Analytic Torsion, the Eta Invariant, and Closed Differential Forms on Spaces of Metrics

by

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Dissertation submitted in partial fulfillment of the requirements for the degree of
Doctor of Philosophy in the Department of Mathematics
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Abstract

The central idea of this dissertation is to interpret certain invariants constructed from Laplace spectral data on a compact Riemannian manifold as regularized integrals of closed differential forms on the space of Riemannian metrics, or more generally on a space of metrics on a vector bundle. We apply this idea to both the Ray-Singer analytic torsion and the eta invariant, explaining their dependence on the metric used to define them with a Stokes' theorem argument. We also introduce analytic multi-torsion, a generalization of analytic torsion, in the context of certain manifolds with local product structure; we prove that it is metric independent in a suitable sense.

To my mother and father

Contents

List of Symbols

$\mathbb{Z}$	The integers.
$\mathbb{Z}_{\geqslant 0}$	The nonnegative integers.
$\mathbb{R}$	The real numbers.
$\mathbb{C}$	The complex numbers.
$\operatorname{Re} s$	The real part of a complex number s.
$\overline{s}$	The complex conjugate of a complex number s.
$\operatorname{Diff}(M)$	The group of diffeomorphisms of a manifold M.
S_k	The symmetric group of permutations on k letters.
$\operatorname{sign}(\sigma)$	The sign of a permutation $\sigma \in S_k$.
GL(V)	The general linear group of invertible linear endomorphisms of a vector space V.
$\Omega^k(\mathcal{M}, A)$	The space of A-valued k-forms on $\mathcal{M}$, where A is an algebra and $\mathcal{M}$ is a space of metrics. (See Section 2.1.)
$C^\infty(M, E)$	The space of smooth sections of a vector bundle $E \to M$.
$L^2(M, E)$	The space of L^2 sections of a vector bundle $E \to M$. (See Section 2.2.)
$H^a(M, E)$	The Sobolev space of order $a \in \mathbb{R}$ of sections of a vector bundle $E \to M$. (See Section 2.2.)

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1

Introduction

Analytic torsion and the eta invariant are invariants constructed from eigenvalues of Laplace-type operators on differential forms. Their geometric-analytic nature means that their definition requires a choice of Riemannian metric, but in special cases, and/or with a modified interpretation, they are in fact topological invariants, i.e., they are independent of the choice of metric.

To elaborate on this point: torsion and eta may be considered invariants in the spirit of the heat equation proof of the Atiyah-Singer index theorem. Unlike the index of an elliptic operator, however, torsion and eta are non-local invariants in the sense that they cannot be expressed as the integral of an integrand that is locally computable in terms of the geometry near a point. This is essentially equivalent to the fact that torsion and eta cannot be computed in terms of the small time asymptotics of the heat kernel; knowledge of the heat trace for all positive times is required. Torsion and eta share the property, however, that their variation with respect to a change in Riemannian metric is a locally computable quantity, expressible in terms of only small time heat asymptotics. In some cases, this local quantity vanishes, giving that the original quantity is independent of the metric. More generally, even if the local quantity does not vanish, this property is enough to construct an associated topological invariant (the Ray-Singer metric or the relative eta invariant) that is independent of the choice of Riemannian metric.

The dependence of torsion and eta on the metric used to define them is the subject of this dissertation. Central to our approach is an interpretation involving closed differential forms on the infinite-dimensional manifold consisting of all Riemannian metrics, or more generally the space of all metrics on some vector bundle. We phrase the fact that the variations of torsion and eta are locally computable in terms of a Stokes' theorem argument on the space of metrics. This perspective suggests the possibility of generalizations involving other closed forms on spaces of metrics. We introduce one such generalization called multi-torsion, which we define on manifolds which are quotients of product manifolds by certain actions of finite groups of diffeomorphisms.

A brief introduction to analytic torsion. Analytic torsion may be thought of as an analogue of an Euler characteristic or index for odd-dimensional manifolds, but as noted above, it differs from an index in its non-local nature. The celebrated Cheeger-Müller theorem ([6], [13], [14], [5]), conjectured by Ray-Singer, states the equality of analytic torsion and Reidemeister torsion, a classical combinatorial invariant that is of historical interest for its use in classifying lens spaces. A deep and surprising consequence is that analytic torsion has a connection with torsion in integer cohomology groups; see, for example, the paper of Bergeron-Venkatesh [4].

To be a bit more precise: let M be a compact manifold of dimension n, let g be a Riemannian metric on M, and let F be a flat unitary vector bundle. Using zeta function regularization, one defines the determinants $\det \Delta_q$ of the Laplacians Δ_q on F-valued differential q-forms, for $q = 0, \ldots, n$. Then the associated de Rham analytic torsion $T = T(M, F, g)$ is defined by the weighted alternating sum

$$\log T(M, F, g) := \frac{1}{2} \sum_{q=0}^{n} (-1)^q q \log \det \Delta_q. \tag{1.1}$$

Ray-Singer [16] made the definition (1.1) and proved that if the dimension n is odd and F is acyclic (i.e., there is no F-cohomology), then $T(M, F, g)$ is independent of the Riemannian metric g, i.e., T is a smooth topological invariant of the pair (M, F).

Our perspective on analytic torsion. We discuss analytic torsion in the context of a $\mathbb{Z}$-graded elliptic complex, which generalizes both the de Rham analytic torsion described above and the Dolbeault analytic torsion for complex manifolds, also introduced by Ray-Singer [17]. We generalize the metric independence theorem of Ray-Singer to this context and to a generalized class of metrics. For example, in the setting of the previous paragraph, our framework allows us to consider any metric on the bundle $F \otimes \Lambda^\bullet T^*M$, not just metrics induced as the product of a metric on F and a Riemannian metric on M.

More importantly, we provide a new perspective on the metric independence theorem by interpreting analytic torsion as an integral on the space of metrics. To give a rough idea of this perspective, we note that the logarithm of analytic torsion is the regularization of the following divergent integral:

$$\frac{1}{2} \int_0^\infty \sum_{q=0}^{n} (-1)^q q \operatorname{Tr} e^{-t\Delta_q^g} \frac{dt}{t}. \tag{1.2}$$

(The regularization, involving analytic continuation of a zeta function, is essential, but we suppress the details for now.) The half-line $(0, \infty)$ embeds in the space of metrics by scaling the metric g, i.e., as a curve C_g parametrized by $t \mapsto \frac{1}{t} g$. The one-form $\sum_{q=0}^{n} (-1)^q q \operatorname{Tr} e^{-t\Delta_q^g} \frac{dt}{t}$, defined on $(0, \infty)$, is in fact the pullback to C_g of ω_T, a one-form defined on the entire space of metrics. We prove that ω_T is closed.

The variation of $T(g)$ with respect to a change in metric g is computable by a Stokes' theorem argument. The fact that ω_T is closed reduces the computation, very roughly, to boundary terms: a term at the limit $t \to \infty$ (which vanishes by the decay of the heat kernel) and a term at the limit $t \to 0$. The latter term is locally computable and, furthermore, vanishes under appropriate assumptions.

Similar considerations apply to the eta invariant of Atiyah-Patodi-Singer [1], which we interpret as the integral of another closed one-form ω_η over the same curve C_g on the space of metrics, providing a new perspective on the well-known metric independence of the relative eta invariant.

Multi-torsion. The form ω_T and the Stokes' theorem argument computing the variation of T suggests the possibility of new invariants constructed from other closed forms on the space of metrics. We introduce a two-form ω_{MT} defined on the space of product metrics on the product manifold $M_1 \times M_2$; ω_{MT} is essentially the wedge product of the torsion one-forms $\omega_T^{M_1}$ and $\omega_T^{M_2}$ corresponding to the two factors. We show that ω_{MT} is also defined more generally on the space of metrics inducing a local geometric product structure in a certain sense; we treat in detail the case of the quotient of $M_1 \times M_2$ by a finite group of diffeomorphisms. In that context, the Laplacian Δ and the heat operator $e^{-\Delta}$ depend on two parameters t_1 and t_2, each corresponding to scaling the local product metric in one of the directions. The integral (1.2) over the half-line $(0, \infty) \subset \mathbb{R}$ is replaced by an integral over the quadrant $(0, \infty) \times (0, \infty) \subset \mathbb{R}^2$, which embeds in the space of product metrics. We call the resulting invariant multi-torsion, and we prove that it is independent of the metric in the space of local product metrics.

Outline. We now outline the contents of this dissertation.

Chapter 2 fixes some notational conventions and develops some tools related to the notion of spaces of metrics and differential forms thereon. The impatient reader may wish merely to skim Chapter 2 on a first reading.

Chapter 3 introduces analytic torsion T in the context of a $\mathbb{Z}$-graded elliptic complex. We first review some facts about heat kernels and zeta functions that are necessary to the definition of analytic torsion. We then introduce the one-form ω_T, explain its relevance to analytic torsion, and prove that it is closed. We prove the metric independence of T (under appropriate conditions) using a Stokes' theorem argument. We study alternate methods of regularization and show that analytic torsion is invariant under certain deformations of the curve over which ω_T is integrated. Finally, in Section 3.9, we discuss briefly the history of Ray-Singer's de Rham analytic torsion, the Cheeger-Müller theorem, and how our results fit in to the picture.

Chapter 4 introduces what we call multi-torsion, which generalizes analytic torsion to manifolds with a local product structure. To define multi-torsion, we must develop some analytic tools to treat heat kernels depending on two parameters and their associated multi-zeta functions. Then we introduce the closed two-form (more generally, m-form) ω_{MT} and integrate it to produce multi-torsion. Finally, we study the dependence of multi-torsion on the metric and prove a metric independence theorem.

Chapter 5 applies ideas similar to those of Chapter 3 to the eta invariant, which we interpret as the integral of a closed one-form ω_η on the space of metrics.

2

Vector bundles, metrics, and differential forms on spaces of metrics

This short chapter develops some tools related to the notion of differential forms on the space of metrics on a vector bundle. The final section discusses flat vector bundles.

2.1 Differential forms

Let M be a compact manifold, and let $E \to M$ be a real or complex vector bundle. By a metric on E, we shall mean a smoothly varying inner product on the fibers of E; this inner product is euclidean (if E is a real vector bundle) or hermitian (if E is a complex vector bundle). In this chapter, we will assume that E is a real vector bundle, but our results hold in the complex case with minor modifications.

Let $\mathcal{M}$ denote the set of all metrics on E. $\mathcal{M}$ is canonically viewed as an open subset of the Fréchet space $C^\infty(M, \operatorname{Sym}^2 E^*)$, giving $\mathcal{M}$ the structure of a Fréchet manifold. Thus the tangent space to $\mathcal{M}$ at a metric $h \in \mathcal{M}$ is canonically identifiable with $C^\infty(\operatorname{Sym}^2 E^*)$, and the tangent bundle of $\mathcal{M}$ is canonically identifiable with

$\mathcal{M} \times C^\infty(\mathrm{Sym}^2 E^*)$.

We will now introduce the notion of differential forms on $\mathcal{M}$.

Definition 2.1.1. Let A be an associative $\mathbb{C}$-algebra. For $k \in \mathbb{Z}_{\geqslant 0}$, an A-valued k-form on $\mathcal{M}$ is a smooth map

$$\omega : \mathcal{M} \times C^\infty(\mathrm{Sym}^2 E^*)^{\times k} \to A$$

that is multilinear and alternating in the $C^\infty(\mathrm{Sym}^2 E^*)$ entries, i.e., for every $h \in \mathcal{M}$, $X_1, \ldots, X_k \in C^\infty(\mathrm{Sym}^2 E^*)$, and $\sigma \in S_k$, ω satisfies

$$\omega(h; X_{\sigma(1)}, \ldots, X_{\sigma(n)}) = \mathrm{sign}(\sigma)\omega(h; X_1, \ldots, X_k).$$

The vector space of A-valued k-forms on $\mathcal{M}$ will be denoted by $\Omega^k(\mathcal{M}, A)$.

$\Omega^k(\mathcal{M}, A)$ possesses a product structure generalizing the usual wedge product on differential forms. For simplicity of notation, we will omit the traditional wedge symbol "$\wedge$", i.e., if $\kappa \in \Omega^k(\mathcal{M}, A)$ and $\lambda \in \Omega^l(\mathcal{M}, A)$, their product will be denoted $\kappa\lambda \in \Omega^{k+l}(\mathcal{M}, A)$.

Remark 2.1.2. Since A need not be a commutative algebra, the product of A-valued forms need not be graded commutative, i.e., if $\kappa \in \Omega^k(\mathcal{M}, A)$ and $\lambda \in \Omega^l(\mathcal{M}, A)$, then $\kappa\lambda$ is not equal to $(-1)^{kl}\lambda\kappa$ in general. A certain graded commutativity property does hold, however, when we introduce traces; see Lemma 2.3.5 below.

We will denote by $\delta^{\mathcal{M}}$ the exterior derivative. If $\kappa \in \Omega^k(\mathcal{M}, A)$, then $\delta^{\mathcal{M}}\kappa \in \Omega^{k+1}(\mathcal{M}, A)$. δ^M satisfies $\delta^{\mathcal{M}} \circ \delta^{\mathcal{M}} = 0$ and the Leibniz rule: if $\kappa \in \Omega^k(\mathcal{M}, A)$ and $\lambda \in \Omega^l(\mathcal{M}, A)$, then $\delta^{\mathcal{M}}(\kappa\lambda) = (\delta^{\mathcal{M}}\kappa)\lambda + (-1)^k\kappa(\delta^{\mathcal{M}}\lambda)$. If $\delta^{\mathcal{M}}\omega = 0$, then we will say ω is closed.

We will be primarily interested in the restriction of forms on $\mathcal{M}$ to finite-dimensional submanifolds of $\mathcal{M}$. More precisely:

Let U be an open set in $\mathbb{R}^k$ (or more generally, a k-manifold) and let $\phi : U \to \mathcal{M}$ be a smooth map. If ω is a $\mathbb{C}$-valued form on $\mathcal{M}$, then we may consider $\phi^*\omega$, the pullback of ω by ϕ, which is a differential form on U in the ordinary sense. We have the usual relation that the exterior derivative commutes with pullback:

$$\delta^U(\phi^*\omega) = \phi^*(\delta^{\mathcal{M}}\omega),$$

where δ^U is the de Rham exterior derivative on U. In particular, if ω is closed on $\mathcal{M}$, then $\phi^*\omega$ is closed on U. Conversely, if $\phi^*\omega$ is closed on U for every pair (U, ϕ) with $\phi : U \to \mathcal{M}$, then ω is closed on $\mathcal{M}$. In this sense, to study closed forms on $\mathcal{M}$, it suffices to study their restriction to finite-dimensional submanifolds of $\mathcal{M}$. But we will find the formalism of forms on the infinite-dimensional space $\mathcal{M}$ to be useful.

We will abuse notation and write $\int_U \omega$ for the integral of $\phi^*\omega$ over U.

We recall Stokes' theorem: if U is a (finite-dimensional) manifold with smooth boundary ∂U and ω is a form on U, then

$$\int_U \delta^U \omega = \int_{\partial U} \omega.$$

Of course, Stokes' theorem also holds on the infinite-dimensional manifold $\mathcal{M}$ in the sense that it holds for the pullback of forms to U for every $\phi : U \to \mathcal{M}$. Stokes' theorem will be essential to our study of analytic torsion and the eta invariant.

A related but distinct remark:

Remark 2.1.3. Rather than considering the space of all metrics on a vector bundle, we sometimes consider a smaller space, generally an (infinite-dimensional Fréchet) submanifold $\mathcal{M}_1$ of the space of all metrics $\mathcal{M}$. When this is the case, implicit in all that we do is that a k-form ω is closed on $\mathcal{M}_1$ if and only if

$$\delta^{\mathcal{M}}\omega(h; X_1, \ldots, X_{k+1}) = 0$$

for all $h \in \mathcal{M}_1$ and for all $X_1, \ldots, X_{k+1}$ tangent to $\mathcal{M}_1$.

2.2 Sobolev spaces and algebras of operators

Let us choose a volume form vol_0 on the base manifold M (or a density if M is not orientable). Then each choice of metric $h \in \mathcal{M}$ induces an L^2-inner product on smooth sections of E, $\langle \cdot, \cdot \rangle_h$, defined by

$$\langle \cdot, \cdot \rangle_h := \int_M h(\cdot, \cdot)_x \, \mathrm{vol}_0(x). \tag{2.1}$$

(Here $h(\cdot, \cdot)_x$ denotes the inner product on the fiber E_x.) The L^2 spaces $L^2_h(M, E)$, and more generally the L^2-Sobolev spaces $H^s_h(M, E)$ for $s \in \mathbb{R}$, are defined in the usual way. The inner products on these Hilbert spaces depend on the metric h, but any two metrics induce equivalent norms since M is compact, so that it makes sense to refer to the topological vector spaces $L^2(M, E)$ and $H^s(M, E)$, which are independent of the metric.

Remark 2.2.1. There is a redundancy that means a change to the volume form can always be rephrased as a change to the metric for the purposes of the L^2 inner product. More precisely: in (2.1), replacing vol_0 by a different volume form $\varphi \, \mathrm{vol}_0$, where $\varphi : M \to \mathbb{R}$ is a smooth positive function, induces the same L^2 inner product as replacing the metric h by φh and keeping vol_0 the same. For this reason, we will sometimes decide to fix a volume form vol_0, which we will do for the rest of this chapter.

Definition 2.2.2. Let B denote the unital associative algebra of bounded linear maps $L^2(M, E) \to L^2(M, E)$. (The notion of bounded is independent of the metric in $\mathcal{M}$ since any two metrics induce equivalent norms.)

Definition 2.2.3. Let Ψ denote the unital associative algebra of pseudodifferential operators $C^\infty(M, E) \to C^\infty(M, E)$, and for $s \in \mathbb{R}$, let Ψ^s denote the subset of Ψ consisting of pseudodifferential operators of order (less than or equal to) s.

Note that if $s < t$, then $\Psi^s \subset \Psi^t$. If $s \leqslant 0$, then Ψ^s forms an algebra since it is closed under composition (in general, $\Psi^s \cdot \Psi^t \subset \Psi^{s+t}$). Thus if $s \leqslant 0$, then $\Omega^k(\mathcal{M}, \Psi^s)$ makes sense, with a well-defined product of differential forms.

Remark 2.2.4. Neither B nor Ψ is contained in the other, but if $s \leqslant 0$, $\Psi^s \subset B$. Thus if $s \leqslant 0$, we may view $\Omega^k(\mathcal{M}, \Psi^s)$ as a subset of $\Omega^k(\mathcal{M}, B)$.

We are primarily interested in B-, Ψ-, and $\mathbb{C}$-valued forms on $\mathcal{M}$.

2.3 Adjoints and traces

We now introduce some useful notions related to adjoints and traces.

Definition 2.3.1. Let $h \in \mathcal{M}$ be a metric and let ϕ be an operator either in B or in Ψ. The h-adjoint of ϕ, denoted ϕ^{*_h}, is the operator either in B or in Ψ, respectively, characterized by (for smooth sections a, b of E)

$$\langle \phi a, b \rangle_h = \langle a, \phi^{*_h} b \rangle_h.$$

We remark that if $\phi \in B$, since smooth sections are dense in L^2 sections, this defines ϕ^{*_h} uniquely as an element of B, i.e., as a bounded linear operator on $L^2(M, E)$.

Definition 2.3.2. If $\omega \in \Omega^k(\mathcal{M}, A)$, where $A = B$ or $A = \Psi$, then the adjoint of ω, denoted $\omega^* \in \Omega^k(\mathcal{M}, A)$, is defined by

$$(\omega^*)(h; X_1, \ldots, X_k)(h) := (\omega(h; X_1, \ldots, X_k))^{*_h},$$

where the latter adjoint is as in Definition 2.3.1. We will say that ω is symmetric when $\omega^* = \omega$.

We will use the notion of a trace-class operator $\phi \in B$; for a definition, see, for example, the book of Reed-Simon [18]. The usual definition of the trace depends a priori on a choice of an L^2 inner product—and therefore, in our context, on a choice of a metric $h \in \mathcal{M}$—but by Lidskii's theorem, the trace is independent of this choice. Thus it makes sense to denote the trace of a trace-class operator $\phi \in B$ by $\operatorname{Tr}\phi$, with no reference to a choice of metric $h \in \mathcal{M}$.

Remark 2.3.3. We will find the following well-known fact useful in later chapters. Let n be the dimension of the base manifold M. Then if $s < -n$ and if $\phi \in \Psi^s \subset B$, then ϕ is trace-class.

Definition 2.3.4. We will say that $\omega \in \Omega^k(\mathcal{M}, B)$ is trace-class when for every $h \in \mathcal{M}$ and $X_1, \ldots, X_k \in C^\infty(\mathcal{M}, \operatorname{Sym}^2 E^*)$, $\omega(h; X_1, \ldots, X_k) \in B$ is trace-class. If ω is trace-class, $\operatorname{Tr}\omega \in \Omega^k(\mathcal{M}, \mathbb{C})$ will denote the $\mathbb{C}$-valued k-form on $\mathcal{M}$ defined by $(\operatorname{Tr}\omega)(h; X_1, \ldots, X_k) := \operatorname{Tr}(\omega(h; X_1, \ldots, X_k))$, and $\overline{\operatorname{Tr}}\,\omega \in \Omega^k(\mathcal{M}, \mathbb{C})$ will denote the complex conjugate of $\operatorname{Tr}\omega$, defined by $(\overline{\operatorname{Tr}}\,\omega)(h; X_1, \ldots, X_k) := \overline{\operatorname{Tr}(\omega(h; X_1, \ldots, X_k))}$.

We have the following results concerning the adjoint of a product and the interaction between adjoints and traces:

Lemma 2.3.5. *Let $\kappa \in \Omega^k(\mathcal{M}, A)$ and $\lambda \in \Omega^l(\mathcal{M}, A)$, where $A = B$ or $A = \Psi$. Then*

$$(\kappa\lambda)^* = (-1)^{kl} \lambda^* \kappa^*.$$

κ is trace-class if and only if κ^* is trace-class; in that case, their traces are complex conjugates of each other:

$$\operatorname{Tr}\kappa = \overline{\operatorname{Tr}}\,\kappa^*.$$

If $\kappa\lambda$ is trace-class, then

$$\begin{aligned} \operatorname{Tr}\kappa\lambda &= \overline{\operatorname{Tr}}\,(\kappa\lambda)^* \\ &= (-1)^{kl}\,\overline{\operatorname{Tr}}\,\lambda^*\kappa^*. \end{aligned}$$

If $\kappa\lambda$ and $\lambda\kappa$ are both trace-class, then

$$\operatorname{Tr}\kappa\lambda = (-1)^{kl} \operatorname{Tr}\lambda\kappa.$$

Proof. The identities follow from the corresponding identities for operators and the alternating property of differential forms. □

We have the following useful corollary involving products of 1-forms:

Lemma 2.3.6. *Let* $\kappa_1, \ldots, \kappa_r \in \Omega^1(\mathcal{M}, A)$*, where* $A = B$ *or* $A = \Psi$*. Let* $\epsilon_r = (-1)^{\frac{1}{2}r(r-1)}$*. Then*

$$(\kappa_1 \cdots \kappa_r)^* = \epsilon_r \, \kappa_r^* \cdots \kappa_1^*.$$

If $\kappa_1 \cdots \kappa_r$ *is trace-class, then*

$$\begin{aligned} \operatorname{Tr} \kappa_1 \cdots \kappa_r &= \overline{\operatorname{Tr}} \, (\kappa_1 \cdots \kappa_r)^* \\ &= \epsilon_r \overline{\operatorname{Tr}} \, \kappa_r^* \cdots \kappa_1^*. \end{aligned}$$

Proof. The lemma follows from Lemma 2.3.5 by induction on r. ϵ_r is the sign of the following order-reversing permutation on r letters: $(1, 2, \ldots, r) \mapsto (r, r-1, \ldots, 1)$, which involves $1 + 2 + \cdots + (r-1) = \frac{1}{2}r(r-1)$ transpositions. □

Finally, the operation of taking the trace commutes with the exterior derivative:

Lemma 2.3.7. *If* $\kappa \in \Omega^k(\mathcal{M}, B)$ *is trace-class, then* $\delta^{\mathcal{M}}\kappa$ *is also trace-class, and* $\delta^{\mathcal{M}}(\operatorname{Tr} \kappa) = \operatorname{Tr}(\delta^{\mathcal{M}}\kappa)$.

2.4 Flat vector bundles

In this section we will establish some conventions regarding flat vector bundles.

Let $F \to M$ be a vector bundle. A connection ∇^F on $F \to M$ is said to be flat when its curvature vanishes. The vector bundle $F \to M$ is said to be flat when it admits a flat connection. Henceforth, when we refer to a flat vector bundle, we will assume that a fixed flat connection has been chosen.

It is well-known that flat vector bundles on M are in one-to-one correspondence with finite-dimensional representations of the fundamental group $\pi_1(M)$, i.e., group homomorphisms $\pi_1(M) \to GL(V)$, where V is a finite-dimensional real or complex vector space and $GL(V)$ denotes the group of invertible linear endomorphisms of V. We will now describe this correspondence briefly.

Let $\tilde{M}$ be the universal cover of M, on which $\pi_1(M)$ acts on the left by deck transformations; let $\Pi : \tilde{M} \to M$ be the covering map. Let $\rho : \pi_1(M) \to GL(V)$ be a representation. Associated to ρ is a flat vector bundle F_ρ, the total space of which is $\tilde{M} \times V$ modulo the equivalence relation $(x, v) \sim (\gamma x, \rho(\gamma^{-1})v)$ for $\gamma \in \pi_1(M)$. The trivial connection on the trivial bundle $\tilde{M} \times V \to \tilde{M}$ descends to a flat connection on $F_\rho \to M$.

Conversely, suppose ∇^F is a flat connection on $F \to M$. Then ∇^F lifts to a flat connection on the bundle $\Pi^*F \to \tilde{M}$. Since $\tilde{M}$ is simply connected and the connection is flat, it is not hard to see (via parallel transport) that $\Pi^*F \to \tilde{M}$ is in fact a trivial bundle, and that this construction must be the inverse of a construction of the form described in the previous paragraph.

We will say that a representation $\rho : \pi_1(M) \to GL(V)$ on a real euclidean inner product space V is orthogonal when the image of ρ lies in the group of automorphisms that are orthogonal with respect to the inner product on V. Similarly, we will say that a representation $\rho : \pi_1(M) \to GL(V)$ on a complex hermitian inner product space V is unitary when the image of ρ lies in the group of automorphisms that are unitary with respect to the inner product on V. We will also use these terms for flat vector bundles: we will say that a flat vector bundle $F \to M$ is orthogonal or unitary if the associated representation of $\pi_1(M)$ is orthogonal or unitary, respectively. If $F \to M$ is orthogonal or unitary, then the euclidean or hermitian inner product on V descends to a canonical metric on $F \to M$ with respect to which the flat connection is metric-compatible.

If $F \to M$ is a flat vector bundle, then one may form the F-valued de Rham complex, as we will now describe. For $q = 0, \ldots, \dim M$, consider the vector bundle

$F \otimes \Lambda^q T^* M \to M$, whose sections are F-valued q-forms. (We will sometimes choose to use the more compact notation $\Lambda^q F$.) Then the exterior derivative on M and the flat connection on F induce an exterior derivative operator that we will denote by d. d maps q-forms to $(q+1)$-forms; the identity $d \circ d = 0$ is equivalent to the flatness of the connection on F.

3

Analytic torsion of an elliptic complex

3.1 Heat kernels and zeta functions

In this section, we will discuss some well-known general properties of elliptic operators that are necessary to our treatment of analytic torsion. The results on zeta functions are due to Seeley [20]; for an overview of these and many related ideas, see, for example, the book of Gilkey [7].

Let M be a compact orientable manifold of dimension n. Fix once and for all a volume form vol_0 on M. Let $E \to M$ be a vector bundle. Let L be an elliptic differential operator on sections of E; we will assume L is nonnegative and formally self-adjoint with respect to an L^2 inner product on sections of E induced by a metric h on E. Let m be the order of L; we will assume $m > 0$.

L has nonnegative real eigenvalues $0 \leqslant \lambda_1 \leqslant \lambda_2 \leqslant \cdots \to \infty$. The zeta function associated to L is the function of $s \in \mathbb{C}$ defined for $\operatorname{Re} s$ large by

$$\zeta(s) := \sum_{\lambda_j > 0} \lambda_j^{-s}. \tag{3.1}$$

$\zeta(s)$ may be interpreted as the trace of the operator L^{-s}. The sum in (3.1) converges for $\operatorname{Re} s > \frac{n}{m}$ and defines a holomorphic function there, and $\zeta(s)$ admits a unique meromorphic continuation to all of $\mathbb{C}$ that is holomorphic at $s = 0$. This allows one to define the determinant of L by

$$\det L := \exp\left(- \frac{d}{ds}\bigg|_{s=0} \zeta(s) \right), \tag{3.2}$$

which one should interpret as a regularized product of the nonzero eigenvalues of L.

The zeta function is closely related to the heat operator, as we will now discuss. For $t > 0$, the heat operator e^{-tL} is characterized by, for $u \in L^2(M, E)$,

$$\left(\frac{\partial}{\partial t} + L \right) (e^{-tL} u) = 0 \text{ for } t > 0, \text{ and}$$

$$\lim_{t \to 0^+} \left(e^{-tL} u \right) = u.$$

e^{-tL} is trace-class as an operator $L^2(M, E) \to L^2(M, E)$; furthermore, away from the kernel of L, its trace is what we will call "admissible," in a sense which we will make precise. To do so, we leave our geometric setting and work in the abstract for a moment.

