Limit Theorems for Differential Equations in Random Media

by

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James Beale

Dissertation submitted in partial fulfillment of the requirements for the degree of
Doctor of Philosophy in the Department of Mathematics
in the Graduate School of Duke University
2012

Abstract

(Differential Equations and Stochastic Processes)

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Abstract

Problems in stochastic homogenization theory typically deal with approximating differential operators with rapidly oscillatory random coefficients by operators with homogenized deterministic coefficients. Even though the convergence of these operators in multiple scales is well-studied in the existing literature in the form of a Law of Large Numbers, very little is known about their rate of convergence or their large deviations.

In the first part of this thesis, we establish analytic results for the Gaussian correction in homogenization of an elliptic differential equation with random diffusion in randomly layered media, which can be thought of as second-order approximations for the random solution. We also derive a Central Limit Theorem for a diffusion in a weakly random media.

In the second part of this thesis we devise a technique for obtaining large deviation results for homogenization problems in random media. We consider the special cases of an elliptic equation with random potential, the random diffusion problem and a reaction-diffusion equation with highly oscillatory reaction term.

There are a number of people to whom I am greatly indebted:

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Contents

List of Abbreviations and Symbols

Symbols

$\langle \phi \vert v \rangle$	The action of the distribution ϕ on v.
$\partial f(x_0)$	The set of subdifferentials of the extended real-valued convex function f at x_0, see Definition 2.6.10
∂U	The boundary of U
∇	Gradient
$\nabla \cdot$	Divergence
$\lambda(\varepsilon)$	The LDP rate function, see Definition 2.6.5
$\Phi_f(t)$	The *t-level set* of an extended real-valued function f, see Definition 2.6.2
φ_X	The characteristic function of a random variable X.
τ_x	The shift operator on Ω
Ω	A probability sample space
ω	A random realization of Ω
$\mathcal{A}$	The space of positive definite matrices with $a_{min} \leq a \leq a_{max}$
A, B	Borel subsets of a topological space
$\bar{A}$	The closure of the set A
A°	The interior of the set A
a^T	Transpose of the matrix a
$BUC(X)$	The space of bounded uniformly continuous functions on X
$C(X)$	The space of continuous functions on X

$C^s(X)$	The space of Hölder continuous functions with exponent s on X
$C_b(X)$	The space of bounded continuous functions on X
$\mathcal{D}_f$	The *domain* of an extended real-valued function f, see Definition 2.6.1.
$D_g f$	The Gateaux derivative of f at g, see Definition 2.6.15
$\mathcal{F}$	The σ-algebra for Ω
$\mathfrak{F}$	The Fourier Transform
$\mathcal{F}_A, \mathcal{F}_{q,A}$	The σ-algebra generated by $\{q_x \mid x \in A\}$
$\mathfrak{f}$	The mixing coefficient
$\mathcal{H}$	A Sobolev space with norm $\|v\|_{\mathcal{H}^s(U)} = \|\mathcal{L}^{s/2} v\|_{L^2(U)}$.
H	A convex function
$H^s(U)$	Hilbert space of functions with square-integrable derivatives of order s
$H^s_w(U)$	$H^s(U)$ space with the weak topology.
$\mathcal{L}$	self-adjoint elliptic differential operator in divergence form
L	The convex dual of H
$L^p(U), L^p(U; \mathbb{R}^m)$	Space of p-integrable functions $f : U \to \mathbb{R}^m$.
$L^p_w(U), L^p_w(U; \mathbb{R}^m)$	L^p space with the weak topology (if $1 \le p < \infty$) or the weak-* topology (if $p = \infty$)
$\mathbb{N}$	The set of natural numbers $0, 1, 2, \dots$
P	A probability measure
$\mathbb{Q}$	The set of rational numbers
q	A random field
$\mathbb{R}$	The set of real numbers
R, R_q	The correlation function of q
$\Re$	The real part of a complex number
$S, S_{q^\varepsilon}, S_k, S_X$	The LDP action functional, see Definition 2.6.4

$\mathbb{T}^n$	The torus $\mathbb{T} = [0,1)^n$.
Tf, $T_{q^\varepsilon}f$	The asymptotic logarithmic moment generating function of a sequence of random fields, see (3.5)
T_G, T_{G,q^ε}	The asymptotic exponential G-moment of a sequence of random fields, see (3.6b)
$T_{p,\alpha}(\kappa)$, $T_{p,\alpha,q^\varepsilon}(\kappa)$	The asymptotic exponential moment of a sequence of random fields q^ε at κ, see 3.6a
U	A connected open bounded set in $\mathbb{R}^n$ with smooth boundary
$W^{s,p}(U)$	Sobolev space of functions with p-integrable derivatives of order s
$W^{s,p}_w(U)$	Sobolev space $W^{s,p}_w(U)$ with the weak topology
X, Y	Topological spaces
$\mathbb{Z}$	The set of integers
$\mathfrak{z}_k$	Sequence of random variables

Abbreviations

CLT	Central Limit Theorem
iid	independent and identically distributed
LDP	Large Deviation Principle see (2.18)
MA	Moving Average
ODE	Ordinary Differential Equation
PDE	Partial Differential Equation
PPP	Poisson Point Process
SDE	Stochastic Differential Equation
SPDE	Stochastic Partial Differential Equation

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1

Introduction

1.1 Homogenization and Averaging Background

Many models in physics and engineering consist of stochastic processes and partial differential equations with parameters that oscillate on a scale smaller than the scale of the domain on which the equation is solved. In material science, for example, we may consider a conductor occupying some bounded region $U \subset \mathbb{R}^n$ in space which is composed of a mixture of several materials with different conductivities. At a macroscopic level, the conductivity changes rapidly as the position varies over lengths comparable to the size of U. To incorporate this feature into the model, we introduce a variable $\varepsilon > 0$ which corresponds to the ratio of the microscopic scale associated with the variations in the conductivity to the the typical length scale of U. In this setting, we may take the thermal conductivity to be a function of the form $a(\frac{x}{\varepsilon}), x \in \mathbb{R}^n$. Thus, the temperature $u^\varepsilon \in H^1_0(U)$ satisfies the PDE

$$-\nabla \cdot a(\frac{x}{\varepsilon})\nabla u^\varepsilon(x) = f(x). \tag{1.1}$$

Here, we assume that the matrix a is strictly positive definite and bounded, ∂U is smooth and that this problem is well-posed by imposing appropriate boundary conditions (Dirichlet, Neumann, etc.). A downside of this formulation is that classical numerical schemes are not efficient as the discrepancy of scales leads to a mesh size of order $\frac{1}{\varepsilon}$.

