Long-Time Behavior of Some ODEs with Partial Damping

Owen Huber
Advisor: Kyle Liss

A senior thesis presented for
Graduation with Distinction

Department of Mathematics
Duke University
Durham, North Carolina

April 23, 2024

Abstract

This thesis examines some partially damped ODEs with a conservative bilinear term, a damping matrix term with a nontrivial kernel, and a deterministic forcing term. We prove that, when forcing is absent, the condition that the bilinear term has no invariant sets in the kernel of the damping term is sufficient to show convergence of all solutions to the origin. We then consider the case that invariant sets exist in the kernel of the damping term and include forcing to escape the invariant sets. We show that solutions diverge under certain symmetries and give a partial proof of boundedness with hyperbolic equilibria in the kernel of the damping term.

Contents

1 Introduction

In this thesis, we consider the problem of nonlinear energy transfer in some ordinary differential equations (ODEs). Specifically, we consider the long-time behavior of the following ODE on $\mathbb{R}^n$:

$$\frac{\mathrm{d}\mathbf{x}}{\mathrm{d}t} = B(\mathbf{x},\mathbf{x}) - A\mathbf{x} + f \tag{1}$$

for which $B : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ is a bilinear conservative term such that $\langle B(\mathbf{x},\mathbf{x}), \mathbf{x}\rangle = 0$ for all $\mathbf{x} \in \mathbb{R}^n$, meaning that solutions to $\dot{\mathbf{x}} = B(\mathbf{x},\mathbf{x})$ remain a fixed distance from the origin for all time, $A \in \mathbb{R}^{n\times n}$ is a damping term which is a symmetric positive semidefinite matrix with a nontrivial kernel, and $f \in \mathbb{R}^n$ is a deterministic forcing term.

The transfer of energy for a conserved quantity through varying degrees of freedom arises in many physical phenomena. Although a linear example, in the advection-diffusion equation with an incompressible drift, the effect of transport is to transfer the energy from low Fourier frequencies to high Fourier frequencies [7]. This can interact with diffusion to greatly enhance the dissipation rate of energy. For three dimensional turbulent flows, nonlinear effects result in energy cascade from large scales to small scales, where the dissipation rate is significantly lower than at larger scales [5].

When studying turbulent flows in certain settings, the Navier-Stokes equation has a Fourier series representation that satisfies a system of SDEs [4]. When this is truncated into a finite dimensional SDE, the laplacian can be translated into matrix damping, and since the laplacian has a factor of k^2 in front of the Fourier series representation for a frequency k, the damping can be ignored for lower frequencies. This translates to a nontrivial null-space in the damping term which offers some freedom from energy transfer. In this translation the nonlinear term in the Navier-Stokes equation becomes our bilinear B term.

This concept was explored in the stochastic setting on the $2D$-torus by Brendan Williamson in [6], and explored for the Lorenz 63 model in [3]. Equations of a similar form to (1) with stochastic forcing were also studied in [1]. We choose to study this transfer of energy in the deterministic setting. Partial damping in the deterministic setting has been studied in the $2D$ incompressible Euler equations in [2].

Motivation for studying the transfer of energy in our deterministic setting lies in the prospect of transferring methods used in this case to understanding behavior in more complicated physical phenomena, for instance turbulent flows. In this thesis we measure "energy" as the distance from the solution to the origin, and study the long-time behavior of this quantity for different systems of the form (1).

A solution experiences no damping when it is in the kernel of A, and when forcing is left out of (1) and the solution is in the kernel of A, the solution remains at a constant energy. We refer to a solution to (1) at time t with initial position of x_0 as $\phi_t(x_0)$, and when forcing is absent, it can be calculated that if $\phi_t(x_0) \in \ker A$ at some time t, then

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}|\phi_t(x_0)|^2 &= 2\langle \phi_t(x_0),\ \frac{\mathrm{d}}{\mathrm{d}t}\phi_t(x_0)\rangle \\ &= 2\langle \phi_t(x_0),\ B(\phi_t(x_0), \phi_t(x_0)) - A\phi_t(x_0)\rangle \\ &= 2\langle \phi_t(x_0),\ B(\phi_t(x_0), \phi_t(x_0))\rangle \\ &= 0\end{aligned}$$

The question arises whether the damping that A provides necessitates that all trajectories must converge to the origin. A counter example to this is immediately available: if there exist invariant solutions to the ODE

$$\frac{\mathrm{d}\mathbf{x}}{\mathrm{d}t} = B(\mathbf{x}, \mathbf{x}) \tag{2}$$

inside the kernel of A, then there exists some initial conditions to the unforced version of (1) for which the conservative term keeps the solution in the kernel of A for all time, and so the solution remains at a constant energy and does not converge to zero. The question then becomes whether the absence of such invariant sets necessitates that all solutions converge to the origin, and we formulate the following theorem:

Theorem 1 *All solutions to*

$$\frac{\mathrm{d}\mathbf{x}}{\mathrm{d}t} = B(\mathbf{x}, \mathbf{x}) - A\mathbf{x}$$

converge to the origin if and only if no solutions to (2) are invariant in the kernel of A. Again, B is a conservative bilinear term and A is a symmetric positive semidefinite matrix with a nontrivial kernel.

We will prove this theorem in section 2.2.

In physical systems, often it is not the case that there are no invariant sets of (2) in the kernel of A, so accordingly we adjust our studying of energy transfer. It is a natural next step to add forcing as many nonlinear system with energy transfer are kept in motion by forcing, and if solutions to (2) are invariant in the kernel of A, the forcing can move the solution out of the kernel. With deterministic forcing present the origin is no longer a fixed point, so instead of proving convergence to the origin we seek to prove boundedness of solutions.

A powerful tool used in proving related theorems is the scaling of (1). If $\mathbf{x}(t)$ is a solution to (1) and we use the notation $\mathbf{x}(0) = \mathbf{x}_0$, then

$$\widetilde{\mathbf{x}}(t) = \frac{\mathbf{x}(t)}{|\mathbf{x}_0|}$$

is a solution to the ODE

$$\frac{\mathrm{d}\widetilde{\mathbf{x}}}{\mathrm{d}t} = |\mathbf{x}_0| B(\widetilde{\mathbf{x}}, \widetilde{\mathbf{x}}) - A\widetilde{\mathbf{x}} + \frac{1}{|\mathbf{x}_0|} f$$

This tells us that we should expect the B term to dominate at high energies, and that, as we will apply in proofs of boundedness with forcing present, when we project solutions at high energies onto the unit sphere, forcing and damping have a minor impact on the solution near the kernel of A compared to the conservative term.

We will first prove the convergence of solutions for an unforced example in $\mathbb{R}^2$, then we will prove the general case of theorem 1.

For the forced case we will define hyperbolic and elliptic fixed points in the kernel of A, show that some solutions of (1) with hyperbolic equilibria in the kernel of A diverge under certain symmetries, then we will sketch a proof that the solutions to an example of the form (1) with hyperbolic equilibria in the kernel of A will remain bounded. We will conclude with a brief discussion of the relationship between fixed points outside of the kernel of A and boundedness for solutions of ODEs with elliptic equilibria.

2 Unforced Case

In this section, we will prove that solutions to

$$\frac{\mathrm{d}\mathbf{x}}{\mathrm{d}t} = F(\mathbf{x}) = B(\mathbf{x},\mathbf{x}) - A\mathbf{x}$$

with the same conditions on B and A as in (1) and the added condition that there are no solutions to (2) that are invariant in the kernel of A must converge to the origin. We will begin with an example in $\mathbb{R}^2$ and then prove the general case.

2.1 Planar Convergence

Theorem 2 *All solutions to the ODE*

$$F(x,y) = \begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} -xy \\ x^2 - y \end{pmatrix} \tag{3}$$

converge to zero.

We notice that this ODE follows the form of (4) with

$$B(\mathbf{x},\mathbf{x}) = \begin{pmatrix} -xy \\ x^2 \end{pmatrix}, \quad A = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

Also, the line $x = 0$ forms a nullcline with $\dot{x} = 0$, so each trajectory will stay on whichever side of the y-axis that it begins on. We also have the symmetry

$$F(-x,y) = \begin{pmatrix} -\dot{x} \\ \dot{y} \end{pmatrix}$$

which, when paired with the fact that we have a nullcline at $x = 0$, tells us that the solutions $(x(t), y(t))$ and $(\widetilde{x}(t), \widetilde{y}(t))$ to (3) with initial conditions $(x_0, y_0) = (x(0), y(0))$ and $(\widetilde{x_0}, \widetilde{y_0}) = (-x(0), y(0))$ respectively satisfy $y(t) = \widetilde{y}(t)$ and $x(t) = -\widetilde{x}(t)$ due to the symmetry across the y-axis. So, if we prove that all initial conditions with $x_0 \geq 0$ converge to the origin, we will have proven our claim that all initial conditions in $\mathbb{R}^2$ converge to the origin.

Also, we can show that the line $x = 0$ is a stable subset for the origin. As it is a nullcline with $\dot{x} = 0$, every solution on the subset will remain on the subset, and as $\dot{y} = -y$ is a sink, we can conclude that every solution on the subset will converge to the origin. So to prove our claim we need only consider initial conditions with $x_0 > 0$.

We will complete this proof by partitioning $\{(x,y) \in \mathbb{R}^2 \,|\, x > 0\}$ into three sets defined as

$$\begin{aligned}
A &= \{(x,y) \in \mathbb{R}^2 \,|\, x > 0 \text{ and } y \geq x^2\} \\
B &= \{(x,y) \in \mathbb{R}^2 \,|\, x > 0 \text{ and } y \geq 0 \text{ and } y < x^2\} \\
C &= \{(x,y) \in \mathbb{R}^2 \,|\, x > 0 \text{ and } y < 0\}
\end{aligned}$$

Then we will prove our claim in three parts: First, we will prove A is a positively invariant set, and that all trajectories in A will converge to the origin. Then we will prove that all trajectories in B will enter

A in finite time and at some $x < \infty$ and $y < \infty$, and finally we will conclude the proof by showing that all trajectories in C will enter B in finite time and at some $x < \infty$.

i. *A is a positively invariant set, and all trajectories in A will converge to the origin.*

First, we can quickly show that every solution on the line $y = x^2$ will immediately enter int(A). On this line, $\dot{y} = 0$ and $\dot{x} = -xy = -x^3 < 0$, so every solution on the line $y = x^2$, except at the origin, must leave this line immediately, and we just must prove that solutions in the interior of A must converge to the origin.