Definition 3.1.1. A $\mathbb{C}$-valued function $h(t)$, $t > 0$, will be said to be admissible when it satisfies both of the following conditions:

(A1) $h(t)$ admits an asymptotic expansion for small t in the sense that there exist real powers $p_0 < p_1 < p_2 < \cdots \to \infty$ and complex coefficients $a_{p_0}, a_{p_1}, a_{p_2}, \ldots$ such that for every real number K, there exists J_K and C_K such that

$$h(t) = \sum_{j=0}^{J_K} a_{p_j} t^{p_j} + r_K(t), \tag{3.3}$$

where the remainder $r_K(t)$ satisfies, for $0 < t < 1$, $|r_K(t)| \leqslant C_K t^K$. In this case, we will write $h(t) \sim \sum_{j \geqslant 0} a_{p_j} t^{p_j}$. (For each q such that t^q does not appear in this asymptotic expansion, our convention is to set $a_q = 0$.)

(A2) There exist $C, \epsilon > 0$ such that, for $t \geqslant 1$, $|h(t)| \leqslant Ce^{-\epsilon t}$.

The utility of this notion of admissibility is that is sufficient to define an associated zeta function via the Mellin transform:

Definition 3.1.2. We define the zeta function associated to $h(t)$ by the following, for $\operatorname{Re} s$ large:

$$\zeta(s) := \frac{1}{\Gamma(s)} \int_0^\infty t^s\, h(t)\, \frac{dt}{t}. \tag{3.4}$$

The following result is well-known:

Lemma 3.1.3. *Suppose $h(t)$ is admissible. Then the integral defining $\zeta(s)$ in (3.4) converges for $\operatorname{Re} s > -p_0$ and defines a holomorphic function there. Furthermore, $\zeta(s)$ admits a unique meromorphic extension to all of $\mathbb{C}$; the extension, which we also denote by $\zeta(s)$, is holomorphic at the origin, where its value is $\zeta(0) = a_0$, i.e., the coefficient on t^0 in the asymptotic expansion of (A1).*

Now we return to our geometric setting. We have the following well-known fundamental result:

Theorem 3.1.4. *Let $\alpha \in C^\infty(\operatorname{End}(E))$. Then $\operatorname{Tr} \alpha e^{-tL}$ has an asymptotic expansion in the sense of (A1) taking the form*

$$\operatorname{Tr} \alpha e^{-tL} \sim \sum_{j \geqslant 0} a_{\frac{-n+2j}{m}}\, t^{\frac{-n+2j}{m}}. \tag{3.5}$$

(The reader will recall that n is the dimension of M and m is the order of L.) To obtain the decay as $t \to \infty$, as in $(A2)$, we must remove the kernel of L in the following sense:

Definition 3.1.5. Let $\Pi_{\ker L} : L^2(M, E) \to \ker \Delta \subset L^2(M, E)$ denote the orthogonal projection onto $\ker L$. For a trace-class operator A, let

$$\operatorname{Tr}' A := \operatorname{Tr} A \left(I - \Pi_{\ker L}\right).$$

Then we have:

Theorem 3.1.6. *Let $\alpha \in C^\infty(M, \mathrm{End}(E))$. Then $\mathrm{Tr}'\, \alpha e^{-tL}$ is an admissible function of $t > 0$, i.e., it satisfies conditions (A1) and (A2). Furthermore, the asymptotic expansion of (A1) takes the form*

$$\mathrm{Tr}'\, \alpha e^{-tL} \sim -\operatorname{tr} \alpha \Pi_{\ker L} + \sum_j a_{\frac{-n+2j}{m}} t^{\frac{-n+2j}{m}}, \tag{3.6}$$

i.e., the coefficients are the same as in (3.5), *except that the coefficient on t^0 differs from that of* (3.5) *by* $(-\operatorname{tr} \alpha \Pi_{\ker L})$.

This allows us to make the definition:

Definition 3.1.7. The zeta function associated to the elliptic operator L and $\alpha \in C^\infty(\mathrm{End}(E))$ is defined as in (3.4) with $h(t) = \mathrm{Tr}'\, \alpha e^{-tL}$, i.e.,

$$\zeta(s; L, \alpha) := \frac{1}{\Gamma(s)} \int_0^\infty t^s\ \mathrm{Tr}'\, \alpha e^{-tL}\, \frac{dt}{t}. \tag{3.7}$$

In the case when α is the identity, we set $\zeta(s; L) := \zeta(s; L, I)$.

Remark 3.1.8. To provide motivation, we remark that $\zeta(s; L, \alpha)$ generalizes the zeta function of (3.1) to include the auxiliary operator α. Indeed, the elementary identity

$$\lambda^{-s} = \frac{1}{\Gamma(s)} \int_0^\infty t^s\, e^{-t\lambda}\, \frac{dt}{t}$$

shows that $\zeta(s; L, \alpha)$ may be rewritten as, for $\operatorname{Re} s$ large,

$$\zeta(s; L, \alpha) = \sum_{\lambda > 0} \lambda^{-s} \operatorname{tr} \alpha \Pi_\lambda,$$

where the sum is over distinct positive eigenvalues of L and

Lemma 3.1.3 implies immediately:

Theorem 3.1.9. *The integral defining* $\zeta(s; L, \alpha)$ *in* (3.7) *converges for* $\operatorname{Re} s > -\frac{n}{m}$ *and defines a holomorphic function there. Furthermore,* $\zeta(s; L, \alpha)$ *admits a unique meromorphic extension to all of* $\mathbb{C}$*; the extension, which we also denote by* $\zeta(s; L, \alpha)$*, is holomorphic at the origin, where its value is*

$$\zeta(0; L, \alpha) = a_0 - \operatorname{tr} \alpha \Pi_{\ker L}, \tag{3.8}$$

i.e., the coefficient on t^0 *in the asymptotic expansion in* (3.6)*. (If there is no* t^0 *term in* (3.5)*, we set* $a_0 = 0$*.)*

We record also the useful consequence:

Corollary 3.1.10. *If the dimension* n *is odd and* $\ker L$ *is trivial, then for any* α*,* $\zeta(0; L, \alpha) = 0$*.*

Proof. We use (3.8). The asymptotic expansion of $\operatorname{Tr} \alpha e^{-tL}$ in (3.5) has terms of the form $a_{\frac{-n+2j}{m}} t^{\frac{-n+2j}{m}}$ for $j \in \mathbb{Z}_{\geqslant 0}$; if the dimension n is odd, then there is no t^0 term, i.e., $a_0 = 0$. Furthermore, if also $\ker L$ is trivial, then of course $\operatorname{tr} \alpha \Pi_{\ker L} = 0$. □

3.2 Elliptic complexes

We will discuss analytic torsion in the context of a $\mathbb{Z}$-graded elliptic complex, which we will now define.

Suppose that the vector bundle $E \to M$ is $\mathbb{Z}$-graded of finite length in the sense that $E = \bigoplus_{q=0}^{r} E^q$ for some vector bundles $E^0, \ldots, E^r$. We assume that there is a differential operator $d : C^\infty(M, E) \to C^\infty(M, E)$ such that for each q, $d : C^\infty(M, E^q) \to C^\infty(M, E^{q+1})$; the restriction of d to $C^\infty(M, E^q)$ we denote by d_q. We assume that $d \circ d = 0$, i.e., we have a chain complex

$$0 \longrightarrow C^\infty(M, E^0) \xrightarrow{d_0} C^\infty(M, E^1) \xrightarrow{d_1} \cdots \xrightarrow{d_{r-1}} C^\infty(M, E^r) \longrightarrow 0.$$

Let $\mathcal{M}$ denote the space of metrics on E such that the subbundles E^q are mutually orthogonal. As discussed in Section 2.2, for each metric $h \in \mathcal{M}$ (and having fixed the volume form), we have an associated L^2 inner product $\langle \cdot, \cdot \rangle_h$. Let d^{*_h} be the formal adjoint of d with respect to $\langle \cdot, \cdot \rangle_h$.

Definition 3.2.1. We will say the complex (E, d) is an elliptic complex when for some choice of metric h, $d + d^{*_h}$ is elliptic.

Remark 3.2.2. In fact, if $d + d^{*_h}$ is elliptic for some metric h, then $d + d^{*_{h'}}$ is elliptic for every metric h'. Furthermore, this condition is equivalent to an exactness condition on the symbol of d. For details, see Gilkey [7].

Definition 3.2.3. We now define some useful endomorphisms of E. Let Q act on sections of E^q as multiplication by q. Let $(-1)^Q$ act on sections of E^q as multiplication by 1 if q is even and -1 if q is odd. More generally, if $f : \mathbb{Z}_{\geqslant 0} \to \mathbb{C}$ is a function on the nonnegative integers, let $f(Q)$ act on sections of E^q as multiplication by $f(q)$.

Definition 3.2.4. The Laplacian Δ^h associated to a metric $h \in \mathcal{M}$ is

$$\begin{aligned} \Delta^h :=& (d + d^{*_h})^2 \\ =& dd^{*_h} + d^{*_h} d. \end{aligned}$$

$\Delta^h_q = d_{q-1} d^{*_h}_{q-1} + d^{*_h}_q d_q$ denotes the restriction of Δ^h to sections of E^q. Let m be the order of Δ as a differential operator. (m is twice the order of d.)

Remark 3.2.5. We use the symbol Δ, which is traditional for Laplacians and is suggestive of the examples below. In general, though, we need not assume that Δ is a "Laplace-type operator" in the sense of Gilkey [7]. In particular, the order m need not be 2.

Δ^h is elliptic and, with respect to $\langle \cdot, \cdot \rangle_h$, formally self-adjoint and nonnegative. By elliptic regularity, Δ^h has a finite-dimensional kernel, which is the intersection $\ker d \cap \ker d^{*_h}$ and which we may identify with the cohomology, as we now discuss.

Definition 3.2.6. For $q = 0, \dots, r$, the qth cohomology vector space $H^q(M, E)$ is the quotient

$$H^q(M, E) := \frac{\ker d_q}{\operatorname{im} d_{q-1}}.$$

We have the Hodge theorem (for a proof, see, for example, the book of Gilkey [19]):

Theorem 3.2.7. *For each metric h, the map $\ker \Delta_q^h \to H^q(M, E)$ induced by the inclusion $\ker \Delta_q^h \hookrightarrow \ker d_q$ is an isomorphism of vector spaces.*

Definition 3.2.8. The elliptic complex (E, d) will be said to be acyclic when the cohomology is trivial, i.e, when $H^q(M, E) = 0$ for $q = 0, \dots, r$, or equivalently, when Δ^h has trivial kernel.

Examples.

1. *The de Rham complex with a flat coefficient bundle.* (Recall our discussion of flat vector bundles in Section 2.4.) Let $F \to M$ be a flat vector bundle and let $E^q = F \otimes \Lambda^q T^* M$ be the bundle of F-valued q-forms. Then the flat connection on F and the usual exterior derivative induce an operator d, giving an elliptic complex. This was the context in which Ray-Singer first introduced analytic torsion [16]. For further comments on this important example, see Section 3.9.

2. *The Dolbeault complex with a holomorphic coefficient bundle.* This is the analogue of the previous example in the complex category. Suppose that M is a complex manifold and let $F \to M$ be a holomorphic vector bundle. Let $E^{p,q} = F \otimes \Lambda^{p,q} T^*_{\mathbb{C}} M$ be the bundle of F-valued (p, q)-forms. The boundary operator d in this context is $\bar{\partial} : C^\infty(M, E^{p,q}) \to C^\infty(M, E^{p,q+1})$, giving an elliptic complex for each fixed p. Ray-Singer also introduced analytic torsion in this context [17].

Remark 3.2.9. The signature operator and the spinor Dirac operator do *not* fit into this framework because there is no natural complex, i.e., no natural operator d such that $d \circ d = 0$.

Remark 3.2.10. Mathai-Wu [11] have defined analytic torsion for elliptic complexes that are only $\mathbb{Z}_2$-graded, which is more general than our setting. But our interpretation of analytic torsion as an integral over a curve in the space of metrics does not make sense in the $\mathbb{Z}_2$-graded context. See Remark 3.9.2 for further comments.

3.3 Analytic torsion

We are now prepared to define analytic torsion. Recall the operators Q and $(-1)^Q$ from Definition 3.2.3.

Definition 3.3.1. The analytic torsion $T(M, E, h)$ associated to the elliptic complex E and a metric $h \in \mathcal{M}$, with associated Laplacian Δ^h, is defined by

$$\log T(M, E, h) := \frac{1}{2} \frac{d}{ds}\bigg|_{s=0} \zeta\left(s; \Delta^h, (-1)^Q Q\right). \tag{3.9}$$

Note that by the definition of the zeta-function, we have that

$$\log T(M, E, h) := \frac{1}{2} \frac{d}{ds}\bigg|_{s=0}^{\mathrm{AC}} \frac{1}{\Gamma(s)} \int_0^\infty t^{s-1} \operatorname{Tr}'(-1)^Q Q e^{-t\Delta}\, dt \tag{3.10}$$

$$= \frac{1}{2} \sum_{q=0}^{r} (-1)^q q \frac{d}{ds}\bigg|_{s=0} \zeta(s; \Delta_q), \tag{3.11}$$

where the superscript AC indicates that analytic continuation is implied.

Remark 3.3.2. We may also phrase the formula for the analytic torsion in terms of determinants. Recall the definition of the determinant of an elliptic operator from (3.2). We see that (3.11) implies

$$\log T(M, E, h) = -\frac{1}{2} \sum_{q=0}^{r} (-1)^q q \log \det \Delta_q, \tag{3.12}$$

i.e, the analytic torsion is a weighted alternating product of determinants.

There is also a notion of determinant for the operators $d_q d_q^*$ and $d_q^* d_q$ (which are not elliptic but are nonnegative) that requires projecting away from their infinite-dimensional kernels. Let $\zeta_{d_q^* d_q}(s)$ be defined for $\operatorname{Re} s$ large by

$$\zeta_{d_q^* d_q}(s) := \frac{1}{\Gamma(s)} \int_0^\infty t^s \ \operatorname{Tr}' e^{-t d_q^* d_q} \frac{dt}{t},$$

where, in an abuse of notation, we mean that

$$\operatorname{Tr}' e^{-t d_q^* d_q} := \operatorname{Tr} e^{-t\Delta_q} \Pi_{\operatorname{im} d_q^*}, \tag{3.13}$$

where $\Pi_{\operatorname{im} d_q^*}$ denotes the orthogonal projection onto the image of d_q^*, on which $d_q^* d_q$ is strictly positive. Then we can define the determinant analogously to (3.2):

$$\log \det d_q^* d_q := - \left. \frac{d}{ds} \right|_{s=0} \zeta_{d_q^* d_q}(s).$$

We define $\zeta_{d_q d_q^*}(s)$ and $\det d_q d_q^*$ similarly. Recall the fact that for any $t > 0$,

$$\operatorname{Tr}' e^{-t\, d_q d_q^*} = \operatorname{Tr}' e^{-t\, d_q^* d_q}, \tag{3.14}$$

which follows because $d_q d_q^*$ and $d_q^* d_q$ have the same nonzero eigenvalues. We also have $\operatorname{Tr}' e^{-t\Delta_q} = \operatorname{Tr}' e^{-t d_q^* d_q} + \operatorname{Tr}' e^{-t d_{q-1} d_{q-1}^*}$, and these facts imply the identities

$$\zeta_q(s) = \zeta_{d_q^* d_q}(s) + \zeta_{d_{q-1} d_{q-1}^*}(s)$$

$$\zeta_{d_q^* d_q}(s) = \zeta_{d_q d_q^*}(s).$$

Substituting these into (3.11) shows that the torsion may be written in terms of only dd^* or only d^*d:

$$\log T(M, E, h) = \frac{1}{2} \sum_{q=0}^{r} (-1)^q \log \det d_q d_q^* \tag{3.15}$$

$$= -\frac{1}{2} \sum_{q=0}^{r} (-1)^q \log \det d_{q-1}^* d_{q-1}. \tag{3.16}$$

Note the absence of the weighting factor q in (3.15) and (3.16) as compared to (3.12).

We remark that the Reidemeister torsion, the classical combinatorial analogue of analytic torsion, has similar expressions in terms of determinants of combinatorial Laplacians. This is explained in the paper of Ray-Singer [16] and seems to have been the motivation for Ray-Singer's definition of analytic torsion.

We are interested in the dependence of $T(h)$ on the metric $h \in \mathcal{M}$. The main result of this section is the following generalization of Ray-Singer's fundamental result, Theorem 2.1 of [16]:

Theorem 3.3.3. *Let $h(u)$ be a smooth one-parameter family of metrics in $\mathcal{M}$. Let $T(h(u)) = T(M, E, h(u))$ be the associated analytic torsion. Then*

$$\frac{d}{du} \log T(h(u)) = a_0'(u), \tag{3.17}$$

where $a_0'(u)$ is the t^0 term in the asymptotic expansion of $\mathrm{Tr}'(-1)^Q \beta_u^{-1} \frac{d\beta_u}{du} e^{-t\Delta^{h(u)}}$.

For the sake of exposition, we will wait until the next section to define β_u, which is a certain endomorphism of E associated to the metric. For now, we merely emphasize the following important corollary:

Corollary 3.3.4. *If the dimension n is odd and E is acyclic, then the analytic torsion $T(h)$ is independent of the metric h, for h in the generalized class of metrics $\mathcal{M}$.*

Proof. This follows immediately from Theorem 3.3.3 and Corollary 3.1.10. □

Remark 3.3.5. Ray-Singer proved Theorem 3.3.3 and Corollary 3.3.4 in the contexts of Examples 1 and 2 above, in the special case that the metric on E is induced as the product of a metric on the coefficient bundle F and a Riemannian metric on M. We emphasize that our results apply to any metric h in the generalized class $\mathcal{M}$ of all metrics on E. For further discussion of this, see Section 3.9.

Moreover, our proof of Theorem 3.3.3 involves a novel interpretation of analytic torsion as the integral of a certain closed differential form on a curve in $\mathcal{M}$. This interpretation is the subject of Section 3.4.

Remark 3.3.6. In the case when n is even and/or E is not acyclic, $T(h)$ may depend on the metric h, but (3.17) can be used to show that an object called the Ray-Singer metric is a topological invariant, independent of the metric h. The Ray-Singer metric is a certain metric on the determinant line of the cohomology $H^\bullet(M, E)$. See [5], for example, for the details.

3.4 A closed form for analytic torsion

We will now introduce ω_T, a certain closed one-form on the space of metrics, and we will interpret analytic torsion as the integral of ω_T over a certain curve in the space of metrics. In this section and the next, the tools of Chapter 2 will be important.

3.4.1 An operator parametrizing the space of metrics

Recall that $\mathcal{M}$ denotes the space of metrics on E such that the subbundles E^q are mutually orthogonal. We have fixed a volume form vol_0 on M. Now we will fix also a metric $h_0 \in \mathcal{M}$ that we will consider to be a "basepoint" metric. Having chosen h_0, we have the following construction: If $h \in \mathcal{M}$ is another metric, then there exists a unique operator $\beta = \beta_h$, an automorphism of E, such that for all $\phi, \psi \in C^\infty(M, E)$,

$$h(\phi, \psi) = h_0(\phi, \beta_h \psi).$$

We will view β as a endomorphism-valued function (i.e., a 0-form) on the space of metrics:

$$\beta : \mathcal{M} \to C^\infty(\mathrm{End}(E))$$

In the language of Definition 2.1.1, $\beta \in \Omega^0(\mathcal{M}, \Psi^0) \subset \Omega^0(\mathcal{M}, B)$.

We now study some properties of β.

Lemma 3.4.1. *β_h is a pointwise symmetric operator with respect to h_0, i.e., on each fiber E_x, for $u, v \in E_x$ we have*

$$h_0(u, \beta_h v)_x = h_0(\beta_h u, v)_x. \tag{3.18}$$

Proof. We use the fact that h and h_0 are euclidean or hermitian inner products on each fiber to compute

$$\begin{aligned} h_0(u, \beta_h v)_x &= h(u, v)_x \\ &= \overline{h(v, u)_x} \\ &= \overline{h_0(v, \beta_h u)_x} \\ &= h_0(\beta_h u, v)_x, \end{aligned}$$

which proves the claim. □

Lemma 3.4.2. *If h_u is a smooth one-parameter family of metrics with corresponding operators $\beta_u := \beta_{h_u}$, then $\beta_u^{-1}\dot{\beta}_u$ is a pointwise symmetric operator with respect to h_u.*

Proof. Differentiating the equality (3.18) gives that

$$\begin{aligned} h_0(f, \beta_u\beta_u^{-1}\dot{\beta}_u g)_x &= h_0(\beta_u\beta_u^{-1}\dot{\beta}_u f, g)_x, \text{ i.e.,} \\ h_u(f, \beta_u^{-1}\dot{\beta}_u g)_x &= h_u(\beta_u^{-1}\dot{\beta}_u f, g)_x. \end{aligned}$$

□

Recall that for each metric $h \in \mathcal{M}$, d^{*_h} (or just d^* if there is no possibility of confusion) denotes the formal adjoint of d with respect to $\langle \cdot, \cdot \rangle_h$. Denote by d^{*_0} the formal adjoint of d with respect to $\langle \cdot, \cdot \rangle_0$. We have the following:

Lemma 3.4.3. *$d^{*_h} = \beta^{-1} d^{*_0} \beta$.*

Proof. $\langle f, \beta^{-1} d^{*_0} \beta g \rangle_h = \langle f, \beta\beta^{-1} d^{*_0} \beta \rangle_0 = \langle f, d^{*_0} \beta g \rangle_0 = \langle df, \beta g \rangle_0 = \langle df, g \rangle_h$. □

Recall that $\delta^{\mathcal{M}}$ denotes the exterior derivative on differential forms on $\mathcal{M}$.

Definition 3.4.4. Let b denote the following "tautological" one-form on $\mathcal{M}$ with values in $C^\infty(\operatorname{End} E)$:

$$b := \beta^{-1}\delta^{\mathcal{M}}\beta.$$

Although β depends on the choice of basepoint metric h_0, a short computation shows that b is well-defined independent of the choice of h_0:

Lemma 3.4.5. *b does not depend on the choice of basepoint metric h_0.*

Proof. Suppose h_0 and h_0' are two choices of basepoint metric; let β and β' correspond to h_0 and h_0' respectively, i.e., for any metric $h \in \mathcal{M}$, we have $h(f,g) = h_0(f, \beta_h g) = h_0'(f, \beta_h' g)$. But then we must have $h_0'(f, \beta_h' g) = h_0(f, \beta_{h_0'}\beta_h' g)$, which implies $\beta_h = \beta_{h_0'}\beta_h'$. $\beta_{h_0'}$ may be viewed as a constant map $\mathcal{M} \to C^\infty(\operatorname{End} E)$, which shows

$$\begin{aligned} \beta^{-1}\delta^{\mathcal{M}}\beta &= (\beta_{h_0'}\beta_h')^{-1}\delta^{\mathcal{M}}(\beta_{h_0'}\beta_h') \\ &= (\beta_h')^{-1}\beta_{h_0'}^{-1}\beta_{h_0'}\delta^{\mathcal{M}}\beta_h' \\ &= (\beta_h')^{-1}\delta^{\mathcal{M}}\beta_h', \end{aligned}$$

which proves the claim. □

Lemma 3.4.2 implies

Lemma 3.4.6. *b is symmetric in the sense of Definition 2.3.2.*

3.4.2 The closed one-form on the space of metrics

Definition 3.4.7. Let $\omega_T = \operatorname{Tr}'(-1)^Q e^{-\Delta} b$.

For the sake of concreteness, we note the following: Consider a curve in $\mathcal{M}$ parametrized by $h(u)$ for u in some interval in $\mathbb{R}$. Let β_u be the associated operator.

Then ω_T pulled back to the curve, which we also denote by ω_T, is

$$\omega_T = \mathrm{Tr}'(-1)^Q e^{-\Delta^{h(u)}} \beta_u^{-1} \frac{d\beta_u}{du}\, du$$

$$= \sum_{q=0}^{r} (-1)^q \,\mathrm{Tr}'\, e^{-\Delta_q^{h(u)}} \beta_u^{-1} \frac{d\beta_u}{du}\, du$$

We will now explain the significance of ω_T. The idea is that the analytic torsion T may be interpreted as a regularized integral of ω_T over a certain curve in the space of metrics $\mathcal{M}$. More precisely:

Let $h \in \mathcal{M}$ be a metric on E. For each degree p, let h^p be the restriction of h to the subbundle E^p.

Remark 3.4.8. Consider the special case in which $E = F \otimes \Lambda^\bullet T^*M$ is the bundle of F-valued forms, where F is a flat unitary bundle. Suppose that h, a metric on $F \otimes \Lambda^\bullet T^*M$, is induced by a Riemannian metric g^{TM} and the canonical metric on F. Suppose we scale g^{TM} by $\frac{1}{t}$, i.e., let $g_t^{TM} = \frac{1}{t} g^{TM}$, and let h_t be the induced metric on $F \otimes \Lambda^\bullet T^*M$. Then a short computation shows that $h_t^p = t^p h^p$. (Here, as in the rest of this chapter, we consider the volume form fixed.)

This motivates the following, which applies to any metric $h \in \mathcal{M}$. For $t \in (0, \infty)$, let h_t be the metric on E whose restriction to E^p is $t^p h^p$. This parametrizes a curve C_h in $\mathcal{M}$. The associated operator β_{h_t} is given by t^p on E^p. Thus the one-form $b = \beta^{-1} \delta^{\mathcal{M}} \beta$ pulled back to C_h is, on E^p,

$$b = p \frac{dt}{t}.$$

We see that $d^* = \beta^{-1} d^{*_0} \beta$ and Δ are given at time t by

$$d^*(t) = t d^{*_0}$$

$$\Delta(t) = t \Delta^h$$

Finally, we have that ω_T pulled back to C_h is

$$\omega_T = \mathrm{Tr}'(-1)^Q Q e^{-t\Delta^h} \frac{dt}{t},$$

which, modulo the regularizing factor t^s, is precisely the integrand in the zeta function defining the analytic torsion T. Thus we have proven

Lemma 3.4.9. *For* $\operatorname{Re} s > \frac{n}{m}$,

$$\zeta(s; \Delta^h, (-1)^Q Q) = \frac{1}{\Gamma(s)} \int_{C_h} t^s \omega_T,$$

where here ω_T *denotes the one-form of Definition 3.4.7 pulled back to* C_h.

Recall that the analytic torsion $T(h)$ is defined as $\frac{1}{2} \frac{d}{ds}\big|_{s=0} \zeta\left(s; \Delta^h, (-1)^Q Q\right)$. Using that $\frac{1}{\Gamma(s)} = s + O(s^2)$ as $s \to 0$ and Lemma 3.4.9, we have the heuristic

$$2T(h) = \left(\int_{C_h} t^s \omega_T\right)\Bigg|_{s=0}^{\mathrm{AC}},$$

where the superscript AC indicates that we must analytically continue the function in parentheses, which is defined for $\operatorname{Re} s_1, s_2$ large, to the origin. (This heuristic is only literally true if the aforementioned analytic continuation is holomorphic at the origin.) Formally, setting $s = 0$ on the right-hand side leaves $\int_{C_h} \omega_T$; this is purely formal because the analytic continuation is necessary for the integral to converge. This suggests the following heuristic interpretation of analytic torsion:

> The analytic torsion $T(h)$ may be interpreted as the (regularized) integral of ω_T over the curve C_h in the space of metrics $\mathcal{M}$.

In the next section, we will prove that ω_T is closed on $\mathcal{M}$. This fact is the essential tool in our proof of the metric variation formula, Theorem 3.3.3. The idea of the proof is simple: If C_h were a closed curve, Stokes' theorem applied to the closed form ω_T

would imply that $T(h)$ would be independent of h. The problem, of course, is the fact that C_h is not closed (not to mention the fact that regularization is necessary). This means that a boundary term enters the computation. It turns out that this boundary term is just a coefficient in a certain $t \to 0$ heat kernel asymptotic expansion.

We give the proof of Theorem 3.3.3 in Section 3.6. First, though, in Section 3.5, we prove that ω_T is closed.

3.5 Proof of closedness

This section consists of a proof of:

Theorem 3.5.1. *ω_T is closed on $\mathcal{M}$, i.e., $\delta^{\mathcal{M}}\omega_T = 0$.*

For each complex number z that is not an eigenvalue of Δ, let R_z denote the resolvent $(\Delta - z)^{-1}$.

Remark 3.5.2. The resolvent is a pseudodifferential operator of order $-m$, i.e., $R_z \in \Psi^{-m}$. (Recall that $m > 0$ is the order of Δ.) In particular, R_z defines a bounded linear map $L^2(M, E) \to L^2(M, E)$, so in the language of Chapter 2, we view $R_z \in \Omega^0(\mathcal{M}, B)$. Furthermore, if N is an integer greater than $\frac{n}{m}$, then $R_z^N \in \Psi^{-Nm}$ is trace-class by Remark 2.3.3.

We will work not with the heat operator $e^{-\Delta}$ but with the resolvent R_z for technical reasons (essentially, the resolvent is easier to differentiate). That is, we will work not with ω_T but with $\tilde{\omega}_T$, which we now define.