In order to make the numerical analysis of this equation more tractable, it is desired to replace u^ε by a *homogenized* or *effective* temperature u^0, which can be done provided the coefficient $a(\frac{x}{\varepsilon})$ oscillates rapidly enough. For instance, it can be shown (Pavliotis and Stuart, 2000, Chapter 12) that if a is periodic, say with period 1, and u^ε has Dirichlet boundary conditions, $\|u^\varepsilon - u^0\|_{L^2(U)} \to 0$ where $u^0 \in H_0^1(U)$ solves the boundary value problem

$$-\nabla \cdot a^0 \nabla u^0(x) = f(x). \tag{1.2}$$

Here, a^0 is called the *homogenized conductivity* and can be computed in terms of a. Specifically, $a^0 = \int_{\mathbb{T}^n} a((\nabla\chi)^T + I)\, dy$, where $\chi \in L^2(\mathbb{T}^n; \mathbb{R}^n)$ is the unique solution to the *cell problem*

$$-\nabla \cdot (a((\nabla\chi)^T + I)) = 0 \tag{1.3}$$

with periodic boundary conditions satisfying $\chi(0) = 0$. In fact, we have the stronger result $\|u^\varepsilon - u^0 - \chi^\varepsilon \cdot \nabla u^0\|_{H^1(U)} \to 0$.

In practice, however, the properties of the highly oscillatory coefficient are usually only known at the statistical level. Hence, it is often convenient to model the diffusion coefficient a as a random field. In the articles Kozlov (1980); Papanicolaou and Varadhan (1981), the notion of stochastic homogenization is made rigorous by showing that $E\|u^\varepsilon - u^0\|_{L^2(U)} \to 0$ under the assumption that the random field is stationary and ergodic, in which case a^0 and u^0 are deterministic. In this case, however, the homogenized conductivity is given by the formula $a^0 = Ea((\nabla\chi)^T + I)$, and χ is the unique solution $\chi \in L^2(\mathbb{R}^n; \mathbb{R}^n)$ for (1.3) satisfying $\nabla\chi$ is stationary and $\chi(0) = 0$. In fact, in analogy to the periodic case, it is also shown the stronger result $E\|u^\varepsilon - u^0 - \chi^\varepsilon \cdot \nabla u^0\|_{H^1(U)} \to 0$. In this setting, we can think of homogenization as a functional law of large numbers where the solution converges to a deterministic quantity as $\varepsilon \to 0$.

While problem (1.1) presents some of the most interesting features of the theory of homogenization we will also introduce the diffusion equation with random potential

$$-\Delta u^\varepsilon(x) + (b - q(\frac{x}{\varepsilon}))u^\varepsilon(x) = f(x) \text{ in } U, \tag{1.4}$$

where $b > 0$, $f \in L^2(U)$ and q is a mean-zero ergodic stationary random field on $\mathbb{R}^n$. In order to make the problem well posed, we assume an appropriate boundary condition (Dirichlet, Neumann) and that $q < b$ uniformly as to avoid the spectrum of $-\Delta$. If q is bounded from below or does not grow too fast (cf. Section 5.4.1 and Bal (2008)) it can be shown that $E\|u^\varepsilon - u^0\|_{H^1(U)} \to 0$ as $\varepsilon \to 0$, where u^0 solves the *averaged* problem

$$-\Delta u^0(x) + bu^0(x) = f(x) \text{ in } U \tag{1.5}$$

with the same boundary conditions as (1.4). We remark that due to the singular perturbation of the diffusion term in (1.1), the effective equation does not arise from merely averaging $a(\frac{x}{\varepsilon})$ and its analysis presents further challenges than in the averaging case (1.4).

Many research papers on stochastic homogenization have arisen over the last two decades. One such work is the article Caffarelli et al. (2005), where a homogenization result is proved for some fully nonlinear differential equations with ergodic coefficients, using the theory of viscosity solutions.

Another interesting feature is that the limiting process may not necessarily be a deterministic one. For instance, Bal (2010) shows that a family of parabolic equations with *large-amplitude potentials* converges to the solution of a stochastic partial differential equation.

Among the nonlinear problems in averaging theory, we shall consider the case of the reaction-diffusion Boundary-Value Problem

$$\frac{\partial}{\partial t}u^{\varepsilon} = \Delta u^{\varepsilon} + b(u^{\varepsilon}, q(\frac{x}{\varepsilon})) \qquad \text{in } U \tag{1.6}$$

$$u^{\varepsilon} = 0 \qquad \text{on } \partial U \times [0,T]$$

$$u^{\varepsilon} = g \in W_0^{1,p} \cap C^1(U) \qquad \text{on } U \times \{t=0\}$$

For simplicity, let us assume that b is a bounded function that is Lipschitz on its first argument, i.e.

$$|b(v_1,\varphi) - b(v_2,\varphi)| \leq K|v_1 - v_2|, \tag{1.7}$$

The Banach fixed point theorem can be applied to u_v^{ε} satisfying the BVP

$$\frac{\partial}{\partial t}u_v^{\varepsilon} = \Delta u_v^{\varepsilon} + b(v, q(\frac{x}{\varepsilon})) \qquad \text{in } U \tag{1.8}$$

$$u_v^{\varepsilon} = 0 \qquad \text{on } \partial U \times [0,T]$$

$$u_v^{\varepsilon} = g \in W_0^{1,p} \cap C^1(U) \qquad \text{on } U \times \{t=0\}$$

to show existence and uniqueness of u^{ε} (Evans, 2010, Section 7.1). Using the techniques of Freidlin and Wentzell (1998), it can be shown (see Lemma 5.4.4) that u^{ε} converges to u^0 in the mean-square sense provided that

$$\lim_{r\to\infty} \frac{1}{|B(0,r)|} \int_{B(0,r)} [b(v(y), q(\frac{y}{\varepsilon})) - \bar{b}(v(y))] f(y)\, dy = 0 \tag{1.9}$$

