depends only on τ.

Thus the sum of the global integral is majorized by a finite sum of a sum of integrals of the form

$$\sum_{y \in \mathbb{P}Y'(F)} \zeta_F(\Re(s)+c_\tau) \int_{\mathbb{A}_F^\times} \int_{\mathbb{A}_F^\times} |f''(a_1y_1, a_2y_2)||a_1|^{\Re(s)+\frac{d_1-n-1}{2}-c_0}$$

$$\times |a_2|^{\Re(s)+\frac{d_2-n-1}{2}-c_0} d^\times a_1 d^\times a_2,$$

which, when $\Re(s)$ large, is majorized by

$$\sum_{y\in\mathbb{P}V(F)}\zeta_F(\Re(s)+c_\tau)\int_{\mathbb{A}_F^\times}\int_{\mathbb{A}_F^\times}|f'(a_1y_1,a_2y_2)||a_1|^{\Re(s)+\frac{d_1-n-1}{2}-c_0}|a_2|^{\Re(s)+\frac{d_2-n-1}{2}-c_0}$$

$$\times\, d^\times a_1 d^\times a_2$$

$$=\sum_{y\in V(F)}\zeta_F(\Re(s)+c_\tau)\int_{F^\times\backslash\mathbb{A}_F^\times}\int_{F^\times\backslash\mathbb{A}_F^\times}|f'(a_1y_1,a_2y_2)||a_1|^{\Re(s)+\frac{d_1-n-1}{2}-c_0}$$

$$\times\,|a_2|^{\Re(s)+\frac{d_2-n-1}{2}-c_0}d^\times a_1 d^\times a_2.$$

This is a product of mirabolic Eisenstein series which converge absolutely for $\Re(s)$ large enough (see [JS81a]). □

Chapter 5

Unramified Computation

In this chapter we give the computation for the unramified local factor of the global integral $I(f, W_{\rho_{\tau,s}})$. The results of this section shed light on the nature of the local integrals $I(f, W_{\rho_{\tau,s}})$.

5.1 Unramified computation

We assume all data are unramified, i.e. the local field F is absolutely unramified, and the character ψ is unramified. Let τ be an irreducible unramified generic representation of $\mathrm{GL}_n(F)$. We denote $K_G = G(\mathcal{O})$. We assume that the matrices of $\mathcal{Q}$ and $\mathcal{Q}'$ are invertible and that the residual characteristic is not 2. We also assume that $d_2 > d_1$.

Let $f \in \mathcal{S}(V(F) \times F^\times)$ be $f(v, u) = \mathbb{1}_{V(\mathcal{O}) \times \mathcal{O}^\times}(v, u)$ for $v \in V(F), u \in F^\times$. Let ρ denote the local Weil representation of $G(F)$. Then $\rho(k)f = f$ for $k \in K_G$.

Let

$$W_{\rho_{\tau,s}} \in \rho_{\tau,s} = \mathrm{Ind}_{Q_n(F)}^{H(F)}(\mathcal{W}(\tau, \psi_0) \otimes |\det|^{s-\frac{1}{2}}) \tag{5.1.1}$$

be the unique spherical vector satisfying $W_{\rho_{\tau,s}}(1,1) = 1$. Here, as the notation indicates, we are viewing the induced representation as a space of smooth functions taking values in the Whittaker model.

The Satake parameter for $\rho_{\tau,s}$ is defined (up to a permutation) by

$$t_{\rho_{\tau,s}} = \mathrm{diag}(\chi_{1,s}(\varpi), \ldots, \chi_{n,s}(\varpi), \chi_{n,s}(\varpi)^{-1}, \ldots, \chi_{1,s}(\varpi)^{-1}) \in \mathrm{Sp}_{2n}(\mathbb{C}).$$

Here each $\chi_i : F^\times \to \mathbb{C}^\times$ is an unramified character and

$$\chi_{i,s} := \chi_i |\cdot|^{s-\frac{1}{2}}.$$

We set

$$\chi_s := \chi_{1,s} \otimes \cdots \otimes \chi_{n,s} : (F^\times)^n \longrightarrow \mathbb{C}^\times.$$

For an unramified character μ of split $\mathrm{SO}_2(F)$, the Satake parameter is

$$t_\mu = \mathrm{diag}(\mu(\varpi), \mu(\varpi)^{-1}) \in \mathrm{SO}_2(\mathbb{C}).$$

We then have the local integral

$$I_s(y) := I(f, W_{\rho_{\tau,s}})(y) = \int_{U_2(F)\backslash G(F)} \rho(g) f(y,1) \int_{N^\circ(F)} W_{\rho_{\tau,s}}(wu\iota(g), a_y)\psi_1(u) du dg, \tag{5.1.2}$$

as the unramified local component of the global integral $I(f, W_{\rho_{\tau,s}})(y)$ by construction. We assume y is integral. In order to compute the local integral I_s we will adapt a procedure in [Kap12]. This requires us to write the function $\rho(g)f(y,1)$ in terms of functions lying in an appropriate Whittaker model.

Let

$$\alpha(y_1, y_2) = \prod_{i=1}^{2} (2 + \sum_{k'=1}^{\mathrm{val}(y_i)} q^{k'} (1 - (q-1) \sum_{k=1}^{\mathrm{val}(y_i)} q^{(\frac{d_i}{2}-2)k}))^{-1}. \tag{5.1.3}$$

Let $H_{1,y}, H_{2,y} \in C^\infty(T_G(F))$ be such that

$$H_{1,y}\left(\begin{pmatrix} a_1 & \\ & a_1^{-1} \end{pmatrix}, \begin{pmatrix} a_2 & \\ & a_2^{-1} \end{pmatrix}\right) = |a_1|^{\frac{d_1}{2}-2}|a_2|^{\frac{d_2}{2}-2},$$

$$H_{1,y}\left(\begin{pmatrix} 1 & \\ & d \end{pmatrix}, \begin{pmatrix} 1 & \\ & d^{-1} \end{pmatrix}\right) = \alpha(y_1, y_2),$$

where $a_i, d \in F^\times$.

Using the Iwasawa decomposition with respect to the Borel subgroup of $G(F)$ consisting of lower triangular matrices in $G(F)$, we define functions $\Phi_{1,y}, \Phi_{2,y} \in C^\infty(\overline{U_2}(F)\backslash G(F)/K_G)$ (where $\overline{U_2}(F)$ is the unipotent radical of the lower Borel subgroup of $G(F)$) by

$$\Phi_{1,y}(g) : G(F) \to \mathbb{C}$$

$$\left(\begin{pmatrix} 1 & 0 \\ t_1 & 1 \end{pmatrix}\begin{pmatrix} a & \\ & d \end{pmatrix}\kappa_1, \begin{pmatrix} 1 & 0 \\ t_2 & 1 \end{pmatrix}\begin{pmatrix} a' & 0 \\ 0 & (ada')^{-1} \end{pmatrix}\kappa_2\right) \mapsto H_{1,y}\left(\begin{pmatrix} a & \\ & d \end{pmatrix}, \begin{pmatrix} a' & 0 \\ 0 & (ada')^{-1} \end{pmatrix}\right),$$

and

$$\Phi_{2,y}(g) : G(F) \to \mathbb{C}$$

$$\left(\begin{pmatrix} 1 & 0 \\ t_1 & 1 \end{pmatrix}\begin{pmatrix} a & \\ & d \end{pmatrix}\kappa_1, \begin{pmatrix} 1 & 0 \\ t_2 & 1 \end{pmatrix}\begin{pmatrix} a' & 0 \\ 0 & (ada')^{-1} \end{pmatrix}\kappa_2\right) \mapsto H_{2,y}\left(\begin{pmatrix} a & \\ & d \end{pmatrix}, \begin{pmatrix} a' & 0 \\ 0 & (ada')^{-1} \end{pmatrix}\right).$$

Here $a, a', d \in F^\times$, $t_i \in F$, and $(\kappa_1, \kappa_2) \in K_G$.

We let $\Phi_y = \Phi_{1,y}\Phi_{2,y}$, and $H_y = H_{1,y}H_{2,y}$. We first show that an appropriate integral of Φ_y is $\rho(g)f(y,1)$ in the following lemma:

Lemma 5.1.1. *We have*

$$\rho(g)f(y,1) = \int_{U_2(F)} \Phi_y(ng)\psi_{U_2,\mathcal{Q}}(n)^{-1}dn, \tag{5.1.4}$$

where $\psi_{U_2,\mathcal{Q}}(n) = \psi(n_1\mathcal{Q}(y_1) + n_2\mathcal{Q}'(y_2))$ *for* $n = \left(\begin{pmatrix} 1 & n_1 \\ 0 & 1 \end{pmatrix}\right), \left(\begin{pmatrix} 1 & n_2 \\ 0 & 1 \end{pmatrix}\right) \in U_2(F)$.

Proof. As a function of g, both sides of the equality are invariant under K_G on the right and both transform via the same character under $U_2(F)$ on the left. Thus it suffices to verify the equality Eq. (5.1.4) for

$$g_1 = \left(\begin{pmatrix} a_1 & \\ & a_1^{-1} \end{pmatrix}, \begin{pmatrix} a_2 & \\ & a_2^{-1} \end{pmatrix}\right),$$

$$g_2 = \left(\begin{pmatrix} 1 & \\ & d \end{pmatrix}, \begin{pmatrix} 1 & \\ & d^{-1} \end{pmatrix}\right).$$

We have

$$\begin{aligned}
\rho\left(\begin{pmatrix} a_1 & \\ & a_1^{-1} \end{pmatrix}, \begin{pmatrix} a_2 & \\ & a_2^{-1} \end{pmatrix}\right) f(y,1) &= |a_1|^{\frac{d_1}{2}} |a_2|^{\frac{d_2}{2}} \mathbb{1}_{V(\mathcal{O})}(a_1 y_1, a_2 y_2) \\
&= |a_1|^{\frac{d_1}{2}} |a_2|^{\frac{d_2}{2}} \mathbb{1}_{\varpi^{-\mathrm{val}(y_1)}\mathcal{O}}(a_1) \mathbb{1}_{\varpi^{-\mathrm{val}(y_2)}\mathcal{O}}(a_2), \\
\rho\left(\begin{pmatrix} 1 & \\ & d \end{pmatrix}, \begin{pmatrix} 1 & \\ & d^{-1} \end{pmatrix}\right) f(y,1) &= \mathbb{1}_{\mathcal{O}^\times}(d) \mathbb{1}_{\mathcal{O}^\times}(d^{-1}) |d|^{\frac{d_2-d_1}{4}} \\
&= \mathbb{1}_{\mathcal{O}^\times}(d).
\end{aligned}$$

On the other hand, the right hand side of Eq. (5.1.4) in the two cases are

$$\int_{F^2} \Phi_y\left(\begin{pmatrix} 1 & n_1 \\ & 1 \end{pmatrix}\begin{pmatrix} a_1 & \\ & a_1^{-1} \end{pmatrix}, \begin{pmatrix} 1 & n_2 \\ & 1 \end{pmatrix}\begin{pmatrix} a_1 & \\ & a_1^{-1} \end{pmatrix}\right) \psi(n_1 \mathcal{Q}(y_1) + n_2 \mathcal{Q}'(y_2)) dn_1 dn_2,$$

$$\int_{F^2} \Phi_y\left(\begin{pmatrix} 1 & n_1 \\ & 1 \end{pmatrix}\begin{pmatrix} 1 & \\ & d \end{pmatrix}, \begin{pmatrix} 1 & n_2 \\ & 1 \end{pmatrix}\begin{pmatrix} 1 & \\ & d^{-1} \end{pmatrix}\right) \psi(n_1 \mathcal{Q}(y_1) + n_2 \mathcal{Q}'(y_2)) dn_1 dn_2.$$

Let $\Phi_y(g_1, g_2) = \Phi'_y(g_1)\Phi''_y(g_2)$ where $(g_1, g_2) \in G(F)$, and

$$\Phi'_y, \Phi''_y \in C^\infty(\overline{U_2}(F) \backslash \mathrm{SL}_2(F) / \mathrm{SL}_2(\mathcal{O}_F)).$$

By symmetry, it suffices to verify the equalities

$$\mathbb{1}_{\varpi^{-\mathrm{val}(y_1)}\mathcal{O}_F}(a)|a|^{\frac{d_1}{2}} = \int_F \Phi'_y\left(\left(\begin{smallmatrix}1 & n\\ & 1\end{smallmatrix}\right)\left(\begin{smallmatrix}a & \\ & a^{-1}\end{smallmatrix}\right)\right)\psi(n\mathcal{Q}(y_1))dt,$$

$$\mathbb{1}_{\mathcal{O}_F^\times}(d) = \int_F \Phi'_y\left(\left(\begin{smallmatrix}1 & n\\ & 1\end{smallmatrix}\right)\left(\begin{smallmatrix}1 & \\ & d\end{smallmatrix}\right)\right)\psi(n\mathcal{Q}(y_1))dt$$

for $a, d \in F^\times$.