Fix once and for all an integer N greater than $\frac{n}{m}$. Remark 3.5.2 ensures that $(-1)^Q R_z^N b$ is trace-class.

Definition 3.5.3. Let $\tilde{\omega}_T \in \Omega^1(\mathcal{M}, \mathbb{C})$ be the following $\mathbb{C}$-valued one-form on $\mathcal{M}$:

$$\tilde{\omega}_T := \mathrm{Tr}(-1)^Q R_z^N b. \tag{3.19}$$

$\tilde{\omega}_T$ depends on a complex number z, but we will usually suppress this dependence. ω_T and $\tilde{\omega}_T$ are related by the Cauchy integral formula: let C be a contour in the complex plane surrounding the interval $[\lambda_1, \infty) \subset \mathbb{R} \subset \mathbb{C}$, where λ_1 is the first nonzero eigenvalue of Δ. Then we have

$$e^{-\Delta}\,(I - \Pi_{\ker \Delta}) = \frac{1}{2\pi i}\frac{1}{(N-1)!}\int_C e^{-z}\, R_z^N\, dz,$$

and therefore also

$$(-1)^Q e^{-\Delta} b\,(I - \Pi_{\ker \Delta}) = \frac{1}{2\pi i}\frac{1}{(N-1)!}\int_C e^{-z}\,(-1)^Q R_z^N b\, dz.$$

Since the trace and $\delta^{\mathcal{M}}$ commute with the integral, we have

$$\omega_T = \frac{1}{2\pi i}\frac{1}{(N-1)!}\int_C e^{-z}\,\tilde{\omega}_T\, dz \text{ and}$$

$$\delta^{\mathcal{M}}\omega_T = \frac{1}{2\pi i}\frac{1}{(N-1)!}\int_C e^{-z}\,\left(\delta^{\mathcal{M}}\tilde{\omega}_T\right)\, dz. \tag{3.20}$$

Remark 3.5.4. Of course, λ_1 depends on the metric, and therefore the choice of C depends on the metric. But this does not affect our computations since on any compact set of metrics $\mathcal{K} \subset \mathcal{M}$, λ_1 is bounded away from zero. Thus if we restrict our attention to metrics in $\mathcal{K}$, we may choose C independently of the metric for the purposes of the contour integrals above.

For similar reasons, our computations are also not affected by the fact that the set of z for which R_z and $\tilde{\omega}_T$ are defined depends on the metric.

We will prove that $\tilde{\omega}_T$ is closed; by (3.20), this is sufficient to prove that ω_T is closed. First, we must prove some intermediate results.

Lemma 3.5.5. *We may rewrite* $\tilde{\omega}_T = \mathrm{Tr}(-1)^Q R_z^N b$ *in either of the following equivalent ways:*

$$\tilde{\omega}_T = \overline{\mathrm{Tr}}\,(-1)^Q R_{\bar{z}}^N b \tag{3.21}$$

$$\tilde{\omega}_T = \frac{1}{2}\,\mathrm{Tr}(-1)^Q R_z^N b + \frac{1}{2}\overline{\mathrm{Tr}}\,(-1)^Q R_{\bar{z}}^N b \tag{3.22}$$

Proof. The second equality follows from the first, so it suffices to prove the first. We simply apply Lemma 2.3.5 relating the trace and the adjoint:

$$\begin{aligned}\tilde{\omega}_T = \operatorname{Tr}(-1)^Q R_z^N b &= \overline{\operatorname{Tr}}\left((-1)^Q R_z^N b\right)^* \\ &= \overline{\operatorname{Tr}}\, b R_{\bar{z}}^N (-1)^Q \\ &= \overline{\operatorname{Tr}}\, (-1)^Q R_{\bar{z}}^N b,\end{aligned}$$

where the last equality follows from the cyclicity of the trace and the fact that $(-1)^Q$ commutes with the resolvent. □

For operators or forms a_1 and a_2, let $[a_1, a_2] = a_1 a_2 - a_2 a_1$ and $\{a_1, a_2\} = a_1 a_2 + a_2 a_1$ denote their commutator and anticommutator, respectively. We have the following:

Lemma 3.5.6. *For a smooth one-parameter family of metrics h_u $(u \in I \subset \mathbb{R})$ with associated operators $\beta_u := \beta_{h_u}$, $d^{*_u} := d^{*_{h_u}}$, $\Delta^u := \Delta^{h_u}$, and $R_z^u := R_z^{h_u}$, we have*

$$\begin{aligned}\frac{\partial}{\partial u} d^{*_u} &= \left[d^{*_u}, \beta_u^{-1} \frac{\partial \beta_u}{\partial u}\right] \\ \frac{\partial}{\partial u} \Delta^u &= \left\{d, \left[d^{*_u}, \beta_u^{-1} \frac{\partial \beta_u}{\partial u}\right]\right\} \\ \frac{\partial}{\partial u} R_z^u &= R_z^u \left(\frac{\partial}{\partial u} \Delta^u\right) R_z^u = R_z^u \left\{d, \left[d^{*_u}, \beta_u^{-1} \frac{\partial \beta_u}{\partial u}\right]\right\} R_z^u\end{aligned}$$

Proof. The assertions follow from the identity $d^{*_h} = \beta^{-1} d^{*_0} \beta$ (Lemma 3.4.3) and the standard fact that $\frac{\partial}{\partial u}(\beta^{-1}) = -\beta^{-1} \frac{\partial \beta}{\partial u} \beta^{-1}$ (and similarly for the derivative of $R_z^u = (z - \Delta^u)^{-1}$), which follows from differentiating the identity $\beta \beta^{-1} = I$.

□

This immediately implies:

Lemma 3.5.7. $\delta^{\mathcal{M}} \Delta = \{d, [d^{*_u}, b]\}$ *and* $\delta^{\mathcal{M}} R_z = R_z \left(\delta^{\mathcal{M}} \Delta\right) R_z$.

We will also need the results:

Lemma 3.5.8. $\delta^{\mathcal{M}} b = -b\, b$.

Proof. Since $\left(\delta^{\mathcal{M}}\right)^2 = 0$, we have

$$\begin{aligned} \delta^{\mathcal{M}} b &= \delta^{\mathcal{M}}(\beta^{-1}\delta^{\mathcal{M}}\beta) \\ &= \delta^{\mathcal{M}}(\beta^{-1})\, \delta^{\mathcal{M}}\beta \\ &= -\beta^{-1}(\delta^{\mathcal{M}}\beta)\beta^{-1}\, \delta^{\mathcal{M}}\beta \\ &= -b\, b. \end{aligned}$$

□

Lemma 3.5.9. *Let $a_1, a_2, a_3 \in \Omega^\bullet(\mathcal{M}, A)$, where $A = B$ or $A = \Psi$. Suppose that d and d^* commute with each of a_1, a_2, and a_3, and that the forms below are trace-class. Then we have the identity*

$$\mathrm{Tr}(-1)^Q a_1 b a_2 (\delta^{\mathcal{M}}\Delta)^* a_3 = \mathrm{Tr}(-1)^Q a_1 (\delta^{\mathcal{M}}\Delta) a_2 b a_3.$$

Proof. We use the cyclicity of the trace and that d and d^* commute with a_1, a_2, and a_3 and anticommute with $(-1)^Q$ to compute:

$$\mathrm{Tr}(-1)^Q a_1 b a_2 (b d d^*) a_3 = \mathrm{Tr}(-1)^Q a_1 (d d^* b) a_2 b a_3$$

$$\mathrm{Tr}(-1)^Q a_1 b a_2 (-d^* d b) a_3 = \mathrm{Tr}(-1)^Q a_1 (-b d^* d) a_2 b a_3$$

$$\mathrm{Tr}(-1)^Q a_1 b a_2 (-d b d^* + d^* b d) a_3 = \mathrm{Tr}(-1)^Q a_1 (d^* b d - d b d^*) a_2 b a_3$$

Summing the three identities above gives the result. □

Lemma 3.5.10. $\delta^{\mathcal{M}} R_z^N = \sum_{i=1}^{N} R_z^i \left(\delta^{\mathcal{M}}\Delta\right) R_z^{N-i+1}$.

Proof. This follows from Lemma 3.5.7. □

Now we are prepared to prove the main result of this section.

Lemma 3.5.11. *The $\mathbb{C}$-valued one-form $\tilde{\omega}_T = \operatorname{Tr}(-1)^Q R_z^N b$ is closed.*

Proof. Recall the identity (3.22):

$$2\tilde{\omega}_T = \operatorname{Tr}(-1)^Q R_z^N b + \overline{\operatorname{Tr}}\,(-1)^Q R_{\bar{z}}^N b.$$

We will look for cancellation between terms in the derivative of the first term on the right-hand-side and terms in the derivative of the second term on the right-hand side. We have on the one hand

$$\begin{aligned}\delta^{\mathcal{M}}\left(\operatorname{Tr}(-1)^Q R_z^N b\right) &= \operatorname{Tr}(-1)^Q(\delta^{\mathcal{M}} R_z^N)b \;+\; \operatorname{Tr}(-1)^Q R_z^N(\delta^{\mathcal{M}} b)\\ &= \sum_{i=1}^{N} \operatorname{Tr}(-1)^Q R_z^i \,\left(\delta^{\mathcal{M}}\Delta\right)\, R_z^{N-i+1} b \;-\; \operatorname{Tr}(-1)^Q R_z^N bb,\end{aligned}$$

and on the other hand

$$\begin{aligned}\delta^{\mathcal{M}}\left(\overline{\operatorname{Tr}}\,(-1)^Q R_{\bar{z}}^N b\right) &= \operatorname{Tr}(-1)^Q(\delta^{\mathcal{M}} R_{\bar{z}}^N)b \;+\; \operatorname{Tr}(-1)^Q R_{\bar{z}}^N(\delta^{\mathcal{M}} b)\\ &= \sum_{i=1}^{N} \overline{\operatorname{Tr}}\,(-1)^Q R_{\bar{z}}^i \,\left(\delta^{\mathcal{M}}\Delta\right)\, R_{\bar{z}}^{N-i+1} b \;-\; \overline{\operatorname{Tr}}\,(-1)^Q R_{\bar{z}}^N bb.\end{aligned}$$

Note that $(bb)^* = -b^*b^* = -bb$ since b is a symmetric one-form. Now compute

$$\begin{aligned}\operatorname{Tr}(-1)^Q R_z^N bb &= \overline{\operatorname{Tr}}\,((-1)^Q R_z^N bb)^*\\ &= -\overline{\operatorname{Tr}}\, bbR_{\bar{z}}^N(-1)^Q\\ &= -\overline{\operatorname{Tr}}\, R_{\bar{z}}^N(-1)^Q bb\\ &= -\overline{\operatorname{Tr}}\,(-1)^Q R_{\bar{z}}^N bb,\end{aligned}$$

where we have used the cyclic property of the trace and that $(-1)^Q$ commutes with R_z.

It remains to show that the following vanishes:

$$\sum_{i=1}^{N}\left[\operatorname{Tr}(-1)^Q R_z^i \,\left(\delta^{\mathcal{M}}\Delta\right)\, R_z^{N-i+1} b + \overline{\operatorname{Tr}}\,(-1)^Q R_{\bar{z}}^i \,\left(\delta^{\mathcal{M}}\Delta\right)\, R_{\bar{z}}^{N-i+1} b\right].$$

Denote the ith summand by A_i. We claim that for each i, $A_i = 0$. To see this, we compute

$$\begin{aligned}
&\mathrm{Tr}(-1)^Q R_z^i \,(\delta^{\mathcal{M}}\Delta)\, R_z^{N-i+1} b \\
&\quad = \overline{\mathrm{Tr}}\, ((-1)^Q R_z^i \,(\delta^{\mathcal{M}}\Delta)\, R_z^{N-i+1} b)^* \\
&\quad = -\overline{\mathrm{Tr}}\, b R_{\bar{z}}^{N-i+1} \,(\delta^{\mathcal{M}}\Delta)^*\, R_{\bar{z}}^i (-1)^Q \\
&\quad = -\overline{\mathrm{Tr}}\, (-1)^Q R_{\bar{z}}^i b R_{\bar{z}}^{N-i+1} \,(\delta^{\mathcal{M}}\Delta)^* \\
&\quad = -\overline{\mathrm{Tr}}\, (-1)^Q R_{\bar{z}}^i \,(\delta^{\mathcal{M}}\Delta)\, R_{\bar{z}}^{N-i+1} b \text{ by Lemma 3.5.9.}
\end{aligned}$$

This proves that $A_i = 0$ and proves the lemma. □

This proves that ω_T is closed by (3.20), completing the proof of Theorem 3.5.1.

3.6 Proof of the metric variation formula

This section consists of a proof of Theorem 3.3.3, which we now restate for convenience.

Theorem 3.6.1. *Let $h(u)$, $u \in [0,1]$, be a smooth one-parameter family of metrics in $\mathcal{M}$. Let $T(h(u)) = T(M, E, h(u))$ be the associated analytic torsion. Then*

$$\frac{d}{du} \log T(h(u)) = a_0'(u), \tag{3.23}$$

where $a_0'(u)$ is the t^0 term in the asymptotic expansion of $\mathrm{Tr}'(-1)^Q \beta_u^{-1} \frac{d\beta_u}{du} e^{-t\Delta^{h(u)}}$.

Proof. The idea of the argument is simple: we apply Stokes' theorem to the one-form ω. $a_0'(u)$ arises essentially as a boundary term. Of course, we must be careful with the details of the regularization and analytic continuation.

Rather than differentiating the analytic torsion, it suffices to compute the difference in analytic torsions associated to $h(0)$ and $h(1)$. (As a result, we will use the usual Stokes' theorem rather than an infinitesimal version.)

For $A > \epsilon > 0$, let $\Sigma = \Sigma_{A,\epsilon}$ denote the surface with boundary in $\mathcal{M}$ parametrized by $(u,t) \mapsto h(u)_t$, for $u \in [0,1]$ and $t \in (\epsilon, A)$. (We use the subscript t as in Subsection 3.4.2.) The boundary of Σ consists of four curves, described respectively by $u = 0$, $u = 1$, $t = \epsilon$, and $t = A$.

ω pulled back to $\partial\Sigma$, in the coordinates (u,t), is

$$\begin{aligned}\omega = &\operatorname{Tr}'(-1)^Q Q e^{-t\Delta(u)} \frac{dt}{t} \\ &+ \operatorname{Tr}'(-1)^Q e^{-t\Delta(u)} \beta^{-1} \frac{\partial \beta}{\partial u} du.\end{aligned}$$

Since ω is closed, $\delta^{\mathcal{M}}(t^s\omega)$ pulled back to Σ is

$$\begin{aligned}\delta^{\mathcal{M}}(t^s\omega) &= st^{s-1}\, dt\, \omega \\ &= st^{s-1} \operatorname{Tr}'(-1)^Q e^{-t\Delta(u)} \beta^{-1} \frac{\partial \beta}{\partial u}\, dt\, du.\end{aligned}$$

Let $\zeta_u(s) = \zeta(s; \Delta^{h(u)}, (-1)^Q Q)$. Let $\zeta_{u,A,\epsilon}(s)$ be defined for $\operatorname{Re} s > \frac{n}{m}$ by

$$\Gamma(s)\zeta_{u,A,\epsilon}(s) := \int_{C_{h(u),A,\epsilon}} t^s \omega.$$

Stokes' theorem gives, for $\operatorname{Re} s > \frac{n}{m}$,

$$\begin{aligned}\Gamma(s)\left[\zeta_{0,A,\epsilon}(s) - \zeta_{1,A,\epsilon}(s)\right] = &\int_0^1 \int_\epsilon^A st^{s-1} \operatorname{Tr}'(-1)^Q e^{-t\Delta(u)} \beta^{-1} \frac{\partial \beta}{\partial u}\, dt\, du \\ &+ \int_0^1 t^s \operatorname{Tr}'(-1)^Q e^{-\epsilon\Delta(u)} \beta^{-1} \frac{\partial \beta}{\partial u} du \\ &- \int_0^1 t^s \operatorname{Tr}'(-1)^Q e^{-A\Delta(u)} \beta^{-1} \frac{\partial \beta}{\partial u} du\end{aligned}$$

In the limits $A \to \infty$ and $\epsilon \to 0$, the latter two terms tend to zero for $\operatorname{Re} s$ large by the admissibility conditions (A1) and (A2). Thus we obtain, for $\operatorname{Re} s > \frac{n}{m}$,

$$\begin{aligned}\zeta_0(s) - \zeta_1(s) &= \frac{1}{\Gamma(s)} \int_0^1 \int_0^\infty st^{s-1} \operatorname{Tr}'(-1)^Q e^{-t\Delta(u)} \beta^{-1} \frac{\partial \beta}{\partial u}\, dt\, du \\ &= \int_0^1 s\zeta\left(s; \Delta(u), (-1)^Q \beta^{-1} \frac{\partial \beta}{\partial u}\right) du\end{aligned}$$

We have shown that this equality holds for $\operatorname{Re} s$ large, but both sides possess unique meromorphic continuations, so in fact the equality holds for all s. Differentiating, we obtain the following expression for the difference in (logarithms of) analytic torsions associated to the two metrics $h(0)$ and $h(1)$:

$$\log T(h(0)) - \log T(h(1)) = \int_0^1 \frac{d}{ds}\bigg|_{s=0} \left(s\zeta\left(s; \Delta(u), (-1)^Q \beta^{-1} \frac{\partial \beta}{\partial u}\right)\right) du$$

But the derivative in the integrand is just the value of the zeta function at $s = 0$:

$$\frac{d}{ds}\bigg|_{s=0} \left(s\zeta\left(s; \Delta(u), (-1)^Q \beta^{-1} \frac{\partial \beta}{\partial u}\right)\right) = \zeta\left(0; \Delta(u), (-1)^Q \beta^{-1} \frac{\partial \beta}{\partial u}\right).$$

Theorem 3.1.9 gives that the latter is given precisely by the coefficient on t^0 in the asymptotic expansion of $\operatorname{Tr}'(-1)^Q e^{-t\Delta(u)} \beta^{-1} \frac{\partial \beta}{\partial u}$. This proves the theorem. □

3.7 Regularization methods

Let $h(t)$ be an admissible function, satisfying $|h(t)| \leqslant Ce^{-\epsilon t}$ for $t \geqslant 1$ and $h(t) = \sum_{j=0}^{J} a_{p_j} t^{p_j} + O(t)$ as $t \to 0$. Recall that the Mellin transform gives the zeta function $\zeta(s)$ defined by, for $\operatorname{Re} s > -p_0$,

$$\zeta(s) = \frac{1}{\Gamma(s)} \int_0^\infty t^s h(t) \frac{dt}{t}. \tag{3.24}$$

Let us assume that the $t \to 0$ asymptotic expansion contains no integer powers of t, i.e., none of the p_j's is an integer. (This is the case, for example, if $h(t) = \operatorname{Tr} e^{-tL}$ for a strictly positive elliptic operator L on an odd-dimensional manifold.) Then $\zeta(0) = 0$, and $\frac{d\zeta}{ds}(0)$ is the value at $s = 0$ of the function $z(s) := \Gamma(s)\zeta(s)$; this value may be interpreted as a regularization of the divergent integral $\int_0^\infty h(t) \frac{dt}{t}$.

In this section, we will consider a class of alternative methods of regularizing the aforementioned divergent integral, given by replacing t^s in (3.24) by $\phi(t)^s$, where $\phi(t)$

is a more general function. We now discuss the assumptions we will place on $\phi(t)$, which we do not claim to be sharp.

Assume $\phi(t)$ is positive and smooth for $t > 0$. Assume that $\phi(t)$ is $O(t^b)$ as $t \to \infty$ for some b. Assume that there exist $a > 0$, $p > 0$, and a function $u(t)$ such that

$$\phi(t) = at^p u(t), \tag{3.25}$$

where $u(t)$ is smooth on $[0, 1]$ and $u(0) = 1$. If $\phi(t)$ satisfies these properties, then we will say that $\phi(t)$ is an "allowable regularizing function".

Let $f(t)$ be an admissible function whose $t \to 0$ asymptotic expansion contains no integer powers of t. Consider the function $z_\phi(s)$ defined by

$$z_\phi(s) = \int_0^\infty \phi(t)^s f(t)\, dt, \tag{3.26}$$

whose value at $s = 0$ (if it is finite) may be interpreted as a regularization of the divergent integral $\int_0^\infty f(t)\, dt$. Note that if we take $f(t) = \frac{h(t)}{t}$ (which is admissible if and only if $h(t)$ is) and $\phi(t) = t$, then $z_\phi(s)$ is the function $z(s)$ mentioned in the first paragraph of this section. Our goal is to study the dependence of $z_\phi(0)$ on the function $\phi(t)$.

First, we prove the following lemma involving Taylor approximations:

Lemma 3.7.1. *Suppose $u(t)$ is a smooth positive function such that $u(0) = 1$. Let $K > 0$ be given. Then for $s \in \mathbb{C}$, we may write*

$$u(t)^s = 1 + \sum_{j=1}^{K-1} s^j \alpha_j(t) + r(s,t), \tag{3.27}$$

where each $\alpha_j(t)$ is a polynomial in t, $r(s,t)$ is holomorphic in s, and for each compact set $U \subset \mathbb{C}$, there exists C such that for $s \in U$ and $0 < t < 1$, $|r(s,t)| \leqslant C|s|t^K$.

Proof. $\ln u(t)$ is a smooth function with $\ln u(0) = 0$; we will use its Taylor approximation for t near 0, i.e., for any L, we may write as $t \to 0$

$$\ln u(t) = \sum_{l=1}^{L} a_l t^l + O(t^{L+1}). \tag{3.28}$$

Now we expand $u(t)^s = e^{s \ln u(t)}$ in powers of $s \ln u(t)$:

$$u(t)^s = 1 + \sum_{j=1}^{K-1} s^j (\ln u(t))^j + \tilde{r}(s,t) \tag{3.29}$$

where $\tilde{r}(s,t)$ is holomorphic in s, and for s in any compact set and $0 < t < 1$, $|\tilde{r}(s,t)| \leqslant C_2 |s|^K |\ln u(t)|^K$, which implies $|\tilde{r}(s,t)| \leqslant C_3 |s|^K t^K$ since $\ln u(t)$ is $O(t)$ as $t \to 0$. (Of course, the constants C_2 and C_3 depend on the compact set.) (3.28) shows that for each j, $(\ln u(t))^j$ may be written as a polynomial in t, which we define to be $\alpha_j(t)$, plus a remainder term $r_j(t)$ that is $O(t^K)$ as $t \to 0$. The desired remainder term $r(s,t)$ is the sum

$$r(s,t) = \sum_{j=1}^{K-1} s^j r_j(s,t) + \tilde{r}(s,t),$$

which satisfies the desired estimate, proving the lemma. □

Now we return to our study of z_ϕ. We may decompose z_ϕ as follows:

$$\begin{aligned} z_\phi(s) &= \int_0^1 \phi(t)^s f(t)\,dt + \int_1^\infty \phi(t)^s f(t)\,dt \\ &=: z_{\phi,0}(s) + z_{\phi,\infty}(s). \end{aligned}$$

$z_{\phi,\infty}(s)$ is holomorphic in s, and its value at $s = 0$ is

$$z_{\phi,\infty}(0) = \int_1^\infty f(t)\,dt,$$

which converges and does not depend on the choice of $\phi(t)$.

Now we will study $z_{\phi,0}(s)$. We substitute the following truncation of the asymptotic expansion of $f(t)$ into the integral defining $z_{\phi,0}(s)$:

$$f(t) = \sum_{j=0}^{J} a_{p_j} t^{p_j} + R_J(t),$$

where for $0 < t < 1$, $R_J(t) \leqslant Ct$. This implies that $R_J(t)$ is integrable on $[0,1]$ and therefore the integral

$$\int_0^1 \phi(t)^s R_J(t)\, dt$$

converges and is holomorphic at $s = 0$, and its value at $s = 0$ does not depend on $\phi(t)$.

It remains to study for each j the function of s

$$\int_0^1 \phi(t)^s a_{p_j} t^{p_j}\, dt. \tag{3.30}$$

Recall that we have assumed each p_j is not an integer. For clarity of notation, we will replace p_j by a number q that is not an integer and ignore the constant a_{p_j}, i.e., let us consider the function

$$\xi_\phi(s) := \int_0^1 \phi(t)^s t^q\, dt.$$

Lemma 3.7.1 shows that we may write $\phi(t)^s$ as

$$\phi(t)^s = a^s t^{ps} \left(1 + \sum_{j=1}^{J} s^j \alpha_j(t) + r(s,t)\right),$$

where each $\alpha_j(t)$ is a polynomial in t and $r(s,t)$ is holomorphic in s and $|r(s,t)| \leqslant$

$C|s|t^{-q}$ for s in some neighborhood of 0 and $0 < t < 1$. Thus we have

$$\xi_\phi(s) = a^s \int_0^1 t^{ps+q}\, dt \tag{3.31}$$

$$+ a^s \sum_{j=1}^{J} s^j \int_0^1 t^{ps+q} \alpha_j(t)\, dt \tag{3.32}$$

$$+ a^s \int_0^1 t^{ps+q} r(s,t)\, dt. \tag{3.33}$$

We have that for any integer k and for $\operatorname{Re} s$ sufficiently large,

$$a^s \int_0^1 t^{ps+q} t^k = a^s \frac{1}{ps+q+k+1},$$

which extends to a meromorphic function that is holomorphic at $s = 0$ because q is not an integer. This implies that the meromorphic continuation of each of the summands in line (3.32) vanishes at $s = 0$ because of the factors of s^j for $j \geqslant 1$. The integral in line (3.33) converges for $\operatorname{Re} s > -1$ and vanishes at $s = 0$ by the estimate $|r(s,t)| \leqslant C|s|t^{-q}$. This leaves the right-hand side of line (3.31), which we can compute explicitly for $\operatorname{Re} s$ large, obtaining

$$a^s \frac{1}{ps+q+1},$$

the value of whose meromorphic extension at $s = 0$ is $\frac{1}{q+1}$, which clearly does not depend on ϕ.

We have proven that the value of $\xi_\phi(s)$ at $s = 0$ does not depend on ϕ, which proves that the same is true for every term in $z_{\phi,0}(s)$ of the form (3.30).

To summarize, we have proven the following:

Proposition 3.7.2. *If $f(t)$ is an admissible function with no integer powers of t in its $t \to 0$ asymptotic expansion, then the value at $s = 0$ of the meromorphic extension of*

$$\int_0^\infty \phi(t)^s f(t)\, dt$$

is independent of the choice of allowable regularizing function $\phi(t)$*.*

Applying this to analytic torsion, we obtain:

Corollary 3.7.3. *If the dimension of* M *is odd and* E *is an acyclic elliptic complex, then the associated analytic torsion is given by*

$$2\ln T(M,E,h) = \left(\int_{C_h} \phi(t)^s \omega_T\right)\Bigg|_{s=0}^{AC},$$

where $\phi(t)$ *is any allowable regularizing function, viewed as a function* $C_h \to \mathbb{R}$*.*

3.8 Varying the curve

The interpretation of the analytic torsion $T(h)$ as a regularized integral of ω_T over the curve C_h raises the question of whether there are other interesting curves in the space of metrics over which to integrate ω_T. In this section, we will show that, assuming odd dimension and acylicity, this regularized integral is constant on a certain family of curves that includes C_h. This should not be surprising, as Stokes' theorem applied to the closed form ω_T gives that the integral is invariant under compactly supported perturbations of C_h. We will now show that the same is true for more general perturbations of C_h.

Fix a metric $h \in \mathcal{M}$. Recall that the curve C_h is parametrized by $t \mapsto h_t$, where on E^q, we scale the metric according to $h_t^q = t^q h^q$. This motivates the following generalization in which we scale the metric on E^q by a more general function of t. Let $f = f(t,q) : (0,\infty) \times \{0,\dots,r\} \to (0,\infty)$ be a function that is smooth in t. We will take the operator β to be multiplication by $f(t,q)$ on E^q; that is, let $h_{f,t}$ be the metric on E such that its restriction to E^q is

$$h_{f,t}^q := f(t,q)h^q.$$

In this notation, h_t is $h_{f,t}$ if we take f to be $f(t,q) = t^q$. Let $C_{h,f}$ be the curve in $\mathcal{M}$ parametrized by $t \in (0,\infty) \mapsto h_{f,t}$.

For ease of notation, for $1 \leqslant q \leqslant r$, let $\rho(t,q)$ be the ratio

$$\rho(t,q) := \frac{f(t,q)}{f(t,q-1)}. \tag{3.34}$$

We will use the notation $\dot\rho(t,q)$ and $\dot f(t,q)$ for derivatives with respect to t. Let us impose the assumptions for each q that $\dot\rho(t,q) > 0$ (so that $\rho(t,q)$ is an invertible function of t with smooth inverse $\rho^{-1}(\cdot,q)$) and that $\lim_{t\to 0^+} \rho(t,q) = 0$ and $\lim_{t\to\infty} \rho(t,q) = \infty$.

