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Seifert surfaces distinguished by sutured Floer homology but not its Euler characteristic

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In this paper we find a family of knots with trivial Alexander polynomial, and construct two non-isotopic Seifert surfaces for each member in our family. In order to distinguish the surfaces we study the sutured Floer homology invariants of the sutured manifolds obtained by cutting the knot complements along the Seifert surfaces. Our examples provide the first use of sutured Floer homology, and not merely its Euler characteristic (a classical torsion), to distinguish Seifert surfaces. Our technique uses a version of Floer homology, called "*longitude Floer homology*" in a way that enables us to bypass the computations related to the *SFH* of the complement of a Seifert surface.

1. Introduction

It is known that every knot in S^3 bounds a Seifert surface. Seifert surfaces play an important role in knot theory and low dimensional topology in general. The minimum genus taken over all oriented surfaces that a knot K bounds is called the genus of K. It is natural to wonder whether or not a given minimal genus Seifert surface for a knot is unique. To make sense of this question, we should be clear on the notion of equivalence between surfaces. We consider two surfaces R and R' to be equivalent if there is an isotopy of S^3 taking R to R'. Fiberedness of a knot is known as a sufficient condition for which its minimal genus Seifert surface is unique (see [4]). However, there are many known examples of knots with non-isotopic Seifert surfaces. See for instance [1,2,6,13,17–20,24].

Many classical tools in distinguishing surfaces deal with the surfaces' complements in S^3. These tools include, for instance, Seifert forms and the fundamental group of the surfaces' complements. They are quite powerful, but, they can potentially lead to tedious algebraic computations. There are also examples beyond the scope of classical tools.

In this paper we find knots with trivial Alexander polynomial and two distinguished Seifert surfaces for each. The idea is that we plumb two untwisted annuli. Then we tie arbitrary nontrivial knots, K_1 and K_2

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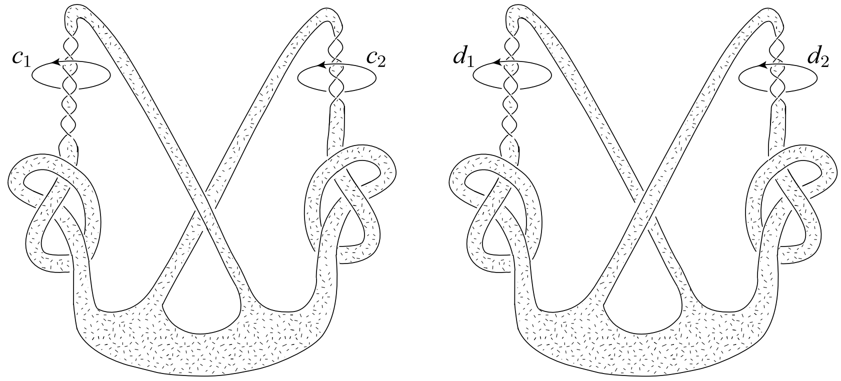


Fig. 1. The above pictures are over/under plumbings of two twisted annuli, R and R' respectively, both are bounded by the same knot, $P(K_1, K_2)$, where K_1 is the right handed trefoil and K_2 is the left handed trefoil. These lead to two distinguished Seifert surfaces R and R', up to equivalence (i.e. there is an isotopy of S^3 taking R to R'), for the knot $P(K_1, K_2)$. The simple closed curves, c_1, c_2, d_1 and d_2 are basis elements for H_1 of the complement of these surfaces inside S^3.

in each of the annuli. We produce some twists in each annulus in such a way that the (blackboard) framings are l and 0, respectively, where l is an arbitrary non-zero integer. We will see in Section 3 that R and R' both are bounded by the same knot $P_l(K_1, K_2)$ (denoted by $P(K_1, K_2)$ if l is understood from the context). Fig. 1 shows an example. Our main theorem shows that the two surfaces are inequivalent, provided that one of the twisting parameters is zero and the other is non-zero. The above strategy works regardless of the knots one ties to these annuli.

We are now in a position to state the main theorem.

Theorem 1. *Let $P_l(K_1, K_2)$ be the knot obtained by plumbing two annuli with arbitrary knots K_1 and K_2 as in Fig. 1, with framings l and 0, respectively, $l \neq 0$. Changing the plumbing as in Fig. 1 results in the same knot, but two inequivalent Seifert surfaces, R and R'.*

Our technique begins by noting that the surfaces' complements have a particular structure called a sutured manifold (see [8]). Assigned to sutured manifolds there is an invariant called sutured Floer homology (denoted by *SFH*) introduced by Juhász in [14]. One cannot possibly use only the rank of *SFH* of minimal genus Seifert surfaces' complements to distinguish them, since the rank in this case depends only on the knot (see [15, Theorem 1.5]). Therefore, we would need to know the structure of *SFH* as a Spin^c-graded group if we ever were able to use it to know two surfaces are not equivalent. Combining the sutured Floer homology of $(S^3(R), \gamma)$ with the Seifert form turns out to be a useful tool in distinguishing different Seifert surfaces (see [13]). Altman in [2] gives an example of using only the sutured Floer homology polytope to distinguish two Seifert surfaces for a knot (for related definitions see Section 2).

The reason we are interested in the particular knots here is twofold. First, the classical methods fail in distinguishing the two Seifert surfaces. Second, the polytopes of the surfaces' complements are the same. Our theorem produces the first examples where the (SFH) + (Seifert form) technique is successful, but $\chi(SFH)$ alone wouldn't have sufficed. Indeed, anytime the twisting of one of the annuli is zero, the torsions associated to the surfaces constructed would be the same provided that the knots we tie have trivial Alexander polynomial. Therefore, χ cannot be used to distinguish the surfaces. We refer the reader to [7, Section 3] for a detailed discussion about the identification of the Euler characteristic of the sutured Floer homology with a type of Turaev torsion polynomial.

We close this section by mentioning that there are other notions of equivalence one could consider. The one we work with throughout the paper (so called *weak equivalence*) is the same as regarding two surfaces R and R' to be equivalent if there is an orientation preserving diffeomorphism between the pairs (S^3, R) and (S^3, R') (see [11]). There is also a more restrictive notion called *strong equivalence*, that considers two

Seifert surfaces for a knot K as equivalent if they are ambient isotopic to each other in $S^3 \setminus n(K)$, where $n(K)$ is a neighborhood of the knot K inside S^3. While it was known that the Seifert surfaces we construct via over/under plumbings for the knots in our examples produce two distinguished Seifert surfaces for the knots up to strong equivalence (see [10, Corollary 3.2]), what we show in the paper is stronger, that is, the two Seifert surfaces are not weakly equivalent. From now on, we will not make any further references to strong equivalence.

2. Background

Sutured manifolds were introduced by Gabai in [8]. Sutured Floer homology is a generalization of Ozsváth–Szabó Floer homology to an invariant of sutured manifolds, and is defined in [14].