For a solution with any initial condition y_0, we can form a trapping region with the area enclosed by the lines $y = y_0$, $y = x^2$, and $x = 0$. We have already show that the line $x = 0$ is a nullcline with $\dot{x} = 0$ and on the line $y = x^2$ that $\dot{y} = 0$ and $\dot{x} < 0$, so it just remains to show that on our line $y = y_0$ that $\dot{y} \leq 0$. Since in A we have the inequality $y_0 \geq x_0^2 \geq 0$, we can conclude that $y_0 \leq 0$ and A must be an invariant set.

Next we will show that all solutions with initial conditions $(x_0, y_0) \in$ int(A) must converge to the origin. Since in int(A) we have the inequality $\dot{x} < 0$ and $\dot{y} < 0$, the solutions $x(t)$ and $y(t)$ are monotone decreasing, and are bounded by the trapping region, so by the monotone convergence theorem we have that

$$(x(t), y(t)) \to (x^*, y^*) \in A$$

We will prove by contradiction that (x^*, y^*) must equal $(0, 0)$.

By the monotone convergence theorem, for all $\epsilon > 0$, there exists some $T > 0$ such that for all $t > T$,

$$x(t) - x^* < \epsilon$$

But we also note that $\dot{x} < -x^* y^*$, which tells us that for all t our solution is bounded by $x(t) < -x^* y^* t + x(T)$. Then for $t = \epsilon/(x^* y^*)$, it must be that $|(x(t), y(t))| < |(x^*, y^*)|$. This is a contradiction, and concludes the proof that all trajectories in A will converge to the origin.

ii. *All trajectories in B will enter A in finite time and at some $x < \infty$ and $y < \infty$*

We follow the same method as above, and consider the solution to (3) that passes through some (x_0, y_0) in B. First, we note that solutions such that $y_0 = 0$ have the properties $\dot{x} = 0$ and $\dot{y} = x^2 > 0$ besides the origin, so the solutions must enter int(B) immediately, and we only need consider solutions with $(x_0, y_0) \in$ int(B) in this section.

In int(B), $\dot{x} < 0$ and $\dot{y} > 0$. Since $t_2 > t_1$ implies that $x(t_2) < x(t_1)$, it must be that for all t, the solution $y(t) < x_0^2$ in int(B). If this was not the case, and that $y(t) \geq x_0^2$ for some t, then either $(x(t), y(t)) \in B$ and $x(t) \geq x_0$, which is a contradiction, or $(x(t), y(t)) \in A$. So, if our solution enters A it must do so at some $y < x_0^2 < \infty$.

Then it follows that in B, our solution $y_0 \leq y(t) \leq x_0^2$ for all t. Then it must also be true that $x(t) > \sqrt{y_0}$ for all t while in B because $x(t)$ is monotone decreasing and $y(t) \geq y_0$, so if $x(t) \leq \sqrt{y_0}$, then $x(t)^2 > y(t)$ and the solution must be in A. So we can bound $\dot{x}$ as $\dot{x} < -y_0^{3/2}$, and since x_0 must travel at most the distance of $x_0 - \sqrt{y_0}$ in the x direction to reach A, the solution with initial conditions (x_0, y_0) will cross into A for some $t < t^*$ where $t^* = \frac{x_0 - \sqrt{y_0}}{y_0^{3/2}}$ and at some point (x^*, y^*) where $y_0 < y^* < x_0^2$ and $x^* < x_0$.

iii. *All trajectories in C will enter B in finite time and at some $x < \infty$*

Finally, let the trajectory with coordinates (x_0, y_0) be in C. First, we notice that since $y < 0$ for all y in C, $\dot{y} > x(t)^2$, and since $\dot{x} > 0$ in C, we have that $\dot{y} > x_0^2$ for the given solution. This also gives us the bound for the solution $y(t)$ with $y(0) = y_0$ of $y(t) > y_0 + x_0^2 t$, so the trajectory will enter B in some $t < t^*$ with $t^* = -y_0/x_0^2$. And since the x component of velocity is continuous, we know that it will not blow up in finite time, and thus all trajectories in C will enter B in finite time at some finite position on the x-axis.

Since every solution with $x_0 > 0$ must enter the invariant set A in finite time, and every solution in A must converge to the origin, we conclude that all solutions to (3) must converge to the origin.

□

This concludes the proof, and next we aim to prove convergence for the general case.

2.2 General Case

Theorem 1 *All solutions to*

$$\frac{\mathrm{d}x}{\mathrm{d}t} = F(x) = B(x,x) - Ax \tag{4}$$

for which $x \in \mathbb{R}^n$, $B : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ is a bilinear conservative term, and $A \in \mathbb{R}^{n \times n}$ is a symmetric positive semidefinite matrix with a nontrivial kernel converge to the origin if and only if no solutions to

$$\frac{\mathrm{d}x}{\mathrm{d}t} = B(x,x) \tag{5}$$

are invariant in the kernel of A.

First we will show that (5) having no solutions invariant in the kernel of A necessitates convergence of (4). We will write solutions to (4) as the flow $\phi_t(x_0) = x(t)$ with $x(0) = x_0$, and decompose our solutions into its projections onto the kernel of A and onto the perpendicular space to the kernel of A. We write

$$\phi_t(x_0) = x(t) = \pi_{\ker A} x(t) + \pi_{\ker A^\perp} x(t)$$

and simplify notation by calling $u(t)$ the projection of $\phi_t(x_0)$ onto $\ker A^\perp$ and $v(t)$ the projection of $\phi_t(x_0)$ onto $\ker A$.

We can calculate the change in energy $\frac{\mathrm{d}}{\mathrm{d}t}|x(t)|^2$ of the system in the following manner

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}|x(t)|^2 &= 2\langle x(t), F(x(t))\rangle \\ &= 2\langle x(t), B(x(t), x(t)) - Ax(t)\rangle \\ &= -2\langle Ax(t), x(t)\rangle\end{aligned}$$

Since A is positive semi-definite, $\langle Ax(t), x(t)\rangle$ must be non-negative, thus the derivative of the "energy" is non-positive. This tells us, by the monotone convergence theorem, that $|x(t)|^2$ must converge to some E^2. Our theorem is proven once we can show that $E^2 = 0$, and we will do so by showing the contradiction that arises when it is assumed that $E^2 > 0$.

Our first claim is that for all initial conditions x_0, $\lim_{t\to\infty} u(t) = \mathbf{0}$. We do so by examining the derivative

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}|x(t)|^2 &= -2\langle Ax(t), x(t)\rangle \\ &= -2\langle Au(t) + Av(t), u(t) + v(t)\rangle \\ &= -2\langle Au(t), u(t)\rangle - 2\langle Au(t), v(t)\rangle \\ &= -2\langle Au(t), u(t)\rangle - 2\langle u(t), A^T v(t)\rangle\end{aligned}$$

Since A is symmetric, $A = A^T$ and thus $A^T v(t)$ vanishes. To continue this proof we use the following lemma:

Lemma 1 *Denote $\{\lambda_i\}$ as the eigenvalues of a symmetric positive semi-definite matrix $A \in \mathbb{R}^{n\times n}$. For any $x \in \mathbb{R}^n$,*

$$\langle x, Ax\rangle \geq \lambda_{min}|x|^2$$

with $\lambda_{min} = \min\{\lambda_i \,|\, \lambda_i > 0\}$.

Proof : Since A is symmetric, its eigenvectors form an orthogonal basis $\{v_i\}$ that correspond to the eigenvalues $\{\lambda_i\}$ such that $Av_i = \lambda_i v_i$. We can then decompose x as

$$x = \sum_{i=1}^{n} c_i v_i$$

for some scalars c_i. Also, we note that $\langle v, Au\rangle = v^T Au = u^T Av$ for all u, v. Now we write

$$\begin{aligned}\langle x, Ax\rangle &= \left(\sum_{i=1}^{n} c_i v_i^T\right) A \left(\sum_{j=1}^{n} c_j v_j\right)\\ &= \sum_{i=1}^{n}\sum_{j=1}^{n} c_i c_j \lambda_j \langle v_i, v_j\rangle\end{aligned}$$

Since $\{v_i\}$ is an orthogonal basis, for all $i \neq j$, we have that $\langle v_i, v_j\rangle = 0$, so our sum becomes

$$\begin{aligned}\langle x, Ax\rangle &= \sum_{j=1}^{n} c_j^2 \lambda_j |v_j|^2\\ &\geq \lambda_{min}|x|^2\end{aligned}$$

concluding the proof of this lemma

□

Now, let $\lambda_{\min}$ be the smallest non-zero eigenvalue of A. By the previous lemma and calculations we can write

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}|x(t)|^2 &= -2\langle Au(t), u(t)\rangle\\ &\leq -2\lambda_{\min}|u(t)|^2\end{aligned}$$

Then for any t_1, t_2 such that $t_2 > t_1$, we have

$$\begin{aligned}|x(t_2)|^2 - |x(t_1)|^2 &= \int_{t_1}^{t_2} \frac{\mathrm{d}}{\mathrm{d}t}|x(t)|^2 dt\\ &\leq -2\lambda_{\min}\int_{t_1}^{t_2} |u(t)|^2 dt\end{aligned}$$

Lemma 2 *If a function $f : \mathbb{R} \to \mathbb{R}$ is uniformly continuous, $f(t) \geq 0$ for all $t \in \mathbb{R}$, and*

$$\lim_{T\to\infty}\int_0^T f(t)\, dt < \infty$$

then $\lim\limits_{t\to\infty} f(t) = 0$

Proof : Suppose that $\lim_{t\to\infty} f(t) \neq 0$. Then for some $\alpha > 0$, there exists a subsequence $\{t_n\}$ such that $\lim_{n\to\infty} t_n = \infty$, $t_n > t_{n-1} + 1$, and $f(t_n) > \alpha$. Since our function is uniformly continuous, there exists some $0 < \delta < 1$ such that

$$|f(t) - f(s)| < \frac{\alpha}{2} \qquad \text{when} \qquad |t - s| < \delta$$

Then for $t \in [t_n - \delta/2, t_n + \delta/2]$,

$$f(t) > f(t_n) - \alpha/2 > \alpha/2$$

and so

$$\int_{t_n-\alpha/2}^{t_n+\alpha/2} f(t)\,dt > \frac{\alpha\delta}{2}$$

and,

$$\int_0^{t_n} f(t)\,dt > \sum_{i=0}^{n} \frac{\alpha\delta}{2}$$

the sum diverges to positive infinity as n goes to infinity, offering a contradiction and concluding the proof of this lemma.