We compute the following operators pulled back to $C_{h,f}$, acting on sections of E^q:

$$\begin{aligned} b &= \frac{\dot f(t,q)}{f(t,q)}\, dt \\ d^*(t) &= \rho(t,q) d^* \\ \Delta &= \rho(t,q) dd^* + \rho(t,q+1) d^* d, \end{aligned}$$

where $d^* = d^{*_h}$ is associated to the original metric h. We obtain that the one-form ω_T pulled back to $C_{h,f}$ is

$$\omega_T = \sum_{q=0}^{r} (-1)^q \frac{\dot f(t,q)}{f(t,q)} \left(\mathrm{Tr}'\, e^{-\rho(t,q) d_{q-1} d_{q-1}^*} + \mathrm{Tr}'\, e^{-\rho(t,q+1) d_q^* d_q} \right) dt,$$

where we are using the notation of (3.13). Recall from (3.14) that for any $c > 0$,

$$\mathrm{Tr}'\, e^{-c\, d_q d_q^*} = \mathrm{Tr}'\, e^{-c\, d_q^* d_q},$$

Thus we may rewrite ω_T as

$$\omega_T = \sum_{q=0}^{r} (-1)^q \left(\frac{\dot f(t,q)}{f(t,q)} - \frac{\dot f(t,q-1)}{f(t,q-1)} \right) \mathrm{Tr}'\, e^{-\rho(t,q) d_{q-1} d_{q-1}^*}\, dt.$$

We will make a change of variables. Let $\tau_q = \rho(t,q)$, i.e., $t = \rho^{-1}(\tau_q, q)$. A short computation, using $dt = \frac{1}{\rho'(t,q)} d\tau_q$ and the definition (3.34) of $\rho(t,q)$, shows that under this change of variables, we have the equality of one-forms

$$\left(\frac{\dot{f}(t,q)}{f(t,q)} - \frac{\dot{f}(t,q-1)}{f(t,q-1)} \right) dt = \frac{d\tau_q}{\tau_q}.$$

This implies that ω_T transforms under this change of variables to

$$\omega_T = \sum_{q=0}^{r} (-1)^q \operatorname{Tr}' e^{-\tau_q d_{q-1} d_{q-1}^*} \frac{d\tau_q}{\tau_q}. \tag{3.35}$$

Proposition 3.7.2 shows that, assuming acyclicity and odd dimension, the method of regularization is irrelevant. So we may choose any method to regularize the integral of $\frac{1}{2}\omega_T$ over $C_{h,f}$; (3.35) gives that we obtain

$$\sum_{q=0}^{r} (-1)^q \ln \det(d_{q-1} d_{q-1}^*).$$

But recall the identity (3.15), which shows that this is precisely the usual analytic torsion.

3.9 Some historical remarks

This section contains some remarks both on our results and on the history of analytic torsion, including discussion of generalized metrics, Morse variations, and the Cheeger-Müller theorem.

Let us now specialize to the case when the elliptic complex (E, d) is the de Rham complex associated to a flat vector bundle $F \to M$, which was introduced briefly in Example 1 on page 21. Recall that this means that $E = F \otimes \Lambda^\bullet T^*M$, and d is induced by the flat connection on F and the usual exterior derivative on ordinary differential forms. In this section, we will give a mostly non-technical, brief (and far from exhaustive) account of the history of the study of analytic torsion in this context, and we will relate some of our results to this history. We will discuss spaces of metrics, the Cheeger-Müller theorem, and in particular Bismut-Zhang's proof of the Cheeger-Müller theorem, which uses a special case of our closed form ω_T and a metric variation involving a Morse function.

In this context, the space of metrics $\mathcal{M}$ that we have considered throughout this chapter is the space of all metrics on $F \otimes \Lambda^\bullet$ such that the $F \otimes \Lambda^q$'s are mutually orthogonal for $q = 0, \ldots, n$. We will now introduce a distinguished subset of $\mathcal{M}$.

Definition 3.9.1. Let $\mathcal{M}^{\text{Riem}}$ be the set of metrics on E that are induced as the tensor product of some metric h^F on F and some Riemannian metric on M, which induces a metric on the exterior bundle $\Lambda^\bullet T^*M$.

$\mathcal{M}^{\text{Riem}}$ is strictly contained in $\mathcal{M}$. To explain this rather loosely, for a metric in $\mathcal{M}^{\text{Riem}}$, the induced inner products on q-forms for $q = 0, \ldots, n$ are related in the sense that they are all induced by a single Riemannian metric. But for a generic metric in $\mathcal{M}$, the inner products on q-forms need not be related in any way; we are allowed to pick metrics on E^q for each q independently.

Ray-Singer defined the analytic torsion T in the case when F is orthogonal (so that F possesses a canonical metric) and proved that T is independent of the Riemannian metric, assuming acyclicity and odd dimension; their results trivially extend to unitary F. Motivated by the formal similarity between T and the Reidemeister torsion τ and by computations of T and τ on lens spaces by Ray [15], Ray-Singer conjectured the equality of T and τ in the orthogonal case. This conjecture, now known as the Cheeger-Müller theorem, was proved independently by Cheeger [6] and Müller [13] using different techniques. Later, Müller [14] generalized the theorem to unimodular bundles using an extension of Cheeger's techniques. To our knowledge, this paper of Müller was the first to observe that analytic torsion $T(h)$ may be defined for any metric $h \in \mathcal{M}^{\mathrm{Riem}}$ (i.e., for any metric on F and for any Riemannian metric). Bismut-Zhang [5] proved a further generalization of the Cheeger-Müller theorem; we will comment a bit further on their proof later in this section.

Now we will make a a few comments about our results. We emphasize that the closedness of ω_T (Theorem 3.5.1), the variation of T with respect to the metric (Theorem 3.3.3), and the metric independence theorem (Corollary 3.3.4) apply on the entire space of metrics $\mathcal{M}$ and not just on $\mathcal{M}^{\mathrm{Riem}}$. In that sense, the latter two results generalize the metric independence theorem of Ray-Singer [16] and the metric anomaly formula of Bismut-Zhang [5], in which they considered only metrics in the distinguished subset $\mathcal{M}^{\mathrm{Riem}}$.

Remark 3.9.2. Mathai-Wu [11] have introduced analytic torsion for $\mathbb{Z}_2$-graded elliptic complexes. This is more general than our $\mathbb{Z}$-graded context, since a $\mathbb{Z}$-graded complex can always be "rolled up" into a $\mathbb{Z}_2$-graded complex. In particular, Mathai-Wu's twisted de Rham torsion ([9], [10]) and twisted Dolbeault torsion [11], involving a closed flux form H of odd degree, do not fit into our $\mathbb{Z}$-graded framework unless the degree of H is one. Mathai-Wu consider an appropriate analogue of our space of all metrics $\mathcal{M}$, and they prove metric independence of the torsion on this space under appropriate assumptions.

An examination of our definition of the form ω_T and the proof of its closedness shows that they extend without change to the $\mathbb{Z}_2$-graded setting. But crucially, our interpretation of analytic torsion as a regularized integral of ω_T over a curve in the space of metrics does not make sense in the $\mathbb{Z}_2$-graded setting, since the curve C_h relies in an essential way on the $\mathbb{Z}$-grading. For this reason, we have chosen not to discuss $\mathbb{Z}_2$-graded complexes in detail.

We have not yet found an interesting application of our generalized metric independence theorem, but we note that the generalization from orthogonal and unitary bundles to arbitrary metrics on F is essential to Bismut-Zhang's proof of the Cheeger-Müller theorem [5]. Their proof involves a "conformal" perturbation of the metric on F involving a Morse function. We will now discuss their techniques a bit further and relate them to our closed form ω_T.

Following Bismut-Zhang, let $f : M \to \mathbb{R}$ be a smooth function. Let g^F be a metric on the coefficient bundle F. For $T \in \mathbb{R}$, consider the metric $g^{F,T}$ on F defined by

$$g^{F,T} := e^{-2Tf} g^F.$$

Note that we obtain the original metric g^F if we take $T = 0$, i.e., $g^F = g^{F,0}$. Let g^{TM} be a Riemannian metric on M. Then g^{TM} and $g^{F,T}$ induce a metric on $F \otimes \Lambda^\bullet T^*M$; we will denote this metric by h^T.

Remark 3.9.3. The metric h^T is "conformal on F" in the sense that it involves multiplication of g^F by a positive function, which scales the metric on F-valued forms by the same factor in all degrees. It is not conformal in the usual sense because it does not involve a conformal change (or any change) to the Riemannian metric, which would scale the metric on F-valued forms differently in each degree.

The operator $\beta = \beta^T$ associated to h^T is $\beta^T = e^{-2Tf}$, i.e., multiplication by the function e^{-2Tf}. We compute the following operators and forms pulled back to the curve parametrized by $T \mapsto h^T$:

$$\begin{aligned} b &= -2f\,dT \\ d^{*,T} &= e^{2Tf} d^{*,0} e^{-2Tf} \\ \Delta^T &= d e^{2Tf} d^{*,0} e^{-2Tf} + e^{2Tf} d^{*,0} e^{-2Tf} d \\ \omega_T &= -2\,\mathrm{Tr}(-1)^Q f e^{-\Delta^T}\,dT, \end{aligned}$$

where f and $e^{\pm 2Tf}$ denote the operators defined by multiplication by f and $e^{\pm 2Tf}$, respectively.

Remark 3.9.4. As Bismut-Zhang discuss, there is an equivalent alternative approach. Rather than considering the metric h^T, one could instead fix the metric and vary the operator d as follows: let $\tilde{d}^T := e^{-Tf} d e^{Tf}$. Then $(\tilde{d}^T)^* := e^{Tf} d^* e^{-Tf}$ is the adjoint of d^T with respect to the L^2 inner product induced by the fixed metric g^F. The associated Laplacian is $\tilde{\Delta}^T := \tilde{d}^T(\tilde{d}^T)^* + (\tilde{d}^T)^*\tilde{d}^T$. (These operators were introduced by Witten, who related the asymptotics of $e^{-\tilde{\Delta}^T}$ as $T \to \infty$ to Morse theory [23].) It is easy to see that $\tilde{\Delta}^T$ and Δ^T are conjugate ($\tilde{\Delta}^T = e^{-Tf}\Delta^T e^{Tf}$), which implies that they have the same eigenvalues. Thus these two approaches are equivalent for our purposes; we will choose to phrase what follows in terms of Δ^T.

We can also scale the Riemannian metric in the usual way, i.e., by a factor $\frac{1}{t}$ for $t > 0$. Let $h(t,T)$ be the metric on $F \otimes \Lambda^\bullet T^*M$ induced by $\frac{1}{t}g^{TM}$ and $g^{F,T} = e^{-2Tf}g^F$; the map $(t,T) \mapsto h(t,T)$, for $t \in (0,\infty)$ and $T \in \mathbb{R}$, parametrizes a surface $\Sigma = \Sigma(g^{TM}, g^F, f)$ in the space of metrics $\mathcal{M}^{\text{Riem}} \subset \mathcal{M}$. Recall that the scaling $t \mapsto \frac{1}{t}g^{TM}$ scales d^* and Δ by a factor of t; we obtain that the torsion one-form ω_T pulled back to Σ, which we will denote by α (following Bismut-Zhang), is

$$\alpha = \operatorname{Tr}(-1)^Q Q e^{-t\Delta^T}\, dt \;-\; 2\operatorname{Tr}(-1)^Q f e^{-t\Delta^T}\, dT.$$

Bismut-Zhang prove that α is closed on the surface Σ, although they do not make our more general observation that α is the pullback to Σ of a closed form defined on all of $\mathcal{M}$. The closedness of α is essential to the argument of Bismut-Zhang that computes the difference between the analytic torsion T and the Reidemeister torsion τ. We will now give a very rough outline of some of the ideas of their argument.

Let f be a Morse function. By the metric independence theorem for analytic torsion, the computation is not affected by choosing a special Riemannian metric g^{TM} that satisfies (among other things) that the associated gradient vector field of f satisfies the Smale transversality conditions. This gives a Thom-Smale complex whose cohomology may be identified with $H^\bullet(M,F)$; the so-called Milnor metric, the details of which we will omit, is computed from this Thom-Smale complex and is known to equal the Reidemeister metric in the unimodular case.

Rather than directly comparing T and τ (or more precisely, the Ray-Singer metric and the Reidemeister metric), Bismut-Zhang compare the Ray-Singer metric and the Milnor metric. Their basic approach is similar to our approach to proving Theorem 3.3.3 (cf. page 35): they integrate α over the curve Γ that is the boundary of the rectangle $R = R_{\epsilon,A,T_0}$, where R is the subset of Σ on which $\epsilon \leqslant t \leqslant A$ and $0 \leqslant T \leqslant T_0$. Crucially, by Stokes' theorem, this integral vanishes. By taking the limits $\epsilon \to 0$, $T_0 \to \infty$, and $A \to \infty$ and carefully keeping track of the rates of divergence of the integrals along each of the four components of Γ, Bismut-Zhang compute the difference between the Ray-Singer metric and the Milnor metric. They prove that this difference vanishes in the unimodular case, thus generalizing the Cheeger-Müller theorem.

4

Multi-torsion on manifolds with local product structure

The interpretation of analytic torsion as the integral of a closed one-form naturally raises the question of the existence of new invariants constructed from other closed differential forms on the space of metrics, possibly of degree greater than one. In this chapter, we introduce one such closed m-form ω_{MT} on the space of metrics inducing a local geometric product structure on certain manifolds. ω_{MT} is inspired by the m-fold product of ω_T in the following sense: on the space of product metrics on a global product manifold $M_1 \times \cdots \times M_m$, $\omega_{MT} = \omega_T^{M_1} \cdots \omega_T^{M_m}$, where for each j, $\omega_T^{M_j}$ denotes the torsion one-form associated to M_j. ω_{MT} also makes sense on manifolds with only a local product structure; we treat in detail the case when $m = 2$ and the base manifold is a quotient of $M_1 \times M_2$ by a finite group of diffeomorphisms, in which case we define a quantity that we call multi-torsion.

Multi-torsion is essentially a regularized integral of the two-form ω_{MT} over a two-dimensional submanifold of the space of metrics. The regularization involves taking a second derivative of a certain multi-zeta function, a function of two complex variables which may be of independent interest. Under appropriate assumptions, we prove the independence of multi-torsion in the class of local product metrics using a Stokes' theorem argument.

4.1 Motivation: Product manifolds

Our goal in this chapter will be to define multi-torsion on certain manifolds with a local product structure. In this section, we begin with some motivating observations in the case of a global product.

4.1.1 Products of zeta functions

We start with some simple observations. Let ζ_1 and ζ_2 be two zeta functions, each related to an admissible function h_j by the Mellin transform; i.e., for $\operatorname{Re} s$ large and for $j = 1, 2$, we have:

$$\zeta_j(s) = \frac{1}{\Gamma(s)} \int_0^\infty t^s h_j(t)\, \frac{dt}{t}.$$

Define $\zeta(s_1, s_2)$ as the following product:

$$\zeta(s_1, s_2) := \zeta_1(s_1)\zeta_2(s_2).$$

Observe that

$$\left.\frac{\partial^2 \zeta}{\partial s_1 \partial s_2}\right|_{(s_1,s_2)=(0,0)} = \left.\frac{d\zeta_1}{ds_1}\right|_{s_1=0} \left.\frac{d\zeta_2}{ds_2}\right|_{s_2=0}. \tag{4.1}$$

We may interpret $\zeta(s_1, s_2)$ as a "multi-Mellin transform," i.e., as an integral over the quadrant $(0, \infty) \times (0, \infty) \subset \mathbb{R}^2$ by Fubini's theorem:

$$\begin{aligned} \zeta(s_1, s_2) &= \zeta_1(s)\zeta_2(s) \\ &= \frac{1}{\Gamma(s_1)\Gamma(s_2)} \int_0^\infty \int_0^\infty t_1^{s_1-1} t_2^{s_2-1} h(t_1, t_2)\, dt_1 dt_2 \end{aligned} \tag{4.2}$$

where $h(t_1, t_2) := h_1(t_1)h_2(t_2)$.

This raises the question:

Question 4.1.1. For which functions $h(t_1, t_2)$ does the integral (4.2) define a meromorphic function $\zeta(s_1, s_2)$, and in which interesting geometric situations do such $h(t_1, t_2)$ arise?

In this chapter, we will provide a partial answer to this question. In particular, we will observe that the product case suggests a condition on what we will call the "multi-asymptotics" of $h(t_1, t_2)$ that is sufficient to define a meromorphic function $\zeta(s_1, s_2)$; we discuss this condition in Section 4.2.

4.1.2 Product manifolds

We now describe a simple geometric setting in which the zeta functions of the previous section might arise. Consider two compact manifolds M_1 and M_2 of dimensions n_1 and n_2, respectively. For $j = 1, 2$, denote by π_j the projection $M_1 \times M_2 \to M_j$. Let $E_1 \to M_1$ and $E_2 \to M_2$ be vector bundles endowed with Laplace-type operators Δ_1 and Δ_2, respectively. Assume that each Δ_j has trivial kernel. Consider the product vector bundle $E := \pi_1^* E_1 \otimes \pi_2^* E_2 \to M_1 \times M_2$. Δ_1 and Δ_2 induce commuting sub-elliptic operators, which we will also denote by Δ_1 and Δ_2, on sections of E. Define $\Delta(t_1, t_2)$, an elliptic operator depending on two positive parameters t_1 and t_2, by

$$\Delta(t_1, t_2) := t_1 \Delta_1 + t_2 \Delta_2 \tag{4.3}$$

To match the notation of Section 4.1.1, for $j = 1, 2$, let $h_j(t_j) := \operatorname{Tr} e^{-t_j \Delta_j}$, and let $\zeta_j(s_j)$ be the associated zeta function. Let $h(t_1, t_2) := \operatorname{Tr} e^{-\Delta(t_1, t_2)}$. Then

$$h(t_1, t_2) = h_1(t_1) h_2(t_2). \tag{4.4}$$

The "multi-zeta function" $\zeta(s_1, s_2) := \zeta_1(s_1)\zeta_2(s_2)$ may be interpreted as follows:

$$\zeta(s_1, s_2) = \frac{1}{\Gamma(s_1)\Gamma(s_2)} \int_0^\infty \int_0^\infty t_1^{s_1 - 1} t_2^{s_2 - 1} \, h(t_1, t_2) \, dt_1 dt_2. \tag{4.5}$$

4.1.3 Analytic torsion

We will now make a simple observation about analytic torsion in this context of product manifolds. Suppose that $F_1 \to M_1$ and $F_2 \to M_2$ are flat, acyclic, unitary vector bundles. To match the notation of the previous subsection, for $j = 1, 2$, let $E_j = F_j \otimes \Lambda^\bullet T^* M_j$. A choice of Riemannian metrics on M_1 and M_2 induces Laplace operators Δ_1 and Δ_2 on sections of E_1 and E_2, respectively. In this context, the analytic torsions $T_j := T(M_j, F_j)$ are defined.

We may consider the product vector bundle $F = \pi_1^* F_1 \otimes \pi_2^* F_2 \to M_1 \times M_2$, which is flat. Let $E = F \otimes \Lambda^\bullet T^*(M_1 \times M_2)$; then E is canonically isomorphic to $\pi_1^* E_1 \otimes \pi_2^* E_2$. The grading (by degree of forms) on each E_j induces a bigrading on E in the following sense: $E = \bigoplus_{q_1,q_2} E^{q_1,q_2}$. where $E^{q_1,q_2} := \pi_1^* E_1^{q_1} \otimes \pi_2^* E_2^{q_2}$ and $E_j^{q_j} := F_j \otimes \Lambda^{q_j} T^* M_j$ denotes the bundle of F_j-valued q_j-forms on M_j. E also possesses a grading by total degree of forms: let $E^q = F \times \Lambda^q T^*(M_1 \times M_2) \cong \bigoplus_{q_1+q_2=q} E^{q_1,q_2}$. As usual, let $(-1)^Q$ be multiplication by $(-1)^q$ on E^q. For $j = 1, 2$, let Q_j and $(-1)^{Q_j}$ be multiplication by q_j and $(-1)^{q_j}$, respectively, on E^{q_1,q_2}. Note that $(-1)^Q = (-1)^{Q_1}(-1)^{Q_2}$.

We have

$$(\mathrm{Tr}(-1)^{Q_1} Q_1 e^{-t_1\Delta_1})(\mathrm{Tr}(-1)^{Q_2} Q_2 e^{-t_2\Delta_2}) = \mathrm{Tr}(-1)^Q Q_1 Q_2 e^{-\Delta(t_1,t_2)}.$$

Let $\zeta_T(s_1, s_2)$ be defined for $\mathrm{Re}\, s_1$ and $\mathrm{Re}\, s_2$ large by

$$\zeta_T(s_1, s_2) = \frac{1}{\Gamma(s_1)\Gamma(s_2)} \int_0^\infty \int_0^\infty t_1^{s_1} t_2^{s_2}\ \mathrm{Tr}(-1)^Q Q_1 Q_2 e^{-\Delta(t_1,t_2)}\, \frac{dt_1}{t_1}\frac{dt_2}{t_2}.$$

Then the observation (4.1) shows that

$$\left.\frac{\partial^2}{\partial s_1 \partial s_2}\right|_{(s_1,s_2)=(0,0)} \zeta_T(s_1, s_2) = (\log T_1)(\log T_2), \tag{4.6}$$

the product of the logarithms of the respective analytic torsions of the factors F_1 and F_2. If n_1 and n_2 are both odd, then this expression is independent of the metrics on F_1 and F_2. (Recall that we assumed both F_1 and F_2 to be acyclic.)

The above considerations motivate a generalization to geometric settings more general than product manifolds. We will consider one such setting in which a multi-zeta function is defined: a quotient of a product manifold by a finite group. The essential feature is a local product structure inducing a Laplacian $\Delta(t_1, t_2)$ depending on two parameters, as in (4.3). In the next section, we will discuss the relevant properties of $h(t_1, t_2) = \operatorname{Tr} e^{-\Delta(t_1,t_2)}$.

4.2 Heat multi-asymptotics and multi-zeta functions

We retain the notation of the previous section, except that we will use α to index the two factors to avoid confusion in the notation. For $\alpha = 1, 2$, we have an admissible function h_α, each admitting an asymptotic expansion for small t_α.

$$h_\alpha(t_\alpha) \sim \sum_{j \geqslant 0} a^\alpha_{p^\alpha_j} (t_\alpha)^{p^\alpha_j}, \tag{4.7}$$

Each h_α also satisfies the estimate, for $t_\alpha \geqslant 1$,

$$|h_\alpha(t_\alpha)| \leqslant e^{-\epsilon_\alpha t_\alpha}. \tag{4.8}$$

"Multiplication" of these conditions implies that the product $h(t_1, t_2) = h_1(t_1)h_2(t_2)$ has certain "multi-asymptotic" properties, which we now discuss.

Since there are now two variables, there are now four cases to consider. We will label the four conditions on $h(t_1, t_2)$ as (MA1)-(MA4):

(MA1) *(Both t_1 and t_2 small.)* For every real number K, we may write

$$\begin{aligned} h(t_1, t_2) = &\sum_{j_1=0}^{J^1_K} \sum_{j_2=0}^{J^2_K} (t_1)^{p^1_{j_1}} (t_2)^{p^2_{j_2}} a_{p^1_{j_1}, p^2_{j_2}} \\ &+ \sum_{i_1=0}^{I^1_K} (t_1)^{p^1_{i_1}} b^1_{p^1_{i_1}} (t_2) + \sum_{i_2=0}^{I^2_K} (t_2)^{p^2_{i_2}} b^2_{p^2_{i_2}} (t_1) \\ &+ r_K(t_1, t_2), \end{aligned}$$

where for $0 < t_1, t_2 < 1$, $|r_K(t_1,t_2)| \leqslant C_K t_1^K t_2^K$, $|b^1_{p^1_{i_1}}(t_2)| \leqslant C_{K,i_1}(t_2)^K$, and $|b^2_{p^2_{i_2}}(t_1)| \leqslant C_{K,i_2}(t_1)^K$.

(MA2) *(t_1 small, t_2 large.)* For every real number K we may write

$$h(t_1,t_2) = \sum_{i_1=0}^{I^1_K} (t_1)^{p^1_{i_1}}\, c^1_{p^1_{i_1}}(t_2) + s^1_K(t_1,t_2)$$

where for $0 < t_1 < 1$ and $t_2 \geqslant 1$, $|c^1_{p^1_{i_1}}(t_2)| \leqslant e^{-\epsilon_2 t_2}$ and $|s^1_K(t_1,t_2)| \leqslant C^1_K(t_1)^K e^{-\epsilon_2 t_2}$.

(MA3) *(t_1 small, t_2 large.)* (This is (MA2) with the roles of t_1 and t_2 reversed.) For every real number K we may write

$$h(t_1,t_2) = \sum_{i_2=0}^{I^2_K} (t_2)^{p^2_{i_2}}\, c^2_{p^2_{i_2}}(t_1) + s^2_K(t_1,t_2)$$

where for $0 < t_2 < 1$ and $t_1 \geqslant 1$, $|c^2_{p^2_{i_2}}(t_1)| \leqslant e^{-\epsilon_1 t_1}$ and $|s^2_K(t_1,t_2)| \leqslant C^2_K(t_2)^K e^{-\epsilon_1 t_1}$.

(MA4) *(Both t_1 and t_2 large.)* If $t_1, t_2 \geqslant 1$,

$$|h(t_1,t_2)| \leqslant C e^{-\epsilon_1 t_1} e^{-\epsilon_2 t_2}$$

Definition 4.2.1. We will say that a smooth $\mathbb{C}$-valued function $h(t_1,t_2)$ is "multi-admissible" when it satisfies conditions (MA1)-(MA4) above.

The utility of this notion of multi-admissible is that it is sufficient to define an associated multi-zeta function.

Definition 4.2.2. We define the multi-zeta function associated to $h(t_1,t_2)$ by the following, for $\operatorname{Re} s_1$ and $\operatorname{Re} s_2$ large:

$$\zeta(s_1,s_2) := \frac{1}{\Gamma(s_1)\Gamma(s_2)} \int_0^\infty \int_0^\infty t_1^{s_1} t_2^{s_2}\, h(t_1,t_2)\, \frac{dt_1}{t_1}\frac{dt_2}{t_2}. \tag{4.9}$$

Lemma 4.2.3. *Suppose $h(t_1, t_2)$ is multi-admissible. Then the integral defining $\zeta(s_1, s_2)$ in (4.9) converges for $\operatorname{Re} s_1 > -p_0^1$ and $\operatorname{Re} s_2 > -p_0^2$ and defines a holomorphic function there. $\zeta(s_1, s_2)$ admits a unique meromorphic extension to all of $\mathbb{C}^2$; the extension, which we also denote by $\zeta(s_1, s_2)$, is holomorphic at the origin, where its value is $\zeta(0,0) = a_{0,0}$. Furthermore, we have the vanishing results:*

1. *If no $(t_1)^0$ terms appear in (MA1) or in (MA2), then $\frac{\partial \zeta}{\partial s_2}(0,0) = 0$.*

2. *If no $(t_2)^0$ terms appear in (MA1) or in (MA3), then $\frac{\partial \zeta}{\partial s_1}(0,0) = 0$.*

Proof. Let $f(s_1, s_2)$ be the integral in (4.9):

$$f(s_1, s_2) := \int_0^\infty \int_0^\infty t_1^{s_1} t_2^{s_2}\, h(t_1, t_2)\, \frac{dt_1}{t_1} \frac{dt_2}{t_2}.$$

Then

$$\zeta_\alpha(s_1, s_2) = \frac{1}{\Gamma(s_1)\Gamma(s_2)} f(s_1, s_2). \tag{4.10}$$

The gamma function Γ is meromorphic and nonvanishing, so $\frac{1}{\Gamma(s_1)\Gamma(s_2)}$ is holomorphic everywhere. Since $\Gamma(z) = \frac{1}{z} + O(1)$ as $z \to 0$,

$$\frac{1}{\Gamma(s_1)\Gamma(s_2)} = s_1 s_2 + O(|(s_1, s_2)|^3) \text{ as } (s_1, s_2) \to (0,0). \tag{4.11}$$

Thus it suffices to show that $f(s_1, s_2)$ admits a meromorphic extension and to study its behavior near $(0,0)$.

We write the domain of the integral in (4.9) as the union of the sets

1. where $0 < t_1, t_2 < 1$,

2. where $0 < t_1 < 1$ and $t_2 \geqslant 1$,

3. where $0 < t_2 < 1$ and $t_1 \geqslant 1$, and

4. where $t_1, t_2 \geqslant 1$.