for all $f \in L^1(U)$. Here, u^0 solves the BVP

$$\frac{\partial}{\partial t}u^0 = \Delta u^0 + \bar{b}(u^0) \qquad \text{in } U \tag{1.10}$$

$$u^0 = 0 \qquad \text{on } \partial U \times [0,T]$$

$$u^0 = g \qquad \text{on } U \times \{t=0\}.$$

1.1.1 Fluctuations

Even though most of the current research has focused on understanding the convergence of u^{ε} to the homogenized solution u^0, not much has been done in trying to understand the asymptotic behavior of the random error $w^\varepsilon = u^\varepsilon - u^0$. During the last decade, there have been numerous attempts to study convergence rates in homogenization of differential equations whose coefficients are highly oscillatory or present a fine-scale structure. In most problems, it is expected that the fluctuations (or rate of convergence) of w^ε are of order ε^n, and that $\frac{1}{\varepsilon^n} w^\varepsilon$ can be approximated by a Gaussian random field as $\varepsilon > 0$, shrinks to zero. This amounts to deriving a Central Limit Theorem on a suitable function space.

In the works Bal (2008); Bal and Jing (2011), for example, the fluctuations for (1.4) with Dirichlet boundary conditions are characterized as Gaussian random fields and their statistical properties are given. This is done as follows. By subtracting (1.5) from (1.4), we see that the random corrector $w^\varepsilon = u^\varepsilon - u^0$ may be expressed as

$$\frac{1}{\varepsilon^{\frac{n}{2}}} w^\varepsilon = \mathcal{G}\left[\frac{1}{\varepsilon^{\frac{n}{2}}} q^\varepsilon (w^\varepsilon + u^0)\right],$$

where $q^\varepsilon(x) = q(\frac{x}{\varepsilon})$ and $\mathcal{G} = (-\Delta + b)^{-1}$ is the Green's operator of $-\Delta + b$ with Dirichlet boundary conditions. By repeated iteration of this equation we get the following equation

$$\frac{1}{\varepsilon^{\frac{n}{2}}} w^\varepsilon = \mathcal{G}\left[\frac{1}{\varepsilon^{\frac{n}{2}}} q^\varepsilon u^0\right] + \varepsilon^{\frac{n}{2}} \mathcal{G}\left[\frac{1}{\varepsilon^{\frac{n}{2}}} q^\varepsilon \mathcal{G}\left[\frac{1}{\varepsilon^{\frac{n}{2}}} q^\varepsilon u^\varepsilon\right]\right]. \tag{1.11}$$

For $n \leq 3$, it can be shown that the last term converges to zero in probability. Hence, disregarding the last term, it can be shown that the mapping from $\frac{1}{\varepsilon^n} q^\varepsilon$ to u^ε is weakly continuous from $L^2_w(U)$ to $L^2(U)$, whence a Limit Theorem can be found. Indeed, the random fluctuations can be approximated by the random oscillatory integral $\mathcal{G}\left[\frac{1}{\varepsilon^{\frac{n}{2}}} q^\varepsilon u^0\right] = \int_U G(x,y) \frac{1}{\varepsilon^{\frac{n}{2}}} q^\varepsilon(y) u^0(y)\, dy$ which is easily shown to converge to a Gaussian process. For larger n, more iterations can be performed until the last term is shown to vanish in probability as $\varepsilon \to 0$.

Using similar techniques, Bal and Jing (2010); Bal (2010) obtained a fluctuation result for parabolic and transport equations, and Bourgeat and Piatnitski (1999); Bal et al. (2008) works out the problem (1.1) in the one-dimensional case.

1.1.2 Large Deviations

While Central Limit approximations only give us information about moderate deviations (of order $\varepsilon^{\frac{n}{2}}$), there are some instances, however, in which one is interested in finding the asymptotics of probabilities of large deviations of w^ε from zero. Some applications include hypothesis testing, risk theory and uncertainty quantification, where one is interested in quantifying probabilities of rare events of the form $P(\|w^\varepsilon - v\| < \ell)$ and $P(\|w^\varepsilon\| > \ell)$ for $\ell \sim O(1)$. Here, v represents an unlikely realization of the corrector and $\|\cdot\|$ is an suitably chosen norm. While the theory of large deviations has grown considerably in the cases of an SDE with a small diffusion term and multiple scales with periodic coefficients, or multiple scales in a one-dimensional random media, not much has been done in the context of stochastic multiple scales in higher dimensions.

Under certain assumptions on the logarithmic moment generating function of the random field, the Large Deviation Theory asserts that these probabilities are upper or lower bounded by $C_1 e^{-\lambda(\varepsilon) C_2}$ for small $\varepsilon > 0$, with $\lambda(\varepsilon) \to 0$ as $\varepsilon \to 0$. In fact, the theory characterizes the rate function $\lambda(\varepsilon)$ and the optimal C_2, which can be written as the infimum of the action functional over the desired set. It turns out that the large deviation probabilities (of order 1) that one obtains from the LDP for w^ε are actually different from the ones we obtain from the Gaussian probabilites coming from the oscillatory terms (of order $\varepsilon^{\frac{n}{2}}$). As a consequence, we cannot neglect terms in the Neumann series (1.11) when considering large deviations.

Let us now review some of the known results of large deviations in the context of equations in multiple scales. One first attempt to study large deviations for equations in multiple scales appears in the work of (Freidlin and Wentzell, 1998, Chapter

7), where the authors study the LDP on the so called "averaging principle" for the ODE

$$\frac{d}{dt}u^\varepsilon(t) = b(u(t), q(t/\varepsilon)), u^\varepsilon(0) = x. \tag{1.12}$$

Here, b is assumed to be a uniformly bounded function with a bounded gradient and q is a stochastic process such that $b(x, q_s)$ is ergodic for all x, i.e.

$$\bar{b}(x) = \lim_{T\to 0} \frac{1}{T}\int_0^T b(x, q(s))\, ds \quad \text{a.s.}$$

The averaging principle for (1.12) states that, for small $\varepsilon > 0$, u^ε is uniformly approximated on the time interval $[0, T]$ by trajectories of the averaged system

$$\frac{d}{dt}u^0(t) = \bar{b}(u^0(t)), \quad u^\varepsilon(0) = x.$$

Under conditions reminiscent of (A1)-(A3) in Section 3, the authors prove the LDP for (1.12) on the space of continuous functions on the time interval $[0, T]$. For fixed $1 \le p \le \infty$, $0 \le x \le T$ and $v \in C([0, T])$,

$$A_1^v = \{\varphi \in C([0,T]) \,|\, \|\varphi - v\|_{L^p(0,T)} > \delta\},$$
$$A_2^x = \{\varphi \in C([0,T]) \,|\, |\varphi(x) - a| > \delta\} \text{ and}$$
$$A_3 = \{\varphi \in C([0,T]) \,|\, \text{ the modulus of continuity of } \varphi > \delta\}$$

are all important Borel subsets of $C([0, T])$, so the Large Deviations give upper and lower bounds for $P(u^\varepsilon \in A_j)$.

