A circle quotient of a G_2 cone

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Abstract. A study is made of $\mathbb{R}^6$ as a singular quotient of the conical space $\mathbb{R}^+ \times \mathbb{CP}^3$ with holonomy G_2, with respect to an obvious action by $U(1)$ on $\mathbb{CP}^3$ with fixed points. Closed expressions are found for the induced metric, and for both the curvature and symplectic 2-forms characterizing the reduction. All these tensors are invariant by a diagonal action of $SO(3)$ on $\mathbb{R}^6$, which can be used effectively to describe the resulting geometrical features.

Contents

1. Introduction

Let $\mathscr{C}$ denote the real 7-dimensional manifold $\mathbb{R}^+ \times \mathbb{CP}^3$ endowed with its $SO(5)$-invariant conical metric whose holonomy group is conjugate to G_2. We shall denote this Riemannian metric by h_2. Its isolated singularity can be smoothed by passing to a complete G_2 metric on the total space of half the bundle of 2-forms over S^4 (as two of the authors showed in [9] and others in [19]). Restricting to unit 2-forms defines the Penrose twistor fibration

$$\pi\colon\ \mathbb{CP}^3 \longrightarrow S^4,$$

which also plays a key role in understanding the conical G_2 structure. The latter is most easily defined by means of a closed 3-form φ, which determines h_2, and a closed 4-form $*\varphi$ (with $*$ defined by h_2 and ultimately φ).

We shall study a quotient

$$Q\colon\ \mathscr{C} \longrightarrow \mathscr{M} \tag{1.1}$$

of $\mathscr{C}$ by a natural circle subgroup $U(1)$ with fixed points, and identify the metric g_2 induced on the 6-dimensional base $\mathscr{M}$. The group $U(1)$ is induced by left multiplication by $e^{i\theta}$ on

$\mathbb{H}^2 = \mathbb{C}^4$, commuting with the projectivization to $\mathbb{CP}^3$. The same action can be obtained from rotation of two coordinates of $\mathbb{R}^5 = \mathbb{R}^2 \oplus \mathbb{R}^3$, since this lifts to the one we want via π. Modulo the origin, $\mathscr{M}$ is a $U(1)\times U(1)$ quotient of $\mathbb{R}^8$ and, applying the Gibbons-Hawking ansatz [18], one can identify $\mathscr{M}$ with $\mathbb{R}^3 \times \mathbb{R}^3$ in which each 'axis' $\mathbb{R}^3$ arises from the fixed point set of the respective $U(1)$.

A point of $\mathscr{M}$ will be represented by a 'bivector' $(\mathbf{u}, \mathbf{v})$ with $\mathbf{u} \in \mathbb{R}^3$ and $\mathbf{v} \in \mathbb{R}^3$. (Our terminology acknowledges that of [20], in which a bivector is the vector part of a quaternion with complex coefficients, in our case a hyperkähler moment map.) This description is closely related to the Kähler quotient

$$\mathbb{CP}^3 /\!/ U(1) \;\cong\; \mathbb{CP}^1 \times \mathbb{CP}^1.$$

The Kähler picture, and an associated metric g_1 on $\mathscr{M}$, will provide a useful comparison for some of our results. However, we are primarily interested in tensors arising from G_2, which explains why we add an action by $\mathbb{R}^+$ rather than remove one. Inside $\mathbb{CP}^3$, the circle action fixes two projective lines, which are the twistor lifts of the 2-sphere $\mathbb{S}^2 = S^4 \cap \mathbb{R}^3$. They are swapped by an anti-linear involution j, and are contained in a unique $U(2)$-invariant complex quadric in $\mathbb{CP}^3$.

Each $U(1)$ (or more effectively, $SO(2)$) orbit on S^4 is specified by a unique point of norm at most one in $\mathbb{R}^3$, so we can identify $S^4/SO(2)$ with the closed unit ball D^3. This gives rise to a commutative diagram in which ϖ is induced by π and an $\mathbb{R}^+$ quotient:

$$\begin{array}{ccccc} & & \mathbb{H}^2 \setminus \mathbf{0} & & \\ & {\scriptstyle\rho}\swarrow & & \searrow & \\ \mathscr{C} = \mathbb{R}^+ \times \mathbb{CP}^3 & & \stackrel{Q}{\longrightarrow} & & \mathbb{R}^6 \setminus \mathbf{0} = \mathscr{M} \\ & \searrow & & \swarrow {\scriptstyle\varpi} & \\ & & D^3 & & \end{array}$$

Figure 1: Quotients described by spaces of dimension 7, 6 and 3

The rich geometry underlying this construction in the context of special holonomy was first highlighted by Atiyah and Witten, who exploited the duality between M-theory and Type IIA superstring theory [4]. The fact that the fixed points of the $U(1)$ action on the G_2 manifold occur in codimension 4 enables M-theory on $\mathscr{C}$ to be identified with Type IIA theory on $\mathscr{M}$, and the fixed point set with $D6$-branes in $\mathscr{M}$.

By analogy to the families of metrics with G_2 holonomy interpolating between highly-collapsed metrics and those asymptotic to the cone over $S^3 \times S^3$ in [15], one might expect $\mathscr{M}$ to acquire a Calabi-Yau metric and the singular $\mathbb{R}^3$'s to be special Lagrangian. In our situation, there is no such collapsed limit because $\mathscr{C}$ has no finite circles at infinity, and our work shows that the picture painted in [4] is somewhat of an oversimplification.

However, we do show that the induced symplectic form σ is very easy to describe on $\mathscr{M}$ and that the singular $\mathbb{R}^3$'s are Lagrangian.

Our aim is to describe the $SU(3)$ structure $(g_2, \mathbb{J}, \sigma)$ induced on the smooth locus $\mathscr{M}'$ of $\mathscr{M}$. Contrary to the assumption adopted in [3], the quotient is not Kähler, but one can rescale g_2 so that σ has constant norm and we are dealing with an *almost Kähler structure*. There is a residual diagonal action of $SO(3)$ on $\mathscr{M}'$ that preserves the tensors $g_2, \mathbb{J}, \sigma$. The singular nature of the quotient makes our initial formulae complicated, as they involve radii functions that are not smooth across $\mathbb{R}^3 \cup \mathbb{R}^3$. Part of our task is to find coordinates on $\mathbb{R}^6$, or subvarieties thereof, that are better adapted to σ and g_2.

Glossary of notation		
tensor	defined on/in	description
e	$\mathbb{R}^8$	Euclidean metric
R		Euclidean norm squared
X		Killing vector field
$\widehat{h}_1$	$\mathbb{CP}^3$	Kähler metric
$\widehat{h}_2$		nearly-Kähler metric
ω		nearly-Kähler 2-form
Υ		nearly-Kähler $(3,0)$-form
h_2	$\mathscr{C}$	G_2 metric
h_c		more general conical metric
h_{BS}		complete G_2 metric
φ		G_2 3-form
$*\varphi$		G_2 4-form
Θ_c		connection 1-form
g_c	$\mathscr{M}$	metric induced from h_c
$\widehat{g}_c$		restriction of g_c to $R=1$
$F_c = d\Theta_c$		curvature 2-form
σ		symplectic 2-form
$\mathbb{J}$		almost complex structure
$\Psi = \psi^+ + i\psi^-$		$(3,0)$-form
$\mathscr{F}_+, \mathscr{F}_-$		$SO(3)$-invariant subvarieties
$\mathscr{M}(\mathbf{n})$		$\mathbb{J}$-holomorphic subvarieties

We introduce the Gibbons-Hawking ansatz in Section 2, and apply it to a baby model of a circle quotient of a G_2 structure. To analyse our curved example, we pull h_2 and other tensors back (via ρ) to $\mathbb{H}^2 \cong \mathbb{R}^8$ in Section 3, and exploit the ambient hyperkähler structure. We consider commuting circle actions in Section 4, and describe the G_2 structure on $\mathscr{C}$ purely in terms of Euclidean coordinates on $\mathbb{R}^8$. This is motivated by work of the first author [1], and we hope to use our methods subsequently to understand circle actions with different weights on $\mathbb{R}^8$. Using the 2-torus action on $\mathbb{R}^8$ and the map Q, we identify the metric g_1 on $\mathscr{M}$ arising from the Fubini-Study metric of $\mathbb{CP}^3$, and the more complicated metric g_2 induced from the G_2 structure of $\mathscr{C}$ (Theorems 4.7 and 4.10).

The diagonal action of $SO(3)$ on $\mathscr{M} \subset \mathbb{R}^6$ enables us to use the bivector formalism to describe invariant tensors in Section 5. We show that the curvature 2-forms F_1, F_2, are determined by their restrictions to $S^2 \times S^2$ (Theorem 5.5). In Section 6, we focus on two 4-dimensional $SO(3)$-invariant submanifolds $\mathscr{F}_\pm$ in $\mathscr{M}$ such that $\mathscr{F}_+$ projects to ∂D^3, while the circle fibres of Q are horizontal over $\mathscr{F}_-$. As an application, we use the twistor fibration to describe a foliation of $\mathscr{C}$ by coassociative submanifolds discovered by Karigiannis and Lotay [22] (Theorem 6.6).

In Section 7, we show that Darboux coordinates for σ can be expressed remarkably simply in terms of the bivector $(\mathbf{u}, \mathbf{v})$, though this result (Theorem 7.2) was by no means obvious. It contrasts with the difficulty in describing the almost complex structure $\mathbb{J}$, though we compute a compatible $(3,0)$-form on $\mathscr{M}$ and verify that $\mathbb{J}$ is non-integrable. The nature of this 3-form leads us to exhibit a family of 4-dimensional pseudo-holomorphic linear subvarieties $\mathscr{M}(\mathbf{n})$ parametrized by $\mathbb{RP}^2$ that exhaust $\mathscr{M}$.

We investigate the metrics g_1 and g_2 in Section 8, and distinguish subvarieties on which they are flat. In particular, we determine their restriction to $\mathscr{F}_+$ and $\mathscr{F}_-$ (Theorem 8.3), and highlight geometrical aspects that 'ignore' the singularities of $\mathscr{M}$.

Acknowledgments. The authors are supported by the Simons Collaboration on Special Holonomy in Geometry, Analysis, and Physics (#488569 Bobby Acharya, #347349 Robert Bryant, #488635 Simon Salamon). They are grateful to Spiro Karigiannis and Jason Lotay for sharing some results from [22] discussed in Section 6. The third author acknowledges useful exchanges of ideas with Benjamin Aslan, Udhav Fowdar, Wendelin Lutz, and Corvin Paul.

2. Preliminaries

This section serves both to motivate the more technical work that follows, and to introduce the Gibbons-Hawking ansatz in a simple G_2 context.

In the study of Ricci-flat metrics with special holonomy, there is a well-established connection between structures defined by the Lie groups $SU(3)$, G_2 and $Spin(7)$ in dimensions 6, 7 and 8. Hand in hand with the condition of *reduced holonomy* is that of *weak holonomy*; the former is characterized by the existence of a non-zero *parallel* spinor, the latter by a *Killing* spinor. If an n-dimensional manifold M^n (with $n = 5, 6, 7$) has a Riemannian metric g with weak holonomy, then the cone $dr^2 + r^2 g$ has reduced holonomy on $M^n \times \mathbb{R}^+$ [5], and the sine-cone $dr^2 + (\sin r)^2 g$ has weak holonomy on $M^n \times (0, \pi)$ [2]. Examples of such metrics permeate this paper, though our focus will be on quotienting a 7-dimensional manifold by a circle action.

In this paper, we restrict attention to G_2 holonomy in seven dimensions, and $SU(3)$ structures (invariably without reduced holonomy) in six dimensions. A G_2 structure on a

7-manifold M is determined by a 'positive' non-degenerate 3-form φ that satisfies

$$d\varphi = 0 \quad \text{and} \quad d{*}\varphi = 0.$$

Here $*$ is Hodge star for the Riemannian metric h_2 uniquely determined by (i) the formula

$$h_2(X,Y)\upsilon = \tfrac{1}{6}(X \lrcorner \varphi) \wedge (Y \lrcorner \varphi) \wedge \varphi,$$

and (ii) the condition that υ be the volume form of h_2 with an appropriate orientation choice [8]. It is then the case that $\nabla\varphi = 0$, where ∇ is the Levi-Civita connection for h_2.

An analogous description of $SU(3)$ ('Calabi-Yau') holonomy consists of a symplectic 2-form σ and a complex closed 3-form $\Psi = \psi^+ + i\psi^-$ satisfying $\Psi \wedge \sigma = 0$ and

$$-i\,\Psi \wedge \overline{\Psi} = \tfrac{4}{3}\sigma^3$$

in the notation of [10]. The real and imaginary components of Ψ must be *stable* in the sense that the stabilizer of either in $GL(6,\mathbb{R})$ is conjugate to $SL(3,\mathbb{C})$, in order that ψ^+ determine an almost complex structure J and $\psi^- = J\psi^+$ [21]. The 2-form σ will necessarily have type $(1,1)$ relative to J, but we also require that the non-degenerate bilinear form $g_2 = \sigma(J\cdot,\cdot)$ be positive definite.

More general $SU(3)$ structures are defined by merely relaxing the closure conditions on σ and/or $\psi^\pm$. When an $SU(3)$ structure arises as a hypersurface of a manifold with holonomy G_2, its torsion $\mathfrak{T}$ 'loses' half of its 42 components. For such an embedding $M^6 \hookrightarrow M^7$, one defines

$$\left\{\begin{array}{rcl} \psi^+ &=& i^*\varphi,\\ \frac{1}{2}\sigma^2 &=& i^*(*\varphi), \end{array}\right.$$

and it is the closure of these two differential forms is the half-flat condition.

Now consider the quotient situation. Suppose (M^7,φ) has holonomy in G_2 and that $U(1)$ acts freely on M^7 with associated Killing vector field X. Then $\mathscr{L}_X\varphi = 0$ and

$$\sigma = X \lrcorner \varphi$$

is closed. Let l be the positive function defined by

$$l^{-4} = h_2(X,X), \tag{2.1}$$

so that $l^{-2} = \|X\|$ measures the size of the $U(1)$ fibres. Let $\theta = l^4 X \lrcorner h_2$ so that $X \lrcorner \theta = 1$. Following [3] (where t corresponds to l^2 and the signs of $\psi^\pm$ are swapped), one can write

$$\begin{array}{rcl} \varphi &=& \theta \wedge \sigma + l^3\psi^-,\\ *\varphi &=& \theta \wedge (l\,\psi^+) + \frac{1}{2}(l^2\sigma)^2. \end{array}$$

The 1-form $i\theta$ defines a connection on the $U(1)$ bundle, and $F = d\theta$ equals ($-i$ times) its curvature. The latter is constrained by the residual torsion:

Lemma 2.1. *The differential forms σ and $\Psi = \psi^+ + i\psi^-$ define an $SU(3)$ structure on $M^7/U(1)$ with $d\sigma = 0$ and $d(l\psi^+) = 0$. Moreover,*

$$\begin{array}{rcl} F \wedge \sigma &=& -d(l^3\psi^-)\\ F \wedge \psi^+ &=& -2l^2\,dl \wedge \sigma^2. \end{array}$$

Proof. The required algebraic properties of the exterior forms follow from the well-known linear algebra linking $SU(3)$ and G_2 structures, so we confine ourselves to understanding the exponents of l in the expressions for φ and $*\varphi$ above the lemma. We want the four terms to have constant norm relative to h_2 on M^7. This implies that $l^2\sigma$ should have constant norm, since $\|l^{-2}\theta\| = 1$, and the 3-form ψ^- is scaled by $(l^2)^{3/2} = l^3$ for consistency. For the same reason, $(l^{-2}\theta)(l^3\psi^+)$ has constant norm.

Since $X \lrcorner *\varphi = l\psi^+$, the latter is indeed closed. The equations involving F follow immediately by differentiation. □

Remark 2.2. One is free to scale the metric induced on $M^7/U(1)$ by any function of l, and the property $d\sigma = 0$ characterizes the choice of an *almost Kähler* metric. However, the metric g_2 for which

$$(M^7, h_2) \longrightarrow (M^7/U(1), g_2)$$

is a Riemannian submersion corresponds to the re-scaled $SU(3)$ structure $(l^2\sigma,\, l^3\Psi)$. The almost complex structure J is integrable if and only if $d(l\psi^-) = 0$, and in this case it was shown in [3] that (i) the Ricci form of the Kähler metric equals $i\partial\overline{\partial}\log s$, and (ii) a new Killing vector field U is defined by $U \lrcorner\, \sigma = -d(l^2)$, and one can further quotient to 4 dimensions. Other reductions leading to triples of 2-forms and Monge-Ampère equations can be imposed with extra symmetry [13].

We shall rely repeatedly on the first part of Lemma 2.1 in the sequel, though the function $N = l^{-4}$ will be more relevant computationally, and we shall only use the symbol l in this section. We conclude it by applying the theory above to a $U(1)$ quotient of the flat G_2 structure on $\mathbb{R}^7$, specified by means of the constant 3-form

$$\varphi = dx_{014} - dx_{234} + dx_{025} - dx_{315} + dx_{036} - dx_{126} + dx_{456}, \tag{2.2}$$

using coordinates x_i with $0 \leqslant i \leqslant 6$. This defines an inclusion $G_2 \subset SO(7)$, for which the orthogonal group fixes the Euclidean metric $e = \sum_{i=0}^{6} dx_i^2$. We further distinguish the subspace $\mathbb{R}^4 = \mathbb{R}^4_{0123}$, and consider the action of $U(1)$ on this subspace giving rise to the Killing vector field

$$X = -x_1\partial_0 + x_0\partial_1 - x_3\partial_2 + x_2\partial_3,$$

where $\partial_0 = \partial/\partial x_0$ etc. There is an associated 1-form

$$\xi = X^\flat = X \lrcorner\, e = -x_1 dx_0 + x_0 dx_1 - x_3 dx_2 + x_2 dx_3,$$

and

$$u_0 = X \lrcorner\, \xi = \sum_{i=0}^{3} x_i^2$$

is the norm squared of both X and ξ. If we set $\theta = \xi/u_0$, then $i\theta$ is a connection form for the smooth circle bundle over $\mathbb{R}^4 \setminus \mathbf{0}$.

We next identify $\mathbb{R}^4 \cong \mathbb{H}$ by means of the quaternionic coordinate

$$q = x_0 + x_1 i + x_2 j + x_3 k.$$

This defines a hyperkähler structure on $\mathbb{R}^4_{0123}$, relative to which the action of $U(1)$ is triholomorphic. The associated moment mapping is

(2.3) $$q \longmapsto \overline{q}\,i\,q = u_1 i - u_3 j + u_2 k,$$

where (to suit the authors' conventions, cf. (5.1))

$$\begin{cases} u_1 &= x_0^2 + x_1^2 - x_2^2 - x_3^2 \\ u_2 &= 2(x_0x_2 + x_1x_3) \\ u_3 &= 2(x_0x_3 - x_1x_2). \end{cases}$$

It is invariant by the $U(1)$ action $q \mapsto e^{i\theta}q$, and defines a homeomorphism $\mathbb{R}^4/U(1) \cong \mathbb{R}^3$.