The respective conditions (MA1)-(MA4) ensure that the integral over each of these four sets converges absolutely if $\operatorname{Re} s_j > \frac{n_j}{2}$ for $j = 1, 2$. Computing the integrals give that for any real number K we have

$$
\begin{aligned}
f(s_1, s_2) = & \sum_{j_1=0}^{J_K^1} \sum_{j_2=0}^{J_K^2} \frac{1}{(s_1 - p_{j_1}^1)(s_2 - p_{j_2}^2)} a_{p_{j_1}^1, p_{j_2}^2} \\
& + \sum_{i_1=0}^{I_K^1} \frac{1}{s_1 - p_{i_1}^1} \tilde{b}_{p_{i_1}^1}^1(s_2) \; + \; \sum_{i_2=0}^{I_K^2} \frac{1}{s_2 - p_{i_2}^2} \tilde{b}_{p_{i_2}^2}^2(s_1) \\
& + \tilde{r}_K(s_1, s_2) \\
& + \sum_{i_1=0}^{I_K^1} \frac{1}{s_1 - p_{i_1}^1} \tilde{c}_{p_{i_1}^1}^1(s_2) \; + \; \tilde{s}_K^1(s_1, s_2) \\
& + \sum_{i_2=0}^{I_K^2} \frac{1}{s_2 - p_{i_2}^2} \tilde{c}_{p_{i_2}^2}^2(s_2) \; + \; \tilde{s}_K^2(s_1, s_2) \\
& + \tilde{h}(s_1, s_2)
\end{aligned}
$$

where the new functions of s_1, s_2 (marked by tildes), each of which is holomorphic for $\operatorname{Re} s_1, \operatorname{Re} s_2 > -K$, are defined in terms of the functions in (MA1)-(MA4) by

$$\tilde{b}^1_{p^1_{i_1}}(s_2) := \int_0^1 (t_2)^{s_2}\, b_{p^1_{i_1}}(t_2)\, \frac{dt_2}{t_2}$$

$$\tilde{b}^2_{p^2_{i_2}}(s_1) := \int_0^1 (t_1)^{s_1}\, b_{p^2_{i_2}}(t_1)\, \frac{dt_1}{t_1}$$

$$\tilde{r}_K(s_1, s_2) := \int_0^1 \int_0^1 (t_1)^{s_1} (t_2)^{s_2}\, r_K(t_1, t_2)\, \frac{dt_1}{t_1} \frac{dt_2}{t_2}$$

$$\tilde{c}^1_{p^1_{i_1}}(s_2) := \int_1^\infty t_2^{s_2}\, c_{p^1_{i_1}}(t_2)\, \frac{dt_2}{t_2}$$

$$\tilde{s}^1_K(s_1, s_2) := \int_1^\infty \int_0^1 (t_1)^{s_1} (t_2)^{s_2}\, s^1_K(t_1, t_2)\, \frac{dt_1}{t_1} \frac{dt_2}{t_2}$$

$$\tilde{c}^2_{p^2_{i_2}}(s_1) := \int_1^\infty t_1^{s_1}\, c_{p^2_{i_2}}(t_1)\, \frac{dt_1}{t_1}$$

$$\tilde{s}^2_K(s_1, s_2) := \int_0^1 \int_1^\infty (t_1)^{s_1} (t_2)^{s_2}\, s^2_K(t_1, t_2)\, \frac{dt_1}{t_1} \frac{dt_2}{t_2}$$

$$\tilde{h}(s_1, s_2) := \int_0^1 \int_1^\infty (t_1)^{s_1} (t_2)^{s_2}\, h(t_1, t_2)\, \frac{dt_1}{t_1} \frac{dt_2}{t_2}.$$

This uniquely defines the meromorphic extension of f to the set where $\operatorname{Re} s_1, \operatorname{Re} s_2 > -K$. Since K is arbitrary, this proves that f has a unique meromorphic extension to all of $\mathbb{C}^2$. We see that as $(s_1, s_2) \to (0, 0)$,

$$f(s_1, s_2) = a_{0,0} \frac{1}{s_1 s_2} + \left(\tilde{b}^1_0(0) + \tilde{c}^1_0(0) \right) \frac{1}{s_1} + \left(\tilde{b}^2_0(0) + \tilde{c}^2_0(0) \right) \frac{1}{s_2} + O\left(1\right).$$

Using (4.10) and (4.11), this shows

$$\zeta(0, 0) = a_{0,0}$$

$$\frac{\partial \zeta}{\partial s_2}(0, 0) = \tilde{b}^1_0(0) + \tilde{c}^1_0(0)$$

$$\frac{\partial \zeta}{\partial s_1}(0, 0) = \tilde{b}^2_0(0) + \tilde{c}^2_0(0).$$

The final vanishing assertions of the theorem follow. □

Remark 4.2.4. There is a natural generalization to functions $h(t_1, \ldots, t_m)$ of m variables, for which one may attempt to define a multi-zeta function $\zeta(s_1, \ldots, s_m)$ via an Mellin-type transform involving an integral over $(0, \infty)^m \subset \mathbb{R}^m$. One should then replace the four conditions (MA1)-(MA4) with 2^m conditions, corresponding to whether each of the m variables $t_1, \ldots, t_m$ is greater than or less than one. We will not pursue this further.

Remark 4.2.5. The multi-zeta functions we consider here bear some formal resemblance to the zeta functions of Shintani [22], but we do not know if there is any deeper connection.

4.3 The geometry of finite quotients of product manifolds

We will now generalize the setting of Subsection 4.1.3, in which we considered a product of acyclic, unitary, flat bundles $F = \pi_1^* F_1 \otimes \pi_2^* F_2 \to M_1 \times M_2$. We will now consider quotients by certain actions of finite groups.

Definition 4.3.1. Let $V \to N$ be a vector bundle. We will say ϕ is a V-diffeomorphism when $\phi : N \to N$ is a diffeomorphism that lifts to a diffeomorphism (also called ϕ) $\phi : V \to V$ on the total space that acts as a linear isomorphism on fibers, i.e., for each fiber V_x, $\phi_x := \phi|_{V_x} : V_x \to V_{\phi(x)}$ is a linear isomorphism. If V is endowed with a metric and $\phi|_{V_x} : V_x \to V_{\phi(x)}$ is in addition an orthogonal map with respect to that metric, then we will say ϕ is a V-isometry.
Let $\mathrm{Diff}(V \to N)$ denote the group of V-diffeomorphisms, and let $\mathrm{Isom}(V \to N)$ denote the group of V-isometries.

A V-diffeomorphism ϕ induces a pullback operator $\phi^* : C^\infty(N, V) \to C^\infty(N, V)$ defined by $\phi^*(\sigma)(x) = (\phi_x)^{-1}\sigma(\phi(x))$. This extends also to a pullback operator $\phi^* : C^\infty(N, \Lambda^\bullet V) \to C^\infty(N, \Lambda^\bullet V)$.

Let Γ be a finite subgroup of $\mathrm{Isom}(F_1 \to M_1) \times \mathrm{Isom}(F_2 \to M_2)$, which we view as a subgroup of $\mathrm{Isom}(F \to M_1 \times M_2)$. For $j = 1, 2$, let Γ_j denote the projection of Γ onto $\mathrm{Isom}(F_j \to M_j)$. We remark that each Γ_j is a subgroup of $\mathrm{Isom}(F_j \to M_j)$, but Γ need not be the product group $\Gamma_1 \times \Gamma_2$.

We will assume Γ, viewed as a group of diffeomorphisms of $M_1 \times M_2$, acts properly discontinuously in the sense that for all $\tilde{x} \in M_1 \times M_2$, there exists an open neighborhood $U \subset M_1 \times M_2$ containing $\tilde{x}$ such that $\gamma(U) \cap U = \varnothing$ for all $\gamma \in \Gamma \backslash \mathrm{Id}$.

We will now discuss the possibility of fixed points. The properly discontinuous assumption implies that for $\gamma = (\gamma_1, \gamma_2) \in \Gamma \backslash \{I\}$, if $\gamma_1 \in \mathrm{Diff}(M_1)$ has a fixed point, then $\gamma_2 \in \mathrm{Diff}(M_2)$ must not have a fixed point (and vice versa). But we will allow exactly zero or one of γ_1 and γ_2 to have fixed points.

Recall the notion of a nondegenerate fixed point:

Definition 4.3.2. Let N be a manifold and let $\phi : N \to N$ be a diffeomorphism. A fixed point $x \in N$ of ϕ is said to be nondegenerate when $I - d\phi_x : T_x N \to T_x N$ is an isomorphism.

Theorem 4.4.8 involves the assumption that for $j = 1, 2$, every $\gamma_j \in \Gamma_j \backslash \{I\}$ has nondegenerate fixed points when viewed as a diffeomorphism of M_j, but we need not assume this in general.

Let M_Γ be the quotient $M_\Gamma := (M_1 \times M_2)/\Gamma$. The properly discontinuous assumption implies that M_Γ is a smooth manifold of dimension $n := n_1 + n_2$. The isometric action of Γ on F induces a quotient vector bundle $F_\Gamma \to M_\Gamma$ with an induced metric. Furthermore, for every $x \in M_\Gamma$, there is an open neighborhood $U \subset M_\Gamma$ containing x such that U is diffeomorphic to a product $\tilde{U}_1 \times \tilde{U}_2$ for some $\tilde{U}_1$ and $\tilde{U}_2$, open sets in M_1 and M_2, respectively; the restriction of F_Γ to U is isometric to the product of the restrictions $\pi^*_{\tilde{U}_1} F_1|_{\tilde{U}_1} \otimes \pi^*_{\tilde{U}_2} F_2|_{\tilde{U}_2}$.

This local product structure gives a bigrading of F_Γ-valued forms, as in Subsection

4.1.3. We will mimic the notation of that subsection: let $E_\Gamma := F_\Gamma \otimes \Lambda^\bullet T^* M_\Gamma \to M_\Gamma$. Then $E_\Gamma = \bigoplus_{q_1,q_2} E_\Gamma^{q_1,q_2}$, where $E_\Gamma^{q_1,q_2}$ is locally isometric to $(\pi_1^* F_1 \otimes \Lambda^{q_1} T^* M_1) \otimes (\pi_2^* F_2 \otimes \Lambda^{q_2} T^* M_2)$. Then we have the operators Q_1, Q_2, $(-1)^{Q_1}$, $(-1)^{Q_2}$, and $(-1)^Q$, defined as in Subsection 4.1.3.

Let d be the exterior derivative on sections of E_Γ. The local product structure ensures that there is a global decomposition of d: $d = d_1 + d_2$, where d_1 maps (q_1, q_2)-forms to $(q_1 + 1, q_2)$-forms, and d_2 maps (q_1, q_2)-forms to $(q_1, q_2 + 1)$-forms. d_1 and d_2 commute.

We would also like a global decomposition of d^* and Δ. To do so, we need to consider metrics that induce a local *geometric* product structure on M_Γ. We will now discuss the relevant space of metrics.

Let $\mathcal{M}_{\Gamma_j}$ denote the set of Γ_j-invariant Riemannian metrics on M_j. A choice of metrics $h_1 \in \mathcal{M}_{\Gamma_1}$ and $h_2 \in \mathcal{M}_{\Gamma_1}$ induces a product Riemannian metric $h_1 \times h_2$ on $M_1 \times M_2$. Let $\mathcal{M}_{\text{prod}}$ denote the set of metrics arising in this way; $\mathcal{M}_{\text{prod}}$ is identifiable with $\mathcal{M}_{\Gamma_1} \times \mathcal{M}_{\Gamma_2}$.

Consider a metric $h = h_1 \times h_2 \in \mathcal{M}_{\text{prod}}$. h is Γ-invariant, and therefore descends to a well-defined metric, which we will also call h, on the quotient M_Γ. Furthermore, h induces a local geometric product structure in the following sense: the identification $U \cong \tilde{U}_1 \times \tilde{U}_2$ from above is a Riemannian isometry, where $\tilde{U}_1 \times \tilde{U}_2 \subset M_1 \times M_2$ is given the product metric induced by h_1 and h_2. In addition to the decomposition $d = d_1 + d_2$, under these hypotheses we have that $d^* = d_1^* + d_2^*$ and

$$\Delta = \Delta_1 + \Delta_2, \text{ where} \tag{4.12}$$

$$\Delta_j = d_j d_j^* + d_j^* d_j \text{ for } j = 1, 2.$$

Our assumptions apply the vanishing of the following (anti)commutators:

$$\{d_1, d_2\} = 0 \tag{4.13}$$

$$\{d_1, d_2^*\} = 0 \tag{4.14}$$

$$\{d_2, d_1^*\} = 0 \tag{4.15}$$

$$[d_i, \Delta_j] = 0 \tag{4.16}$$

$$[d_i^*, \Delta_j] = 0 \tag{4.17}$$

$$[\Delta_1, \Delta_2] = 0 \tag{4.18}$$

where (4.16) and (4.17) hold for $i, j = 1, 2$.

Remark 4.3.3. The conditions outlined above are essentially what is required to define the closed form ω_{MT}, which we will introduce in Section 4.6.

4.4 Multi-zeta functions and multi-torsion

The decomposition in (4.12) of the Laplacian on F_Γ-valued forms on M_Γ suggests the consideration of the following generalized Laplacian:

Definition 4.4.1. For $t_1, t_2 > 0$, let $\Delta(t_1, t_2) := t_1\Delta_1 + t_2\Delta_2$.

$\Delta(t_1, t_2)$ is elliptic and nonnegative (in fact, strictly positive by our acyclicity assumption), so its associated heat operator $e^{-\Delta(t_1,t_2)}$ is defined. In Section 4.6, we will interpret $\Delta(t_1, t_2)$ as the Laplacian associated to a scaled metric.

Recall the definition of multi-admissible from Definition 4.2.1. We will show that the heat trace $\operatorname{Tr} e^{-\Delta(t_1,t_2)}$ is multi-admissible. More generally, to study the dependence of the multi-torsion on the metric, we will also need to consider certain auxiliary operators of the following form: For $j = 1, 2$, let $\alpha_j \in C^\infty(M_j, \operatorname{End}\Lambda^\bullet F_j)$, and suppose that α_j commutes with the action of Γ_j. Then α_1 and α_2 induce an operator $\alpha = \alpha_1 \otimes \alpha_2 \in C^\infty(M_1 \times M_2, \Lambda^\bullet F)$ that commutes with the action of Γ, and α descends to an operator $\alpha_\Gamma \in C^\infty(M_\Gamma, \operatorname{End}\Lambda^\bullet F_\Gamma)$.

We have the following fundamental result:

Theorem 4.4.2. *Let* α_Γ *be as above. Then* $\operatorname{Tr} \alpha_\Gamma e^{-\Delta(t_1,t_2)}$ *is multi-admissible.*

We will prove this theorem in Section 4.5, where we will also elaborate on the details of the multi-asymptotics of $\operatorname{Tr} \alpha_\Gamma e^{-\Delta(t_1,t_2)}$.

This allows us to define the associated zeta function using (4.9), i.e.:

Definition 4.4.3. Let $\alpha_\Gamma \in C^\infty(M_\Gamma, \operatorname{End} \Lambda^\bullet F_\Gamma)$ be as above. For $\operatorname{Re} s_1 > \frac{n_1}{2}$ and $\operatorname{Re} s_2 > \frac{n_2}{2}$, define $\zeta(s_1, s_2; \Delta(t_1,t_2), \alpha_\Gamma)$ by the following:

$$\begin{aligned}&\zeta(s_1, s_2; \Delta(t_1,t_2), \alpha_\Gamma)\\ &:= \frac{1}{\Gamma(s_1)\Gamma(s_2)} \int_0^\infty \int_0^\infty t_1^{s_1} t_2^{s_2} \operatorname{Tr} \alpha_\Gamma e^{-\Delta(t_1,t_2)} \frac{dt_1}{t_1} \frac{dt_2}{t_2}. \end{aligned} \tag{4.19}$$

We immediately obtain:

Corollary 4.4.4. *Suppose both factors are acyclic. Then the integral defining* $\zeta(s_1, s_2; \Delta(t_1,t_2), \alpha)$ *in* (4.19) *converges on the set of points* $(s_1, s_2) \in \mathbb{C}^2$ *such that* $\operatorname{Re} s_1 > \frac{n_1}{2}$ *and* $\operatorname{Re} s_2 > \frac{n_2}{2}$ *and defines a holomorphic function there. Furthermore,* $\zeta(s_1, s_2; \Delta(t_1,t_2), \alpha)$ *admits a unique meromorphic extension to all of* $\mathbb{C}^2$*; the extension is holomorphic at the origin.*

Proof. This follows from Theorem 4.4.2 and Lemma 4.2.3. □

Remark 4.4.5. Recall that if $\ker \Delta$ is nontrivial, the Definition 3.1.7 of the usual zeta function via the Mellin transform requires the use of $\operatorname{Tr}' e^{-t\Delta}$ (as opposed to $\operatorname{Tr} e^{-t\Delta}$) to ensure decay as $t \to \infty$. In defining the multi-zeta function, we have assumed acyclicity in both factors, which means that we need not worry about projecting away from $\ker \Delta(t_1,t_2)$. Moreover, this "double acyclicity" assumption seems quite essential without extensive modification, as it is essential to the decay of $\operatorname{Tr} e^{-\Delta(t_1,t_2)}$ in the limits $t_1 \to \infty$ (for fixed t_2) and $t_2 \to \infty$ (for fixed t_1). (This decay is encoded in the multi-admissible conditions (MA2)-(MA4), and it follows essentially from the triviality of the kernels of *both* Δ_1 and Δ_2.) This decay, in turn, is essential to the integrability of $\operatorname{Tr} e^{-\Delta(t_1,t_2)}$ in the integral in (4.19).

Definition 4.4.6. For $(h_1, h_2) \in \mathcal{M}_{\text{prod}}$, we define multi-torsion $MT = MT(M_\Gamma, F_\Gamma, h_1, h_2)$ by

$$MT(M_\Gamma, F_\Gamma, h_1, h_2) = \left.\frac{\partial^2}{\partial s_1 \partial s_2}\right|_{(s_1,s_2)=(0,0)} \zeta(s_1, s_2; \Delta(t_1,t_2), (-1)^Q Q_1 Q_2)$$

If we wish to emphasize only the dependence of the multi-torsion on the metrics h_1 and h_2, we will denote it by $MT(h_1, h_2)$.

We see that the multi-zeta function $\zeta(s_1, s_2; \Delta(t_1,t_2), (-1)^Q Q_1 Q_2)$ appearing in Definition 4.4.6 may be written as (for $\operatorname{Re} s_1 > \frac{n_1}{2}$ and $\operatorname{Re} s_2 > \frac{n_2}{2}$)

$$\begin{aligned}
&\zeta(s_1, s_2; \Delta(t_1,t_2), (-1)^Q Q_1 Q_2) \\
&= \frac{1}{\Gamma(s_1)\Gamma(s_2)} \int_0^\infty \int_0^\infty t_1^{s_1} t_2^{s_2} \operatorname{Tr}(-1)^Q Q_1 Q_2 e^{-\Delta(t_1,t_2)} \frac{dt_1}{t_1}\frac{dt_2}{t_2} \\
&= \frac{1}{\Gamma(s_1)\Gamma(s_2)} \int_0^\infty \int_0^\infty t_1^{s_1} t_2^{s_2} \sum_{q_1,q_2} (-1)^{q_1+q_2} q_1 q_2 \operatorname{Tr} e^{-(t_1\Delta_1^{q_1} + t_2\Delta_2^{q_2})} \frac{dt_1}{t_1}\frac{dt_2}{t_2},
\end{aligned} \tag{4.20}$$

where $t_1\Delta_1^{q_1} + t_2\Delta_2^{q_2}$ denotes the restriction of $t_1\Delta_1 + t_2\Delta_2$ to (q_1, q_2)-forms.

(4.6) shows that in the case when Γ is the trivial group, i.e., when M_Γ is simply the product $M_1 \times M_2$, MT is the product of the logarithms of the analytic torsions of the factors. In that case, if n_1 and n_2 are both odd, then MT is trivially independent of the metric by Corollary 3.3.4.

We will now study the dependence of MT on the metric in the general case. First we have the following vanishing result:

Proposition 4.4.7. *For $j = 1, 2$, if n_j is even and Γ_j acts by orientation-preserving diffeomorphisms, then $MT(h_1, h_2) = 0$ for any $(h_1, h_2) \in \mathcal{M}_{prod}$.*

Proof. Without loss of generality, we may assume $j = 1$ for the sake of clarity. The Hodge star on M_1 induces an operator $*_1$ on sections of $F \otimes \Lambda^\bullet T^*(M_1 \times M_2)$. Since Γ_1 acts by orientation-preserving isometries, $*_1$ descends to a well-defined invertible operator on the quotient, which we also denote by $*_1$. $*_1$ maps (q_1, q_2)-forms to $(n_1 - q_1, q_2)$-forms. Thus we have the commutation relations

$$\begin{aligned} *_1 Q_1 &= (n_1 I - Q_1) *_1 \\ *_1 Q_2 &= Q_2 *_1 \\ *_1 (-1)^Q &= (-1)^{n_1} (-1)^Q *_1 \\ *_1 \Delta(t_1, t_2) &= \Delta(t_1, t_2) *_1 \end{aligned}$$

The last implies that $*_1$ commutes also with $e^{-\Delta(t_1,t_2)}$. We compute the following, using the cyclicity of the trace:

$$\begin{aligned} \operatorname{Tr}(-1)^Q Q_1 Q_2 e^{-\Delta(t_1,t_2)} &= \operatorname{Tr} *_1^{-1} (-1)^Q Q_1 Q_2 e^{-\Delta(t_1,t_2)} *_1 \\ &= (-1)^{n_1} \operatorname{Tr}(-1)^Q (n_1 I - Q_1) Q_2 e^{-\Delta(t_1,t_2)}. \end{aligned}$$

In the case when n_1 is even, this implies

$$\operatorname{Tr}(-1)^Q Q_1 Q_2 e^{-\Delta(t_1,t_2)} = \frac{n_1}{2} \operatorname{Tr}(-1)^Q Q_2 e^{-\Delta(t_1,t_2)},$$

which we claim vanishes. This vanishing shows that the zeta function of (4.20) vanishes identically, which proves the proposition.

To prove the claim that $\operatorname{Tr}(-1)^Q Q_2 e^{-\Delta(t_1,t_2)} = 0$, we will use what is essentially a standard argument that can be used to prove the McKean-Singer formula [12] relating the index to the heat equation. We start by differentiating with respect to t_1. Since $\frac{\partial}{\partial t_1} \Delta(t_1, t_2) = \Delta_1$, which commutes with Δ and therefore with the resolvent $(\Delta - z)^{-1}$, we obtain

$$\begin{aligned} \frac{\partial}{\partial t_1} \operatorname{Tr}(-1)^Q Q_2 e^{-\Delta(t_1,t_2)} &= -\operatorname{Tr}(-1)^Q Q_2 e^{-\Delta(t_1,t_2)} \Delta_1. \\ &= -\operatorname{Tr}(-1)^Q Q_2 e^{-\Delta(t_1,t_2)} d_1 d_1^* - \operatorname{Tr}(-1)^Q Q_2 e^{-\Delta(t_1,t_2)} d_1^* d_1. \end{aligned}$$

But this is zero for the following reason: the cyclicity of the trace and the fact that d_1 commutes with Q_2 and $e^{-\Delta(t_1,t_2)}$ and anticommutes with $(-1)^Q$ imply that

$$\operatorname{Tr}(-1)^Q Q_2 e^{-\Delta(t_1,t_2)} d_1 d_1^* = -\operatorname{Tr}(-1)^Q Q_2 e^{-\Delta(t_1,t_2)} d_1^* d_1.$$

We have shown that for each $t_2 > 0$, $\operatorname{Tr}(-1)^Q Q_2 e^{-\Delta(t_1,t_2)}$ is constant in t_1. But Theorem 4.4.2 gives that $\operatorname{Tr}(-1)^Q Q_2 e^{-\Delta(t_1,t_2)} \to 0$ as $t_1 \to \infty$, so in fact $\operatorname{Tr}(-1)^Q Q_2 e^{-\Delta(t_1,t_2)}$ vanishes for all $t_1, t_2 > 0$. □

Thus the interesting case (at least under the orientation-preserving assumption) is when both dimensions n_1 and n_2 are odd. The metric independence of analytic torsion, Corollary 3.3.4, generalizes to the following, which is the main result of this chapter.

Theorem 4.4.8. *For $j = 1, 2$, if n_j is odd and if for every $\gamma_j \in \Gamma_j$, γ_j is either orientation-preserving or has nondegenerate fixed points as a diffeomorphism of M_j, then $MT(h_1, h_2)$ is independent of the metric h_j.*

We will give the proof in Section 4.8. The proof involves a Stokes' theorem argument and relies crucially on two things:

- Our interpretation of MT as an integral of a certain closed two-form ω_{MT} over a surface in the space of metrics, which we discuss in Section 4.6; and
- The results of Subsection 4.5.5, in which we explain the relevance of the hypotheses of Theorem 4.4.8 to the vanishing of constant terms in certain multi-asymptotic expansions. These constant terms arise essentially as boundary terms in the Stokes' theorem argument.

4.5 Proof of heat kernel multi-asymptotics

In this section we will study the multi-asymptotics of the heat kernel on M_Γ. Our primary goal will be to prove Theorem 4.4.2. In the last part of this section (Subsection 4.5.5), we will also make some observations about constant terms in certain multi-asymptotic expansions that are essential to the metric independence theorem, Theorem 4.4.8.

A note on notation: We will use letters such as x and y to denote points in M_Γ. Tildes will denote points in $M_1 \times M_2$, e.g., if $x, y \in M$, then $\tilde{x}, \tilde{y} \in M_1 \times M_2$ will denote points lying over x, y, respectively. Subscripts will denote projections onto the factors M_1 and M_2, i.e., if $\tilde{x} \in M_1 \times M_2$, then $\tilde{x}_1 = \pi_1(\tilde{x})$ and $\tilde{x}_2 = \pi_2(\tilde{x})$.

4.5.1 Heat kernel on global product

First we must understand the heat kernel on $M_1 \times M_2$. For $j = 1, 2$, let $k^{M_j}(t; \tilde{x}_j, \tilde{y}_j) \in \mathrm{Hom}((\Lambda^\bullet F_j)_{\tilde{y}_j}, (\Lambda^\bullet F_j)_{\tilde{x}_j})$ denote the heat kernel on F_j-valued forms on M_j. Recall that F denotes the flat product bundle $\pi_1^* F_1 \otimes \pi_2^* F_2 \to M_1 \times M_2$. The heat kernel $k^{M_1 \times M_2}(t_1, t_2; \tilde{x}, \tilde{y})$ on F-valued forms is the tensor product of the factor heat kernels in the following sense:

$$k^{M_1 \times M_2}(t_1, t_2; \tilde{x}, \tilde{y}) = k^{M_1}(t_1; \tilde{x}_1, \tilde{y}_1) \otimes k^{M_2}(t_2; \tilde{x}_2, \tilde{y}_2). \tag{4.21}$$

Implicit in the notation is the canonical identification of fibers $(\Lambda^\bullet F)_{\tilde{z}} \cong (\Lambda^\bullet F_1)_{\tilde{z}_1} \otimes (\Lambda^\bullet F_2)_{\tilde{z}_2}$.

4.5.2 Heat kernel on the quotient: Selberg principle

Suppose $\pi : \tilde{X} \to X$ is a finite-sheeted Riemannian cover, i.e., a finite-sheeted smooth covering map that is a local isometry. Suppose also that $\tilde{W} \to \tilde{X}$ and $W \to X$ are unitary flat bundles, and that π extends to a smooth covering map of the total spaces $\tilde{W} \to W$ that is a linear isometry on fibers. For $x \in X$ and $\tilde{x} \in \pi^{-1}(x)$, let $\pi_{\tilde{x}}^* : (\Lambda^\bullet \tilde{W})_x \to (\Lambda^\bullet W)_{\tilde{x}}$ be the induced identification of fibers. Let G be the group of deck transformations of the cover; G is a finite group that acts by $\tilde{W}$-isometries. For each $g \in G$, let $g_{\tilde{x}}^* : (\Lambda^\bullet \tilde{W})_{g(\tilde{x})} \to (\Lambda^\bullet \tilde{W})_{\tilde{x}}$ be the induced identification of fibers.

Then we have the following relationship between the heat kernels on N and $\tilde{N}$, which is a simple version of the Selberg principle [21]:

$$k^N(t;x,y) = \sum_{g\in G} (\pi_{\tilde{x}}^*)^{-1} \circ g_{\tilde{x}}^* \circ k^{\tilde{N}}(t; g(\tilde{x}), \tilde{y}) \circ \pi_{\tilde{y}}^*,$$

where $\tilde{x} \in \pi^{-1}(x)$ and $\tilde{y} \in \pi^{-1}(y)$ are any lifts of $x, y \in X$.