Applying the Iwasawa decompostion with respect to the lower Borel subgroup of $\mathrm{SL}_2(F)$ to $\left(\begin{smallmatrix}1 & n\\ & 1\end{smallmatrix}\right)\left(\begin{smallmatrix}a & \\ & a^{-1}\end{smallmatrix}\right)$ and $\left(\begin{smallmatrix}1 & n\\ & 1\end{smallmatrix}\right)\left(\begin{smallmatrix}1 & \\ & d\end{smallmatrix}\right)$, we have

$$\left(\begin{smallmatrix}1 & n\\ & 1\end{smallmatrix}\right)\left(\begin{smallmatrix}a & \\ & a^{-1}\end{smallmatrix}\right) = \begin{cases}\left(\begin{smallmatrix}a & \\ & a^{-1}\end{smallmatrix}\right)\left(\begin{smallmatrix}1 & a^{-2}n\\ 0 & 1\end{smallmatrix}\right) & \text{for} \quad |a^{-2}n| \leq 1\\ \left(\begin{smallmatrix}1 & \\ n^{-1} & 1\end{smallmatrix}\right)\left(\begin{smallmatrix}a^{-1}n & \\ & an^{-1}\end{smallmatrix}\right)\left(\begin{smallmatrix}a^2n^{-1} & 1\\ -1 & 0\end{smallmatrix}\right) & \text{for} \quad |a^{-2}n| > 1\end{cases},$$

$$\left(\begin{smallmatrix}1 & n\\ & 1\end{smallmatrix}\right)\left(\begin{smallmatrix}1 & \\ & d\end{smallmatrix}\right) = \begin{cases}\left(\begin{smallmatrix}1 & \\ & d\end{smallmatrix}\right)\left(\begin{smallmatrix}1 & dn\\ 0 & 1\end{smallmatrix}\right) & \text{for} \quad |dn| \leq 1\\ \left(\begin{smallmatrix}1 & \\ n^{-1} & 1\end{smallmatrix}\right)\left(\begin{smallmatrix}dn & \\ & n^{-1}\end{smallmatrix}\right)\left(\begin{smallmatrix}(dn)^{-1} & 1\\ -1 & 0\end{smallmatrix}\right) & \text{for} \quad |dn| > 1\end{cases}.$$

Then it suffices to verify the equalities

$$\begin{aligned}\mathbb{1}_{\varpi^{-\mathrm{val}(y_1)}\mathcal{O}_F}(a)|a|^{\frac{d_1}{2}} = &\int_{|n|\leq|a|^2} H'_y\left(\left(\begin{smallmatrix}a & \\ & a^{-1}\end{smallmatrix}\right)\right)\psi(n\mathcal{Q}(y_1))^{-1}dn\\ &+\int_{|a|^2<|n|} H'_y\left(\left(\begin{smallmatrix}a^{-1}n & \\ & an^{-1}\end{smallmatrix}\right)\right)\psi(n\mathcal{Q}(y_1))^{-1}dn\end{aligned}, \tag{5.1.5}$$

$$\begin{aligned}\mathbb{1}_{\mathcal{O}_F^\times}(d) = &\int_{|n|\leq|d|^{-1}} H'_y\left(\left(\begin{smallmatrix}1 & \\ & d\end{smallmatrix}\right)\right)\psi(n\mathcal{Q}(y_1))^{-1}dn\\ &+\int_{|d|^{-1}<|n|} H'_y\left(\left(\begin{smallmatrix}dn & \\ & n^{-1}\end{smallmatrix}\right)\right)\psi(n\mathcal{Q}(y_1))^{-1}dn\end{aligned}. \tag{5.1.6}$$

We first verify Eq. (5.1.5).

For $|a| > q^{\text{val}(y_1)}$, $H_y(a) = 0$ and $H_y(a^{-1}n) = 0$ when $|a|^2 < n$, thus the right hand side of Eq. (5.1.5) is 0, while the left hand side is also 0. In particular, we have the equality for $|a| > q^{\text{val}(y_1)}$.

For $|a| = q^{\text{val}(y_1)}$, $H_y(a^{-1}n) = 0$ when $|a|^2 < n$, thus we have that the right hand side of Eq. (5.1.5) is

$$\begin{aligned}
\int_{|n|\le|a|^2} H_y(a)\psi(n\mathcal{Q}(y_1))^{-1}dn &= H_y(a)\int_{|n|\le q^{2\text{val}(y_1)}} \psi(n\mathcal{Q}(y_1))^{-1}dn \\
&= q^{2\text{val}(y_1)}H_y(a)\int_{\mathcal{O}_F} \psi(n\varpi^{-2\text{val}(y_1)}\mathcal{Q}(y_1))^{-1}dn \\
&= q^{2\text{val}(y_1)}H_y(a) \\
&= q^{2\text{val}(y_1)}|a|^{\frac{d_1}{2}-2}\mathbb{1}_{\varpi^{-\text{val}(y_1)}\mathcal{O}_F}(a) \\
&= q^{\frac{d_1}{2}\text{val}(y_1)},
\end{aligned}$$

so Eq. (5.1.5) is valid in this case.

For $|a| < q^{\text{val}(y_1)}$, we have that the left hand side of Eq. (5.1.5) is

$$\begin{aligned}
&H_y(a)\int_{|n|\le|a|^2} \psi(n\mathcal{Q}(y_1))^{-1}dn + \int_{|a|^2<|n|} H_y(a^{-1}n)\psi(n\mathcal{Q}(y_1))^{-1}dn \\
&= q^{-2\text{val}(a)}H_y(a) + \int_{q^{-\text{val}(a)}<|n|\le q^{\text{val}(y_1)}} H_y(n)\psi(mn\mathcal{Q}(y_1))^{-1}q^{-\text{val}(a)}dn \\
&= q^{-2\text{val}(a)}H_y(a) + \sum_{k=0}^{\text{val}(y_1)+\text{val}(a)-1} q^{-2\text{val}(a)+k}(q-1)H_y(\varpi^{\text{val}(a)-1+k}).
\end{aligned}$$

On the other hand the right hand side is $q^{-\frac{d_1}{2}\text{val}(a)}$. One can show by induction on $\text{val}(a)$, for $\text{val}(a) > \text{val}(y_1)$, that these are equal.

For Eq. (5.1.6), the right hand side is

$$\int_{|n|\leq|d|^{-1}} H'_y(\begin{pmatrix}1 & \\ & d\end{pmatrix})\psi(n\mathcal{Q}(y_1))^{-1}dn + \int_{|d|^{-1}<|n|} H'_y(\begin{pmatrix}dn & \\ & n^{-1}\end{pmatrix})\psi(n\mathcal{Q}(y_1))^{-1}dn$$

$$=H'_y(\begin{pmatrix}1 & \\ & d\end{pmatrix})(\int_{|n|\leq|d|^{-1}} \psi(n\mathcal{Q}(y_1))^{-1}dn + \int_{|d|^{-1}<|n|} H'_y(\begin{pmatrix}dn & \\ & (dn)^{-1}\end{pmatrix})\psi(n\mathcal{Q}(y_1))^{-1}dn).$$

When $d \in \mathcal{O}^\times$, the above expression is

$$\frac{1+\int_{|dn|>1} H'_y(\begin{pmatrix}dn & \\ & (dn)^{-1}\end{pmatrix})\psi(n\mathcal{Q}(y_1))^{-1}dn)}{2+\sum_{k'=1}^{\mathrm{val}(y_1)} q^{k'}(1-(q-1)\sum_{k=1}^{\mathrm{val}(y_1)} q^{(\frac{d_1}{2}-2)k})} = 1.$$

This is equal to the left hand side of Eq. (5.1.6).

Finally, when $d \notin \mathcal{O}^\times$, both the left hand side and right hand side of Eq. (5.1.6) are 0. This exhausts the cases and proves the lemma. □

Inserting the result of Lemma 5.1.1 to the local integral I, we obtain

$$I_s(y) = \int_{U_2(F)\backslash G(F)} \int_{U_2(F)} \Phi_y(ng)\psi_{U_2,Q}(n)^{-1}dn \int_{N^\circ(F)} W_{\rho_{\tau,s}}(wu\iota(g), a_y)\psi_1(u)dudg.$$

Then by collapsing the integrals we get

$$I_s(y) = \int_{G(F)} \Phi_y(g) \int_{N^\circ(F)} W_{\rho_{\tau,s}}(wu\iota(g), a_y)\psi_1(u)dudg. \tag{5.1.7}$$

We justify this step by showing the above integral converges absolutely for $\Re(s)$ large in Lemma 5.2.2 below.

Since $\Phi_y \in C^\infty(\overline{U_2}(F)\backslash G(F)/K_G)$, applying the Iwasawa decomposition with respect to the lower Borel subgroup $\overline{B}_G(F)$ of $G(F)$ we get

$$
\begin{aligned}
&I_s(y)\\
&=\int_{T_G(F)}\Phi_y(g)\delta_{\overline{B}_G}^{-1}(g)\int_{\overline{U_2}(F)}\int_{N^\circ(F)}W_{\rho_{\tau,s}}(wuy\iota(g),a_y)\psi_1(u)dudydg\\
&=\int_{A_1(F)}\int_{M_1(F)}\int_{G_1(F)}\Phi_y(amx)\delta_{\overline{B}_G}^{-1}(amx)\\
&\times\int_{\overline{U_2}(F)}\int_{N^\circ(F)}W_{\rho_{\tau,s}}(wuy\iota(amx),a_y)\psi_1(u)dudydadmdx\\
&=\int_{F^\times}\int_{F^\times}\int_{F^\times}\Phi_y(\left(\begin{smallmatrix}a&\\&a^{-1}(mb)^{-1}\end{smallmatrix}\right),\left(\begin{smallmatrix}m&\\&b\end{smallmatrix}\right))|am|^2\\
&\times\int_{\overline{U_2}(F)}\int_{N^\circ(F)}W_{\rho_{\tau,s}}\left(wuy\iota_2\begin{pmatrix}am&&&\\&ab&&\\&&(ab)^{-1}&\\&&&(am)^{-1}\end{pmatrix},a_y\right)\psi_1(u)dudyd^\times ad^\times md^\times b\\
&=\int_{F^\times}H'_y(\begin{pmatrix}a_1&\\&a_1^{-1}\end{pmatrix})|a_1|^2\int_{F^\times}\int_{F^\times}H_{1,y}(\left(\begin{smallmatrix}1&\\&(mb)^{-1}\end{smallmatrix}\right),\left(\begin{smallmatrix}m&\\&b\end{smallmatrix}\right))H_{2,y}(\left(\begin{smallmatrix}1&\\&(mb)^{-1}\end{smallmatrix}\right),\left(\begin{smallmatrix}m&\\&b\end{smallmatrix}\right))|m|^2\\
&\times\int_{\overline{U_2}(F)}\int_{N^\circ(F)}W_{\rho_{\tau,s}}\left(wuy\iota_2\begin{pmatrix}a_1m&&&\\&a_1b&&\\&&(a_1b)^{-1}&\\&&&(a_1m)^{-1}\end{pmatrix},a_y\right)\psi_1(u)dudyd^\times a_1d^\times md^\times b.
\end{aligned}
$$

Let $I_F=[-\frac{2\pi i}{\log q},\frac{2\pi i}{\log q}]$, $\sigma_1,\sigma_2\in\mathbb{R}$, and

$$
c_q=(\frac{\log q}{2\pi i})^2. \tag{5.1.8}
$$

Let

$$
\chi_{s_1,s_2}\left(\left(\begin{smallmatrix}1&\\&a_1\end{smallmatrix}\right),\left(\begin{smallmatrix}a_2&\\&(a_1a_2)^{-1}\end{smallmatrix}\right)\right)=|a_1|^{s_1}|a_2|^{s_2}
$$

for $a_i\in F^\times$, and

$$
H_{y,s_2}(1)=\int_{F^{\times2}}H_{2,y}\left(\left(\begin{smallmatrix}1&\\&a_1\end{smallmatrix}\right),\left(\begin{smallmatrix}a_2&\\&(a_1a_2)^{-1}\end{smallmatrix}\right)\right)\chi_{s_1,s_2}^{-1}\left(\left(\begin{smallmatrix}1&\\&a_1\end{smallmatrix}\right),\left(\begin{smallmatrix}a_2&\\&(a_1a_2)^{-1}\end{smallmatrix}\right)\right)d^\times a_1d^\times a_2.
$$

By Mellin inversion, for $\left(\left(\begin{smallmatrix}1 & \\ & (mb)^{-1}\end{smallmatrix}\right), \left(\begin{smallmatrix}m & \\ & b\end{smallmatrix}\right)\right) \in M_1(F)G_1(F)$, we have

$$H_{2,y}\left(\left(\begin{smallmatrix}1 & \\ & (mb)^{-1}\end{smallmatrix}\right), \left(\begin{smallmatrix}m & \\ & b\end{smallmatrix}\right)\right) = c_q \int_{iI_F+\sigma_1}\int_{iI_F+\sigma_2} H_{y,s_2}(1)\chi_{s_1,s_2}\left(\left(\begin{smallmatrix}1 & \\ & m^{-1}b\end{smallmatrix}\right), \left(\begin{smallmatrix}m & \\ & b^{-1}\end{smallmatrix}\right)\right) ds_1 ds_2.$$

