DISCRETE AND CONTINUOUS
DYNAMICAL SYSTEMS SERIES S
Volume **8**, Number **1**, February **2015**

doi:10.3934/dcdss.2015.8.55

pp. **55–76**

HOMOGENIZATION OF THERMAL-HYDRO-MASS TRANSFER PROCESSES

Shixin Xu

Department of Mathematics, Soochow University
Suzhou 215006, China
and
Department of Mathematics, University of Science and Technology of China
Hefei 230026, China

Xingye Yue

Department of Mathematics, Soochow University
Suzhou 215006, China

Abstract. In the repository, multi-physics processes are induced due to the long-time heat-emitting from the nuclear waste, which is modeled as a nonlinear system with oscillating coefficients. In this paper we first derive the homogenized system for the thermal-hydro-mass transfer processes by the technique of two-scale convergence, then present some error estimates for the first order expansions.

1. **Introduction.** Accompanied by the developing of the nuclear power, more and more nuclear waste is produced. The half-life period of radioactivity of nuclear waste is usual very long, especially for the high radioactive waste, which may be several tens of thousands years long. The safe disposal of nuclear waste is an important problem.

Due to the long-term heat emitted by radioactivity, the rock and the underwater in the repository will be heated up. Since the water and rock have different thermal expansivity, thermal input may cause significant pore pressure change which will induce convective flow in porous media. The temperature builds up to a certain level and then decreases. It will take 15-100 years to attain the peak of temperature near the waste repository and 200-1000 years for the far field. That means the thermal-hydro-mechanical processes will last a very long time. A lot of researches have been done to the thermal-hydro-mechanical processes (see [6, 7, 14, 21, 23, 24, 25]). There also exits the possibility of the leakage of nuclear waste. If so, the nuclear waste will be dissolved in water and transferred to the far field by the underwater flow. Then the diffusion-convection processes of radioactive nuclear waste must be considered and it leads to the thermal-hydro-mass transfer processes in porous media. In order to describe these processes mathematically, We first show some notations that will be used.

Suppose a cubic domain $\Omega \subset R^3$ is occupied by the porous media which is adjacent to the waste at the left boundary Γ_1, through which the heat and waste comes into the porous media.

2010 *Mathematics Subject Classification.* Primary: 35B27, 35B40; Secondary: 35M30.

Key words and phrases. Homogenization, two-scale convergence, first order expansion, thermal-hydro-mass transfer processes.

This work is supported in part by NSF of China under the Grants 10871190 and 11271281.

ρ_s	density of solid	$\widetilde{D}_T^\varepsilon$	heat conductivity coefficient
c_s	specific heat of solid	K_ε	permeability coefficient
ρ_f	density of fluid	$\widetilde{D}_c^\varepsilon$	diffusion coefficient of waste in porous media
c_f	specific heat of fluid	β_1	heat exchange coefficient
ϕ_ε	porosity	β_2	mass exchange coefficient
v_ε	Darcy velocity of fluid	T_{out}	temperature outside left boundary
T_ε	temperature	C_{out}	concentration of waste outside left boundary
p_ε	pressure	T_r	reference temperature (constant)
C_ε	concentration of waste	ρ_0	initial density of fluid

TABLE 1. Notations. Here $0 < \varepsilon \ll 1$ is the ratio of the characteristic length of micro scale to the whole field

By the conservation of energy, the thermal process is governed by

$$\frac{\partial}{\partial t}(T_\varepsilon \overline{\rho c}) - \nabla \cdot (\widetilde{D}_T^\varepsilon \nabla T_\varepsilon) = 0, \tag{1}$$

with $\overline{\rho c} = \rho_s c_s + \phi_\varepsilon(\rho_f c_f - \rho_s c_s)$.

By the conservation law of mass of fluid, we have

$$\frac{\partial(\rho_f \phi_\varepsilon)}{\partial t} + \nabla \cdot (v_\varepsilon \rho_f) = 0. \tag{2}$$

By the conservation law of mass of nuclear waste, the mass transfer process is controlled by

$$\frac{\partial}{\partial t}(\rho_f C_\varepsilon \phi_\varepsilon) - \nabla \cdot (\widetilde{D}_c^\varepsilon \nabla C_\varepsilon) + \nabla \cdot (v_\varepsilon \rho_f C_\varepsilon) = 0, \tag{3}$$

As a constitution relation, the Darcy's law is needed:

$$v_\varepsilon = -K_\varepsilon \nabla p_\varepsilon. \tag{4}$$

To close the above model, we still need the following state equation by Boussinesq approximation to the density of fluid ([11]):

$$\rho_f = \rho_0(1 - \alpha(T_\varepsilon - T_r)), \tag{5}$$

where ρ_0 is constant standing for the density of fluid at initial temperature T_r and α is the thermal expansion coefficient of water. In reality, α will be very small, i.e. $0 < \alpha \ll 1$ and the thermal expansion coefficient of rock matrix is even much smaller than α. And note that the small change of the density of fluid only has significant effect on the pressure of fluid, not directly on the temperature and concentration. So we take some assumptions on the density:

1. ρ_s is a constant in this study;
2. $\rho_f = \rho_0$ in equations (1) and (3).

Then the thermal-hydro-mass transport processes can be described as:

$$\begin{cases} (1 + \phi_\varepsilon \gamma)\frac{\partial T_\varepsilon}{\partial t} - \nabla \cdot (D_T^\varepsilon \nabla T_\varepsilon) = 0, & \text{in } \Omega_T, \\ -\nabla \cdot ((1 + \alpha(T_r - T_\varepsilon))K^\varepsilon \nabla p_\varepsilon) = \alpha \phi_\varepsilon \frac{\partial T_\varepsilon}{\partial t}, & \text{in } \Omega_T, \\ \frac{\partial C_\varepsilon}{\partial t}\phi_\varepsilon - \nabla \cdot (D_c^\varepsilon \nabla C_\varepsilon) - \nabla \cdot (K^\varepsilon \nabla p_\varepsilon C_\varepsilon) = 0, & \text{in } \Omega_T, \end{cases} \tag{6}$$

with $\Omega_T = \Omega \times (0, T]$, $\gamma = \dfrac{\rho_0 c_f - \rho_s c_s}{\rho_s c_s}$, $D_T^\varepsilon = \dfrac{\widetilde{D}_T^\varepsilon}{\rho_s c_s}$ and $D_c^\varepsilon = \dfrac{\widetilde{D}_c^\varepsilon}{\rho_0 c_f}$.

As to the boundary conditions, Robin boundary conditions for T_ε and C_ε are assumed on Γ_1 due to the heat dissipation and leakage of waste. For pressure p_ε, we assume that the fluid is impermeable on Γ_1. On the right boundary Γ_2, which is far away from the waste, the Dirichlet boundary conditions are applied. On the other boundaries $\Gamma_3 = \partial\Omega \setminus (\Gamma_1 \cup \Gamma_2)$, impermeable conditions for T_ε, C_ε and p_ε are imposed. We also need the initial values of T_ε and C_ε, which we assume to be zero, the same as the far field data.

$$\begin{cases} (D_T^\varepsilon \nabla T_\varepsilon)\cdot\nu = \beta_1(T_{out} - T_\varepsilon),\ (K^\varepsilon\nabla p_\varepsilon)\cdot\nu = 0, & \text{on } \Gamma_1\times(0,T], \\ (D_c^\varepsilon \nabla C_\varepsilon)\cdot\nu = \beta_2(C_{out} - C_\varepsilon), & \text{on } \Gamma_1\times(0,T], \\ T_\varepsilon = p_\varepsilon = C_\varepsilon = 0, & \text{on } \Gamma_2\times(0,T], \\ (D_T^\varepsilon \nabla T_\varepsilon)\cdot\nu = (K^\varepsilon\nabla p_\varepsilon)\cdot\nu = (D_c^\varepsilon \nabla C_\varepsilon)\cdot\nu = 0, & \text{on}\Gamma_3\times(0,T], \\ T_\varepsilon(x,0)=0,\ C_\varepsilon(x,0)=0, & \text{in } \Omega, \end{cases} \tag{7}$$

where ν is the unit outward normal vector.

System (6) is a nonlinear partial differential system with high oscillating coefficients. From a numerical point of view, resolving the microscopic details of (6) using typical numerical methods would require at least a cost of $O(\varepsilon^{-n})$ (n=3) or more. This often becomes prohibitively expensive since $\varepsilon \ll 1$. One way of avoiding this is to solve instead the homogenized equation of the problem (6). The general theory of homogenization can be found in [4, 5, 10, 26] for simple model problems. This paper is devoted to establish the homogenization theory of thermal-hydro-mass transfer processes (6). Two-scale convergence method is employed to deal with the coupled nonlinear terms and the weak convergence for p_ε. Two-scale convergence was first introduced by G. Allaire [1, 2] and G. Nguetseng [18]. In [3, 15], two-scale convergence for time dependent problem was considered. At the same time, the error estimate between the solutions of original problem and their first order expansions is also very import in homogeneous theory. The usual error estimate method for elliptic problems can be found in [10] and [26]. In order to reduce the regularity assumptions for the homogenized problems, the skew-symmetric matrix technique [26] and boundary correctors are always used. For parabolic problems, the initial value corrector is also needed [8]. For problem (6)-(7), it is more complicate since the mixed Robbin-Dirichlet boundary conditions are imposed. Following the idea of [9], we derive the error estimates between the solutions of problem (6)-(7) and their first order expansions. There is one thing also worth pointing out that the homogenized behavior of Equation (3) is different with a single diffusion-convection equation with a prescribed multiscale velocity field. For a single diffusion-convection equation with a prescribed multiscale velocity field, the homogenized velocity is determined not only by micro velocity itself, but also by the micro diffusion coefficient [26]. But here in a system, the homogenized velocity is only determined by the micro pressure equation (2), not related with the micro diffusion coefficient $\widetilde{D}_c^\varepsilon$.

Throughout the paper, we also use some mathematical notations such as $W^{k,p}(\Omega)$, $H^k(\Omega) = W^{k,2}(\Omega)$ for general Sobolev spaces, $V = \{v \in H^1(\Omega), v = 0, \text{ on } \Gamma_2\}$, $W = \{v \in L^2(0,T;V); \frac{\partial v}{\partial t} \in L^2(0,T;H^{-1}(\Omega))\}$ and $C^\infty_{\Gamma_2}(\Omega) = \{v \in C^\infty(\Omega);\ v = 0 \text{ on } \Gamma_2\}$. $C > 0$ is a general constant independent of ε, which may be different at different occurrences. Einstein summation convention is also applied. The outline of paper is as follows: in §2 we first get some a priori estimates of the system (6);

in §3 we use two-scale method to derive the homogenized system; in §4 we present some error estimates between the solutions of (6) and their first order expansions. As to the multiscale numerical method for problem (6), we will show it in another paper.

2. **A priori estimates.** In this section, some a priori estimates of the problem (6) will be derived. We need some assumptions on the coefficients in order to ensure the regularity.

- (**H**0): The exchange coefficients β_1 and β_2, the density ρ_s, the specific heats c_s and c_f are positive constants. The conductive coefficients D_T^ε, D_c^ε and permeability coefficient K^ε are in the forms of $D_T^\varepsilon(x) = D_T(x, \frac{x}{\varepsilon})$, $D_c^\varepsilon(x) = D_c(x, \frac{x}{\varepsilon})$, $K^\varepsilon(x) = K(x, \frac{x}{\varepsilon})$ and $D_T(x,y)$, $K(x,y)$, $D_c(x,y)$ satisfy that
  - symmetric,
  - continuous and periodic with respect to $y \in Y = [0,1]^n$,
  - uniformly elliptic, i.e. there exist positive constants λ_i and Λ_i $(i = 1,2,3)$, such that

$$\begin{cases} \lambda_1|\xi|^2 \le \xi^T D_T(x,y)\xi \le \Lambda_1|\xi|^2, & \forall \xi \in R^n, x \in \Omega, y \in Y, \\ \lambda_2|\xi|^2 \le \xi^T K(x,y)\xi \le \Lambda_2|\xi|^2, & \forall \xi \in R^n, x \in \Omega, y \in Y, \\ \lambda_3|\xi|^2 \le \xi^T D_c(x,y)\xi \le \Lambda_3|\xi|^2, & \forall \xi \in R^n, x \in \Omega, y \in Y. \end{cases} \tag{8}$$

  For porosity ϕ_ε, we give the following assumption,
- (**H**1): $0 < c \le \phi^\varepsilon(x) \le 1, \phi^\varepsilon = \phi(x, \frac{x}{\varepsilon})$ and $\phi(x,y)$ is continuous and periodic with respect to $y \in Y = [0,1]^n$.