A second article worth mentioning is Bal et al. (2011), where the "pointwise" LDP for u^ε, the solution to the one-dimensional Boundary Value Problem

$$\frac{d}{dx}a(\frac{x}{\varepsilon})\frac{d}{dx}u^\varepsilon(x) = f(x), \quad u^\varepsilon(0) = u^e(1) = 0, \tag{1.13}$$

is derived. By "pointwise" we mean that the LDP for $u^\varepsilon(x)$ is derived for fixed x, that is, the bounds for the probabilities $P(u^\varepsilon \in A_j)$ are only proved when $j = 2$.

It is also worth mentioning the article Freidlin (1985), where the asymptotics of a reaction-diffusion equation with a highly oscillatory reaction term are obtained from its stochastic representation via the Feynman-Kac Formula. The LDP for this stochastic functional yields the asymptotics for the original equation.

1.2 Overview of the Results

The objective of this thesis is to present techniques which are useful to obtain Large Deviation and Central Limit Theorems for differential equations with multiple scales in random media. In particular, these techniques are used to derive some results for the examples presented in Section 1.1.

Chapter 2 presents the mathematical machinery necessary to be able to state and prove the CLTs and LDPs.

In Chapter 3, we state the main results rigorously and discuss their proofs. The main results which are proved are:

1. CLT for (1.1) in a weakly random media
2. CLT for (1.1) in layered media.
3. LDP for (1.4)
4. LDP for (1.1)
5. LDP for (1.6)

Chapters 4 and 5 present the proofs for the CLT and the LDP, respectively. The assumptions required to prove these results mainly rely on the statistical properties of the random field, such as ergodicity, strong mixing and the existence of exponential moments.

Finally, Chapter 6 presents some concluding remarks and gives an outline of future research.

2

Preliminaries

2.1 Notation

Let $v : \mathbb{R}^m \to \mathbb{R}^m$ The gradient of v is the $m \times m$ matrix such that $(\nabla v)_{i,j} = \frac{\partial}{\partial x_j} v_i$ and the divergence of v is the scalar $\nabla \cdot v = \sum_{j=1}^m \frac{\partial}{\partial x_j} v_j$. For an $m \times m$ matrix a, the divergence is the vector such that $(\nabla \cdot a)_j = \sum_{i=1}^m \frac{\partial}{\partial x_j} a_{i,j}$.

Fix constants $0 < a_{min} < a_{max}$. Define $\mathcal{A}$ to be the set of functions with values on the positive $n \times n$ definite matrices a such that $as \cdot s \geq a_{min}|s|^2$ and $|as| \leq a_{max}|s|$.

2.2 Sobolev Spaces

Denote by $W^{k,p}(\mathbb{R}^n)$ the Sobolev space of functions $u \in L^p(\mathbb{R}^n)$ such that for every multi-index α with $|\alpha| \leq k$, the weak partial derivative $D^\alpha u$ belongs to $L^p(\mathbb{R}^n)$ with norm

$$\|u\|_{W^{k,p}(\mathbb{R}^n)} := \begin{cases} \left(\sum_{|\alpha|\leq k} \|D^\alpha u\|^p_{L^p(\mathbb{R}^n)}\right)^{1/p}, & 1 \leq p < +\infty; \\ \sum_{|\alpha|\leq k} \|D^\alpha u\|_{L^\infty(\mathbb{R}^n)}, & p = +\infty. \end{cases}$$

Define the Fourier Transform of $g \in L^1(\mathbb{R}^n)$ by

$$\hat{g}(\xi) = \frac{1}{(2\pi)^{\frac{n}{2}}} \int_{\mathbb{R}^n} e^{-ix\cdot\xi} g(x)\, dx. \tag{2.1}$$

For $g, h \in L^2(\mathbb{R}^n)$, Plancherel's Identity states that

$$\int_{\mathbb{R}^n} g_y \bar{h}_y\, dy = \int_{\mathbb{R}^n} \hat{g}_\xi \bar{\hat{h}}_\xi\, d\xi. \tag{2.2}$$

In order to obtain sharp results, we will need to work with fractional Sobolev spaces. We define $W^{k+r,p}(\mathbb{R}^n)$ to be the space of functions whose norm

$$\|u\|^p_{W^{k+r,p}(\mathbb{R}^n)} = \|u\|^p_{W^{k,p}(\mathbb{R}^n)} + \sum_{|\alpha|=k} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{|D^\alpha u(x) - D^\alpha u(y)|^p}{|x-y|^{n+pr}}\, dx\, dy,$$

is finite, where $1 \le p < \infty$, k is a positive integer and $0 < r < 1$. The corresponding spaces for negative k can be obtained by duality. If $p = 2$, we define $H^s(\mathbb{R}^n) = W^{s,2}(\mathbb{R}^n)$ and the norm is equivalent to

$$\|u\|_{H^s(\mathbb{R}^n)} = \left\| \mathfrak{F}^{-1}[((1 + |\cdot|^{2s})^{s/2} \hat{u}] \right\|_{L^2(\mathbb{R}^n)}. \tag{2.3}$$

for all $s \in \mathbb{R}$. Here, $\mathfrak{F}^{-1}$ denotes the inverse Fourier transform. We denote the closure of $C_0^\infty(\mathbb{R}^n)$ in $W^{s,p}(\mathbb{R}^n)$ by $W_0^{s,p}(\mathbb{R}^n)$. The notation $W_w^{s,p}(\mathbb{R}^n)$ represents the topological space $W^{s,p}(\mathbb{R}^n)$ with its weak topology. A number of continuous embeddings can be found among these spaces. The proofs of these depend on the Littlewood-Paley decomposition and can be found on Runst and Sickel (1996).