As a consequence, we have the following in our setting:

$$k^{M_\Gamma}(t_1,t_2;x,y) = \sum_{\gamma\in\Gamma} (\pi_{\tilde{x}}^*)^{-1} \circ \gamma_{\tilde{x}}^* \circ k^{M_1\times M_2}(t_1,t_2;\gamma(\tilde{x}),\tilde{y}) \circ \pi_{\tilde{y}}^*, \tag{4.22}$$

where $\tilde{x} = (\tilde{x}_1, \tilde{x}_2)$ and $\tilde{y} = (\tilde{y}_1, \tilde{y}_2)$ are any lifts of x and y. Using (4.21) and (4.22), we obtain the following, which we record as a proposition:

Proposition 4.5.1.

$$\begin{aligned} &k^{M_\Gamma}(t_1,t_2;x,y) \\ &= \sum_{\gamma=(\gamma_1,\gamma_2)\in\Gamma} (\pi_{\tilde{x}}^*)^{-1} \circ \big(\gamma_1^* k^{M_1}(t_1;\gamma_1(\tilde{x}_1),\tilde{y}_1) \otimes \gamma_2^* k^{M_2}(t_2;\gamma_2(\tilde{x}_2),\tilde{y}_2)\big) \circ \pi_{\tilde{y}}^* \end{aligned} \tag{4.23}$$

4.5.3 The trace of the heat kernel with an auxiliary operator

We now study the heat kernel on the diagonal. Proposition 4.5.1 implies that at a point (x,x) on the diagonal of $M_\Gamma \times M_\Gamma$, we have

$$\begin{aligned} &k^{M_\Gamma}(t_1,t_2;x,x) \\ &= \sum_{\gamma=(\gamma_1,\gamma_2)\in\Gamma} (\pi_{\tilde{x}}^*)^{-1} \circ \big(\gamma_1^* k^{M_1}(t_1;\gamma_1(\tilde{x}_1),\tilde{x}_1) \otimes \gamma_2^* k^{M_2}(t_2;\gamma_2(\tilde{x}_2),\tilde{x}_2)\big) \circ \pi_{\tilde{x}}^* \end{aligned}$$

Let $\alpha = \alpha_1 \otimes \alpha_2$ and α_Γ be as in the hypotheses of Theorem 4.4.2. Then the integral kernel of $\alpha_\Gamma e^{-\Delta(t_1,t_2)}$ has the value at (x, x)

$$
\begin{aligned}
&(\alpha_\Gamma)_x k^{M_\Gamma}(t_1, t_2; x, x) \\
&= (\alpha_\Gamma)_x \sum_{\gamma=(\gamma_1,\gamma_2)\in\Gamma} (\pi^*_{\tilde{x}})^{-1} \circ \left(\gamma_1^* k^{M_1}(t_1; \gamma_1(\tilde{x}_1), \tilde{x}_1) \otimes \gamma_2^* k^{M_2}(t_2; \gamma_2(\tilde{x}_2), \tilde{x}_2)\right) \circ \pi^*_{\tilde{x}} \\
&= \sum_{\gamma=(\gamma_1,\gamma_2)\in\Gamma} (\pi^*_{\tilde{x}})^{-1} \circ \Big((\alpha_1)_{\tilde{x}_1} \gamma_1^* k^{M_1}(t_1; \gamma_1(\tilde{x}_1), \tilde{x}_1) \otimes \\
&\qquad (\alpha_2)_{\tilde{x}_2} \gamma_2^* k^{M_2}(t_2; \gamma_2(\tilde{x}_2), \tilde{x}_2)\Big) \circ \pi^*_{\tilde{x}}
\end{aligned}
$$

since $(\alpha_\Gamma)_x (\pi^*_{\tilde{x}})^{-1} = (\pi^*_{\tilde{x}})^{-1} \alpha_{\tilde{x}}$ and $\alpha = \alpha_1 \otimes \alpha_2$.

We now take the trace on the fiber at x. By the cyclicity of the trace and the multiplicativity of the trace of a tensor product, we obtain

$$
\begin{aligned}
&\mathrm{tr}(\alpha_\Gamma)_x k^{M_\Gamma}(t_1, t_2; x, x) \\
&= \sum_{\gamma=(\gamma_1,\gamma_2)\in\Gamma} \left(\mathrm{tr}\,(\alpha_1)_{\tilde{x}_1} \gamma_1^* k^{M_1}(t_1; \gamma_1(\tilde{x}_1), \tilde{x}_1)\right) \left(\mathrm{tr}\,(\alpha_2)_{\tilde{x}_2} \gamma_2^* k^{M_2}(t_2; \gamma_2(\tilde{x}_2), \tilde{x}_2)\right).
\end{aligned}
$$

To prove Theorem 4.4.2, it suffices to prove that for each γ,

$$
\begin{aligned}
&h_\gamma(t_1, t_2) := \\
&\int_{M_\Gamma} \left(\mathrm{tr}\,(\alpha_1)_{\tilde{x}_1} \gamma_1^* k^{M_1}(t_1; \gamma_1(\tilde{x}_1), \tilde{x}_1)\right) \left(\mathrm{tr}\,(\alpha_2)_{\tilde{x}_2} \gamma_2^* k^{M_2}(t_2; \gamma_2(\tilde{x}_2), \tilde{x}_2)\right) \mathrm{vol}_{M_\Gamma}(x)
\end{aligned}
$$

is a multi-admissible function of t_1, t_2. Let $D \subset M_1 \times M_2$ be a fundamental domain for the cover $M_1 \times M_2 \to M_\Gamma$. Then we have

$$
\begin{aligned}
&h_\gamma(t_1, t_2) = \\
&\int_{D} \left(\mathrm{tr}\,(\alpha_1)_{\tilde{x}_1} \gamma_1^* k^{M_1}(t_1; \gamma_1(\tilde{x}_1), \tilde{x}_1)\right) \left(\mathrm{tr}\,(\alpha_2)_{\tilde{x}_2} \gamma_2^* k^{M_2}(t_2; \gamma_2(\tilde{x}_2), \tilde{x}_2)\right) \mathrm{vol}_{M_\Gamma}(x).
\end{aligned}
$$

But by the Γ-invariance of all the relevant operators, this is the same (up to the size of Γ, which is the number of sheets in the cover) as integrating over all of $M_1 \times M_2$, which we can decompose as a product:

$$\begin{aligned}
h_\gamma(t_1,t_2) =& \frac{1}{|\Gamma|}\int_{M_1\times M_2} \big(\mathrm{tr}\,(\alpha_1)_{\tilde{x}_1}\,\gamma_1^* k^{M_1}(t_1;\gamma_1(\tilde{x}_1),\tilde{x}_1)\big) \\
&\qquad\qquad \big(\mathrm{tr}\,(\alpha_2)_{\tilde{x}_2}\,\gamma_2^* k^{M_2}(t_2;\gamma_2(\tilde{x}_2),\tilde{x}_2)\big)\ \mathrm{vol}_{M_1\times M_2}(\tilde{x}) \\
=& \frac{1}{|\Gamma|}\left(\int_{M_1} \big(\mathrm{tr}\,(\alpha_1)_{\tilde{x}_1}\,\gamma_1^* k^{M_1}(t_1;\gamma_1(\tilde{x}_1),\tilde{x}_1)\big)\,\mathrm{vol}_{M_1}(\tilde{x}_1)\right) \\
&\quad \left(\int_{M_2} \big(\mathrm{tr}\,(\alpha_2)_{\tilde{x}_2}\,\gamma_2^* k^{M_2}(t_2;\gamma_2(\tilde{x}_2),\tilde{x}_2)\big)\,\mathrm{vol}_{M_2}(\tilde{x}_2)\right)
\end{aligned}$$

All that remains is to prove that for $j = 1, 2$, the following is an admissible function of t_j:

$$\int_{M_j} \Big(\mathrm{tr}\,(\alpha_j)_{\tilde{x}_j}\,\gamma_j^* k^{M_1}(t_1;\gamma_j(\tilde{x}_j),\tilde{x}_j)\Big)\,\mathrm{vol}_{M_j}(\tilde{x}_j). \tag{4.24}$$

Note that this is precisely the trace of the operator $\alpha_j\gamma_j^* e^{-t_j\Delta_j}$, viewed as an operator $L^2(M_j,\Lambda^\bullet F_j) \to L^2(M_j,\Lambda^\bullet F_j)$.

4.5.4 Equivariant heat kernels

In this subsection, we will consider expressions of the form (4.24). We will now introduce new notation to emphasize the generality in which the following results hold.

Consider a compact Riemannian manifold N of dimension l and a flat unitary vector bundle $V \to N$. Let ϕ be an isometry of N that extends to a V-isometry; we need not assume that ϕ has nondegenerate fixed points. Let $\Lambda^\bullet V := V \otimes \Lambda^\bullet T^*N$ denote the bundle of V-valued differential forms. ϕ induces a map ϕ^* on sections of $\Lambda^\bullet V$. Let Δ be the Laplacian on sections of $\Lambda^\bullet V$. Let $\sigma \in C^\infty(\mathrm{End}(\Lambda^\bullet V))$.

Recall the notion of an admissible function of $t > 0$ from Definition 3.1.1. The following fundamental fact is well-known:

Lemma 4.5.2. *Under the assumptions from above,* $\mathrm{Tr}'\,\sigma\,\phi^*\,e^{-t\Delta}$ *is admissible.*

For a proof, see Gilkey [7]. We remark that we have stated the lemma in terms of $\mathrm{Tr}'\,\sigma\,\phi^*\,e^{-t\Delta}$ to ensure decay as $t \to \infty$, but the essential content of the lemma is that $\mathrm{Tr}\,\sigma\,\phi^*\,e^{-t\Delta}$ admits a $t \to 0$ asymptotic expansion. In the case that we are interested in, our acyclicity assumption implies that we need not worry about the kernel of Δ, i.e., $\mathrm{Tr}'\,\sigma\,\phi^*\,e^{-t\Delta} = \mathrm{Tr}\,\sigma\,\phi^*\,e^{-t\Delta}$.

Lemma 4.5.2 shows that the terms of the form (4.24) are all admissible. This proves that for each $\gamma \in \Gamma$, $h_\gamma(t_1, t_2)$ is multi-admissible. Summing over the group elements $\gamma \in \Gamma$ completes the proof of Theorem 4.4.2, which we restate here:

Theorem 4.5.3. *For $j = 1, 2$, let $\alpha_j \in C^\infty(M_j, \mathrm{End}\,\Lambda^\bullet F_j)$, and suppose that α_j commutes with the action of Γ_j. Then α_1 and α_2 induce an operator $\alpha = \alpha_1 \otimes \alpha_2 \in C^\infty(M, F_1)$ that commutes with the action of Γ, and α descends to an operator $\alpha_\Gamma \in C^\infty(M_\Gamma, \mathrm{End}\,\Lambda^\bullet F_\Gamma)$. Then $\mathrm{Tr}\,\alpha_\Gamma e^{-\Delta(t_1,t_2)}$ is multi-admissible.*

The reader will recall that the multi-admissibility of $\mathrm{Tr}\,\alpha_\Gamma e^{-\Delta(t_1,t_2)}$ was essential to our definitions of the multi-zeta function and multi-analytic torsion.

4.5.5 Vanishing of constant terms

Having proven the multi-admissibility of $\mathrm{Tr}\,\alpha_\Gamma e^{-\Delta(t_1,t_2)}$, we now study the constant terms in its multi-asymptotic expansion. The results of this subsection will be essential in proving the metric independence result of Theorem 4.4.8, whose proof we complete in Section 4.8.

Theorem 4.4.8 involves the assumptions that the dimension n_j of M_j is odd and that each $\gamma_j \in \Gamma_j$, viewed as a diffeomorphism of M_j, either is orientation-preserving or has nondegenerate fixed points. We will now explain the relevance of these assumptions.

We retain the notation of the previous subsection. First, we discuss the orientation-preserving assumption and what it has to do with dimension.

Lemma 4.5.4. *Suppose ϕ is an isometry of the Riemannian manifold N. Let X_0 be a submanifold of N that is a connected component of the fixed point set of ϕ. Let c be the codimension of X_0 in N. Then the parity of c depends on whether ϕ is orientation-preserving, i.e.:*

- *ϕ is orientation-preserving if and only if c is even, and equivalently,*
- *ϕ is orientation-reversing if and only if c is odd.*

Lemma 4.5.5. *Suppose ϕ is orientation-preserving and l is odd. Then there is no t^0 term in the asymptotic expansion of* $\operatorname{Tr} \sigma\, \phi^* \, e^{-t\Delta}$.

We omit the proof, but the result follows essentially from an examination of the approximate heat kernel of (4.26) below, using Lemma 4.5.4. For a proof, see the proof of Proposition 2 in [8].

We will now study the case when ϕ has nondegenerate fixed points, implying the codimension of each fixed point in N is l, the dimension of N. Lemma 4.5.4 shows that in that case, l is even if and only if ϕ is orientation-preserving.

If ϕ has nondegenerate fixed points, the ϕ-heat trace $\operatorname{Tr} \sigma\, \phi^* \, e^{-t\Delta}$ is in fact bounded as $t \to 0$. We will need the following result computing the constant term in that case:

Lemma 4.5.6. *Suppose that ϕ has nondegenerate fixed points. Let* $\operatorname{Fix}(\phi)$ *denote the finite set of fixed points in N. Let $\alpha \in C^\infty(N, \operatorname{End} \Lambda^\bullet V)$. Then as $t \to 0$,*

$$\operatorname{Tr} \alpha\phi^* e^{-t\Delta} = \sum_{x_0 \in \operatorname{Fix}(\phi)} \frac{\operatorname{tr} \alpha_{x_0} \phi^*_{x_0}}{|\det(I - d\phi_{x_0})|} + O\left(t^{\frac{1}{2}}\right).$$

Proof. Let $\phi^*_x : (\Lambda^\bullet V)_{\phi(x)} \to (\Lambda^\bullet V)_x$ be the linear map between fibers such that $(\phi^*(\sigma))(x) = \phi^*_x(\sigma(\phi(x)))$. The operator $\alpha\, \phi^* \, e^{-t\Delta}$ has an integral kernel whose value at (x, x) on the diagonal of $M \times M$ is

$$\alpha_x \, \phi^*_x \, k(t; \phi(x), x) \in \operatorname{End}((\Lambda^\bullet V)_x), \tag{4.25}$$

Recall that we may approximate the heat kernel k near the diagonal by an approximate heat kernel p^K, where p^K has the form

$$p^K(t;x,y) = (4\pi t)^{-l/2} \exp\left(-\frac{\operatorname{dist}^2(x,y)}{4t}\right) \sum_{j=0}^{K} t^j u_j(x,y), \tag{4.26}$$

where $u_0(x,x) = I$. For any a, by choosing K sufficiently large, we may replace k by p^K in (4.25) at the expense of an error term whose trace is of order t^a, which we may ignore for the purposes of proving the lemma. We have

$$\begin{aligned}
&\operatorname{tr} \alpha_x\, \phi_x^*\, p^K(t;\phi(x),x) \\
&= (4\pi t)^{-l/2} \exp\left(-\frac{\operatorname{dist}^2(\phi(x),x)}{4t}\right) \sum_{j=0}^{K} t^j \ \operatorname{tr} \alpha_x\, \phi_x^*\, u_j(\phi(x),x).
\end{aligned}$$

We wish to integrate the above expression times the volume form over M. The Gaussian factor ensures that for any $\epsilon > 0$, the points x such that $\operatorname{dist}(\phi(x),x) > \epsilon$ do not contribute to the asymptotic expansion. Thus it suffices to integrate over an arbitrarily small neighborhood of each fixed point of ϕ.

Let $x_0 \in M$ be a fixed point of ϕ. Let $T = d\phi_{x_0}$. Choose normal coordinates $x = (x^1, \ldots, x^l)$ centered at x_0. In these coordinates, we have

$$\begin{aligned}
\operatorname{dist}^2(\phi(x),x) &= |Tx - x|^2 + r(x), \ \text{where } r(x) \text{ is } O\left(|x|^3\right) \\
\alpha_x\, \phi_x^*\, u_0(\phi(x),x) &= \alpha_{x_0}\phi_{x_0}^* + O(|x|) \\
\operatorname{vol}(x) &= (1 + O(|x|))\, dx
\end{aligned}$$

where $dx = dx^1 \cdots dx^l$. Thus

$$\begin{aligned}
&\operatorname{tr} \alpha_x\, \phi_x^*\, p^K(t;\phi(x),x) \operatorname{vol}(x) \\
&= (4\pi t)^{-l/2} \exp\left(-\frac{|Tx-x|^2 + r(x)}{4t}\right) \left(\operatorname{tr} \alpha_{x_0}\, \phi_{x_0}^* + O(|x|) + O(t)\right)\, dx.
\end{aligned} \tag{4.27}$$

Since $\det(I - T) \neq 0$, there exists $\delta_1 > 0$ such that $|Tx - x|^2 \geqslant \delta_1 |x|^2$. Using this and the fact that $r(x)$ is $O(|x|^3)$, there exists $\epsilon > 0$ and $\delta > 0$ such that for $|x| \leqslant \epsilon$,

$$|Tx - x|^2 + r(x) \geqslant \delta |x|^2. \tag{4.28}$$

For this ϵ, we will integrate (4.27) over the set on which $|x| < \epsilon$ and show that we obtain

$$\frac{\operatorname{tr} \alpha_{x_0} \phi^*_{x_0}}{|\det(I-T)|} + O\left(t^{\frac{1}{2}}\right),$$

which suffices to prove the lemma.

A standard computation shows the key fact that

$$\int_{|x|<\epsilon} (4\pi t)^{-l/2} \exp\left(-\frac{|Tx-x|^2}{4t}\right) \operatorname{tr} \alpha_{x_0} \phi^*_{x_0}\, dx = \frac{\operatorname{tr} \alpha_{x_0} \phi^*_{x_0}}{|\det(I-T)|} + O\left(t^{\frac{1}{2}}\right).$$

(For a proof, see [19]. The $O\left(t^{\frac{1}{2}}\right)$ is not sharp.) The change of variables $x = t^{\frac{1}{2}}y$ shows that the L^1 norm of $(4\pi t)^{-l/2} \exp\left(\frac{-\delta|x|^2}{4t}\right) x^a t^b$ is $O\left(t^{\frac{1}{2}a+b}\right)$, which implies that we may ignore the terms of the form $O(|x|)$ and $O(|t|)$ in (4.27). Thus it suffices to show that the following is $O\left(t^{\frac{1}{2}}\right)$:

$$\int_{|x|<\epsilon} (4\pi t)^{-l/2} \exp\left(-\frac{|Tx-x|^2}{4t}\right) \left(\exp\left(-\frac{r(x)}{4t}\right) - 1\right) dx \tag{4.29}$$

We write the set on which $|x| < \epsilon$ as the union of the sets A and B, defined by:

$$A := \{x \in \mathbb{R}^n : t^{\frac{1}{3}} \leqslant |x| < \epsilon\}$$

$$B := \{x \in \mathbb{R}^n : |x| < t^{\frac{1}{3}}\}.$$

(We may assume $t^{\frac{1}{3}} < \epsilon$.)

For $x \in A$, we use the estimate (4.28), which gives

$$\int_A (4\pi t)^{-l/2} \exp\left(-\frac{|Tx-x|^2}{4t}\right) \left|\exp\left(-\frac{r(x)}{4t}\right) - 1\right| dx$$

$$\leqslant 2 \int_{|x| \geqslant t^{\frac{1}{3}}} (4\pi t)^{-l/2} \exp\left(\frac{-\delta|x|^2}{4t}\right) dx,$$

which is $O\left(\exp\left(-\frac{\delta}{8} t^{-\frac{1}{3}}\right)\right)$, which is in particular $O\left(t^{\frac{1}{2}}\right)$.

Let $x \in B$, i.e., $|x| < t^{\frac{1}{3}}$. Then $\frac{|x|^3}{t} < 1$; thus since $r(x)$ is $O(|x|^3)$, $\left|\frac{r(x)}{t}\right| < C$. Taylor's theorem implies $\left|\exp\left(-\frac{r(x)}{4t}\right) - 1\right| \leqslant C_1 \frac{|r(x)|}{4t} \leqslant C_2 \frac{|x|^3}{t}$, so

$$\int_B (4\pi t)^{-l/2} \exp\left(-\frac{|Tx - x|^2}{4t}\right) \left|\exp\left(-\frac{r(x)}{4t}\right) - 1\right| dx$$

$$\leqslant C_2 \int_B (4\pi t)^{-l/2} \exp\left(\frac{-\delta_1 |x|^2}{4t}\right) \frac{|x|^3}{t}\, dx,$$

which is $O\left(t^{\frac{1}{2}}\right)$. This proves the lemma. □

We have the following vanishing result in the case in which the operator α of Lemma 4.5.6 takes a certain special form:

Corollary 4.5.7. *Suppose that ϕ has nondegenerate fixed points. Let $g(u)$ be a smooth one-parameter family of ϕ-invariant Riemannian metrics on N.*
Then $\mathrm{Tr}(-1)^Q *^{-1} \frac{d*}{du} \phi^* e^{-t\Delta^{g(u)}}$ *is $O(t^{1/2})$ as $t \to 0$.*

Proof. We will show that for each fixed point x_0,

$$\mathrm{tr}(-1)^Q *_{x_0}^{-1} \frac{d*_{x_0}}{du} \phi^*_{x_0} = 0, \tag{4.30}$$

which suffices to prove the claim by Lemma 4.5.6.

We simply conjugate the left-hand side of (4.30) by $*_{x_0}$, using the facts that $*(-1)^Q = (-1)^l (-1)^Q *$, $* \frac{d*}{du} = -\frac{d*}{du} *$ (since $*$ squares to a constant), and $*\phi^* = \epsilon_{\text{or.}} \phi^* *$ where $\epsilon_{\text{or.}} = 1$ if ϕ is orientation-preserving and $\epsilon_{\text{or.}} = -1$ if ϕ is orientation-reversing. But since ϕ has nondegenerate fixed points, ϕ is orientation-preserving if and only if l is even, which means $\epsilon_{\text{or.}} = (-1)^l$. The cyclicity of the trace and the facts above show that

$$\begin{aligned} \mathrm{tr}(-1)^Q *_{x_0}^{-1} \frac{d*_{x_0}}{du} \phi^*_{x_0} &= \mathrm{tr}\, *_{x_0} (-1)^Q *_{x_0}^{-1} \frac{d*_{x_0}}{du} \phi^*_{x_0} *_{x_0}^{-1} \\ &= -\,\mathrm{tr}(-1)^Q *_{x_0}^{-1} \frac{d*_{x_0}}{du} \phi^*_{x_0}, \end{aligned}$$

which proves (4.30) and completes the proof of the corollary. ∎

The following proposition concerns constant terms in multi-asymptotic expansions. It is essential to Theorem 4.4.8, whose proof we will give in Section 4.8.

Proposition 4.5.8. *Suppose the dimension n_1 is odd and for every $\gamma_1 \in \Gamma_1$, γ_1 is either orientation-preserving or has nondegenerate fixed points as a diffeomorphism of M_1. Suppose α_1 is of the special form $\alpha_1 = *_1 \frac{d*_1}{du}$ for a smooth one-parameter family of metrics $h_1(u)$ on M_1. Then there are no $(t_1)^0$ terms in the multi-asymptotic expansions of (MA1)-(MA3) for the multi-admissible function*

$$\mathrm{Tr}(-1)^Q \alpha_\Gamma e^{-\Delta(t_1,t_2)}.$$

A similar statement holds with the roles of 1 *and* 2 *reversed.*

Proof. By the discussion in Subsections 4.5.3 and 4.5.4, it suffices to prove the claim that for each $\gamma_1 \in \Gamma_1$, $\mathrm{Tr}(-1)^{Q_1}\alpha_1\gamma_1^* e^{-t_1\Delta_1}$ has no $(t_1)^0$ term in its small t_1 asymptotic expansion. There are two cases:

First, if γ_1 is orientation-preserving, then the claim follows from Lemma 4.5.5 since n_1 is odd.

Second, if γ_1 has nondegenerate fixed points, then the claim follows from Corollary 4.5.7. ∎

We obtain the following corollary:

Corollary 4.5.9. *Retain the assumptions of Proposition 4.5.8, including the special form of α_1. Then*

$$\left.\frac{\partial}{\partial s_2}\right|_{(s_1,s_2)=(0,0)} \zeta\left(s_1, s_2; -\Delta(t_1,t_2,u), (-1)^Q \alpha_\Gamma\right) = 0.$$

A similar statement holds with the roles of 1 *and* 2 *reversed.*

Proof. This follows from Proposition 4.5.8 and Lemma 4.2.3. ∎

4.6 A closed form for multi-torsion

In this section we will introduce the closed form ω_{MT} and interpret multi-torsion as the integral of ω_{MT} over a surface in the space of metrics.

The form ω_{MT} makes sense in a somewhat more general setting than that in which we defined multi-torsion. What is essential is that there is a sufficiently nice decomposition of d, d^*, Δ, and $\delta^{\mathcal{M}}$. More precisely, we make the following assumptions:

Let M be a compact manifold of dimension n. (In the setting of Section 4.3, $M = M_\Gamma$ and $n = n_1 + n_2$.) Let (E, d) be an elliptic complex on M, and let $(-1)^Q$ be the grading operator as usual. We assume that d admits a decomposition

$$d = d_1 + \cdots + d_m,$$

such that for all i, j, d_i and d_j commute and d_i anticommutes with $(-1)^Q$. This implies that for any metric on E, we have a decomposition of the associated d^*:

$$d^* = d_1^* + \cdots + d_m^*.$$

We assume that our space of allowable metrics $\mathcal{M}$ satisfies the following: for any metric in $\mathcal{M}$, the associated d_i^*'s satisfy that for all i, j, d_i^* and $d_j^\star$ commute and that if $i \neq j$, $d_i^\star$ and d_j commute. Then we also have the decomposition of the Laplacian $\Delta = \sum_{k=1}^m \Delta_k$, where $\Delta_k := d_k d_k^* + d_k^* d_k$. d_k and d_k^* commute with Δ and therefore with functions of Δ, in particular the resolvent $R = R_z = (z - \Delta)^{-1}$.

Furthermore, we require that the $\mathcal{M}$-exterior derivative $\delta^{\mathcal{M}}$ decomposes as

$$\delta^{\mathcal{M}} = \delta_1^{\mathcal{M}} + \cdots \delta_m^{\mathcal{M}}$$

such that $\delta_i^{\mathcal{M}} d_j^* = 0$ for $i \neq j$.

Fix a basepoint metric $h_0 \in \mathcal{M}$ and a volume form vol_0; let $\langle \cdot, \cdot \rangle_0$ be the induced L^2-inner product. As in Subsection 3.4.1, let $\beta \in \Omega^0(\mathcal{M}, B)$ be the operator defined by, for $h \in \mathcal{M}$, $\langle \cdot, \cdot \rangle_h = \langle \cdot, \beta_h \cdot \rangle_0$. The 1-form $b = \beta^{-1}\delta^{\mathcal{M}}\beta$ now decomposes as

$$b = b_1 + \cdots + b_m,$$

where $b_j = \beta^{-1}\delta_j^{\mathcal{M}}\beta$. We obtain that $\delta^{\mathcal{M}} d_j^* = \delta_j^{\mathcal{M}} d_j^* = [d_j^*, b_j]$. We assume that b_j commutes with d_i and d_i^* for $i \neq j$, and that each b_j is symmetric in the sense of Definition 2.3.2. (Thus we assume that a generalization of Lemma 3.4.6 holds. We need not assume this in the geometric setting of Section 4.3, since in that setting this fact holds automatically for the same reason that Lemma 3.4.6 holds.)

We will now introduce ω_{MT} and $\tilde{\omega}_{MT}$, which are m-forms that generalize the one-forms ω_T and $\tilde{\omega}_T$, respectively, of Chapter 3.

Recall that R_z denotes the resolvent $(\Delta - z)^{-1}$. Fix an integer N such that $(\text{order}\,\Delta)N > n$, where n is the dimension of the base manifold M. (order Δ denotes the order of Δ as a differential operator; in Chapter 3, we referred to this integer as m, but in this chapter, we reserve m for the degree of the forms ω_{MT} and $\tilde{\omega}_{MT}$.) Then by Remark 2.3.3, $R_z^N \in \Psi^{-(\text{order}\,\Delta)N}$ is trace-class, which ensures the operator-valued forms introduced below are trace-class. (See also Remark 3.5.2.)

We must now introduce some rather cumbersome combinatorics and associated notation. Let $\mathcal{C}_{N,m}$ denote the finite set of compositions (i.e., ordered partitions) of the positive integer N into m positive integer summands; i.e, $\mathcal{C}_{N,m}$ is the set of multiindices $I = (i_1, \ldots, i_m)$ such that each i_j is a strictly positive integer and $N = i_1 + \cdots + i_m$. We will view S_{m-1} as the permutation group of the set $\{2, \ldots, m\}$.

For $\sigma \in S_{m-1}$, $I = (i_1, \ldots, i_m) \in \mathcal{C}_{N,m}$, and z a complex number not in the spectrum of Δ, let $\mathcal{R}_{z,\sigma,I}$ be defined by

$$\mathcal{R}_{z,\sigma,I} := R_z^{i_1} b_1 \, R_z^{i_2} b_{\sigma(2)} \cdots R_z^{i_m} b_{\sigma(m)}.$$

Let $T_{\sigma,I}$ and $\overline{T}_{\sigma,I}$ be defined by

$$T_{\sigma,I} := \operatorname{sign}\sigma \;\operatorname{Tr}(-1)^Q \mathcal{R}_{z,\sigma,I},$$

$$\overline{T}_{\sigma,I} := \operatorname{sign}\sigma \;\overline{\operatorname{Tr}}\,(-1)^Q \mathcal{R}_{\overline{z},\sigma,I}.$$

($T_{\sigma,I}$ and $\overline{T}_{\sigma,I}$ depend on z, but we suppress this. Note that $T_{\sigma,I}$ is *not* necessarily equal to the complex conjugate of $\overline{T}_{\sigma,I}$.)

Our m-form $\tilde{\omega}_{MT}$ is defined by

$$\tilde{\omega}_{MT} := \frac{1}{|\mathcal{C}_{N,m}|(m-1)!} \sum_{\sigma \in S_{m-1}, I \in \mathcal{C}_{N,m}} \frac{1}{2} \left(T_{\sigma,I} + \overline{T}_{\sigma,I} \right),$$

where $|\mathcal{C}_{N,m}|$ denotes the number of elements in the finite set $\mathcal{C}_{N,m}$, so that the normalizing constant $|\mathcal{C}_{N,m}|(m-1)!$ is the number of summands in the sum. ($(m-1)!$ is the number of elements in S_{m-1}; we have not bothered to compute $|\mathcal{C}_{N,m}|$.) Again, we suppress the dependence on z. We have symmetrized in a sense over all orderings of the factors $2, \ldots, m$; we could have also chosen to symmetrize over all orderings of all the factors $1, \ldots, m$, which would have been equivalent by the graded cyclicity of the trace.