So we have $\phi^\varepsilon \rightharpoonup \phi_0(x) = \int_Y \phi(x,y)dy$ weakly $*$ in $L^\infty(\Omega)$.

The system (6) is not fully coupled in the sense that one computes first T_ε, then p_ε and finally C_ε. By the weak maximum principle [16], there exists a constant $C > 0$ such that

$$\|T_\varepsilon\|_{L^\infty(\Omega_T)} + \|C_\varepsilon\|_{L^\infty(\Omega_T)} \le C. \tag{9}$$

By the basic theory of elliptic and parabolic equations [12, 13, 16], we can get the existence and uniqueness of solutions,

Theorem 2.1. *Suppose hypotheses (**H**0), (**H**1) hold. If T_{out}, $C_{out} \in L^2((0,T)\times\Gamma_1)$ and α is sufficiently small, then there exist uniqueness solutions $T_\varepsilon \in W$, $p_\varepsilon \in L^2(0,T;V)$ and $C_\varepsilon \in W$ to system (6) and (7).*

Next we show some a priori estimates for the solutions.

Theorem 2.2. *Let T_ε, p_ε and C_ε be the solutions of problem (6) and (7). If (**H**0) and (**H**1) hold, $T_{out} \in L^\infty(0,T;L^2(\Gamma_1))$, $\frac{\partial T_{out}}{\partial t} \in L^2(0,T;H^{-\frac12}(\Gamma_1))$, $C_{out} \in L^2((0,T)\times\Gamma_1)$ and α is sufficiently small, then there exists a constant $C > 0$ independent of ε, such that*

$$\begin{aligned} &\|\frac{\partial T_\varepsilon}{\partial t}\|_{L^2(\Omega_T)} + \sup_{0\le t\le T}\|T_\varepsilon\|_{H^1(\Omega)} \\ &\quad\le C\left(\|T_{out}\|_{L^\infty(0,T;L^2(\Gamma_1))} + \|\frac{\partial T_{out}}{\partial t}\|_{L^2(0,T;H^{-\frac12}(\Gamma_1))}\right), \\ &\|p_\varepsilon\|_{L^2(0,T;V)} \le C\left(\|T_{out}\|_{L^\infty(0,T;L^2(\Gamma_1))} + \|\frac{\partial T_{out}}{\partial t}\|_{L^2(0,T;H^{-\frac12}(\Gamma_1))}\right), \end{aligned}$$

$$\|\phi_\varepsilon \frac{\partial C_\varepsilon}{\partial t}\|_{L^2(0,T;H^{-1}(\Omega))} + \sup_{0\le t\le T}\|C_\varepsilon\|_{L^2(\Omega)} + \|\nabla C_\varepsilon\|_{L^2(\Omega_T)}$$
$$\le C\left(\|C_{out}\|_{L^2((0,T)\times\Gamma_1)} + \|T_{out}\|_{L^\infty(0,T;L^2(\Gamma_1))} + \|\frac{\partial T_{out}}{\partial t}\|_{L^2(0,T;H^{-\frac{1}{2}}(\Gamma_1))}\right).$$

Consequently, we get that T_ε is bounded in W. By Aubin-Lions lemma, there exists a compact injection $W \subset L^2(\Omega_T)$ [10], so there exists a subsequence of T_ε converges strongly in $L^2(\Omega_T)$. C_ε is bounded in $L^\infty(0,T;L^2(\Omega)) \cap L^2(0,T;V)$ and $\phi_\varepsilon \frac{\partial C_\varepsilon}{\partial t}$ is bounded in $L^2(0,T;H^{-1}(\Omega))$. By a variant of Aubin-Lions lemma [20, 17], we also have a subsequence of C_ε converging strongly in $L^2(\Omega_T)$. In conclusion, we get the following lemma.

Lemma 2.3. *Let T_ε, p_ε and C_ε be the solutions of problem (6). Under the same conditions of Theorem 2.2, there exist T_0, C_0 and $p_0 \in L^2(0,T;V)$ such that up to a subsequence,*

$$\begin{cases} (1)\ T_\varepsilon \rightharpoonup T_0(x,t) & \textit{weakly in } L^2(0,T;V),\\ (2)\ \frac{\partial T_\varepsilon}{\partial t} \rightharpoonup \frac{\partial T_0}{\partial t} & \textit{weakly in } L^2(0,T;H^{-1}(\Omega)),\\ (3)\ T_\varepsilon(t,x) \to T_0 & \textit{strongly in } L^2(\Omega_T),\\ (4)\ C_\varepsilon \rightharpoonup C_0(x,t) & \textit{weakly in } L^2(0,T;V),\\ (5)\ C_\varepsilon(t,x) \to C_0 & \textit{strongly in } L^2(\Omega_T),\\ (6)\ p_\varepsilon \rightharpoonup p_0(x,t) & \textit{weakly in } L^2(0,T;V). \end{cases} \tag{10}$$

Remark 1. Please note that the temperature T_ε and concentration C_ε converge strongly in $L^2(\Omega_T)$ while the pressure p_ε only converges weakly in $L^2(0,T;V)$ since there is no information on its derivative with respect to time. This leads to some difficulties to deal with the nonlinear coupled terms, which can be overcome by the method of two-scale convergence.

Before we prove Theorem 2.2, we first present the weak forms of system (6) and (7), which will be needed later. Find $T_\varepsilon \in W$, $p_\varepsilon \in L^2(0,T;V)$ and $C_\varepsilon \in W$ such that

$$\langle(1+\phi_\varepsilon\gamma)\frac{\partial T_\varepsilon}{\partial t}, v\rangle_{H^{-1}(\Omega),H^1(\Omega)} + \int_\Omega D_T^\varepsilon \nabla T_\varepsilon \nabla v dx + \int_{\Gamma_1} \beta_1 T_\varepsilon v ds$$
$$= \int_{\Gamma_1} \beta_1 T_{out} v ds, \quad \forall v \in V; \tag{11}$$

$$\int_\Omega K^\varepsilon \nabla p_\varepsilon \nabla v dx = \alpha \int_\Omega \frac{\partial T_\varepsilon}{\partial t}\phi_\varepsilon v dx + \alpha \int_\Omega (T_\varepsilon - T_r) K^\varepsilon \nabla p_\varepsilon \nabla v dx,\ \forall v \in V; \tag{12}$$

$$\langle \phi_\varepsilon \frac{\partial C_\varepsilon}{\partial t}, v\rangle_{H^{-1}(\Omega),H^1(\Omega)} + \int_\Omega D_c^\varepsilon \nabla C_\varepsilon \nabla v dx + \int_\Omega K^\varepsilon \nabla p_\varepsilon C_\varepsilon \nabla v dx$$
$$= \beta_2 \int_{\Gamma_1} (C_{out} - C_\varepsilon) v ds,\ \forall v \in V. \tag{13}$$

Proof of Theorem 2.2. **Step 1.** Taking $v = T_\varepsilon(\cdot,t)$ in (11) and using (H0) and (H1), we have

$$\frac{d}{dt}\|\sqrt{1+\phi_\varepsilon\gamma}T_\varepsilon\|^2_{L^2(\Omega)} + \lambda_1\|\nabla T_\varepsilon\|^2_{L^2(\Omega)} + \beta_1\|T_\varepsilon\|^2_{L^2(\Gamma_1)} \le C\|T_{out}\|^2_{L^2(\Gamma_1)}. \tag{14}$$

Here we have used Trace's Theorem, Hölder's inequality and Poincaré's inequality for $T_\varepsilon(\cdot,t) \in V$ in the above estimate. Integrating over $(0,t)$ with $t \in [0,T]$ on both sides, we have that

$$\sup_{0\le t\le T}\|T_\varepsilon\|^2_{L^2(\Omega)}+\|\nabla T_\varepsilon\|^2_{L^2(\Omega_T)}+\|T_\varepsilon\|^2_{L^2((0,T)\times\Gamma_1)}\le C\|T_{out}\|^2_{L^2((0,T)\times\Gamma_1)}. \tag{15}$$

Multiplying the first equation of system (6) by $\frac{\partial T_\varepsilon}{\partial t}$ and integrating by part, it yields

$$\begin{aligned}\int_\Omega(1+\phi_\varepsilon\gamma)|\frac{\partial T_\varepsilon}{\partial t}|^2dx+\frac{1}{2}\frac{d}{dt}\int_\Omega D_T^\varepsilon|\nabla T_\varepsilon|^2dx+\frac{\beta_1}{2}\frac{d}{dt}\int_{\Gamma_1}|T_\varepsilon|^2ds\\ =\beta_1\int_{\Gamma_1}T_{out}\frac{\partial T_\varepsilon}{\partial t}dx\\ =\beta_1\frac{d}{dt}\left(\int_{\Gamma_1}T_{out}T_\varepsilon ds\right)-\beta_1\langle\frac{\partial T_{out}}{\partial t},T_\varepsilon\rangle_{H^{-\frac12}(\Gamma_1),H^{\frac12}(\Gamma_1)}.\end{aligned} \tag{16}$$

Integrating over $(0,t)$ with $t\in[0,T]$ and using the Trace Theorem, it follows

$$\begin{aligned}&\|\tfrac{\partial T_\varepsilon}{\partial t}\|_{L^2(\Omega_T)}+\sup_{0\le t\le T}\|T_\varepsilon\|_{H^1(\Omega)}\\ \le\ & C\left(\|T_{out}\|_{L^\infty(0,T;L^2(\Gamma_1))}+\|\tfrac{\partial T_{out}}{\partial t}\|_{L^2(0,T;H^{-\frac12}(\Gamma_1))}\right).\end{aligned} \tag{17}$$

Step 2. Estimate $\|\nabla p_\varepsilon\|_{L^2(\Omega_T)}$. Multiplying p_ε on both sides of the second equation of (6), we have by integration by parts

$$\int_\Omega K^\varepsilon\nabla p_\varepsilon\nabla p_\varepsilon dx=\alpha\int_\Omega\frac{\partial T_\varepsilon}{\partial t}\phi_\varepsilon p_\varepsilon dx+\alpha\int_\Omega(T_\varepsilon-T_r)K_\varepsilon\nabla p_\varepsilon\nabla p_\varepsilon dx.$$

Thanks to the estimate of (H0) and (H1), we obtain

$$\begin{aligned}\lambda_2\|\nabla p_\varepsilon\|^2_{L^2(\Omega)}\ \le\ &\alpha\|\frac{\partial T_\varepsilon}{\partial t}\|_{L^2(\Omega)}\|p_\varepsilon\|_{L^2(\Omega)}\\ &+\alpha\Lambda_2(\|T_r\|_{L^\infty(\Omega)}+\|T_\varepsilon\|_{L^\infty(\Omega)})\|\nabla p_\varepsilon\|^2_{L^2(\Omega)}.\end{aligned}$$

For sufficiently small $\alpha>0$, $\alpha\Lambda_2(\|T_r\|_{L^\infty(\Omega)}+\|T_\varepsilon\|_{L^\infty(\Omega)})\le\frac{\lambda_2}{2}$ and it follows from the Poincare's inequality and (17) that

$$\begin{aligned}\|\nabla p_\varepsilon\|^2_{L^2(\Omega_T)}\ &\le\ C\left(\frac{\alpha}{\lambda_2}\right)^2\|\frac{\partial T_\varepsilon}{\partial t}\|^2_{L^2(\Omega_T)}\\ &\le\ C\left(\|T_{out}\|_{L^\infty(0,T;L^2(\Gamma_1))}+\|\frac{\partial T_{out}}{\partial t}\|_{L^2(0,T;H^{-\frac12}(\Gamma_1))}\right).\end{aligned} \tag{18}$$

So the second inequality of Theorem 2.2 is proved.