In this section we begin by briefly recalling some basic notions about sutured manifolds. We then discuss the structure of *SFH* as a group and how it behaves under decomposing a sutured manifold along an embedded surface. A key tool that will be used in Section 2.3 is a generalization of the Thurston norm to an invariant of relative homology classes in a sutured manifold called the Sutured-Thurston norm. We end this section by recalling a fact about the behavior of *SFH* under a Murasugi sum of two surfaces.

2.1. Sutured manifolds

The following with more details is contained in [14]. See also [8].

Definition 2.1. A sutured manifold (M, γ) is a compact oriented 3-manifold with boundary, together with a set $\gamma \subset \partial M$ consisting of annuli $A(\gamma)$ and tori $T(\gamma)$. Furthermore, the interior of each component of $A(\gamma)$ contains a suture, i.e., a homologically non-trivial simple closed curve. The union of the sutures is denoted by $s(\gamma)$.

We then take $R(\gamma) = \partial M \setminus \text{int}(\gamma)$. Define $R_+(\gamma)$ $(R_-(\gamma))$ to be those components of $R(\gamma)$ whose normal vector points out of (into) M. The orientation of $R(\gamma)$ must be coherent with respect to $s(\gamma)$, i.e., if δ is a component of $\partial R(\gamma)$ and is given the boundary orientation, then δ must represent the same homology class in $H_1(\gamma)$ as some suture. (M, γ) is called balanced if M has no closed components, $\chi(R_-(\gamma)) = \chi(R_+(\gamma))$, and the map from $\pi_0(A(\gamma))$ to $\pi_0(\partial M)$ is surjective.

Let $S^3(R_i) = S^3 \setminus \text{int}(R_i \times I)$ denote the complement of our Seifert surface. We equip this with a suture $\gamma = \partial R_i \times \{1/2\}$.

Definition 2.2. A sutured manifold (M, γ) is called taut if M is irreducible and $R(\gamma)$ is incompressible and Thurston norm minimizing in its homology class in $H_2(M, \gamma)$.

In general, for a Seifert surface R of a knot K it follows that $S^3(R)$ is taut if and only if $g(R) = g(K)$. In particular, both $S^3(R)$ and $S^3(R')$ in our examples are taut sutured manifolds.

2.2. Sutured Floer homology polytope and decomposition of a sutured manifold along an embedded surface

In this subsection we consider the *SFH* of a sutured manifold as a Spin^c-graded group. We recall the definition of the sutured Floer homology polytope and then describe how the shape of the polytope changes when one decomposes a sutured manifold along an embedded surface. Throughout, we use the notation of [16]. We do not define all the terms here and refer the reader to [14,15] and [16] for related definitions and also more details.

Let (M, γ) be a balanced sutured manifold. Let also v_0 be a nowhere vanishing vector field along ∂M pointing into M along $R_-(\gamma)$, pointing out of M along $R_+(\gamma)$ and which restricts to γ to be the gradient of a height function $s(\gamma) \times I \to I$. We point out that the space of such vector fields is contractible [16].

To a sutured manifold (M, γ), one can assign a Heegaard diagram $(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta})$ where Σ is a compact oriented surface with boundary and $\boldsymbol{\alpha} = \{\alpha_1, \alpha_2, \dots, \alpha_d\}$ and $\boldsymbol{\beta} = \{\beta_1, \beta_2, \dots, \beta_d\}$ are two sets of pairwise disjoint simple closed curves in $\mathrm{int}(\Sigma)$. Take $\mathbb{T}_{\boldsymbol{\alpha}} = \alpha_1 \times \alpha_2 \times \dots \times \alpha_d$ and $\mathbb{T}_{\boldsymbol{\beta}} = \beta_1 \times \beta_2 \times \dots \times \beta_d$ as subsets of the symplectic manifold $\mathrm{Sym}^g(\Sigma)$ (see [14] for details). To every point $\mathrm{x} \in \mathbb{T}_{\boldsymbol{\alpha}} \cap \mathbb{T}_{\boldsymbol{\beta}}$, one can associate a relative Spin^c structure, $\mathfrak{s}(\mathrm{x}) \in \underline{\mathrm{Spin}}^c(M, \gamma)$ as follows.

First pick a Morse function which determines the Heegaard diagram whose gradient vector field agrees with v_0 along ∂M. Next, modify the vector field in a neighborhood of the flowlines specified by x. This produces a non-vanishing vector field v that agrees with v_0 on ∂M. The homology class of v specifies a relative Spin^c structure which we denote by $\mathfrak{s}(\mathrm{x})$. It turns out that $\underline{\mathrm{Spin}}^c(M, \gamma)$ is an affine space over $H^2(M, \partial M; \mathbb{Z})$. Therefore, it makes sense to talk about the difference of two relative Spin^c structure, $\mathfrak{s}(\mathrm{x}) - \mathfrak{s}(\mathrm{y}) \in H^2(M, \partial M; \mathbb{Z})$. We denote $PD^{-1}[\mathfrak{s}(\mathrm{x}) - \mathfrak{s}(\mathrm{y})]$ by $\epsilon(\mathrm{x}, \mathrm{y})$ which is an element of $H_1(M; \mathbb{Z})$.

Definition 2.3. Let (M, γ) be a balanced sutured manifold. The support of sutured Floer homology of (M, γ) is

$$S(M, \gamma) = \{\mathfrak{s} \in \mathrm{Spin}^c(M, \gamma) : SFH(M, \gamma, \mathfrak{s}) \neq 0\}.$$

$SFH(M, \gamma)$ is a finitely generated abelian group (see [14]), thus, $S(M, \gamma)$ is finite. Moreover, if (M, γ) is taut then $S(M, \gamma) \neq \emptyset$ by [15, Theorem 1.4].

It turns out that $v_0^\perp$ is a trivial vector bundle over ∂M, provided that (M, γ) is balanced in each component. Let us denote the set of all trivializations of $v_0^\perp$ by $T(M, \gamma)$. For $\mathfrak{t} \in T(M, \gamma)$, let $c_1(\mathfrak{s}, \mathfrak{t}) \in H^2(M, \partial M; \mathbb{Z})$ be the relative Euler class of the vector bundle $v^\perp$ with respect to the trivialization $\mathfrak{t}$, where v is a nowhere zero vector field along M which agrees with v_0 on ∂M. In other words, $c_1(\mathfrak{s}, \mathfrak{t})$ is the obstruction to extending $\mathfrak{t}$ from ∂M to a trivialization of $v^\perp$ over M.

Definition 2.4. Fix $\mathfrak{t} \in T(M, \gamma)$. Define

$$C(M, \gamma, \mathfrak{t}) = \{i(c_1(\mathfrak{s}, \mathfrak{t})) : \mathfrak{s} \in S(M, \gamma)\} \subset H^2(M, \partial M; \mathbb{R}),$$

where $i : H^2(M, \partial M; \mathbb{Z}) \to H^2(M, \partial M; \mathbb{R})$ is the map induced by the natural embedding $\mathbb{Z} \hookrightarrow \mathbb{R}$.