□

We note that the change in the energy of $x(t)$ must be finite if it converges to some value greater than zero, so then the integral of $|u(t)|^2$ must be finite as well. The energy of u is globally Lipschitz as it is continuous and $\frac{\mathrm{d}}{\mathrm{dt}}|u(t)|^2$ is bounded, so we can conclude that

$$\lim_{t\to\infty} u(t) = \mathbf{0}$$

Recall that we use the notation $x(t) = \phi_t(x_0)$ where $x_0 = x(t)|_{t=0}$ and define $K = \{x \in \mathbb{R}^n \mid E^2 \leq |x|^2 \leq E^2 + 1\}$. We can assume that x_0 and $\phi_t(x_0) \in K$ without loss of generality, so all trajectories must enter K in finite time and K must be an invariant set.

Lemma 3 *There exists some $0 < T < \infty$ and $c > 0$ such that*

$$\sup_{0\leq t\leq T} |\pi_{\ker A^\perp} F(\phi_t(x_0))| \geq c \qquad \textit{for } \forall x_0 \in K$$

Proof : Fix some $x_0 \in K$, and let $D(x_0) = \{t \,:\, |\pi_{\ker A^\perp} F(\phi_t(x_0))| > 0\}$. First, we will prove that $D(x_0)$ is non-empty by considering two possible cases. We note that in order for $|\pi_{\ker A^\perp} F(\phi_t(x_0))|$ to be zero for all time, the solution $\phi_t(x_0)$ must either stay exclusively in the kernel of A or exclusively outside the kernel of A for all time, as it cannot have a component of velocity that is perpendicular to the kernel of A at any time. We prove that each case of $D(x_0)$ being empty is impossible by offering two contradictions.

First, suppose that $\phi_t(x_0) \in \ker A$ for all time. Then, $A\phi_t(x_0) = 0$ for all time, but B has no invariant sets in the kernel of A, so there exists some t^* so that $B(\phi_{t^*}(x_0), \phi_{t^*}(x_0)) \notin \ker A$, so then $|\pi_{\ker A^\perp} F(\phi_{t^*}(x_0))| = |\pi_{\ker A^\perp} B(\phi_{t^*}(x_0), \phi_{t^*}(x_0))| > 0$ at some time t^*.

Now suppose that $\phi_t(x_0) \notin \ker A$. Then $\pi_{\ker A^\perp}\phi_t(x_0) \neq \mathbf{0}$ and is constant for all t as there is no change of the solution with respect to the perpendicular space to the kernel of A. Then, there exists some $\beta > 0$ so

$$\frac{\mathrm{d}}{\mathrm{d}t}|\phi_t(x_0)|^2 \leq -2\lambda_{min}|\pi_{\ker A^\perp}\phi_t(x_0)|^2 \leq -\beta$$

which contradicts that the energy of x must remain non-negative, thus we conclude that $D(x_0)$ is nonempty. We define

$$T(x_0) = \inf D(x_0)$$

T is a continuous function of x_0, so it follows from the extreme value theorem that $T < \infty$. Then, let

$$T = 1 + \sup_{x_0\in K} T(x_0)$$

Now we define

$$M(x_0) = \sup_{t<T} ||\pi_{\ker A^\perp} F(\phi_t(x_0))|| > 0$$

Since $\pi_{\ker A^\perp} F(\phi_t(x_0))$ is continuous with respect to x_0, we see that $M(x_0)$ must also be continuous with respect to x_0, and since it is defined on a compact domain, $M(x_0)$ must reach its infimum on the domain. This tells us that there exists a positive minimum of $|\pi_{\ker A^\perp} F(\phi_t(x_0))|$ on the domain, which is c, and concludes the proof of this lemma.

□

Now, we define $c > 0$ as the lower bound of the supremum of $|\pi_{\ker A^\perp} F(\phi_t(x_0))|$ for every $x_0 \in K$ on the time interval $(0, T)$ with T defined as in Lemma 3. We know that $|\pi_{\ker A^\perp} F(\phi_t(x_0))|$ is uniformly continuous, so there exists a $\delta > 0$ such that for each t' that satisfies $|\pi_{\ker A^\perp} F(\phi_{t'}(x_0))| > c$, the interval $I = [t' - \delta, t']$ is such that for all $t \in I$,

$$|\pi_{\ker A^\perp} F(\phi_{t'}(x_0)) - \pi_{\ker A^\perp} F(\phi_t(x_0))| \leq \frac{c}{2} \tag{6}$$

By the previous lemma, we may also choose a $T_1 > 0$ so that for every $x_0 \in K$ and all $t > T_1$,

$$|\pi_{\ker A^\perp} \phi_t(x_0)| < \frac{c\delta}{4} \tag{7}$$

for which δ is the same value defined in the formulation of (6). Again using the T defined in (6) and lemma 2, letting $T_2 = \max\{T, T_1\} + 1$, we see that for any $x_0 \in K$, there exists some $t_* < T_2$ such that both (6) and (7) are satisfied. Now, we can finish the proof with the following lemma.

Lemma 4 *If there exists some $v_* \in \mathbb{R}^n$ and some α such that $|v_*| > \alpha > 0$, and the continuous function $v : \mathbb{R} \to \mathbb{R}^n$ defined on the interval $I = [0, T]$ satisfies $|v(t) - v_*| \leq \alpha/2$ for all $t \in I$, then*

$$\left| \int_I v(t) dt \right| > \frac{T\alpha}{2}$$

Proof : Let

$$y(t) = \int_0^t v(s) ds$$

and $y'(t) = v(t)$. We note that

$$|y(t)| \geq |\text{proj}_{v_*} y(t)|$$

when we use the notation that $\text{proj}_u w$ is the projection of the vector w onto the vector u. Now we can calculate

$$\begin{aligned} |\text{proj}_{v_*} y(T)| &= \left| \text{proj}_{v_*} \int_I y'(s)\, ds \right| \\ &= \left| \int_I \text{proj}_{v_*} v(s)\, ds \right| \end{aligned}$$

Next, we notice that there exists some function $b \in C^1(\mathbb{R})$ such that $b(s) \geq 1/2$ for all $s \in \mathbb{R}$, and

$$b(s) v_* = \text{proj}_{v_*} v(s)$$

This function b exists because $v(t) \in B_r(v_*)$ where B is a ball of radius r around v_* with $r = \frac{|v_*|}{2}$, as indicated by the fact that $|v(t) - v_*| \leq \alpha/2 \leq |v_*|/2$. Then it follows that

$$\begin{aligned} |\text{proj}_{v_*} y(T)| &= \left| \int_I b(s) v_*\, ds \right| \\ &\geq \frac{1}{2} \left| \int_I v_* ds \right| \\ &\geq \frac{T\alpha}{2} \end{aligned}$$

Putting this all together, $|y(T)| \geq |\text{proj}_{v_*} y(T)| \geq \frac{T\alpha}{2}$ and we have concluded our proof of lemma 4. □

Using lemma 4 and combining our figures from (6), we reach the result,

$$\begin{aligned} |u(t_*) - u(t_* - \delta)| &= \left| \int_{t_*-\delta}^{t_*} u'(t)dt \right| \\ &= \left| \int_{t_*-\delta}^{t_*} \pi_{\ker A^\perp} F(\phi_t(x_0))dt \right| \\ &> \frac{\delta c}{2} \end{aligned}$$

However, we specified in (7) that for all $x_0 \in K$ and all $t > T_2$, that $|u(t)| < \frac{c\delta}{4}$. Since $t_* - \delta > T_2$ and $t_* > T_2$, it is a contradiction that the energy of the change in u

$$|u(t_*) - u(t_* - \delta)| > \frac{c\delta}{2} > 2 \max_{t>T_2} |u(t)|$$

Now it only remains to prove the other direction of the if and only if statement.

To prove that all solutions to (4) converging to the origin necessitates no solutions to (5) being invariant in the kernel of A, we prove the contrapositive as follows. Suppose that $\phi_t(x_0)$ is a solution to (5) that is invariant in the kernel of A, $x_0 \in \ker A$, and $|x_0| > 0$. Then for all time,

$$\frac{\mathrm{d}|\phi_t(x_0)|^2}{\mathrm{d}t} = -2\langle \phi_t(x_0), A\phi_t(x_0) \rangle = 0$$

Therefore $|\phi_t(x_0)| = |x_0| > 0$ for all time, and the solution does not converge to the origin. This concludes the proof. □

3 Forced Case

Now, ignoring the condition that there are no invariant solutions to (2) inside the kernel of A, to observe the long-time behavior of this system we add deterministic forcing. Otherwise, it would be trivially true that not all solutions converge to the origin, namely those that have a B term invariant in the kernel of A. As mentioned in the introduction, the analog of convergence in this case is boundedness. Though we would like to study this behavior in $\mathbb{R}^n$, at this point we have only considered the case of $\mathbb{R}^3$. To further simplify the problem, we suppose that $A \in \mathbb{R}^{3\times3}$ is a diagonal matrix of the form

$$A = \begin{pmatrix} d_1 & 0 & 0 \\ 0 & d_2 & 0 \\ 0 & 0 & d_3 \end{pmatrix}$$

with $d_i \geq 0$ and the kernel

$$\ker A = \bigoplus_{\{j|d_j=0\}} \mathbf{e}_j$$

Note that if $\dim(\ker(A)) = 2$ in $\mathbb{R}^3$, then there is only damping on one axis and the kernel of A is a plane. This offers more degrees of freedom in the kernel of A than if the kernel of A would have only dimension one. We recognize that boundedness relates to whether the solution can escape to higher energies inside the kernel of A, so if a system with a kernel of dimension one is unbounded, we imagine the same behavior would take place in an identical system with a kernel of higher dimension and more degrees of freedom. So, before considering all possible damping matrices we seek to prove boundedness for $\dim(\ker(A)) = 1$.