Let

$$c(q^{s_2}, y_2) = 1 - q^{s_2} + (q-1)q^{-\mathrm{val}(y_2)s_2}. \tag{5.1.9}$$

We have

$$\begin{aligned}
&H_{y,s_2}(1)\\
&= \int_{F^{\times 2}} H_{2,y}\left(\left(\begin{smallmatrix}1 & \\ & a_1\end{smallmatrix}\right), \left(\begin{smallmatrix}a_2 & \\ & (a_1a_2)^{-1}\end{smallmatrix}\right)\right) |a_1|^{-s_1}|a_2|^{-s_2} d^\times a_1 d^\times a_2\\
&= \int_{F^{\times 2}} \mathbb{1}_{\mathcal{O}_F^\times}(a_1)(\mathbb{1}_{\varpi^{-\mathrm{val}(y_1)}\mathcal{O}_F}(a_2) - \sum_{k=1}^{\infty} q^{(\frac{d_2}{2}-2)k}(q-1)\mathbb{1}_{\varpi^{k-\mathrm{val}(y_1)}\mathcal{O}_F}(a_2))|a_2|^{-s_2}d^\times a_2\\
&= \int_{F^\times} \mathbb{1}_{\varpi^{-\mathrm{val}(y_2)}\mathcal{O}_F}(a_2)|a_2|^{-s_2}d^\times a_2 + (q-1)\sum_{k=1}^{\infty} q^{(\frac{d_2}{2}-2)k}\\
&\times \int_{F^\times} \mathbb{1}_{\varpi^{i-\mathrm{val}(y_2)}\mathcal{O}_F}(a_2)|a_2|^{-s_2}d^\times a_2\\
&= \sum_{i=-\mathrm{val}(y_2)}^{\infty} q^{is_2} + (q-1)\sum_{k=1}^{\infty}(\sum_{i=k-\mathrm{val}(y_2)}^{\infty} q^{is_2})\\
&= c(q^{s_2}, y_2)\zeta_v(-s_2)^2.
\end{aligned}$$

We denote $\chi'_{s_1,s_2} = \chi_{s_1,s_2} H_{1,y}\delta^{-1}_{\overline{B_G}}$. Let

$$I_{s,s_1,s_2}(y) := \int_{F^\times}\int_{F^\times} |m|^{-s_1+s_2+\frac{d_2}{2}}|b|^{-s_1}$$
$$\times \int_{\overline{U_2}(F)}\int_{N^\circ(F)} W_{\rho_{\tau,s}}\left(wuy\iota_2\begin{pmatrix} m & & & \\ & b & & \\ & & b^{-1} & \\ & & & m^{-1}\end{pmatrix}, a_y\right)\psi_1(u)dudyd^\times md^\times b.$$

Then

$$I_s(y)$$
$$= c_q\int_{F^\times} H'_y\begin{pmatrix} a_1 & \\ & a_1^{-1}\end{pmatrix}|a_1|^{-2}\int_{F^\times}\int_{F^\times}\int_{iI_F+\sigma_1}\int_{iI_F+\sigma_2} H_{y,s_2}(1)\chi'_{s_1,s_2}\left(\begin{pmatrix} 1 & \\ & (mb)^{-1}\end{pmatrix},\begin{pmatrix} m & \\ & b\end{pmatrix}\right)$$
$$\times\int_{\overline{U_2}(F)}\int_{N^\circ(F)} W_{\rho_{\tau,s}}\left(wuy\iota_2\begin{pmatrix} a_1m & & & \\ & a_1b & & \\ & & (a_1b)^{-1} & \\ & & & (a_1m)^{-1}\end{pmatrix}, a_y\right)$$
$$\times\psi_1(u)dudyd^\times a_1d^\times md^\times bds_1ds_2$$
$$= c_q\alpha(y_1,y_2)\int_{iI_F+\sigma_1}\int_{iI_F+\sigma_2} H_{y,s_2}(1)\int_{F^\times} H'_y\begin{pmatrix} a_1 & \\ & a_1^{-1}\end{pmatrix}|a_1|^{-s_2}d^\times a_1$$
$$\times\int_{F^\times}\int_{F^\times}|m|^{-s_1+s_2+\frac{d_2}{2}}|b|^{-s_1}\int_{\overline{U_2}(F)}\int_{N^\circ(F)} W_{\rho_{\tau,s}}\left(wuy\iota_2\begin{pmatrix} m & & & \\ & b & & \\ & & b^{-1} & \\ & & & m^{-1}\end{pmatrix}, a_y\right)\psi_1(u)$$
$$\times dudyd^\times md^\times bds_1ds_2$$
$$= c_q\alpha(y_1,y_2)\int_{iI_F+\sigma_1}\int_{iI_F+\sigma_2} H_{y,s_2}(1)\int_{F^\times} H'_y\begin{pmatrix} a_1 & \\ & a_1^{-1}\end{pmatrix}|a_1|^{-s_2}d^\times a_1 I_{s,s_1,s_2}(y)ds_1ds_2.$$

We show in Lemma 5.2.3 that I_{s,s_1,s_2} converges when $\Re(s_1),\Re(-s_2)$ large and s in some vertical strip in the right-half plane depends on $\Re(s_1),\Re(-s_2)$.

We have now written $I_s(y)$ in terms of Whittaker functions as mentioned below Eq. (5.1.2). We now make use of the local functional equation for the local Rankin-

Selberg integral on $\mathrm{GL}_1 \times \mathrm{GL}_n$ to simplify $I_s(y)$. This requires the introdution of certain dual local integrals. Let

$$r(a) = \left(\begin{smallmatrix} 0 & 1 & 0 \\ 0 & 0 & I_{n-2} \\ 1 & 0 & a \end{smallmatrix}\right) \in \mathrm{GL}_n(F).$$

We construct the dual integral

$$I'_s(y) =$$

$$\int_{T_G(F)} \Phi_y(g)\delta_{\overline{B}_G}^{-1}(g) \int_{\overline{U_2}(F)} \int_{N^\circ(F)} \int_{M_{1\times(n-2)}(F)} W_{\rho_{\tau,s}}(wuy\iota(g), r(a)a_y)\psi_1(u)dadudydg.$$

Applying the same process as for I_s, we have that

$$I'_s(y)$$

$$= c_q\alpha(y_1,y_2) \int_{iI_F+\sigma_1} \int_{iI_F+\sigma_2} \int_{F^\times} H'_y \begin{pmatrix} a_1 & \\ & a_1^{-1} \end{pmatrix} |a_1|^{-s_2} d^\times a_1 \int_{F^\times} \int_{F^\times} |m|^{-s_1+s_2+\frac{d_2}{2}} |b|^{-s_1}$$

$$\times \int_{\overline{U_2}(F)} \int_{N^\circ(F)} \int_{M_{1\times(n-2)}(F)} W_{\rho_{\tau,s}} \left(wuy\iota_2 \begin{pmatrix} m & & & \\ & b & & \\ & & b^{-1} & \\ & & & m^{-1} \end{pmatrix}, r(a)a_y \right)$$

$$\times \psi_1(u)dadudyd^\times md^\times bds_1ds_2.$$

We denote

$$I'_{s,s_1,s_2}(y) = \int_{F^\times} \int_{F^\times} |m|^{-s_1+s_2+\frac{d_2}{2}} |b|^{-s_1}$$

$$\times \int_{\overline{U_2}(F)} \int_{N^\circ(F)} \int_{M_{1\times(n-2)}(F)} W_{\rho_{\tau,s}} \left(wuy\iota_2 \begin{pmatrix} m & & & \\ & b & & \\ & & b^{-1} & \\ & & & m^{-1} \end{pmatrix}, r(a)a_y \right)$$

$$\times \psi_1(u)dadudyd^\times md^\times b.$$

Remark. We show in Lemma 5.2.4 that I'_{s,s_1,s_2} converges when

$$\Re(s_1)+\Re(-s_2)\gg 1 \text{ and } A(s_1,s_2)\geq \Re(s)\geq B(s_1,s_2) \tag{5.1.10}$$

for some $A(s_1,s_2)>B(s_1,s_2)$. The convergence region of I'_{s,s_1,s_2} intersects the convergence region of I_{s,s_1,s_2} when $\Re(-s_2)$ in some vertical strip.

We now relate I'_{s_1,s_2} to our integral I_{s,s_1,s_2}.

Lemma 5.1.2. *In the region of convergence of I_{s,s_1,s_2} (see Lemma 5.2.3), we have*

$$I_{s,s_1,s_2}(y)=\gamma(s-s_1+s_2+\frac{d_2+1}{2},\tau)^{-1}I'_{s,s_1,s_2}(y),$$

where $\gamma(s-s_1+s_2+\frac{d_2+1}{2},\tau)$ is the $\mathrm{GL}_1\times\mathrm{GL}_n$ gamma factor defined in [JPSS83].

Proof. We let χ_1,χ_2 be characters on $F^\times$ such that $\chi_1=|\cdot|^{-s_1+s_2+\frac{d_2}{2}}$, $\chi_2=|\cdot|^{-s_1}$.

Let μ be a character of $\mathrm{SO}_2(F)$, and let $\pi_\zeta=\mathrm{Ind}_{\overline{B}_{G'(F)}}^{G'(F)}(\chi\otimes\mu)$. Let $\varphi_\zeta(g,m,I_2)\in V_{\pi_\zeta}$. Let

$$I_1=\int_{\mathrm{GL}_1(F)\backslash G'(F)}\int_{N^\circ(F)}\int_{\mathrm{GL}_1(F)}\varphi_\zeta(g,m,I_2)$$
$$\times W_{\rho_{\tau,s}}(wu\iota_2(g),\mathrm{diag}(m,I_{n-1}))|\mathrm{det}m|^{s-\zeta-\frac{n-1}{2}}\psi_1(u)dmdudg,$$

and

$$I_2=\int_{\mathrm{GL}_1(F)\backslash G'(F)}\int_{N^\circ(F)}\int_{\mathrm{GL}_1(F)}\varphi_\zeta(g,m,I_2)$$
$$\times\int_{M_{1\times(n-2)}(F)}W_{\rho_{\tau,s}}(wuy\iota_2(g),r(a))|\mathrm{det}m|^{s-\zeta-\frac{n-1}{2}}\psi_1(u)dadmdudg.$$

By [Kap12, Claim 4.1], one has

$$I_1 = \gamma(s-\zeta, \chi_1 \otimes \tau) I_2.$$

Let

$$I_1' = \int_{G'(F)} \varphi_\zeta(g, I_1, I_2) \int_{N^\circ(F)} W_{\rho_{\tau,s}}(wuy\iota(g), 1)\psi_1(u) du dy dg,$$

and

$$I_2' = \int_{G'(F)} \varphi_\zeta(g, I_1, I_2) \int_{N^\circ(F)} \int_{M_{1\times(n-2)}(F)} W_{\rho_{\tau,s}}(wuy\iota(g), r(a))\psi_1(u) da du dy dg.$$

By [Kap12, Proof of Lemma 4.1], one has

$$I_1 = I_1', I_2 = I_2'.$$

By the definition of φ_ξ as element of an induced representation and since we are in the unramified case, we have

$$\begin{aligned} I_1' &= \int_{T_{G'}(F)} \chi(g) \int_{\overline{U_2}(F)} \int_{N^\circ(F)} W_{\rho_{\tau,s}}(wuy\iota(g), 1)\psi_1(u) du dy dg \\ &= \int_{F^\times} \int_{F^\times} \chi_1(m)\chi_2(b)|m|^{-\frac{1}{2}} \int_{\overline{U_2}(F)} \int_{N^\circ(F)} W_{\rho_{\tau,s}}\left(wuy\iota_2\begin{pmatrix} m & & & \\ & b & & \\ & & b^{-1} & \\ & & & m^{-1}\end{pmatrix}, 1\right) \\ &\times \psi_1(u) du dy d^\times m d^\times b \end{aligned}$$

and

$$
\begin{aligned}
I_2' &= \int_{T_{G'}(F)} \chi(g) \int_{\overline{U_2}(F)} \int_{N^\circ} \int_{M_{1\times(n-2)}(F)} W_{\rho_{\tau,s}}(wuy\iota(g), r(a))\psi_1(u) dadudydg \\
&= \int_{F^\times} \int_{F^\times} \chi_1(m)\chi_2(b)|m|^{-\frac{1}{2}} \\
&\times \int_{\overline{U_2}(F)} \int_{N^\circ(F)} \int_{M_{1\times(n-2)}(F)} W_{\rho_{\tau,s}} \left(wuy\iota_2 \begin{pmatrix} m & & & \\ & b & & \\ & & b^{-1} & \\ & & & m^{-1} \end{pmatrix}, r(a) \right) \psi_1(u) \\
&\times dadudyd^\times m d^\times b.
\end{aligned}
$$

Here

$$
\chi \begin{pmatrix} m & & & \\ & b & & \\ & & b^{-1} & \\ & & & m^{-1} \end{pmatrix} = \chi_1(m)\chi_2(b)|m|^{-\frac{1}{2}},
$$

where χ_1, χ_2 are characters on $F^\times$.

Then we have

$$
I_1' = \gamma(s-\zeta, \chi_1 \otimes \tau) I_2'.
$$

□

Inserting the result to our local integral I_s we get

$$
\begin{aligned}
I_s(y) &= c_q \alpha(y_1, y_2) \int_{iI_F+\sigma_1} \int_{iI_F+\sigma_2} H_{y,s_2}(1) \int_{F^\times} H_y' \begin{pmatrix} a_1 & \\ & a_1^{-1} \end{pmatrix} |a_1|^{-s_2} d^\times a_1 \\
&\times \gamma(s - s_1 + s_2, \chi' \otimes \tau)^{-1} I'_{s,s_1,s_2} ds_1 ds_2 \\
&= c_q \alpha(y_1, y_2) \int_{iI_F+\sigma_1} \int_{iI_F+\sigma_2} \frac{c(q^{s_2}, y_2)\zeta_v(-s_2 - \frac{d_1}{2} + 2)^2 H_{y,s_2}(1)}{\gamma(s - s_1 + s_2, \chi' \otimes \tau)} I'_{s,s_1,s_2} ds_1 ds_2.
\end{aligned}
$$

Remark. Since $c(q^{s_2}, y_2)\zeta_v(-s_2 - \frac{d_1}{2} + 2)^2 H_{y,s_2}(1)$ converges when

$$\Re(-s_2) > \frac{d_1}{2} + 3,$$

$$c(q^{s_2}, y_2)\zeta_v(-s_2 - \frac{d_1}{2} + 2)^2 H_{y,s_2}(1)\gamma(s - s_1 + s_2, \chi' \otimes \tau)^{-1} I'_{s,s_1,s_2}$$

converges when

$$\max(\frac{d_1}{2} + 3, -n + \frac{d_2}{2} + 8 + c_1) < \Re(-s_2) < n + \frac{d_2}{2} + 2 + c_1 + c_2,$$

$\Re(s_1) + \Re(-s_2)$ large, and $\Re(s)$ bounded in some vertical strip depends on $\Re(s_1) + \Re(-s_2)$ as in Lemma 5.2.3. Since we set $d_2 > d_1$ the region for $\Re(-s_2)$ is non-empty.