Step 3. Estimate C_ε. Multiplying C_ε on both sides of the third equation of (6) and integrating by parts, it yields,

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\int_\Omega\phi_\varepsilon|C_\varepsilon|^2dx+\int_\Omega D_c^\varepsilon|\nabla C_\varepsilon|^2dx+\beta_2\int_{\Gamma_1}|C_\varepsilon|^2dx\\ =\ &\beta_2\int_{\Gamma_1}C_{out}C_\varepsilon ds+\int_\Omega K^\varepsilon\nabla p_\varepsilon C_\varepsilon\nabla C_\varepsilon dx\\ \le\ &C\|C_{out}\|^2_{L^2(\Gamma_1)}+\frac{\beta_2}{2}\|C_\varepsilon\|^2_{L^2(\Gamma_1)}+C\|\nabla p_\varepsilon\|^2_{L^2(\Omega)}+\frac{\lambda_3}{2}\|\nabla C_\varepsilon\|^2_{L^2(\Omega)},\end{aligned} \tag{19}$$

where we have used the fact that $\|C_\varepsilon\|_{L^\infty}$ is uniformly bounded from the maximum principle (9). Integrating over $(0,t)$ with $t\in[0,T]$ on both sides and combining (18), we have that

$$\begin{aligned}&\sup_{0\le t\le T}\|C_\varepsilon\|_{L^2(\Omega)}+\|\nabla C_\varepsilon\|_{L^2(\Omega_T)}\\ \le\ & C\left(\|T_{out}\|_{L^\infty(0,T;L^2(\Gamma_1))}+\|\tfrac{\partial T_{out}}{\partial t}\|_{L^2(0,T;H^{-\frac12}(\Gamma_1))}+\|C_{out}\|_{L^2((0,T)\times\Gamma_1)}\right).\end{aligned}$$

Combining the weak form of C_ε(13) and estimate (18), we can get the H^{-1} norm estimate of $\phi_\varepsilon \frac{\partial C_\varepsilon}{\partial t}$,

$$\begin{aligned} &\|\phi_\varepsilon \frac{\partial C_\varepsilon}{\partial t}\|_{L^2(0,T;H^{-1}(\Omega))} \\ \leq\ & C\left(\|T_{out}\|_{L^\infty(0,T;L^2(\Gamma_1))} + \|\frac{\partial T_{out}}{\partial t}\|_{L^2(0,T;H^{-\frac{1}{2}}(\Gamma_1))} + \|C_{out}\|_{L^2((0,T)\times\Gamma_1)}\right). \end{aligned}$$

The proof is completed. □

3. **Homogenization.**

In this section, we will derive the homogenized equations for the limit T_0, C_0 and p_0 by the method of two-scale convergence for time dependent problem. Before stating the main results, we first review the concept on two-scale convergence.

Definition 3.1. ([2, 22])A sequence $u_\varepsilon(t,x) \in L^2(\Omega_T)$ two-scale converges to $u(t,x,y) \in L^2(\Omega_T \times Y)$ and we write $u_\varepsilon \overset{2}{\rightharpoonup} u_0(t,x,y)$, if for $\forall\ \varphi(t,x,y) \in L^2(\Omega_T, C_{per}(Y))$, we have

$$\lim_{\varepsilon\to 0}\int_0^T\int_\Omega u_\varepsilon(t,x)\varphi(t,x,\frac{x}{\varepsilon})dxdt = \int_0^T\int_\Omega\int_Y u_0(t,x,y)\varphi(t,x,y)dydxdt.$$

Proposition 1. *([22])*

- *Let $u_\varepsilon(t,x)$ be a bounded sequence in $L^2(\Omega_T)$. Then there exists a subsequence, still denoted by u_ε, and a function $u_0 \in L^2(\Omega_T \times Y)$ such that*

$$u_\varepsilon \overset{2}{\rightharpoonup} u_0(t,x,y).$$

  Moreover, u_ε converges weakly in $L^2(\Omega_T)$ to the average of the two-scale limit

$$u_\varepsilon \rightharpoonup v(t,x) = \int_Y u_0(t,x,y) \quad \textit{weakly in } L^2(\Omega_T).$$

- *Let u_ε be a bounded sequence in $L^2(0,T;V)$ such that*

$$u_\varepsilon \rightharpoonup u_0(x,t) \quad \textit{weakly in } L^2(0,T;V).$$

  Then u_ε two-scale converges to u_0 in $L^2(\Omega_T)$. In addition there exists a $u_1 \in L^2(\Omega_T; H_{per}(Y))$, up to a subsequence, still denoted by u_ε, such that

$$\nabla u_\varepsilon \overset{2}{\rightharpoonup} \nabla_x u_0 + \nabla_y u_1.$$

One of our main results on the homogenization of thermal-hydro-mass transfer processes is as follows

Theorem 3.2. *Let p_ε, T_ε, C_ε be the solutions of problem (6). If (**H**0), (**H**1) hold, $T_{out}, \in L^\infty(0,Y;L^2(\Gamma_1))$, $\frac{\partial T_{out}}{\partial t} \in L^2(0,T;H^{-\frac{1}{2}}(\Gamma_1))$, $C_{out} \in L^2((0,T)\times\Gamma_1)$ and α is sufficiently small, then*

$$\begin{cases} p_\varepsilon \overset{2}{\rightharpoonup} p_0, & \textit{in } L^2(\Omega_T\times Y), \\ T_\varepsilon \overset{2}{\rightharpoonup} T_0, & \textit{in } L^2(\Omega_T\times Y), \\ C_\varepsilon \overset{2}{\rightharpoonup} C_0, & \textit{in } L^2(\Omega_T\times Y), \end{cases} \tag{20}$$

and their two-scale limits p_0, T_0 and C_0 satisfy the following homogenized system:

$$\begin{cases} (1+\phi_0\gamma)\frac{\partial T_0}{\partial t} - \nabla\cdot(D_T^0(x)\nabla T_0) = 0, & in\ \Omega_T, \\ -\nabla\cdot((1+\alpha(T_r - T_0))K^0(x)\nabla p_0) = \alpha\phi_0\frac{\partial T_0}{\partial t}, & in\ \Omega_T, \\ \frac{\partial C_0}{\partial t}\phi_0 - \nabla\cdot(D_c^0(x)\nabla C_0) - \nabla\cdot(K^0(x)\nabla p_0 C_0) = 0, & in\ \Omega_T, \\ (D_T^0\nabla T_0)\cdot\nu = \beta_1(T_{out} - T_0),\ (K^0\nabla p_0)\cdot\nu = 0, & on\ \Gamma_1\times(0,T], \\ (D_c^0\nabla C_0)\cdot\nu = \beta_2(C_{out} - C_0), & on\ \Gamma_1\times(0,T], \\ (D_T^0\nabla T_0)\cdot\nu = (K^0\nabla p_0)\cdot\nu = (D_c^0\nabla C_0)\cdot\nu = 0, & on\ \Gamma_3\times(0,T], \\ T_0 = p_0 = C_0 = 0, & on\ \Gamma_2\times(0,T], \\ T_0(x,0) = 0,\ C_0(x,0) = 0, & in\ \Omega, \end{cases} \tag{21}$$

where

$$\begin{aligned} (D_T^0(x))_{ij} &= \int_Y (D_T)_{ij}(x,y) + (D_T)_{ik}(x,y)\frac{\partial N^j}{\partial y_k}dy \\ &\triangleq \left\langle (D_T)_{ij}(x,y) + (D_T)_{ik}(x,y)\frac{\partial N^j}{\partial y_k}\right\rangle_Y, \end{aligned} \tag{22}$$

with $N^j(x,y)$ being the solution of the following cell problem

$$\begin{cases} -\nabla_y\cdot(D_T(x,y)\nabla_y N^j(x,y)) = \nabla_y\cdot(D_T(x,y)e_j), & in\ Y, \\ N^j(x,y)\ is\ Y\text{-}periodic,\ and\ \langle N^j(x,y)\rangle_Y = 0; \end{cases} \tag{23}$$

$(K^0(x))_{ij} = \left\langle K_{ij}(x,y) + K_{ik}(x,y)\frac{\partial\chi^j}{\partial y_k}\right\rangle_Y$ *with $\chi^j(x,y)$ solving the cell problem*

$$\begin{cases} -\nabla_y\cdot(K(x,y)\nabla_y\chi^j(x,y)) = \nabla_y\cdot(K(x,y)e_j), & in\ Y, \\ \chi^j(x,y)\ is\ Y\text{-}periodic,\ and\ \langle\chi^j(x,y)\rangle_Y = 0, \end{cases} \tag{24}$$

and $(D_c^0(x))_{ij} = \left\langle (D_c)_{ij}(x,y) + (D_c)_{ik}(x,y)\frac{\partial\pi^j}{\partial y_k}\right\rangle_Y$ *with $\pi^j(x,y)$ solving the cell problem*

$$\begin{cases} -\nabla_y\cdot(D_c(x,y)\nabla_y\pi^j(x,y)) = \nabla_y\cdot(D_c(x,y)e_j), & in\ Y, \\ \pi^j(x,y)\ is\ Y\text{-}periodic,\ and\ \langle\pi^j(x,y)\rangle_Y = 0. \end{cases} \tag{25}$$

Furthermore, there exist p_1, T_1 and C_1, which are in form as

$$p_1 = \chi_\varepsilon^k\frac{\partial p_0}{\partial x_k},\ T_1 = N_\varepsilon^k\frac{\partial T_0}{\partial x_k},\ C_1 = \pi_\varepsilon^k\frac{\partial C_0}{\partial x_k}, \tag{26}$$

such that

$$\begin{aligned} \nabla p_\varepsilon &\overset{2}{\rightharpoonup} \nabla_x p_0(t,x) + \nabla_y p_1(t,x,y) \quad in\ [L^2(\Omega_T\times Y)]^n, \\ \nabla T_\varepsilon &\overset{2}{\rightharpoonup} \nabla_x T_0(t,x) + \nabla_y T_1(t,x,y) \quad in\ [L^2(\Omega_T\times Y)]^n, \\ \nabla C_\varepsilon &\overset{2}{\rightharpoonup} \nabla_x C_0(t,x) + \nabla_y C_1(t,x,y) \quad in\ [L^2(\Omega_T\times Y)]^n, \end{aligned}$$

where $\chi_\varepsilon^k := \chi(x,\frac{x}{\varepsilon})$, $N_\varepsilon^k := N^k(x,\frac{x}{\varepsilon})$, and $\pi_\varepsilon^k := \pi^k(x,\frac{x}{\varepsilon})$.

Remark 2. The third equation of (21) is a homogenized equation of a convection-diffusion problem, wherein the homogenized velocity $v_0 = -K^0\nabla p_0$ is totally determined by the micro-scale velocity $v_\varepsilon = -K^\varepsilon\nabla p_\varepsilon$. This is interesting and different with the behavior of the homogenization for a single equation of convection-diffusion problem with prescribed multiscale velocity, where the homogenized velocity may be determined by both the micro-scale velocity v_ε and the micro-scale diffusion coefficient D_c^ε [26], when the velocity field is not divergence-free.

The weak forms of system (21) are as follows. Find $T_0 \in W$, $p_0 \in L^2(0,T;V)$ and $C_0 \in W$ such that

$$\int_\Omega (1+\phi_0\gamma)\frac{\partial T_0}{\partial t} v dx + \int_\Omega D_T^0 \nabla T_0 \nabla v dx + \int_{\Gamma_1} \beta_1 T_0 v ds = \int_{\Gamma_1} \beta_1 T_{out} v ds, \forall v \in V; \tag{27}$$

$$\int_\Omega K^0 \nabla p_0 \nabla v dx = \alpha \int_\Omega \frac{\partial T_0}{\partial t}\phi_0 v dx + \alpha \int_\Omega (T_0 - T_r) K^0 \nabla p_0 \nabla v dx, \forall v \in V; \tag{28}$$

$$\langle \phi_0 \frac{\partial C_0}{\partial t}, v\rangle_{H^{-1}(\Omega),H^1(\Omega)} + \int_\Omega D_c^0 \nabla C_0 \nabla v dx + \int_\Omega K^0 \nabla p_0 C_0 \nabla v dx = \beta_2 \int_{\Gamma_1} (C_{out} - C_0) v ds,\ \forall v \in V. \tag{29}$$

Theorem 3.3. *Suppose (**H**0) and (**H**1) hold. If T_{out}, $C_{out} \in L^2((0,T)\times\Gamma_1)$ and α is sufficiently small, then there exist uniqueness solutions $T_0 \in W$, $p_0 \in L^2(0,T;V)$ and $C_0 \in W$ for system (21).*

By Theorem 2.2, Lemma 2.3 and Proposition 1, we can get the following two-scale convergence results.