(a) $W^{s_1,p_1}(\mathbb{R}^n) \subset W^{s_2,p_2}(\mathbb{R}^n)$ if $s_1 - \frac{n}{p_1} \ge s_2 - \frac{n}{p_2}$, $p_1 \le p_2$ and $s_2 < s_1$.

(b) $W^{s_1,p}(\mathbb{R}^n) \subset C^{s_2}(\mathbb{R}^n)$ if $s_1 - \frac{n}{p_1} = s_2 > 0$, $1 < p < \infty$ and s_2 not an integer.

(c) $W^{s,p_1}(U) \subset W^{s,p_2}(U)$ for $p_2 < p_1$.

(d) $W^{s+\theta,p}(U) \subset\subset W^{s,p}(U)$ and $C^{s+\theta}(U) \subset\subset C^s(U)$ are compact for all $\theta > 0$.

The first two embeddings above are also valid for domains other than $\mathbb{R}^n$ in an obvious way. For the last two embeddings, U must be an open, bounded set with smooth boundary. All other embeddings can be obtained by combining these results.

2.3 Partial Differential Equations

In this section we cite the main regularity results we shall use throughout the thesis. Even though the results are stated in terms of Dirichlet boundary conditions, they are true for any boundary condition such which admits a unique weak solution. We will always assume U is an open, bounded set with smooth boundary. The next result is a consequence of (Gilbarg and Trudinger, 1998, Theorem 9.11).

Lemma 2.3.1. *Suppose that $b(x) > 0$ for all $x \in U$. If $f \in L^p(U)$ there is a unique solution $u \in W^{2,p} \cap H_0^1(U)$ to the equation $-\Delta u + bu = f$ which satisfies*

$$\|u\|_{W^{2,p}(U)} \leq C\|f\|_{L^p(U)}, \tag{2.4}$$

where C only depends on $\|b\|_{L^\infty(U)}$ and diameter(U).

Lemma 2.3.2. *Consider the parabolic BVP*

$$\begin{aligned} &\frac{\partial}{\partial t}u = \Delta u + f && \text{in } U \\ &u = 0 && \text{on } \partial U \times [0,T] \\ &u = g \in W_0^{1,p} \cap C^1(U) && \text{on } U \times \{t=0\} \end{aligned}$$

If $f \in L^p(U)$, there is a unique solution $u \in W^{1,p}$ which satisfies the estimates

$$\|u\|_{L^p(0,T;W^{2,p}(U))} + \|\frac{\partial}{\partial t}u^\varepsilon\|_{L^p(0,T;L^p(U))} \leq C(\text{diameter}(U), n)[\|f\|_{L^p(0,T;L^p(U)} + \|g\|_{W_0^{1,p}(U)}]. \tag{2.5}$$

Furthermore, if $f \in L^\infty(U)$ and g is Hölder continuous, the solution is Hölder continuous and for every $0 < s < 2$ there is $0 < a < 1$ such that

$$\|u\|_{C^a(0,T;H^{2,s}(U))} \leq C(\text{diameter}(U), n)[\|f\|_{L^\infty(0,T;L^\infty(U)} + \|g\|_{C^a(U)}]. \tag{2.6}$$

Proof. This result follows from (Lieberman, 2005, Theorem 12.14) and the interpolation inequalities in Runst and Sickel (1996). □

For $a \in \mathcal{A}$, consider the self-adjoint elliptic operator $\mathcal{L} = -\nabla \cdot a\nabla + b$, with $b > 0$. By the Lax-Milgram Theorem, there is a unique $u \in H_0^1(U)$ solving $\mathcal{L}u = f$, and $\|u\|_{H_0^1(U)} \leq C\|f\|_{L^2(U)}$, where C does not depend on u or f. Let $\mathcal{G} = (\mathcal{L})^{-1}$ be the Green's operator of $\mathcal{L}$ with Dirichlet boundary conditions, so that $u = \mathcal{G}f$. It is well-known that $\mathcal{G}$ is a self-adjoint, strictly positive definite compact operator, whence there is a sequence of eigenvalues $0 < \lambda_1 \leq \lambda_2 \leq \lambda_3 \leq \cdots$ of $\mathcal{L}$ such that $\lambda_k \to \infty$ as $k \to \infty$, with corresponding eigenvectors $e_k \in L^2(U)$ which form an orthonormal basis in $L^2(U)$.

For $s \in \mathbb{R}$, define the Sobolev Space

$$\mathcal{H}^s(R) = \left\{ v = \sum_{j,k=1}^{\infty} v_k e_k \mid \|v\|_{\mathcal{H}^s(R)} = \sum_{k=1}^{\infty} \lambda_k^s |v_k|^2 < \infty \right\}.$$

If $v = \sum_{j,k=1}^{\infty} v_k e_k$, define the power $(\mathcal{L})^{s/2} v = \sum_{k=1}^{\infty} \lambda_k^s e_k$. By the functional calculus of elliptic operators (cf. Reed and Simon (1980)), $(\mathcal{L})^{s/2}$ is a positive definite self-adjoint operator with compact resolvent mapping $\mathcal{H}^{s+t}(R)$ into $\mathcal{H}^t(R)$. Furthermore,

$$\begin{aligned} \|v\|_{\mathcal{H}^s(R)} &= \sum_{k=1}^{\infty} \lambda_k^s |v_k|^2 \\ &= \sum_{k=1}^{\infty} \lambda_k^s \left| \int_R v_k e_k \, dx \right|^2 \\ &= \sum_{k=1}^{\infty} \left| \int_U v_k \mathcal{L}^{s/2} e_k \, dx \right|^2 \\ &= \|\mathcal{L}^{s/2} v\|_{L^2(R)}^2. \end{aligned}$$

Since the inner product of the basis vectors e_j and e_k on $\mathcal{H}^s$ is

$$(e_j, e_k)_{\mathcal{H}^s} = (\mathcal{L}^{\frac{s}{2}} e_j, \mathcal{L}^{\frac{s}{2}} e_k)_{L^2(U)} = \lambda_j^{s/2} \lambda_k^{s/2} \delta_{j,k},$$

$\{\lambda_k^{-s/2} e_k\}_{k=1}^{\infty}$ forms an orthonormal basis for $\mathcal{L}$ on $\mathcal{H}^s$.