Let $P(M, \gamma, \mathfrak{t})$ be the polytope obtained as the convex hull of $C(M, \gamma, \mathfrak{t})$ inside $H^2(M, \gamma; \mathbb{R})$. Thus if (M, γ) is taut and $\alpha \in H_2(M, \partial M)$, then,

$$c(\alpha, \mathfrak{t}) = \min\{\langle c, \alpha\rangle : c \in C(M, \gamma, \mathfrak{t})\},$$

is a well-defined number.

Definition 2.5. Fix $\mathfrak{t} \in T(M, \gamma)$. For $\alpha \in H_2(M, \partial M)$, let

$$H_\alpha = \{x \in H^2(M, \partial M; \mathbb{R}) : \langle x, \alpha\rangle = c(\alpha, \mathfrak{t})\}.$$

In addition, we take

$$P_\alpha = H_\alpha \cap P(M, \gamma, \mathfrak{t}),$$

and we also use the notation,

$$SFH_\alpha(M,\gamma) = \bigoplus_{\{\mathfrak{s}\in \mathrm{Spin}^c(M,\gamma): i(c_1(\mathfrak{s},\mathfrak{t}))\in P_\alpha(M,\gamma,\mathfrak{t})\}} SFH(M,\gamma,\mathfrak{s}).$$

It turns out that this is independent of $\mathfrak{t}$ [16, Lemma 3.12]. (See also the discussion after [16, Definition 4.12].)

The following useful fact is contained in [16, Proposition 4.13]. See [16] for the definitions of a nice decomposing surface and a strongly balanced sutured manifold. We just point out that our setup fits within the framework of the following:

Proposition 2.6. *Let the sutured manifold (M,γ) be taut and strongly balanced. Fix an element $\alpha \in H_2(M,\partial M)$. Then $P_\alpha(M,\gamma,\mathfrak{t})$ is a face of the polytope $P(M,\gamma,\mathfrak{t})$. If S is a nice decomposing surface that results in a taut decomposition $(M,\gamma) \overset{S}{\rightsquigarrow} (M',\gamma')$ and $[S]=\alpha$, then,*

$$SFH(M',\gamma') \cong SFH_\alpha(M,\gamma).$$

2.3. Sutured-Thurston norm and depth of a sutured manifold

We now recall the definitions of different norms assigned to a sutured manifold and discuss how they are related. All is contained in [7].

Definition 2.7. Let (M,γ) be a sutured manifold. Given a properly embedded, compact connected oriented surface $S \subset M$, let,

$$x^s(S) = \max\left\{0, \frac{1}{2}\left|S\cap s(\gamma)\right| - \chi(S)\right\},$$

and extend this definition to disconnected surfaces by taking the sum over the components.

Now for $\alpha \in H_2(M,\partial M)$, take,

$$x^s(\alpha) = \min\left\{x^s(S) : [S,\partial S] = \alpha\right\}.$$

where the minimum is taken over all properly embedded surfaces $S \subset M$.

Example 2.8. Let $K \subset S^3$ be a knot and let $(M,\gamma) = (S^3 \setminus n(K),\gamma)$ where $n(K)$ is a neighborhood of the knot K and γ consists of two meridional sutures. If $\alpha \in H_2(M,\partial M)$ is a generator, then $x^s(\alpha) = 2g(K)$. Note that this differs from the usual Thurston norm x of M, which satisfies $x(\alpha) = 2g(K) - 1$ for a nontrivial knot.

Definition 2.9. Let $S(M,\gamma)$ be the support of $SFH(M,\gamma)$. If $\alpha \in H_2(M,\partial M;\mathbb{R})$, we define,

$$z(\alpha) = \max\left\{\langle \mathfrak{s}-\mathfrak{t},\alpha\rangle : \mathfrak{s},\mathfrak{t} \in S(M,\gamma)\right\}.$$

We remind the reader that the surfaces in Fig. 1 are obtained from the plumbing of two knotted annuli. The following key proposition helps us to bypass the computations of the SFH of the complements of the mentioned annuli.

Proposition 2.10. *([7, Proposition 7.7]) Let (M,γ) be an irreducible balanced sutured manifold such that all boundary components of M are tori. Then $z = x^s$.*

We do not aim to give a precise proof of the proposition here; however, the proof uses a key fact which is helpful during the course of computations we will do. The detailed proof is contained in [7, Proposition 7.7].

Sketch of the proof. It turns out that we can assume each of the components of ∂M consists of two sutures. For (M, γ), there is a link, L, in a 3-manifold Y, where Y is obtained by Dehn filling ∂M such that the μ_i's are meridians of the tori. For each $\mathfrak{s} \in \mathrm{Spin}^c(M, \gamma)$ we obtain a relative first Chern class $c_1(\mathfrak{s}) \in H^2(M, \partial M)$ such that the set $\{c_1(\mathfrak{s}) : \mathfrak{s} \in S(M, \gamma)\}$ is symmetric about the origin. Then, due to [15, Remark 8.5], for every $h \in H_2(M, \partial M)$,

$$\max\bigl\{\bigl\langle c_1(\mathfrak{s}), h\bigr\rangle : \mathfrak{s} \in S(M, \gamma)\bigr\} = x(h) + \sum_{i=1}^{l} \bigl|\langle h, \mu_i\rangle\bigr|. \tag{1}$$

Since the image of $S(M, \gamma)$ is symmetric and $\mathrm{Spin}^c(M, \gamma)$ is an affine space over $H^2(M, \partial M)$, this is equivalent to saying

$$\max\bigl\{\langle \mathfrak{s} - \mathfrak{t}, h\rangle : \mathfrak{s}, \mathfrak{t} \in S(M, \gamma)\bigr\} = x(h) + \sum_{i=1}^{l} \bigl|\langle h, \mu_i\rangle\bigr|. \tag{2}$$

Note that the left side of (2) is $z(h)$ and the right side is $x^s(h)$ which completes the proof. □

2.4. SFH of the Murasugi sum of two manifolds

In [14], Juhász found a formula that governs the behavior of *SFH* under a plumbing of two annuli. We recall [14, Remark 10.8] as a proposition here.