Lemma 3.4. *If p_ε, T_ε and C_ε solve problem (6), then up to a subsequence, still denoted by p_ε, T_ε, C_ε, there exist $p_1(t,x,y)$, $T_1(t,x,y)$ and $C_1(t,x,y) \in L^2(\Omega_T; H_{per}(Y))$ such that*

$$\begin{cases} p_\varepsilon \xrightarrow{2} p_0(t,x) & \text{in } L^2(\Omega_T\times Y), \\ \nabla p_\varepsilon \xrightarrow{2} \nabla_x p_0(t,x) + \nabla_y p_1(t,x,y) & \text{in } [L^2(\Omega_T\times Y)]^n, \end{cases} \tag{30}$$

$$\begin{cases} T_\varepsilon \xrightarrow{2} T_0(t,x) & \text{in } L^2(\Omega_T\times Y), \\ \nabla T_\varepsilon \xrightarrow{2} \nabla_x T_0(t,x) + \nabla_y T_1(t,x,y) & \text{in } [L^2(\Omega_T\times Y)]^n, \end{cases} \tag{31}$$

$$\begin{cases} C_\varepsilon \xrightarrow{2} C_0(t,x) & \text{in } L^2(\Omega_T\times Y), \\ \nabla C_\varepsilon \xrightarrow{2} \nabla_x C_0(t,x) + \nabla_y C_1(t,x,y) & \text{in } [L^2(\Omega_T\times Y)]^n, \end{cases} \tag{32}$$

Where p_0, T_0 and C_0 are defined in (10).

Let us check the convergence of the coupled term $K^\varepsilon \nabla p_\varepsilon C_\varepsilon$.

Lemma 3.5. *If p_ε and C_ε are the solutions of problem (6), then up to a subsequence, still denoted by p_ε, C_ε, for every $\varphi_\varepsilon(t,x) = \varphi(t,x) + \varepsilon\varphi_1(t,x,y)$ with $\varphi(t,x) \in C^\infty_{\Gamma_2}(\Omega_T)$ and $\varphi_1(t,x,y) \in C^\infty_{\Gamma_2}(\Omega_T; C^\infty_{per}(Y))$, we have*

$$\lim_{\varepsilon\to 0} \int_0^T\int_\Omega K^\varepsilon \nabla p_\varepsilon C_\varepsilon \cdot \nabla\varphi_\varepsilon(t,x) dxdt = \int_0^T\int_\Omega\int_Y K(x,y)(\nabla_x p_0 + \nabla_y p_1) C_0 \cdot (\nabla_x \varphi + \nabla_y \varphi_1) dydxdt,$$

where p_0, C_0, p_1, and C_1 are defined in Theorem 3.2.

Proof. For any $\varphi_\varepsilon(t,x)=\varphi(t,x)+\varepsilon\varphi_1(t,x,\frac{x}{\varepsilon})$ with $\varphi(t,x)\in C^\infty_{\Gamma_2}(\Omega_T)$, $\varphi_1(t,x,y)\in C^\infty_{\Gamma_2}(\Omega_T;C^\infty_{per}(Y))$,

$$\int_0^T\!\!\int_\Omega K^\varepsilon\nabla p_\varepsilon C_\varepsilon\cdot\nabla\varphi_\varepsilon(t,x)dxdt = \int_0^T\!\!\int_\Omega K^\varepsilon\nabla p_\varepsilon(C_\varepsilon-C_0)\cdot\nabla\varphi_\varepsilon(t,x)dxdt + \int_0^T\!\!\int_\Omega K^\varepsilon\nabla p_\varepsilon C_0\cdot\nabla\varphi_\varepsilon(t,x)dxdt.$$

The first term at the right-hand side tends to 0 since we have from Lemma 2.3 that $C_\varepsilon(t,x)\to C_0$ strongly in $L^2(\Omega_T)$ as $\varepsilon\to 0$. So using Lemma 3.4, we obtain

$$\lim_{\varepsilon\to0}\int_0^T\!\!\int_\Omega K^\varepsilon\nabla p_\varepsilon C_\varepsilon\cdot\nabla\varphi_\varepsilon(t,x)dxdt=\lim_{\varepsilon\to0}\int_0^T\!\!\int_\Omega K^\varepsilon\nabla p_\varepsilon C_0\cdot\nabla\varphi_\varepsilon(t,x)dxdt$$
$$=\int_0^T\!\!\int_\Omega\!\int_Y K(x,y)(\nabla_x p_0+\nabla_y p_1)C_0\cdot(\nabla_x\varphi+\nabla_y\varphi_1)dydxdt.$$

□

Remark 3. In the above proof, only the two scale convergence result (2.3) is used. So for temperature T_ε, we also have

$$\lim_{\varepsilon\to0}\int_0^T\!\!\int_\Omega K^\varepsilon\nabla p_\varepsilon T_\varepsilon\cdot\nabla\varphi_\varepsilon(t,x)dxdt$$
$$=\int_0^T\!\!\int_\Omega\!\int_Y K(x,y)(\nabla_x p_0+\nabla_y p_1)T_0\cdot(\nabla_x\varphi+\nabla_y\varphi_1)dydxdt,$$

where $p_0,\ T_0,\ p_1,$ and T_1 are defined in Theorem 3.2.

Now we give the proof of Theorem 3.2.

Proof of Theorem 3.2. For the temperature equation, it is a direct result of homogenization theory for parabolic equation [10, 20].

We begin with the pressure equation in (6). For any $\varphi_\varepsilon(x)=\varphi(x)+\varepsilon\varphi_1(x,\frac{x}{\varepsilon})$ with $\varphi(x)\in C^\infty_{\Gamma_2}(\Omega)$, $\varphi_1\in C^\infty_{\Gamma_2}(\Omega;C^\infty_{per}(Y))$ and any $\psi(t)\in D([0,T))$,

$$\int_0^T\!\!\int_\Omega K^\varepsilon\nabla p_\varepsilon(t,x)\nabla\varphi_\varepsilon(x)\psi(t)dxdt=-\alpha\int_0^T\!\!\int_\Omega\phi_\varepsilon T_\varepsilon\frac{\partial\psi(t)}{\partial t}\varphi_\varepsilon(x)dxdt$$
$$+\alpha\int_0^T\!\!\int_\Omega(T_\varepsilon-T_r)K^\varepsilon\nabla p_\varepsilon\nabla\varphi_\varepsilon(x)\psi(t)dxdt. \quad (33)$$

Thanks to Lemma 2.3, Lemma 3.4 and Remark 3, up to a subsequence, the above formula converges in two-scale to

$$\int_0^T\!\!\int_\Omega\!\int_Y K(x,y)(\nabla_x p_0(x)+\nabla_y p_1)(\nabla_x\varphi(x)+\nabla_y\varphi_1(x,y))\psi(t)dydxdt$$
$$=-\alpha\int_0^T\!\!\int_\Omega\phi_0T_0\frac{\partial\psi(t)}{\partial t}\varphi(x)dxdt+\alpha\int_0^T\!\!\int_\Omega\!\int_Y(T_0-T_r)K(x,y)(\nabla_x p_0(x)+\nabla_y p_1)(\nabla_x\varphi(x)$$
$$+\nabla_y\varphi_1(x,y))\psi(t)dydxdt. \quad (34)$$

Setting $\varphi(x)=0$ and $\varphi_1=\mu(x)\mu_1(y)$ for any $\mu\in C^\infty_{\Gamma_2}(\Omega),\mu_1\in C^\infty_{per}(Y))$ in (34), we obtain

$$\int_0^T\!\!\int_\Omega\!\int_Y(1-\alpha(T_0-T_r))K(x,y)(\nabla_x p_0+\nabla_y p_1)\nabla_y\mu_1(y)\mu(x)\psi(t)dydxdt=0.$$

Due to the arbitrariness of $\mu(x)$ and $\psi(t)$, $p_1 = p_1(t,x,y)$ satisfies that

$$\int_Y K(x,y)(\nabla_x p_0(x) + \nabla_y p_1)\nabla_y \mu_1(y)dy = 0, \quad \forall \mu_1(y) \in C^\infty_{per}(Y). \tag{35}$$

Comparing with the cell problem (24), we have $p_1 = \chi^j_\varepsilon \frac{\partial p_0}{\partial x_j}$.

Setting $\varphi_1 = 0$ in (34), we obtain

$$\begin{aligned}\int_0^T\int_\Omega\int_Y &(1-\alpha(T_0-T_r))K(x,y)(\nabla_x p_0 + \nabla_y p_1)\nabla_x\varphi(x)\psi(t)dydxdt\\ &= -\alpha\int_0^T\int_\Omega \phi_0 T_0 \frac{\partial\psi(t)}{\partial t}\varphi(x)dxdt,\end{aligned} \tag{36}$$

i.e.

$$\begin{aligned}\int_0^T\int_\Omega &(1-\alpha(T_0-T_r))K^0(x)\nabla_x p_0 \nabla\varphi(x)\psi(t)dxdt\\ &= \alpha\int_0^T\int_\Omega \phi_0 \frac{\partial T_0(t)}{\partial t}\varphi(x)\psi(t)dxdt + \alpha\int_\Omega \phi_0 T_0(0,x)\psi(0)\varphi(x)dx.\end{aligned}$$

Hence we obtain the homogenized equation for the pressure

$$-\nabla_x\cdot(1-\alpha(T_0-T_r))K^0(x)\nabla_x p_0) = \alpha\phi_0\frac{\partial T_0(t)}{\partial t} \tag{37}$$

with $K^0(x)\nabla_x p_0\cdot\nu = 0$ on $\partial\Omega\backslash\Gamma_2$ and $T_0|_{t=0} = 0$.

For the concentration equation in (6), choosing the same test functions as above, we also have

$$\begin{aligned}&-\int_0^T\int_\Omega C_\varepsilon(t,x)\phi_\varepsilon\varphi_\varepsilon(x)\frac{\partial\psi}{\partial t}(t)dxdt + \int_0^T\int_\Omega D_c^\varepsilon\nabla C_\varepsilon\nabla\varphi_\varepsilon\psi(t)dxdt\\ &+\int_0^T\int_\Omega K^\varepsilon\nabla p_\varepsilon C_\varepsilon\nabla\varphi_\varepsilon(x)\psi(t)dxdt = \beta_2\int_0^T\int_{\Gamma_1}(C_{out}-C_\varepsilon)\varphi_\varepsilon\psi(t)dsdt.\end{aligned}$$

By Lemmas 3.4 and 3.5, up to a subsequence, the above formula converges in two-scale to

$$\begin{aligned}&-\int_0^T\int_\Omega C_0(t,x)\phi_0\varphi(x)\frac{\partial\psi}{\partial t}(t)dxdt\\ &+\int_0^T\int_\Omega\int_Y D_c(x,y)(\nabla_x C_0 + \nabla_y C_1(x,y))(\nabla_x\varphi + \nabla_y\varphi_1(x,y))\psi(t)dydxdt\\ &+\int_0^T\int_\Omega\int_Y K(x,y)(\nabla_x p_0 + \nabla_y p_1)C_0(\nabla_x\varphi(x) + \nabla_y\varphi_1(x,y))\psi(t)dydxdt\\ = \;&\beta_2\int_0^T\int_{\Gamma_1}(C_{out}-C_0)\varphi(x)\psi(t)dsdt.\end{aligned} \tag{38}$$

Setting $\varphi(x) = 0$ and noting that the cell problem (24) for the pressure, we get

$$\int_0^T\int_\Omega\int_Y D_c(x,y)(\nabla_x C_0 + \nabla_y C_1(x,y))\nabla_y\varphi_1(x,y)\psi(t)dydxdt = 0.$$

Comparing with the cell problem (25) for concentration, we have $C_1 = \pi_\varepsilon^j \frac{\partial C_0}{\partial x_j}$. Choosing $\varphi_1 = 0$ in (38), we obtain

$$\begin{aligned}
& -\int_0^T\int_\Omega C_0(t,x)\phi_0\varphi(x)\frac{\partial\psi}{\partial t}(t)dxdt \\
& +\int_0^T\int_\Omega\int_Y D_c(x,y)(\nabla_x C_0+\nabla_y C_1(x,y))\nabla_x\varphi(x)\psi(t)dydxdt \\
& +\int_0^T\int_\Omega\int_Y K(x,y)(\nabla_x p_0+\nabla_y p_1)C_0\nabla_x\varphi(x)\psi(t)dydxdt \\
= \;& \beta_2\int_0^T\int_{\Gamma_1}(C_{out}-C_0)\varphi(x)\psi(t)dsdt,
\end{aligned}$$

i.e.

$$\begin{aligned}
& \int_0^T\int_\Omega \phi_0\frac{\partial C_0}{\partial t}\varphi(x)\psi(t)dxdt+\int_0^T\int_\Omega D_c^0(x)\nabla C_0\nabla\varphi\psi(t)dxdt \\
& +\int_0^T\int_\Omega K^0(x)\nabla_x p_0 C_0\nabla\varphi\psi(t)dxdt \\
= \;& \beta_2\int_0^T\int_{\Gamma_1}(C_{out}-C_0)\varphi\psi(t)dsdt-\int_\Omega\phi_0C_0(0,x)\psi(0)\varphi(x)dx.
\end{aligned}$$

So we get the homogenized equation for the concentration

$$\phi_0\frac{\partial C_0}{\partial t}-\nabla\cdot(D_c^0(x)\nabla C_0)-\nabla\cdot(K^0(x)\nabla p_0C_0)=0,\quad \text{in } \Omega_T, \tag{39}$$

with $C_0|_{t=0}=0$, $D_c^0\nabla C_0\cdot\nu=\beta_2(C_{out}-C_0)$ on Γ_1 and $D_c^0\nabla C_0\cdot\nu=0$ on $\partial\Omega\backslash(\Gamma_1\cup\Gamma_2)$.