The hyperkähler structure is specified by the anti-self-dual (ASD) 2-forms

$$dx_{01} - dx_{23}, \quad dx_{02} - dx_{31}, \quad dx_{03} - dx_{12},$$

where dx_{ij} is shorthand for $x_i \wedge dx_j$. The curvature can be expressed in terms of this basis and the self-dual curvature form $d\xi = 2(dx_{01} + dx_{23})$:

$$d\theta = \frac{1}{u_0} d\xi - \frac{1}{u_0^2}(u_1\omega_1 - u_3\omega_2 + u_2\omega_3).$$

The u_i provide a smooth structure on $\mathbb{R}^4$, and we can express $U(1)$ invariant quantities in terms of these coordinates. In particular,

$$u_0^2 = u_1^2 + u_2^2 + u_3^2,$$

so that u_0 is the radius and u_0^{-1} is harmonic in the u_i coordinates. The Euclidean metric on $\mathbb{R}^4$ can then be recovered by means of the Gibbons-Hawking ansatz; it is

$$u_0\theta^2 + \tfrac{1}{4}u_0^{-1}\sum_{i=1}^{3} du_i^2.$$

The second summand on the right equals the metric induced submersively on the quotient $\mathbb{R}^3$. The factor of 1/4 (and 1/2 in the lemma below) could be eliminated by halving the coordinates u_i, but that would be inconvenient later.

Lemma 2.3.

$$d\theta = \tfrac{1}{2}u_0^{-3}\big(u_1\,du_2 \wedge du_3 + u_2\,du_3 \wedge du_1 + u_3\,du_1 \wedge du_2\big).$$

This result is well known (up to the factor of 1/2), since the right-hand side equals

$$\tfrac{1}{2}u_0^{-3} * (u_1du_1 + u_2du_2 + u_3du_3) = -\tfrac{1}{2} * d(u_0^{-1}).$$

In the sequel, we shall denote this expression by $\frac{1}{4}u_0^{-3}\{\mathbf{u}, d\mathbf{u}, d\mathbf{u}\}$ (an extra factor of 2 converts the 1/4 into 1/2, see Notation 5.3). Such triple products will be used to express various tensors. If we restrict to the 2-sphere $u_0 = 1$ and adopt spherical coordinates ϕ (latitude) and θ (longitude), then

$$-2\,d\theta = \cos\phi\ d\theta \wedge d\phi$$

is the area 2-form, whose integral equals 4π. It follows that the circle bundle has first Chern class $\mathbf{c}_1 = -1$ over S^2; it is the Hopf bundle. For contrasting applications of the Gibbons-Hawking ansatz in four dimensions, see [26, 12].

Thus far, we have dealt only with 4-dimensional geometry. Given that $U(1)$ acts trivially on $\mathbb{R}^3_{456}$, the quotient of $\mathbb{R}^7$ is

$$\frac{\mathbb{R}^4}{U(1)} \times \mathbb{R}^3 \;\cong\; \mathbb{R}^6,$$

with coordinates $(u_1, u_2, u_3; x_4, x_5, x_6)$. We we can now identify the structure induced from the 3-form (2.2):

Proposition 2.4. *The quotient $\mathbb{R}^6$ has an induced $SU(3)$ structure with $u_0 = l^{-4}$,*

$$\begin{aligned} \sigma &= -\tfrac{1}{2}(du_1 \wedge dx_4 - du_3 \wedge dx_5 + du_2 \wedge dx_6), \\ \Psi &= -(\tfrac{1}{2}l\,du_1 + il^{-1}dx_4) \wedge (-\tfrac{1}{2}l\,du_3 + il^{-1}dx_5) \wedge (\tfrac{1}{2}l\,du_2 + il^{-1}dx_6). \end{aligned}$$

Proof. The $SU(3)$ structure is completely determined by σ and ψ^+. The first equation follows from the definition $\sigma = X \lrcorner\, \varphi$. In accordance with (2.3),

$$\begin{aligned} l\psi^+ &= X \lrcorner *\varphi \\ &= \tfrac{1}{2}(du_1 \wedge dx_{56} - du_3 \wedge dx_{64} + du_2 \wedge dx_{45}) - x_0 dx_{023} - x_1 dx_{123} - x_2 dx_{012} - x_3 dx_{013} \\ &= \tfrac{1}{2}(du_1 \wedge dx_{56} - du_3 \wedge dx_{64} + du_2 \wedge dx_{45}) - \tfrac{1}{8}l^4 du_{123}. \end{aligned}$$

The last line equals l times the real part of the simple 3-form Ψ specified by the proposition. Since the stable form ψ^+ determines J, it follows that $\Psi = \psi^+ + i\psi^-$ is a $(3,0)$ form compatible with σ. □

Although the starting metric e is flat, the circle bundle is not, and the torsion $\mathfrak{T}$ of the $SU(3)$ structure above is determined by $d(l\psi^-) \neq 0$. This confirms that the quotient is not Kähler, but provides results that are entirely consistent with Lemma 2.1. We purposely chose a circle subgroup that acts trivially on $\mathbb{R}^3$, though other $SU(3)$ structures can be defined by a different choice of

$$U(1) \subset SO(4) \subset G_2$$

acting on $\mathbb{R}^7 \cong \mathbb{R}^4 \oplus \Lambda^2_-(\mathbb{R}^4)$. Proposition 2.4 exhibits the simplest model for quotients of metrics with holonomy G_2, including the one on $\mathscr{C}$ that is the main focus of this paper. The equations to set up the quotient are identical, but we shall inevitably struggle to find such simple expressions for the induced differential forms in a non-flat situation.

3. Metrics with holonomy G_2

At this point, we need to refresh notation, and establish our choice of real, complex and quaternionic coordinates on

$$\mathbb{R}^8 = \mathbb{C}^4 = \mathbb{H}^2$$

that will persist for the remainder of the paper. We shall consider $\mathbb{R}^8$ as a module for the Lie group $Sp(2)Sp(1)$, with the group $Sp(2)$ of quaternionic matrices acting by *left* multiplication, and the group $Sp(1)$ of unit quaternion scalars acting on the *right*. We set

$$\begin{aligned} q_0 &= x_0 + x_1 i + x_2 j + x_3 k &= z_0 + j z_1, \\ q_1 &= x_4 + x_5 i + x_6 j + x_7 k &= z_2 + j z_3, \end{aligned} \tag{3.1}$$

so that

$$z_0 = x_0 + i x_1, \quad z_1 = x_2 - i x_3, \quad z_2 = x_4 + i x_5, \quad z_3 = x_6 - i x_7. \tag{3.2}$$

Then z_i and q_j become homogeneous coordinates for $\mathbb{CP}^3$ and $\mathbb{HP}^1$ respectively, consistent with the choice of *right* multiplication by $\mathbb{H}^*$.

The twistor projection $\pi\colon \mathbb{CP}^3 \to \mathbb{HP}^1$ is represented by

$$[z_0, z_1, z_2, z_3] \mapsto [q_0, q_1] = [1,\ q_1 q_0^{-1}],$$

in which the point at infinity is defined by $q_0 = 0$. Away from this point,

$$\begin{aligned} q_1 q_0^{-1} \;=\; \frac{1}{|q_0|^2} q_1 \overline{q_0} &= \frac{1}{|z_0|^2 + |z_1|^2} (z_2 + j z_3)(\overline{z}_0 - j z_1) \\ &= \frac{1}{|z_0|^2 + |z_1|^2} \Big(z_2 \overline{z}_0 + \overline{z}_3 z_1 + (z_0 \overline{z}_3 - \overline{z}_1 z_2) j \Big). \end{aligned} \tag{3.3}$$

This convention will determine the chirality of the Killing vector fields defined below.

We denote by

$$R = \sum_{i=0}^{7} x_i^2$$

the radius squared for the Euclidean metric

$$e = \sum_{i=0}^{7} dx_i \otimes dx_i.$$

The vector fields $\partial_0 = \partial/\partial x_1, \ldots, \partial_7 = \partial/\partial x_7$ constitute an orthonormal basis for e.

The right action of $Sp(1)$ determines Killing vector fields

$$\begin{aligned} Y_1 &= -x_1\partial_0 + x_0\partial_1 + x_3\partial_2 - x_2\partial_3 - x_5\partial_4 + x_4\partial_5 + x_7\partial_6 - x_6\partial_7 \\ Y_2 &= -x_2\partial_0 + x_0\partial_2 + x_1\partial_3 - x_3\partial_1 - x_6\partial_4 + x_4\partial_6 + x_5\partial_7 - x_7\partial_5 \\ Y_3 &= -x_3\partial_0 + x_0\partial_3 + x_2\partial_1 - x_1\partial_2 - x_7\partial_4 + x_4\partial_7 + x_6\partial_5 - x_5\partial_6, \end{aligned}$$

tangent to the fibres of the principal bundle

$$S^7 = \frac{Sp(2)}{Sp(1)} \longrightarrow \frac{Sp(2)}{Sp(1) \times Sp(1)} = S^4.$$

With the conventions below, we shall identify the associated rank 3 vector bundle with the bundle $\Lambda^2_- T^* S^4$ of *anti-self-dual* 2-forms.

Consider the 1-forms $\alpha_i = Y_i \lrcorner\, g$, such as

$$\alpha_1 = -x_1 dx_0 + x_0 dx_1 + x_3 dx_2 - x_2 dx_3 - x_5 dx_4 + x_4 dx_5 + x_7 dx_6 - x_6 dx_7.$$

Observe that

$$\tfrac{1}{2}d\alpha_1 = dx_{01} - dx_{23} + dx_{45} - dx_{67}$$

is the sum of two ASD 2-forms on separate $\mathbb{R}^4$'s. Now consider the action by the group $\mathbb{R}^+$ of positive real scalars on $\mathbb{R}^8$, so that $x_i \mapsto \lambda x_i$ for $\lambda \in \mathbb{R}^+$. The 1-forms

$$\hat{\alpha}_i = \frac{\alpha_i}{R}$$

are invariant by this action, and (by interpreting Y_i as elements of $\mathfrak{so}(3)$)

$$\vartheta = \sum_{i=1}^{3} \hat{\alpha}_i \otimes Y_i$$

is a connection form on the principal $\mathbb{H}^*$ bundle. Its curvature equals

$$\begin{aligned} d\vartheta + [\vartheta, \vartheta] &= (d\hat{\alpha}_1 + 2\hat{\alpha}_2 \wedge \hat{\alpha}_3) \otimes Y_i + \cdots \\ &= \sum_{i=1}^{3} \tau_i \otimes Y_i, \end{aligned}$$

where

$$\begin{cases} \tau_1 = d\hat{\alpha}_1 + 2\hat{\alpha}_{23} \\ \tau_2 = d\hat{\alpha}_2 + 2\hat{\alpha}_{31} \\ \tau_3 = d\hat{\alpha}_3 + 2\hat{\alpha}_{12} \end{cases}$$

We can check that the coefficients and signs are correct by verifying that

$$Y_i \lrcorner\, \tau_j = 0 \qquad i, j = 1, 2, 3,$$

which follows from equations such as $Y_i \lrcorner\, \hat{\alpha}_i = 1$ and $Y_1 \lrcorner\, d\hat{\alpha}_2 = 2\hat{\alpha}_3$. This implies that the τ_i are semi-basic over S^4. Bearing in mind that

$$\tau_i \wedge \tau_i \wedge \alpha_{123} \wedge dR = -16\, dx_{01234567},$$

we shall shoose orientations on S^7 and S^4 so that $\{\tau_1, \tau_2, \tau_3\}$ a basis of *anti*-self-dual 2-forms over S^4.

Fix $i = 1$ and consider the subgroup $U(1)_1$ generated by Y_1. This will be our choice for defining the complex projective space

$$\mathbb{CP}^3 = \frac{S^7}{U(1)_1}.$$

The 1-form $\hat{\alpha}_1$ determines a connection on the $U(1)_1$ bundle $S^7 \to \mathbb{CP}^3$, with curvature 2-form proportional to $d\hat{\alpha}_1$. The rescaling of α_1 ensures that

$$Y_1 \lrcorner\, d\hat{\alpha}_1 = \mathscr{L}_{Y_1}\hat{\alpha}_1 - d(X_1 \lrcorner\, \hat{\alpha}_1) = 0,$$

so that $d\hat{\alpha}_1$ passes to $\mathbb{CP}^3$. It is well known that $d\hat{\alpha}_1$ is the Kähler form for (a suitably normalized) Fubini-Study metric on $\mathbb{CP}^3$. We denote the standard integrable complex structure on $\mathbb{CP}^3$ by J_1.

The nearly-Kähler structure of $\mathbb{CP}^3$ is compatible with the non-integrable almost complex structure J_2, obtained from J_1 by reversing sign on the twistor fibres [14]. We shall denote the $SU(3)$ structure on $(\mathbb{CP}^3, J_2)$ by a 2-form ω and a $(3,0)$-form Υ, neither of which are closed. (We shall reserve σ and Ψ to describe the $SU(3)$ structure of the quotient $\mathbb{R}^+ \times \mathbb{CP}^3/SO(2)$, see the Glossary of Notation on page 3.)

Lemma 3.1.

$$\begin{aligned} \omega &= \hat{\alpha}_{23} + \tau_1 = d\hat{\alpha}_1 + 3\hat{\alpha}_{23} \\ \Upsilon &= (\hat{\alpha}_2 + i\hat{\alpha}_3) \wedge (\tau_2 - i\tau_3) \end{aligned}$$

Proof. Since

$$d\hat{\alpha}_1 = \tau_1 - 2\hat{\alpha}_{23}$$

is Kähler form on $(\mathbb{CP}^3, J_1)$, we must have $\omega = \tau_1 + \lambda\hat{\alpha}_{23}$ for some $\lambda > 0$. Using the fact that τ_i^2 defines the same volume form on S^4 independent of i, one verifies that the nearly-Kähler identities, namely

$$d\omega = 3\,\mathrm{Im}\,\Upsilon \quad \text{and} \quad d\Upsilon = 2\omega^2,$$

hold for $\lambda = 1$. The formula for the $(3,0)$-form Υ follows from the fact that $J_2 = -J_1$ when restricted to the fibres over S^4. □

If a 6-manifold M carries a nearly-Kähler metric then the conical metric on $\mathbb{R}^+ \times M$ has holonomy contained in G_2. In particular, when M is the twistor space $(\mathbb{CP}^3, J_2)$, with isometry group $SO(5)$, the conical metric has holonomy *equal* to G_2 [9]. We choose as radial parameter the Euclidean norm squared

$$R = \|\mathbf{q}\|^2 = |q_0|^2 + |q_1|^2;$$

this will ensure that $Q \circ \rho$ arises from a quadratic map in the commutative Figure 1.

Proposition 3.2. *The conical G_2 structure on $\mathscr{C}$ is characterized by the exact forms*

$$\varphi = d(\tfrac{1}{3}R^3\omega) \quad \textit{and} \quad *\varphi = d(\tfrac{1}{4}R^4\,\mathrm{Re}\,\Upsilon).$$

Proof. Each 1-form on $\mathbb{CP}^3$ is weighted with the radial parameter R with respect to the conical metric, so

$$\begin{aligned} \varphi &= dR \wedge R^2\omega + R^3\,\mathrm{Im}\,\Upsilon, \\ *\varphi &= dR \wedge R^3\,\mathrm{Re}\,\Upsilon + \tfrac{1}{2}(R^2\omega)^2. \end{aligned}$$

The formulae follow. □

Let $\widehat{h}_1$ denote the Kähler metric of $(\mathbb{CP}^3, J_1)$ corresponding to the 2-form $d\hat{\alpha}_1$. Let $\widehat{h}_2$ denote the nearly-Kähler metric of $(\mathbb{CP}^3, J_2)$ determined by Lemma 3.1. The next result describes the pullbacks of the conical metrics

$$h_c = dR^2 + R^2\widehat{h}_c, \qquad c = 1, 2, \tag{3.4}$$

to $\mathbb{R}^8$, in term of the Euclidean metric e and the 1-forms α_i defined previously. We are mainly interested in h_2 since this has holonomy G_2, but related metrics will be useful for comparison purposes.

Proposition 3.3. *The metrics h_1 and h_2 on $\mathscr{C}$ belong to the one-parameter family of bilinear forms*

$$h_c = \tfrac{1}{2}dR^2 + 2Re - 2\alpha_1^2 + (1-c)\big(\alpha_2^2+\alpha_3^2\big). \tag{3.5}$$

These forms define Riemannian metrics on $\mathscr{C}$ provided $c<3$.

Proof. The Euclidean metric is given by

$$e = dr^2 + r^2 s_7,$$

where $R=r^2$,

$$2s_7 = \pi^* s_4 + 2\sum_{i=1}^{3}\hat\alpha_i^2,$$

and s_7, s_4 are standard metrics on S^7, S^4. Starting from e, we can define s_7 and s_4 by these formulae and verify that $Y_i \lrcorner\, s_4 = 0$ for $i=1,2,3$. We have inserted a '2' in the definition of s_7 to match the Kähler metric

$$2(s_7 - \hat\alpha_1^2) = \pi^* s_4 + 2(\hat\alpha_2^2+\hat\alpha_3^2)$$

on $\mathbb{CP}^3$ corresponding to the 2-form $d\hat\alpha_1 = \tau_1 + 2\hat\alpha_{23}$ used in the proof of Lemma 3.1.

The bilinear from defined by the proposition can now be expressed as

$$h_c = dR^2 + R^2\pi^* s_4 + (3-c)\big(\alpha_2^2+\alpha_3^2\big). \tag{3.6}$$

When $c=2$, this matches the metric $\pi^* s_4 + \hat\alpha_2^2 + \hat\alpha_3^2$ inherent in Lemma 3.1. The bilinear form h_c contains Y_1 in its kernel since $\alpha_1 = Y_1 \lrcorner\, e$ and Y_1 has zero contraction with dR, α_2, α_3. Note that

$$h_3 = dR^2 + R^2\pi^* s_4, \tag{3.7}$$

but if $c<3$ then h_c has rank 7 and is positive definite on the quotient $\mathbb{R}^8/\langle Y_1\rangle$. $\square$

Remark 3.4. The underlying family of metrics on $\mathbb{CP}^3$ described in the proof is well known: it arises from Riemannian submersions $\mathbb{CP}^3 \to S^4$ by varying the scaling on the fibres. The cases $c=1$ and $c=2$ correspond to the two $Sp(2)$-invariant Einstein metrics on $\mathbb{CP}^3$ [6, 32]. In the former case, the conical metric can be simply expressed as

$$h_1 = dR^2 + 2R^2(s_7 - \hat\alpha_1^2),$$

and has six equal eigenvalues. The novelty in our approach consists of expressing everything in Euclidean terms on $\mathbb{R}^8$, which will have computational advantages.