Proposition 2.11. *If a surface R is a Murasugi sum of two subsurfaces R_1 and R_2, then over any field $\mathbb{F}$, we have,*

$$SFH\bigl(S^3(R); \mathbb{F}\bigr) \cong SFH\bigl(S^3(R_1); \mathbb{F}\bigr) \otimes SFH\bigl(S^3(R_2); \mathbb{F}\bigr).$$

The above formula is an isomorphism of Spin^c-graded groups. In the sense of our examples, on the right hand side we have two one dimensional polytopes, each of which is a Spin^c-graded group. (See Proposition 2.10.) See also the discussion in the beginning of Section 3.3. (To verify that the sutured Floer homology polytopes of the complementary surfaces on the right hand side are one dimensional, it should be noted that the first singular homology of each of the manifolds is isomorphic to $\mathbb{Z}$. See [16, Theorem 3].) Proposition 2.11 tells us that the sutured Floer homology group of the complement of a Murasugi sum of two surfaces R_1 and R_2, is obtained from a tensor product of the sutured Floer homology groups of the complements of R_1 and R_2 respectively. Note also that a plumbing is a special case of a Murasugi sum (see also [21]). As a matter of fact, what we have in Fig. 1 is a Murasugi sum of two annuli along a 4-gon, so this proposition enables us to compute the *SFH* of the complement of each of those annuli and then simply take their tensor product.

3. Using $SFH(Y(R))$ to distinguish Seifert surfaces

This section will be devoted to using the sutured Floer homology invariants in order to distinguish inequivalent Seifert surfaces of knots in the form of Fig. 1. We keep using the same notation as in Section 1. First, we show that different plumbings of the knotted annuli have the same boundaries. Then, we will prove, the over/under plumbings lead to two distinguished Seifert surfaces for their common boundary.

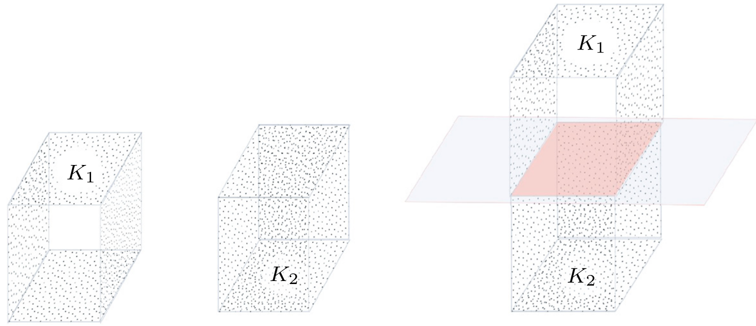


Fig. 2. The first two pictures are showing two knotted annuli. The third picture shows they are plumbed along a red 4-gon where the 4-gon is on the sphere, one annulus is inside the sphere and the other annulus is outside of it. The light blue plane is part of the sphere. (For interpretation of the references to color in this figure, the reader is referred to the web version of this article.)

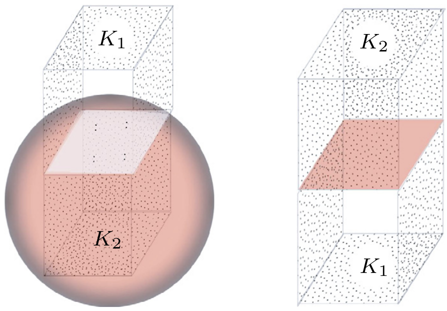


Fig. 3. Another way of plumbing the knotted annuli where we switch from the plumbing region D in Fig. 2 to a different 4-gon D'. We take D' to be the whole sphere except that the 4-gon, D, is removed from it. By a series of isotopies consisting of "switching" the knotted annuli and isotoping the plumbing region, D', we get the picture on the right hand side.

3.1. R and R′ bound the same knot

The goal of this subsection is to prove that the knotted annuli in the form of Fig. 1, where the right handed trefoil and the left handed trefoil are replaced by arbitrary knots, K_1 and K_2, both are bounded by the same knot, $P_l(K_1, K_2)$. We would like to mention here that, it is possible to do isotopies in Fig. 1 to go from one presentation of $P_l(K_1, K_2)$ to the other. However, we take a different route. Let us recall the definition of a plumbing here (see [9,10,18] and [19] for more details). Given compact oriented surfaces $S_1, S_2 \subset S^3$; if there are 3-balls $V_1, V_2 \subset S^3$ satisfying the following properties:

$$V_1 \cup V_2 = S^3, \qquad V_1 \cap V_2 = \partial V_1 = \partial V_2 = S^2, \qquad S_i \subset V_i \quad (i = 1, 2),$$
$$R = S_1 \cup S_2 \quad \text{and} \quad D = S_1 \cap S_2 \text{ is a 4-gon},$$

then R is called a plumbing of S_1 and S_2. In our examples S_1, S_2 and their plumbing are shown in Fig. 2. Put $P(K_1, K_2) = \partial R$. Note that $R' = (R - D) \cup D'$ is an oriented surface with $\partial R' = P(K_1, K_2)$ where $D' = S^2 - \text{int}(D)$. We will say that R' is a *dual* of R. Notice that R' is also a plumbing of ${S'}_1$ and ${S'}_2$ where ${S'}_i = (S_i - D) \cup D'$ $(i = 1, 2)$. We still need to argue that the two different plumbings we talked about above are the same as the plumbings in Fig. 1. Notice that if we pull down the K_2-knotted annulus in the left picture of Fig. 1, we obtain the representation of a plumbing in the sense of the right picture in Fig. 2. Then, with the above notation, if we change the plumbing region, D to D', we will have a hole in the sphere as the left side of Fig. 3. The goal is to show that the left and the right sides of Fig. 3 are isotopic. To see this, start from the left side of Fig. 3, and shrink D' smaller and smaller to have the hole (which was the 4-gon D before) filled in with the 4-gon D'. Simultaneously, we must push the K_2-knotted annulus "upward" in order to prevent the possible self-intersections in the plumbing diagram. The result

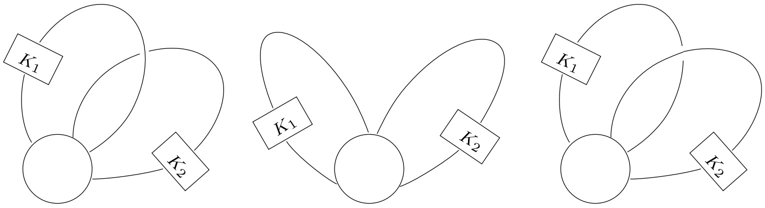


Fig. 4. We encode the surfaces in this picture with planar diagrams where the circles and the bands are representing the zero cells and the one cells, respectively.

then will be a disk (namely D') with two knotted 1-handles attached to it where the K_1-knotted annulus is located on the top of the K_2-knotted annulus. Now, by pulling the K_1-knotted annulus (in the "downward" direction and) around the K_2-knotted annulus and the disk D', missing the possible self-intersections of the plumbing diagram, we get to the right side of Fig. 3.

Remark 3.1. The above argument essentially shows that if we push the Seifert surfaces R and R' inside the four-ball and keep the knot, $P(K_1, K_2)$, in S^3, they are isotopic. Pushing the plumbing disk, D, of the surface R inside the four-ball, enables us to go from the right picture in Fig. 2 to the right picture in Fig. 3 and simultaneously avoid all intersections that could possibly occur. Therefore, we obtain an isotopy between R and R' inside the four-ball.