ω_{MT} is defined in terms of $\tilde{\omega}_{MT}$ via the following contour integral:

$$\omega_{MT} := \frac{1}{(N-1)!} \frac{1}{2\pi i} \int_C e^{-z} \, \tilde{\omega}_{MT} \, dz, \tag{4.31}$$

where C is a contour in $\mathbb{C}$ enclosing $[0, \infty)$. We remark that there is not an obvious simpler formula for ω_{MT} involving the heat operator (analogous to Definition 3.4.7) because $b_1, \ldots, b_m$ do not commute with the resolvent R_z. But in the most important special case, the contour integral in (4.31) is easy to compute, giving a heat operator; see (4.32) below.

The main result of this section is

Theorem 4.6.1. *Under the assumptions above, $\tilde{\omega}_{MT}$ and ω_{MT} are closed on $\mathcal{M}$.*

The next section will be dedicated to proving that $\tilde{\omega}_{MT}$ is closed, which suffices to prove Theorem 4.6.1. First, though, we remark on the significance of ω_{MT} in the context of multi-torsion:

We note that the geometric setting of Section 4.3 satisfies the assumptions above, with $m = 2$ and with $\mathcal{M} = \mathcal{M}_{\text{prod}}$. In that setting, we have a bigrading of E: $E = \bigoplus_{q_1,q_2} E^{q_1,q_2}$. Let $h = h_1 \times h_2$ be a metric on E. Let h^{q_1,q_2} be the restriction of h to E^{q_1,q_2} and let $\Delta^h = \Delta_1^h + \Delta_2^h$ be the associated Laplacian. For $t_1, t_2 > 0$, let h_{t_1,t_2} be the metric whose restriction to E^{q_1,q_2} is $t_1^{q_1} t_2^{q_2} h^{q_1,q_2}$, so that $\beta(t_1,t_2) = t_1^{q_1} t_2^{q_2}$ on E^{q_1,q_2}. Then we see that for $j = 1, 2$, we have, on the surface Σ_h in $\mathcal{M}_{\text{prod}}$ parametrized by $(t_1, t_2) \mapsto h_{t_1,t_2}$,

$$b_j = q_j \frac{dt_j}{t_j}$$

$$d_j^*(t_j) = t_j d^*(h)$$

$$\Delta(t_1, t_2) = t_1 \Delta_1^h + t_2 \Delta_2^h$$

Pulled back to Σ_h, every summand in the sum in (4.6) is the same; i.e., we have for every σ, I that

$$\frac{1}{2}\left(T_{\sigma,I} + \overline{T}_{\sigma,I}\right) = \operatorname{Tr}(-1)^Q Q_1 Q_2 (t_1 \Delta_1^h + t_2 \Delta_2^h - z)^{-N} \frac{dt_1}{t_1}\frac{dt_2}{t_2},$$

which follows from the graded cyclicity of the trace, the adjoint-trace relation of Lemma 2.3.5, and the fact that Q_1 and Q_2 commute with the resolvent R_z. Thus the 2-form $\tilde{\omega}_{MT}$ pulled back to Σ_h is

$$\tilde{\omega}_{MT} = \operatorname{Tr}(-1)^Q \, Q_1 Q_2 \, (t_1 \Delta_1^h + t_2 \Delta_2^h - z)^{-N} \frac{dt_1}{t_1}\frac{dt_2}{t_2}.$$

Performing the contour integral gives in turn that ω_{MT} pulled back to Σ_h is

$$\omega_{MT} = \operatorname{Tr}(-1)^Q \, Q_1 Q_2 \, e^{-(t_1 \Delta_1^h + t_2 \Delta_2^h)} \frac{dt_1}{t_1}\frac{dt_2}{t_2}. \tag{4.32}$$

Thus we have proven the following generalization of Lemma 3.4.9:

Lemma 4.6.2. *For* $\operatorname{Re} s_1, \operatorname{Re} s_2$ *large,*

$$\zeta(s_1, s_2; \Delta^h(t_1,t_2), (-1)^Q Q_1 Q_2) = \frac{1}{\Gamma(s_1)\Gamma(s_2)} \int_{\Sigma_h} t_1^{s_1} t_2^{s_2} \omega_{MT}.$$

Using that $\frac{1}{\Gamma(s)} = s + O(s^2)$ as $s \to 0$ and Lemma 4.6.2, and recalling Definition 4.4.6 of the multi-torsion $MT(h_1, h_2)$, we have shown that

$$MT(h_1, h_2) = \left(\int_{\Sigma_h} t_1^{s_1} t_2^{s_2} \omega_{MT} \right) \Bigg|_{(s_1,s_2)=(0,0)}^{\text{AC}},$$

where the superscript AC indicates that we must analytically continue the function in parentheses to the origin. Formally, setting $s_1 = s_2 = 0$ on the right-hand side leaves $\int_{\Sigma_h} \omega_{MT}$; this is purely formal because the regularization is necessary for the integral to converge. This suggests the following heuristic interpretation of multi-torsion, generalizing our interpretation of analytic torsion:

> The multi-torsion $MT(h)$ may be interpreted as a regularized integral of ω_{MT} over the surface Σ_h in the space of metrics $\mathcal{M}_{\text{prod}}$.

4.7 Proof of closedness

This section is dedicated to the proof of Theorem 4.6.1, which we restate for convenience:

Theorem 4.7.1. *Under the assumptions above, $\tilde{\omega}_{MT}$ and ω_{MT} are closed on $\mathcal{M}$.*

Without loss of generality, it suffices to prove that $\delta_1^{\mathcal{M}} \tilde{\omega}_{MT} = 0$, which we will do. Recall that

$$\delta_1^{\mathcal{M}} \Delta = d_1 d_1^* b_1 - d_1 b_1 d_1^* + d_1^* b_1 d_1 - b_1 d_1^* d_1$$

$$(\delta_1^{\mathcal{M}} \Delta)^* = b_1 d_1 d_1^* - d_1 b_1 d_1^* + d_1^* b_1 d_1 - d_1^* d_1 b_1$$

We will find the following lemma, which generalizes Lemma 3.5.9, useful:

Lemma 4.7.2. *Consider operator-valued forms A, B, and C. Assume that d_1 and d_1^* commute with each of A, B, and C, and that the forms below are trace-class. Then we have the identity*

$$\text{Tr}(-1)^Q A b_1 B (\delta_1^{\mathcal{M}} \Delta)^* C = \text{Tr}(-1)^Q A (\delta_1^{\mathcal{M}} \Delta) B b_1 C.$$

Proof. We use the cyclicity of the trace and that d_1 and d_1^* commute with A, B, and C and anticommute with $(-1)^Q$ to compute:

$$\mathrm{Tr}(-1)^Q Ab_1B(b_1d_1d_1^*)C = \mathrm{Tr}(-1)^Q A(d_1d_1^*b_1)Bb_1C$$

$$\mathrm{Tr}(-1)^Q Ab_1B(-d_1^*d_1b_1)C = \mathrm{Tr}(-1)^Q A(-b_1d_1^*d_1)Bb_1C$$

$$\mathrm{Tr}(-1)^Q Ab_1B(-d_1b_1d_1^* + d_1^*b_1d_1)C = \mathrm{Tr}(-1)^Q A(d_1^*b_1d_1 - d_1b_1d_1^*)Bb_1C$$

Summing the three identities above gives the result. □

We will now compute the derivatives of $T_{\sigma,I}$ and $\overline{T}_{\sigma,I}$ in the 1-direction:

$$\begin{aligned}\delta_1^{\mathcal{M}}(T_{\sigma,I}) = &- \operatorname{sign}\sigma\, \mathrm{Tr}(-1)^Q R_z^{i_1} b_1 b_1 R_z^{i_2} b_{\sigma(2)} \dots R_z^{i_m} b_{\sigma(m)} \\ &+ \sum_{\alpha=1}^{i_1} \operatorname{sign}\sigma\, \mathrm{Tr}(-1)^Q R_z^{\alpha}(\delta_1^{\mathcal{M}}\Delta) R_z^{i_1-\alpha+1} b_1 R_z^{i_2} b_{\sigma(2)} \cdots R_z^{i_m} b_{\sigma(m)} \\ &+ \sum_{j=2}^{m}\sum_{\beta=1}^{i_j} T_{\sigma,I;j,\beta}\end{aligned}$$

where

$$T_{\sigma,I;j,\beta} := (-1)^{j-1} \operatorname{sign}\sigma\, \mathrm{Tr}(-1)^Q R_z^{i_1} b_1 \cdots b_{\sigma(k-1)} R_z^{\beta}(\delta_1^{\mathcal{M}}\Delta) R_z^{i_j-\beta+1} b_{\sigma(j)} \cdots R_z^{i_m} b_{\sigma(m)}.$$

Similarly,

$$\begin{aligned}\delta_1^{\mathcal{M}}(\overline{T}_{\sigma,I}) = &- \operatorname{sign}\sigma\, \overline{\mathrm{Tr}}\,(-1)^Q R_{\overline{z}}^{i_1} b_1 b_1 R_{\overline{z}}^{i_2} b_{\sigma(2)} \dots R_{\overline{z}}^{i_m} b_{\sigma(m)} \\ &+ \sum_{\alpha=1}^{i_1} \operatorname{sign}\sigma\, \overline{\mathrm{Tr}}\,(-1)^Q R_{\overline{z}}^{\alpha}(\delta_1^{\mathcal{M}}\Delta) R_{\overline{z}}^{i_1-\alpha+1} b_1 R_{\overline{z}}^{i_2} b_{\sigma(2)} \cdots R_{\overline{z}}^{i_m} b_{\sigma(m)} \\ &+ \sum_{j=2}^{m}\sum_{\beta=1}^{i_j} \overline{T}_{\sigma,I;j,\beta}\end{aligned}$$

where

$$\overline{T}_{\sigma,I;j,\beta} := (-1)^{j-1} \operatorname{sign}\sigma\, \overline{\mathrm{Tr}}\,(-1)^Q R_{\overline{z}}^{i_1} b_1 \cdots b_{\sigma(k-1)} R_{\overline{z}}^{\beta}(\delta_1^{\mathcal{M}}\Delta) R_{\overline{z}}^{i_j-\beta+1} b_{\sigma(j)} \cdots R_{\overline{z}}^{i_m} b_{\sigma(m)}.$$

We will now show $\delta_1^{\mathcal{M}}\tilde{\omega}_{MT} = 0$ via several lemmas.

Lemma 4.7.3.

$$\begin{aligned}&\sum_{\sigma\in S_{m-1},I\in\mathcal{C}_{N,m}} \operatorname{sign}\sigma\ \mathrm{Tr}(-1)^Q R_z^{i_1} b_1 b_1 R_z^{i_2} b_{\sigma(2)}\cdots R_z^{i_m} b_{\sigma(m)}\\ &+\sum_{\sigma\in S_{m-1},I\in\mathcal{C}_{N,m}} \operatorname{sign}\sigma\,\overline{\mathrm{Tr}}\,(-1)^Q R_{\overline{z}}^{i_1} b_1 b_1 R_{\overline{z}}^{i_2} b_{\sigma(2)}\cdots R_{\overline{z}}^{i_m} b_{\sigma(m)} \;=\; 0.\end{aligned}$$

Proof.

$$\begin{aligned}&\operatorname{sign}\sigma\ \mathrm{Tr}(-1)^Q R_z^{i_1} b_1 b_1 R_z^{i_2} b_{\sigma(2)}\cdots R_z^{i_m} b_{\sigma(m)}\\ &=\operatorname{sign}\sigma\,\overline{\mathrm{Tr}}\,\left((-1)^Q R_z^{i_1} b_1 b_1 R_z^{i_2} b_{\sigma(2)}\cdots R_z^{i_m} b_{\sigma(m)}\right)^*\\ &=\operatorname{sign}\sigma(-1)^{\frac{1}{2}m(m+1)}\overline{\mathrm{Tr}}\, b_{\sigma(m)} R_{\overline{z}}^{i_m}\cdots b_{\sigma(2)} R_{\overline{z}}^{i_2} b_1 b_1 R_{\overline{z}}^{i_1}(-1)^Q\\ &=\operatorname{sign}\sigma(-1)^{\frac{1}{2}m(m+1)}\overline{\mathrm{Tr}}\,(-1)^Q R_{\overline{z}}^{i_2} b_1 b_1 R_{\overline{z}}^{i_1} b_{\sigma(m)} R_{\overline{z}}^{i_m}\cdots b_{\sigma(2)}.\end{aligned}$$

Let r be the permutation reversing the order of $(2,\dots,m)$. Note that $\operatorname{sign} r = (-1)^{\frac{1}{2}(m-2)(m-1)} = (-1)^{1+\frac{1}{2}m(m+1)}$, so that $\operatorname{sign}(r\circ\sigma) = (-1)^{1+\frac{1}{2}m(m+1)}\operatorname{sign}\sigma$. Summing over $\sigma \in S_{m-1}$ and $I \in \mathcal{C}_{N,m}$, we obtain the lemma. □

Lemma 4.7.4.

$$\begin{aligned}&\sum_{\sigma\in S_{m-1},I\in\mathcal{C}_{N,m}}\sum_{\alpha=1}^{i_1} \operatorname{sign}\sigma\ \mathrm{Tr}(-1)^Q R_z^{\alpha}(\delta_1^{\mathcal{M}}\Delta) R_z^{i_1-\alpha+1} b_1 R_z^{i_2} b_{\sigma(2)}\cdots R_z^{i_m} b_{\sigma(m)}\\ &+\sum_{\sigma\in S_{m-1},I\in\mathcal{C}_{N,m}}\sum_{\alpha=1}^{i_1} \operatorname{sign}\sigma\,\overline{\mathrm{Tr}}\,(-1)^Q R_{\overline{z}}^{\alpha}(\delta_1^{\mathcal{M}}\Delta) R_{\overline{z}}^{i_1-\alpha+1} b_1 R_{\overline{z}}^{i_2} b_{\sigma(2)}\cdots R_{\overline{z}}^{i_m} b_{\sigma(m)} \;=\; 0\end{aligned}$$

Proof. We compute:

$$\begin{aligned}&\operatorname{sign}\sigma\ \mathrm{Tr}(-1)^Q R_z^{\alpha}(\delta_1^{\mathcal{M}}\Delta) R_z^{i_1-\alpha+1} b_1 R_z^{i_2} b_{\sigma(2)}\cdots R_z^{i_m} b_{\sigma(m)}\\ &=\operatorname{sign}\sigma\,\overline{\mathrm{Tr}}\,\left((-1)^Q R_z^{\alpha}(\delta_1^{\mathcal{M}}\Delta) R_z^{i_1-\alpha+1} b_1 R_z^{i_2} b_{\sigma(2)}\cdots R_z^{i_m} b_{\sigma(m)}\right)^*\\ &=\operatorname{sign}\sigma(-1)^{\frac{1}{2}m(m+1)}\overline{\mathrm{Tr}}\, b_{\sigma(m)} R_{\overline{z}}^{i_m}\cdots b_{\sigma(2)} R_{\overline{z}}^{i_2} b_1 R_{\overline{z}}^{i_1-\alpha+1}(\delta_1^{\mathcal{M}}\Delta)^* R_{\overline{z}}^{\alpha}(-1)^Q\\ &=\operatorname{sign}\sigma(-1)^{\frac{1}{2}m(m+1)}\overline{\mathrm{Tr}}\, b_{\sigma(m)} R_{\overline{z}}^{i_m}\cdots b_{\sigma(2)} R_{\overline{z}}^{i_2}(\delta_1^{\mathcal{M}}\Delta) R_{\overline{z}}^{i_1-\alpha+1} b_1 R_{\overline{z}}^{\alpha}(-1)^Q\\ &=\operatorname{sign}\sigma(-1)^{\frac{1}{2}m(m+1)}\overline{\mathrm{Tr}}\,(-1)^Q R_{\overline{z}}^{i_2}(\delta_1^{\mathcal{M}}\Delta) R_{\overline{z}}^{i_1-\alpha+1} b_1 R_{\overline{z}}^{\alpha} b_{\sigma(m)} R_{\overline{z}}^{i_m}\cdots b_{\sigma(2)},\end{aligned}$$

which proves the lemma by an argument similar to the previous lemma. □

Lemma 4.7.5. *For each j $(2 \leqslant j \leqslant m)$,*

$$\sum_{\sigma \in S_{m-1}, I \in \mathcal{C}_{N,m}} \sum_{\beta=1}^{i_j} \left(T_{\sigma,I;j,\beta} + \overline{T}_{\sigma,I;j,\beta}\right) = 0.$$

Proof. We compute

$$\begin{aligned}
T_{\sigma,I;j,\beta} &= (-1)^{j-1} \operatorname{sign} \sigma \\
&\qquad \operatorname{Tr}(-1)^Q R_z^{i_1} b_1 \cdots b_{\sigma(j-1)} R_z^{\beta} (\delta_1^{\mathcal{M}} \Delta) R_z^{i_j - \beta + 1} b_{\sigma(j)} \cdots R_z^{i_m} b_{\sigma(m)} \\
&= (-1)^{j-1} \operatorname{sign} \sigma \\
&\qquad \overline{\operatorname{Tr}} \left((-1)^Q R_z^{i_1} b_1 \cdots b_{\sigma(j-1)} R_z^{\beta} (\delta_1^{\mathcal{M}} \Delta) R_z^{i_j - \beta + 1} b_{\sigma(j)} \cdots R_z^{i_m} b_{\sigma(m)}\right)^* \\
&= (-1)^{j-1} \operatorname{sign} \sigma (-1)^{\frac{1}{2} m(m+1)} \\
&\qquad \overline{\operatorname{Tr}}\, b_{\sigma(m)} R_{\overline{z}}^{i_m} \cdots b_{\sigma(j)} R_{\overline{z}}^{i_j - \beta + 1} (\delta_1^{\mathcal{M}} \Delta)^* R_{\overline{z}}^{\beta} b_{\sigma(j-1)} \cdots b_1 R_{\overline{z}}^{i_1} (-1)^Q \\
&= (-1)^{j-1} \operatorname{sign} \sigma (-1)^{\frac{1}{2} m(m+1)} \\
&\qquad \overline{\operatorname{Tr}}\, b_{\sigma(m)} R_{\overline{z}}^{i_m} \cdots b_{\sigma(j)} R_{\overline{z}}^{i_j - \beta + 1} b_1 R_{\overline{z}}^{\beta} b_{\sigma(j-1)} \cdots b_2 R_{\overline{z}}^{i_2} (\delta_1^{\mathcal{M}} \Delta) R_{\overline{z}}^{i_1} (-1)^Q \\
&= (-1)^{j-1} \operatorname{sign} \sigma (-1)^{\frac{1}{2} m(m+1) + j(m-j+1)} \\
&\qquad \overline{\operatorname{Tr}}\, (-1)^Q R_{\overline{z}}^{i_j - \beta + 1} b_1 R_{\overline{z}}^{\beta} b_{\sigma(j-1)} \cdots b_2 R_{\overline{z}}^{i_2} (\delta_1^{\mathcal{M}} \Delta) R_{\overline{z}}^{i_1} b_{\sigma(m)} R_{\overline{z}}^{i_m} \cdots b_{\sigma(j)}.
\end{aligned}$$

Summing over $\sigma \in S_{m-1}$, $I \in \mathcal{C}_{N,m}$, and $\beta \in \{1, \ldots, i_j\}$ proves the lemma.

□

The previous three lemmas prove that $\delta_1^{\mathcal{M}} \tilde{\omega}_{MT} = 0$. The same argument shows that $\delta_k^{\mathcal{M}} \tilde{\omega}_{MT} = 0$ for all k. Thus we have proven Theorem 4.6.1.

4.8 Proof of metric independence

We will now prove Theorem 4.4.8 concerning the metric independence of multi-torsion under appropriate assumptions. We restate the theorem for convenience:

Theorem 4.8.1. *For $j = 1, 2$, if n_j is odd and if for every $\gamma_j \in \Gamma_j$, γ_j is either orientation-preserving or has nondegenerate fixed points as a diffeomorphism of M_j, then $MT(h_1, h_2)$ is independent of the metric h_j.*

Proof. We will mimic the proof of Theorem 3.3.3 in Section 3.6. Again, the essential idea is to apply Stokes' theorem. The hypotheses ensure that the boundary terms vanish.

Without loss of generality, we assume $j = 1$. So we suppose that n_1 is odd. We fix a metric h_2 on M_2 and consider a curve $h_1(u)$, $u \in [0, 1]$, of metrics on M_1. For $\epsilon_1, \epsilon_2, A_1, A_2 > 0$, let $B = B_{\epsilon_1,\epsilon_2,A_1,A_2}$ be the cube in M parametrized by $(t_1, t_2, u) \mapsto h(u)_{t_1,t_2}$, for $\epsilon_1 \leqslant t_1 \leqslant A_1$, $\epsilon_2 \leqslant t_2 \leqslant A_2$, $0 \leqslant u \leqslant 1$. ω_{MT} pulled back to ∂B, in the coordinates (t_1, t_2, u), is

$$\begin{aligned}\omega_{MT} = \operatorname{Tr}(-1)^Q e^{-\Delta(t_1,t_2,u)} \frac{Q_1}{t_1} \frac{Q_2}{t_2}\, dt_1 dt_2 \\ + \operatorname{Tr}(-1)^Q e^{-\Delta(t_1,t_2,u)} \beta_1^{-1} \frac{\partial \beta_1}{\partial u} \frac{Q_2}{t_2}\, du dt_2.\end{aligned}$$

(Note that there is no $du dt_1$ component.) $\delta(t_1^{s_1} t_2^{s_2} \omega_{MT})$ is

$$\delta(t_1^{s_1} t_2^{s_2} \omega_{MT}) = s_1 t_1^{s_1-1} t_2^{s_2} \operatorname{Tr}(-1)^Q e^{-\Delta(t_1,t_2,u)} \beta_1^{-1} \frac{\partial \beta_1}{\partial u} \frac{Q_2}{t_2}\, du dt_1 dt_2$$

Assume that $\operatorname{Re} s_1, \operatorname{Re} s_2$ are both large. We apply Stokes' theorem on the cube B to obtain

$$\iiint_B \delta_1^{\mathcal{M}}(t_1^{s_1} t_2^{s_2} \omega_{MT}) = \iint_{\partial B} t_1^{s_1} t_2^{s_2} \omega_{MT}, \tag{4.33}$$

The integral over B is

$$\iiint_B \delta_1^{\mathcal{M}}(t_1^{s_1} t_2^{s_2} \omega_{MT}) \tag{4.34}$$

$$= s_1 \int_0^1 \int_{\epsilon_2}^{A_2} \int_{\epsilon_1}^{A_1} t_1^{s_1-1} t_2^{s_2} \operatorname{Tr}(-1)^Q e^{-\Delta(t_1,t_2,u)} \beta_1^{-1} \frac{\partial \beta_1}{\partial u} \frac{Q_2}{t_2}\, dt_1\, dt_2\, du. \tag{4.35}$$

The boundary ∂B is a cube with six faces, described by $u = 0$, $u = 1$, $t_1 = \epsilon_1$, $t_1 = A_1$, $t_2 = \epsilon_2$, and $t_2 = A_2$. Since ω_{MT} vanishes on the latter two faces, the integral over ∂B consists of four terms:

$$\int_{\epsilon_2}^{A_2}\int_{\epsilon_1}^{A_1} t_1^{s_1}t_2^{s_2}\,\mathrm{Tr}(-1)^Q e^{-\Delta(t_1,t_2,1)}\frac{Q_1}{t_1}\frac{Q_2}{t_2}\,dt_1\,dt_2 \qquad (u=1) \tag{4.36}$$

$$-\int_{\epsilon_2}^{A_2}\int_{\epsilon_1}^{A_1} t_1^{s_1}t_2^{s_2}\,\mathrm{Tr}(-1)^Q e^{-\Delta(t_1,t_2,0)}\frac{Q_1}{t_1}\frac{Q_2}{t_2}\,dt_1\,dt_2 \qquad (u=0) \tag{4.37}$$

$$+\int_0^1\int_{\epsilon_2}^{A_2} \epsilon_1^{s_1}t_2^{s_2}\,\mathrm{Tr}(-1)^Q e^{-\Delta(\epsilon_1,t_2,u)}\beta_1^{-1}\frac{\partial\beta_1}{\partial u}\frac{Q_2}{t_2}\,dt_2\,du \qquad (t_1=\epsilon_1) \tag{4.38}$$

$$+\int_0^1\int_{\epsilon_2}^{A_2} A_1^{s_1}t_2^{s_2}\,\mathrm{Tr}(-1)^Q e^{-\Delta(A_1,t_2,u)}\beta_1^{-1}\frac{\partial\beta_1}{\partial u}\frac{Q_2}{t_2}\,dt_2\,du \qquad (t_1=A_1) \tag{4.39}$$

For fixed ϵ_2 and A_2, we will now study the limits $\epsilon_1 \to 0^+$ and $A_1 \to \infty$. By the heat kernel estimates, the term (4.38) tends to 0 as $\epsilon_1 \to 0^+$, and the term (4.39) tends to 0 as $A_1 \to \infty$. Thus from (4.33) we obtain

$$\begin{aligned} &s_1\int_0^1\int_{\epsilon_2}^{A_2}\int_0^\infty t_1^{s_1-1}t_2^{s_2}\,\mathrm{Tr}(-1)^Q e^{-\Delta(t_1,t_2,u)}\beta_1^{-1}\frac{\partial\beta_1}{\partial u}\frac{Q_2}{t_2}\,du dt_2 dt_1 \\ &= \int_{\epsilon_2}^{A_2}\int_0^\infty t_1^{s_1}t_2^{s_2}\,\mathrm{Tr}(-1)^Q e^{-\Delta(t_1,t_2,1)}\frac{Q_1}{t_1}\frac{Q_2}{t_2}\,dt_1 dt_2 \\ &- \int_{\epsilon_2}^{A_2}\int_0^\infty t_1^{s_1}t_2^{s_2}\,\mathrm{Tr}(-1)^Q e^{-\Delta(t_1,t_2,0)}\frac{Q_1}{t_1}\frac{Q_2}{t_2}\,dt_1 dt_2. \end{aligned}$$

We may now take the limits $\epsilon_2 \to 0^+$ and $A_2 \to \infty$ to obtain

$$\begin{aligned} &s_1\int_0^1\int_0^\infty\int_0^\infty t_1^{s_1-1}t_2^{s_2}\,\mathrm{Tr}(-1)^Q e^{-\Delta(t_1,t_2,u)}\beta_1^{-1}\frac{\partial\beta_1}{\partial u}\frac{Q_2}{t_2}\,du dt_2 dt_1 \\ &= \int_0^\infty\int_0^\infty t_1^{s_1}t_2^{s_2}\,\mathrm{Tr}(-1)^Q e^{-\Delta(t_1,t_2,1)}\frac{Q_1}{t_1}\frac{Q_2}{t_2}\,dt_1 dt_2 \\ &- \int_0^\infty\int_0^\infty t_1^{s_1}t_2^{s_2}\,\mathrm{Tr}(-1)^Q e^{-\Delta(t_1,t_2,0)}\frac{Q_1}{t_1}\frac{Q_2}{t_2}\,dt_1 dt_2. \end{aligned}$$

Multiplying both sides by $\frac{1}{\Gamma(s_1)\Gamma(s_2)}$ gives that

$$s_1 \int_0^1 \zeta(s_1, s_2; -\Delta(t_1, t_2, u), (-1)^Q \beta_1^{-1} \frac{\partial \beta_1}{\partial u} Q_2) du$$
$$= \zeta(s_1, s_2; \Delta(t_1, t_2, 1), (-1)^Q Q_1 Q_2) - \zeta(s_1, s_2; \Delta(t_1, t_2, 1), (-1)^Q Q_1 Q_2).$$

We have shown that the equality holds for $\operatorname{Re} s_1$ and $\operatorname{Re} s_2$ large, but both sides possess unique meromorphic continuations to all of $\mathbb{C}^2$, so in fact the equality must hold everywhere for the meromorphic continuations. In particular, using the Definition 4.4.6 of multi-torsion, we have the following equality of derivatives at the origin:

$$\left.\frac{\partial^2}{\partial s_1 \partial s_2}\right|_{(s_1,s_2)=(0,0)} s_1 \int_0^1 \zeta(s_1, s_2; -\Delta(t_1, t_2, u), (-1)^Q \beta_1^{-1} \frac{\partial \beta_1}{\partial u} Q_2) du$$
$$= MT(1) - MT(0).$$

The left-hand side is equal to

$$\int_0^1 \left.\frac{\partial}{\partial s_2}\right|_{(s_1,s_2)=(0,0)} \zeta(s_1, s_2; -\Delta(t_1, t_2, u), (-1)^Q \beta_1^{-1} \frac{\partial \beta_1}{\partial u} Q_2) du.$$

By Corollary 4.5.9, the integrand vanishes if n_1 is odd. This proves $MT(1) - MT(0) = 0$ and proves the theorem. □

5

The eta invariant

In this chapter we study the eta invariant, a regularized signature of a certain square root of the Laplacian on manifolds of dimension congruent to 3 modulo 4. We show that the eta invariant allows for an interpretation similar to that for analytic torsion, as the integral of a closed one-form on the space of Riemannian metrics. We phrase the well-known fact that the relative eta invariant is metric independent in terms of essentially the same Stokes' theorem argument that computes the dependence of analytic torsion on the metric.

We consider only the eta invariant as defined by Atiyah-Patodi-Singer in the first of the series of papers [1], [2], [3], in which they studied the index theorem for manifolds with boundary.