The previous proof related expressions for metrics in Euclidean coordinates on $\mathbb{R}^8$ to those arising from Riemannian submersions from $\mathbb{CP}^3$ to S^4. We can apply the same technique to the differential forms defining G_2 structures. Consider first the 2-form

$$\tau_0 = dR\wedge\alpha_1 - \alpha_2\wedge\alpha_3$$

defined on $\mathbb{R}^8$. In contrast to τ_1,τ_2,τ_3, its restriction to the fibres of $\mathbb{H}^2\setminus\mathbf{0}\to\mathbb{HP}^1$ is non-degenerate. Since

$$\tfrac{1}{3}R^3\omega + R\tau_0 = \tfrac{1}{3}R^3 d\hat\alpha_1 + R\,dR\wedge\alpha_1 = d(\tfrac{1}{3}R^3\hat\alpha_1)$$

is exact, it follows from Proposition 3.2 that the 2-form $-R\tau_0$ is an alternative primitive for φ. Moreover,

$$\begin{aligned} d\tau_0 &= d(RdR\wedge\hat\alpha_1 - R^2\hat\alpha_2\wedge\hat\alpha_3) \\ &= -RdR\wedge d\hat\alpha_1 - 2RdR\wedge\hat\alpha_{23} - R^2(\tau_2\wedge\hat\alpha_3 - \hat\alpha_2\wedge\tau_3) \\ &= (-RdR)\wedge\tau_1 + (-R\alpha_3)\wedge\tau_2 + (R\alpha_2)\wedge\tau_3. \end{aligned} \tag{3.8}$$

Denote this form by γ_2 for consistency with [9, page 842], where (with different notation) it is paired with the simple 3-form

$$\gamma_1 = (-RdR)\wedge(-R\alpha_3)\wedge(R\alpha_2) = -R^3 dR\wedge\alpha_{23}.$$

At any given point of $\mathscr{C}$, the 3-form α_{123} generates the cotangent space to the S^3 fibres of $S^7\to S^4$, whereas γ_1 generates the $\mathbb{R}^3$ fibres.

We have

$$-\varphi = d(R\tau_0) = R^{-3}\gamma_1 + R\gamma_2, \tag{3.9}$$

which matches [9, Case iii, page 844] with $r=R^4$, $\kappa=1/2$, and an overall change of orientation. The description (3.9) enables us to modify φ when the conical metric h_2 is deformed to the complete 'Bryant-Salamon' metric. The latter has the effect of smoothing the vertex of the cone, which is replaced by the zero section S^4 in $\Lambda^2_- T^*S^4$. The following theorem is well known [9, 19], but serves to record the G_2 structure in a novel way:

Theorem 3.5. *The total space $\widetilde{\mathscr{C}}$ of $\Lambda^2_- T^*S^4$ admits a complete metric $h_{\rm BS}$ with holonomy equal to G_2, which (if the scalar curvature of S^4 equals $1/6$) equals*

$$h_{\rm BS} = (R^2+R^{-2})^{1/2}h_2 + (R^8+R^4)^{-1/2}(dR^2+\alpha_2^2+\alpha_3^2),$$

and is associated to the 3-form $\varphi_{\rm BS} = -d\big((R^4+1)^{1/4}\tau_0\big)$.

Proof. The assumption on the scalar curvature means $\kappa=1/2$, then [9, page 844] tells us that the 3-form associated to the complete G_2 metric equals

$$(R^4+1)^{-3/4}\gamma_1 + (R^4+1)^{1/4}\gamma_2,$$

which coincides with minus the 3-form $\varphi_{\rm BS}$ defined above. The metric $h_{\rm BS}$ can be gleaned from the proof of Proposition 3.3 with $c=2$. It is represented by

$$(R^4+1)^{-1/2}R^2(dR^2+\alpha_2^2+\alpha_3^2) + (R^4+1)^{1/2}\pi^*s_4,$$

relative to the fibration $\Lambda^2_- T^*S^4\to S^4$, and one can then express π^*s_4 in terms of h_2. □

4. A 2-torus action on $\mathbb{R}^8$

Left multiplication by $U(1)$ on $\mathbb{R}^8$ gives a Killing vector field

$$X = X_1 = -x_1\partial_0 + x_0\partial_1 - x_3\partial_2 + x_2\partial_3 - x_5\partial_4 + x_4\partial_5 - x_7\partial_6 + x_6\partial_7.$$

Given the sign changes in passing from Y_1 to X, we have

$$\begin{aligned} \tfrac12(X+Y_1) &= -x_1\partial_0 + x_0\partial_1 - x_5\partial_4 + x_4\partial_5, \\ \tfrac12(X-Y_1) &= -x_3\partial_2 + x_2\partial_3 - x_7\partial_6 + x_7\partial_6. \end{aligned}$$

so that these combinations define standard $U(1)$ actions on the two summands

$$\mathbb{R}^4_{0145} = \{(x_0, x_1, x_4, x_5)\}, \quad \mathbb{R}^4_{2367} = \{(x_2, x_3, x_6, x_7)\}.$$

We now consider the associated moment maps.

Although $\mathbb{R}^4_{0145}$ is not a quaternionic subspace of $\mathbb{H}^2$ as given, we can identify it with $\mathbb{H}$ by setting

$$q = x_0 + x_1 i + x_4 j + x_5 k.$$

This enables us to apply the analysis from Section 2, merely replacing the indices $2, 3$ by $4, 5$. The $U(1)$ action corresponding to $\frac{1}{2}(X + Y_1)$ is triholomorphic, and the moment mapping is

$$(x_0, x_1, x_4, x_5) \longmapsto (u_1, u_2, u_3),$$

where

$$\left\{ \begin{array}{lllll} u_1 & = & x_0^2 + x_1^2 - x_4^2 - x_5^2 & = & |z_0|^2 - |z_1|^2 \\ u_2 & = & 2(x_0 x_4 + x_1 x_5) & = & \mathrm{Re}(z_0 \overline{z}_2) \\ u_3 & = & 2(x_0 x_5 - x_1 x_4) & = & -\mathrm{Im}(z_0 \overline{z}_2), \end{array} \right. \tag{4.1}$$

We denote (u_1, u_2, u_3) by $\mathbf{u}$. There is a bijection

$$\mathbb{R}^4_{0145}/U(1) \cong \Lambda^2_-(\mathbb{R}^4_{0145}) \cong \mathbb{R}^3,$$

which is a diffeomorphism away from the respective origins. We set

$$u = x_0^2 + x_1^2 + x_4^2 + x_5^2,$$

so that

$$u^2 = \sum_{i=1}^{3} u_i^2 = |\mathbf{u}|^2.$$

The function u was denoted u_0 in Section 2; it will be convenient to omit the subscript since no confusion should arise in this printed document with vector $\mathbf{u} \in \mathbb{R}^3$.

Similarly, the hyperkähler moment map for $\frac{1}{2}(X - Y_1)$ on $\mathbb{R}^4_{2367}$ can be identified with (v_1, v_2, v_3), where

$$\left\{ \begin{array}{lllll} v_1 & = & x_2^2 + x_3^2 - x_6^2 - x_7^2 & = & |z_1|^2 - |z_3|^2 \\ v_2 & = & 2(x_2 x_6 + x_3 x_7) & = & \mathrm{Re}(z_1 \overline{z}_3) \\ v_3 & = & 2(x_2 x_7 - x_3 x_6) & = & -\mathrm{Im}(z_1 \overline{z}_3). \end{array} \right. \tag{4.2}$$

We also set $\mathbf{v} = (v_1, v_2, v_3)$ and $v = |\mathbf{v}|$, so that

$$u + v = \sum_{i=0}^{7} x_i^2 = R.$$

Our real 6-dimensional quotient space is

$$\mathscr{M} \;=\; \frac{\mathbb{CP}^3}{SO(2)} \times \mathbb{R}^+ \;\cong\; \frac{\mathbb{R}^8 \setminus \mathbf{0}}{U(1) \times U(1)} \;\cong\; \mathbb{R}^6 \setminus \mathbf{0},$$

and we shall work with the coordinates

$$(\mathbf{u}, \mathbf{v}) = (u_1, u_2, u_3; v_1, v_2, v_3)$$

on $\mathscr{M}$. Note that the 'radii' u and v are not everywhere smooth in these coordinates.

Remark 4.1. The appearance of complex conjugates in the moment maps is a consequence of the choice (3.2). In fact, $\mathscr{M}$ is ambidextrous: as a T^2 quotient of $\mathbb{R}^8$ the two $U(1)$ factors have equal status. Switching factors amounts to swapping X and Y_1, and changing the sign of $\mathbf{v}$. This is achieved by replacing z_1 by $-\overline{z}_3$ and z_3 by $\overline{z}_1$. However, it is the additional structure that we impose that breaks the symmetry. Referring to the definitions at the start of Section 3, we see that a change from *right* to *left* quaternionic multiplication will replace z_1, z_3 by their conjugates and change the sign of v_3 (but not v_1, v_2).

The interaction of X with the underlying quaternionic structure of $\mathbb{R}^8$ gives rise to functions $\mu_j = X \lrcorner\, \alpha_j$, explicitly

$$\left\{\begin{array}{rcl} \mu_1 &=& x_0^2 + x_1^2 - x_2^2 - x_3^2 + x_4^2 + x_5^2 - x_6^2 - x_7^2 \\ \mu_2 &=& 2(-x_0x_3 + x_1x_2 - x_4x_7 + x_5x_6) \\ \mu_3 &=& 2(x_0x_2 + x_3x_1 + x_4x_6 + x_5x_7). \end{array}\right. \tag{4.3}$$

These functions will be needed to define an appropriate $Sp(2)$ invariant connection on the $U(1)$ bundle Q. Observe that $\mu_1 = u - v$, whereas μ_2, μ_3 will be used (in Corollary 6.3) to characterize those points of $\mathscr{M}$ over which the circle fibres are *horizontal* in $\mathbb{CP}^3$.

Fix $c < 3$. The Riemannian metric h_c defined on $\mathscr{C}$ by Proposition 3.3 can be used to define a 1-form

$$\Theta_c = \frac{2}{N_c} X \lrcorner\, h_c, \tag{4.4}$$

where

$$N_c = h_c(X, X) = 2(R^2 - \mu_1^2) + \tfrac{1}{2}(1 - c)\big(\mu_2^2 + \mu_3^2\big). \tag{4.5}$$

By design, Θ_c annihilates the orthogonal complement of X and $X \lrcorner\, \Theta_c = 2$. Obviously, $\mathscr{L}_X\Theta_c = 0$, and so we also have $\mathscr{L}_X(d\Theta_c) = -d(X \lrcorner\, \Theta_c) = 0$. This confirms that $d\Theta_c$ passes to the quotient $\mathscr{M}$. We may regard Θ_c as the connection defined by the respective metric on the total space of Q, and $F = d\Theta_c$ as its curvature. Strictly speaking both should be multiplied by $i \in \mathfrak{u}(1)$, but will work with the real forms. These forms do not change when the metric is rescaled by a constant.

The factor '2' has been inserted in the definition of Θ_c to reflect the fact that $U(1)$ acts on $\mathbb{R}^8$, while $SO(2) = U(1)/\mathbb{Z}_2$ acts effectively on $\mathbb{CP}^3$ and S^4. (Up to now, we have blurred this distinction.) Left multiplication by e^{it} is only effective on $\mathbb{CP}^3$ for $t \in [0, \pi)$, and the connection 1-form is normalized so that the integral of Θ_c over each circle fibre of $\mathscr{C}$ equals 2π. This will be important in a subsequent discussion of Chern classes.

Returning to Proposition 3.3 and the definitions (4.4) and (4.5), we infer

Proposition 4.2. *Assuming $c < 3$, we have*

$$\begin{array}{rcl} N_c &=& 2(5 - c)uv + 2(1 - c)\mathbf{u} \cdot \mathbf{v} \\ N_c\Theta_c &=& 4RX \lrcorner\, e - 4\mu_1\alpha_1 + 2(1 - c)(\mu_2\alpha_2 + \mu_3\alpha_3). \end{array}$$

Recall that the Riemannian metric

$$h_c = dR^2 + R^2\widehat{h}_c$$

on the cone $\mathscr{C} = \mathbb{R}^+ \times \mathbb{CP}^3$ is defined for $c < 3$, and has holonomy G_2 when $c = 2$.

Definition 4.3. *The* pushdown *of h_c is the Riemannian metric g_c on $\mathscr{M}$ defined by setting*

$$h_c = Q^* g_c + \tfrac{1}{4} h_c(X, X)\,\Theta_c \otimes \Theta_c.$$

To obtain g_c one 'subtracts' the component of h_c tangent to the circle fibres, and

$$Q \colon (\mathscr{C}, h_2) \longrightarrow (\mathscr{M}, g_2)$$

is a Riemannian submersion on an open subset of its domain.

Note that g_c is $U(1)$-invariant and 'horizontal' in the sense that $X \lrcorner\, (Q^* g_c) = 0$. It follows that g_c can (for c fixed) be expressed as

$$\sum_{i=1}^{3} A_i du_i^2 + \sum_{i,j=1}^{3} B_{ij} du_i dv_j + \sum_{j=1}^{3} C_j dv_j^2, \tag{4.6}$$

where the coefficients are rational functions of $u_1, \ldots, v_3$. Our aim is to describe the metrics g_1 and g_2 in this way, but it is instructive first to write down the Euclidean metric e on $\mathbb{R}^8$ in terms of our T^2 quotient, using

Notation 4.4. Write

$$\alpha_i = \alpha_i' - \alpha_i'', \qquad X \lrcorner\, e = \alpha_i' + \alpha_i'',$$

where

$$\alpha_i' \in \Lambda^2(\mathbb{R}^4_{0145})^*, \quad \alpha_i'' \in \Lambda^2(\mathbb{R}^4_{2367})^*.$$

Observe that

$$X \lrcorner\, \alpha_1 = u - v = \mu_1,$$

whereas $e(X, X) = u + v = R$.

Lemma 4.5. *The Euclidean metric on $\mathbb{R}^8$ can be written*

$$e = R^{-1}\big(\alpha_1{}^2 + uv\Theta_1{}^2\big) + \tfrac{1}{4}\big(u^{-1}|d\mathbf{u}|^2 + v^{-1}|d\mathbf{v}|^2\big).$$

Proof. Using the expression for the Gibbons-Hawking metric just above Lemma 2.3 and Notation 4.4, we have

$$e = u^{-1}(\alpha_1')^2 + v^{-1}(\alpha_1'')^2 + \tfrac{1}{4}\big(u^{-1}|d\mathbf{u}|^2 + v^{-1}|d\mathbf{v}|^2\big).$$

We deduce that

$$X \lrcorner\, h_1 = 4(u\alpha_1'' + v\alpha_1'),$$

which implies that $\Theta_1 = u^{-1}\alpha_1' + v^{-1}\alpha_1''$ is the sum of two connection 1-forms of the type defined in Section 2 for the 4-dimensional case. The formula for e now follows. $\square$

Remark 4.6. The lemma provides a non-flat instance of the Gibbons-Hawking ansatz to supplement our treatment in Section 2. If we regard $\mathbb{R}^8=\mathbb{H}^2$ as a hyperkähler space defined by the *left* action of $Sp(1)$ then $\mathbf{u}-\mathbf{v}\colon\mathbb{R}^6\to\mathbb{R}^3$ is a moment mapping for the triholomorphic action of $U(1)_1$ generated by Y_1. It is well known that the resulting hyperkähler quotient

$$H(\mathbf{m})=\frac{\{\mathbf{x}\in\mathbb{R}^8:\mathbf{u}-\mathbf{v}=\mathbf{m}\}}{U(1)_1}$$

is an Eguchi-Hanson space, provided $\mathbf{m}\neq\mathbf{0}$. Its Ricci-flat metric can be obtained from Lemma 4.5 by subtracting the vertical term $R^{-1}{\alpha_1}^2$ and setting $d\mathbf{v}=-d\mathbf{u}$. It equals

$$e-R^{-1}{\alpha_1}^2=R^{-1}uv{\Theta_1}^2+\tfrac14(u^{-1}+v^{-1})|d\mathbf{u}|^2=V^{-1}{\Theta_1}^2+\tfrac14 V|d\mathbf{u}|^2,$$

where

$$V=u^{-1}+v^{-1}=\frac{1}{|\mathbf{u}|}+\frac{1}{|\mathbf{u}-\mathbf{m}|}.$$

By comparison with the formula just before Lemma 2.3, we see that this is a Gibbons-Hawking potential V defined by poles at $\mathbf{0}$ and $\mathbf{m}$ in $\mathbb{R}^3$.

In conclusion, we can say that each affine subvariety

$$H(\mathbf{m})/U(1)=\{(\mathbf{u},\mathbf{v}):\mathbf{u}-\mathbf{v}=\mathbf{m}\neq\mathbf{0}\}\subset\mathscr{M} \tag{4.7}$$

(with $\mathbf{m}$ fixed) is the base of an Eguchi-Hanson space. The corresponding subset of the (u,v)-quadrant is the semi-infinite rectangle

$$\{(u,v):-m\leqslant u-v\leqslant m\leqslant u+v\}\subset\mathbb{R}^2$$

where $m=|\mathbf{m}|$. The function $\mathbf{u}=(u_1,u_2,u_3)$ is a moment mapping for the action of $U(1)$ generated by X. In the description of $H(\mathbf{m})$ as the cotangent bundle of a 2-sphere S^2, $U(1)$ rotates S^2 and the poles of V are the fixed points of S^2 at the zero section of T^*S^2. For $m=0$, the space $H(\mathbf{0})$ can be identified with a cone over S^3 which the Hopf map projects to $\mathbb{S}^2$, the fixed point set of $U(1)$ acting S^4. This cone is resolved to T^*S^2 when $\mathscr{C}$ is deformed into $\widetilde{\mathscr{C}}$ (cf. Theorem 3.5).

Returning to Definition 4.3, the Kähler case $c=1$ is easy to describe, since the splitting $\mathbb{R}^8=\mathbb{R}^4_{0145}\oplus\mathbb{R}^4_{2367}$ is preserved:

Theorem 4.7.

$$g_1=\tfrac12 dR^2+\tfrac12 R\big(u^{-1}|d\mathbf{u}|^2+v^{-1}|d\mathbf{v}|^2\big),$$

where $|\mathbf{A}|^2$ *denotes* $\mathbf{A}\cdot\mathbf{A}$.

Proof. It now suffices from Proposition 3.3 to verify that

$$h_1(X,X){\Theta_1}^2=2R\big(u^{-1}(\alpha_1')^2+v^{-1}(\alpha_1'')^2\big)-2\alpha_1^2,$$

and this follows from equations in the proof of Lemma 4.5. □

The case $c = 2$ is more complicated, and some preliminary definitions will render the result more transparent. Working on $\mathscr{M}$ away from the locus $uv = 0$, we define vectors

$$\mathbf{A}_\pm = v\mathbf{u} \mp u\mathbf{v} \tag{4.8}$$

and scalars

$$a_\pm = uv \mp \mathbf{u}\cdot\mathbf{v}. \tag{4.9}$$

If we denote by 2θ the angle between $\mathbf{u}$ and $\mathbf{v}$ (for $0 \leqslant \theta \leqslant \pi/2$) so that $\mathbf{u}\cdot\mathbf{v} = uv\cos 2\theta$, then

$$\begin{array}{lcl} \mathbf{A}_+\cdot\mathbf{A}_+ = 2uv\,a_+ & \Rightarrow & |\mathbf{A}_+| = 2uv\sin\theta \\ \mathbf{A}_-\cdot\mathbf{A}_- = 2uv\,a_- & \Rightarrow & |\mathbf{A}_-| = 2uv\cos\theta, \end{array}$$

Moreover, $\mathbf{A}_+\cdot\mathbf{A}_- = 0$.