To compute our final result, we remain to compute $I'_{s,s_1,s_2}(y)$. By the properties of $W_{\rho_{\tau,s}}$, we have

$$\begin{aligned}
&I'_{s,s_1,s_2}(y) \\
&= |4\mathcal{Q}'(y_2)|^{-\Re(s)+\frac{1}{2}-\frac{n}{2}} \int_{F^\times}\int_{F^\times} |m|^{-s_1+s_2+\frac{d_2}{2}}|b|^{-s_1} \\
&\times \int_{\overline{U_2}(F)}\int_{N^\circ(F)}\int_{M_{1\times(n-2)}(F)} W_{\rho_{\tau,s}}\left(r(a)wuy\iota_2\begin{pmatrix} -4\mathcal{Q}'(y_2)m & & & \\ & b & & \\ & & b^{-1} & \\ & & & (-4\mathcal{Q}'(y_2)m)^{-1}\end{pmatrix}, 1\right) \\
&\times \psi_1(u)dadudyd^\times md^\times b.
\end{aligned}$$

Since $\overline{U_2}(F)$ normalizes $N^\circ(F)$ and ψ_1, we can interchange uy to yu in the integral.

Denoting ${}^{(r(a)w_0)^{-1}}(yu) = (r(a)w_0)(yu)(r(a)w_0)^{-1}$, we have that

$$
\begin{aligned}
&I'_{s,s_1,s_2}(y)\\
&= |4\mathcal{Q}'(y_2)|^{-\Re(s)+\frac{1}{2}-\frac{n}{2}} \int_{F^\times}\int_{F^\times} |m|^{-s_1+s_2+\frac{d_2}{2}}|b|^{-s_1} \int_{\overline{U_2}(F)}\int_{N^\circ(F)}\\
&\times \int_{M_{1\times(n-2)}(F)} W_{\rho_{\tau,s}}\left({}^{(r(a)w_0)^{-1}}(yu)r(a)w\iota_2\begin{pmatrix} -4\mathcal{Q}'(y_2)m &&& \\ & b && \\ && b^{-1} & \\ &&& (-4\mathcal{Q}'(y_2)m)^{-1}\end{pmatrix}, 1\right)\\
&\times \psi_1(u)dadudyd^\times md^\times b.
\end{aligned}
$$

Let $\tilde{w_0} = \begin{pmatrix} && I_n \\ & (-1)^n & \\ I_n && \end{pmatrix}$ and let V be the subgroup

$$
V(R) = \left\{ \begin{pmatrix} I_{n-1} & 0 & y_2 & y_3 & s \\ & 1 & 0 & 0 & y_3' \\ && 1 & 0 & y_2' \\ &&& 1 & 0 \\ &&&& I_{n-1} \end{pmatrix} \in H(R) \right\}.
$$

By [Kap12, Claim 4.2], denoting ${}^{\tilde{w_0}}V = (\tilde{w_0})^{-1}V\tilde{w_0}$ we have

$$
\begin{aligned}
&I'_{s,s_1,s_2}(y)\\
&= |4\mathcal{Q}'(y_2)|^{-\Re(s)+\frac{1}{2}-\frac{n}{2}} \int_{F^\times}\int_{F^\times} |m|^{-s_1+s_2+\frac{d_2}{2}}|b|^{-s_1}\\
&\times \int_{M_{1\times(n-2)}(F)}\int_{{}^{\tilde{w_0}}V(F)} W_{\rho_{\tau,s}}\left(vu(a)w\iota_2\begin{pmatrix} -4\mathcal{Q}'(y_2)m &&&\\ & b &&\\ && b^{-1} &\\ &&& (-4\mathcal{Q}'(y_2)m)^{-1}\end{pmatrix},1\right)\\
&\times \psi_1(u)dvdad^\times m d^\times b\\
&= |4\mathcal{Q}'(y_2)|^{-\Re(s)+\frac{1}{2}-\frac{n}{2}} \sum_{k\in\mathbb{Z}} |\varpi^k|^{-s_1+s_2+\frac{d_2}{2}}\\
&\times \int_{M_{1\times(n-2)}(F)}\int_{\mathrm{SO}_2(F)}\int_{{}^{\tilde{w_0}}V(F)} W_{\rho_{\tau,s}}\left(vu(a)wx\iota_2\begin{pmatrix} -4\mathcal{Q}'(y_2)\varpi^k &&&\\ & 1 &&\\ && 1 &\\ &&& (-4\mathcal{Q}'(y_2)\varpi^k)^{-1}\end{pmatrix},1\right)\\
&\times \mu_{s_1}(x)\psi_1(u)dvdxda.
\end{aligned}
$$

Here $\mu_{s_1}(x) = |b|^{-s_1}$ is a character for $x \in \iota(G_1(F)) \cong \mathrm{SO}_2(F)$ (see Lemma 3.3.2).

As in [Kap12, Page 161], by the structure of $x \in \iota(G_1(F))$, we have ${}^{(u(a)w_0)^{-1}}x = {}^{w'\tilde{w_0}}x$, where $w' = \mathrm{diag}(I_n, -1, I_n)$. Thus we have

$$
\begin{aligned}
&I'_{s,s_1,s_2}(y)\\
&= |4\mathcal{Q}'(y_2)|^{-\Re(s)+\frac{1}{2}-\frac{n}{2}} \sum_{k\in\mathbb{Z}} |\varpi^k|^{-s_1+s_2+\frac{d_2}{2}}\\
&\times \int_{M_{1\times(n-2)}(F)}\int_{{}^{w'\tilde{w_0}}\mathrm{SO}_2(F)}\int_{{}^{\tilde{w_0}}V(F)} W_{s,}\left(vxu(a)w\iota_2\begin{pmatrix} -4\mathcal{Q}'(y_2)\varpi^k &&&\\ & 1 &&\\ && 1 &\\ &&& (-4\mathcal{Q}'(y_2)\varpi^k)^{-1}\end{pmatrix},1\right)\\
&\times \mu_{s_1}(x)\psi_1(u)dvdxda.
\end{aligned}
$$

We regard the $dvdx$ integral over ${}^{\tilde{w_0}}V(F) \times {}^{w'\tilde{w_0}}\mathrm{SO}_2(F)$ as a function on $H(F)$:

$$B_{W_{\rho_{\tau,s}}}(h) := \int_{{}^{w'\tilde{w_0}}\mathrm{SO}_2(F)} \int_{{}^{\tilde{w_0}}V(F)} W_{\rho_{\tau,s}}(vxh,1)\psi_1(u)\mu_{s_1}(x)dadudmdx. \tag{5.1.11}$$

Let

$$\psi_1' := \psi\left(\sum_{i=1}^{n-1} v_{i,i+1} + \frac{1}{2}v_{2n,1}\right) \quad (v \in {}^{\tilde{w_0}}Z_n(F) \ltimes {}^{\tilde{w_0}}V(F))$$

be a character.

Similar to the Bessel function in [Kap12, Page 162], our function $B_{W_{\rho_{\tau,s}}}$ is an unramified Bessel function which corresponds to the Bessel functional defined for the subgroup

$$({}^{\tilde{w_0}}Z_n(F) \ltimes {}^{\tilde{w_0}}V(F)) \rtimes {}^{w'\tilde{w_0}}\mathrm{SO}_2(F) = {}^{\tilde{w_0}}R^\circ(F)$$

(for ${}^{w'\tilde{w_0}}\mathrm{SO}_2(F)$ split) and representations $\rho_{\tau,s}$, ψ_1' and μ_{s_1}.

For $\delta = (\delta_1, \ldots, \delta_n) \in \mathbb{Z}^n$, we denote

$$\varpi_H^\delta = \mathrm{diag}(\varpi^{\delta_1}, \ldots, \varpi^{\delta_n}, 1, \varpi^{-\delta_1}, \ldots, \varpi^{-\delta_n}) \in T_H(F).$$

Substituting the function to our local integral we get

$$I'_{s,s_1,s_2}(y) = \sum_{k\in\mathbb{Z}} |\varpi^k|^{-s_1+s_2+\frac{d_2}{2}} \int_{M_{1\times(n-2)}(F)} B_{W_{\rho_{\tau,s}}}(\tilde{w_0}u(a)w\varpi_H^{\tilde{\delta}_{k+\mathrm{val}(4\mathcal{Q}'(y_2))}})da,$$

where $\tilde{\delta}_k = (0_{n-2}, k, 0) \in \mathbb{Z}^n$.

We have the following equality using the same argument as that proving [Kap12, Claim 4.3]:

Lemma 5.1.3.

$$\int_{M_{1\times(n-2)}(F)} B_{W_{\rho_{\tau,s}}}(\tilde{w_0}u(a)w\varpi_H^{\tilde{\delta}_k})da = B_{W_{\rho_{\tau,s}}}(\varpi_H^{\delta_{-k}})|\varpi^k|^{n-2},$$

where $\delta_k = (k, 0_{n-1}) \in \mathbb{Z}^n$.

□

Thus we get

$$I'_{s,s_1,s_2}(y) = |4\mathcal{Q}'(y_2)|^{-\Re(s)+\frac{1}{2}-\frac{n}{2}} \sum_{k\in\mathbb{Z}} q^{-(-s_1+s_2+n-2+\frac{d_2}{2})k} B_{W_{\rho_{\tau,s}}}(\varpi_H^{\delta_{-k-\mathrm{val}(4\mathcal{Q}'(y_2))}}).$$

Since $B_{W_{\rho_{\tau,s}}}$ is an unramified Bessel function, by the vanishing condition of unramified Bessel function, $B_{W_{\rho_{\tau,s}}}(\varpi_H^{\delta_{-k-\mathrm{val}(4\mathcal{Q}'(y_2))}}) = 0$ unless $k \le -\mathrm{val}(4\mathcal{Q}'(y_2))$. We have

$$I'_{s,s_1,s_2}(y) = |4\mathcal{Q}'(y_2)|^{-\Re(s)+\frac{1}{2}} \sum_{k\le-\mathrm{val}(4\mathcal{Q}'(y_2))} q^{-(-s_1+s_2+n-2+\frac{d_2}{2})k} B_{W_{\rho_{\tau,s}}}(\varpi_H^{\delta_{-k-\mathrm{val}(4\mathcal{Q}'(y_2))}}).$$

We proceed to state the main theorem of this section. Consider

$$\frac{c(q^{s_2}, y_2)\zeta_v(-s_2-\frac{d_1}{2}+2)^2\zeta_v(-s_2)^2}{\gamma(s-s_1+s_2, \chi'\otimes\tau)} B_{W_{\rho_{\tau,s}}}(\varpi_H^{\delta_{k-\mathrm{val}(4\mathcal{Q}'(y_2))}}) \tag{5.1.12}$$

as a product of Laurent series in q^{s_1} and q^{s_2}, where $c(q^{s_2}, y_2)$ is as defined in Eq. (5.1.9).

Let

$$C_{k,s}(y) :=$$

$$c_q \int_{iI_F+\sigma_1} \int_{iI_F+\sigma_2} q^{(-s_1+s_2)k} \frac{c(q^{s_2}, y_2)\zeta_v(-s_2 - \frac{d_1}{2} + 2)^2 \zeta_v(-s_2)^2}{\gamma(s - s_1 + s_2, \chi' \otimes \tau)} B_{W_{\rho_{\tau,s}}}(\varpi_H^{\delta_{k-\mathrm{val}(4\mathcal{Q}'(y_2))}}), \tag{5.1.13}$$

where c_q is as defined in Eq. (5.1.8). This is nothing but the product of the $-k$-th coefficient in q^{s_1} and the k-th coefficient in q^{s_2} of (5.1.12).

Theorem 5.1.4. *For all the data unramified and $\Re(s)$ large, we have*

$$I_s(y) = \alpha(y_1, y_2)|4Q'(y_2)|^{-\Re(s)+\frac{1}{2}-\frac{n}{2}} \sum_{k=\mathrm{val}(4\mathcal{Q}'(y_2))}^{\infty} q^{(n-2+\frac{d_2}{2})k} C_{k,s}(y),$$

where $\alpha(y_1, y_2)$ is as defined in Eq. (5.1.3).