Next we define some terminology used throughout this section. We refer to equilibria as the fixed points of (1) ignoring the forcing, or solutions to

$$B(\mathbf{x},\mathbf{x}) - A\mathbf{x} = 0$$

If we let $F(\mathbf{x}) = B(\mathbf{x},\mathbf{x}) - A\mathbf{x}$, we can define the linearization $DF(\mathbf{x}^*) : \mathbb{R}^3 \to \mathbb{R}^{3\times 3}$ for equilibria $\mathbf{x}^* \in \mathbb{R}^3$, and then define hyperbolic equilibria as $\mathbf{x}^*$ such that the eigenvalues of $DF(\mathbf{x}^*)$ are all real, and one is positive, one is negative, and one is zero.

An example of an ODE with hyperbolic equilibria that we will use is

$$\frac{d}{dt}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} ayz \\ bxz \\ -(a+b)xy \end{pmatrix} - \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} + \begin{pmatrix} f_1 \\ f_2 \\ f_3 \end{pmatrix} \tag{8}$$

with $a, b > 0$. Note that our equilibria in the kernel of A are the line $x = y = 0$. Also, equilibria in the kernel of A are the simplest examples of when B is invariant in the kernel of A. When we consider any cross-section of the xy-plane for some $z = z_0 > 0$, we are left with the system

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} -1 & az_0 \\ bz_0 & -1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$$

which is a saddle and has the eigenvalues $\lambda = -1 \pm \sqrt{ab}z_0$, one of which will be positive and the other negative for the z_0 at high energies. If we were to find the eigenvalues of the linearization in $\mathbb{R}^3$ we would find that we would have an additional eigenvalue of zero. We choose to use the example of the linearization at a cross-section as it illuminates how the ODE behaves near the kernel of A.

An interesting property of (8) is that for any set of parameters, it has exactly two fixed points of the whole system outside of the kernel of A. The elliptic case does not have this property, and except for cases in which certain symmetries exist, numerics suggest that all solutions to (8) converge to one of the two fixed points of the system. We can solve for fixed points and find that they exist when

$$y = \frac{-f_3}{x(a+b)}, \qquad z = \frac{-f_3 d_2}{bx^2(a+b)} - \frac{f_2}{bx}$$

and x is defined by the roots of

$$\frac{f_3^2 a d_2}{b(a+b)^2} + \frac{f_3 f_2 x}{b(a+b)} + f_1 x^3 - d_1 x^4 = 0 \tag{9}$$

We notice that for every x value given by the roots of (9), there is exactly one y value and exactly one z value such that (x, y, z) a fixed point. So there are exactly as many fixed points as there are roots to (9). Now, since $-d_1 < 0$ and $(f_3^2 a d_2)b^{-1}(a+b)^{-2} > 0$, (9) must have at least two roots, and thus at least two fixed points in (8).

Now we define elliptic equilibrium as those who have complex eigenvalues for the linearization. Examples of ODEs with elliptic equilibria in the kernel of A that we will use will be of the form

$$\frac{d}{dt}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} ayz \\ bxz \\ -(a+b)xy \end{pmatrix} - \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} + \begin{pmatrix} f_1 \\ f_2 \\ f_3 \end{pmatrix} \tag{10}$$

again with $a, b > 0$. Note that the only difference between this equation and the hyperbolic is what axis are damped and accordingly what axis the kernel of A is on. Instead of hyperbolic equilibria in the kernel of A, we now have elliptic equilibria on the kernel of A. The equilibria in the kernel of A are when $x = z = 0$, and when we consider any cross-section of the xz-plane for some $y = y_0 > 0$, we are left with the system

$$\begin{pmatrix} \dot{x} \\ \dot{z} \end{pmatrix} = \begin{pmatrix} -1 & ay_0 \\ -y_0(a+b) & -1 \end{pmatrix}\begin{pmatrix} x \\ z \end{pmatrix}$$

The eigenvalues will be $\lambda = -1 \pm iy_0\sqrt{a(a+b)}$. We notice that for $|y_0| >> 1$ the imaginary term dominates.

3.1 Diverging Conditions for Hyperbolic Equilibria

We expect that a system will remain bounded when there exist only hyperbolic equilibria in the kernel of A, but this is not the case when certain symmetries are present. To prove that there are initial conditions for which this system diverges, we will consider the following ODE

$$\frac{d}{dt}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} ayz \\ axz \\ -2axy \end{pmatrix} - \begin{pmatrix} d & 0 & 0 \\ 0 & d & 0 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} + \begin{pmatrix} f_1 \\ -f_1 \\ f_3 \end{pmatrix} \tag{11}$$

for $a, d, f_1, f_3 > 0$, $x(0) = -y(0)$, and $z(0) > 0$. We will prove that (11) diverges by showing that there exists an invariant set in which solutions must diverge to infinity.

Consider the new variable $\psi(t) = x(t)+y(t)$ with $\psi(0) = x(0)+y(0) = 0$. We calculate $\dot{\psi} = \dot{x}+\dot{y}$ as follows

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}\psi &= ayz + axz - dx - dy + f_1 - f_1 \\ &= az(x+y) - d(x+y) \\ &= \psi(t)(az(t) - d) \end{aligned}$$

Then we have a fixed point at $\psi = 0$, which happens to be our initial condition, telling us that $\psi(t) = 0$ and $x(t) = -y(t)$ for all t. Note that this is only possible as our forcing is in the plane of the stable subset, which is $x = -y$. Then, we notice that $x(t)y(t) \leq 0$, therefore $\dot{z} = -2axy + f_3 \geq f_3 > 0$ for all t, so $z(t) \geq f_3 t + z_0$ for all t, and as z blows up, (11) must diverge to infinity. We can use some graphs and numerics to visualize how this symmetry causes divergence.

Figure 1 depicts a cross section of the xy-plane for our $z(0) = z_0$ when we are on the positive z-axis. We have shown that for all cross sections in the xy-plane along this axis, the trajectory remains on the stable subset M_s, namely $x(t) = -y(t)$, while the z-value of the trajectory is strictly increasing.

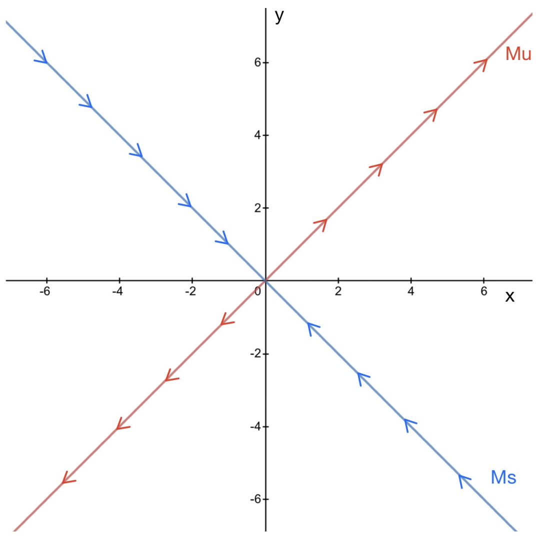
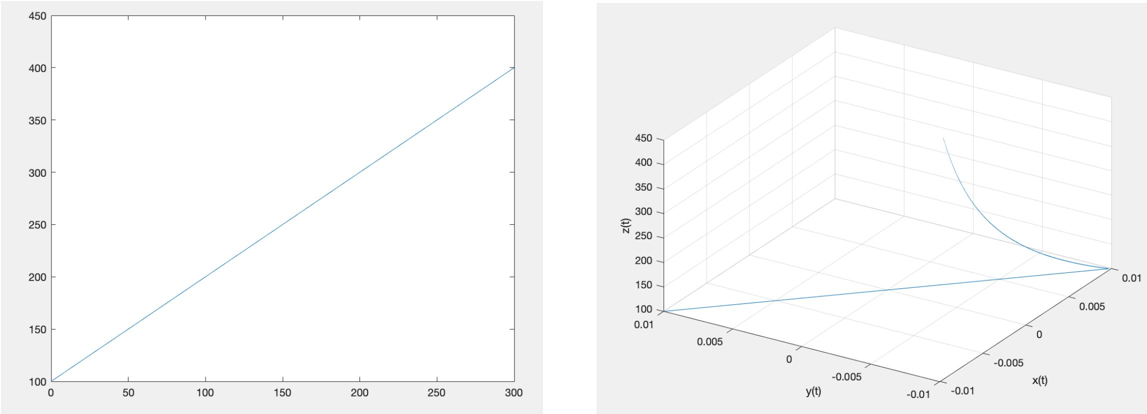


Figure 1: Cross-section for $z = z_0$

We can confirm that our solution with these symmetries remains on M_s and diverges towards positive infinity along the z-axis numerically. Also, we can gain some intuition as to the behavior of the solution as it diverges. We let $(x_0, y_0, z_0) = (-0.01, 0.01, 100)$ with $d = 1$, $a = 1$, $f_1 = 1$, $f_3 = 1$, and run the simulation from $t = 0$ to $t = 300$ using MATLAB.

Figure 2 shows the plot of the trajectory and the energy over time. In plot (b) the solution begins in the bottom left part of the plot, and shoots through the kernel of A before approaching the kernel again asymptotically as z increases to infinity.



(a) Energy of the trajectory over time

(b) Plot of trajectory

Figure 2

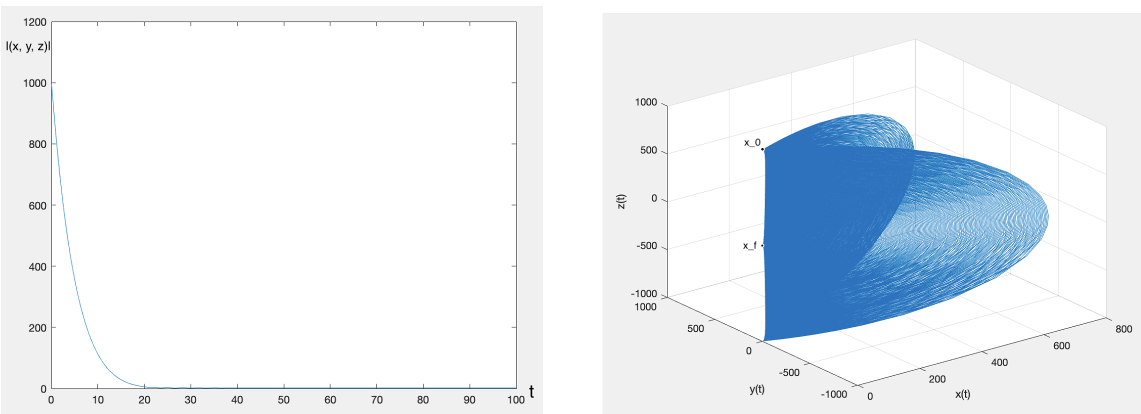
We see that the trajectory stays on the stable subset, and the energy grows at an almost constant rate defined by the magnnitude of f_3. This can be understood through some simple calculations.

$$\frac{\mathrm{d}}{\mathrm{d}t}|\mathbf{x}| = \frac{\langle \dot{\mathbf{x}}, \mathbf{x} \rangle}{|\mathbf{x}|} = \frac{\langle (-x+1, x+1, 1), (x, -x, z) \rangle}{|\mathbf{x}|} = \frac{2(-x^2+x)+z}{|\mathbf{x}|}$$

We notice that as the solution diverges, the solution remains close to the z-axis and thus x is very small, so the change in energy becomes $\frac{\mathrm{d}}{\mathrm{d}t}|\mathbf{x}| \sim z/|\mathbf{x}| \sim 1$ as supported by figure 2. It can be shown that for all divergent cases with hyperbolic or elliptic equilibria, the solution diverges at a constant rate defined by the magnitude of the forcing term. This makes sense as the forcing term is the only term pumping energy into the system, and it is constant.