The next definition will be exploited repeatedly in the sequel:

Definition 4.8. *Set*

$$\mathscr{F}_\pm = \{(\mathbf{u},\mathbf{v}) \in \mathscr{M} : \mathbf{A}_\pm = \mathbf{0}\}.$$

Note that $\mathscr{F}_+ \cap \mathscr{F}_- = \{(\mathbf{u},\mathbf{v}) \in \mathscr{M} : uv = 0\}$, which (modulo the origin of $\mathbb{R}^6$) is $\mathbb{R}^3 \cup \mathbb{R}^3$. The formulae above make it clear that, away from their intersection, $\mathscr{F}_+$ (resp. $\mathscr{F}_-$) consists of points $(\mathbf{u},\mathbf{v})$ for which $\mathbf{u},\mathbf{v}$ are parallel and aligned (resp. anti-aligned). This explains our choice of opposing signs in the definitions (4.8) and (4.9).

For the purpose of describing g_2, we also define a vector-valued 1-form

$$\mathbf{B} = u\,d\mathbf{v} - v\,d\mathbf{u} \tag{4.10}$$

and scalar 1-forms

$$\Gamma_\pm = \frac{1}{uv}\mathbf{A}_\pm\cdot\mathbf{B},$$

so that $\Gamma_\pm$ vanishes on $\mathscr{F}_\pm$.

Lemma 4.9.

$$\begin{array}{rcccl} \Gamma_+ &=& u\,dv + v\,du - \mathbf{u}\cdot d\mathbf{v} - \mathbf{v}\cdot d\mathbf{u} &=& da_+ \\ \Gamma_- &=& u\,dv - v\,du + \mathbf{u}\cdot d\mathbf{v} - \mathbf{v}\cdot d\mathbf{u} &=& -2(\mu_2\alpha_3 - \mu_3\alpha_2). \end{array}$$

Proof. The 1-form $\mu_2\alpha_3 - \mu_3\alpha_2$ is initially defined on $\mathscr{C}$, but it is $U(1)$-invariant and has zero contraction with X, so passes to $\mathscr{M}$. The final equality now follows from a direct computation, whilst the others are more elementary. □

These definitions enable us to state and prove

Theorem 4.10.

$$g_2 = \tfrac{1}{2}dR^2 + \tfrac{1}{2}\left|d\mathbf{u} + d\mathbf{v}\right|^2 + \frac{2}{N}|\mathbf{B}|^2 + \frac{1}{2N}\Gamma_+^2 - \frac{1}{4N}\Gamma_-^2,$$

where $N = N_2 = 6uv - 2\mathbf{u}\cdot\mathbf{v}$.

Note that g_2 is, as its definition requires, well defined where $uv \neq 0$. The latter actually implies that $N_c > 0$ for $c < 3$; this follows from Proposition 4.2 since $5 - c > |1 - c|$.

Theorem 4.10 was originally derived by solving equations for A_i, B_{ij}, C_j in (4.6) with the help of Maple. We shall present a rigorous proof based on a formula for g_c which is less obviously non-singular:

Lemma 4.11. *The metric g_c induced on $\mathscr{M}$ by the conical metric h_c on $\mathscr{C}$ equals*

$$g_1 + (1-c)\left[\frac{1}{8a_-}{\Gamma_-}^2 + \frac{1}{uvN_c a_-}\{\mathbf{B}, \mathbf{u}, \mathbf{v}\}^2\right],$$

where curly brackets indicate triple product.

Proof. It will be convenient to define the 1-form $\eta = \mu_2\alpha_2 + \mu_3\alpha_3$ on $\mathscr{C}$ for the scope of the proof, so

$$N_c\Theta_c = 8uv\Theta_1 + 2(1-c)\eta.$$

A computation also gives $\mu_2^2 + \mu_3^2 = 2a_-$, so Lemma 4.9 implies

$$\begin{aligned} \eta^2 + \tfrac{1}{4}{\Gamma_-}^2 &= \eta^2 + (\mu_2\alpha_3 - \mu_3\alpha_2)^2 \\ &= (\mu_2^2 + \mu_3^2)(\alpha_2^2 + \alpha_3^2) \\ &= 2a_-(\alpha_2^2 + \alpha_3^2). \end{aligned} \tag{4.11}$$

Using Proposition 3.3,

$$\begin{aligned} g_c &= h_c - \tfrac{1}{4}h_c(X,X)\Theta_c^2 \\ &= h_1 + (1-c)(\alpha_2^2 + \alpha_3^2) - \tfrac{1}{4}N_c\Theta_c^2 \\ &= g_1 + 2uv\Theta_1^2 + (1-c)(\alpha_2^2 + \alpha_3^2) - \frac{1}{N_c}\big(4uv\Theta_1 + (1-c)\eta\big)^2. \end{aligned}$$

Using (4.11) to eliminate $\alpha_2^2 + \alpha_3^2$, and the equation $N_c = 8uv + 2(1-c)a_-$, we obtain

$$g_c = g_1 + \frac{1}{8a_-}(1-c){\Gamma_-}^2 + (1-c)\frac{4uv}{a_-N_c}(a_-\Theta_1 - \eta)^2.$$

The 1-form $a_-\Theta_1 - \eta$ has zero contraction with X and is $U(1)$ invariant, and must therefore be expressible in terms of $\mathbf{u}$ and $\mathbf{v}$. A direct calculation yields

$$2(a_-\Theta_1 - \eta) = \frac{1}{uv}\{\mathbf{B}, \mathbf{u}, \mathbf{v}\}, \tag{4.12}$$

using Notation 5.3 from the next section. □

Proof of Theorem 4.10. In the light of Theorem 4.7, we need to show that

$$g_1 - g_2 = \frac{1}{2uv}|\mathbf{B}|^2 - \frac{2}{N}|\mathbf{B}|^2 - \frac{1}{2N}{\Gamma_+}^2 + \frac{1}{4N}{\Gamma_-}^2,$$

or equivalently:

$$4Nu^2v^2(g_1 - g_2) = 4uv\,a_+|\mathbf{B}|^2 - 2(\mathbf{A}_+\cdot\mathbf{B})^2 + (\mathbf{A}_-\cdot\mathbf{B})^2.$$

By Lemma 4.11, it suffices to show that the right-hand side of this last equation equals

$$\frac{N}{2}(\mathbf{A}_-\cdot\mathbf{B})^2 + \frac{4}{uv}\{\mathbf{B}, \mathbf{u}, \mathbf{v}\}^2$$

divided by a_-, or equivalently:

$$4uv\,a_+a_-|\mathbf{B}|^2 - \{\mathbf{B},\mathbf{u},\mathbf{v}\}^2 = a_-(\mathbf{A}_+\cdot\mathbf{B})^2 + a_+(\mathbf{A}_-\cdot\mathbf{B})^2. \tag{4.13}$$

This identity can be verified by inspection of its components with respect to B_1^2 and B_1B_2, where $\mathbf{B} = (B_1, B_2, B_3)$. (The B_i are linearly independent 1-forms provided $uv \neq 0$.) Dividing by $4uv$, this gives the two equations

$$\begin{aligned} |\mathbf{u}\times\mathbf{v}|^2 - (u_2v_3 - u_3v_2)^2 &= u^2v_1^2 + v^2u_1^2 - 2u_1v_1\mathbf{u}\cdot\mathbf{v},\\ -(u_2v_3 - u_3v_2)(u_3v_1 - u_1v_3) &= u^2v_1v_2 + v^2u_1u_2 - (u_1v_2 + v_1u_2)\mathbf{u}\cdot\mathbf{v}, \end{aligned}$$

whose validity is readily checked. □

Lemma 4.11 tells us that $g_1 - g_2$ belongs pointwise to the 6-dimensional space spanned by the quadratic forms

$$B_iB_j = u^2dv_idv_j - uv(du_idv_j + dv_idu_j) + v^2du_idu_j.$$

We shall exploit Theorem 4.10 in Sections 7 and 8.

5. SO(3) invariance

The previous results have revealed the evident $SO(3)$ symmetry inherent in the bivector formalism with $(\mathbf{u},\mathbf{v}) \in \mathbb{R}^6$. In this section, we investigate the effect of this and other symmetries, and study the curvature of the $U(1)$ bundle defined by (1.1) away from the singular locus $\mathbb{R}^3\cup\mathbb{R}^3$ of $\mathbb{R}^6$.

The G_2 structure on $\mathscr{C} = \mathbb{R}^+\times\mathbb{CP}^3$ is invariant by $SO(5)$, which is double covered by the action of $Sp(2)$ on $\mathbb{H}^2$. The diagonal $U(1)$ in $Sp(2)$ commutes with $SU(2)$, which acts on $\mathbb{H}^2$ by

$$\begin{pmatrix} q_0\\ q_1\end{pmatrix} \longmapsto A\begin{pmatrix} q_0\\ q_1\end{pmatrix}, \qquad A = \begin{pmatrix} a & b\\ -\bar b & \bar a\end{pmatrix},\ |a|^2+|b|^2 = 1,$$

and on $\mathbb{C}^4$ by

$$\begin{pmatrix} z_0 & -z_3\\ z_2 & z_1\end{pmatrix} \longmapsto A\begin{pmatrix} z_0 & -z_3\\ z_2 & z_1\end{pmatrix}.$$

The determinant $z_0z_1 + z_2z_3$ will play a key role in the sequel (see Proposition 6.2).

The map (1.1) is induced from mapping $\mathbf{q} = (q_0,\, q_1)^\top$ to the pair of Hermitian matrices

$$\mathbf{U} = \begin{pmatrix} u_1 & u_2 - iu_3\\ u_2 + iu_3 & -u_1\end{pmatrix}, \quad \mathbf{V} = \begin{pmatrix} v_1 & v_2 - iv_3\\ v_2 + iv_3 & -v_1\end{pmatrix}. \tag{5.1}$$

This representation helps to explain our convention in the definition of the Gibbons-Hawking coordinates u_i, v_j in Section 4. In any case,

$$\mathbf{U}(A\mathbf{q}) = A\mathbf{U}A^{-1}, \quad \mathbf{V}(A\mathbf{q}) = A\mathbf{V}A^{-1},$$

either of which induces the usual double covering $SU(2)\to SO(3)$. These facts tell us that the residual subgroup $SO(3) = SU(2)/\mathbb{Z}_2$ acts diagonally on $\mathscr{M}\subset\mathbb{R}^3\times\mathbb{R}^3$. As a subgroup of $SO(5)$, it acts trivially on a 2-dimensional subspace $\mathbb{R}^2$ in $\mathbb{R}^5$ that (we shall see) corresponds to the subset $\{\mathbf{u} = -\mathbf{v}\}$ of $\mathscr{M}$.

The right action of j on $\mathbb{H}^2$ induces an anti-linear involution of $(\mathbb{CP}^3, J_1)$ without fixed points; it acts as the antipodal map on each S^2 fibre. It is the so-called *real* structure, and S^4 can be defined as the set of *real* (meaning j-invariant) projective lines in $\mathbb{CP}^3$. Since Y_2 is the Killing vector field generated by the action of e^{jt}, we can compute the action of j at a given point by applying the associated rotation by $\pi/2$. If we define

$$j^*\mathbf{z} = \mathbf{z}\circ j^{-1} = -\mathbf{z}\circ j,$$

then (3.1) tells us that

$$\begin{array}{c} j^*(z_0,z_1,z_2,z_3) = (\overline{z}_1, -\overline{z}_0, \overline{z}_3, -\overline{z}_2) \\ j^*(x_0,x_1,x_2,x_3,x_4,x_5,x_6,x_7) = (x_2,x_3,-x_0,-x_1,x_6,x_7,-x_4,-x_5). \end{array} \tag{5.2}$$

The left $U(1)$ action commutes with j, and we have

Lemma 5.1. *The involution j passes to an isometry $\hat{\jmath}\colon \mathscr{M}\to\mathscr{M}$ that interchanges the coordinates $\mathbf{u}\leftrightarrow\mathbf{v}$. Under this operation, the curvature 2-form $F_c = d\Phi_c$ is symmetric. As regards the tensors in Lemma 2.1, the symplectic form σ is anti-symmetric, and the $(3,0)$-form Ψ maps to its complex conjugate $\overline{\Psi}$.*

Proof. The fact that $\hat{\jmath}^*\mathbf{u} = \mathbf{v}$ follows from (5.2). Since j leaves invariant X and

$$j^*\alpha_1 = -\alpha_1,\quad j^*\alpha_2 = \alpha_2,\quad j^*\alpha_3 = -\alpha_3,$$

it has the same $(-,+,-)$ effect on the functions μ_i. Hence $j^*\Theta_c = \Theta_c$, $j^*\omega = -\omega$, and $j^*\Upsilon = \overline{\Upsilon}$. (Note that j must change the sign of ω on $\mathbb{CP}^3$ because it is an isometry for both the Kähler and nearly-Kähler metric on $\mathbb{CP}^3$, and yet changes the sign of both J_1, J_2.) We conclude that

$$j^*\varphi = -\varphi, \qquad j^*(*\varphi) = *\varphi,$$

and the rest follows. $\square$

Consider the connections described by Proposition 4.2. We shall first show the curvature 2-form

$$F_c = d\Theta_c = d\Big(\frac{2}{N_c} X \lrcorner\, h_c\Big) \tag{5.3}$$

is completely determined (for any c) by its restriction to $S^2\times S^2$ in $\mathbb{R}^6$. Consider the action of $\mathbb{R}^+\times\mathbb{R}^+$ on $\mathbb{R}^4_{0145}\times\mathbb{R}^4_{2367}$ given by

$$(x_0,x_1,x_2,x_3,x_4,x_5,x_6,x_7)\longmapsto(\lambda x_0,\lambda x_1,\mu x_2,\mu x_3,\lambda x_4,\lambda x_5,\mu x_6,\mu x_7),$$

with $\lambda,\mu>0$. We denote this representation by $\mathbb{R}^{++}$. The infinitesimal action is generated by the vector fields

$$\begin{array}{rcccl} \partial_u &=& u\frac{\partial}{\partial u} &=& x_0\partial_0 + x_1\partial_1 + x_4\partial_4 + x_5\partial_5,\\ \partial_v &=& v\frac{\partial}{\partial v} &=& x_2\partial_2 + x_3\partial_3 + x_6\partial_6 + x_7\partial_7, \end{array}$$

Note that $\partial_u + \partial_v$ is dual to dR relative to the Euclidean metric on $\mathbb{R}^8$.

The invariance of the curvature form by $\mathbb{R}^{++}$ derives from that of its primitive:

Lemma 5.2. *For any c, the 1-form Θ_c is invariant by $\mathbb{R}^{++}$.*

Proof. It suffices to consider the action of λ above, and to do this we use Notation 4.4. The action on $\mathbb{R}^4_{0145}$ has the following effect on tensors:

$$\begin{array}{rcl} R & \mapsto & \lambda^2 u + v \\ X \lrcorner\, e & \mapsto & \lambda^2\alpha_1 + \alpha_2 \\ N_c & \mapsto & \lambda^2 N_c, \quad c = 1, 2, \end{array}$$

$$\mu_1 \mapsto \lambda^2 u - v, \qquad \alpha_1 \mapsto \lambda^2\alpha_1' - \alpha_1''$$
$$\mu_2, \mu_3 \mapsto \lambda\mu_2, \lambda\mu_3, \qquad \alpha_2, \alpha_3 \mapsto \lambda\alpha_2, \lambda\alpha_3.$$

It now follows that both $R(X \lrcorner\, e) - \mu_1\alpha_1$ and $\mu_2\alpha_2 + \mu_3\alpha_3$ scale homogeneously by λ^2, which is cancelled out by dividing by N_c. □

The *Kähler quotient* of $\mathbb{CP}^3$ by $U(1)$ is constructed by first identifying the moment mapping f defined by

$$df = X \lrcorner\, d\hat\alpha_1 = -d(X \lrcorner\, \hat\alpha_1),$$

$d\hat\alpha_1$ being the Kähler form. Hence,

$$-f = \frac{\mu_1}{R} = \frac{u - v}{u + v} = \frac{|z_0|^2 - |z_1|^2 + |z_2|^2 - |z_3|^2}{|z_0|^2 + |z_1|^2 + |z_2|^2 + |z_3|^2}$$

(see (4.3)). If we regard $\mathbb{CP}^3$ as the hypersurface of $\mathscr{C}$ defined by $R = 1$ then we can identify the quotient at level $\ell \in [-1, 1]$ with

$$\frac{f^{-1}(\ell)}{U(1)} = \{(\mathbf{u}, \mathbf{v}) \in \mathscr{M} : -u{+}v = \ell,\ u{+}v = 1\},$$

On the open interval this is $S^2 \times S^2$, with the respective sphere collapsing as $c \to \pm 1$. A product Kähler 2-form on $S^2 \times S^2$ then pulls back to the restriction of $d\hat\alpha_1$ to $f^{-1}(\ell)$, whereas

$$J_1 d\mu_1 = \tfrac{1}{2} X \lrcorner\, h_1 = 2(v\alpha_1' + u\alpha_1''),$$

in the notation of Proposition 5.2.

By general principles, we can identify a generic Kähler quotient by a compact Lie group G with a stable holomorphic quotient by G^c [24]. In our case, the $U(1)$ action on $\mathbb{CP}^3$ obviously extends to

$$\{[z_0, z_1, z_2, z_3] \mapsto [\zeta z_0, \zeta^{-1} z_1, \zeta z_2, \zeta^{-1} z_3], \quad \zeta \in \mathbb{C}^*.$$

Mapping $[z_0, z_1, z_2, z_3]$ to $[z_0 z_1, z_2 z_3, z_1 z_2, z_0 z_3]$ then realizes the $\mathbb{C}^*$ quotient as a quadric biholomorphic to $\mathbb{CP}^1 \times \mathbb{CP}^1$.

Next we shall express F_c in terms of $SO(3)$ invariant quantities manufactured from the coordinates $(\mathbf{u}, \mathbf{v})$ using scalar and triple products. We had originally carried this out only for $c = 2$, but the general case enables us to express the relationship with F_1, by analogy to Lemma 4.11. The formulae will also include the radii u, v, which (as we have remarked) are not smooth over $\mathbb{R}^3 \cup \mathbb{R}^3$.