On the Fourier side, after extending v to be 0 outside U, this norm has the equivalent form

$$\|v\|_{\mathcal{H}^s(R)} = \int_{\mathbb{R}^n} (a\xi\cdot\xi + b)^{s/2}|\hat{v}|^2\, d\xi.$$

In particular, if $H^s(R)$ denotes the space with norm (2.3) then $\|v\|_{\mathcal{H}^s(R)} \leq C\|v\|_{H^s(R)}$.
By Poincaré Inequality,

$$\|v\|^2_{H^1_0(U)} \leq C\int_U \mathcal{L}v\, v\, dx = \|v\|_{\mathcal{H}^1(U)},$$

so $\mathcal{H}^1(U)$ and $H^1_0(U)$ are equivalent. By duality, the same is true for $H^{-1}(U)$ and $L^2(U)$.

The asymptotic behavior of the eigenvalues of $\mathcal{L}$ is given by Weyl's Formula (cf. Safarov and Vassiliev (1991))

$$\lim_{j\to\infty} \frac{\lambda_j^{\frac{n}{2}}}{j} = C, \tag{2.7}$$

where C depends on the coefficients of $\mathcal{L}$ and the domain U.

We say that the distribution $G \in \mathcal{D}'(U\times U)$ is the Green's function of $\mathcal{L}$ if, for fixed $y \in U$, it satisfies the homogeneous Boundary Value Problem

$$\mathcal{L}_x G(x,y) = \delta(x-y) \text{ in } U$$
$$G(x,y) = 0 \qquad \text{for } x \in \partial U, y \in U.$$

The Green's function is unique and continuously differentiable away from the diagonal $x = y$. In fact, it is the integral kernel of the Green's operator $\mathcal{G}$, that is, the solution to the problem $\mathcal{L}u = f$ can be written as $u(x) = \mathcal{G}f(x) = \int_U G(x,y)f(y)\, dy$.

2.4 Ergodicity

In this section we introduce the main tools in ergodic theory we shall need throughout our study. The setup is mostly standard and can be found, for instance, in (Papanicolaou and Varadhan, 1981, Section 2) or (Durrett, 2010, Chapter 7). Let

$(\Omega, \mathcal{F}, P)$ be a probability space. Suppose that for every $x \in \mathbb{R}^n$ there exists an operator $\tau_x : \Omega \to \Omega$ such that

(a) The map $(x, \omega) \to \tau_x(\omega)$ is $(\mathbb{R}^n \times \Omega, B(\mathbb{R}^n) \times \mathcal{F})$-measurable.

(b) $P(\tau_x A) = P(A)$ for all $A \in \mathcal{F}, x \in \mathbb{R}^n$.

(c) $\tau_{x+y} = \tau_x \circ \tau_y, \tau_0 \omega = \omega$.

The set $\{\tau_x\}_{x \in \mathbb{R}^n}$ is said to be a *measure-preserving transformation* group acting on Ω. For any $q_0 \in L^p(\Omega), 1 \leq p \leq \infty$, the random field $q(x, \omega) = q_0(\tau_x \omega)$ is said to be stationary and has the property that the vector $(q(x_1), q(x_2), \ldots, q(x_r))$ has the same law as the shifted vector $(q(x_1 - y), q(x_2 - y), \ldots, q(x_r - y))$.
A set $A \subset \mathcal{F}$ is τ-invariant if $\tau_x A = A$ for all $x \in \mathbb{R}^n$. The measure-preserving transformations τ_x (or a stationary random field q) are said to be ergodic if all the τ-invariant sets have probability 0 or 1. The importance of ergodicity relies on the Birkhoff Ergodic Theorem (cf. (Durrett, 2010, Theorem 7.2.1), which provides a law of large numbers for stationary random fields. In this work, ergodicity is the essential tool in understanding the asymptotics of oscillatory integrals.
If $U \subset \mathbb{R}^n$ is an open bounded set and $g \in L^p(\mathbb{T}^n)$ is a periodic function then $g^\varepsilon(x) = g(\frac{x}{\varepsilon})$ converges to $\int_{\mathbb{T}^n} g(y)\, dy$ weakly in $L^p(U)$ if $1 \leq p < \infty$ and in the weak-* topology in $L^\infty(U)$ if $p = \infty$ (Pavliotis and Stuart, 2000, Theorem 2.29). A more general result can be proved if g is replaced with an ergodic random field $q : \Omega \times U \to \mathbb{R}^m$. Throughout, we adopt the notation $q^\varepsilon(x) = q(\frac{x}{\varepsilon})$ for $\varepsilon > 0$.

Theorem 2.4.1. *Let q be a uniformly bounded stationary ergodic random field and $f \in L^1(U)$. Then, the sequence of integrals $\int_U f_y \cdot q_{y/\varepsilon}\, dy$ converges to $\int_U f_y \cdot E[q_0]\, dy$ a.s. as $\varepsilon \to 0$.*

Proof. Let $T_y : \Omega \to \Omega$ denote the shift operator $T_x q(\omega) = q(\tau_x \omega)$. The sequence $\{T_x\}_{x \in \mathbb{R}^n}$ forms a strongly continuous unitary group on $L^2(\Omega)$, i.e. $T_{x+y} = T_x \circ$

$T_y, T_0 = I$ and the transpose of T_x on $L^2(\Omega)$ is T_{-x}. Hence, by Stone's Theorem (Reed and Simon, 1980, Section 8.4) there is a spectral projection-valued measure π on $\mathbb{R}^n$ such that $\tau_y = \int_{\mathbb{R}^n} e^{iy\cdot\xi}\,\pi(d\xi)$. From Fubini's Theorem we have that

$$\int_U f_y \cdot q_{y/\varepsilon}\,dy = \int_U f_y \cdot \int_{\mathbb{R}^n} e^{iy\cdot\xi/\varepsilon}\,\pi(d\xi)\,q(\omega)\,dy = \int_{\mathbb{R}^n}\int_U e^{iy\cdot\xi/\varepsilon} f_y\,dy\cdot\,\pi(d\xi)\,q(\omega).$$