Proof. Combining all the above results we get

$$
\begin{aligned}
&I_s(y) \\
&= c_q\alpha(y_1,y_2)\int_{iI_F+\sigma_1}\int_{iI_F+\sigma_2}\frac{c(q^{s_2},y_2)\zeta_v(-s_2-\frac{d_1}{2}+2)^2H_{y,s_2}(1)}{\gamma(s-s_1+s_2,\chi'\otimes\tau)}I'_{s,s_1,s_2}ds_1ds_2 \\
&= c_q\alpha(y_1,y_2)\int_{iI_F+\sigma_1}\int_{iI_F+\sigma_2}\frac{c(q^{s_2},y_2)\zeta_v(-s_2-\frac{d_1}{2}+2)^2\zeta_v(-s_2)^2}{\gamma(s-s_1+s_2,\chi'\otimes\tau)} \\
&\times \sum_{k=\mathrm{val}(4\mathcal{Q}'(y_2))}^{\infty} q^{(-s_1+s_2+n-2+\frac{d_2}{2})k}B_{W_{\rho_{\tau,s}}}(\varpi_H^{\delta_{k-\mathrm{val}(4\mathcal{Q}'(y_2))}})ds_1ds_2 \\
&= c_q\alpha(y_1,y_2)\sum_{k=\mathrm{val}(4\mathcal{Q}'(y_2))}^{\infty} q^{(n-2+\frac{d_2}{2})k} \\
&\times \int_{iI_F+\sigma_1}\int_{iI_F+\sigma_2}\frac{c(q^{s_2},y_2)\zeta_v(-s_2-\frac{d_1}{2}+2)^2\zeta_v(-s_2)^2}{\gamma(s-s_1+s_2,\chi'\otimes\tau)} \\
&\times q^{(-s_1+s_2)k}B_{W_{\rho_{\tau,s}}}(\varpi_H^{\delta_{k-\mathrm{val}(4\mathcal{Q}'(y_2))}})ds_1ds_2.
\end{aligned}
$$

We will make the above manipulations rigorous in Lemma 5.2.5 by showing that the infinite sum

$$
\frac{c(q^{s_2},y_2)\zeta_v(-s_2-\frac{d_1}{2}+2)^2\zeta_v(-s_2)^2}{\gamma(s-s_1+s_2,\chi'\otimes\tau)}\sum_{k=\mathrm{val}(4\mathcal{Q}'(y_2)}^{\infty} q^{(-s_1+s_2+n-2+\frac{d_2}{2})k}B_{W_{\rho_{\tau,s}}}(\varpi_H^{\delta_{k-\mathrm{val}(4\mathcal{Q}'(y_2))}})
$$

converges absolutely for $\Re(s_1),\Re(-s_2)$ large and $\Re(s)$ lies in some region in the right plane that depends on $\Re(s_1),\Re(-s_2)$. $\square$

We give an explicit expression for the unramified Bessel function $B_{W_{\rho_{\tau,s}}}$ following

[Kap12] below. Let

$$B^\circ \tag{5.1.14}$$

be the unique Bessel function corresponding to the data $\rho_{\tau,s}$, ψ_1' and μ_{s_1} such that $B^\circ_{\psi_1', s_1}(1) = 1$. Let us recall the formula for B° from [BFF97, Theorem 1.6]. We first define three functions depending on the characters $\chi_{i,s}$ and the character μ_{s_1}:

$$\Delta(\chi) = \prod_{i=1}^{n} \chi_{i,s}(\varpi)^{-1+i-n}(1-\chi_{i,s}(\varpi)^2) \prod_{1\le i\le j\le n}(1-\chi_{i,s}(\varpi)\chi_{j,s}(\varpi))(1-\chi_{i,s}(\varpi)\chi_{j,s}(\varpi)^{-1}),$$

$$D(\chi_s, \mu_{s_1}) = \prod_{i=1}^{n} \chi_{i,s}(\varpi)^{-(n+1-i)} \prod_{i=1}^{n}(1-\chi_{i,s}(\varpi)\mu_{s_1}(\varpi)q^{-\frac{1}{2}})(1-\chi_{i,s}(\varpi)\mu_{s_1}(\varpi)^{-1}q^{-\frac{1}{2}}),$$

and for $t \in T_H(F)$,

$$B^\circ(t) = \frac{\delta_H(t)^{1/2}}{(1-q^{-1})\Delta(\chi_s)} \sum_{w\in W} \operatorname{sgn}(w) D({}^{w}\chi_s, \mu_{s_1})\, {}^{w}\chi_s(t)^{-1}. \tag{5.1.15}$$

The function $\Delta(\chi)$ is (up to a sign) the denominator in the Weyl character formula for the symplectic group Sp_{2n}. The equality (5.1.15) is the analogue for unramified Bessel functions of the Casselman-Shalika formula for Whittaker functions (see [BFF97, Theorem 1.6]).

Lemma 5.1.5. *Let W be the Weyl group of H. Let t_τ be the Satake parameter for τ. We denote the local L-function for $\mu_{s_1} \times \tau$ and symmetric square L-function for τ*

as

$$L(s, \mu_{s_1} \times \tau) = \det(1 - (t_\mu \otimes t_\tau) q^{-s})^{-1}$$

$$= \prod_{i=1}^{n} (1 - \chi_i(\varpi)\mu_{s_1}(\varpi) q^{-s})^{-1} (1 - \chi_i(\varpi)\mu_{s_1}(\varpi)^{-1} q^{-s})^{-1},$$

$$L(2s, \tau, \mathrm{Sym}^2) = \prod_{1 \leq i \leq j \leq n} (1 - \chi_i(\varpi)\chi_i(\varpi) q^{-2s})^{-1} \prod_{i=1}^{n} (1 - \chi_i(\varpi) q^{-2s})^{-1}.$$

For $t = \varpi_H^{\delta_{-k-\mathrm{val}(4\mathcal{Q}'(y_2)}} \in T_H(F)$ *such that* $k \leq -\mathrm{val}(4\mathcal{Q}'(y_2))$, *we have*

$$B_{W_{\rho_{\tau,s}}}(t) = \frac{L(s, \mu_{s_1} \times \tau)}{L(2s, \tau, \mathrm{Sym}^2)} B^\circ(t).$$

Proof. Let $f_0 \in \rho_{\tau,s}$ be such that $f_0(K) = 1$. Let B_{f_0} be the unramified Bessel function associated with f_0. Since we normalize $W_{\rho_{\tau,s}}$ by $W_{\rho_{\tau,s}}(1,1) = 1$, we have $B_{W_{\rho_{\tau,s}}} = W_{\tau_s}(1)^{-1} B_{f_0}(1) B^\circ$, where

$$W_{\tau_s}(1) = \sum_{1 \leq i \leq j \leq n} (1 - \chi_{i,s}(\varpi)\chi_{i,s}(\varpi)^{-1} q^{-1})$$

(see [CS80a]), and by [Kap12, Theorem 1],

$$B_{f_0}(1) = \frac{\prod_{1 \leq i \leq j \leq n} (1 - \chi_{i,s}(\varpi)\chi_{j,s}(\varpi) q^{-1})(1 - \chi_{i,s}(\varpi)\chi_{j,s}(\varpi)^{-1} q^{-1}) \prod_{i=1}^{n} (1 - \chi_{i,s}(\varpi) q^{-1})}{\prod_{i=1}^{n} (1 - \chi_{i,s}(\varpi)\mu_{s_1}(\varpi) q^{-\frac{1}{2}})(1 - \chi_{i,s}(\varpi)\mu_{s_1}(\varpi)^{-1} q^{-\frac{1}{2}})}.$$

Using the explicit formula for the unramified Bessel functional (see [BFF97], [Kap12, Section 2.5]), we deduce the result using similar notations as in [Kap12, Section 2.5]. □

5.2 Convergence of the local integrals in the unramified case

Let F be a non-Archimedean local field. In this section we prove absolute convergence for various local integrals that appear in the unramified computations in Section 5.1. We point out that the bounds we prove in this section are not used in our global considerations.

The points of the group $wN^{\circ}\overline{U_2}'w^{-1}$ in an F-algebra R are

$$wN^{\circ}\overline{U_2}'(R)w^{-1} =$$

$$\left\{ \begin{pmatrix} 1 &&&&&& \\ c_1 & 1 &&&&& \\ && I_{n-2} &&&& \\ -\alpha c_2 & 0 & v_3' & 1 &&& \\ v_1 & v_2 & T & v_3 & I_{n-2} && \\ -\alpha c_1 & 0 & v_2' & 0 && 1 & \\ \alpha c_1^2 - \frac{1}{2}c_2^2 & \alpha c_1 & v_1' & \alpha c_2 && -c_1 & 1 \end{pmatrix} : c_1, c_2 \in R, v_1, v_2, v_3 \in R^{n-2}, T \in M_{(n-2)}(R) \right\}.$$

Here $\overline{U_2}' = \iota(\overline{U_2})$ and $\alpha = \frac{1}{2}$.

Lemma 5.2.1. *For $\Re(s)$ large, the integral*

$$\int_{wN^{\circ}(F)\overline{U_2}'(F)w^{-1}} W_{\rho_{\tau,s}}(uv, 1)du \tag{5.2.1}$$

converges, where v is a unipotent element of $H(F)$ defined as in Eq. (4.2.2).

Remark. We remark that this is different from Lemma 4.2.1. The integral is over the group $wN^{\circ}(F)\overline{U_2}'(F)w^{-1}$ whereas the integral in Lemma 4.2.1 is over $wN^{\circ}(F)w^{-1}$.

Proof. For $u \in wN^{\circ}(F)\overline{U_2}'(F)w^{-1}$, we have

$$uv = \begin{pmatrix} 1 & & & & & & \\ c_1 & 1 & & c & & -\frac{1}{2}c^2 & \\ & & I_{n-2} & & & & \\ -\alpha c_2 & 0 & v_3' & 1 & & -c & \\ v_1 & v_2 & T & v_2c+v_3 & I_{n-2} & -\frac{1}{2}v_2c^2-v_3c & \\ -\alpha c_1 & 0 & v_2' & 0 & & 1 & \\ \alpha c_1^2-\frac{1}{2}c_2^2 & \alpha c_1 & v_1' & \alpha c_1c+\alpha c_2 & & -c_1 & 1 \end{pmatrix},$$

where $c \in F$.

Considering the Iwasawa decomposition of $u \in wN^{\circ}(F)\overline{U_2}'(F)w^{-1}$ in H (using the notation as in Lemma 4.2.1), we have

$$uv = na'k.$$

Here $(n, a', k) \in N_n(F) \times T_H(F) \times K_H$ and $a' = \mathrm{diag}(a, 1, w_0a^{-1}w_0)$ for $a \in \mathrm{GL}_n(F)$. We denote the i-th line of uv as $(uv)_i$. By a similar argument as in Lemma 4.2.2, we have

$$[\mathcal{L}(uv)]^{-2j} \le |\frac{a_j}{a_{j+1}}| \le [\mathcal{L}(uv)]^{2j}.$$

Here $j = 1, \ldots, n$ and $[\mathcal{B}(uv)] = \max\{1, \|\mathcal{B}(uv)\|\}$ and $\|\cdot\|$ is the sup-norm, and

$$[\mathcal{L}(uv)]^{-n} \le D(uv) = |\det(a)| \le [\mathcal{L}(uv)]^{-1}.$$

Note that by arguing as in Lemma 4.2.1 and Lemma 4.2.2, the integral Eq. (5.2.1) is majorized by

$$\sum_{j=1}^{\nu} c_{j,s}[v]^C \int_{wN^{\circ}\overline{U_2}'(F)w^{-1}} [u]^{-\Re(s)-\frac{n-1}{2}+C} du, \tag{5.2.2}$$

where C is a positive constant which depends only on τ. $\square$

Lemma 5.2.2. *The integral*

$$I_s(y) = \int_{G(F)} \Phi_y(g) \int_{N^\circ(F)} W_{\rho_{\tau,s}}(wu\iota(g), a_y)\psi_1(u)dudg \tag{5.2.3}$$

converges absolutely for $\Re(s)$ *large enough.*

Proof. By the Iwasawa decomposition with respect to the lower Borel subgroup of $G(F)$, we have

$$I_s(y) = \int_{\overline{U_2}(F)} \int_{T_G(F)} \Phi_y(vg)\delta_{\overline{B}_G}^{-1}(g) \int_{N^\circ(F)} W_{\rho_{\tau,s}}(wuv\iota(g), a_y)\psi_1(u)dudvdg.$$

Since Φ_y is invariant under $\overline{U_2}(F)$, we have

$$\begin{aligned} I_s(y) &= \int_{T_G(F)} \Phi_y(g)\delta_{\overline{B}_G}^{-1}(g) \int_{\overline{U_2}(F)} \int_{N^\circ(F)} W_{\rho_{\tau,s}}(wuv\iota(g), a_y)\psi_1(u)dudvdg \\ &= \int_{G_1(F)} \int_{A_1(F)} \int_{A_2(F)} \Phi_y(xa_1a_2)\delta_{\overline{B}_G}^{-1}(xa_1a_2) \\ &\times \int_{\overline{U_2}(F)} \int_{N^\circ(F)} W_{\rho_{\tau,s}}(wuv\iota(xa_1a_2), a_y)\psi_1(u)dudvdxda_1da_2. \end{aligned}$$