Additionally, assuming that $z(t) >> 0$, it follows that $(az(t) - d) > 0$, so $\psi(0) > 0$ implies that $\frac{\mathrm{d}}{\mathrm{d}t}\psi(t) > 0$ for all t, and $\psi(0) < 0$ implies that $\frac{\mathrm{d}}{\mathrm{d}t}\psi(t) < 0$ for all t. This tells us that the fixed point at $\psi = 0$ is unstable.

Since the equilibria of $\psi = 0$ is unstable, in order for divergence to occur the initial conditions must be exactly on the stable subset of the cross-section. This tells us that our claim of convergence for hyperbolic examples should remain true for initial conditions and forcing without this symmetry. Below in figure 3 we plot the energy of the system over time on the left and the trajectory on the right of (11) with $z(0) = 1000, x(0) = 1, y(0) = -0.99999, f_1 = 1, f_3 = 1, a = 1$ from $t = 0$ to $t = 100$.



(a) Energy of the trajectory over time

(b) Plot of trajectory

Figure 3

Now, the trajectory begins at x_0 and ends at x_f. The trajectory oscillates while exponentially losing energy and then remains at a constant low energy in finite time as predicted.

3.2 Boundedness for Hyperbolic Systems

We will sketch a proof that the ODE

$$\frac{d}{dt}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} yz - x + 1 \\ xz - y \\ -2xy + 1 \end{pmatrix} \tag{12}$$

remains bounded. Also, throughout this section we will use the notation

$$\mathbf{x} = \mathbf{x}(t) = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

and unless otherwise noted, x, y, and z are single-variable functions dependant on t. We choose this example to prove as it is a simple case for which hyperbolic equilibria exist in the kernel of A and $f \notin M_s$ allowing us to avoid the symmetries present in the last example. That there is no forcing in the y-axis makes some calculations easier, but with more work this method of proof can be extended to all hyperbolic cases in $\mathbb{R}^3$. There remains steps to prove this statement for all hyperbolic examples in $\mathbb{R}^n$, and there still remains an unfinished approximation argument to complete this proof.

We begin our sketch of the proof by showing that a certain condition being satisfied is equivalent to proving that all solutions to the ODE remain bounded.

Lemma 5 *Suppose that there exists some $E \geq 1$ such that for every $|\mathbf{x}_0| > E$, there is some $T(\mathbf{x}_0) \leq 1$ such that $|\mathbf{x}(T(\mathbf{x_0}))| < E$. Then our solution $\mathbf{x}(t)$ remains bounded*

Proof: Suppose that for every $|\mathbf{x}_0| > E$, there is some $T(\mathbf{x}_0) \leq 1$ such that $|\mathbf{x}(T(\mathbf{x_0}))| < E$. Since

$$\frac{\mathrm{d}}{\mathrm{d}t}|\mathbf{x}(t)|^2 = -2x(t)^2 - 2y(t)^2 + 2x(t) + 2z(t)$$

it follows that

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}|\mathbf{x}(t)| &= \frac{-x(t)^2 - y(t)^2 + x(t) + z(t)}{|\mathbf{x}(t)|} \\ &\leq \frac{x(t) + z(t)}{|\mathbf{x}(t)|} \\ &\leq 2 \end{aligned}$$

Then on our interval $(0, T(\mathbf{x_0}))$, the energy can achieve a maximum of $|\mathbf{x}_0|+2$. Continuing this process, we have the upper bound

$$\limsup_{t\to\infty}|\mathbf{x}(t)| < E + 2$$

or that our solution remains bounded.

□

The following sketch of a proof aims to show that the conditions of lemma 5 are met, but lacks the approximation argument to complete the proof.

We first partition $\mathbb{R}^3$ into the three sets $M_1 \cup M_2 \cup M = \mathbb{R}^3$ as follows

$$\begin{aligned} M &= \{\mathbf{x} \in \mathbb{R}^3 \,:\, |x|^2 + |y|^2 \geq \delta|z|^2\} \\ M_1 &= \{\mathbf{x} \in \mathbb{R}^3 \,:\, |x|^2 + |y|^2 \leq |z|^{-2}\} \\ M_2 &= (M \cup M_1)^c \end{aligned}$$

for some δ that will be chosen in the approximation argument. Then, we will prove that for some $E > 0$ and all points in M such that $|\mathbf{x_0}| > E$, there exists some $T_3 < 1/3$ such that for some $t < T_3$, $|\mathbf{x}(t)| < E$. Then we will show that for all $\mathbf{x}_0 \in M_1$, the solution will enter M_2 in some $t < T_1 < 1/3$, and that for all $\mathbf{x}_0 \in M_2$, the solution will enter M in some time $t < T_2 < 1/3$. Together, this will prove that for all $|\mathbf{x}_0| > E$ and some $t^* < T_1 + T_2 + T_3 < 1$, the condition of lemma 5 that $|\mathbf{x}(t^*)| < E$ is met and we can conclude the proof.

Without loss of generality, we let

$$|x_0| \geq |y_0|$$

and note that in M:

$$\begin{aligned} E &\leq \sqrt{|x_0|^2 + |y_0|^2 + |z_0|^2} \\ &\leq \sqrt{|x_0|^2 + |x_0|^2 + \frac{2}{\delta}|x_0|^2} \\ &\leq |x_0|\sqrt{2 + \frac{2}{\delta}} \end{aligned}$$

And denoting $c_\delta = (2 + \frac{2}{\delta})^{-1}$, we have that

$$|x_0| \geq c_\delta E$$

First, we will prove the following lemma:

Lemma 6 *There exists some $\epsilon > 0$ that is dependent only on δ such that if $t \leq \frac{\epsilon}{E}$, then $|x(t)| \geq \frac{1}{2}|x_0|$*

Proof: We will prove this using a Taylor expansion. For some $s \in (0, t)$, we can write $x(t)$ as follows.

$$\begin{aligned} x(t) &= x_0 + x'(0)t + \frac{x''(s)t^2}{2} \\ x(t) &= x_0 + t(y_0z_0 - x_0 + 1) + \frac{t^2}{2}(-y(s)z(s) + x(s) - 1) \\ &\quad + \frac{t^2}{2}(y(s)(-2x(s)y(s) + 1) + z(s)(x(s)z(s) - y(s))) \\ |x(t)| &\geq |x_0| - t(|y_0z_0 - x_0 + 1|) - \frac{t^2}{2}(|y(s)z(s) + x(s) - 1|) \\ &\quad - \frac{t^2}{2}(|y(s)(-2x(s)y(s) + 1) + z(s)(x(s)z(s) - y(s))|) \\ &\geq |x_0| - tc_1E^2 - t^2c_2E^3 \end{aligned}$$

For some c_1, c_2, $c_3 > 0$ as we know that $s < \frac{\epsilon}{E} << 1$. Then, using $|x_0| \geq c_\delta E$, we have that

$$\begin{aligned} |x(t)| &\geq |x_0| - c_1\epsilon E - c_2\epsilon^2 E \\ &\geq |x_0| - c_3\epsilon c_\delta |x_0| \\ &\geq |x_0|(1 - c_3\epsilon c_\delta) \end{aligned}$$

Then, there must be a ϵ dependant only on δ such that $1 - c_3\epsilon c_\delta < 1/2$, concluding the proof of our lemma.

□

We can follow the same process as in the proof of this lemma in our y-variable, and choose our ϵ such that for $t \in (0, \epsilon/E)$, we have that $|y(t)| > \frac{|y_0|}{2}$ and $|x(t)| > \frac{|x_0|}{2}$. Now for $t \in I$ where $I = (0, \frac{\epsilon}{E})$, we can bound the change in energy

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}|\mathbf{x}|^2 &= \frac{1}{2}(-x(t)^2 - y(t)^2 + x(t) + z(t)) \\
&\leq -\frac{x(t)^2}{2} + |x(t)| + |z(t)| \\
&\leq -\frac{x(t)^2}{2} + |x(t)| \left(1 + \frac{\delta^{-1/2}}{2}\right) \\
&\leq |x(t)| \left(\frac{-|x(t)|}{2} + 1 + \frac{\delta^{-1/2}}{2}\right) \\
&\leq \frac{|x(t)|}{2} \left(\frac{-|x_0|}{4} + 1 + \frac{\delta^{-1/2}}{2}\right) \\
&\leq \frac{|x(t)|}{2} \left(\frac{-c_\delta E}{4} + 1 + \frac{\delta^{-1/2}}{2}\right)
\end{aligned}
$$

Now, supposing that our choices of E and δ are such that $\frac{-c_\delta E}{4} + 1 + \frac{\delta^{-1/2}}{2} < 0$, we can conclude with the inequality

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}|\mathbf{x}|^2 &\leq \frac{|x_0|}{4} \left(\frac{-c_\delta E}{4} + 1 + \frac{\delta^{-1/2}}{2}\right) \\
&\leq \frac{c_\delta E}{4} \left(\frac{-c_\delta E}{4} + 1 + \frac{\delta^{-1/2}}{2}\right) < 0
\end{aligned}
$$

Then we can calculate the lower bound for energy loss on the interval I as follows

$$
\begin{aligned}
\left|\mathbf{x}\left(\frac{\epsilon}{E}\right)\right|^2 - |\mathbf{x}_0|^2 &\leq \int_I \frac{c_\delta E}{4} \left(\frac{-c_\delta E}{4} + 1 + \frac{\delta^{-1/2}}{2}\right) dt \\
&\leq \frac{c_\delta \epsilon}{4} \left(\frac{-c_\delta E}{4} + 1 + \frac{\delta^{-1/2}}{2}\right)
\end{aligned}
$$

and we conclude that on our interval of I, our energy must decrease by something of the order $\sqrt{E}$, which is very large. And since for every $\frac{\epsilon}{E} << 1$ we expect the energy to decrease on the scale of $\sqrt{E}$, we further expect that for some $t < 1/3$,

$$|\mathbf{x}(t)| < E$$

when we complete the appropriate approximation argument.