The Riemann-Lebesgue Lemma yields that $\int_U e^{iy\cdot\xi/\varepsilon} f_y\,dy$ vanishes for $\xi \neq 0$ as $\varepsilon \to 0$. Hence, this integral converges to $\int_U f_y \cdot \pi(0)q(\omega)\,dy$ by the Dominated Convergence Theorem. But $\pi(0)$ is the projection operator into the τ-invariant functions, which are constants by the ergodicity assumption. Thus, $\pi(0)q(\omega) = E[q_0]$, and we are done. $\square$

If q is not uniformly bounded, but $q \in L^p(U)$ a.s., then a similar result holds provided $f \in L^{p'}(U)$, where p' denotes the Hölder conjugate of p. In practice, this result arises in the following way. Suppose that the random field q is an ergodic mean-zero random field in $L^p_{loc}(U)$ and $f : L^p_w(U) \to X$ is continuous in the weak topology for some topological vector space X. Then, the sequence $f \circ q^\varepsilon$ converges to $f(0)$ a.s. by Theorem 2.4.1. For example, if u^ε is as in (1.4), with q a bounded random field, it can be shown that the function mapping $q(\frac{x}{\varepsilon})$ to u^ε is continuous from $L^\infty_w(U)$ to $H^1_0(U)$, whence u^ε converges a.s. to u^0 in (1.5) by Theorem 2.4.1. We will generalize this example and we will study (1.1) in Chapter 5.

2.5 Central Limits in Random Media

We now wish to study the rate of convergence in homogenization, which may be described in terms of a Gaussian random process. The corresponding fluctuation theory often requires additional assumptions on the correlation function of the random field in addition to ergodicity, namely, a strong mixing condition.

2.5.1 Central Limits on Hilbert Spaces

In this subsection we give a brief overview of the techniques we use to prove central limit theorems on Hilbert Spaces. For a deeper treatment of the results presented herein, the reader is encouraged to consult Billingsley (1999).

Definition 2.5.1. Let $q^\varepsilon, q : \Omega \to X$ be random fields. We say that the law of q^ε converges to q in X if and only if $\lim_{\varepsilon\to 0} Eg(q^\varepsilon) = Eg(v)$ for all $g \in C_b(X)$. Similarly, the laws of the measures μ_ε on X induced by q^ε converge to μ in X if and only if they converge in the weak-* topology on the set of Borel measures of X. It is well known that $C_b(X)$ can be replaced by $BUC(X)$ in this definition.

By defining $\mu_\varepsilon(B) = P(q^\varepsilon \in B)$ for any Borel $B \subset X$ we will sometimes interchange the notions of convergence in distribution between random fields and measures when appropriate.

Definition 2.5.2. A set M of measures on X is precompact if and only if every sequence of measures in M has a convergent subsequence whose law coverges in X. A set M of measures on X is said to be tight if and only if for every $\theta > 0$ there is a compact set $K \subset X$ such that $\mu(X \setminus K) < \theta$ for all $\mu \in M$.

Theorem 2.5.3 (Prokhorov)**.** *If a set M of measures on X is tight then it is precompact in distribution. The converse is true if X is a Polish space.*

Corollary 2.5.4. *If μ_ε is a sequence of measures such that every subsequence has a subsequence which converges in distribution to μ then the whole sequence μ_ε converges to μ in distribution.*

In what follows X will denote a separable Hilbert Space with the strong norm topology, and X_w will mean X with the weak topology. It is well-known that the Borel σ-algebras of both topologies are the same, so the concept of measure is not different in these two topologies. Throughout, we will let $B = \{e_1, e_2, \ldots\}$ be an orthonormal basis for X.

Lemma 2.5.5. *The law of a sequence of X-valued random variables $q^\varepsilon : \Omega \to X$ converges to q on X (respectively on X_w) if and only if*

1. *the law of the sequence of scalar-valued random variables $(q^\varepsilon, w)_X$ converges to $(q, w)_X$ for all $w \in X$, and*

2. *the sequence of measures induced by the q^ε is precompact in X (respectively in X_w).*

Recall that a sequence v_k converges (weakly) in X_w if and only if (v_k, e_j) converges as $k \to \infty$ for all $k \in \mathbb{N}$ and $\|v_k\|_X$ is uniformly bounded. Thus, a subset A of X_w is (weakly) compact if and only if $\sup_{v \in A} \|v\| < \infty$. It is also easy to verify that a subset A of X is compact if and only if

$$\sup_{v \in A} \|v\| < \infty \quad \text{and} \quad \lim_{K \to \infty} \sup_{v \in A} \sum_{k=K}^{\infty} |(v, e_k)_X|^2 = 0.$$

The following two results are consequences of Lemma 2.5.5 and the characterization of compact spaces on X and X_w.

Corollary 2.5.6. *The law of a sequence of X-valued random variables $q^\varepsilon : \Omega \to X$ converges on X_w to q if and only if*

1. *the law of the sequence of scalar-valued random variables $(q^\varepsilon, w)_X$ converges to $(q, w)_X$ for all $w \in X$, and*

2. *there is $0 < \varepsilon_1 < \varepsilon_0$ such that $\lim\limits_{N \to \infty} \sup\limits_{0 < \varepsilon < \varepsilon_1} P(\|q^\varepsilon\|_X \geq N) = 0$.*

In particular, from Chebyshev Inequality, the convergence in law on X_w holds if condition 1 is true and there is $0 < \varepsilon_1 < \varepsilon_0$ such that $\sup\limits_{0 < \varepsilon < \varepsilon_1} E\|q^\varepsilon\|_X^2 < \infty$.

Corollary 2.5.7. *The law of a sequence of X-valued random variables $q^\varepsilon : \Omega \to X$ converges on X to q if and only if*

1. *the law of the sequence of scalar-valued random variables $(q^\varepsilon, w)_X$ converges to $(q, w)_X$ for all $w \in X$, and*

2. *there is $0 < \varepsilon_1 < \varepsilon_0$ such that $\lim_{N\to\infty} \sup_{0<\varepsilon<\varepsilon_1} P(\|q^\varepsilon\|_X \geq N) = 0$, and*

3. *there is $0 < \varepsilon_1 < \varepsilon_0$ such that*

$$\lim_{K\to 0} \sup_{0<\varepsilon<\varepsilon_1} P\left(\sum_{k=K}^{\infty} |(q^\varepsilon, e_k)_X|^2 \geq \delta\right) = 0 \text{ for all } \delta > 0.$$

In particular, by Chebyshev Inequality, the convergence in law on X holds if condition 1 *is true and there is $0 < \varepsilon_1 < \varepsilon_0$ such that*

$$\sup_{0<\varepsilon<\varepsilon_1} E\|q^\varepsilon\|_X^2 = \sup_{0<\varepsilon<\varepsilon_1} \sum_{k=1}^{\infty} E|(q^\varepsilon, e_k)_X|^2 < \infty \quad \text{and} \tag{2.8}$$

$$\lim_{K\to\infty} \sup_{0<\varepsilon<\varepsilon_1} \sum_{k=K}^{\infty} E|(q^\varepsilon, e_k)_X|^2 = 0.$$

For more refined results of the CLT in Hilbert spaces we refer to Merkle (1989). We shall use the following two results to prove Central Limit results. The first one is similar to the Contraction Principle and allows us to extend the CLT between topological spaces.