Notation 5.3. In order to state results in this section (and adjacent ones), we exploit various triple products combining functions and 1-forms. Our convention is that each triple product has $3! = 6$ terms in which a wedge product counts as one. This is exemplified by the following dictionary:

$$\begin{array}{rcl}
\{\mathbf{u}, \mathbf{v}, d\mathbf{u}\} &=& (u_2 v_3 - u_3 v_2) du_1 + \cdots \\
\{\mathbf{u}, d\mathbf{u}, d\mathbf{u}\} &=& 2u_1 du_2 \wedge du_3 + \cdots \\
\{d\mathbf{u}, d\mathbf{u}, d\mathbf{u}\} &=& 6 du_1 \wedge du_2 \wedge du_3 \\
\{\mathbf{u}, d\mathbf{u}, d\mathbf{v}\} &=& u_1(du_2 \wedge dv_3 - dv_3 \wedge du_2) + \cdots \\
\{\mathbf{u}, d\mathbf{v}, d\mathbf{v}\} &=& 2u_1 dv_2 \wedge dv_3 + \cdots \\
\{d\mathbf{u}, d\mathbf{v}, d\mathbf{v}\} &=& 2 du_1 \wedge dv_2 \wedge dv_3 + \cdots
\end{array}$$

This is a consistent scheme, in the sense that (for example) the substitution $\mathbf{v} = \mathbf{u}$ in $\{\mathbf{u}, d\mathbf{u}, d\mathbf{v}\}$ yields $\{\mathbf{u}, d\mathbf{u}, d\mathbf{u}\}$. Moreover,

$$\begin{array}{rcl}
\{d\mathbf{u}, d\mathbf{u}, d\mathbf{u}\} &=& d\{\mathbf{u}, d\mathbf{u}, d\mathbf{u}\} \\
\{d\mathbf{u}, d\mathbf{v}, d\mathbf{v}\} &=& d\{\mathbf{u}, d\mathbf{v}, d\mathbf{v}\} \\
\{\mathbf{u}, d\mathbf{u}, d\mathbf{v}\} &=& d\{\mathbf{u}, \mathbf{v}, d\mathbf{u}\} + 2\{\mathbf{v}, d\mathbf{u}, d\mathbf{u}\}.
\end{array}$$

On a separate matter,

Notation 5.4. In the light of Lemma 5.1, and to reduce the length of our displays, we shall use

$$\begin{array}{lll}
X \circeq Y & \text{as shorthand for} & X = Y + \widetilde{Y}, \\
X \mathrel{\widehat{\circeq}} Y & \text{as shorthand for} & X = Y - \widetilde{Y},
\end{array}$$

where $\widetilde{Y}$ denotes Y with $\mathbf{u}$ and $\mathbf{v}$ (and u and v) interchanged. The second symbol will only be used in Theorem 7.7.

The conventions above enable us to state

Theorem 5.5. *Assuming $c < 3$, the curvature 2-form F_c defined by* (5.3) *is given by*

$$N_c^2 F_c \circeq 8(3-c)\frac{v^2}{u}\{\mathbf{u}, d\mathbf{u}, d\mathbf{u}\} + 4(1-c)\big[du \wedge \{\mathbf{u}, \mathbf{v}, d\mathbf{v}\} + u\{\mathbf{v}, d\mathbf{u}, d\mathbf{v}\}\big].$$

Proof. From the proof of Theorem 4.7,

$$\Theta_1 = u^{-1}\alpha_1' + v^{-1}\alpha_1''.$$

We conclude that

$$F_1 = \tfrac{1}{4}u^{-3}\{\mathbf{u}, d\mathbf{u}, d\mathbf{u}\} + \tfrac{1}{4}v^{-3}\{\mathbf{v}, d\mathbf{v}, d\mathbf{v}\} \circeq \tfrac{1}{4}u^{-3}\{\mathbf{u}, d\mathbf{u}, d\mathbf{u}\}.$$

This is the sum of the two 4-dimensional curvatures encountered in Lemma 2.3, and (since $N_1 = 8uv$) agrees with the theorem's statement.

Lemma 5.2 tells us that F_c is invariant by the action $(\mathbf{u}, \mathbf{v}) \mapsto (\lambda^2\mathbf{u}, \mu^2\mathbf{v})$ on $\mathbb{R}^6$. A computation shows that the restriction $\widehat{F}_c$ of F_c to $S^2 \times S^2$ (so $u = 1 = v$) satisfies

$$N_c^2 \widehat{F}_2 \circeq 8(3-c)\{\mathbf{u}, d\mathbf{u}, d\mathbf{u}\} + 4(1-c)\{\mathbf{u}, d\mathbf{u}, d\mathbf{v}\}.$$

To find the expression for F_c on (an open set of) $\mathbb{R}^6$ stated in the theorem, we replace $(\mathbf{u}, \mathbf{v})$ by $(u^{-1}\mathbf{u}, v^{-1}\mathbf{v})$ in $\widehat{F}_c$. $\square$

Definition 5.6. *Given a unit vector* $\mathbf{n}$ *in* $\mathbb{R}^3$, *set*

$$\mathscr{M}(\mathbf{n}) = \{(\mathbf{u},\mathbf{v}) \in \mathscr{M} : \mathbf{u}\cdot\mathbf{n} = 0 = \mathbf{v}\cdot\mathbf{n},\ uv \neq 0\}.$$

Thus, $\mathscr{M}(\mathbf{n})$ is an open subset of the 4-dimensional subspace of $\mathbb{R}^6$ that satisfies the vector equation $(\mathbf{u}\times\mathbf{v})\times\mathbf{n} = \mathbf{0}$. Note that $\mathscr{M}(\mathbf{n}_1) = \mathscr{M}(\mathbf{n}_2)$ if and only if $\mathbf{n}_1 = \pm\mathbf{n}_2$ and that $\mathscr{M}(\mathbf{n}_1)\cap\mathscr{M}(\mathbf{n}_2)$ lies in a plane parametrized by $(\pm u, \pm v)$ otherwise. The curvature form F_c and the induced $SU(3)$ structure are well defined at all points of $\mathscr{M}(\mathbf{n})$, and this subset will play an important role in both Sections 7 and 8.

Corollary 5.7. F_c *vanishes on each linear subvariety* $\mathscr{M}(\mathbf{n})$ *for any* c*. It also vanishes on the subset* $\{\mathbf{u} = -\mathbf{v}\}$ *of* $\mathscr{M}$ *(which by Definition 4.8 lies in* $\mathscr{F}_-$*).*

Proof. These statements are consequences of Theorem 5.5. In the first case, $\mathbf{u}$ and $\mathbf{v}$ are constrained to lie in a common 2-dimensional subspace of $\mathbb{R}^3$. It follows that all the triple products vanish. In the second case, all the triple products change sign (or are zero) when $\mathbf{u}$ and $\mathbf{v}$ are interchanged. □

The first Chern class of the bundle Q over $S^2\times S^2$ is given by

$$\mathbf{c}_1(Q) = \frac{1}{2\pi}[d\Theta_c] = \frac{1}{2\pi}[F_c].$$

Now take $c=1$. From the discussion after Lemma 2.3, we know that $\frac14\{\mathbf{u}, d\mathbf{u}, d\mathbf{u}\}$ integrates to -2π over S^2. The bidegree of $\mathbf{c}_1(Q)$ over $S^2\times S^2$ is therefore $(-1,-1)$. The same conclusion must be valid for other values of c, and we can do a consistency check by restricting the formula for F_2 to the diagonal sphere $\Delta = \{(\mathbf{u},\mathbf{u}) : \mathbf{u}\in S^2\}$. Bearing in mind Notation 5.3, we obtain $F_2|_\Delta = \frac12\{\mathbf{u}, d\mathbf{u}, d\mathbf{u}\}$, so

$$\frac{1}{2\pi}\int_\Delta F_2 = -2.$$

This is what one expects since the associated complex line bundle over $\Delta\cong\mathbb{CP}^1$ is isomorphic to $\mathscr{O}(-1)\otimes\mathscr{O}(-1) = \mathscr{O}(-2)$.

Since F_1 and F_c are equal in cohomology, their difference must be exact, and this is made explicit by the next result, whose proof we omit:

Proposition 5.8. *On* $S^2\times S^2$,

$$\widehat{F}_c \ \circleddash\ \tfrac14\{\mathbf{u}, d\mathbf{u}, d\mathbf{u}\} + d\Big(\frac{1-c}{(3-c)N_c}\{\mathbf{u},\mathbf{v}, d\mathbf{u}\}\Big).$$

The restriction of the circle bundle Q to $S^2\times S^2$ is homeomorphic to the cone over $S^2\times S^3$, the basic Sasaki-Einstein manifold $T^{1,1}$ (see [17, Appendix A]).

Remark 5.9. If we choose to identify $\mathscr{M}$ with $\mathbb{C}^3\setminus\mathbf{0}$ by $(\mathbf{u},\mathbf{v})\mapsto\mathbf{u}+i\mathbf{v}$, we can define an $SO(3)$-equivariant mapping $\mathscr{M}\to\mathbb{CP}^2$. A slice to the $U(1)$ orbits is given by

$$\{(\mathbf{u},\mathbf{v})\in\mathscr{M} : \mathbf{u}\cdot\mathbf{v} = 0\},$$

and the equation $u = v$ then determines a conic curve $C \cong \mathbb{CP}^1$. This set-up was used by Li to construct a new $SO(3)$-invariant Kähler-Einstein metric on $\mathbb{CP}^2 \setminus C$ with cone angle $2\pi/3$ along C [27] (see also [11, 28, 29]). It gives rise to a Sasaki-Einstein metric of cohomogeneity-one on the link of the singularity $z_1^2 + z_2^2 + z_3^2 + z_4^3 = 0$, but this example is not compatible with our geometry; in particular the action of $U(1)$ on $\mathbb{C}^3$ is not an isometry for any c.

6. The reduced twistor fibration

We shall continue to analyse the action of $SO(3)$ in this section, but in relation to the twistor fibration $\pi\colon \mathbb{CP}^3 \to S^4$. Consider the lower part of Figure 1 in the Introduction, in which $D^3 \cong S^4/SO(2)$ is the closed unit ball in the subspace $\mathbb{R}^3$ of $\mathbb{R}^5$ fixed by the action of $SO(2) = U(1)/\mathbb{Z}_2$. We may identify its boundary 2-sphere ∂D^3 with the fixed point set $\mathbb{S}^2$ of $SO(2)$ in S^4. The map ϖ can be expressed as a composition

$$\varpi\colon \mathscr{M} \longrightarrow \frac{\mathscr{M}}{\mathbb{R}^+} \cong \frac{\mathbb{CP}^3}{SO(2)} \longrightarrow D^3,$$

in which the second map is a reduction of π. It is $SO(3)$-equivariant, and symmetric in $\mathbf{u}, \mathbf{v}$ since π commutes with j. In fact, it could not be simpler:

Proposition 6.1.

$$\varpi(\mathbf{u}, \mathbf{v}) = \frac{\mathbf{u} + \mathbf{v}}{u + v}.$$

Proof. As an alternative to (3.3), π can be computed via the isomorphism $\Lambda^2_0(\mathbb{C}^4) \cong \mathbb{C}^5$ that defines the double cover $Sp(2) \to SO(5)$. Indeed,

$$[\mathbf{z}] \longmapsto -\frac{1}{\|\mathbf{z}\|^2}\mathbf{z} \wedge (j^*\mathbf{z})$$

defines an $Sp(2)$-equivariant and j-invariant mapping from $\mathbb{CP}^3$ to $S^4 \subset \mathbb{R}^5$, and must therefore coincide with π up to an isometry (the minus sign is for convenience). For simplicity, suppose that $\mathbf{z}$ is a unit vector. Using (5.2) and an obvious basis for the exterior product, we have

$$\begin{aligned} \pi[\mathbf{z}] &= -(z_0, z_1, z_2, z_3) \wedge (\overline{z}_1, -\overline{z}_0, \overline{z}_3, -\overline{z}_2) \\ &= (|z_0|^2{+}|z_1|^2,\ -z_0\overline{z}_3{+}z_2\overline{z}_1,\ z_0\overline{z}_2{+}z_3\overline{z}_1,\ -z_1\overline{z}_3{-}z_2\overline{z}_0,\ z_1\overline{z}_2{-}z_3\overline{z}_0,\ |z_2|^2{+}|z_3|^2). \end{aligned}$$

Four of the six coordinates in the last line are $U(1)$-invariant, and $\Lambda^2_0\mathbb{C}^4$ is the subspace orthogonal to the component represented by $\sum |z_i|^2$ $(= 1)$. The image $\varpi([\mathbf{z}])$ is therefore represented in a real orthogonal basis by

$$\begin{aligned} |z_0|^2 + |z_1|^2 - |z_2|^2 - |z_3|^2 &= u_1 + v_1 \\ \mathrm{Re}(z_0\overline{z}_2 + z_3\overline{z}_1) &= u_2 + v_2 \\ -\mathrm{Im}(z_0\overline{z}_2 + z_3\overline{z}_1) &= u_3 + v_3. \end{aligned}$$

The choice of signs is dictated by the fact that ϖ must be $SO(3)$-invariant. If $\mathbf{z}$ is not a unit vector, one merely has to divide everything by $\|\mathbf{z}\|^2 = u + v$. □

Observe that

$$\varpi(\mathbf{u},\mathbf{v}) = \frac{u}{u+v}\boldsymbol{\sigma} + \frac{v}{u+v}\boldsymbol{\tau}$$

lies on the chord with unit endpoints $\boldsymbol{\sigma} = \mathbf{u}/u$ and $\boldsymbol{\tau} = \mathbf{v}/v$ in ∂D^3. The generic fibre of ϖ has dimension 3, but reduces to dimension 2 over over ∂D^3. For future reference, we note that the radius

$$s = |\varpi(\mathbf{u},\mathbf{v})| = \frac{1}{R}|\mathbf{u}+\mathbf{v}| \tag{6.1}$$

in ∂D^3 defines a function $S^4 \to [0,1]$. It vanishes on the circle $\mathbb{S}^1 = S^4 \cap \mathbb{R}^2$, which is the fixed point set of the $SO(3)$ action (and maximal $SO(2)$ orbit), and reaches the extreme value 1 on the fixed point set $\mathbb{S}^2$ of the $SO(2)$ action.

Definition 4.8 underlies many properties of ϖ and Q, and of the tensors defined on the spaces that feature in Figure 1. It follows from Proposition 6.1 that

$$\mathscr{F}_+ = \varpi^{-1}(\partial D^3), \tag{6.2}$$

and $Q^{-1}(\mathscr{F}_+)$ is a cone over the inverse image

$$\pi^{-1}(\mathbb{S}^2) \cong S^2 \times S^2 \subset \mathbb{CP}^3$$

that we shall next realize as a non-holomorphic quadric. By contrast, $\mathscr{F}_-$ projects *onto* S^4 and D^3, and arises from a *holomorphic* quadric intimately connected to the $SO(2)$ action on the 4-sphere.

Proposition 6.2. *Set* $f_+ = z_0\overline{z}_3 - \overline{z}_1 z_2$ *and* $f_- = z_0 z_1 + z_2 z_3$. *Then*

$$Q^{-1}(\mathscr{F}_\pm) = \big\{[\mathbf{z}] \in \mathscr{C} : f_\pm = 0\big\}.$$

Proof. Square brackets in the line above represent the $U(1)_1$ quotient $\mathbb{C}^4\backslash\mathbf{0} \to \mathscr{C}$ generated by Y_1. We shall in fact prove that

$$2|f_\pm|^2 = uv \mp \mathbf{u}\cdot\mathbf{v}, \tag{6.3}$$

a quantity that was denoted by $a_\pm$ in (4.9). This clearly suffices, and we only need to verify the equation for f_+ since the other then follows from (4.2).

Using the Euclidean coordinates on $\mathbb{R}^8$ defined by (3.1), we abbreviate $x_i x_j$ by x_{ij} and $x_i x_j x_k x_l$ by x_{ijkl}. As a first step, $uv - u_1 v_1$ equals

$$(x_{00}+x_{11}+x_{44}+x_{55})(x_{22}+x_{33}+x_{66}+x_{77}) - (x_{00}+x_{11}-x_{44}-x_{55})(x_{22}+x_{33}-x_{66}-x_{77}),$$

which is twice

$$\begin{aligned} &x_{0066} + x_{0077} + x_{1166} + x_{1177} + x_{4422} + x_{4433} + x_{5522} + x_{5533} \\ &\quad = (x_{06} - x_{17})^2 + (x_{16} + x_{07})^2 + (x_{24} - x_{35})^2 + (x_{25} + x_{34})^2. \end{aligned}$$

On the other hand, $u_2 v_2 + u_3 v_3$ equals 4 times

$$(x_{04}+x_{15})(x_{26}+x_{37})+(x_{05}-x_{14})(x_{27}-x_{36}) = (x_{06}-x_{17})(x_{24}-x_{35})+(x_{16}+x_{07})(x_{25}+x_{34}).$$

Therefore $uv - \mathbf{u}\cdot\mathbf{v}$ equals twice

$$(x_{06} - x_{17} - x_{24} + x_{35})^2 + (x_{16} + x_{07} - x_{25} - x_{34})^2,$$

and the two terms in parentheses are the real and imaginary components of f_+. □

The almost complex structures J_1, J_2 (only the first integrable) and the Einstein metrics $\widehat{h}_1, \widehat{h}_2$ on $\mathbb{CP}^3$ coincide on the horizontal distribution

$$D = \langle \alpha_2, \alpha_3 \rangle^{\mathrm{o}},$$

which at each point is the orthogonal complement to the vertical tangent space to the fibration $\pi\colon \mathbb{CP}^3 \to S^4$, with respect to any of the metrics $\widehat{h}_c$. The next result follows from the equation

$$X \lrcorner (\alpha_2 - i\alpha_3) = \mu_2 - i\mu_3 = 2i(z_0 z_1 + z_2 z_3)$$

(see (4.3)) and Proposition 6.2:

Corollary 6.3. *$\mathscr{F}_-$ is the locus of points in $\mathscr{M}$ over which the circle fibres are horizontal relative to π.*

A quadratic form in the z_i's can be regarded as a holomorphic section of

$$H^0(\mathbb{CP}^3, \mathscr{O}(2)) \cong \mathfrak{sp}(2, \mathbb{C}),$$

which can in turn be identified with the complexification of the Lie algebra

$$\mathfrak{sp}(2) = \mathfrak{u}(2) \oplus \mathfrak{m} = \mathfrak{su}(2) \oplus \mathfrak{u}(1) \oplus \mathfrak{m}$$

of Killing vector fields on S^4. Here $\mathfrak{u}(1)$ is spanned by the vector field $X_* = \pi_* X$ on S^4 that generates our circle action, and f_- is invariant by $U(2)$. The associated quadric Z_- in $\mathbb{CP}^3$ is the divisor defined by X_*, whose equation is determined by the self-dual component of the 2-form ∇X_* [6, ch. 13].

Now that we have identified the $\mathfrak{u}(1)$ summand, it is easy to see that Lie subalgebra $\mathfrak{su}(2)$ is generated by the quadrics

$$\begin{array}{rcl} Z_1 &=& \{z_0 z_1 - z_2 z_3 = 0\} \\ Z_2 &=& \{z_0 z_3 + z_1 z_2 = 0\} \\ Z_3 &=& \{z_0 z_3 - z_1 z_2 = 0\}. \end{array}$$

All four equations representing Z_-, Z_1, Z_2, Z_3 are j-invariant, which is equivalent to asserting that they define *real* elements in $\mathfrak{sp}(2, \mathbb{C})$. Each quadric defines a *non-constant* orthogonal complex structure on

$$S^4 \setminus S^1 \;\cong\; S^2 \times \mathbb{C}^+,$$

compatible with a scalar flat Kähler metric on the product [30, 31]. In each case, the discriminant locus is a circle consisting of points in S^4 whose twistor fibres lie in Z_i. For Z_-, we have denoted this circle by $\mathbb{S}^1$.

An analogue of Proposition 6.2 can be proved in the same way:

Proposition 6.4. *We have $Z_i = Q^{-1}(\mathscr{F}_i)$, where*

$$\mathscr{F}_i = \big\{(\mathbf{u}, \mathbf{v}) \in \mathscr{M} : \mathbf{u} \cdot \mathbf{v} = uv + 2u_i v_i\big\}.$$

For example, $\mathscr{F}_3$ has equation $uv = u_1 v_1 + u_2 v_2 - u_3 v_3$. We shall use this in Section 8.