Next we aim to show that solutions with initial conditions very close to the kernel of A, which we define as $\mathbf{x}_0 \in M_1$, must leave that region in short time and enter M_2.

It would be convenient to rescale time and project our solution onto the unit sphere to show that our solution must leave M_1 in a short time. However, when we use the fact that in M_1, $|xy| << 1$ and then rescale the ODE, we get a fixed $z(t) = z_0$ and are left with the ODE in the xy-plane:

$$
\begin{cases}
\dot{x} = yz_0 - x + 1 \\
\dot{y} = xz_0 - y
\end{cases}
$$

which has a fixed point at

$$(x^*, y^*, z^*) = \left(\frac{-1}{z_0^2 - 1}, \frac{-1}{z_0 - \frac{1}{z_0}}, z_0\right)$$

With a fixed point in the approximation on the plane, it could be that there is an invariant set near the kernel that the solution can stay in while diverging. So, we are unable to use this method to show that solutions leave M_1.

Instead, for $\mathbf{x}_0$ very close to the z-axis we must approximate $\dot{z}(t) = 1$ which gives us the solution $z(t) = z_0 + t$. This makes our approximation the non-autonomous ODE

$$\begin{cases} \dot{x} = yz_0 + ty - x + 1 \\ \dot{y} = xz_0 + tx - y \end{cases} \tag{13}$$

Now we aim to prove the claim that initial conditions $\mathbf{x}_0 \in M_1$ must leave M_1 in a small time $T_1 < 1/3$. We will do so by introducing a new variable, finding an explicit solution to (13) using an integrating factor, and then approximating the resulting integral with a Taylor expansion.

As we did in section 3.1, let

$$\psi(t) = x(t) + y(t)$$

then, substituting this into (13), we have the ODE

$$\dot{\psi}(t) = (z_0 - 1 + t)\psi(t) + 1 \tag{14}$$

Note that unlike the example in section 3.1 $\psi_0 = 0$ is not a fixed point. Now we rewrite (14) as

$$\dot{\psi}(t) + (-z_0 + 1 - t)\psi(t) = 1$$

and get our integrating factor of $\exp\left(\int -(z_0 - 1 + t)dt\right) = \exp\left(t(-z_0 + 1) - t^2/2\right)$. We multiply both sides by the integrating factor and integrate to get

$$\int e^{t(-z_0+1)-t^2/2}\psi'(t)dt + \int e^{t(-z_0+1)-t^2/2}\psi(t)(-z_0 + 1 - t)dt = \int e^{t(-z_0+1)-t^2/2}dt$$

Then we can integrate the first term on the left hand side by parts to get

$$\int e^{t(-z_0+1)-t^2/2}\psi'(t)dt = e^{t(-z_0+1)-t^2/2}\psi(t) - \psi(0) - \int e^{t(-z_0+1)-t^2/2}\psi(t)(-z_0 + 1 - t)dt$$

which, when we substitute back into the previous expression yields

$$e^{t(-z_0+1)-t^2/2}\psi(t) = \int e^{t(-z_0+1)-t^2/2}dt + \psi(0)$$

and with some manipulation yields the expression

$$\psi(t) = e^{t(z_0-1)+t^2/2}\left(\int_0^t e^{s(-z_0+1)}e^{-s^2/2}ds + \psi(0)\right)$$

Now since we are working on a timescale which is very small, we replace the term $e^{-s^2/2}$ with its Taylor expansion around $s = 0$, giving us the approximation $e^{-s^2/2} = 1 - \frac{s^2}{2} + O(s^4)$. We ignore the terms greater than or equal to order four, and substituting the Taylor expansion into our equation we have

$$\psi(t) = e^{t(z_0-1)+t^2/2}\int_0^t e^{s(-z_0+1)}\left(1 - \frac{s^2}{2}\right)ds + \psi(0)e^{t(z_0-1)+t^2/2}$$

which we can integrate by parts as follows

$$\int_0^t e^{s(-z_0+1)}\left(1 - \frac{s^2}{2}\right)ds = \left[e^{s(-z_0+1)}\left(\frac{1}{-z_0+1}\right)\left(1 - \frac{s^2}{2}\right)\right]\Bigg|_{s=0}^t + \int_0^t se^{s(-z_0+1)}ds$$

Then we evaluate

$$\int_0^t se^{s(-z_0+1)}ds = \left[\frac{s}{1-z_0}e^{s(-z_0+1)}\right]\Big|_{s=0}^{t} - \int_0^t \frac{1}{1-z_0}e^{s(1-z_0)}ds$$
$$= \frac{t}{1-z_0}e^{t(-z_0+1)} - \frac{1}{(1-z_0)^2}\left(e^{t(1-z_0)}-1\right)$$

Which, when we plug back into the earlier equation, we get

$$\int_0^t e^{s(-z_0+1)}\left(1-\frac{s^2}{2}\right)ds = \frac{e^{t(1-z_0)}\left(1-\frac{t^2}{2}\right)-1}{1-z_0} + \frac{t}{1-z_0}e^{t(-z_0+1)} - \frac{\left(e^{t(1-z_0)}-1\right)}{(1-z_0)^2}$$

Finally, we have the solution

$$\psi(t) = \frac{e^{t(z_0-1)+t^2/2}}{1-z_0}\left(e^{t(1-z_0)}\left(1-\frac{t^2}{2}\right) - 1 + te^{t(-z_0+1)} - \frac{\left(e^{t(1-z_0)}-1\right)}{1-z_0} + \psi(0)(1-z_0)\right)$$

Now, we can perform more Taylor expansions to yield $e^{t(z_0-1)+t^2/2} = e^{t(z_0-1)}(1+t^2/2)$, which when substituted into the above equation yields

$$\psi(t) = \frac{e^{t(z_0-1)}(1+t^2/2)}{1-z_0}\left(e^{t(1-z_0)}\left(1-\frac{t^2}{2}\right) - 1 + te^{t(-z_0+1)} - \frac{\left(e^{t(1-z_0)}-1\right)}{1-z_0} + \psi(0)(1-z_0)\right)$$
$$= (1-z_0)^{-1}\left(1 - t^4/4 - e^{t(z_0-1)}(1+t^2/2) + t(1+t^2/2) - \frac{(1+t^2/2)(1-e^{t(z_0-1)})}{1-z_0}\right)$$
$$+ e^{t(z_0-1)}(1+t^2/2)\psi(0)$$

For small time, we notice that the first term on the RHS is of a lower order than the second as it is multiplied by $(1-z_0)^{-1}$. But, we also note that it is this term that ensures that there is no fixed point for $\psi(0)=0$ that may lead to divergence as seen in section 3.1. So we know that we will move off of any potential stable subset, and can assume that $\psi(0)\neq 0$ in our approximations.

Also, we note that since the forcing is in the positive z-direction, our solution can only diverge along the z-axis to $z=+\infty$, so we are assuming that $z_0>0$. The case that forcing is negative in the z-direction would follow the same method of proof only considering $z_0<0$. With these properties in mind, we can assume $\psi(0)\neq 0$ and make the following approximations for a very small $t>0$.

$$|\psi(t)| \geq \frac{e^{t(z_0-1)}|\psi(0)|}{2}$$

In order to leave M_1 at some t, we must have

$$|x(t)|+|y(t)| \geq |\psi(t)| \geq |z|^{-1} \geq E^{-1}$$

which occurs at some $t<T_1$ when

$$E^{-1} = \frac{e^{T_1(z_0-1)}|\psi(0)|}{2}$$

we evaluate that our solution must leave M_1 at some $t<T_1$ for

$$T_1 = \frac{\ln\left(\frac{2}{E|\psi(0)|}\right)}{z_0-1} << 1/3$$

Now that we have show that all solutions leave M_1 in short time, we will show that all solutions with $|\mathbf{x_0}|>E$ such that $\mathbf{x_0}\in M_2$ will leave M_2 in short time. Also, note that by how we defined M_2 either $|x_0|\geq\frac{2}{E}$, $|y_0|\geq\frac{2}{E}$, or both.

Finally, we must prove that solutions in M_2 enter M in a small time. As we did in our proof that the unforced case must converge to the origin, we let $u(t) = \pi_{\ker A^{\perp}}\mathbf{x}(t)$, so that $u(t)$ is the distance from the solution to the z-axis, or $u^2 = x^2 + y^2$, and again we choose $z_0 > E$. Now with our assumptions that $|x_0|$ and $|y_0|$ leave the region very close to the kernel of A and do not fall into a fixed point, we can study the dynamics of the projection of this system onto the unit sphere. In doing so we must also scale time. We introduce $\tau = |\mathbf{x}_0|t \sim z_0 t$ and $\widetilde{\mathbf{x}} = \frac{\mathbf{x}}{|\mathbf{x}|}$. Then our ODE becomes

$$\frac{\mathrm{d}}{\mathrm{d}\tau}\widetilde{\mathbf{x}} = B(\widetilde{\mathbf{x}}, \widetilde{\mathbf{x}}) - \frac{1}{z_0}A\widetilde{\mathbf{x}} + \frac{1}{z_0^2}f \tag{15}$$

with $\widetilde{z_0} \sim 1$. We also notice that with scaling M_2 remains unchanged, and so does the result we are trying to prove that we must leave M_2 in short time.