The whole sequence two-scale convergence comes from the uniqueness of the solution for the homogenized system (21). Hence we complete the proof of Theorem 3.2. □

4. **Error estimates for first order expansions.**

In this section, we will present the error estimates between p_ε, T_ε, C_ε and their first order expansions

$$p_1^\varepsilon=p_0+\varepsilon\chi_\varepsilon^k\frac{\partial p_0}{\partial x_k},\; T_1^\varepsilon=T_0+\varepsilon N_\varepsilon^k\frac{\partial T_0}{\partial x_k},\; C_1^\varepsilon=C_0+\varepsilon\pi_\varepsilon^k\frac{\partial C_0}{\partial x_k}. \tag{40}$$

To this purpose, we need some regularity assumptions on homogenized problem (21)

$$(\mathbf{H}2):\left\{\begin{array}{l}
p_0(t,x)\in L^2(0,T,W^{2,\infty}(\Omega)),\\
\;T_0(t,x),\; C_0(t,x)\in L^2(0,T;W^{1,\infty}(\Omega))\bigcap L^2(0,T;H^2(\Omega)),\\
\nabla(\frac{\partial T_0}{\partial t}),\;\nabla(\frac{\partial C_0}{\partial t})\in(L^2(\Omega_T))^n,\\
\chi^i(x,y),\; N^i(x,y),\; \pi^i(x,y)\in L^\infty(\Omega,W^{1,\infty}_{per}(Y))\cap L^\infty(Y;H^1(\Omega)),\\
\;\; i=1,\cdots,n.
\end{array}\right. \tag{41}$$

In the pressure equation of system (6), since ϕ_ε only weakly* converges to ϕ_0 in $L^\infty(\Omega)$, we can only get that the right hand side $\phi_\varepsilon\frac{\partial T_\varepsilon}{\partial t}$ weakly converges in $L^2(0,T;H^{-1}(\Omega))$. Here due to the special structure of right hand side, we have got the convergence results in Theorem 3.2. In order to get the error estimate for first order expansion, further assumption has to be imposed either on the porosity ϕ_ε, wherein we may consider the situation of porosity with only small change in amplitude $\phi_\varepsilon(x) = \phi_0 + \varepsilon\phi_1\left(\frac{x}{\varepsilon}\right)$ or on the temperature field T_ε, wherein we need more regularity on T_ε. Here we choose the second case and assume that

$$(\mathbf{H}3): \begin{cases} T_{out} \text{ satisfies some more compatibility conditions such that} \\ \frac{\partial}{\partial t}T_\varepsilon \in C^0([0,T];L^2(\Omega)), \\ \frac{\partial}{\partial t}T_0(t,x) \in L^2(0,T;W^{1,\infty}(\Omega)) \bigcap L^2(0,T;H^2(\Omega)), \\ \nabla(\frac{\partial^2 T_0}{\partial t^2}) \in (L^2(\Omega_T))^n. \end{cases} \tag{42}$$

The following theorem on the error estimate for the first order expansions is also one of our main results.

Theorem 4.1. *Let p_ε, T_ε, C_ε be the solutions of problem (6) and p_1^ε, T_1^ε, C_1^ε be defined in (40). If (**H**0), (**H**1) and (**H**2) hold, $T_{out} \in L^\infty(0,T;L^2(\Gamma_1))$, $\frac{\partial T_{out}}{\partial t} \in L^2(0,T;H^{-\frac{1}{2}}(\Gamma_1))$ and α is sufficiently small, then there exists constant $C > 0$ independent of ε, such that*

$$\sup_{0\le t\le T} \|T_\varepsilon - T_1^\varepsilon\|_{L^2(\Omega)} + \|\nabla(T_\varepsilon - T_1^\varepsilon)\|_{L^2(\Omega_T)} \le C\varepsilon^{\frac{1}{2}}. \tag{43}$$

*Furthermore, if (**H**3) also holds and $C_{out} \in L^2((0,T)\times\Gamma_1)$, then*

$$\sup_{0\le t\le T} \|p_\varepsilon - p_1^\varepsilon\|_{H^1(\Omega)} \le C\varepsilon^{\frac{1}{2}}, \tag{44}$$

$$\sup_{0\le t\le T} \|C_\varepsilon - C_1^\varepsilon\|_{L^2(\Omega)} + \|\nabla(C_\varepsilon - C_1^\varepsilon)\|_{L^2(\Omega_T)} \le C\varepsilon^{\frac{1}{2}}. \tag{45}$$

Before proving the theorem, several lemmas are needed.

Lemma 4.2. [9] *Let $\theta^\varepsilon \in V$ be the boundary corrector satisfying the following problem*

$$\begin{cases} -\nabla\cdot(A^\varepsilon\nabla\theta^\varepsilon) = 0, & \text{in } \Omega, \\ (A^\varepsilon\nabla\theta^\varepsilon)\cdot\nu = \frac{\partial}{\partial x_k}(Q_{ik}^j \frac{\partial u}{\partial x_j})\cdot\nu_i, & \text{on } \partial\Omega\setminus\Gamma_2, \\ \theta^\varepsilon = 0, & \text{on } \Gamma_2, \end{cases} \tag{46}$$

where $A^\varepsilon(x) = A(x,\frac{x}{\varepsilon})$ is positive definite and uniformly bounded, $u \in H^2(\Omega)\bigcap W^{1,\infty}(\Omega)$ and $Q_{ik}^j(x,\frac{x}{\varepsilon}) \in L^\infty(\Omega;W^{1,\infty}(Y)) \cap L^\infty(Y;H^1(\Omega))$ is skew-symmetric matrix $Q_{ik}^j = -Q_{ki}^j$, then

$$\|\varepsilon\nabla\theta^\varepsilon\|_{L^2(\Omega)} \le C\varepsilon(|u|_{H^1(\Omega)} + |u|_{H^2(\Omega)}) + C\varepsilon^{\frac{1}{2}}|u|_{W^{1,\infty}(\Omega)}. \tag{47}$$

This boundary corrector problem is slightly different with that defined in [9], where the Dirichlet boundary condition is missing and Q_{ik}^j is a problem specific matrix. However, the idea used in [9] still works. One can prove the above lemma similarly.

Lemma 4.3. *If $\eta_\varepsilon \in V$ is the boundary corrector satisfying the following problem*

$$\begin{cases} (1+\phi_\varepsilon\gamma)\frac{\partial\eta_\varepsilon}{\partial t} - \nabla\cdot(D_T^\varepsilon\nabla\eta_\varepsilon) = 0, & \text{in } \Omega_T, \\ (D_T^\varepsilon\nabla\eta_\varepsilon)\cdot\nu = -\beta_1\eta_\varepsilon, & \text{on } \Gamma_1\times(0,T], \\ (D_T^\varepsilon\nabla\eta_\varepsilon)\cdot\nu = 0, & \text{on } \Gamma_3\times(0,T], \\ \eta_\varepsilon = -N_\varepsilon^j\frac{\partial T_0}{\partial x_j}, & \text{on } \Gamma_2\times(0,T], \\ \eta_\varepsilon|_{t=0} = 0, & \text{in } \Omega, \end{cases} \tag{48}$$

*and (**H**0) - (**H**2) hold, then*

$$\sup_{0\le t\le T} \|\varepsilon\eta^\varepsilon\|_{L^2(\Omega)} \le C\varepsilon, \quad \|\varepsilon\nabla\eta^\varepsilon\|_{L^2(\Omega_T)} \le C\varepsilon^{\frac{1}{2}}. \tag{49}$$

We omit the proof of this lemma since it is similar to Theorem 3.1 in [8].

Lemma 4.4. *Let C_1^ε be defined as in (40) and C_0 be the solution of problems (21), respectively. If (**H**0) and (**H**2) hold, then there exists a positive constant C independent of ε such that, for any $\varphi \in V$,*

$$\left| \int_\Omega (D_c^\varepsilon \nabla C_1^\varepsilon - D_c^0 \nabla C_0) \nabla \varphi dx \right| \le C\varepsilon^{\frac{1}{2}} \|\nabla \varphi\|_{L^2(\Omega)}.$$

Proof. For $\phi \in V$, according to the definition of C_1^ε, after some simple calculation, we have

$$\begin{aligned} J_1 &\equiv \int_\Omega (D_c^\varepsilon \nabla C_1^\varepsilon - D_c^0 \nabla C_0) \nabla \varphi dx \\ &= \int_\Omega g_i^j \frac{\partial C_0}{\partial x_j} \frac{\partial \varphi}{\partial x_i} + \int_\Omega R_1^\varepsilon \nabla \varphi dx, \end{aligned} \tag{50}$$

with

$$\begin{cases} g_i^j(x,y) = (D_c^\varepsilon)_{ij} + \sum_k \left((D_c^\varepsilon)_{ik} \dfrac{\partial \pi_\varepsilon^j}{\partial y_k} \right) - (D_c^0)_{ij}, \\ (R_1^\varepsilon)_i = \sum_{j,k} \left(\varepsilon (D_c^\varepsilon)_{ij} \pi_\varepsilon^k \dfrac{\partial^2 C_0}{\partial x_j \partial x_k} + \varepsilon (D_c^\varepsilon)_{ij} \dfrac{\partial \pi_\varepsilon^k}{\partial x_j} \dfrac{\partial C_0}{\partial x_k} \right). \end{cases} \tag{51}$$

Since $\langle g_i^j \rangle_Y = 0$ and $\nabla_y \cdot g^j = 0$, there exists a skew-symmetric matrix ([26]) $G_{ik}^j(x,y) \in L^\infty(\Omega; W^{1,\infty}(Y)) \cap L^\infty(Y; H^1(\Omega))$ such that $\sum_k \frac{\partial G_{ik}^j}{\partial y_k} = g_i^j$. With this notation, we can rewrite

$$g_i^j \frac{\partial C_0}{\partial x_j} = \sum_k \left(\frac{\partial}{\partial x_k} (\varepsilon G_{ik}^j \frac{\partial C_0}{\partial x_j}) - \varepsilon \frac{\partial G_{ik}^j}{\partial x_k} \frac{\partial C_0}{\partial x_j} - \varepsilon G_{ik}^j \frac{\partial^2 C_0}{\partial x_j \partial x_k} \right). \tag{52}$$

Then we have

$$J_1 = \int_\Omega \frac{\partial}{\partial x_k} (\varepsilon G_{ik}^j \frac{\partial C_0}{\partial x_j}) \frac{\partial \varphi}{\partial x_i} + \int_\Omega R_1^\varepsilon \nabla \varphi dx - \int_\Omega R_2^\varepsilon \nabla \varphi dx, \tag{53}$$

with $R_2^\varepsilon = \sum_{j,k} \left(\varepsilon \frac{\partial G_{ik}^j}{\partial x_k} \frac{\partial C_0}{\partial x_j} + \varepsilon G_{ik}^j \frac{\partial^2 C_0}{\partial x_j \partial x_k} \right)$. Let θ_c^ε be the boundary corrector as a solution of the following problem:

$$\begin{cases} -\triangle \theta_c^\varepsilon = 0, & \text{in}\Omega, \\ (\nabla \theta_c^\varepsilon) \cdot \nu = \frac{\partial}{\partial x_k} (G_{ik}^j \frac{\partial C_0}{\partial x_j}) \cdot \nu_i, & \text{on } \partial\Omega \setminus \Gamma_2, \\ \theta_c^\varepsilon = 0, & \text{on } \Gamma_2. \end{cases} \tag{54}$$

From the skew-symmetry of matrix G_{ik}^j, the weak form of problem (54) reads

$$\int_\Omega \nabla \theta_c^\varepsilon \nabla \varphi dx = \int_\Omega \frac{\partial}{\partial x_k} (G_{ik}^j \frac{\partial C_0}{\partial x_j}) \frac{\partial \varphi}{\partial x_i}.$$