Using the Iwasawa decomposition of $\mathrm{SO}_3(F)$ as in Lemma 4.2.3, we have

$$
\begin{aligned}
&|I_s(y)| \\
&\leq \int_{F^\times}\int_{F^\times}\int_{F^\times} |\Phi_y| \left(\begin{pmatrix} a_1 & \\ & ba_1^{-1} \end{pmatrix}, \begin{pmatrix} a_2 & \\ & b^{-1}a_2^{-1} \end{pmatrix} \right) |a_1a_2|^{\Re(s)-n+5} |\lfloor ba_1a_2^{-1} \rfloor|^{\Re(s)-n+3} \\
&\times \int_{\overline{U_2}(F)} \int_{N^\circ(F)} |W_{\rho_{\tau,s}}|((wuvn', \mathrm{diag}(-4\mathcal{Q}'(y_2)a_1a_2, \lfloor b^{-1}a_1a_2^{-1} \rfloor, I_{n-2})) \\
&\times du dv d^\times b d^\times a_1 d^\times a_2 \\
&= \int_{F^\times}\int_{F^\times}\int_{F^\times} |\Phi_y| \left(\begin{pmatrix} a_1 & \\ & ba_1^{-1} \end{pmatrix}, \begin{pmatrix} a_2 & \\ & b^{-1}a_2^{-1} \end{pmatrix} \right) |a_1a_2|^{\Re(s)-n+5} |\lfloor ba_1a_2^{-1} \rfloor|^{\Re(s)-n+3} \\
&\times \int_{\overline{U_2}(F)} \int_{N^\circ(F)} |W_{\rho_{\tau,s}}|((wuvw^{-1})(wn'w^{-1}), \mathrm{diag}(-4\mathcal{Q}'(y_2)a_1a_2, \lfloor b^{-1}a_1a_2^{-1} \rfloor, I_{n-2})) \\
&\times du dv d^\times b d^\times a_1 d^\times a_2 \\
&= \int_{F^\times}\int_{F^\times}\int_{F^\times} |\Phi_y| \left(\begin{pmatrix} a_1 & \\ & ba_1^{-1} \end{pmatrix}, \begin{pmatrix} a_2 & \\ & b^{-1}a_2^{-1} \end{pmatrix} \right) |a_1a_2|^{\Re(s)-n+5} |\lfloor ba_1a_2^{-1} \rfloor|^{\Re(s)-n+3} \\
&\times \int_{wN^\circ\overline{U_2}'(F)w^{-1}} |W_{\rho_{\tau,s}}|(u(wn'w^{-1}), \mathrm{diag}(-4\mathcal{Q}'(y_2)a_1a_2, \lfloor b^{-1}a_1a_2^{-1} \rfloor, I_{n-2})) \\
&\times du d^\times b d^\times a_1 d^\times a_2,
\end{aligned}
$$

where n' is as defined in Eq. (4.2.15).

By Eq. (5.2.2) in Lemma 5.2.1 and a similar argument as in Lemma 4.2.3, the above integral is majorized by a finite sum of integrals of the form

$$
\begin{aligned}
&\int_{F^\times}\int_{F^\times}\int_{F^\times} |\Phi_y| \left(\begin{pmatrix} a_1 & \\ & ba_1^{-1} \end{pmatrix}, \begin{pmatrix} a_2 & \\ & b^{-1}a_2^{-1} \end{pmatrix} \right) |a_1a_2|^{\Re(s)-n+5} |\lfloor ba_1a_2^{-1} \rfloor|^{\Re(s)-n+3-c_1} \\
&\qquad \times \left(\int_{wN^\circ\overline{U_2}'(F)w^{-1}} [u]^{-\Re(s)-\frac{n-1}{2}+c_2} du \right) d^\times b d^\times a_1 d^\times a_2,
\end{aligned}
$$

where $c_1, c_2 > 0$ are constants depend only on τ.

Substituting the above result into our local integral while using the exiplicit formula for Φ_y, we have that for $\Re(s)$ large the above integral is majorized by

$$\int_{F^\times}\int_{F^\times}\int_{F^\times} |H_{1,y}| \left(\begin{pmatrix} a_1 & \\ & ba_1^{-1} \end{pmatrix}, \begin{pmatrix} a_2 & \\ & b^{-1}a_2^{-1} \end{pmatrix} \right) |H_{2,y}|(\begin{pmatrix} a_1 & \\ & ba_1^{-1} \end{pmatrix}, \begin{pmatrix} a_2 & \\ & b^{-1}a_2^{-1} \end{pmatrix})$$

$$\times |a_1 a_2|^{\Re(s)-n+5} d^\times b d^\times a_1 d^\times a_2$$

$$= \int_{F^\times}\int_{F^\times} |a_1|^{\Re(s)+\frac{d_1-n+3}{2}} |a_2|^{\Re(s)+\frac{d_2-n+3}{2}}$$

$$\times (\mathbb{1}_{\varpi^{-ky_1}\mathcal{O}_F}(a_1) - \sum_{k=1}^{\infty} q^{(\frac{d_1}{2}-2)k}(q-1)\mathbb{1}_{\varpi^{k-ky_1}\mathcal{O}_F}(a_1))$$

$$\times (\mathbb{1}_{\varpi^{-ky_2}\mathcal{O}_F}(a_2) - \sum_{k=1}^{\infty} q^{(\frac{d_1}{2}-2)k}(q-1)\mathbb{1}_{\varpi^{k-ky_2}\mathcal{O}_F}(a_2)) da_1^\times da_2^\times$$

which converges absolutely for $\Re(s)$ large by a similar computation as for $H_{y,s_1,s_2}(1)$ in Section 5.1. □

Lemma 5.2.3. *There exists positive integers $C_1 < C_2, C_3, C_4$ which depend on (τ, d_1, d_2, n) such that*

$$I_{s,s_1,s_2}(y) = \int_{F^\times}\int_{F^\times} \chi'_{s_1,s_2}(\begin{pmatrix} 1 & \\ & (mb)^{-1} \end{pmatrix}, \begin{pmatrix} m & \\ & b \end{pmatrix}) \int_{\overline{U_2}(F)} \int_{N^\circ(F)}$$

$$\times W_{\rho_{\tau,s}} \left(wuy\iota_2 \left(\begin{pmatrix} m & & & \\ & b & & \\ & & b^{-1} & \\ & & & m^{-1} \end{pmatrix} \right), a_y \right) \psi_1(u) du dy d^\times m d^\times b$$

converges absolutely for

$$\Re(s_1) + \Re(-s_2) + C_1 < \Re(s) < \Re(s_1) + 2\Re(-s_2) + C_2,$$

$$C_3 < \Re(s_1) + \Re(-s_2), C_4 < \Re(-s_2).$$

Proof. As in Lemma 5.2.2, I_{s,s_1,s_2} is majorized by a finite sum of integrals of the form

$$\int_{F^\times} \int_{F^\times} |m|^{-s_1+s_2+\frac{d_2}{2}} |b|^{-s_1} |m|^{\Re(s)-n+5} |\lfloor b \rfloor|^{\Re(s)-n+3-c_1}$$
$$\times \int_{wN^\circ \overline{U_2}'(F)w^{-1}} [u]^{-\Re(s)-\frac{n-1}{2}+c_2} du d^\times m d^\times b,$$

where $c_1, c_2 > 0$ are constants depend only on τ.

Also, as in Lemma 4.2.3, we have

$$|m| \leq c'[u] |\lfloor b \rfloor|^{-2}.$$

Here c' is a conatant depends only on τ.

Thus the integral is majorized by a finite sum of integrals of the form

$$\int_{F^\times} |b|^{-s_1} |\lfloor b \rfloor|^{\Re(s)-n+3-c_1} d^\times b \int_{wN^\circ \overline{U_2}'(F)w^{-1}} \int_{|m| \leq c'[u] |\lfloor b \rfloor|^{-2}} |m|^{-s_1+s_2+\frac{d_2}{2}+\Re(s)-n+5}$$
$$\times [u]^{-\Re(s)-\frac{n-1}{2}+c_2} d^\times m du.$$

The integral

$$\int_{|m| \leq c'[u] |\lfloor b \rfloor|^{-2}} |m|^{-s_1+s_2+\frac{d_2}{2}+\Re(s)-n+5} d^\times m$$

converges absolutely when

$$-\Re(s_1) + \Re(s_2) + \frac{d_2}{2} + \Re(s) - n + 5 > 0.$$

Thus when $\Re(s) > \Re(s_1) - \Re(s_2) - \frac{d_2}{2} + n - 5$, the integral I'_{s,s_1,s_2} is majorized by a finite sum of integrals of the form

$$\int_{F^\times} |b|^{-s_1} \lfloor b^2 \rfloor^{-\Re(s)+2s_1-2s_2+d_2+n-7-c_1} d^\times b \int_{wN^\circ \overline{U_2}'(F)w^{-1}} \lfloor u \rfloor^{-s_1+s_2+\frac{d_2-3n+11}{2}+c_2} du.$$

This integral converges absolutely when

$$-\Re(s_1)+\Re(s_2)+\frac{d_2-3n+11}{2}+c_2<-C,$$

$$-\Re(s)+2\Re(s_1)-2\Re(s_2)+d_2+n-7-c_1-\Re(s_1)>0,$$

$$\Re(s)-2\Re(s_1)+2\Re(s_2)-d_2-n+7+c_1-\Re(s_1)<0.$$

Here C is a positive integers which depends on τ.

Summarizing, the convergence region is equivalent to

$$\Re(s_1)+\Re(-s_2)-\frac{d_2}{2}+n-5<\Re(s)$$

$$<\min(\Re(s_1),3\Re(s_1))+2\Re(-s_2)+d_2+n-7-c_1,$$

$$\Re(s_1)+\Re(-s_2)>\frac{d_2-3n+11}{2}+c_2+C.$$

This is non-empty when

$$\Re(s_1)+\Re(-s_2)-\frac{d_2}{2}+n-5<\min(\Re(s_1),3\Re(s_1))+2\Re(-s_2)+d_2+n-7-c_1,$$

which is equvalent to

$$\Re(-s_2)>-\frac{3d_2}{2}+2+c_1.$$

Thus we get the non-empty region in the statement of the lemma. □

Let $X_1 \subset H$ be the unipotent subgroup of H whose points in an F-algebra R are

$$X_1(R) =$$

$$\left\{ \begin{pmatrix} 1 &&&&&& \\ & I_{n-2} &&&&& \\ && 1 &&&& \\ 0 & v_3' & -\alpha c_2 & 1 &&& \\ -\alpha c_1 & v_1' & \frac{1}{2}c_2^2 & \alpha c_2 & 1 && \\ v_2 & T & v_1 & v_3 && I_{n-2} & \\ 0 & v_2' & -\alpha c_1 & 0 &&& 1 \end{pmatrix} : c_1, c_2 \in F, v_1, v_2, v_3 \in F^{n-2}, T \in M_{n-2}(F) \right\},$$

where $\alpha = \frac{1}{2}$.

Lemma 5.2.4. *There are constants $C_0, C'', c_2 > 0$, c_1 which depend on (τ, d_1, d_2, n) such that*

$$I'_{s,s_1,s_2}(y) = \int_{F^\times} \int_{F^\times} \chi'_{s_1,s_2}\left(\left(\begin{smallmatrix} 1 & \\ & (mb)^{-1} \end{smallmatrix}\right), \left(\begin{smallmatrix} m & \\ & b \end{smallmatrix}\right)\right)$$

$$\times \int_{\overline{U_2}(F)} \int_{N^\circ(F)} \int_{M_{1\times(n-2)}(F)} W_{\rho_{\tau,s}}(wuy\iota(g), r(e)a_y)\psi_1(u) de du dy d^\times m d^\times b$$

converges absolutely for

$$\Re(s_1) + \Re(-s_2) - \frac{d_1}{2} - 5 - c_1 > \Re(s)$$

$$> \max(\frac{C_0}{C_0+1}(\Re(s_1) + \Re(-s_2) - \frac{d_2}{2} - 5 - c_1) - \frac{n-1-C''}{C_0+1},$$

$$\Re(s_1) + \frac{2C_0}{2C_0+1}(\Re(-s_2) - \frac{d_2}{2} - 5 - c_1) - \frac{n-3+c_2}{2C_0+1},$$

$$\Re(s_1) + \frac{2C_0}{2C_0-1}(\Re(-s_2) - \frac{d_2}{2} - 5 - c_1) - \frac{n-3+c_2}{2C_0-1}).$$

$$-n+\frac{d_2}{2}+8+c_1<\Re(-s_2)<n+\frac{d_2}{2}+2+c_1+c_2,$$

$$c_0(-n+3-c_2)+C''<\Re(s_1)+\Re(-s_2).$$

Proof. We use the approach outlined in the proof of [Sou93, Proposition 11.16].

As in Lemma 5.2.3, we have

$$\begin{aligned}
&|I'_{s,s_1,s_2}| \\
&\leq \int_{F^\times}\int_{F^\times} |m|^{-s_1+s_2+\frac{d_2}{2}+\Re(s)+5}|b|^{-s_1}|\lfloor b\rfloor|^{\Re(s)-n+3} \\
&\times \int_{\overline{U_2}(F)}\int_{N^\circ(F)}\int_{M_{1\times(n-2)}(F)} |W_{\rho_{\tau,s}}|(wuyn', r(e)\mathrm{diag}(-4\mathcal{Q}'(y_2)m, \lfloor b\rfloor, I_{n-2})) \\
&\times\, dedudyd^\times md^\times b,
\end{aligned}$$

where

$$r(e)=\begin{pmatrix}0&1&0\\0&0&I_{n-1}\\1&0&e\end{pmatrix}.$$

Since

$$\begin{pmatrix}0&1&0\\0&0&I_{n-1}\\1&0&e\end{pmatrix}\begin{pmatrix}m&&\\&\lfloor b\rfloor&\\&&I_{n-2}\end{pmatrix}=\begin{pmatrix}\lfloor b\rfloor&&\\&I_{n-2}&\\&&m\end{pmatrix}\begin{pmatrix}0&1&0\\0&0&I_{n-2}\\1&0&m^{-1}e\end{pmatrix},$$

we have

$$
\begin{aligned}
&|I'_{s,s_1,s_2}(y)| \\
&\leq |-4\mathcal{Q}'(y_2)|^{s_1-s_2-\frac{d_2}{2}-\Re(s)-5} \int_{F^\times} \int_{F^\times} |m|^{-s_1+s_2+\frac{d_2}{2}+\Re(s)+5} |b|^{-s_1} |\lfloor b \rfloor|^{\Re(s)-n+3} \\
&\times \int_{\overline{U_2}(F)} \int_{N^\circ(F)} \int_{M_{1\times(n-2)}(F)} |W_{\rho_{\tau,s}}|(v(r(m^{-2}e))wuyn', \mathrm{diag}(\lfloor b \rfloor, I_{n-2}, m)) \\
&\times dedudyd^\times md^\times b.
\end{aligned}
$$

Here the symbol $v(r(m^{-2}e)) \in M_n(F)$ is as defined in Eq. (3.1.4).