Since $z_0 >> 1$ and $\widetilde{x_0}, \widetilde{y_0} << 1$, we approximate the dynamics as

$$\begin{cases} \dot{\widetilde{x}} = \widetilde{y}\widetilde{z} \\ \dot{\widetilde{y}} = \widetilde{x}\widetilde{z} \\ \dot{\widetilde{z}} = -2\widetilde{x}\widetilde{y} \end{cases} \tag{16}$$

in which we ignore both forcing and damping due to the scaling. Then we make the next approximation that $\widetilde{x}\widetilde{y} \sim 0$ inside M, leaving us with $\dot{\widetilde{z}} \sim 0$ and $\widetilde{z}(t) = \widetilde{z_0} \sim 1$. Then our new system is

$$\begin{cases} \dot{\widetilde{x}} &= \widetilde{y} \\ \dot{\widetilde{y}} &= \widetilde{x} \end{cases} \tag{17}$$

which is just a saddle on the plane $z = 1$ tangent to the unit sphere.

It remains to show that these approximations can be made. Also, the proof can be completed by using the integrating factor from the previous part to show that ψ becomes large enough that the solution leaves M_2 as well as M_1. For a very high energy E and a small δ, when the solution begins far away enough from a stable subset of the kernel, the bilinear term dominates in such a way that we imagine the projection onto the unit sphere to simplify to (17). We do this simply to illustrate that once the fixed points present in M_1 are avoided such approximations describe the system's behavior.

Solving (17) yields the solution

$$\begin{cases} \widetilde{x}(\tau) = \frac{\widetilde{x}_0}{2}(e^{\tau} + e^{-\tau}) + \frac{\widetilde{y}_0}{2}(e^{\tau} - e^{-\tau}) \\ \widetilde{y}(\tau) = \frac{\widetilde{x}_0}{2}(e^{\tau} - e^{-\tau}) + \frac{\widetilde{y}_0}{2}(e^{\tau} + e^{-\tau}) \end{cases} \tag{18}$$

on the plane. Now, we utilize the result from our last section that solutions will stay off of the stable subspace, as we have that $x + y$ is not close to zero. Then it follows that

$$\begin{aligned} |\widetilde{u}(\tau)|^2 &= |\widetilde{x}(\tau)|^2 + |\widetilde{y}(\tau)|^2 \\ &= \frac{\widetilde{x}_0^2 + \widetilde{y}_0^2}{2}(e^{2\tau} + e^{-2\tau}) \\ &= \frac{\widetilde{u}_0^2}{2}(e^{2\tau} + e^{-2\tau}) \\ &> \frac{\widetilde{u}_0^2}{2}(e^{2\tau}) \end{aligned}$$

Now, we want to find some τ^* such that $\widetilde{\mathbf{x}}$ leaves M_2, which will happen when $|\widetilde{u}(\tau^*)|^2 > \delta|z_0|^2$, or at

$$\begin{aligned} \frac{(\widetilde{u}_0 e^{\tau^*})^2}{2} &= \delta|z_0|^2 \\ \tau^* &= \ln\left(\frac{\sqrt{2}\delta|z_0|}{|\widetilde{u}_0|}\right) \end{aligned}$$

Noting that $|\widetilde{u}_0| > |z_0|^{-1}$, we are left with

$$\begin{aligned}\tau^* &< \ln\left(\sqrt{2}\delta|z_0|^2\right) \\ &< 2\ln\left(2^{1/4}\sqrt{\delta}|z_0|\right)\end{aligned}$$

Finally rescaling time, we have that $\mathbf{x}$ must leave M_2 at some time $t = T^*$ such that

$$T^* < \frac{2\ln\left(2^{1/4}\sqrt{\delta}|z_0|\right)}{|z_0|} << 1$$

Then we can define $T_2 = T^* < 1/3$, concluding that if $t^* < T_1 + T_2 + T_3 < 1$, and $|\mathbf{x_0}| > E$, then at some time $t < t^*$ our solution $|\mathbf{x}(t)| < E$ and thus remains bounded by lemma 5.

3.3 Boundedness for Elliptic Systems

In this section, we will consider the ODE

$$\frac{d}{dt}\begin{pmatrix}x\\y\\z\end{pmatrix} = \begin{pmatrix}ayz\\bxz\\-(a+b)xy\end{pmatrix} - \begin{pmatrix}d_1 & 0 & 0\\0 & 0 & 0\\0 & 0 & d_3\end{pmatrix}\begin{pmatrix}x\\y\\z\end{pmatrix} + \begin{pmatrix}f_1\\f_2\\f_3\end{pmatrix} \tag{19}$$

with $a, b, d_1, d_3 > 0$. As discussed earlier in this section, for $|y_0| >> 1$ we expect the dynamics in the cross sections of the xz-plane to exhibit spiral sink behavior around the y-axis. At large energies the conservative term dominates and these equilibria resemble centers while at lower energies as the damping is relatively strong, the equilibria more closely resemble spiral sinks. Figure 4 below depicts the expected behavior contributed from the conservative term B.

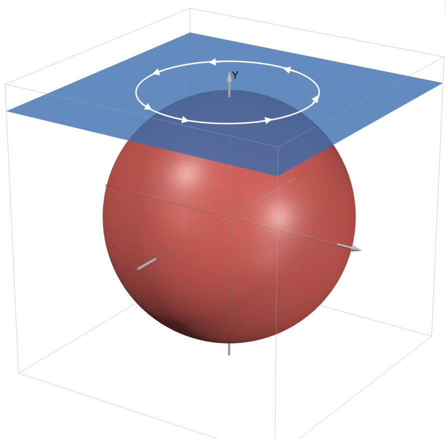


Figure 4: The linearization of an elliptic equilibria on the y-axis

Note that the cone for which the energy of the system is increasing will be similar in shape to that used in the previous subsection on hyperbolic fixed points, only now it is oriented on the y-axis and is dependant on our matrix A and the forcing. As the y-axis hosts stable equilibria, we assume that there will be trajectories in which the conservative term cause the trajectory to spiral inside the cone surrounding the kernel of A while gaining energy from the forcing term, eventually diverging to infinity. We will show an interesting relationship between the presence of fixed points of (19) and boundedness.

We can find the fixed points for the whole system of (19) by first finding the conditions for which $\dot{y} = 0$, which happens when

$$x = \frac{-f_2}{bz}$$

Then, in order for $\dot{x} = 0$, our parameters must satisfy $ayz = d_1x - f_1$. When our result from $\dot{y} = 0$ is substituted in, we get the relationship

$$y = \left(\frac{-d_1f_2 - f_1b}{ab}\right)\frac{1}{z^2}$$

Finally, substituting the prior results into $-(a+b)xy + f_3 = d_3z$, we get an expression for the z-values of fixed points

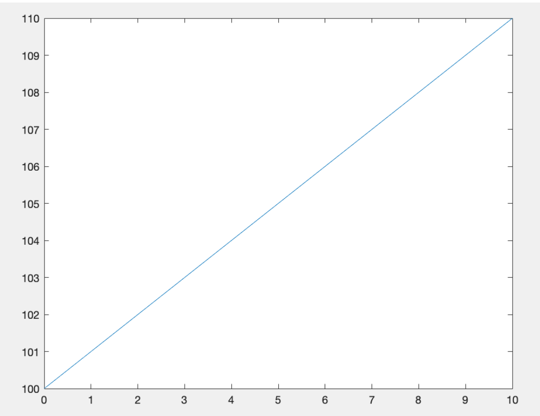
$$\frac{-f_2(a+b)(d_1f_2 + f_1b)}{ab^2} + f_3z^3 - d_3z^4 = 0 \tag{20}$$

To simplify the above equation, we can let $c_1 = d_3$, $c_2 = f_3$, and $c_3 = -f_2(a+b)(d_1f_2 + f_1b)/(ab^2)$ so that

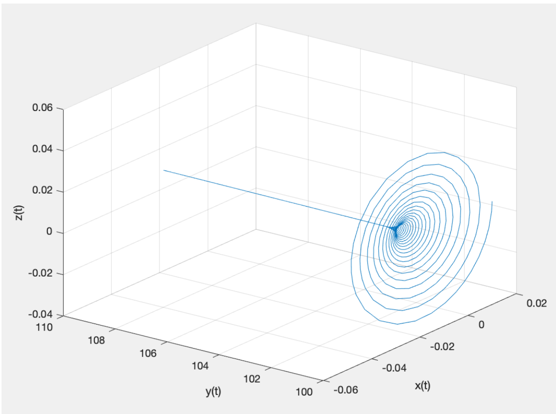
$$0 = -c_1z^4 + c_2z^3 + c_3$$

Note that $c_1 > 0$, $c_2 \neq 0$, and c_3 has no restrictions. Also, note that for every solution z, there is exactly one pair of (x, y) such that (x, y, z) is a fixed point, as indicated in our process of finding (20).

Graphing $f(z) = -c_1z^4 + c_2z^3 + c_3$ we notice that any set of parameters (c_1, c_2, c_3) yields either zero, one, or two roots. An example of when this equation has zero real roots is when $c_1 = 5$, $c_2 = -4$, $c_3 = -2$. For ease of calculation, let $a = 1$ and $b = 1$ so that $c_3 = -2f_2(d_1f_2 + f_1)$. To satisfy $c_3 = -1$, we let $f_2 = 1$, $d_1 = 2$ and $f_1 = -1$. Then to satisfy the requirements for c_2 and c_1 we let $d_3 = 5$ and $f_3 = -4$. Now, as we are trying to find unbounded behavior for the system, we choose initial conditions with a large y_0 and very small x_0, z_0. We will choose $x_0 = z_0 = 0.01$ and $y_0 = 100$. The results are as follows.



(a) Energy of the trajectory over time

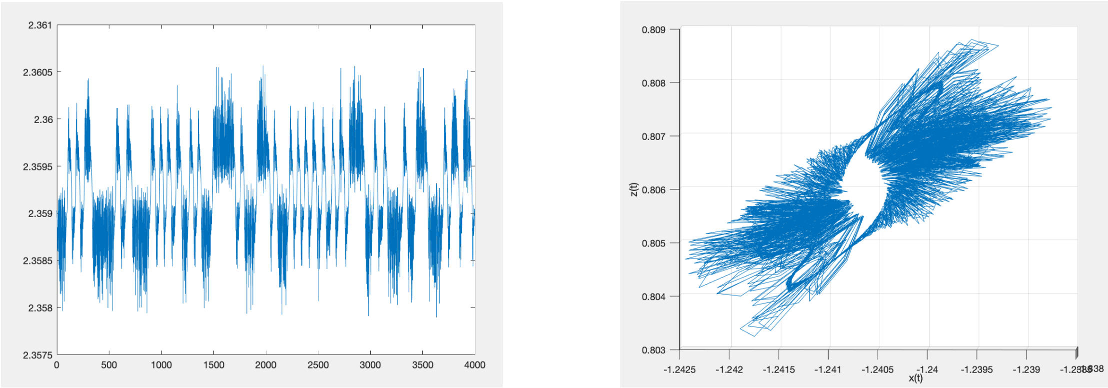


(b) Plot of trajectory

Figure 5: Plots for zero fixed points

The trajectory begins at the outer edge of the spiral in plot b and then enters the kernel of A where it diverges to infinity. We choose $y_0 > 0$ as the forcing is in the positive y-direction, so the solution must diverge in the positive direction of the y-axis. Plots for longer time confirm that this behavior continues, and the intervals of time that these computations run for are chosen based on the software's ability. Also, as the energy is only being increased by the forcing, it will resemble a linear graph as seen in figure a. Now, we will show that when one fixed point is present and the solution exists near the fixed point, this behavior ceases.