5.1 The eta invariant

All the results of this section are due to Atiyah-Patodi-Singer [1].

Let M be a compact oriented manifold of dimension $n = 4k - 1$. Let $F \to M$ be a flat unitary vector bundle with flat connection ∇^F and metric h^F. Let $\Lambda^\bullet F = F \otimes \Lambda^\bullet T^*M$ be the bundle of F-valued forms.

Let $\mathcal{M}^{TM}$ denote the space of Riemannian metrics on M i.e., metrics on the tangent bundle $TM \to M$. A Riemannian metric $g \in \mathcal{M}^{TM}$ induces a metric, which in an abuse of notation we will also call g, on the exterior bundle $\Lambda^\bullet T^*M$. We will consider only metrics on $\Lambda^\bullet F$ that are induced as tensor products of the fixed metric h^F and a Riemannian metric g.

As usual, F's flat connection and the exterior derivative on M induce an operator d on sections of $\Lambda^\bullet F$. A metric $g \in \mathcal{M}^{TM}$ determines operators d^* and Δ and (having fixed an orientation of M) a Hodge star operator $* = *_g$, which extends to an endomorphism of $\Lambda^\bullet F$.

Let $(-1)^Q$ be as usual. Let $\sigma = (-1)^p$ on forms of degree $2p$ or $2p-1$, i.e., on q-forms,

$$\sigma = \begin{cases} 1, & q \equiv 0 \pmod 4 \\ -1, & q \equiv 1 \pmod 4 \\ -1, & q \equiv 2 \pmod 4 \\ 1, & q \equiv 3 \pmod 4 \end{cases}$$

Let B be the first order operator

$$B := \left(*d(-1)^Q - d* \right) \sigma.$$

B is a self-adjoint operator and $B^2 = \Delta$. Therefore B has real eigenvalues, and for each eigenvalue λ of B, λ^2 is an eigenvalue of Δ; conversely, every eigenvalue of Δ arises in this way.

Recall the definition (3.7) of the zeta function $\zeta(s; L, P)$. The eta function $\eta(s; B, P)$ is defined by

$$\eta(s; B, P) := \zeta\left(\frac{s+1}{2}; B^2, PB\right). \tag{5.1}$$

In the special case when P is the identity, we set

$$\eta(s; B) := \eta(s; B, I). \tag{5.2}$$

The eta invariant $\eta = \eta(M, F, g)$ is defined by

$$\eta(M, F, g) := \eta(0; B(g)), \tag{5.3}$$

where here our notation emphasizes the dependence of B on g. We must explain why $\eta(s; B)$ is finite at $s = 0$. First, though, we make a couple of observations:

Remark 5.1.1. Note that by the definition of the zeta function, we have for $\operatorname{Re} s$ large that

$$\eta(s; B, P) = \frac{1}{\Gamma\left(\frac{s+1}{2}\right)} \int_0^\infty t^{\frac{s+1}{2}} \operatorname{Tr} PBe^{-tB^2} \frac{dt}{t}. \tag{5.4}$$

Since B annihilates harmonic forms, we need not project away from the harmonic forms, i.e., "Tr′" need not appear in (5.4).

Remark 5.1.2. The identity for $\operatorname{Re} s$ large

$$\operatorname{sign}(\lambda)\lambda^{-s} = \frac{1}{\Gamma\left(\frac{s+1}{2}\right)} \int_0^\infty t^{\frac{s+1}{2}} \lambda e^{-t\lambda^2} \frac{dt}{t}$$

shows that for $\operatorname{Re} s$ large

$$\eta(s; B) = \sum_{\lambda \neq 0} \operatorname{sign}(\lambda)\lambda^{-s},$$

where the sum is over the nonzero eigenvalues of B. This shows that $\eta(0; B)$ may be interpreted as a regularized signature of B, i.e., a regularized difference between the number of positive eigenvalues of B and the number of negative eigenvalues of B.

Lemma 5.1.3. *$\zeta\left(\frac{s+1}{2}; \Delta, P\right)$ has a simple pole at $s = 0$ whose residue is $\frac{2}{\Gamma\left(\frac{1}{2}\right)} a_{-1/2}$, where $a_{-1/2}$ is the coefficient on $t^{-1/2}$ in the $t \to 0$ asymptotic expansion of* $\operatorname{Tr} Pe^{-t\Delta}$.

Thus a priori, $\eta(s; B)$ has a simple pole at $s = 0$ whose residue is the coefficient on $t^{-\frac{1}{2}}$ in the $t \to 0$ asymptotic expansion of $\operatorname{Tr} Be^{-tB^2}$. But in fact, by a theorem of Atiyah-Patodi-Singer [1], that coefficient vanishes, and furthermore, $\operatorname{Tr} Be^{-tB^2}$ is $O\left(t^{\frac{1}{2}}\right)$, so that $\eta(s)$ is holomorphic for $s > -\frac{1}{2}$. This ensures that the definition (5.3) of the eta invariant makes sense.

Remark 5.1.4. In fact, for our purposes, we do not need to assume the aforementioned theorem of Atiyah-Patodi-Singer; rather, we could simply take the definition of the eta invariant to be the finite part of $\eta(s; B)$ at $s = 0$ (i.e., ignore the potential pole).

The eta invariant is not a topological invariant, but we do have the following result of Atiyah-Patodi-Singer [1], a proof of which we will give in Section 5.4 using the closed form introduced in Section 5.2:

Theorem 5.1.5. *For a smooth one-parameter family of Riemannian metrics $g(u)$, the derivative of the eta invariant is*

$$\frac{d}{du}\eta(M, F, g(u)) = \frac{2}{\Gamma\left(\frac{1}{2}\right)} a_{1/2}(g(u)), \tag{5.5}$$

where $a_{-1/2}(g(u))$ denotes the $t^{-1/2}$ coefficient in the $t \to 0$ asymptotic expansion of $\operatorname{Tr} * \left[d, *^{-1}\frac{d*}{du}\right] e^{-t\Delta^{g(u)}}$.

In particular, the variation of the eta invariant is a locally computable quantity. This implies the following, which states that a relative version of the eta invariant, also known as the rho invariant, is topological:

Corollary 5.1.6. *Let $\eta(M, \mathbb{C}^r, g)$ denote the eta invariant associated to the trivial flat rank r bundle with total space $M \times \mathbb{C}^r$. Let F be a rank r unitary flat bundle. Then the relative eta invariant $\eta_{rel}(M, F, g)$ defined by*

$$\eta_{rel}(M, F, g) := \eta(M, F, g) - \eta(M, \mathbb{C}^r, g) \tag{5.6}$$

is independent of the metric $g \in \mathcal{M}^{TM}$.

Proof. The coefficient $a_{1/2}$ is the integral over M of an integrand $f(x)$ that depends only on the local geometry near $x \in M$. The bundle F is locally isometric to the trivial bundle $M \times \mathbb{C}^r$ of the same rank, so their coefficients $a_{1/2}$ agree. This fact and Theorem 5.1.5 imply that $\frac{d}{du}\eta_{\mathrm{rel}}(M, F, g(u)) = 0$, proving the claim. □

5.2 A closed form for the eta invariant

In this section we will interpret the eta invariant as an integral on the space of Riemannian metrics $\mathcal{M}^{TM}$. Henceforth, δ will denote the exterior derivative on $\mathcal{M}^{TM}$.

First, we will rephrase the formula for $\eta(s;B)$ in a more convenient way. Note that $*d$ maps q-forms to $(4k-2-q)$-forms, implying that the only contribution to $\operatorname{Tr} *d(-1)^Q e^{-t\Delta}$ is from $(2k-1)$-forms; similarly, $d*$ maps q-forms to $(4k-q)$-forms, implying that the only contribution to $\operatorname{Tr} d * e^{-t\Delta}$ is from $(2k)$-forms. Furthermore, conjugation by $*$ (recall that $*^{-1} = *$ since the dimension n is odd) gives the equality of traces

$$\begin{aligned}\operatorname{Tr} *d(-1)^Q e^{-t\Delta_{2k-1}} &= -\operatorname{Tr} *de^{-t\Delta_{2k-1}} \\ &= -\operatorname{Tr} * * de^{-t\Delta_{2k-1}} * \\ &= -\operatorname{Tr} d * e^{-t\Delta_{2k}} \\ &= -\operatorname{Tr} d * e^{-t\Delta}.\end{aligned}$$

This implies

$$\begin{aligned}-\frac{1}{2}\operatorname{Tr} Be^{-t\Delta} &= \operatorname{Tr} *de^{-t\Delta} \\ &= \operatorname{Tr} d * e^{-t\Delta}\end{aligned}$$

so that we may rewrite (5.4) (with $P = I$) as the following identity, for $\operatorname{Re} s$ large:

$$\eta(s;B) = \frac{-2}{\Gamma\left(\frac{s+1}{2}\right)} \int_0^\infty t^{\frac{s+1}{2}} \operatorname{Tr} *de^{-t\Delta} \frac{dt}{t}, \tag{5.7}$$

i.e., $\eta(s;B) = -2\zeta\left(\frac{s+1}{2}; \Delta, *d\right)$.

Let $b = *^{-1}\delta *$, a "tautological" one-form that is a special case of the form b of Chapter 3.

Definition 5.2.1. Let ω_η be the following one-form on $\mathcal{M}^{TM}$:

$$\operatorname{Tr} *[d,b]e^{-\Delta}$$

In the next section, we will prove that ω_η is closed.

We will now explain the relevance of ω_η to the eta invariant. Let $g \in \mathcal{M}^{TM}$. Let C_g be the curve in $\mathcal{M}^{TM}$ parametrized by, for $0 < t < \infty$, $t \mapsto \frac{1}{t} g$. We compute that, after pulling back to C_g, we have

$$\begin{aligned} *(t) &= t^{Q - \frac{n}{2}} *_g \\ b &= \left(Q - \frac{n}{2} I \right) \frac{dt}{t} \\ [d, b] &= -d \, \frac{dt}{t} \\ \Delta(t) &= t \Delta^g, \end{aligned}$$

implying

$$\omega_\eta = -t^{-\frac{1}{2}} \operatorname{Tr} *_g d \, e^{-t \Delta^g_{2k-1}} \, dt$$

Using (5.4), we have

$$\eta(s) = \frac{2}{\Gamma\left(\frac{s+1}{2}\right)} \int_{C_h} t^{\frac{s}{2}} \omega_\eta .$$

Thus we have the interpretation:

> The eta invariant $\eta(g)$ may be interpreted as $\frac{2}{\Gamma(1/2)}$ times the (regularized) integral of ω_η over the curve C_g in the space of Riemannian metrics $\mathcal{M}^{TM}$.

In fact, the theorem of Atiyah-Patodi-Singer [1] shows that the regularization is not necessary, i.e., the eta invariant is literally equal to the convergent integral $\frac{2}{\Gamma(1/2)} \int_{C_h} \omega_\eta$.

5.3 Proof of closedness

This section is devoted to a proof of

Theorem 5.3.1. *ω_η is closed on $\mathcal{M}^{TM}$.*

As with the form associated to analytic torsion, to prove this theorem, we will work not directly with the heat operator but with the resolvent $R_z = (\Delta - z)^{-1}$. Fix an integer N so that $2N - 1 > n$, which ensures that $\tilde{\omega}_\eta$, defined by

$$\tilde{\omega}_\eta = \operatorname{Tr} *[d, b] R_z^N,$$

is trace-class. By the Cauchy integral formula, ω_η and $\tilde{\omega}_\eta$ are related by the identity

$$\begin{aligned}\omega_\eta &= \operatorname{Tr} *[d, b] e^{-\Delta} \\ &= \frac{1}{2\pi i} \frac{1}{(N-1)!} \int_C e^{-z}\, \tilde{\omega}_\eta \, dz,\end{aligned}$$

so to prove ω_η is closed, it suffices to prove $\tilde{\omega}_\eta$ is closed, i.e., $\delta \tilde{\omega}_\eta = 0$, which we will now do.

Recall some facts:

$$\begin{aligned}\delta * &= * b \\ \delta b &= -bb \\ \delta R_z &= R_z \{d, [d^*, b]\} R_z \\ \delta [d, b] &= -[d, bb] \\ \{*, b\} &= 0 \\ (*d)^* &= *d \text{ (on } (2k-1)\text{-forms)}\end{aligned}$$

The trace identities from Chapter 2 will of course be useful. Because taking the adjoint introduces a complex conjugate, we will also need the following analogue of Lemma 3.5.5:

Lemma 5.3.2. *We may rewrite $\tilde{\omega}_\eta = \operatorname{Tr} *[d, b] R_z^N$ in either of the following equivalent ways:*

$$\tilde{\omega}_\eta = \overline{\operatorname{Tr}} * [d, b] R_{\overline{z}}^N \tag{5.8}$$

$$\tilde{\omega}_\eta = \frac{1}{2} \operatorname{Tr} *[d, b] R_z^N + \frac{1}{2} \overline{\operatorname{Tr}} * [d, b] R_{\overline{z}}^N \tag{5.9}$$

Proof. The second equality follows from the first, so it suffices to prove the first. Note that since $*$ anticommutes with b, $*[d,b]$ may also be written as the anticommutator $\{*d,b\}$, which is symmetric because both $*d$ and b are symmetric (in the relevant degree). We now apply Lemma 2.3.5 relating the trace and the adjoint:

$$\begin{aligned}\tilde{\omega}_\eta = \operatorname{Tr}\operatorname{Tr} *[d,b]R_z^N &= \overline{\operatorname{Tr}}\left(*[d,b]R_z^N\right)^* \\ &= \overline{\operatorname{Tr}}\, R_{\bar{z}}^N * [d,b] \\ &= \overline{\operatorname{Tr}} * [d,b]R_{\bar{z}}^N,\end{aligned}$$

where the last equality follows from the cyclicity of the trace. □

Using the expression (5.9) for $\tilde{\omega}_\eta$, we compute

$$2\delta\tilde{\omega}_\eta = \operatorname{Tr} *b[d,b]R_z^N - \operatorname{Tr} *[d,bb]R_z^N + \sum_{i+j=N+1} \operatorname{Tr} *[d,b]R_z^i\{d,[d^*,b]\}R_z^j \tag{5.10}$$

$$+ \overline{\operatorname{Tr}} * b[d,b]R_{\bar{z}}^N - \overline{\operatorname{Tr}} * [d,bb]R_{\bar{z}}^N + \sum_{i+j=N+1} \overline{\operatorname{Tr}} * [d,b]R_{\bar{z}}^i\{d,[d^*,b]\}R_{\bar{z}}^j \tag{5.11}$$

We expand the first term on the right-hand side of (5.10) and show cancellation with the corresponding term of line (5.11):

$$\begin{aligned}\operatorname{Tr} *bdbR_z^N &= -\operatorname{Tr} b * dbR_z^N \\ &= -\overline{\operatorname{Tr}}\,(b * dbR_z^N)^* \\ &= \overline{\operatorname{Tr}}\, R_{\bar{z}}^N b * db \\ &= \overline{\operatorname{Tr}}\, b * dbR_{\bar{z}}^N \\ &= -\overline{\operatorname{Tr}} * bdbR_{\bar{z}}^N.\end{aligned}$$

$$\begin{aligned}-\operatorname{Tr} *bbdR_z^N &= -\operatorname{Tr} bb * dR_z^N \\ &= -\overline{\operatorname{Tr}}\,(bb * dR_z^N)^* \\ &= \overline{\operatorname{Tr}}\, R_{\bar{z}}^N bb * d \\ &= \overline{\operatorname{Tr}}\, bb * dR_{\bar{z}}^N.\end{aligned}$$

The second term on the right-hand side of line (5.10) vanishes:

$$\begin{aligned} &-\operatorname{Tr} *dbbR_z^N + \operatorname{Tr} *bbdR_z^N \\ &= -\operatorname{Tr} bbR_z^N * d + \operatorname{Tr} *bbdR_z^N \\ &= -\operatorname{Tr} bb * dR_z^N + \operatorname{Tr} *bbdR_z^N \\ &= -\operatorname{Tr} *bbdR_z^N + \operatorname{Tr} *bbdR_z^N \\ &= 0. \end{aligned}$$

The same argument shows that the second term on line (5.11) also vanishes.

Now we have left to show that

$$\sum_{i+j=N+1} \left(\operatorname{Tr} *[d,b]R_z^i\{d,[d^*,b]\}R_z^j + \overline{\operatorname{Tr}} * [d,b]R_{\overline{z}}^i\{d,[d^*,b]\}R_{\overline{z}}^j\right) = 0.$$

In fact, we'll show that the (i,j)-summand is zero for every i,j. Expanding the three (anti-)commutators in the "Tr" term of (i,j)-summand gives eight terms, three of which are trivially zero since $d^2 = (d^*)^2 = 0$:

$$-\operatorname{Tr} *dbR_z^i dbd^* R_z^j = 0$$

$$-\operatorname{Tr} *bdR_z^i d[d^*,b]R_z^j = 0$$

The same argument shows that the corresponding three terms terms of the "$\overline{\operatorname{Tr}}$" term vanish. Also, we have

$$\begin{aligned} \operatorname{Tr} *dbR_z^i dd^* bR_z^j &= \overline{\operatorname{Tr}}\,(*dbR_z^i dd^* bR_z^j)^* \\ &= -\overline{\operatorname{Tr}}\, R_z^j bdd^* R_z^i b * d \\ &= -\overline{\operatorname{Tr}} * dbR_z^i dd^* bR_z^j \\ &= 0. \end{aligned}$$

We're left with

$$\begin{aligned}\operatorname{Tr}\{*d,b\}R_z^i[d^*,b]dR_z^j &= \operatorname{Tr} *dbR_z^i d^*bdR_z^j \\ &- \operatorname{Tr} *dbR_z^i bd^*dR_z^j \\ &+ \operatorname{Tr} b*dR_z^i d^*bdR_z^j \\ &- \operatorname{Tr} b*dR_z^i bd^*dR_z^j,\end{aligned}$$

plus the corresponding terms from the "$\overline{\operatorname{Tr}}$" term. The four terms on the right-hand side we denote by $A_{ij}, B_{ij}, C_{ij}, D_{ij}$, respectively; the corresponding terms from the "$\overline{\operatorname{Tr}}$" term we denote by $\overline{A}_{ij}, \overline{B}_{ij}, \overline{C}_{ij}, \overline{D}_{ij}$, respectively. (Note that our notation is *not* meant to suggest that $\overline{A}_{ij}$ is the complex conjugate of A_{ij}.) We compute

$$\begin{aligned}B_{ij} &= - \operatorname{Tr} *dbR_z^i bd^*dR_z^j \\ &= -\overline{\operatorname{Tr}}\,(*dbR_z^i bd^*dR_z^j)^* \\ &= \overline{\operatorname{Tr}}\, R_{\overline{z}}^j d^*dbR_{\overline{z}}^i b*d \\ &= -\overline{\operatorname{Tr}}\, b*dR_{\overline{z}}^j d^*dbR_{\overline{z}}^i \\ &= -\overline{\operatorname{Tr}}\, bR_{\overline{z}}^j d^*d*dbR_{\overline{z}}^i \\ &= \overline{\operatorname{Tr}}\, *dbR_{\overline{z}}^i bd^*dR_{\overline{z}}^j \\ &= -\overline{B}_{ij}\end{aligned}$$

where we've used that $*dd^* = d^*d*$.

$$\begin{aligned}D_{ij} &= - \operatorname{Tr} b*dR_z^i bd^*dR_z^j \\ &= -\overline{\operatorname{Tr}}\,(b*dR_z^i bd^*dR_z^j)^* \\ &= \overline{\operatorname{Tr}}\, R_{\overline{z}}^j d^*dbR_{\overline{z}}^i *db \\ &= \overline{\operatorname{Tr}}\, bR_{\overline{z}}^i *dbR_{\overline{z}}^j d^*d \\ &= \overline{\operatorname{Tr}}\, b*dR_{\overline{z}}^i bd^*dR_{\overline{z}}^j \\ &= -\overline{D}_{ij}.\end{aligned}$$

$$\begin{aligned}
A_{ij} &= \operatorname{Tr} *dbR^i_z d^* bdR^j_z \\
&= \overline{\operatorname{Tr}}\,(*dbR^i_z d^* bdR^j_z)^* \\
&= -\overline{\operatorname{Tr}}\, R^j_{\overline{z}} d^* bdR^i_{\overline{z}} b * d \\
&= \overline{\operatorname{Tr}}\, b * dR^j_{\overline{z}} d^* bdR^i_{\overline{z}} \\
&= \overline{C}_{ji}.
\end{aligned}$$

On the other hand,

$$\begin{aligned}
A_{ij} &= -\operatorname{Tr} R^j_z d^* bdR^i_z b * d \\
&= -\operatorname{Tr} R^j_z bdR^i_z b * dd^* \\
&= -\operatorname{Tr} R^j_z bdR^i_z bd^* d* \\
&= -\operatorname{Tr} *R^j_z bdR^i_z bd^* d \\
&= \operatorname{Tr} b * dR^i_z bd^* dR^j_z \\
&= -D_{ij}
\end{aligned}$$

The same arguments show $\overline{A}_{ij} = C_{ji} = -\overline{D}_{ij}$. Since $D_{ij} + \overline{D}_{ij}$ vanishes, so do $A_{ij} + \overline{A}_{ij}$ and $\overline{C}_{ji} + C_{ji}$. This concludes the proof of Theorem 5.3.1.

5.4 Proof of the metric variation formula

In this section, we will prove Theorem 5.1.5, which we restate for convenience:

Theorem 5.4.1. *For a smooth one-parameter family of Riemannian metrics $g(u)$, the derivative of the eta invariant is*

$$\frac{d}{du}\eta(M, F, g(u)) = \frac{2}{\Gamma\left(\frac{1}{2}\right)} a_{-1/2}(g(u)), \tag{5.12}$$

where $a_{-1/2}(g(u))$ denotes the $t^{-1/2}$ coefficient in the $t \to 0$ asymptotic expansion of $\operatorname{Tr} * \left[d, *^{-1}\frac{d*}{du}\right] e^{-t\Delta^{g(u)}}$.

Proof. Suppose $g(u)$ is a smooth family of metrics for $u \in [0,1]$. Our goal will be to prove

$$\eta(M, F, g(1)) - \eta(M, F, g(0)) = \frac{1}{2\Gamma\left(\frac{1}{2}\right)} \int_0^1 a_{1/2}(g(u))\, du, \tag{5.13}$$

which suffices to prove (5.12).

For $A > \epsilon > 0$, let $\Sigma = \Sigma_{A,\epsilon}$ denote the surface with boundary in $\mathcal{M}^{TM}$ parametrized by $(u,t) \mapsto g(u)_t$, for $u \in [0,1]$ and $t \in [\epsilon, A]$. (We use the subscript t as in Subsection 3.4.2.) The boundary of Σ consists of four curves, described by $u = 0$, $u = 1$, $t = \epsilon$, and $t = A$. We compute the following operators pulled back to Σ:

$$\begin{aligned} *(u,t) &= t^{Q-\frac{n}{2}} *_{g(u)} \\ b &= *^{-1}\delta * \\ &= \left(Q - \frac{n}{2} I\right) \frac{dt}{t} + *_{g(u)}^{-1} \frac{d}{du} *_{g(u)}\, du \\ [d,b] &= -d\frac{dt}{t} + \left[d, *_{g(u)}^{-1} \frac{d}{du} *_{g(u)}\right] du \end{aligned}$$

This implies that ω_η pulled back to $\partial\Sigma$ is

$$\begin{aligned} \omega_\eta = &- t^{\frac{1}{2}} \operatorname{Tr} *_{g(u)} d\, e^{-t\Delta^g_{2k-1}} \frac{dt}{t} \\ &+ t^{\frac{1}{2}} \operatorname{Tr} *_{g(u)} \left[d, *_{g(u)}^{-1} \frac{d}{du} *_{g(u)}\right] e^{-t\Delta^g_{2k-1}} du \end{aligned}$$

Since ω_η is closed, $\delta^{\mathcal{M}}\left(t^{\frac{s}{2}}\omega_\eta\right)$ pulled back to Σ is

$$\begin{aligned} \delta^{\mathcal{M}}\left(t^{\frac{s}{2}}\omega_\eta\right) &= \frac{s}{2} t^{\frac{s}{2}-1}\, dt\, \omega \\ &= \frac{s}{2} t^{\frac{s}{2}} t^{-\frac{1}{2}} \operatorname{Tr} *_{g(u)} \left[d, *_{g(u)}^{-1} \frac{d}{du} *_{g(u)}\right] e^{-t\Delta^g_{2k-1}} dt du. \end{aligned}$$

(Recall that only forms of degree $2k-1$ contribute to the trace.)

Assume $\operatorname{Re} s$ is large. We apply Stokes' theorem to obtain

$$\iint_{\Sigma} \delta^{\mathcal{M}}\left(t^{\frac{s}{2}}\omega_\eta\right) = \int_{\partial\Sigma} t^{\frac{s}{2}}\omega_\eta .$$

The integral over Σ is

$$\begin{aligned}
&\iint_{\Sigma} \delta^{\mathcal{M}}\left(t^{\frac{s}{2}}\omega_\eta\right) \\
&= \int_0^1 \int_\epsilon^A \frac{s}{2} t^{\frac{s}{2}} t^{-\frac{1}{2}} \operatorname{Tr} *_{g(u)} \left[d, *_{g(u)}^{-1} \frac{d}{du} *_{g(u)}\right] e^{-t\Delta^g_{2k-1}} dt du.
\end{aligned}$$

The integral over $\partial\Sigma$ consists of four terms:

$$\begin{aligned}
&\int_{\partial\Sigma} t^{\frac{s}{2}}\omega_\eta \\
&\quad = \int_\epsilon^A t^{\frac{s}{2}} (-) t^{\frac{1}{2}} \operatorname{Tr} *_{g(1)} d\, e^{-t\Delta^g(1)_{2k-1}} \frac{dt}{t} \\
&\quad - \int_\epsilon^A t^{\frac{s}{2}} (-) t^{\frac{1}{2}} \operatorname{Tr} *_{g(0)} d\, e^{-t\Delta^g(0)_{2k-1}} \frac{dt}{t} \\
&\quad + \int_0^1 \epsilon^{\frac{s}{2}} \epsilon^{\frac{1}{2}} \operatorname{Tr} *_{g(u)} \left[d, *_{g(u)}^{-1} \frac{d}{du} *_{g(u)}\right] e^{-\epsilon\Delta^g_{2k-1}} du \qquad (5.14) \\
&\quad - \int_0^1 A^{\frac{s}{2}} A^{\frac{1}{2}} \operatorname{Tr} *_{g(u)} \left[d, *_{g(u)}^{-1} \frac{d}{du} *_{g(u)}\right] e^{-A\Delta^g_{2k-1}} du. \qquad (5.15)
\end{aligned}$$

In the limits $\epsilon \to 0$ and $A \to \infty$, the terms (5.14) and (5.15) vanish by the heat kernel estimates. Thus we obtain

$$\begin{aligned}
&\int_0^1 \int_0^\infty \frac{s}{2} t^{\frac{s}{2}} t^{-\frac{1}{2}} \operatorname{Tr} *_{g(u)} \left[d, *_{g(u)}^{-1} \frac{d}{du} *_{g(u)}\right] e^{-t\Delta^g_{2k-1}} dt du \\
&= \int_0^\infty t^{\frac{s}{2}} (-) t^{\frac{1}{2}} \operatorname{Tr} *_{g(1)} d\, e^{-t\Delta^g(1)_{2k-1}} \frac{dt}{t} \\
&\quad - \int_0^\infty t^{\frac{s}{2}} (-) t^{\frac{1}{2}} \operatorname{Tr} *_{g(0)} d\, e^{-t\Delta^g(0)_{2k-1}} \frac{dt}{t}.
\end{aligned}$$

Dividing by $\Gamma\left(\frac{s+1}{2}\right)$ gives the equality

$$\int_0^1 \frac{s}{2}\zeta\left(\frac{s+1}{2};\Delta^{g(u)}, *_{g(u)}\left[d, *_{g(u)}^{-1}\frac{d}{du}*_{g(u)}\right]\right) du = \eta(s;B^{g(1)}) - \eta(s;B^{g(1)}).$$

This equality holds for all s by the uniqueness of the analytic continuations.

We now evaluate at $s = 0$. The right-hand side gives the difference in eta invariants $\eta(M, F, g(1)) - \eta(M, F, g(0))$. The left-hand side gives, because of the factor of s, the residue at $s = 0$ of the function

$$\int_0^1 \frac{1}{2}\zeta\left(s;B^{g(u)}, *_{g(u)}\left[d, *_{g(u)}^{-1}\frac{d}{du}*_{g(u)}\right]\right) du. \tag{5.16}$$

By Lemma 5.1.3, the residue of the integrand at $s = 0$ is precisely $\frac{1}{\Gamma(1/2)}$ times the $t^{-\frac{1}{2}}$ coefficient in the small t asymptotic expansion of $\mathrm{Tr} * \left[d, *^{-1}\frac{d*}{du}\right] e^{-t\Delta^{g(u)}}$. This proves (5.13).

□

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Biography

Phillip Andreae was born December 9, 1988, in Knoxville, Tennessee. In 2006, he graduated from Bearden High School. In 2010, he earned a B.S. with highest honors in mathematics and physics from Emory University. In 2016, he earned a Ph.D. in mathematics from Duke University. Beginning in August 2016, Phillip will be an assistant professor of mathematics at Meredith College in Raleigh, North Carolina.