3.2. Classical methods

In this subsection we observe that classical methods fail in distinguishing the two Seifert surfaces, at least for a subfamily of our examples. Let us start by looking at the Seifert forms. Pick that subset of our examples where the non-zero twisting parameter is 1. Easy computation based on the dual to the basis elements represented in Fig. 1, shows that the Seifert forms are given by $V_R = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}$ and $V_{R'} = \begin{pmatrix} 1 & 0 \\ -1 & 0 \end{pmatrix}$ for R and R', respectively. One can check that if $W = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}$, then we have $V_{R'} = W^T V_R W$. Since, $W \in SL_2(\mathbb{Z})$, V_R and $V_{R'}$ are congruent. Thus, Seifert forms are incapable in distinguishing these particular surfaces. Another effective way to distinguish between surfaces is by looking at the homeomorphism type of their complements. Starting from the left picture in Fig. 1, we take a regular neighborhood of the surface, R, inside S^3 to obtain a handlebody with one zero handle and two one handles. The left picture in Fig. 4 illustrates a schematic diagram for our discussion when we retract the handles to one zero cell and two one cells. Notice that we can do an isotopy of the attaching spheres inside S^3 in the sense that in the schematic diagram, first, we obtain the middle picture in Fig. 4 and then, the right picture, that is a planar diagram for the surface, R'. Hence, $S^3 \setminus R$ and $S^3 \setminus R'$ are homeomorphic. Thus any algebro-topological invariant derived from the homeomorphism type (e.g. π_1) will fail to distinguish the surfaces R and R'.

3.3. Structure of $SFH(S^3(R))$

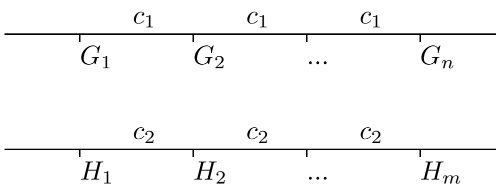
As we discussed in Section 1, we plumb two knotted annuli. Fig. 1 shows an example when the knots, K_1 and K_2, are the left handed and right handed trefoils, respectively. The point is that the complement of each of these annuli in S^3 deformation retracts to the knot complement, regardless of what the framings of those knots are. Thus, we have $H_1(M_i) \cong \mathbb{Z}$ where $M_i = S^3(A(K_i))$ and $A(K_i)$'s are the annuli as in Fig. 5 ($i = 1, 2$). Notice also that we are in a position to use Proposition 2.10. Therefore if we take a minimal genus Seifert surface R_i of $A(K_i)$ we have $x^s(R_i) = z(R_i)$. We have that

$$x^s(R_i) = \max\left\{0, \frac{1}{2}\left|R_i \cap s(\gamma_i)\right| - \chi(R_i)\right\}$$

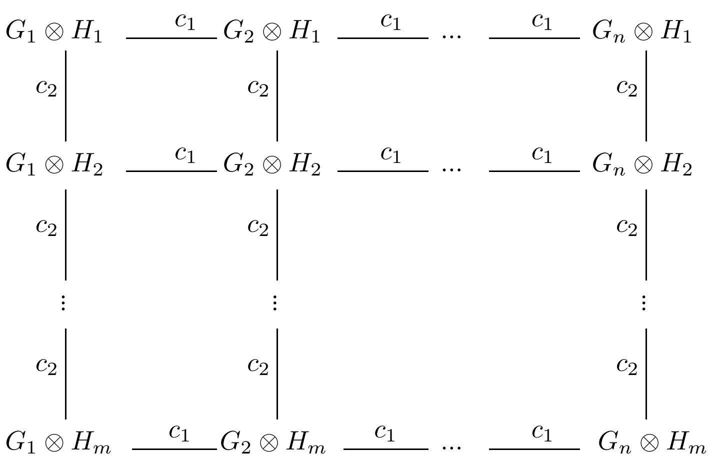
and that $x^s(R_i) = 2g_i$, similar to the computation we did in Example 2.8, where g_i is the genus of K_i. Recall that $SFH(M_i)$ is a Spin^c-graded group, i.e.

$$SFH(M_i) = \bigoplus_{\mathfrak{s} \in \underline{\mathrm{Spin}}^c(M_i)} SFH(M_i, \mathfrak{s}).$$

Recall also that for $\mathfrak{s}_1, \mathfrak{s}_2 \in \underline{\mathrm{Spin}}^c(M_i)$, we have that $\mathfrak{s}_1 - \mathfrak{s}_2 \in H^2(M_i, \partial M_i; \mathbb{Z})$. Note that by Poincaré duality we have that $H^2(M_i, \partial M_i; \mathbb{Z}) \cong H_1(M_i; \mathbb{Z})$. Thus by representing $SFH(M_1, \mathfrak{s}_j)$ (respectively $SFH(M_2, \mathfrak{s}_j)$) via G_j (respectively H_j), we obtain the following polytopes (see also [13, Section 4.3] for a detailed discussion),



for $S^3(A(K_1))$ and $S^3(A(K_2))$, respectively, where c_1 and c_2 are the generators of $H_1(M_i; \mathbb{Z})$. Proposition 2.10 together with the fact that $x^s(A(K_i)) = 2g(K_i)$ and non-triviality of K_i give us that G_1, G_n, H_1 and H_m are all non-zero, and also $m, n \geq 2$. We now plumb the annuli. Due to Proposition 2.11 we get a tensor product formula. Thus, one obtains the polytope shown below for the manifold $S^3(R)$ where c_1 and c_2 are the standard basis elements for $H_1(S^3(R); \mathbb{Z})$, specified in Fig. 1. In addition, for $S^3(R')$, we follow the exact same process that obviously results in the same polytope, except that c_1 and c_2 will be replaced by d_1 and d_2. In summary, we find that the polytopes of $SFH(S^3(R), \gamma)$ and $SFH(S^3(R'), \gamma')$ are rectangular and



at least the *extremal* vertices $G_1 \otimes H_1$, $G_n \otimes H_1$, $G_1 \otimes H_m$, and $G_n \otimes H_m$ in the rectangle are non-zero groups. Roughly speaking, a Spin^c structure is extremal if it defines a vertex of the convex hull of the support of sutured Floer homology. We refer the reader to [3, Section 2.3] for more details.

3.4. Calculation

In this subsection we present a way to calculate the groups $G_1 \otimes H_1$, $G_1 \otimes H_m$, $G_n \otimes H_1$ and $G_n \otimes H_m$ sitting on four of the vertices of the above polytope. We explain the computations needed to obtain G_1 and G_n.