By a change of variable $e \mapsto m^2 e$, we have that

$$
\begin{aligned}
&|I'_{s,s_1,s_2}(y)| \\
&\leq |-4\mathcal{Q}'(y_2)|^{s_1-s_2-\frac{d_2}{2}-\Re(s)-5} \int_{F^\times} \int_{F^\times} |m|^{-s_1+s_2+\frac{d_2}{2}+\Re(s)+5} |b|^{-s_1} |\lfloor b \rfloor|^{\Re(s)-n+3} |m|^n \\
&\times \int_{\overline{U_2}(F)} \int_{N^\circ(F)} \int_{M_{1\times(n-2)}(F)} |W_{\rho_{\tau,s}}|(v(r(e))wuyn', \mathrm{diag}(\lfloor b \rfloor, I_{n-2}, m)) dedudyd^\times md^\times b \\
&= |-4\mathcal{Q}'(y_2)|^{s_1-s_2-\frac{d_2}{2}-\Re(s)-5} \int_{F^\times} \int_{F^\times} |m|^{-s_1+s_2+\frac{d_2}{2}+\Re(s)+5} |b|^{-s_1} |\lfloor b \rfloor|^{\Re(s)-n+3} \\
&\times \int_{N^\circ(F)} \int_{M_{1\times(n-2)}(F)} |W_{\rho_{\tau,s}}|(v(r(e))(wuw^{-1})(wyw^{-1})(wn'w^{-1}), \mathrm{diag}(\lfloor b \rfloor, I_{n-2}, m)) \\
&\times dedudyd^\times md^\times b.
\end{aligned}
$$

Since for $u \in N^\circ(F)$,

$$
wuw^{-1} \in \begin{pmatrix}
1 & & & & & & \\
 & 1 & & & & & \\
 & & I_{n-2} & & & & \\
 & & v_3' & 1 & & & \\
v_1 & v_2 & T & v_3 & I_{n-2} & & \\
0 & 0 & v_2' & & & 1 & \\
0 & 0 & v_1' & & & & 1
\end{pmatrix}
$$

and

$$v(r(e))\begin{pmatrix} 1 & & & & & & \\ & 1 & & & & & \\ & & I_{n-2} & & & & \\ & & v_3' & 1 & & & \\ v_1 & v_2 & T & v_3 & I_{n-2} & & \\ 0 & 0 & v_2' & & & 1 & \\ 0 & 0 & v_1' & & & & 1 \end{pmatrix} = \begin{pmatrix} 1 & & & & & & \\ & I_{n-2} & & & & & \\ & & 1 & & & & \\ 0 & v_3' & 0 & 1 & & & \\ 0 & v_1' & 0 & 0 & 1 & & \\ v_2 & T+e'v_1'-ev_1 & v_1 & v_3 & & I_{n-2} & \\ 0 & v_2' & 0 & 0 & & & 1 \end{pmatrix} v(r(e)),$$

we have

$$|W_{\rho_{\tau,s}}|(v(r(e))(wuw^{-1})(wyw^{-1})(wn'w^{-1}), \mathrm{diag}(\lfloor b \rfloor, I_{n-2}, m))$$

$$=|W_{\rho_{\tau,s}}|(u'v(r(e))(wyw^{-1})(wn'w^{-1}), \mathrm{diag}(\lfloor b \rfloor, I_{n-2}, m)).$$

Here

$$u' = \begin{pmatrix} 1 & & & & & & \\ & I_{n-2} & & & & & \\ & & 1 & & & & \\ 0 & v_3' & 0 & 1 & & & \\ 0 & v_1' & 0 & 0 & 1 & & \\ v_2 & T+e'v_1'-ev_1 & v_1 & v_3 & & I_{n-2} & \\ 0 & v_2' & 0 & 0 & & & 1 \end{pmatrix}.$$

We observe that

$$\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & I_{n-2} \\ 1 & 0 & e \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ c_1 & 1 & 0 \\ 0 & 0 & I_{n-2} \end{pmatrix} = \begin{pmatrix} 1 & -c_1 e & c_1 \\ 0 & I_{n-2} & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & I_{n-2} \\ 1 & 0 & e \end{pmatrix}.$$

Also, for $y \in \overline{U_2}(F)$,

$$
\begin{aligned}
wyw^{-1} &= \begin{pmatrix} 1 & & & & & & \\ c_1 & 1 & & & & & \\ & & I_{n-2} & & & & \\ -\alpha c_2 & 0 & 0 & 1 & & & \\ 0 & 0 & 0 & 0 & 1 & & \\ -\alpha c_1 & 0 & 0 & 0 & & 1 & \\ \alpha c_1^2 - \frac{1}{2}c_2^2 & \alpha c_1 & 0 & \alpha c_2 & & -c_1 & I_{n-2} \end{pmatrix} \\
&= v\left(\begin{pmatrix} 1 & & \\ c_1 & 1 & \\ & & I_{n-2} \end{pmatrix}\right) \begin{pmatrix} 1 & & & & & & \\ & 1 & & & & & \\ & & I_{n-2} & & & & \\ -\alpha c_2 & 0 & 0 & 1 & & & \\ 0 & 0 & 0 & 0 & 1 & & \\ -\alpha c_1 & 0 & 0 & 0 & & 1 & \\ -\frac{1}{2}c_2^2 & \alpha c_1 & 0 & \alpha c_2 & & & I_{n-2} \end{pmatrix},
\end{aligned}
$$

where $c_1, c_2 \in F$, $\alpha = \frac{1}{2}$. This implies that

$$
\begin{aligned}
&|W_{\rho_{\tau,s}}|(u'v(r(e))(wyw^{-1})(wn'w^{-1}), \mathrm{diag}(\lfloor b \rceil, I_{n-2}, m)) \\
=&|W_{\rho_{\tau,s}}|\left(u'v\left(\begin{pmatrix} 1 & -c_1 e & c_1 \\ 0 & I_{n-2} & 0 \\ 0 & 0 & 1 \end{pmatrix}\right) y'(wn'w^{-1})v(r(e)), \mathrm{diag}(\lfloor b \rceil, I_{n-2}, m)\right).
\end{aligned}
$$

Here

$$
y' = \begin{pmatrix} 1 & -c_1 e & c_1 & & & & \\ & I_{n-2} & & & & & \\ & & 1 & & & & \\ 0 & -\alpha c_2 e & \alpha c_2 & 1 & & & \\ \alpha c_1 & \frac{1}{2}c_2^2 e & -\frac{1}{2}c_2^2 & -\alpha c_2 & 1 & & -c_1 \\ -\alpha c_1 e' & e'\frac{1}{2}c_2^2 e & \frac{1}{2}c_2^2 e' & -\alpha c_2 e' & & I_{n-2} & c_1 e' \\ 0 & \alpha c_1 e & \alpha c_1 & 0 & & & 1 \end{pmatrix},
$$

where $e' = -J_{n-2}{}^t e$.

Since

$$
\begin{aligned}
&\begin{pmatrix} 1 & & & & & & \\ & I_{n-2} & & & & & \\ & & 1 & & & & \\ 0 & v_3' & 0 & 1 & & & \\ 0 & v_1' & 0 & 0 & 1 & & \\ v_2 & T+e'v_1'-ev_1 & v_1 & v_3 & & I_{n-2} & \\ 0 & v_2' & 0 & 0 & & & 1 \end{pmatrix} v\left(\begin{pmatrix} 1 & -c_1 e & c_1 \\ 0 & I_{n-2} & 0 \\ 0 & 0 & 1 \end{pmatrix}\right) \\
=&v\left(\begin{pmatrix} 1 & -c_1 e & c_1 \\ 0 & I_{n-2} & 0 \\ 0 & 0 & 1 \end{pmatrix}\right) \begin{pmatrix} 1 & & & & & & \\ & I_{n-2} & & & & & \\ & & 1 & & & & \\ 0 & v_3' & 0 & 1 & & & \\ 0 & v_1'-c_1 v_2' & 0 & 0 & 1 & & \\ v_2 & T+e'v_1'-ev_1-c_1 e v_2 & v_1+c_1 v_2 & v_3 & & I_{n-2} & \\ 0 & v_2' & 0 & 0 & & & 1 \end{pmatrix},
\end{aligned}
$$

we have

$$
\begin{aligned}
&|W_{\rho_{\tau,s}}|\left(u'v\left(\left(\begin{smallmatrix}1 & -c_1e & c_1\\ 0 & I_{n-2} & 0\\ 0 & 0 & 1\end{smallmatrix}\right)\right)y'(wn'w^{-1})v(r(e)),\mathrm{diag}(\lfloor b\rfloor, I_{n-2}, m)\right)\\
=&|W_{\rho_{\tau,s}}|\left(v\left(\left(\begin{smallmatrix}1 & -c_1e & c_1\\ 0 & I_{n-2} & 0\\ 0 & 0 & 1\end{smallmatrix}\right)\right)u''y'(wn'w^{-1})v(r(e)),\mathrm{diag}(\lfloor b\rfloor, I_{n-2}, m)\right)\\
=&|W_{\rho_{\tau,s}}|(u''y'(wn'w^{-1})v(r(e)),\mathrm{diag}(\lfloor b\rfloor, I_{n-2}, m)).
\end{aligned}
$$

Here

$$
u''=\begin{pmatrix}
1 & & & & & \\
 & I_{n-2} & & & & \\
 & & 1 & & & \\
0 & v_3' & 0 & 1 & & \\
0 & v_1'-c_1v_2' & 0 & 0 & 1 & \\
v_2 & T+e'v_1'-ev_1-c_1ev_2 & v_1+c_1v_2 & v_3 & & I_{n-2} \\
0 & v_2' & 0 & 0 & & & 1
\end{pmatrix}.
$$

Thus, using the following change of variables

$$
\begin{aligned}
&v_1\mapsto v_1-c_1v_2-\frac{1}{2}c_2^2e'\\
&v_2\mapsto v_2-\alpha c_1e\\
&v_3\mapsto v_3-\alpha c_2e'\\
&T\mapsto T-e'v_1'+ev_1-\frac{1}{2}c_2^2e'e,
\end{aligned}
$$

we obtain the integral I_{s,s_1,s_2} is bounded by

$$
\begin{aligned}
&\int_{F^\times}\int_{F^\times}|m|^{-s_1+s_2+\frac{d_2}{2}+\Re(s)+5}|b|^{-s_1}|\lfloor b\rfloor|^{\Re(s)-n+3}\\
&\times\int_{X_1(F)}\int_{M_{1\times(n-2)}(F)}|W_{\rho_{\tau,s}}|(x_1(wnw^{-1})v(r(e)),\mathrm{diag}(\lfloor b\rfloor, I_{n-2}, m))dedx_1d^\times md^\times b.
\end{aligned}
$$

Now we proceed to decompose $v(r(e))$.

We have

$$\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & I_{n-2} \\ 1 & 0 & e \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & I_{n-2} & 0 \\ 0 & e & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & I_{n-2} \\ 1 & 0 & 0 \end{pmatrix}.$$

Then we apply the Iwasawa decomposition with respect to the standard Borel subgroup of $\mathrm{GL}_n(F)$

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & I_{n-2} & 0 \\ 0 & e & 1 \end{pmatrix} = n_e t_e k_e.$$

Here $n_e = \mathrm{diag}(I_2, n'_e)$ where n'_e lies in the unipotent radical of the standard Borel subgroup of $\mathrm{GL}_{n-2}(F)$, $t_e = (t_1, \ldots, t_n)$, where $t_1 = t_2 = 1$, $k_e \in \mathrm{GL}_n(\mathcal{O})$.

By the structure of this decomposition, we have

$$[e] \leq |t_n| = |t_3 \cdots t_{n-1}|^{-1}.$$

since $\det(t_e) = 1$.