Since $SO(3)$ acts diagonally on $(\mathbf{u},\mathbf{v})\in\mathscr{M}$, the $SO(3)$-orbit containing $(\mathbf{u},\mathbf{v})\in\mathscr{M}$ is 2-dimensional (a 2-sphere) if and only if $(\mathbf{u},\mathbf{v})$ belongs to $\mathscr{F}_+\cup\mathscr{F}_-$. It follows that any $SO(3)$-orbit in $Q^{-1}(\mathscr{F}_-)$ is a 2-sphere inside a quadric, though this orbit is a fibre of the twistor fibration over any point of $\mathbb{S}^1$. On the other hand, any $SO(3)$ orbit in $Q^{-1}(\mathscr{F}_+\setminus\mathscr{F}_-)$ is 3-dimensional.

The restriction of the G_2 3-form φ (recall Propositon 3.2) to a 3-dimensional $SO(3)$ orbit $\mathscr{O}$ in $\mathscr{C}$ must be a constant multiple of the volume form. The absence of 3-dimensional cohomology in $\mathscr{C}$ implies that this constant multiple must be zero. Recalling (4.9) and (6.3), set

$$a=a_+=2|f_+|^2,\qquad t=\arg f_+. \tag{6.4}$$

Since u,v and $f_+=|f_+|e^{it}$ are constant on $\mathscr{O}$, it follows that

$$\varphi\wedge da\wedge dt\wedge du\wedge dv=0. \tag{6.5}$$

The proof of Proposition 6.1 tells us that the function f_+/R factors through π, so the same is true of t and the $SO(2)$-invariant function a/R^2. In fact, (6.1) implies that

$$1-s^2=\frac{2a}{R^2}, \tag{6.6}$$

and we record without proof

Lemma 6.5. *Let $X^\flat=X_*\lrcorner\, s_4$ denote the 1-form dual to X_* on S^4. Then*

$$X^\flat=(1-s^2)dt.$$

One can regard $t\colon S^4\setminus\mathbb{S}^2\to[0,2\pi)$ as a 'longitude' and $s\colon S^4\to[0,1]$ is sine of 'latitude'.

Our aim is to replace the four functions a,t,u,v in (6.5) by three functions whose constancy defines a coassociative submanifold $\mathscr{W}$ of $\mathscr{C}$, so that $\mathscr{W}$ has dimension 4 and φ pulls back to zero on $\mathscr{W}$. This is possible because each 3-dimensional tangent space $T_o\mathscr{O}$ is contained in a unique coassociative subspace, which is generated by a basis $\{W_1,W_2,W_3\}$ of $T_o\mathscr{O}$ and the 3-fold cross product W_4 defined by

$$(*\varphi)(W_1,W_2,W_3,W)=h_2(W_4,W).$$

We shall in fact show that the desired functions are a,t and

$$b=u^2-v^2=R(u-v). \tag{6.7}$$

The value of b is easily estimated at a point of $\mathscr{C}$ for which $f_-=0$, for then

$$R^2-\frac{b^2}{R^2}=4uv=4|f_+|^2+4|f_-|^2=4|f_+|^2=R^2(1-s^2),$$

and so $b=\pm sR^2$. Note that b vanishes over $\mathbb{S}^1=\{s=0\}$.

Over any point p of $S^4\setminus\mathbb{S}^1$, and for each value of $R>0$, there is a 2-sphere in $\mathscr{C}$ lying over p that has exactly two antipodal 'poles' belonging to the quadric $f_-=0$. Let

$[\mathbf{z}] = [z_0, z_1, z_2, z_3]$ be such a pole, chosen so that $b > 0$. Any point in the same $\mathbb{R}^3$ fibre with the same value of $\|\mathbf{z}\|^2 = R$ must (using (3.3)) equal $[\tilde{z}_0, \tilde{z}_1, \tilde{z}_2, \tilde{z}_3]$, where

$$\begin{aligned} \tilde{z}_0 + j\tilde{z}_1 &= (1+|\lambda|^2)^{-1/2}(z_0 + jz_1)(1+j\lambda), \\ \tilde{z}_2 + j\tilde{z}_3 &= (1+|\lambda|^2)^{-1/2}(z_2 + jz_3)(1+j\lambda), \end{aligned}$$

for some $\lambda \in \mathbb{C} \cup \{\infty\}$. Thus

$$(1+|\lambda|^2)^{1/2}[\tilde{z}_0, \tilde{z}_1, \tilde{z}_2, \tilde{z}_3] = [z_0 - \overline{z}_1\lambda,\ z_1 + \overline{z}_0\lambda,\ z_2 - \overline{z}_3\lambda,\ z_3 + \overline{z}_2\lambda].$$

The value of $b = R(u-v)$ is transformed into $\tilde{b}$ where

$$\begin{aligned} (1+|\lambda|^2)\tilde{b} &= R\big(|z_0 - \overline{z}_1\lambda|^2 - |z_1 + \overline{z}_0\lambda|^2 + |z_2 - \overline{z}_3\lambda|^2 - |z_3 + \overline{z}_2\lambda|^2\big) \\ &= (1-|\lambda|^2)b - 2R(\lambda\overline{z}_0\overline{z}_1 + \overline{\lambda}z_0z_1 + \lambda\overline{z}_2\overline{z}_3 + \overline{\lambda}z_2z_3) \\ &= (1-|\lambda|^2)b, \end{aligned}$$

since $f_- = 0$. Now

$$h = \frac{1-|\lambda|^2}{1+|\lambda|^2} \in [-1, 1]$$

represents normalized height on the 2-sphere under stereographic projection. Therefore

$$\tilde{b} = s\,h\,R^2, \tag{6.8}$$

and the locus of points on a twistor fibre with $s > 0$ and $\tilde{b}$ constant is a pole or a parallel circle, the equator if and only if $b = 0$. It follows from (3.9) that R^2 ($r^{1/2}$ in the notation of [9, §4]) coincides with the norm of self-dual 2-forms, and is therefore the 'natural' radial parameter of the twistor fibres. Equation (6.8) therefore tells us that, for each fixed $s > 0$, the quantity $\tilde{b}$ is a Euclidean coordinate in the fibre.

The equation $s = 1$ distinguishes the 5-dimensional subset $\mathscr{C}_+ = Q^{-1}(\mathscr{F}_+)$ of $\mathscr{C}$ lying over the totally geodesic 2-sphere $\mathbb{S}^2$. The twistor lift of $\mathbb{S}^2$ (in the sense of [14]) distinguishes a pole (where $f_- = 0$) in each twistor fibre in $\mathscr{C}_+ \cong \mathbb{R}^+ \times Z_+$. It follows that the restriction of the G_2 form φ to $\mathscr{C}_+$ is a constant multiple of $d\tilde{b} \wedge \tau_1$, where τ_1 is the self-dual 2-form determined by the relevant point of $\mathbb{S}^2$. Setting $\tilde{b}$ constant therefore defines a coassociative submanifold of $\mathscr{C}_+$ which intersects each $\mathbb{R}^3$ fibre in a plane of constant height. The origin of this plane corresponds to the pole of the twistor 2-sphere touching the plane. The union (over $\mathbb{S}^2$) of these poles forms the unique 2-dimensional $SO(3)$ orbit in the coassociative, which we can identify with TS^2. If $\tilde{b} = 0$, the coassociative is a union of equators, an example that was long recognized [23, 25], though in the present conical context this union is the orbifold $\mathbb{C}^2/\mathbb{Z}_2$ minus its singular point.

At this juncture, we can dispense with the tildes, and use b to denote the value of $u^2 - v^2$ at an arbitrary point of $\mathscr{C}$. We have shown that $\mathscr{C}$ contains a cosassociative submanifold with $s = 1$ and any constant value of $b \in \mathbb{R}$. Karigiannis and Lotay were the first to identify the foliation of $\mathscr{C}$ by coassociatives arising from the action of $SO(3)$ under consideration, and they extend the discussion to the complete G_2 metric on $\Lambda^2_- T^*S^4$ [22]. Our interpretation of the conical situation using bivector and twistor space formalism is summarized by

Theorem 6.6. *Let $a \geqslant 0$, $b \in \mathbb{R}$ and $t \in [0, 2\pi)$. Setting a, b constant and (if $s < 1$) t constant defines a coassociative submanifold of $\mathscr{C}$ diffeomorphic to TS^2 unless $a = b = 0$.*

Proof. We have motivated the significance of the $SO(3)$ invariant functions a, b, t (see (6.4) and (6.7)), or equivalently $uv - \mathbf{u}\cdot\mathbf{v}$, $u^2 - v^2$, t. The fact that these define coassociative manifolds throughout $\mathscr{C}$ follows from the differential relation

$$da \wedge db \wedge dt \neq 0 \quad \text{provided} \quad ab \neq 0, \tag{6.9}$$

and the identity

$$\varphi \wedge da \wedge db \wedge dt = 0, \tag{6.10}$$

which strenghtens (6.5). We have verified both by computer. Topologically, the situation is reminiscent of (4.7), and we describe this next.

Fixing t amounts to restricting attention to a totally geodesic 3-sphere

$$S^3_t = (\mathbb{R}_t \oplus \mathbb{R}^3) \cap S^4,$$

where $\mathbb{R}_t \subset \mathbb{R}^2$ is the line corresponding to $\arg f_+ = t$, and $SO(2)$ acts trivially on $\mathbb{R}^3$. The choice of t will also distinguish a point $p_t \in \mathbb{S}^1$. The configuration of the coassociative manifolds above S^3_t can be understood by reference to the functions

$$\begin{cases} R^2 = \dfrac{2a}{1 - s^2} \\[2ex] R^2\sqrt{1 - h^2} = \dfrac{\sqrt{4a^2s^2 - b^2(1 - s^2)^2}}{s(1 - s^2)}, \end{cases} \tag{6.11}$$

formed by rearranging (6.6) and (6.8), with a, b presumed constant. Whilst R^2 represents the natural radius of the twistor 2-sphere containing the point in question, the second function is the radius of the small circle generated by its $SO(3)$ orbit.

The second equation in (6.11) implies that $|h| = 1$ when $s = s_{\min}$, where

$$1 - s^2_{\min} = 2c(\sqrt{c^2 + 1} - c) \tag{6.12}$$

and $c = a/b$. The right-hand side lies in the interval $(0, 1)$ provided $a > 0$ and $b \neq 0$. In this case, the coassociative manifold projects onto the semi-open annulus in S^3_t defined by $s_{\min} \leqslant s < 1$. It consists of points in $\mathscr{C}$ whose norm squared $\|\mathbf{z}\|^2$ varies in inverse proportion to $\sqrt{1 - s^2}$, and intersects each $\mathbb{R}^3$ fibre over the annulus in a small circle that shrinks to a point $p_{\min}$ over each point of the 'limiting' 2-sphere $\{s = s_{\min}\} \subset S^3_t$. Having fixed a and b, the surface in Figure 2 depicts the union of small circles lying over a geodesic segment in S^4 from a point of $\mathbb{S}^2$ to $p_{\min}$. The associated value R^2 of the equatorial radii are shown (in red) for reference. In this way, the surface represents a fibre in the tangent bundle of this 2-sphere over its vertex $p_{\min}$.

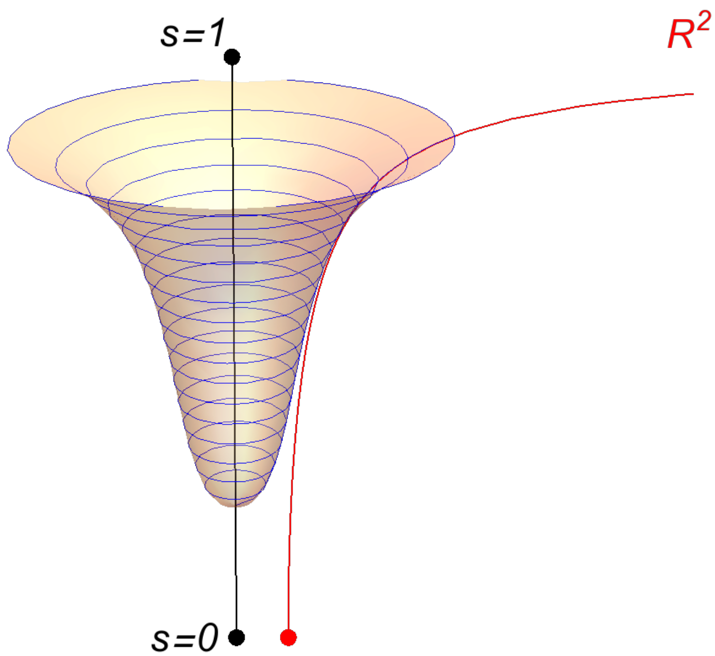

Figure 2: The fibre of a coassociative submanifold of $\mathscr{C}$ with $a = \frac{1}{2}$ and $b = \frac{1}{4}$

If $a > 0$ but $b = 0$ then (6.11) implies that $h \equiv 0$, and the coassociative is the closure of a union of equators in twistor fibres assuming all values of $s \in (0, 1)$. If Figure 2 were redrawn to illustrate this case, the surface would retain a positive radius at $s = 0$, though smoothness is maintained for reasons we now explain. For a fixed value of s close to 0 the equators (of radius R^2 close to $2a$) lie over a tiny 2-sphere close to p_t. As s attains the value 0, the limits of these equators exhaust the twistor fibre over p_t with $R^2 = 2a$, which now plays the role of the limiting 2-sphere.

The case $a = 0$ was discussed before the statement of the theorem. □

7. SU(3) structure

We now turn attention to the symplectic form

$$\sigma = X \lrcorner \varphi$$

obtained by contracting the exact 3-form φ on $\mathscr{C} = \mathbb{R}^+ \times \mathbb{CP}^3$ with the Killing vector field X tangent to the $SO(2)$ fibres. To proceed, one can use either of the descriptions

$$\varphi = d(\tfrac{1}{3}R^3\omega) = -d(R\tau_0)$$

from Section 3, provided we work over $\mathbb{R}^8$. We already know that $\mathscr{L}_X\omega = 0$ since $SO(2)$ is a symmetry of the nearly-Kähler structure of $\mathbb{CP}^3$. But it is also true that

$$\mathscr{L}_X\tau_0 = \mathscr{L}_X(dR \wedge \alpha_1 - \alpha_{23}) = 0.$$

This follows because $\mathscr{L}_X Y = [X, Y_1] = 0$ and $\alpha_1 = Y \lrcorner\, e$ so $\mathscr{L}_X \alpha_1 = 0$, and also $3\alpha_{23} = R^2(\omega - d\alpha_1)$. Therefore

$$\sigma = -X \lrcorner\, d(R\tau_0) = d(RX \lrcorner\, \tau_0).$$

We shall work from this formula, together with

Lemma 7.1.

$$X \lrcorner\, \tau_0 = -u\,du + v\,dv + \tfrac{1}{2}\big[-u\,dv + v\,du + \mathbf{u}\cdot d\mathbf{v} - \mathbf{v}\cdot d\mathbf{u}\big].$$

Proof. Recall that the function μ_i on $\mathscr{C}$ was defined (in Section 4) to be the interior product $X \lrcorner\, \alpha_i$. The definition of τ_0 therefore gives

$$X \lrcorner\, \tau_0 = -\mu_1 dR - \mu_2\alpha_3 + \mu_3\alpha_2 = -\mu_1 dR + \tfrac{1}{2}\Gamma_-.$$

The result follows from Lemma 4.9 after substituting $\mu_1 = u - v$ and $dR = du + dv$. $\square$

Bearing in mind that $u\,du = \sum_{i=1}^{3} u_i du_i$ and $v\,dv = \sum_{i=1}^{3} v_i dv_i$, we can crudely approximate $X \lrcorner\, \tau_0$ by the sum

$$\tfrac{1}{2}\sum_{i=1}^{3}(u_i - v_i)(du_i + dv_i).$$

This leads us to define

$$\text{(7.1)} \qquad \begin{cases} \mathbf{p} &= \mathbf{u} + \mathbf{v}, \\ \mathbf{q} &= (u+v)(\mathbf{u} - \mathbf{v}) = R(\mathbf{u} - \mathbf{v}), \end{cases}$$

and write $\mathbf{p} = (p_1, p_2, p_3)$ and $\mathbf{q} = (q_1, q_2, q_3)$. (The context should make it clear that these q_i are not quaternions!) Then

$$\begin{aligned} \tfrac{1}{2}\sum_{i=1}^{3} q_i dp_i - RX \lrcorner\, \tau_0 &= \tfrac{3}{2}R(u\,du - v\,dv) + \tfrac{1}{2}R(u\,dv - v\,du) \\ &= d(\tfrac{1}{2}R^2\mu_1). \end{aligned}$$

The fact that this 1-form is exact is somewhat of a miracle, since it shows that the vectors $\mathbf{p}, \mathbf{q}$ furnish Darboux coordinates for σ:

Theorem 7.2. *With the above notation,*

$$\sigma = -\tfrac{1}{2}\sum_{i=1}^{3} dp_i \wedge dq_i,$$

and this is non-degenerate away from the origin in $\mathbb{R}^6$.

The realization of the canonical coordinates in the theorem initially came about by observing that setting $\mathbf{p}$ to be a constant vector defines a Lagrangian submanifold of $\mathscr{M}$. This assertion is equivalent to the equation

$$\sigma \wedge dp_1 \wedge dp_2 \wedge dp_3 = 0.$$

The idea of weighting sums and differences of the $(\mathbf{u},\mathbf{v})$ coordinates with powers of the function $R=u+v$ arises from Proposition 6.1. The generic fibres of $\varpi\colon \mathscr{M}\to D^3$ are *not* Lagrangian, but we do have:

Corollary 7.3. *The following maps $\mathscr{M}\to\mathbb{R}^3$ have σ-Lagrangian fibres:*

$$(\mathbf{u},\mathbf{v})\mapsto\mathbf{p},\qquad (\mathbf{u},\mathbf{v})\mapsto\mathbf{q},$$
$$(\mathbf{u},\mathbf{v})\mapsto\mathbf{u}\sqrt{u+v},\qquad (\mathbf{u},\mathbf{v})\mapsto\mathbf{v}\sqrt{u+v}.$$

Proof. The Lagrangian nature of the first two follows immediately from Theorem 7.2. By (skew) symmetry it suffices to verify the third, so set $\mathbf{u}=R^{-1/2}\mathbf{m}$ with (as always) $R=u+v$ and $\mathbf{m}$ a constant vector. The restriction of -2σ to a fibre is

$$\sum_{i=1}^{3}\big(\tfrac12 R^{-3/2}m_i\,dR+dv_i\big)\wedge\big(\tfrac12 R^{-1/2}m_i\,dR+R\,dv_i+v_i\,dR\big)=\Big(\sum_{i=1}^{3}v_i dv_i\Big)\wedge dR.$$

But

$$dR=du+dv=-\tfrac12 R^{-3/2}|\mathbf{m}|dR+dv$$

and $\sum v_i dv_i=v\,dv$, so the right-hand side of the first display vanishes. □

The methods adopted at the start of this section are suited to a study of the symplectic form

$$\sigma_{\mathrm{BS}}=-X\lrcorner\, d\big((R^4+1)^{1/4}\tau_0\big)=d\big((R^4+1)^{1/4}X\lrcorner\,\tau_0\big)$$

induced from the complete G_2 structure of Theorem 3.5. An initial observation is that the three subspaces defined by the respective equations $\mathbf{u}=\mathbf{0}$, $\mathbf{v}=\mathbf{0}$, $\mathbf{u}=\mathbf{v}$ are Lagrangian relative to σ_{BS} (as they are for σ). This fact follows by substituting the equations into Lemma 7.1 and then differentiating, and is equally apparent from

Corollary 7.4. *The 2-form $2(R^4{+}1)^{3/4}\sigma_{\mathrm{BS}}$ equals*

$$-(R^4+2)du\wedge dv+R^3dR\wedge(\mathbf{u}\cdot d\mathbf{v}-\mathbf{v}\cdot d\mathbf{u})+2(R^4{+}1)\sum_{i=1}^{3}du_i\wedge dv_i.$$

Proof. We again use Lemma 7.1 and the same calculations that led to Theorem 7.2. This shows that

$$2(R^4{+}1)^{3/4}\sigma_{\mathrm{BS}}=R^3dR\wedge\Gamma_-+2(R^4+1)\Big(-d\mu_1\wedge dR+du\wedge dv+\sum_{i=1}^{3}du_i\wedge dv_i\Big),$$

which simplifies to the expression stated. □

For the remainder of this section, we shall consider exclusively the metric g_2 described by Theorem 4.10. We shall study the almost complex structure $\mathbb{J}$ on $\mathscr{M}$ defined by

$$\sigma(W,Y)=N_2^{1/2}g_2(\mathbb{J}W,Y),\tag{7.2}$$

in accordance with (2.1) and Remark 2.2. Note that both g_2 and (as must be the case) $\mathbb{J}$ are unaffected by re-scaling the Killing vector field X used in their definition. We shall also identify the complex volume form induced from the G_2 structure of $\mathscr{C}$.