Then we have

$$J_1 = \varepsilon \int_\Omega \nabla \theta_c^\varepsilon \nabla \varphi dx + \int_\Omega R_1^\varepsilon \nabla \varphi dx - \int_\Omega R_2^\varepsilon \nabla \varphi dx, \tag{55}$$

By Lemma 4.2 and the definition of R_1^ε, R_2^ε, we get the estimate:

$$\begin{aligned}|J_1| &\le \varepsilon\|\nabla\theta_c^\varepsilon\|_{L^2(\Omega)}\|\nabla\varphi\|_{L^2(\Omega)} + \|R_1^\varepsilon\|_{L^2(\Omega)}\|\nabla\varphi\|_{L^2(\Omega)} + \|R_2^\varepsilon\|_{L^2(\Omega)}\|\nabla\varphi\|_{L^2(\Omega)}\\ &\le C\varepsilon^{\frac{1}{2}}|C_0|_{W^{1,\infty}(\Omega)}\|\nabla\varphi\|_{L^2(\Omega)} + C\varepsilon(|C_0|_{H^1(\Omega)} + |C_0|_{H^2(\Omega)})\|\nabla\varphi\|_{L^2(\Omega)}\\ &\le C\varepsilon^{\frac{1}{2}}\|\nabla\varphi\|_{L^2(\Omega)} + C\varepsilon\|\nabla\varphi\|_{L^2(\Omega)}.\end{aligned}$$

So we complete the proof. □

Remark 4. Suppose that (**H**0) and (**H**2) be valid. We can obtain in a same way as in Lemma 4.4 that for any $\varphi \in V$,

$$\left|\int_\Omega (D_T^\varepsilon \nabla T_1^\varepsilon - D_T^0\nabla T_0)\nabla\varphi dx\right| \le C\varepsilon^{\frac{1}{2}}\|\nabla\varphi\|_{L^2(\Omega)}.$$

Lemma 4.5. *Let C_ε, p_ε be the solutions of problem (6) and C_0, p_0 be the solutions of problem (21). If (**H**0) and (**H**2) hold, then there exists a positive constant C independent of ε such that for $\forall\varphi \in V$,*

$$\begin{aligned}&\left|\int_\Omega [K^\varepsilon(x)\nabla p_\varepsilon C_\varepsilon - K^0(x)\nabla p_0 C_0]\nabla\varphi dx\right| \le C\varepsilon^{\frac{1}{2}}\|\nabla\varphi\|_{L^2(\Omega)}\\ &\quad +\Lambda_2\|C_\varepsilon\|_{L^\infty(\Omega)}\|\nabla(p_\varepsilon - p_1^\varepsilon)\|_{L^2(\Omega)}\|\nabla\varphi\|_{L^2(\Omega)}\\ &\quad +\Lambda_2\|\nabla p_1^\varepsilon\|_{L^\infty(\Omega)}\|C_\varepsilon - C_1^\varepsilon\|_{L^2(\Omega)}\|\nabla\varphi\|_{L^2(\Omega)}.\end{aligned}$$

Proof. First,

$$\int_\Omega [K^\varepsilon(x)\nabla p_\varepsilon C_\varepsilon - K^0(x)\nabla p_0 C_0]\nabla\varphi dx = J_2 + J_3 \tag{56}$$

with $J_2 = \int_\Omega [K^\varepsilon(x)\nabla p_\varepsilon C_\varepsilon - K^\varepsilon(x)\nabla p_1^\varepsilon C_0]\nabla\varphi dx$ and $J_3 = \int_\Omega [K^\varepsilon(x)\nabla p_1^\varepsilon C_0 - K^0(x)\nabla p_0 C_0]\nabla\varphi dx$.

For the first term, we have

$$\begin{aligned}|J_2| &\le \left|\int_\Omega [K^\varepsilon\nabla(p_\varepsilon - p_1^\varepsilon)C_\varepsilon]\nabla\varphi dx\right| + \left|\int_\Omega K^\varepsilon\nabla p_1^\varepsilon(C_\varepsilon - C_0)\nabla\varphi dx\right|\\ &\le \Lambda_2\|C_\varepsilon\|_{L^\infty(\Omega)}\|\nabla(p_\varepsilon - p_1^\varepsilon)\|_{L^2(\Omega)}\|\nabla\varphi\|_{L^2(\Omega)}\\ &\quad +\Lambda_2\|\nabla p_1^\varepsilon\|_{L^\infty(\Omega)}\|C_\varepsilon - C_1^\varepsilon\|_{L^2(\Omega)}\|\nabla\varphi\|_{L^2(\Omega)}\\ &\quad +C\varepsilon\Lambda_2\|\nabla p_1^\varepsilon\|_{L^\infty(\Omega)}|C_0|_{H^1(\Omega)}\|\nabla\varphi\|_{L^2(\Omega)}.\end{aligned} \tag{57}$$

The term J_3 can be bounded by a similar way used to treat term J_1 in Lemma 4.4. But we need a slightly different corrector problem to treat the coupling with C_0. By simple calculations, we have for $i = 1, \ldots, n$,

$$(K^\varepsilon\nabla p_1^\varepsilon - K^0\nabla p_0)_i = \sum_j h_i^j \frac{\partial p_0}{\partial x_j} + (r_1^\varepsilon)_i, \tag{58}$$

with $h_i^j(x,y) = K_{ij}^\varepsilon + \sum_k K_{ik}^\varepsilon \frac{\partial \chi_\varepsilon^j}{\partial y_k} - K_{ij}^0$ and $(r_1^\varepsilon)_i = \sum_{j,k}\left(\varepsilon K_{ij}^\varepsilon \frac{\partial \chi_\varepsilon^k}{\partial x_j}\frac{\partial p_0}{\partial x_k} + \varepsilon K_{ij}^\varepsilon \chi_\varepsilon^k \frac{\partial^2 p_0}{\partial x_j \partial x_k}\right)$. Since $\langle h_i^j\rangle_Y = 0$ and $\nabla_y \cdot h^j = 0$, there exists a skew-symmetric matrix $H_{ik}^j(x,y) \in L^\infty(\Omega; W^{1,\infty}(Y)) \cap L^\infty(Y; H^1(\Omega))$ such that $\sum_k \frac{\partial H_{ik}^j}{\partial y_k} = h_i^j$.

With this notation, we can rewrite

$$h_i^j \frac{\partial p_0}{\partial x_j} = \sum_k \left(\frac{\partial}{\partial x_k} \left(\varepsilon H_{ik}^j \frac{\partial p_0}{\partial x_j} \right) - \varepsilon \frac{\partial H_{ik}^j}{\partial x_k} \frac{\partial p_0}{\partial x_j} - \varepsilon H_{ik}^j \frac{\partial^2 p_0}{\partial x_j \partial x_k} \right). \tag{59}$$

In summary, we obtain for $i = 1, \ldots, n$,

$$(K^\varepsilon \nabla p_1^\varepsilon - K^0 \nabla p_0)_i = \sum_{j,k} \frac{\partial}{\partial x_k} \left(\varepsilon H_{ik}^j \frac{\partial p_0}{\partial x_j} \right) + (r_1^\varepsilon)_i - (r_2^\varepsilon)_i, \tag{60}$$

with $(r_2^\varepsilon)_i = \sum_{j,k} \left(\varepsilon \frac{\partial H_{ik}^j}{\partial x_k} \frac{\partial p_0}{\partial x_j} + \varepsilon H_{ik}^j \frac{\partial^2 p_0}{\partial x_j \partial x_k} \right)$.

Let the boundary corrector θ_p^ε satisfying the following problem, which is slightly different with the boundary corrected problem (54),

$$\begin{cases} -\triangle \theta_p^\varepsilon = 0, \quad \text{in} \Omega, \\ (\nabla \theta_p^\varepsilon) \cdot \nu = \left(\frac{\partial}{\partial x_k} (H_{ik}^j \frac{\partial p_0}{\partial x_j} C_0) \right) \cdot \nu_i \quad \text{on } \partial\Omega \setminus \Gamma_2, \\ \theta_p^\varepsilon = 0 \quad \text{on } \Gamma_2. \end{cases} \tag{61}$$

By the skew-symmetry of matrix H_{ik}^j, the weak form of the above problem is as follows: Find $\theta_p^\varepsilon \in V$ such that

$$\int_\Omega \nabla \theta_p^\varepsilon \nabla \varphi dx = \int_\Omega \frac{\partial}{\partial x_k} (\varepsilon H_{ik}^j \frac{\partial p_0}{\partial x_j} C_0) \frac{\partial \varphi}{\partial x_i} dx, \ \forall \varphi \in V. \tag{62}$$

By (60), J_3 can be rewritten as

$$\begin{aligned} J_3 &= \int_\Omega \frac{\partial}{\partial x_k} (\varepsilon H_{ik}^j \frac{\partial p_0}{\partial x_j} C_0) \frac{\partial \varphi}{\partial x_i} dx + \int_\Omega r_1^\varepsilon C_0 \nabla \varphi dx - \int_\Omega r_2^\varepsilon C_0 \nabla \varphi dx \\ &\quad - \varepsilon \int_\Omega \frac{\partial C_0}{\partial x_k} H_{ik}^j \frac{\partial p_0}{\partial x_j} \frac{\partial \varphi}{\partial x_i} dx \\ &= \varepsilon \int_\Omega \nabla \theta_p^\varepsilon \nabla \varphi dx + \int_\Omega r_1^\varepsilon C_0 \nabla \varphi dx - \int_\Omega r_2^\varepsilon C_0 \nabla \varphi dx \\ &\quad - \varepsilon \int_\Omega \frac{\partial C_0}{\partial x_k} H_{ik}^j \frac{\partial p_0}{\partial x_j} \frac{\partial \varphi}{\partial x_i} dx. \end{aligned} \tag{63}$$

From Lemma 4.2 and the definitions of r_1^ε, r_2^ε, we obtain the estimate of J_3

$$\begin{aligned} |J_3| \leq\ & C\varepsilon^{\frac{1}{2}} |C_0|_{W^{1,\infty}(\Omega)} \|\nabla \varphi\|_{L^2(\Omega)} + C\varepsilon (|p_0|_{W^{1,2}(\Omega)} + |p_0|_{W^{2,2}(\Omega)}) \|\nabla \varphi\|_{L^2(\Omega)} \\ & + C\varepsilon \|\nabla p_0\|_{L^\infty(\Omega)} |C_0|_{W^{1,2}(\Omega)} \|\nabla \varphi\|_{L^2(\Omega)}. \end{aligned} \tag{64}$$

Combining (57) with (64), we complete the proof. □

Remark 5. If (**H**0) and (**H**2) hold, then we can get, in a same way as the above Lemma, that for any $\varphi \in V$,

$$\begin{aligned} \left| \int_\Omega [K^\varepsilon(x) \nabla p_\varepsilon T_\varepsilon - K^0(x) \nabla p_0 T_0] \nabla \varphi dx \right| &\leq C\varepsilon^{\frac{1}{2}} \|\nabla \varphi\|_{L^2(\Omega)} \\ & + \Lambda_2 \|T_\varepsilon\|_{L^\infty(\Omega)} \|\nabla (p_\varepsilon - p_1^\varepsilon)\|_{L^2(\Omega)} \|\nabla \varphi\|_{L^2(\Omega)} \\ & + \Lambda_2 \|\nabla p_1^\varepsilon\|_{L^\infty(\Omega)} \|T_\varepsilon - T_1^\varepsilon\|_{L^2(\Omega)} \|\nabla \varphi\|_{L^2(\Omega)}. \end{aligned}$$

In the estimates of Lemma 4.5 and Remark 5, T_ε and C_ε are uniformly bounded by (2.2) and ∇p_1^ε is also uniformly bounded since under the assumption (**H**2), there exists a constant $C > 0$ such that

$$\|\nabla p_1^\varepsilon\|_{L^\infty(\Omega)} \le \|\nabla p_0 + \varepsilon \nabla_x \chi_i(x,y)\frac{\partial p_0}{\partial x_i} + \nabla_y \chi_i(x,y)\frac{\partial p_0}{\partial x_i} + \varepsilon \chi_i \nabla \frac{\partial p_0}{\partial x_i}\|_{L^\infty(\Omega)} \le C.$$

We also need the following lemma to deal with the terms containing ϕ_ε.