The integral I_{s_1,s_2} is majorized by

$$\int_{F^\times} \int_{F^\times} |m|^{-s_1+s_2+\frac{d_2}{2}+\Re(s)+5} |b|^{-s_1} |\lfloor b \rfloor|^{\Re(s)-n+3}$$
$$\times \int_{X_1(F)} \int_{M_{1\times(n-2)}(F)} |W_{\rho_{\tau,s}}|(x_1(wnw^{-1})v(n_e), \mathrm{diag}(\lfloor b \rfloor, I_{n-2}, m)t_e) de dx_1 d^\times m d^\times b.$$

By change of variables $x_1 \mapsto v(n_e)(x_1(wnw^{-1}))v(n_e)^{-1}$ the above integral is

$$\int_{F^\times}\int_{F^\times} |m|^{-s_1+s_2+\frac{d_2}{2}+\Re(s)+5}|b|^{-s_1}|\lfloor b\rfloor|^{\Re(s)-n+3}$$
$$\times \int_{X_1(F)}\int_{M_{1\times(n-2)}(F)} |W_{\rho_{\tau,s}}|(x_1(wnw^{-1}), \mathrm{diag}(\lfloor b^2\rfloor, I_{n-2}, m^2)t_e)dedx_1d^\times md^\times b.$$

Thus by similar arguments as in Lemma 5.2.1 the integral is majorized by a finite sum of integrals of the form

$$\int_{F^\times}\int_{F^\times} |m|^{-s_1+s_2+\frac{d_2}{2}+\Re(s)+5}|b|^{-s_1}|\lfloor b\rfloor|^{\Re(s)-n+3-2C}$$
$$\times \int_{X_1(F)}\int_{M_{1\times(n-2)}(F)} [x_1]^{-\Re(s)-\frac{n-1}{2}+C}\eta_j(\mathrm{diag}(\lfloor b^2\rfloor, I_{n-2}, m^2)t_e)dedudyd^\times md^\times b.$$

Here η_j is some positive quasi-character depends on τ and C is some positive integer which depends on τ.

Using the notation as in Lemma 5.2.1, we denote the Iwasawa decomposition of $x_1 \in X_1(F)$ as $x_1 = na'k$. Then we have that $\mathrm{diag}(\lfloor b\rfloor, I_{n-2}, m)at_e$ lies in the support of a gauge on $\mathrm{GL}_n(F)$, we have

$$\left|\frac{a_2}{a_3t_3}\right| \le 1, \quad \left|\frac{a_it_i}{a_{i+1}t_{i+1}}\right| \le 1, \quad \left|\frac{t_{n-1}}{mt_n}\right| \le 1,$$

where $i = 3, \dots, n-2$.

Thus, by similar arguments as in Lemma 5.2.1 we have

$$[e] \le |t_3\cdots t_{n-1}|^{-1} \le [x_1]^{C'}|\lfloor b\rfloor|^{-2C'},$$

where C' is some positive integer. By [Sou93, Proposition 11.15, Lemma 2] we have

$$\max\left\{\left|\frac{t_i}{t_{i+1}}\right|, \left|\frac{t_{i+1}}{t_1}\right|\right\} \leq [e]^{2n} \leq [x_1]^{2nC'}|\lfloor b\rfloor|^{-4nC'}.$$

Thus we have

$$|m| \geq [x_1]^{-C_0}|\lfloor b\rfloor|^{2C_0},$$

where C_0 is a positive integer depends on τ.

Then, the integral I'_{s_1,s_2} is majorized by a finite sum of integrals of the form

$$\int_{F^\times} |b|^{-s_1}|\lfloor b\rfloor|^{\Re(s)-n+3-c_2} \times \int_{X_1(F)}\int_{|m|\geq[x_1]^{-C_0}|\lfloor b\rfloor|^{2C_0}} |m|^{-s_1+s_2+\frac{d_2}{2}+\Re(s)+5+c_1}[x_1]^{-\Re(s)-\frac{n-1}{2}+c_3}dx_1d^\times md^\times b,$$

where $c_2, c_3 > 0$, c_1 are constants depend on τ.

Similar as in Lemma 5.2.3, the above integral converges absolutely when

$$\begin{aligned}
&-\Re(s_1)+\Re(s_2)+\frac{d_2}{2}+\Re(s)+5+c_1<0,\\
&-C_0(-\Re(s_1)+\Re(s_2)+\frac{d_2}{2}+\Re(s)+5+c_1)-\Re(s)-\frac{n-1}{2}+c_3<-C'',\\
&2C_0(-\Re(s_1)+\Re(s_2)+\frac{d_2}{2}+\Re(s)+5+c_1)+\Re(s)-n+3-c_2-\Re(s_1)>0,\\
&-2C_0(-\Re(s_1)+\Re(s_2)+\frac{d_2}{2}+\Re(s)+5+c_1)+\Re(s)-n+3-c_2-\Re(s_1)<0,
\end{aligned} \tag{5.2.4}$$

where C'' is a positive integer which depends on τ. Thus we deduce the lemma.

Then above region (5.2.4) is simplified to

$$\begin{aligned}
\Re(s) <&\Re(s_1) + \Re(-s_2) - \frac{d_1}{2} - 5 - c_1 \\
\Re(s) >&\max(\frac{C_0}{C_0+1}(\Re(s_1) + \Re(-s_2) - \frac{d_2}{2} - 5 - c_1) - \frac{n-1-C''}{C_0+1}, \\
&\Re(s_1) + \frac{2C_0}{2C_0+1}(\Re(-s_2) - \frac{d_2}{2} - 5 - c_1) - \frac{n-3+c_2}{2C_0+1}, \\
&\Re(s_1) + \frac{2C_0}{2C_0-1}(\Re(-s_2) - \frac{d_2}{2} - 5 - c_1) - \frac{n-3+c_2}{2C_0-1}).
\end{aligned} \tag{5.2.5}$$

For this to be non-empty we need

$$\begin{aligned}
&\Re(s_1) - \Re(s_2) - \frac{d_1}{2} - 5 - c_1 \\
&> \max(\frac{C_0}{C_0+1}(\Re(s_1) + \Re(-s_2) - \frac{d_2}{2} - 5 - c_1) - \frac{n-1-C''}{C_0+1}, \\
&\Re(s_1) + \frac{2C_0}{2C_0+1}(\Re(-s_2) - \frac{d_2}{2} - 5 - c_1) - \frac{n-3+c_2}{2C_0+1}, \\
&\Re(s_1) + \frac{2C_0}{2C_0-1}(\Re(-s_2) - \frac{d_2}{2} - 5 - c_1) - \frac{n-3+c_2}{2C_0-1}).
\end{aligned}$$

For the above to be valid we need

$$\begin{aligned}
&-n + \frac{d_2}{2} + 8 + c_1 < \Re(-s_2) < n + \frac{d_2}{2} + 2 + c_1 + c_2, \\
&c_0(-n + 3 - c_2) + C'' < \Re(s_1) + \Re(-s_2).
\end{aligned}$$

Since $n \geq 3$, $-n + \frac{d_2}{2} + 8 + c_1 < n + \frac{d_2}{2} + 2 + c_1 + c_2$, then the above inequalities are valid. Thus region (5.2.5) is non-empty. □

Lemma 5.2.5. *The infinite sum*

$$\sum_{k=\mathrm{val}(4\mathcal{Q}'(y_2)}^{\infty} \frac{c(q^{s_2}, y_2)\zeta_v(-s_2 - \frac{d_1}{2} + 2)^2\zeta_v(-s_2)^2 q^{(-s_1+s_2+n-2+\frac{d_1}{2})k}}{\gamma(s - s_1 + s_2, \chi' \otimes \tau)} B_{\psi_1', s_2}(\varpi_H^{\delta_{k-\mathrm{val}(4\mathcal{Q}'(y_2)}})$$

converges absolutely when

$$\Re(s_1) - \Re(s_2) - 2 - \frac{d_1}{2} \leq \Re(s) \leq \Re(s_1) - \Re(s_2) + n + 1 - \frac{d_1}{2}, \tag{5.2.6}$$

$$\Re(s_1) > C_1, \Re(-s_2) > C_2, \tag{5.2.7}$$

where C_1, C_2 are constants depends on (n, d_1, d_2).

Proof. By the formula of c_{s_2} and B_{ψ_1', s_1, s_2}, it suffices to show

$$\sum_{k=-\mathrm{val}(4\mathcal{Q}'(y_2)}^{\infty} q^{(-s_1+s_2-n-1+\frac{d_2}{2})k}\zeta_v(-s_2 - \frac{d_1}{2} + 2)^2\zeta_v(-s_2)^2\gamma(s - s_1 + s_2, \chi' \otimes \tau)^{-1}$$

$$\times \sum_{w \in W} {}^{w}\chi_s(\varpi^{\delta_k})^{-1}$$

converges absolutely in the region given by 5.2.6 and 5.2.7.

Note that $\zeta_v(-s_2 - \frac{d_1}{2} + 2)^2\zeta_v(-s_2)^2$ converges when

$$\Re(-s_2) > 0, \Re(-s_2 - \frac{d_1}{2} + 2) > 0,$$

and $\gamma(s - s_1 + s_2, \chi' \otimes \tau)^{-1}$ converges when

$$\Re(s - s_1 + s_2 + \frac{d_1 + 5}{2}) \geq \frac{1}{2}$$

which lies in the given region. It remains to show the sum

$$\sum_{k=0}^{\infty} q^{-(s_1 - s_2 + n + 1 - \frac{d_1}{2})k} \sum_{w \in W} {}^{w}\chi_s(\varpi^{\delta_k})^{-1}$$

converges absolutely in the region.

We have

$$|\chi_s(\varpi^{\delta_k})^{-1}| = q^{(s - \frac{1}{2})k} |\chi_1(\varpi^{-k})|.$$

By [JS81c, Corollary 2.5] we have

$$|\,{}^{w}\chi_1(\varpi^{-k})| < q^{\frac{k}{2}}$$

for any $w \in W$ and $k > 0$. Then it suffices to observe that

$$\sum_{k=0}^{\infty} q^{-(s_1 - s_2 + n + 1 - \frac{d_1}{2})k} q^{sk}$$

and

$$\sum_{k=0}^{\infty} q^{-(s_1 - s_2 + n + 1 - \frac{d_1}{2})k} q^{-(s-1)k}$$

converges in the given region because they are convergent geometric series. The convergence region is obviously non-empty since n is positive. □

Chapter 6

Conclusion

6.1 Review of the results

We give a brief summary of the results of this dissertation. Conjectures of Braverman-Kazhdan [BK00], L. Lafforgue [Laf14], Ngô [Ngô14, Ngô20] and Sakellaridis [Sak12] suggest that every affine spherical variety admits a generalized Poisson summation formula. See Chapter 1 for details of the Poisson Summation Conjecture.

In this dissertation we proved a generalized Poisson Summation Formula for pairs of quadratic spaces with an automorphic twist on GL_n. Specifically, we define the global Schwartz space $\mathcal{S}(Y'(\mathbb{A}_F), \tau_v, s)$ as the image of the integral $I(f, W_{\xi_s})$ (see Eq. (2.1.2)). We prove a summation formula for the spherical variety Y' using the intertwinning operator $M(\tau, s)$ (see the formalization in Theorem 2.1.1. This result both enlarges the collection of the proven cases (which is currently very small) and provides perspectives towards proving the analytic properties of higher rank triple product L-functions.

We also compute the local factors of the integrals $I(f, W_{\xi_s})$ at the unramified places, which can be considered as the basic function in the local Schwartz space $\mathcal{S}(Y'(F_v), \tau_v, s)$ for any unramified places v.

6.2 Future work

Theorem 2.1.1 has a different form than predicted by the papers mentioned at the beginning of Chapter 1. It seems reasonable to expect that there exists a spherical variety Z for $\mathrm{O}(V_1) \times \mathrm{O}(V_2) \times \mathrm{GL}_n$ equipped with a Schwartz space $\mathcal{S}(Z(\mathbb{A}_F))$ with

Fourier transform $\mathcal{F}$ and Poisson summation formula

$$\sum_{z\in Z(F)} f(z) = \sum_{z\in Z(F)} \mathcal{F}(f)(z),$$

such that

$$\int_{[\mathrm{GL}_n]} \xi_s(1,g') \sum_{z\in Z(F)} f(z(g,g'))dg' = \sum_{y\in \mathbb{P}Y'(F)} I(f, W_{\xi_s})(gy)$$

and

$$\int_{[\mathrm{GL}_n]} \xi_s(1,g') \sum_{z\in Z(F)} \mathcal{F}(f)(z(g,g'))dg' = \sum_{y\in \mathbb{P}Y'(F)} I(f, M(\tau,s)W_{\xi_s})(gy)$$

for $g \in \mathrm{O}(V_1)(\mathbb{A}_F) \times \mathrm{O}(V_2)(\mathbb{A}_F)$, $g' \in \mathrm{GL}_n(\mathbb{A}_F)$, and ξ_s be a smooth holomorphic section from the (normalized induction) space $\mathrm{Ind}_{Q_n(\mathbb{A}_F)}^{H(\mathbb{A}_F)}(\tau \otimes |\det|^{s-\frac{1}{2}})$.

In the future we plan to construct the spherical variety Z and prove a refined Poisson summation formula for Z.

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Biography

Miao (Pam) Gu graduated with a B.Sc. from the University of British Columbia in Spring 2018, where she majored in mathematics and minored in music. She has coauthored one research paper during her undergraduate studies.

In Fall 2018, Gu enrolled in the mathematics doctoral program at Duke University. Her main research interests lie in the area of automorphic forms and L-functions and related topics, and she is also interested in the area of enriched enumerative geometry. She was rewarded conference travel award, summer research fellowships, and graduate representative travel grant at Duke.

During her graduate studies, Gu has authored 1 research article, which is included in this dissertation, and coauthored 1 research article.

Gu has been invited to give talks at 8 international and national conferences and number theory seminars during her graduate studies, including conferences "Periods, functoriality and L-functions" at Centre International de Rencontres Mathématiques (CIRM) in France and "Junior Number Theory Days" at Johns Hopkins University.

Besides her passion in research, Gu is also enthusiastic as an educator. She participated in the DOMath program (currently known as Math+) at Duke as a mentor in summer 2020, during which she mentored 3 undergraduate student. She has served as TA and instructor at the University of British Columbia and Duke University.

Gu will continue her academic career as a postdoctoral assistant professor at University of Michigan.