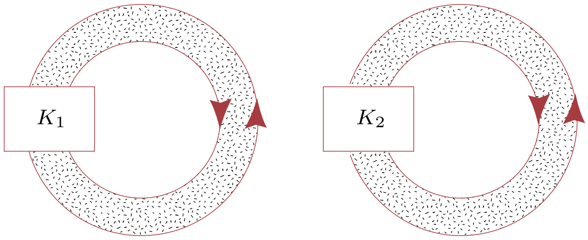


Fig. 5. Knotted annuli with oriented boundary, $A(K_1)$ and $A(K_2)$. We assume that $A(K_1)$ has framing zero, and the framing of $A(K_2)$ is non-zero.

Then, H_1 and H_m could be calculated in a quite similar way. Recall that the discussion in Section 3.3 showed that for non-trivial knots K_1 and K_2 the groups G_1, G_n, H_1, and H_m are all non-trivial. Throughout, we fix the knot K_1 and for the sake of simplicity of notation, we set $K = K_1$.

A priori we deal with the complement of a knotted annulus, with two oriented sutures on the two edges of it (see Fig. 5). Let us denote this manifold by M. Then, M is homeomorphic to the knot complement. As a sutured manifold, however, it is different; the sutures are not meridional. They are rather like oriented longitudes.

Remark 3.2. $SFH(M, \gamma)$ is isomorphic to an invariant of knots called "Longitude Floer Homology" of either the zero or l framed knot. As in the construction of knot Floer homology [22,23] we first find a Heegaard diagram for the knot complement in S^3. While in there we add the meridian of the knot to the set of β curves to obtain a Heegaard diagram for S^3; in the longitude Floer homology case, we add a longitude that results in a Heegaard diagram for $S^3_0(K)$. This subject has been first studied by Eftekhary in [5] for framing zero and later, has been developed by Hedden for arbitrary surgery coefficients in [12].

We end this subsection with the following proposition and its corollary that aim to calculate the groups G_1 and G_n, the extremal vertices for the knotted annulus with framing zero. Let $A(K) = A(K_1)$ be the knotted annulus (as in Fig. 5) with framing zero. Let also $S(K)$ be a Seifert surface for the knot K. The key to obtaining G_1 and G_n is to decompose (M, γ) along $S(K)$. (The resulting manifold is denoted by (M', γ').) Note that $S(K)$ is a nice decomposing surface (see [15] for the definition of a nice decomposing surface) and (M', γ') is taut. Therefore, based on Proposition 2.6, $SFH(M', \gamma')$ is a face of the polytope of $SFH(M, \gamma)$, that is, $SFH(M', \gamma') \cong G_n$. We are now in a position to state the following enlightening proposition. With the above notation:

Proposition 3.3. *If K is a nontrivial knot:*

$$G_n \cong \widehat{HFK}\big(S^3, K, g(K)\big) \oplus \widehat{HFK}\big(S^3, K, g(K)\big).$$

Proof. Let (M, γ) be the knot complement with two oriented longitudes as the sutures (see Fig. 5). We decompose (M, γ) along $S(K)$, a Seifert surface of K, to obtain (M', γ'),

$$(M, \gamma) \overset{S(K)}{\rightsquigarrow} (M', \gamma').$$

Notice that the result is a connected sutured manifold with three parallel sutures. See Fig. 6. The green arc on the right side of Fig. 6 represents an annulus, A, in (M', γ'). We observe that A is a separating annulus since it is boundary parallel. Note that the cross section of A consists of half of a bigon, the other half of which is an interval on the boundary of the tubular neighborhood of K. By decomposing (M', γ') along A we get a sutured manifold which is the disjoint union of $D^2 \times S^1$ with four parallel

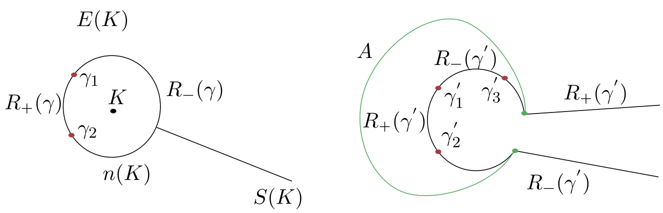


Fig. 6. The left picture represents the knot exterior, $E(K)$, in S^3 with two parallel longitudinal sutures γ_1 and γ_2. Also $\gamma = \{\gamma_1, \gamma_2\}$, $n(K)$ represents a tubular neighborhood of K, and $S(K)$ is a minimal genus Seifert surface for K (represented by the straight line to the right). The right picture is the manifold (M', γ'); the result of decomposing (M, γ) along $S(K)$ where $\gamma' = \{\gamma'_1, \gamma'_2, \gamma'_3\}$. The green arc is a boundary parallel annulus, A, in M' which is separating. (For interpretation of the references to color in this figure, the reader is referred to the web version of this article.)

longitudinal sutures and a manifold which is homeomorphic to $(S^3(S(K)), \partial S(K))$ with one suture. Finally, [15, Theorem 1.5] together with [15, Proposition 8.6] and [15, Proposition 8.10] complete the proof. Note that, after decomposing (M', γ') along A, in order to get the isomorphism between the sutured Floer homology groups using [15, Proposition 8.10], it is required that at least one component of ∂A does not bound a disk in $R(\gamma')$, a requirement not fulfilled in the case of the unknot. □

In order to obtain G_1, we just need to decompose (M, γ) along the same surface, $S(K)$, with the opposite orientation. One can also obtain H_1 and H_m in a similar manner by replacing the knotted annulus $A(K) = A(K_1)$ by $A(K_2)$, and respectively $S(K) = S(K_1)$ by $S(K_2)$. More precisely, if the framing is l (that is assumed to be non-zero), after decomposing the knot complement along the Seifert surface in the proof of Proposition 3.3, we get a sutured manifold which is homeomorphic to the Seifert surface complement. Thus, using [15, Theorem 1.5] and [15, Proposition 8.6], we get that a face of the polytope of $SFH(S^3(A(K_2)))$ is

$$H_m \cong \widehat{HFK}\big(S^3, K, g(K)\big).$$

In particular, $G_1 \otimes H_1$, $G_1 \otimes H_m$, $G_n \otimes H_1$ and $G_n \otimes H_m$ are all non-zero since $\widehat{HFK}(S^3, K_i, g(K_i)) \ncong 0$, $i = 1, 2$, provided that K_i is nontrivial.

Proposition 3.3 can be interpreted in the language of *longitude Floer homology* as well. As we mentioned in Remark 3.2, the sutured Floer homology of the Seifert surface complementary manifold, corresponding to the knot K, is isomorphic to $\widehat{HFL}(K)$.

Corollary 3.4. *For a given non-trivial knot K in S^3,*

$$\widehat{HFL}(K, \mathfrak{s}_{top}) \cong \widehat{HFK}\big(S^3, K, g(K)\big) \oplus \widehat{HFK}\big(S^3, K, g(K)\big)$$

where $\mathfrak{s}_{top}$ is the highest grading in the support of $\widehat{HFL}$, i.e., $\widehat{HFL}(K, i) \cong 0$ for $i > \mathfrak{s}_{top}$.