Lemma 4.6. *([19]) Let $g(x,y) \in L^\infty(\Omega \times R^N)$ be periodic in Y with respect to y and satisfy $\int_Y g(x,y)dy = 0$ for any $x \in \Omega$. Then there exists a constant $C > 0$ independent of ε such that for any $u, v \in H^1(\Omega)$,*

$$\left|\int_\Omega uvg(x,\frac{x}{\varepsilon})dx\right| \le C\varepsilon \|u\|_{H^1(\Omega)}\|v\|_{H^1(\Omega)}. \tag{65}$$

Now we give the proof of Theorem 4.1.

Proof of Theorem 4.1. **Step 1.** For $T_\varepsilon - T_1^\varepsilon$, by the weak forms (11) for T_ε and (27) for T_0, we have

$$\begin{aligned}
&\langle (1+\phi_\varepsilon\gamma)\frac{\partial (T_\varepsilon - T_1^\varepsilon)}{\partial t}, \varphi\rangle_{H^{-1},H^1} + \int_\Omega D_T^\varepsilon \nabla (T_\varepsilon - T_1^\varepsilon))\nabla \varphi dx \\
&\quad + \beta_1 \int_{\Gamma_1} (T_\varepsilon - T_1^\varepsilon)\varphi ds \\
= \;& \langle (1+\phi_\varepsilon\gamma)\frac{\partial (T_0 - T_1^\varepsilon)}{\partial t}, \varphi\rangle_{H^{-1},H^1} + \langle \gamma(\phi_0 - \phi_\varepsilon)\frac{\partial T_0}{\partial t}, \varphi\rangle_{H^{-1},H^1} \\
&+ \int_\Omega (D_T^0 \nabla T_0 - D_T^\varepsilon \nabla T_1^\varepsilon)\nabla\varphi + \beta_1 \int_{\Gamma_1}(T_0 - T_1^\varepsilon)\varphi ds \\
= \;& -\varepsilon \int_\Omega (1+\phi_\varepsilon\gamma) N_\varepsilon^j \frac{\partial}{\partial t}\left(\frac{\partial T_0}{\partial x_j}\right)\varphi dx + \int_\Omega (D_T^0 \nabla T_0 - D_T^\varepsilon \nabla T_1^\varepsilon)\nabla \varphi dx \\
&+ \int_\Omega \gamma(\phi_0 - \phi_\varepsilon)\frac{\partial T_0}{\partial t}\varphi dx - \beta_1 \varepsilon \int_{\Gamma_1} N_\varepsilon^j \frac{\partial T_0}{\partial x_j}\varphi ds.
\end{aligned} \tag{66}$$

Introduce the boundary corrector η_ε defined in problem (48). A weak form for problem (48) reads, for every $\varphi \in V$,

$$\int_\Omega (1+\phi_\varepsilon\gamma)\frac{\partial \eta_\varepsilon}{\partial t}\varphi dx + \int_\Omega D_T^\varepsilon \nabla \eta_\varepsilon \nabla \varphi dx + \int_{\Gamma_1} \beta_1 \eta_\varepsilon \varphi ds = 0. \tag{67}$$

Then $e_T = T_\varepsilon - T_1^\varepsilon - \varepsilon\eta^\varepsilon \in V$. Thanks to (67) and setting $\varphi = e_T$ in (66), we obtain

$$\begin{aligned}
&\langle (1+\phi_\varepsilon\gamma)\frac{\partial e_T}{\partial t}, e_T\rangle_{H^{-1},H^1} + \int_\Omega D_T^\varepsilon \nabla e_T \nabla e_T dx + \beta_1 \int_{\Gamma_1} e_T e_T ds \\
= \;& -\varepsilon \int_\Omega (1+\phi_\varepsilon\gamma) N_\varepsilon^j \frac{\partial}{\partial t}\left(\frac{\partial T_0}{\partial x_j}\right) e_T dx + \int_\Omega (D_T^0 \nabla T_0 - D_T^\varepsilon \nabla T_1^\varepsilon)\nabla e_T dx \\
&+ \int_\Omega \gamma(\phi_0 - \phi_\varepsilon)\frac{\partial T_0}{\partial t} e_T dx - \beta_1 \varepsilon \int_{\Gamma_1} N_\varepsilon^j \frac{\partial T_0}{\partial x_j} e_T ds.
\end{aligned} \tag{68}$$

Using the uniformly elliptic condition, Poincare's inequality, Remark 4 and Lemma 4.6 , we have

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\|\sqrt{1+\phi_\varepsilon\gamma}\ e_T\|^2_{L^2(\Omega)}+\lambda_1\|\nabla e_T\|^2_{L^2(\Omega)}+\beta_1\|e_T\|^2_{L^2(\Gamma_1)}\\&\quad\le C\varepsilon^2+C\varepsilon+\frac{\lambda_1}{2}\|\nabla e_T\|^2_{L^2(\Omega)}+\frac{\beta_1}{2}\|e_T\|^2_{L^2(\Gamma_1)}\\&\quad\le C\varepsilon+\frac{\lambda_1}{2}\|\nabla e_T\|^2_{L^2(\Omega)}+\frac{\beta_1}{2}\|e_T\|^2_{L^2(\Gamma_1)}.\end{aligned}$$

Integrating over $(0,t)$ with $t\in(0,T]$ on both sides of above formula and using Gronwall's inequality, we obtain

$$\sup_{0\le t\le T}\|e_T\|^2_{L^2(\Omega)}+\|\nabla e_T\|^2_{L^2(\Omega_T)}+\beta_1\|e_T\|^2_{L^2((0,T)\times\Gamma_1)}\le C\varepsilon. \tag{69}$$

From the estimates for η_ε in Lemma 4.3, the error estimate for temperature field (43) is obtained.

Step 2. Estimate $p_\varepsilon-p_1^\varepsilon$. By the weak forms (12) for p_ε and (28) for p_0, for every $\varphi\in V$,

$$\begin{aligned}\int_\Omega K^\varepsilon\nabla(p_\varepsilon-p_1^\varepsilon)\nabla\varphi dx&=\int_\Omega(K^\varepsilon\nabla p^\varepsilon-K^0\nabla p_0)\nabla\varphi dx\\&\quad+\int_\Omega(K^0\nabla p_0-K^\varepsilon\nabla p_1^\varepsilon)\nabla\varphi dx\equiv I_1+I_2.\end{aligned}\tag{70}$$

The second term has been treated in the proof of Lemma 4.5. From (60),

$$I_2\ =\ \varepsilon\int_\Omega K^\varepsilon\nabla\theta_1^\varepsilon\nabla\varphi dx+\int_\Omega r_1^\varepsilon\nabla\varphi dx-\int_\Omega r_2^\varepsilon\nabla\varphi dx, \tag{71}$$

where we have used the weak form of the boundary corrector θ_1^ε, which is the solution of the following problem

$$\begin{cases}-\nabla\cdot(K^\varepsilon\nabla\theta_1^\varepsilon)=0, & \text{in }\Omega,\\ (K^\varepsilon\nabla\theta_1^\varepsilon)\cdot\nu=\left(\frac{\partial}{\partial x_k}(H^j_{ik}\frac{\partial p_0}{\partial x_j})\right)\cdot\nu_i, & \text{on }\partial\Omega\setminus\Gamma_2,\\ \theta_1^\varepsilon=0, & \text{on }\Gamma_2\end{cases}\tag{72}$$

and thanks to the skew-symmetry of matrix H^j_{ik}, the weak form is

$$\int_\Omega K^\varepsilon\nabla\theta_1^\varepsilon\nabla\varphi dx=\int_\Omega\frac{\partial}{\partial x_k}\left(H^j_{ik}\frac{\partial p_0}{\partial x_j}\right)\frac{\partial\varphi}{\partial x_i}dx,\quad\forall\ \varphi\in V. \tag{73}$$

We also need another boundary corrector θ_2^ε such that

$$\begin{cases}-\nabla\cdot(K^\varepsilon\nabla\theta_2^\varepsilon)=0, & \text{in }\Omega,\\ (K^\varepsilon\nabla\theta_2^\varepsilon)\cdot\nu=0, & \text{on }\partial\Omega\setminus\Gamma_2,\\ \theta_2^\varepsilon=-\chi_\varepsilon^j\frac{\partial p_0}{\partial x_j}, & \text{on }\Gamma_2,\end{cases}\tag{74}$$

with a weak form as

$$\int_\Omega K^\varepsilon\nabla\theta_2^\varepsilon\nabla\varphi dx=0,\quad\forall\ \varphi\in V.$$

Denote by $e_p=p_\varepsilon-p_1^\varepsilon-\varepsilon\theta_1^\varepsilon-\varepsilon\theta_2^\varepsilon\in V$. Insert (71) into (70). Using the weak form of (74), we have

$$\int_\Omega K^\varepsilon\nabla e_p\nabla\varphi dx=I_1+\int_\Omega r_1^\varepsilon\nabla\varphi dx-\int_\Omega r_2^\varepsilon\nabla\varphi dx. \tag{75}$$

To deal with the first term I_1, from problem (6) and homogenized problem (21), we have

$$\begin{aligned} I_1 &= \int_\Omega (K^\varepsilon \nabla p^\varepsilon - K^0 \nabla p_0)\nabla\varphi dx \\ &= \alpha \int_\Omega (\phi_\varepsilon \frac{\partial T_\varepsilon}{\partial t} - \phi_0 \frac{\partial T_0}{\partial t})\varphi dx - \alpha \int_\Omega (K^0 \nabla p_0 T_0 - K^\varepsilon \nabla p_\varepsilon T_\varepsilon)\nabla\varphi dx \\ &\equiv I_3 + I_4. \end{aligned} \tag{76}$$

The term I_4 has been treated in Remark 5. What's left is to treat term I_3. To this purpose, we need further the assumption (**H**3). Denote by $w_\varepsilon = \frac{\partial}{\partial t} T_\varepsilon$ and by $w_0 = \frac{\partial}{\partial t} T_0$. We have that w_ε satisfies that

$$\begin{cases} (1+\phi_\varepsilon\gamma)\frac{\partial w_\varepsilon}{\partial t} - \nabla\cdot(D_T^\varepsilon \nabla w_\varepsilon) = 0, & \text{in } \ \Omega_T, \\ (D_T^\varepsilon \nabla w_\varepsilon)\cdot\nu = \beta_1(\frac{\partial T_{out}}{\partial t} - w_\varepsilon), & \text{on } \Gamma_1\times(0,T], \\ w_\varepsilon = 0, & \text{on } \Gamma_2\times(0,T], \\ (D_T^\varepsilon \nabla w_\varepsilon)\cdot\nu = 0, & \text{on} \Gamma_3\times(0,T], \\ w_\varepsilon(x,0) = 0, & \text{in } \Omega, \end{cases} \tag{77}$$

The corresponding homogenized problem is

$$\begin{cases} (1+\phi_0\gamma)\frac{\partial w_0}{\partial t} - \nabla\cdot(D_T^0 \nabla w_0) = 0, & \text{in } \ \Omega_T, \\ (D_T^0 \nabla w_0)\cdot\nu = \beta_1(\frac{\partial T_{out}}{\partial t} - w_0), & \text{on } \Gamma_1\times(0,T], \\ w_0 = 0, & \text{on } \Gamma_2\times(0,T], \\ (D_T^0 \nabla w_0)\cdot\nu = 0, & \text{on} \Gamma_3\times(0,T], \\ w_0(x,0) = 0, & \text{in } \Omega, \end{cases} \tag{78}$$

By the argument used in Step 1 to derive the error estimate (43), we have that

$$\sup_{0\le t\le T} \|w_\varepsilon - w_0\|_{L^2(\Omega)} \le C\varepsilon^{\frac{1}{2}}. \tag{79}$$

Now we can deal with the term I_3. By the above result and Lemma 4.6, we have

$$\begin{aligned} |I_3| &= \left|\alpha \int_\Omega (\phi_\varepsilon w_\varepsilon - \phi_0 w_0)\varphi dx\right| \\ &= \left|\alpha \int_\Omega \phi_\varepsilon (w_\varepsilon - w_0)\varphi dx - \alpha\int_\Omega (\phi_\varepsilon - \phi_0) w_0 \varphi dx\right|, \\ &\le C\alpha\varepsilon^{\frac{1}{2}}\|\varphi\|_{L^2(\Omega)} + C\alpha\varepsilon\|w_0\|_{H^1(\Omega)}\|\varphi\|_{H^1(\Omega)}. \end{aligned} \tag{80}$$

Choose $\varphi = e_p$ in equation (75). Using (76) and (80) and Remark 5, we obtain

$$\begin{aligned} \lambda_2\|\nabla e_p\|^2_{L^2(\Omega)} &\le \int_\Omega K^\varepsilon \nabla e_p \nabla e_p dx \\ &\le C_1\alpha\Big((\varepsilon+\varepsilon^{1/2})\|\nabla e_p\|_{L^2(\Omega)} + \|T_\varepsilon - T_1^\varepsilon\|_{L^2(\Omega)}\|\nabla e_p\|_{L^2(\Omega)} \\ &\quad + \|\nabla(p_\varepsilon - p_1^\varepsilon)\|_{L^2(\Omega)}\|\nabla e_p\|_{L^2(\Omega)}\Big) \\ &\quad + C(\|r_1^\varepsilon\|_{L^2(\Omega)}\|\nabla e_p\|_{L^2(\Omega)} + \|r_2^\varepsilon\|_{L^2(\Omega)}\|\nabla e_p\|_{L^2(\Omega)}). \end{aligned} \tag{81}$$

By (43),

$$\|T_\varepsilon - T_1^\varepsilon\|_{L^2(\Omega)} \quad \leq \quad C\varepsilon^{\frac{1}{2}}. \tag{82}$$

By Lemma 4.2,

$$\begin{aligned} \|\nabla(p_\varepsilon - p_1^\varepsilon)\|_{L^2(\Omega)} \quad &\leq \quad \|\nabla e_p\|_{L^2(\Omega)} + \|\varepsilon\nabla\theta_1^\varepsilon\|_{L^2(\Omega)} + \|\varepsilon\nabla\theta_2^\varepsilon\|_{L^2(\Omega)} \\ &\leq \quad \|\nabla e_p\|_{L^2(\Omega)} + C\varepsilon^{1/2}. \end{aligned} \tag{83}$$

Inserting (82)-(83) into (81), we obtain

$$\|\nabla e_p\|^2_{L^2(\Omega)} \leq C\varepsilon, \tag{84}$$

since α is small enough to satisfy that $C_1\alpha \leq \lambda_2/2$.