Now, we can calculate that (20) has exactly two real roots very close to each other when $c_1 = 5$, $c_2 = 6.978$, and $c_3 = -2$. Note that we have only changed c_2, and that for any $c_2 < 6.977$ we have zero real roots. Then, we use the same parameters as in the last computation, but we must adjust f_3. Now, we choose an initial point close to the fixed points, which are near $z^* \sim 0.808$, $x^* \sim 1.24$, $y^* \sim 1.837$. We must truncate these numbers as the simulations in MATLAB fail when so many decimals are included in the initial conditions. The trajectory and energy of the trajectory are plotted below in figure 6.

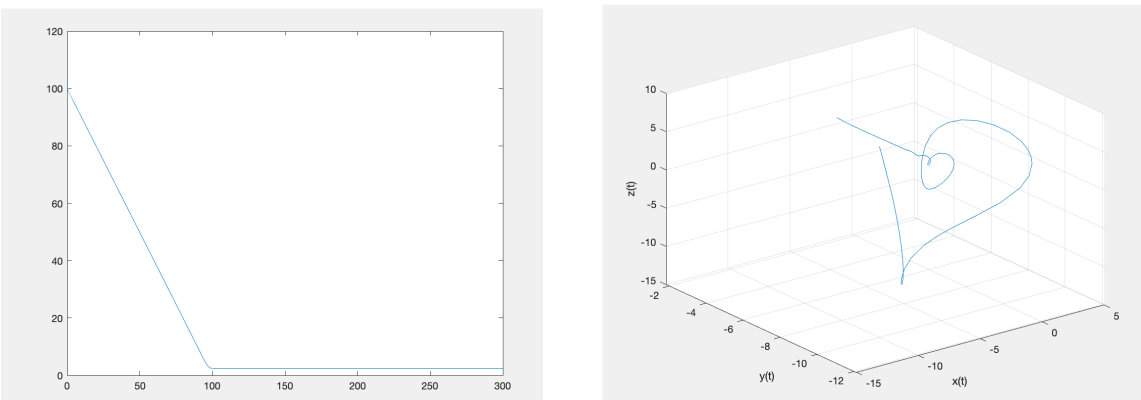


(a) Energy of the trajectory over time (b) Plot of trajectory

Figure 6

The numerics in this case were not expected. In previous plots of both the elliptic and hyperbolic systems, if a solution remained bounded it entered a fixed point remained at that fixed point for all time. The behavior in this case appears bounded and nonperiodic. It is possible that this result is only due to an error of MATLAB for such precise numerical values on a small scale. Either way, in the future I would like to continue studying this specific case and find if analysis agrees with the plots.

With the presence of fixed points near the origin, one may assume that initial conditions far from the fixed point will always diverge. This is not the case as it is the forcing in the positive y-direction that drives the divergence, and if we begin at a high energy with a $y_0 < 0$, in order to diverge the solution must travel to the other side of the y-axis, thus passing the fixed point and falling into a bounded steady state. We numerically show this with the same parameters as before, but now with initial conditions $(x_0, y_0, z_0) = (-10, -100, 12)$. The numerics support boundedness in figure 7

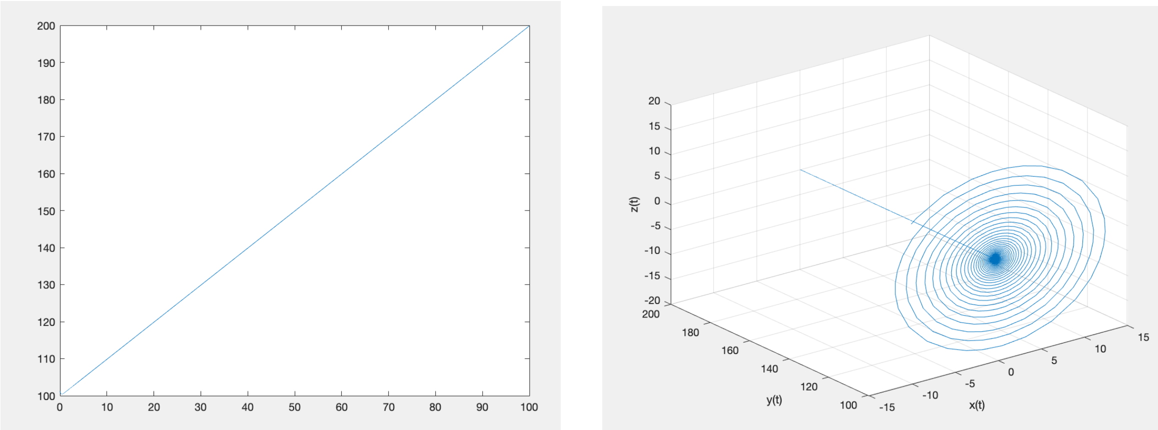


(a) Energy of the trajectory over time (b) Plot of trajectory

Figure 7

In plot b. the trajectory begins in the middle of the plot, shoots downwards in the z-direction and spirals before remaining fixed at the far left end of the line. Similar to the hyperbolic case, the trajectory for $y_0 > 0$ seemingly finds the stable fixed point near the origin. Now to show that it is the negative initial y condition that caused convergence to a fixed point in the previous example, we can run the same simulation while flipping the sign of y_0. Defining the initial conditions $(x_0, y_0, z_0) = (-10, 100, 12)$, the numerics for a short time show that the solution finds an invariant set in the y-axis and diverges towards $\lim_{t\to\infty} y(t) = +\infty$. Supporting plots are in figure 8 below.

From the numerical results we notice that when the solution is not very close to a fixed point, it oscillates around the y-axis while losing energy and finally falls into an invariant set along the y-axis.



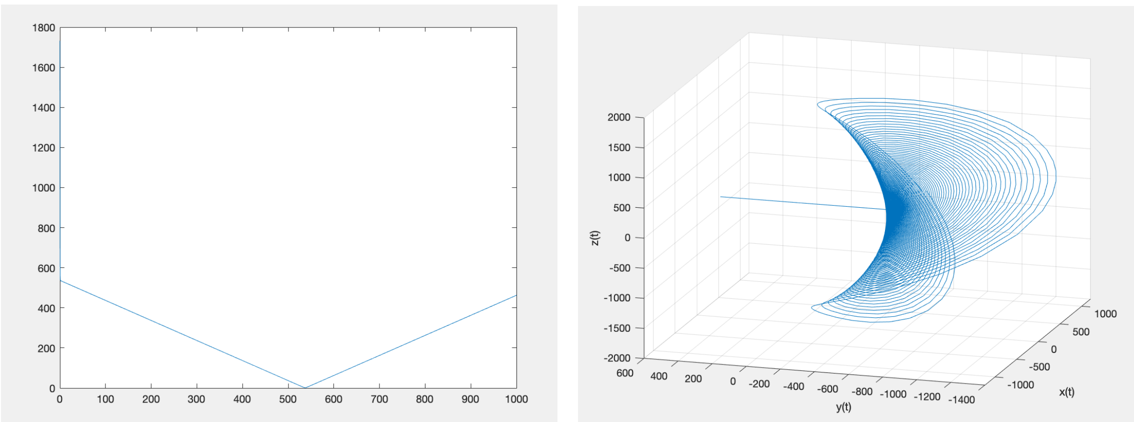
(a) Energy of the trajectory over time

(b) Plot of trajectory

Figure 8

Since the forcing is in the positive y-direction, each solution diverges in the positive y-direction unless it falls into a fixed point.

In figure 7 the solution enters the subset along the y-axis for some negative y, and the fixed point near the origin is also near the subset, so as the solution passes near the fixed point along the subset it falls into the fixed point and remains bounded. However, initial conditions with negative initial y-values do not always exhibit this behavior. Consider an example with the identical parameters and fixed points as before but with $(x_0, y_0, z_0) = (-1000, -1000, 1000)$. The behavior is shown below in figure 9



(a) Energy of the solution over time

(b) Plot of the solution in $\mathbb{R}^3$

Figure 9

The trajectory oscillates at very high energies and then enters an invariant set along the y-axis as expected, but this subset is not close to the fixed points. Because the solution enters this set at such a large energy, it passes the fixed points and diverges to infinity along the y-axis. The question of divergence thus is more subtle than how negative the initial conditions are in the y-direction.

References

[1] Jacob Bedrossian and Kyle Liss. Stationary measures for stochastic differential equations with degenerate damping. *Probability Theory and Related Fields*, pages 1–78, 2024.

[2] Tarek Elgindi, Wenqing Hu, and Vladimir Sverak. On 2d incompressible euler equations with partial damping. *Communications in Mathematical Physics*, 355:145–159, 2017.

[3] Juraj Földes, Nathan E Glatt-Holtz, and David P Herzog. Sensitivity of steady states in a degenerately damped stochastic lorenz system. *Stochastics and Dynamics*, 21(08):2150055, 2021.

[4] Uriel Frisch. *Turbulence: the legacy of AN Kolmogorov*. Cambridge university press, 1995.

[5] Stephen B. Pope. *Turbulent Flows*. Cambridge University Press, 2000.

[6] Brendan Williamson. On sdes with partial damping inspired by the navier-stokes equations, 2019.

[7] Michele Coti Zelati, Gianluca Crippa, Gautam Iyer, and Anna L Mazzucato. Mixing in incompressible flows: transport, dissipation, and their interplay. *arXiv preprint arXiv:2308.00358*, 2023.