3.5. Proof of the main theorem

In this subsection we prove the main theorem of the paper. Having known the results of subsections 3.3 and 3.4, it remains to distinguish R and R'. To do so, we show that $SFH(M, \gamma)$ and $SFH(M', \gamma')$ are not isomorphic, where M and M' are $S^3(R)$ and $S^3(R')$ respectively. We still need to clarify the notion of an isomorphism between two relatively Spin^c-graded groups. The following is [13, Definition 4.1].

Definition 3.5. Two relatively Spin^c-graded groups

$$SFH(M, \gamma) = \bigoplus_{\mathfrak{s} \in \underline{\mathrm{Spin}}^c(M,\gamma)} SFH(M, \gamma, \mathfrak{s}), \qquad SFH\big(M', \gamma'\big) = \bigoplus_{\mathfrak{s} \in \underline{\mathrm{Spin}}^c(M',\gamma')} SFH\big(M', \gamma', \mathfrak{s}\big)$$

are isomorphic if

1. There is an isomorphism, $f^*: H^2(M', \partial M'; \mathbb{Z}) \to H^2(M, \partial M; \mathbb{Z})$.
2. There is a bijection of sets $u : \underline{\mathrm{Spin}}^c(M', \gamma') \to \underline{\mathrm{Spin}}^c(M, \gamma)$.
3. The following diagram commutes

$$\begin{array}{ccc} \underline{\mathrm{Spin}}^c(M', \gamma') \otimes H^2(M', \partial M'; \mathbb{Z}) & \xrightarrow{(u, f^*)} & \underline{\mathrm{Spin}}^c(M, \gamma) \otimes H^2(M, \partial M; \mathbb{Z}) \\ \downarrow & & \downarrow \\ \underline{\mathrm{Spin}}^c(M', \gamma') & \xrightarrow[u]{} & \underline{\mathrm{Spin}}^c(M, \gamma) \end{array}$$

   where the vertical arrows are induced by the action of $H^2(M', \partial M')$ on $\underline{\mathrm{Spin}}^c(M', \gamma')$ and $H^2(M, \partial M)$ on $\underline{\mathrm{Spin}}^c(M, \gamma)$.[1]
4. There are isomorphisms $g_{\mathfrak{s}} : SFH(M', \gamma', \mathfrak{s}) \to SFH(M, \gamma, u(\mathfrak{s}))$ for every $\mathfrak{s} \in \underline{\mathrm{Spin}}^c(M', \gamma')$.

If the surfaces R and R' are weakly equivalent, then there will be a map $(S^3, R) \to (S^3, R')$, representing the equivalence (see the discussion in Section 1), that can further be restricted to a map $f : S^3(R) \to S^3(R')$. Moreover, the sutured Floer homology groups of $S^3(R)$ and $S^3(R')$ will be, in the sense of the above definition, isomorphic [13, Theorem 2.16]. Also, f^* and u in Definition 3.5 will both be obtained by pulling back along the function f. Note that, $f_* : H_1(S^3 \setminus R) \to H_1(S^3 \setminus R')$ must preserve the Seifert form, i.e, $a.b = f_*(a).f_*(b)$ for every $a, b \in H_1(S^3 \setminus R)$. See [13, p. 542] for more details.

Proof of the main theorem. Generalizing the computations done in Section 3.3, it is straightforward to see that the Seifert matrices for the two surfaces R and R' are given by $V_R = \begin{pmatrix} l & 1 \\ 0 & 0 \end{pmatrix}$ and $V_{R'} = \begin{pmatrix} l & 0 \\ -1 & 0 \end{pmatrix}$ based on the basis elements c_i's and d_i's, shown in Fig. 1 $(i = 1, 2)$. Set $M = S^3(R)$ and $M' = S^3(R')$. Using the notation of Section 3, let us mark one generator in each $G_i \otimes H_j$ by x_{ij} and let us also denote the group generated by these elements, by $\langle x_{ij} \rangle$. If the two Seifert surfaces were equivalent, then based on Definition 3.5 we would have $\sigma : SFH(M, \gamma) \to SFH(M', \gamma')$, a bijection from the generators to generators, i.e., it would map every generator of $SFH(M, \gamma)$, say x_{ij}, to a generator $\sigma(x_{ij})$ of $SFH(M', \gamma')$. We also obtain a map $f : (M, \gamma) \to (M', \gamma')$, where f is compatible with taking difference classes, i.e., $\epsilon(a, b).\epsilon(c, d) = f_*\epsilon(a, b).f_*\epsilon(c, d)$ for some isomorphism $f_* : H_1(S^3 \setminus R) \to H_1(S^3 \setminus R')$ where, $a, b, c, d \in H_1(S^3 \setminus R)$.

Suppose such a σ exists. As we mentioned earlier, in the Seifert forms V_R and $V_{R'}$, l is a non-zero integer. Let us first assume that $l > 0$. Then,

$$\epsilon\big(\sigma(x_{11}), \sigma(x_{nm})\big)^2 = \big(f_*\epsilon(x_{11}, x_{nm})\big)^2 = (nc_1 + mc_2)^2 = n^2 l + nm. \tag{3}$$

Since σ sends generators to generators, for some α and β in $\mathbb{Z}$,

$$\epsilon\big(\sigma(x_{11}), \sigma(x_{nm})\big) = \alpha d_1 + \beta d_2. \tag{4}$$

The fact that f_* preserves the Seifert form together with (3) and (4) implies that,

$$(\alpha d_1 + \beta d_2)^2 = n^2 l + nm. \tag{5}$$

On the other hand, using $V_{R'}$, we have

[1] We use the fact that there is a natural action of $H^2(M, \partial M)$ (respectively $H^2(M', \partial M')$) on $\underline{\mathrm{Spin}}^c(M)$ (respectively $\underline{\mathrm{Spin}}^c(M')$). We refer the reader to [13] for more details.

$$(\alpha d_1 + \beta d_2)^2 = \alpha^2 l - \alpha\beta. \tag{6}$$

Notice that x_{11} and x_{nm} have the furthest distance in the polytope of $S^3(R)$. This implies that $n^2l + nm$ is the greatest positive number that can possibly be generated from $\alpha^2 l - \alpha\beta$, where $|\alpha| \leq n$ and $|\beta| \leq m$. Hence, we must have $|\alpha| = n$ and $|\beta| = m$. In addition, α and β must have opposite signs. Thus

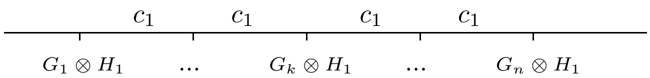
$$\epsilon\big(\sigma(x_{11}), \sigma(x_{nm})\big) = \pm(nd_1 - md_2). \tag{7}$$