Thanks to the estimates on θ_1^ε and θ_2^ε in Lemma 4.2, (84) means that

$$\sup_{0\leq t\leq T} \|p_\varepsilon - p_1^\varepsilon\|_{H^1(\Omega)} \leq C\varepsilon^{\frac{1}{2}}. \tag{85}$$

Step 3. Estimate $C_\varepsilon - C_1^\varepsilon$. By the weak forms (13) for C_ε and (29) for C_0, we have for $\forall v \in V$,

$$\begin{aligned} &\langle \phi_\varepsilon \frac{\partial (C_\varepsilon - C_1^\varepsilon)}{\partial t}, v\rangle_{H^{-1}(\Omega),H(\Omega)} + \int_\Omega D_c^\varepsilon \nabla(C_\varepsilon - C_1^\varepsilon)\nabla v dx + \beta_2 \int_{\Gamma_1} (C_\varepsilon - C_1^\varepsilon) v ds \\ &\quad = \langle (\phi_0 \frac{\partial C_0}{\partial t} - \phi_\varepsilon \frac{\partial C_1^\varepsilon}{\partial t}), v\rangle_{H^{-1}(\Omega),H(\Omega)} + \int_\Omega (D_c^0 \nabla C_0 - D_c^\varepsilon \nabla C_1^\varepsilon)\nabla v dx \\ &\quad\quad + \int_\Omega (K^0 \nabla p_0 C_0 - K^\varepsilon \nabla p_\varepsilon C_\varepsilon)\nabla v dx - \varepsilon\beta_2 \int_{\Gamma_1} \pi_\varepsilon^j \frac{\partial C_0}{\partial x_j} v ds \end{aligned} \tag{86}$$

If we denote $e_c = C_\varepsilon - C_1^\varepsilon - \varepsilon\varsigma_\varepsilon$, where ς_ε is boundary corrector as the solution of the following problem:

$$\begin{cases} \phi_\varepsilon \frac{\partial \varsigma_\varepsilon}{\partial t} - \nabla\cdot(D_c^\varepsilon \nabla\varsigma_\varepsilon) = 0, & \text{in } \Omega_T, \\ (D_c^\varepsilon \nabla\varsigma_\varepsilon)\cdot\nu = -\beta_2\varsigma_\varepsilon, & \text{on } \Gamma_1 \times (0,T], \\ (D_c^\varepsilon \nabla\varsigma_\varepsilon)\cdot\nu = 0, & \text{on } \Gamma_3 \times (0,T], \\ \varsigma_\varepsilon = -\pi_\varepsilon^j \frac{\partial C_0}{\partial x_j}, & \text{on } \Gamma_2 \times (0,T], \\ \varsigma_\varepsilon|_{t=0} = 0, & \text{in } \Omega, \end{cases} \tag{87}$$

then for $e_c = C_\varepsilon - C_1^\varepsilon - \varepsilon\varsigma \in V$ we have,

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\int_\Omega \phi_\varepsilon |e_c|^2 dx + \int_\Omega D_c^\varepsilon |\nabla e_c|^2 dx + \beta_2 \int_{\Gamma_1} e_c^2 dx \\ &\quad = -\varepsilon \int_\Omega \phi_\varepsilon \pi_\varepsilon^j \frac{\partial}{\partial t}(\frac{\partial C_0}{\partial x_j})\nabla e_c dx + \int_\Omega (D_c^0 \nabla C_0 - D_c^\varepsilon \nabla C_1^\varepsilon)\nabla e_c dx \\ &\quad\quad + \int_\Omega (K^0 \nabla p_0 C_0 - K^\varepsilon \nabla p_\varepsilon C_\varepsilon)\nabla e_c dx - \varepsilon\beta_2 \int_{\Gamma_1} \pi_\varepsilon^j \frac{\partial C_0}{\partial x_j} e_c ds \\ &\quad\quad - \int_\Omega (\phi_\varepsilon - \phi_0)\frac{\partial C_0}{\partial t} e_c dx. \end{aligned} \tag{88}$$

Thanks to $\int_Y (\phi(x,y) - \phi_0(x))dy = 0$, we can get by Lemma 4.6 that

$$\left|\int_\Omega (\phi_\varepsilon - \phi_0)\frac{\partial C_0}{\partial t} e_c dx\right| \leq C\varepsilon \|\frac{\partial C_0}{\partial t}\|_{H^1(\Omega)} \|\nabla e_c\|_{L^2(\Omega)}. \tag{89}$$

The second term on the right hand side of (88) can be bounded by Lemma 4.4. The third term on the right hand side can be treated by Lemma 4.5. The fourth term on the right hand side can be handled by Trace Theorem. So we can get the estimate of $C_\varepsilon - C_1^\varepsilon$

$$\sup_{0\le t\le T} \|C_\varepsilon - C_1^\varepsilon\|_{L^2(\Omega)} + \|\nabla(C_\varepsilon - C_1^\varepsilon)\|_{L^2(\Omega_T)} \le C\varepsilon^{\frac{1}{2}}. \tag{90}$$

Now we complete the proof of Theorem 4.1. □

5. **Conclusion.** In summary, a mathematical model is established for the thermal-hydro-mass transfer processes in porous media. Then the corresponding homogenized system is derived with the help of two-scale convergence. Some error estimates are also presented for the first order expansions under some higher regular assumptions.

Acknowledgments. The authors would like to thank the referees for their valuable suggestions to improve the paper. This work is supported in part by NSF of China under the Grants 10871190 and 11271281.

REFERENCES

[1] G. Allaire, Homogénéisation et convergence à deux échelles, application à un problème de convection-diffusion, *C. R. Acad. Sci. Paris*, **312** (1991), 581–586.
[2] G. Allaire, Homogenization and two-scale convergence, *SIAM J. Math. Anal.*, **23** (1992), 1482–1518.
[3] G. Allaire, Homogenization of the unsteady Stokes equations in porous media, in *Progress in partial differential equations: Calculus of variations, applications* (eds. Bandle C, et al.) (Pont-à-Mousson, 1991), Pitman Research Notes in Math. Ser., 267, Longman Higher Education, Harlow, 1992, 109–123.
[4] I. Babuška, Solution of problem with interfaces and singularities, in *Mathematical Aspects of Finite Elements in Partial Differential Equations* (ed. C. de Boor), Academic Press, New York, 1974, 213–277.
[5] A. Bensousan, J. L. Lions and G. Papanicolaou, *Asymptotic Analysis for Periodic Structures*, North-Holland, Amsterdam, 1978.
[6] G. S. Bodvarsson, W. Boyle, R. Patterson and D. Williams, Overview of scientific investigations at Yucca Mountain-the potential repository for high-level nuclear waste, *J. Contam. Hydro.*, **38** (1999), 3–24.
[7] T. A. Buscheck, N. D. Rosenberg, J. A. Blink, Y. Sun and J. Gansemer, Analysis of thermo-hydrologic behavior for above-boiling and below-boiling thermal-operating modes for a repository at Yucca Mountain, *J. Contam. Hydro.*, **62-63** (2003), 441–457.
[8] Z. M. Chen, W. B. Deng and H. Ye, A new upscaling method for the solute transport equations, *Discrete Contin. Dyn. Syst., Ser. A*, **13** (2005), 941–960.
[9] Z. M. Chen and T. Y. Hou, A mixed multiscale finite element method for elliptic problems with oscillating coefficients, *Math. Comp.*, **72** (2003), 541–576.
[10] D. Cioranescu and P. Donato, *An Introduction to Homogenization*, Oxford Lecture Series in Mathematics and Its Applications, 17, Oxford University Press, 1999.
[11] H. I. Ene and E. Sanchez-Palencia, Some thermal problems in flow through a periodic model of porous media, *Int. J. Eng. Sci.*, **19** (1981), 117–127.
[12] L. C. Evans, *Partial Differential Equations*, American Mathematical Society, Providence, 1998.
[13] D. Gilbarg and N. S. Trudinger, *Elliptic Partial Differential Equations of Second Order*, 2^{nd} edition, Berlin Heidelberg New York, 1983.
[14] L. N. Guimaraes, A. Gens, M. Sanchez and S. Olivella, THM and reactive transport analysis of expansive clay barrier in radioactive waste isolation, *Commun. Numer. Meth. Engng.*, **22** (2006), 849–859.
[15] H. Haller, *Verbundwerkstoffe mit Formgedächtnislegierung: Mikromechanische Modellierung und Homogenisierung*, Dissertation, TU-München 1997.
[16] G. M. Lieberman, *Second Order Parabolic Differential Equations*, World Scientific, 1996.

[17] E. Marusic-Paloka and A. Piatnitski, Homogenization of a nonlinear convection- diffusion equation with rapidly oscillating coefficients and strong convection, *J. Lond. Math. Soc.*, **72** (2005), 391–409.

[18] G. Nguestseng, A general convergence result for a functional related to the theory of homogenization, *SIAM J. Math. Anal.*, **20** (1989), 608–623.

[19] O. A. Oleǐnik, A. S. Shamaev and G. A. Yosifian, *Mathematical Problems in Elasticity and Homogenization*, Studies in Mathematics and its Applications, 26, North-Holland Publishing Co., Amsterdam, 1992.

[20] S. B. Otsmane, G. Francfort and F. Murat, Correctors for the homogenization of the wave and heat equations, *J. Math. Pures Appl.*, **71** (1992), 197–231.

[21] D. Patriarche, E. Ledoux, J. L. Michelot, R. S. Coincon and S. Savoye, Diffusion as the main process for mass transport in very low water argillites: 2. Fluid flow and mass transport modeling, *Water Resource Research*, **40** (2004), W01516.

[22] G. A. Pavliotis and A. M. Stuart, *Multiscale Methods Averaging and Homogenization*, Springer, 2008.

[23] J. Rutqvist and C. F. Tsang, Analysis of thermal-hydrologic-mechanical behavior near an emplacement drift at Yucca Mountial, *J. Contam. Hydro.*, **62-63** (2003), 637–652.

[24] H. R. Thomas, Y. He, M. R. Sansom and C. L. W. Li, On the development of a model of the thermo-mechanical-hydraulic behaviour of unsaturated soils, *Engineering Geology*, **41** (1996), 197–218.

[25] C. F. Tsang, O. Stephansson and J. A. Hudso, A discussion of thermo-hydro-mechanical (THM) process associated with nuclear waste repositories, *Int. J. Rock Mech. Min. Sci.*, **37** (2000), 397–402.

[26] V. V. Zhikov, S. M. Kozlov and O. A. Oleǐnik, *Homogenization of Differential Operators and Integral Functionals*, Springer Berlin, 1994.

Received December 2012; revised February 2013.

E-mail address: xsxztr@mail.ustc.edu.cn
E-mail address: xyyue@suda.edu.cn