Consider a tangent vector E in $T_m\mathscr{M}$, it is natural to define dual 1-forms

$$E^\flat = E \lrcorner\, g_2, \qquad E^\natural = E \lrcorner\, \sigma.$$

With the 'endomorphism' sign convention for the action of $\mathbb{J}$ on 1-forms, (7.2) becomes

$$E^\natural = N_2^{1/2}\mathbb{J}E^\flat. \tag{7.3}$$

This suggests the following strategy to try to pin down $\mathbb{J}$. We seek a tangent vector E such that *either* $E^\flat$ *or* $E^\natural$ is as simple as possible, in the hope that the other one is not over complicated.

We give one example of this approach. For this purpose, let ξ denote the 1-form $\frac{1}{2}\sum q_j\,dp_j$ (cf. (7.1)) and let $B_i = u\,dv_i - v\,du_i$ as at the end of Section 4. Then

Proposition 7.5. *For each fixed $i = 1, 2, 3$,*

$$N^{1/2}\mathbb{J}(d(Rp_i)) = 2R^{-1}p_i\xi + 2RB_i - q_i\,dR.$$

Proof. For clarity of notation, we set $i = 3$ and define

$$E = u\frac{\partial}{\partial u_3} + v\frac{\partial}{\partial v_3}.$$

This belongs to the annihilator of the 1-form $\mathbf{B} = (B_1, B_2, B_3)$, and so has zero contraction with $\Gamma_\pm$ (see page 18). Theorem 4.10 then implies that

$$E^\flat = \tfrac{1}{2}R(du_3 + dv_3) + \tfrac{1}{2}(u_3 + v_3)dR = \tfrac{1}{2}d(Rp_3).$$

On the other hand,

$$E^\natural = -\tfrac{1}{2}(u_3 - v_3)R\,dR + \tfrac{1}{2}(u_3 + v_3)\sum_{j=1}^{3}(u_j - v_j)(du_j + dv_j) + RB_3.$$

The result follows from (7.3). □

From the discussion in Section 2, we have

$$N^{-1/4}\psi^+ = X \lrcorner\,(*\varphi) = d\beta,$$

where $\beta = -\frac{1}{4}R^4 X \lrcorner\, \mathrm{Re}\,\Upsilon$. Expressing β in terms of $(\mathbf{u}, \mathbf{v})$ is not hard:

Lemma 7.6. *The 2-form β is given by*

$$16R^{-1}\beta \;\circleddash\; 3\{\mathbf{u}, d\mathbf{v}, d\mathbf{v}\} - uv^{-1}\{\mathbf{v}, d\mathbf{v}, d\mathbf{v}\} + 4\{\mathbf{u}, d\mathbf{u}, d\mathbf{v}\}.$$

To find ψ^- and the $(3,0)$-form $\Psi = \psi^+ + i\psi^-$, one needs to involve Θ_2 more directly. This is achieved in the next result, which we quote without proof. It illustrates the complexity of the $SU(3)$ structure induced on $\mathscr{M}$.

Theorem 7.7. *The space of* $(3,0)$ *forms on* $\mathscr{M}$ *is generated by* $\Psi = \psi^+ + i\psi^-$, *where*

$$\begin{aligned} 8uv\psi^+ \;\text{⊜}\;\; & \tfrac{1}{6}v(N_2+4v^2)\{d\mathbf{u},d\mathbf{u},d\mathbf{u}\} - v(4u^2+3uv+\mathbf{u}\cdot\mathbf{v})\{d\mathbf{v},d\mathbf{u},d\mathbf{u}\} \\ & +\big((u+2v)\mathbf{v}\cdot d\mathbf{v} + v\mathbf{u}\cdot d\mathbf{v}\big)\wedge\{\mathbf{u},d\mathbf{u},d\mathbf{u}\} + (v\mathbf{u}\cdot d\mathbf{v} - u\mathbf{v}\cdot d\mathbf{v})\wedge\{\mathbf{v},d\mathbf{u},d\mathbf{u}\}. \end{aligned}$$

$$\begin{aligned} 4N_2^{1/2}\psi^- \;\widehat{\text{⊜}}\;\; & \tfrac{1}{3}(N_2+4v^2)\{d\mathbf{u},d\mathbf{u},d\mathbf{u}\} + \big((3+uv^{-1})\mathbf{u}\cdot\mathbf{v} - 3u^2 - 5uv\big)\{d\mathbf{u},d\mathbf{v},d\mathbf{v}\} \\ & +2\mathbf{v}\cdot d\mathbf{v}\wedge\{\mathbf{u},d\mathbf{u},d\mathbf{u}\} + 2uv^{-1}\mathbf{v}\cdot d\mathbf{u}\wedge\{\mathbf{v},d\mathbf{v},d\mathbf{v}\} \\ & +\big((1-uv^{-1})\mathbf{v}\cdot d\mathbf{v} + (3+vu^{-1})\mathbf{u}\cdot d\mathbf{v}\big)\wedge\{\mathbf{v},d\mathbf{u},d\mathbf{u}\}. \end{aligned}$$

We have verified by computer that the $(1,0)$-form γ defined by Proposition 7.5 satisfies $\gamma\wedge\Psi = 0$.

Corollary 7.8. *The almost complex structure* $\mathbb{J}$ *induced on* $\mathscr{M}$ *is not integrable.*

Proof. From Remark 2.2, one must verify that $d(N_2^{-1/4}\psi^-) \neq 0$. In fact, to do this, we have computed

$$d(N_2^{-1/4}\psi^-)\wedge du_1\wedge du_2\,\big|_{(2,2,1;2,2,1)} = \tfrac{2}{3}du_{123}\wedge dv_{123},$$

where the left-hand side has been evaluated at $\mathbf{u} = (1,2,2)$ and $\mathbf{v} = (1,2,2)$, so that (conveniently) $u = v = 3$. $\square$

The next result implies that ψ^+ vanishes on the 3-dimensional subspace $\langle du_i, dv_i, dR\rangle^{\mathrm{o}}$ of $T_m\mathscr{M}$, for any m and fixed i:

Lemma 7.9.

$$\psi^+\wedge du_i\wedge dv_i\wedge dR = 0, \qquad i = 1,2,3.$$

Proof. To check the equation, take $i = 3$ again. The exterior product of $du_3\wedge dv_3$ with the *visible part* of $8uv\psi^+$ displayed in Theorem 7.7 equals

$$\begin{aligned} & u(u_3v_1 - v_1v_3) + v(u_1u_3 + u_1v_3 + 2u_3v_1)dv_{31}\wedge du_{123} \\ & \qquad +u(u_3v_2 - v_2v_3) + v(u_2u_3 + u_2v_3 + 2u_3v_2)dv_{32}\wedge du_{123}. \end{aligned}$$

Wedging further with dR yields

$$(u_1u_3v_2 + u_1v_2v_3 - u_2u_3v_1 - u_2v_1v_3)dv_{123}\wedge du_{123},$$

but this is cancelled by the symmetrization implicit in the relation ⊜. $\square$

In contrast to the lemma, one can verify that

$$\psi^-\wedge du_3\wedge dv_3\wedge dR \neq 0,$$

except at points where $u_1 = u_2 = v_1 = v_2 = 0$ or $u_3 = v_3 = 0$. It follows that the subspace $\langle du_i, dv_i, dR\rangle$ of $T_m^*\mathscr{M}$ admits no $\mathbb{J}$-invariant 2-plane for generic m, though the key word is 'generic'. For if we restrict the $(3,0)$ form Ψ of Theorem 7.7 to any 4-dimensional linear subvariety $\mathscr{M}(\mathbf{n})$ (recall Definition 5.6 and Corollary 5.7), the result is zero because all the triple products vanish. This implies

Theorem 7.10. *$\mathscr{M}(\mathbf{n})$ is $\mathbb{J}$-holomorphic for each $\mathbf{n}\in S^2$.*

Proof. We present an argument that does not depend on the calculation of $\psi^\pm$. Take $\mathbf{n}=(1,0,0)$ for definiteness, so that $\mathscr{M}(\mathbf{n})$ lies in

$$\mathbb{R}^4=\{(\mathbf{u},\mathbf{v})\in\mathbb{R}^6 : u_1=0=v_1\}.$$

Consider the tangent vectors

$$\partial_1=\partial/\partial u_1,\qquad \partial_4=\partial/\partial v_1$$

in $\mathbb{R}^6$ defined along this $\mathbb{R}^4$ (they are normal to the $\mathbb{R}^4$ relative to the flat metric.) It follows Theorem 7.2 that

$$\partial_1{}^\natural=R\,dv_1,\qquad \partial_4{}^\natural=-R\,du_1. \tag{7.4}$$

A key reason for this simplicity is that

$$\partial_1\lrcorner\, du=u^{-1}\partial_1\lrcorner\,(u_1du_1),$$

which vanishes along $\mathbb{R}^4$, similarly for $\partial_4\lrcorner\, dv$ and interior products with $dR=du+dv$. Theorem 4.10 implies that

$$\begin{array}{rcl}\partial_1{}^\flat &=& \frac12 N^{-1}\big[(N+4v^2)du_1+(N-4uv)dv_1\big]\\ \partial_4{}^\flat &=& \frac12 N^{-1}\big[(N-4uv)du_1+(N+4u^2)dv_1\big],\end{array} \tag{7.5}$$

where $N=N_2=6uv-2\mathbf{u}\cdot\mathbf{v}$. This time, a key point is that the interior products with Γ_+ and Γ_- vanish along $\mathbb{R}^4$ (as in the proof of Proposition 7.5). It follows from (7.3) that $\mathbb{J}du_1$ and $\mathbb{J}dv_1$ both belong to $\langle du_1,dv_1\rangle$ at all points of $\mathbb{R}^4$ for which $uv\neq 0$. The annihilator of these subspaces are the tangent spaces to $\mathbb{R}^4$, and are therefore $\mathbb{J}$-invariant. □

Remark 7.11. We know from (7.3) that $\mathbb{J}E^\natural=-N^{1/2}E^\flat$ for any tangent vector E. A computation of $\mathbb{J}^2$ involves the determinant

$$(N+4u^2)(N+4v^2)-(N-4uv)^2=4R^2N,$$

and allows us to verify that $\mathbb{J}^2=-\mathbf{1}$. This confirms that the symplectic form σ is correctly normalized in Theorem 7.2. We can also strengthen Corollary 7.8 using the $(1,0)$-forms $\varepsilon_i=(1-i\mathbb{J})d(Rp_i)$ made explicit in Proposition 7.5. A computation shows that

$$d\varepsilon_2\wedge\varepsilon_2\wedge\varepsilon_3\wedge du_1\wedge dv_1\neq 0\quad\text{along}\quad u_1=0=v_1,$$

so the restriction of $\mathbb{J}$ to $\mathscr{M}(\mathbf{n})$ is not integrable. Finally, note that $\mathbb{J}$ degenerates across the locus $N=0$, i.e. when $u=0$ or $v=0$.

Finally, the vanishing of both Ψ and F_2, when restricted to $\mathscr{M}(\mathbf{n})$, is consistent with Lemma 2.1, which describes the intrinsic torsion of the $SU(3)$ structure induced on $\mathscr{M}$.

8. Metrics on subvarieties

We return to consider the family of Riemannian metrics g_c in(tro)duced by Definition 4.3 and described by Lemma 4.11. Restricted to the hypersurface $R = 1$, each is the pushdown of the metric $\widehat{h}_c$ described by Proposition 3.3. Recall that $\widehat{h}_1$ is the Kähler metric on $\mathbb{CP}^3$ and h_2 has holonomy G_2 on the cone $\mathscr{C} = \mathbb{R}^+ \times \mathbb{CP}^3$. The diagonal action of $SO(3)$ on

$$\{(\mathbf{u}, \mathbf{v}) \in \mathbb{R}^6 : \mathbf{u}, \mathbf{v} \in \mathbb{R}^3\}$$

defines an isometry of $(\mathscr{M}, g_c)$ for any $c < 3$.

We first present a result that motivated other results in this section.

Proposition 8.1. *The restriction of g_c to the negative quadrant*

$$\{(\mathbf{u}, \mathbf{v}) = (u, 0, 0; -v, 0, 0) : u, v > 0\} \subset \mathscr{F}_-$$

in $\mathscr{M}$ has first fundamental form

$$\Big(1 + \frac{v}{2u}\Big)du^2 + du\,dv + \Big(1 + \frac{u}{2v}\Big)dv^2$$

independently of $c < 3$, and zero Gaussian curvature.

Proof. This result may be regarded as a corollary of Propositions 3.3 and 4.2. Up to the action of $SO(2)$, we can realize the subset $u_2 = u_3 = v_2 = v_3 = 0$ of $\mathscr{M}$ by taking

$$x_2 = x_3 = x_4 = x_5 = 0, \quad \text{i.e.} \quad z_1 = z_2 = 0,$$

so that

$$u = x_0^2 + x_1^2 = |z_0|^2, \quad v = x_6^7 + x_7^2 = |z_3|^2.$$

Moreover, $\alpha_2 = \alpha_3 = 0$ so $\mu_2 = \mu_3 = 0$, and $\mu_1 = u - v$. Thus h_c and Θ_c are independent of c. We already know from Definition 4.8 that $N = h_c(X, X)$ is independent of c on $\mathscr{F}_-$. The first fundamental form can now be read off from Theorem 4.7: g_1 restricts to

$$ds^2 = \tfrac{1}{2}(du + dv)^2 + \tfrac{1}{2}(u^{-1}du^2 + v^{-1}dv^2),$$

which simplifies to that stated.

To prove that this metric has zero Gaussian curvature where defined, we use the substitution

$$\left\{\begin{array}{l} u = R\cos^2(\phi/2) \\ v = R\sin^2(\phi/2), \end{array}\right. \tag{8.1}$$

with $0 \leqslant \phi \leqslant \pi$. Then

$$\left\{\begin{array}{l} du = \cos^2(\phi/2)dR - R\cos(\phi/2)\sin(\phi/2)d\phi \\ dv = \sin^2(\phi/2)dR + R\cos(\phi/2)\sin(\phi/2)d\phi. \end{array}\right. \tag{8.2}$$

A straightforward calculation reveals that

$$ds^2 = dR^2 + \tfrac{1}{2}R^2 d\phi^2, \tag{8.3}$$

which is the metric on a double cone in $\mathbb{R}^3$ with half angle $\pi/4$, and certainly flat. $\square$

Remark 8.2. Because of the $SO(3)$ invariance, g_c will have an identical nature on any negative 'diagonally linear' quadrant in $\mathbb{R}^6$, and one can also switch the minus sign from v_1 to u_1. It follows from the proof of Proposition 8.1 that such a quadrant lies in the projection of the cone over a complex projective line $\mathbb{CP}^1 \subset Z_-$ tangent to the horizontal distribution D (see Corollary 6.3), and can therefore said to be *superminimal* [7]. The range of the angle ϕ in the proof was restricted to $(0,\pi)$, but can be extended to $\mathbb{R}/(2\pi\mathbb{Z})$ to include the image of $j(\mathbb{CP}^1)$. It also follows from the proof of the proposition that g_1 takes an identical form in the positive quadrant

$$\{(\mathbf{u},\mathbf{v}) = (u,0,0;v,0,0) : u,v>0\} \subset \mathscr{F}_+,$$

though g_c will have a slightly different (albeit, flat) form if $c \neq 1$.

We shall use the substitution (8.1) throughout this section, in order to revert to a conical description

$$g_c = dR^2 + R^2\widehat{g}_c \tag{8.4}$$

of the induced metrics on $\mathscr{M}$, reflecting their origin in $\mathscr{C}$. There are three justifications for squaring the trigonometric functions: (i) it ensures that $R = u+v$ is as before, (ii) it amounts to using polar coordinates for the complex moduli $|z_i|$ in $\mathscr{C}$, and (iii) it simplifies the form of g_c in subsequent statements. The choice of the half-angle is less significant.

In addition, we set

$$\left\{\begin{array}{rcl} \mathbf{u} &=& u\boldsymbol{\sigma},\\ \mathbf{v} &=& v\boldsymbol{\tau}, \end{array}\right.$$

so that $\boldsymbol{\sigma},\boldsymbol{\tau} \in S^2$, and $|d\boldsymbol{\sigma}|^2 = d\boldsymbol{\sigma}\cdot d\boldsymbol{\sigma}$ and $|d\boldsymbol{\tau}|^2 = d\boldsymbol{\tau}\cdot d\boldsymbol{\tau}$ denote the round metrics. Then

$$\left\{\begin{array}{rcl} d\mathbf{u} &=& \boldsymbol{\sigma}\,du + u\,d\boldsymbol{\sigma},\\ d\mathbf{v} &=& \boldsymbol{\tau}\,dv + v\,d\boldsymbol{\tau}, \end{array}\right.$$

and

$$|d\mathbf{u}+d\mathbf{v}|^2 = du^2 + dv^2 + u^2|d\boldsymbol{\sigma}|^2 + v^2|d\boldsymbol{\tau}|^2 + 2uv\cos 2\theta,$$

where $\boldsymbol{\sigma}\cdot\boldsymbol{\tau} = \cos 2\theta$ so that the angle between $\mathbf{u}$ and $\mathbf{v}$ equals 2θ.