Combining (3) and (7), we get the following polytopes that illustrate the placement of the x_{ij}'s and $\sigma(x_{ij})$'s:

$$\begin{array}{ccccccc} \langle x_{11}\rangle & \overset{c_1}{\text{---}} \cdots \overset{c_1}{\text{---}} & \langle x_{k1}\rangle & \overset{c_1}{\text{---}} \cdots \overset{c_1}{\text{---}} & \langle x_{1n}\rangle \\ c_2 \vert & & & & c_2 \vert \\ \vdots & & & & \vdots \\ c_2 \vert & & & & c_2 \vert \\ \langle x_{1m}\rangle & \overset{c_1}{\text{---}} \cdots \overset{c_1}{\text{---}} & \langle x_{km}\rangle & \overset{c_1}{\text{---}} \cdots \overset{c_1}{\text{---}} & \langle x_{nm}\rangle \end{array} \qquad \begin{array}{ccccc} \langle \sigma(x_{ij})\rangle & \overset{d_1}{\text{---}} \cdots \overset{d_1}{\text{---}} & \langle \sigma(x_{k1})\rangle & \overset{d_1}{\text{---}} \cdots \overset{d_1}{\text{---}} & \langle \sigma(x_{11})\rangle \\ d_2 \vert & & & & d_2 \vert \\ \vdots & & . & & \vdots \\ d_2 \vert & & & & d_2 \vert \\ \langle \sigma(x_{nm})\rangle & \overset{d_1}{\text{---}} \cdots \overset{d_1}{\text{---}} & \langle \sigma(x_{km})\rangle & \overset{d_1}{\text{---}} \cdots \overset{d_1}{\text{---}} & \langle \sigma(x_{i'j'})\rangle \end{array}$$

Note that, due to (7), the polytope on the right side is either having the shape in the above picture or a rotation of the right side by 180 degrees. Also there is a similar possibility for the left polytope.[2] In the following, all cases are considered.

Observe that $\sigma(x_{11})$ and $\sigma(x_{nm})$ are located along the other diagonal compared to x_{11} and x_{nm}. Now, in the horizontal direction, take the closest nonzero group $G_k \otimes H_1$ to $\langle x_{11}\rangle$ such that $k > 0$. Such a group exists since, *a priori* $G_n \otimes H_1 \neq 0$. Thus, we can assume that $x_{k1} \neq 0$ for some $k > 0$. Based on our convention, $x_{k1} \in G_k \otimes H_1$ and $k \leq n$. Summarizing our discussion, the following figure shows the first row of the polytope for $SFH(S^3(R))$,



and we have that, for some $1 < k \leq n$, $G_k \otimes H_1 \neq 0$. And that we are choosing a representative x_{k1} in that group. Also, $G_i \otimes H_1$ is trivial for $1 < i < k$. We chose x_{k1} so that

$$\epsilon(x_{11}, x_{k1}) = \pm k c_1. \tag{8}$$

Then, using V_R and (8)

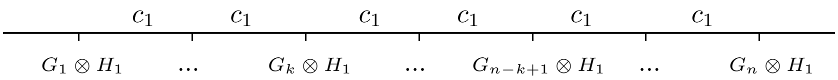
$$\epsilon\big(\sigma(x_{11}), \sigma(x_{k1})\big)^2 = \big(f_*\epsilon(x_{11}, x_{k1})\big)^2 = k^2 c_1^2 = k^2 l. \tag{9}$$

Now, for the rest of the proof we need to locate $\sigma(x_{k1})$ in the corresponding polytope. For some γ and ζ, $\epsilon(\sigma(x_{11}), \sigma(x_{k1})) = \gamma d_1 + \zeta d_2$ which in turn, using (9) and $V_{R'}$, ensures

$$(\gamma d_1 + \zeta d_2)^2 = \gamma^2 l - \gamma\zeta = k^2 l, \tag{10}$$

where $k > 0$, and so $\gamma \neq 0$. Note that *SFH* is symmetric in the sense that if x_{k1}, in the polytope for M, is the closest non-zero element to x_{11} in the first row, then the closest non-zero element to x_{n1}, in the same row, is $x_{(n-k+1)1}$ [7, Proposition 2.18]. In other words, in the polytope for $SFH(S^3(R))$,

[2] Note that (7) only tells us where $\langle \sigma(x_{11})\rangle$ and $\langle \sigma(x_{nm})\rangle$ are located in the right polytope, and not where $\langle \sigma(x_{k1})\rangle$ is. The location of $\langle \sigma(x_{k1})\rangle$ in the right polytope will be discussed later in the proof.



not only that $G_i \otimes H_1$ (respectively G_i) is trivial for $1 < i < k$, but also $G_j \otimes H_1$ (respectively G_j) is trivial for $n - k + 1 < j < n$ (see [7, Proposition 2.18]). The triviality of G_i for $1 < i < k$ and G_j for $n - k + 1 < j < n$, follows from the fact that H_1 is a non-zero group. As a result, using Proposition 2.11, we get that $G_r \otimes H_s$, for both the intervals $1 < r < k$ and also $n - k + 1 < r < n$, is trivial regardless of the integer s. This proves that $|\gamma| \geq k$. Observe that, from where $\sigma(x_{11})$ is located in the polytope, we get that γ and ζ have different signs, i.e. $\gamma\zeta \leq 0$. So $\gamma^2 - \gamma\zeta \geq \gamma^2$. As a result, $\gamma^2 l - \gamma\zeta > k^2 l$ unless $\zeta = 0$, $|\gamma| = k$, and

$$\epsilon\big(\sigma(x_{11}), \sigma(x_{k1})\big) = \pm k d_1. \tag{11}$$

We get a contradiction now. On the one hand, by combining (7) and (11) and using $V_{R'}$

$$\epsilon\big(\sigma(x_{11}), \sigma(x_{nm})\big).\epsilon\big(\sigma(x_{11}), \sigma(x_{k1})\big) = \pm(nd_1 - md_2). \pm kd_1 = \pm(nkl + mk), \tag{12}$$

where $l, m, n, k > 0$. On the other hand, by combining (8) and noticing that $\epsilon(x_{11}, x_{nm}) = \pm(nc_1 + mc_2)$, and also using V_R we get

$$\epsilon(x_{11}, x_{nm}).\epsilon(x_{11}, x_{k1}) = \pm(nc_1 + mc_2). \pm kc_1 = \pm nkl. \tag{13}$$

By comparing (12) and (13) we get the desired contradiction.

For the case $l < 0$, instead of x_{11}, x_{nm} and x_{k1} we take x_{n1}, x_{1m} and $x_{(n-k+1)1}$ for the smallest positive k, where $x_{(n-k+1)1} \neq 0$. Then, a contradiction follows similarly.

These show that $SFH(S^3(R)) \ncong SFH(S^3(R'))$ and so R is not equivalent to R'. □

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