Using the methods of Section 4, one can show that $\widehat{g}_c = \widehat{g}_c(\phi,\sigma,\tau)$ extends to a smooth bilinear form on $\mathbb{R}^6$, though we shall restrict our discussion to the cases $c=1$ and $c=2$. Lemma 4.11 suggests that the second case faithfully reflects the behaviour of g_c for all values of the parameter c with $1<c<3$. In an attempt to identify the two 2-spheres in $\mathbb{R}^3\cup\mathbb{R}^3$ (and thereby bypass singularities), we shall first describe the restriction of g_1 and g_2 to the subvarieties

$$\mathscr{F}_\pm \;\cong\; \mathbb{R}^+ \times [0,\pi] \times S^2. \tag{8.5}$$

of Definition 4.8. The bijection (8.5) is determined by the coordinates $(R,\phi,\boldsymbol{\sigma})$.

It follows from Theorem 4.7 that g_1 is the sum of $\frac{1}{2}R$ times $u|d\boldsymbol{\sigma}|^2 + v|d\boldsymbol{\tau}|^2$ and the first fundamental form (8.3). It follows that

$$\widehat{g}_1 = \cos^2(\phi/2)\big|d\boldsymbol{\sigma}\big|^2 + \sin^2(\phi/2)\big|d\boldsymbol{\tau}\big|^2 + d\phi^2. \tag{8.6}$$

This implies that the restriction of g_1 to *both* $\mathscr{F}_+$ and $\mathscr{F}_-$ equals

$$dR^2 + R^2(\tfrac{1}{2}|d\boldsymbol{\sigma}|^2 + \tfrac{1}{2}d\phi^2).$$

By contrast,

Corollary 8.3. *The restriction of g_2 to $\mathscr{F}_\pm$ equals $dR^2 + R^2\widehat{g}_2$ where*

$$\widehat{g}_2 = \begin{cases} \frac{1}{2}|d\boldsymbol{\sigma}|^2 + \frac{1}{4}d\phi^2 & \text{on } \mathscr{F}_+ \\ \frac{1}{8}(3+\cos 2\phi)|d\boldsymbol{\sigma}|^2 + \frac{1}{2}d\phi^2 & \text{on } \mathscr{F}_-. \end{cases}$$

Proof. We use Theorem 4.10, bearing in mind that $N = N_2 = 6uv - 2\mathbf{u}\cdot\mathbf{v}$ simplifies to $4uv$ on $\mathscr{F}_+$ and $8uv$ on $\mathscr{F}_-$. Suppose that $(\mathbf{u},\mathbf{v}) \in \mathscr{F}_\pm$, so $\mathbf{u} = u\boldsymbol{\sigma}$ and $\mathbf{v} = \pm v\boldsymbol{\sigma}$. Then

$$\begin{aligned} |d\mathbf{u} + d\mathbf{v}|^2 &= \big|(du \pm dv)\boldsymbol{\sigma} + (u \pm v)d\boldsymbol{\sigma}\big|^2 \\ &= \begin{cases} dR^2 + R^2|d\boldsymbol{\sigma}|^2 & \text{on } \mathscr{F}_+ \\ (du - dv)^2 + (u-v)^2|d\boldsymbol{\sigma}|^2 & \text{on } \mathscr{F}_-. \end{cases} \end{aligned}$$

Moreover,

$$\begin{aligned} |\mathbf{B}|^2 &= \big| \pm u(dv\boldsymbol{\sigma} + v\,d\boldsymbol{\sigma}) - v(du\boldsymbol{\sigma} + u\,d\boldsymbol{\sigma})\big|^2 \\ &= \begin{cases} (u\,dv - v\,du)^2 = R^2uv\,d\phi^2 & \text{on } \mathscr{F}_+ \\ (u\,dv + v\,du)^2 + 4u^2v^2|d\boldsymbol{\sigma}|^2 & \text{on } \mathscr{F}_-. \end{cases} \end{aligned}$$

The proof is completed by adding up the various terms, recalling that $\Gamma_\pm = 0$ on $\mathscr{F}_\pm$ (see Definition 4.8). The case of $\mathscr{F}_+$ is easiest, and can also be deduced from Lemma 4.11. For $\mathscr{F}_-$, the non-trivial coefficient of $|d\boldsymbol{\sigma}|^2$ in g_2 arises as $\frac{1}{2}(u^2+v^2)$. □

Remark 8.4. The restriction of g_1 to $\mathscr{F}_\pm$ is invariant by diagonal translation in the (u,v) plane defined by rotating the angle ϕ. This action is not however an isometry for $g_2|_{\mathscr{F}_-}$. Restricted to four dimensions, neither metric degenerates where only one of u,v is zero. Equation (8.6) shows that g_1 is singular on each locus $\{u=0\}$ and $\{v=0\}$, since the respective 2-sphere is shrunk to a point. The same is true for g_2 because

$$\begin{aligned} \lim_{u->0}\widehat{g}_2 &= |d\boldsymbol{\sigma}|^2 + \tfrac{1}{4}(3-\cos 2\theta)d\phi^2, \\ \lim_{v->0}\widehat{g}_2 &= |d\boldsymbol{\tau}|^2 + \tfrac{1}{4}(3-\cos 2\theta)d\phi^2, \end{aligned} \tag{8.7}$$

limits that are verified using Theorem 4.10. Although g_1 has little to do with G_2 holonomy, it *is* associated to a G_2 structure with closed 3-form that arises as the $U(1)$ quotient of the flat $Spin(7)$-structure on $\mathbb{R}^8$ [16].

The 4-dimensional subvarieties $\mathscr{F}_\pm$ of $\mathscr{M}$ are distinguished by their $SO(3)$-invariance. Neither can be $\mathbb{J}$-holomorphic; this follows from

Lemma 8.5. *Fix $\mathscr{F}_+$ or $\mathscr{F}_-$. Each 2-sphere $S^2_{u,v} \subset \mathscr{F}_\pm$ defined by setting u and v equal to positive constants is totally real in $\mathscr{M}$; indeed $\mathbb{J}(TS^2_{u,v}) \perp T\mathscr{F}_\pm$.*

Proof. Let V, W be vectors tangent to $\mathscr{F}_\pm$ with V tangent to one of the 2-spheres $S^2_{u,v}$ at some point. Using identities from the proofs of Theorem 7.2 and Corollary 8.3, one deduces that the pullback of σ equals $d\zeta$ where

$$\zeta = \begin{cases} -\frac{1}{3}R(u\,dv - v\,du) & \text{on } \mathscr{F}_+ \\ \frac{2}{3}R(u\,dv - v\,du) & \text{on } \mathscr{F}_-. \end{cases}$$

Therefore σ equals a constant multiple of $R\,du \wedge dv$ on $\mathscr{F}_\pm$, and

$$g(\mathbb{J}V, W) = \sigma(V, W) = 0,$$

as asserted. □

From (8.5), we see that $S^2_{1,1}$ parametrizes a family of real surfaces inside $\mathscr{F}_\pm$. Up to the $SO(3)$ action, each is equivalent to one of the quadrants of Proposition 8.1 or Remark 8.2. By Corollary 8.3, the tangent spaces of the leaf

$$\{(u\boldsymbol{\sigma}, \pm v\boldsymbol{\sigma}) : u, v > 0\} \subset \mathscr{F}_\pm$$

are orthogonal to the distribution TS^2_{uv} in $T\mathscr{F}_\pm$, so Lemma 8.5 implies that the leaf is $\mathbb{J}$-holomorphic. This is also a corollary of Theorem 7.10, since the leaf is the intersection of two of the $\mathbb{J}$-holomorphic surfaces.

We began this section by restricting g_c to a 2-dimensional subspace $\mathbb{R}^2$ of $\mathbb{R}^6$, and then extended that result to the 4-dimensional subvarieties $\mathscr{F}_+, \mathscr{F}_-$. It is natural to consider too the subvarieties highlighted by Definition 5.6 and Theorem 7.10. For definiteness, we shall take $\mathbf{n} = (0, 0, 1)$ this time, so that

$$\mathscr{M}(\mathbf{n}) = \{(\mathbf{u}, \mathbf{v}) \in \mathscr{M} : u_1 = 0 = v_1,\ uv \neq 0\}. \tag{8.8}$$

Since $\mathbf{n}$ will remain fixed for the remainder of this section, we shall denote the subspace of $\mathbb{R}^6$ containing (8.8) merely by $\mathbb{R}^4$.

The choice of setting u_1, v_1 to zero (contrasting with that of Proposition 8.1) is dictated by their expression as quadratic forms diagonalized by $x_0, \ldots, x_7$, so that $|z_0| = |z_2|$, and $|z_1| = |z_3|$. This enables us to take

$$[\mathbf{z}] = [z_0, z_1, z_2, z_3] = [\lambda e^{i\alpha}, \mu e^{i\beta}, \lambda e^{i\gamma}, \mu e^{i\delta}],$$

with $\lambda, \mu \geqslant 0$ and $\lambda\mu \neq 0$. (Square brackets again represent the $U(1)_1$ quotient.) Then

$$\begin{cases} \mathbf{u} &= \big(0,\ u\cos(\theta + \chi), -u\sin(\theta + \chi)\big), \\ \mathbf{v} &= \big(0,\ v\cos(\theta - \chi),\ \ v\sin(\theta - \chi)\big), \end{cases}$$

where $u = 2\lambda^2$, $v = 2\mu^2$ and $\alpha - \gamma = \theta + \chi$ and $\beta - \delta = \theta - \chi$. Since $\mathbf{u}\cdot\mathbf{v} = uv\cos 2\theta$, the angle between $\mathbf{u}$ and $\mathbf{v}$ is again θ and is independent of χ.

Using the coordinates (R, ϕ) and substituting expressions for $\mathbf{u}, \mathbf{v}$ into g_2 yields

Theorem 8.6. *The restriction of* $\widehat{g}_2$ *to* $\mathbb{R}^4$ *equals*

$$\tfrac{1}{2}d\theta^2 + \tfrac{1}{8}(3 - \cos 2\theta)d\phi^2 + \tfrac{1}{16}(7 + \cos 2\theta + 2\sin^2\theta\cos 2\phi)d\chi^2 \\ + \cos\phi\, d\theta\, d\chi - \tfrac{1}{4}\sin 2\theta \sin\phi\, d\phi\, d\chi.$$

Note that when $\phi = 0$ $(v = 0)$ or $\phi = \pi$ $(u = 0)$ then $\widehat{g}_2$ degenerates to $(d\theta \pm d\phi)^2 + \cdots$.

We now define subsets

$$\begin{aligned} \mathscr{F}'_- &= \{(0, -u\sin\chi, -u\cos\chi;\, 0,\ v\sin\chi,\ v\cos\chi)\} \\ \mathscr{F}'_1 &= \{(0,\ u\cos\chi,\ -u\sin\chi;\ 0,\ v\cos\chi, -v\sin\chi)\}, \end{aligned}$$

corresponding to to $\theta = \pi/2$ and $\theta = 0$, and

$$\begin{aligned} \mathscr{F}'_2 &= \{(0, -u\sin\theta, -u\cos\theta;\ 0,\ v\sin\theta, -v\cos\theta)\} \\ \mathscr{F}'_3 &= \{(0,\ u\cos\theta,\ -u\sin\theta;\ 0,\ v\cos\theta,\ v\sin\theta)\}, \end{aligned}$$

corresponding to $\chi = \pi/2$ and $\chi = 0$. In all cases, $u, v \geqslant 0$ and $0 \leqslant \theta, \chi \leqslant 2\pi$. It follows from the definition in Proposition 6.4 that

$$\mathscr{F}'_- = \mathscr{F}_- \cap \mathbb{R}^4, \quad \text{and} \quad \mathscr{F}'_i = \mathscr{F}_i \cap \mathbb{R}^4 \quad \forall i = 1, 2, 3,$$

to which we can add that $\mathscr{F}'_1 = \mathscr{F}_+ \cap \mathbb{R}^4$, since the equations of Z_+ and Z_1 coincide when $|z_1| = |z_3|$. But there is also a duality between the pairs $\{\mathscr{F}_-, \mathscr{F}'_1\}$ and $\{\mathscr{F}'_2, \mathscr{F}'_3\}$ that derives from Remark 4.1: changing the sign of v_3 swaps the subsets over, though this map is definitely not $SO(3)$-equivariant.

Observe that $\mathscr{F}'_2 = R_{23}(\mathscr{F}'_3)$ where $R_{23} \in SO(2)$ is rotation by $\pi/2$ in the '2–3' plane. Any non-zero vector in $\mathbb{R}^6$ lies in the $SO(3)$ orbit of a point of $\mathscr{F}'_2$ or $\mathscr{F}'_3$ (this orbit is a 2-sphere if θ is a multiple of $\pi/2$). Therefore both $\mathscr{F}'_2$ and $\mathscr{F}'_3$ are *slices* for the $SO(3)$ action, corresponding to the invariants u, v, θ. These isometric slices are characterized by an almost symmetric treatment of $\mathbf{u}$ and $\mathbf{v}$, which leads to a simpler form of the induced metric below. The intersection $\mathscr{F}'_2 \cap \mathscr{F}'_3$ contains the union $\mathbb{R}^2 \cup \mathbb{R}^2$ of the two 2-planes given by $uv = 0$.

Theorem 8.6 and the subsequent discussion yields

Theorem 8.7. *An open subset of $\mathscr{M} = \mathbb{R}^6 \setminus \mathbf{0}$ is foliated by 3-dimensional submanifolds parametrized by $SO(3)$, each member of which is a cone over a cylinder isometric to $\mathscr{F}'_2$ (or $\mathscr{F}'_3$) endowed with the metric $dR^2 + R^2\widehat{g}_2$ where*

$$\widehat{g}_2 = \tfrac{1}{2}d\theta^2 + \tfrac{1}{8}(3 - \cos 2\theta)d\phi^2.$$

Remark 8.8. Since the quadrics Z_1, Z_2, Z_3 are equivalent under $SU(2)$, their images in $\mathscr{M}$ are congruent under $SO(3)$ and thus all isometric. What is more surprising is that $\mathscr{F}_2$ embeds isometrically into $\mathscr{F}_-$ after a change of coordinates $\theta \mapsto \phi + \frac{\pi}{2}$. At first sight this appears to contradict Corollary 8.3, given that $\mathscr{F}_1$ coincides with the *positive* space $\mathscr{F}_+$ over $\mathbb{R}^4$. The problem is that, although $\mathscr{F}_1 = R_{13}(\mathscr{F}_3)$, this relation no longer holds if we apply primes.

Let us try to visualize $\mathscr{F}'_2$ and $\mathscr{F}'_3$. Each is a cone over a cylinder $S^1 \times [0, \pi]$, with $\theta \in S^1$ and $\phi \in [0, \pi]$. If one sets

$$f(\theta) = \tfrac{1}{2}\sqrt{3 - \cos 2\theta},$$

then $2\widehat{g}_2$ is the first fundamental form of the surface of revolution in $\mathbb{R}^3$ parametrized by

$$\big(f(\theta)\cos\phi,\ f(\theta)\sin\phi,\ g(\theta)\big).$$

Here $f'(\theta)^2+g'(\theta)^2=1$, so that the profile curve $(f(\theta),g(\theta))$ has unit speed. Its Gaussian curvature

$$K=-\frac{f''(\theta)}{f(\theta)}=1-\frac{8}{(3-\cos 2\theta)^2}$$

varies between -1 and $\frac{1}{2}$.

Figure 3 represents $(\mathscr{F}_2'\cup\mathscr{F}_3')/\mathbb{R}^+$ topologically as a 2-torus, obtained by identifying the outer boundaries of the curvilinear rectangle in the usual way. We have chosen to represent $\mathscr{F}_2'$ by the blue patch, whose boundaries correspond to $\phi=\pi,0$ (meridians left and right forming the intersection $\mathscr{F}_2'\cap\mathscr{F}_3'$) and to $\theta=0,2\pi$ (semicircles bottom and top that are identified to form the cylinder). Attaching $\mathscr{F}_2'$ (the yellow patch) to $\mathscr{F}_3'$ requires a vertical jump $\theta\mapsto\theta+\pi/2$ (corresponding to $L\in SO(2)$); this is shown schematically for $u=0$ but the combination is not smooth (the parallels are not C^2 in ϕ). The isometry $\hat{\jmath}$ of Lemma 5.1 acts by interchanging $\theta\leftrightarrow 2\pi-\theta$ and $\phi\leftrightarrow\pi-\phi$, and therefore flips the upper and lower halves of each coloured surface.

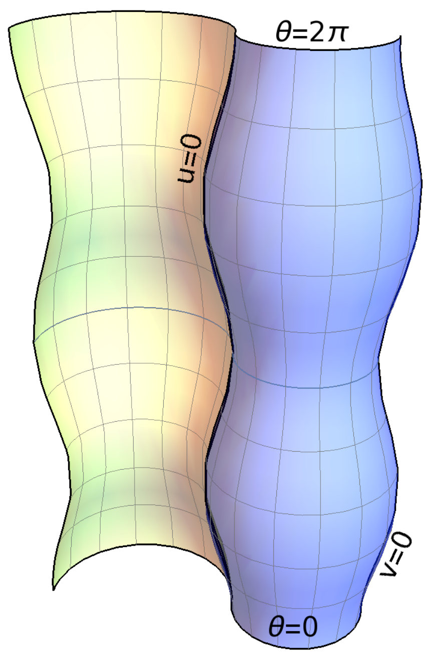

Figure 3: Surfaces of revolution associated to $\mathscr{F}_2'$ and $\mathscr{F}_3'$

Since $\mathscr{F}_2'$ incorporates pairs $(\mathbf{u},\mathbf{v})$ in which the two vectors make an arbitrary angle, we can use its geometry to measure the relative orientation of the two $\mathbb{R}^3$ subspaces defined by $v=0$ and $u=0$. The so-called 'principal angles' between these subspaces are determined by the function

$$\theta\longmapsto\int_0^\pi \tfrac{1}{\sqrt{2}}f(\theta)\,d\phi=\pi\sqrt{\tfrac{3}{8}-\tfrac{1}{8}\cos 2\theta}.$$

Its value varies from $\pi/2$ to $\pi/\sqrt{2} \sim 127^{\circ}$, and the bulges in the surface of revolution reflect the fact that a semicircle of radius $R = 1$ has circumference $\sqrt{2}\pi$ when $\theta = \pi/2$ or $3\pi/2$. The latter corresponds to the situation of Proposition 8.1 in which the two vectors are anti-aligned.

Observe that $Q^{-1}(\mathscr{F}_i')$ is diffeomorphic to a cone over $S^1 \times S^2$ for $i = 1, 2, 3$. Corollary 5.7 tells us that the connection Θ_2 is flat over $\mathbb{R}^4$, so there it equals some exact 1-form $d\psi$. The restriction of the G_2 metric h_2 to $Q^{-1}(\mathscr{F}_2')$ equals

$$\begin{aligned} Q^*g_2 + \tfrac{1}{4}N_c\Theta_2^2 &= Q^*g_2 + 2uv(3 - \cos 2\theta)d\psi^2 \\ &= dR^2 + \tfrac{1}{2}R^2\left[d\theta^2 + \tfrac{1}{4}(3 - \cos 2\theta)\big(d\phi^2 + \sin^2\!\phi\, d\psi^2\big)\right]. \end{aligned}$$

Parallel semi-circles in the surface of revolution are the images by Q of 2-spheres with latitude ψ, which collapses at north and south poles lying over the meridians $u = 0$ and $v = 0$. In this way, the circle quotient over $\mathscr{F}_2'$ is modelled metrically on the height function $S^2 \to [-1, 1]$.

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