Lemma 2.5.8. *Suppose $f : X \to Y$ is a continuous function and there is a sequence of X-valued random processes q^ε converging in law to q on X. Then, the composition $f \circ q^\varepsilon$ converges in law to $f \circ q$ on Y.*

Lemma 2.5.9 (Slutsky)**.** *Suppose that the law of X^ε converges to a measure X and Y^ε converges in probability to a constant a. Then, $X^\varepsilon + Y^\varepsilon$ converges in law to $X + a$ and $X^\varepsilon Y^\varepsilon$ converges to aX^ε.*

This theorem will be used in the following way. Suppose that u^ε and v^ε are solutions to the equations $\mathcal{L}u^\varepsilon = f$ and $\mathcal{L}v^\varepsilon = f + R^\varepsilon$, where $\mathcal{L}$ is a linear operator. If u^ε converges in law to a random field q and R^ε converges to 0 in probability, then v^ε also converges in law to q.

2.5.2 Mixing Conditions

Here we discuss the convergence in law of an oscillatory random field $q^\varepsilon(x) = q(\frac{x}{\varepsilon})$. For a ergodic random field $q : \Omega \times \mathbb{R}^n \to \mathbb{R}$, define the correlation function as $R(x) = E[q(y+x)\eta(y)]$ for all $x, y \in \mathbb{R}^n$. Intuitively, in order to obtain a CLT, it is required that the laws of q_x, q_y are asymptotically independent whenever $|x-y|$ is large. The mixing condition formalizes this notion of rapid decorrelation.

For any Borel $A \subset \mathbb{R}^n$, denote the sub-σ-algebra generated by q restricted on A (i.e. $\{q_x \mid x \in A\}$) by $\mathcal{F}_A$. The random field q satisfies the *strong mixing condition* if there exists a bounded decreasing function $\mathfrak{f} : \mathbb{R}^+ \to \mathbb{R}^+$ such that

$$\frac{|E[\theta\xi] - E[\theta]E[\xi]|}{\sqrt{E[|\theta|^2]E[|\xi|^2]}} \leq \mathfrak{f}(\text{distance}(A, B)) \tag{2.9}$$

for all $\theta \in L^2(\Omega, \mathcal{F}_A, P)$ and $\xi \in L^2(\Omega, \mathcal{F}_B, P)$. We will further assume that $r^{n-1}\mathfrak{f}^{\frac{1}{2}}(r)$ is integrable. One important consequence of this assumption is that the random field q has *short-range correlations*, that is, the correlation function R belongs to $L^1 \cap L^\infty(\mathbb{R}^n)$. A second consequence is the following result for stationary random fields on the lattice $\mathbb{Z}^n$ whose proof is due to Bolthausen (1982).

Theorem 2.5.10. *Let $X_k, k \in \mathbb{Z}^n$ be a real valued stationary random field with mean $\bar{X} = E|X_k|^2 < \infty$ and correlation function $R(k) = E[X_k X_0] - E[X_k]E[X_0]$. Let $Q_r \subset \mathbb{Z}^n$ be a sequence which increases to $\mathbb{Z}^n$ such that*

$$\lim_{r\to\infty} \frac{|\partial Q_r|}{|Q_r|} = 0,$$

where $\partial Q = \{k \in Q \,|\, |k - \tilde{k}| = 1$ *for some* $\tilde{k} \in \mathbb{Z}^n \setminus Q\}$. *If the mixing coefficients of* X_k *satisfy* (2.9), *then the correlation function is absolutely summable and* $\sigma^2 = \sum_{k \in \mathbb{Z}^n} R(k) \geq 0$. *Moreover, if* $\sigma^2 > 0$ *then*

$$\frac{1}{\sigma |Q_r|} \sum_{k \in Q_r} (X_k - E[X_k])$$

converges in law to a standard normal random variable as $r \to \infty$.

This result allows us to prove the CLT for an oscillatory integral due to (Bal, 2008, Theorem 2.10). The idea of the proof consists in approximating the oscillatory integral with a discrete sum on the lattice and invoking Theorem 2.5.10.

Lemma 2.5.11. *Let* $f \in L^2(U)$. *Suppose that the random field* q *satisfies the strong mixing condition* (2.9) *and has short-range correlations (i.e.* $R \in L^\infty(\mathbb{R}^n) \cap L^1(\mathbb{R}^n)$*). Then, the random variable* $I^\varepsilon = \int_U \frac{1}{\varepsilon^{\frac{n}{2}}} q_{y/\varepsilon} f_y \, dx$ *converges in law to a Gaussian random variable with mean zero and variance* $\|R\|_{L^1(\mathbb{R}^n)} \|f\|^2_{L^2(U)}$, *as* $\varepsilon \to 0$. *We will write* $I = \|R\|_{L^1(\mathbb{R}^n)} \int_U f_y \, dW_y$, *where* W *represents a standard Brownian motion.*

Since $E\|q^\varepsilon\|^2_{L^2(U)} = |U| R(0) < \infty$, Corollary 2.5.6 guarantees that the law of $\frac{1}{\varepsilon^{\frac{n}{2}}} q^\varepsilon$ converges to a Gaussian process on $L^2_w(U)$. As we remarked earlier, since the Borel σ-algebras of $L^2_w(U)$ and $L^2(U)$ are equal, this Gaussian process can be thought of as living on $L^2(U)$. By Lemma 2.5.8, we will extend this result to some problems in homogenization. We finish this section by proving the following useful lemma. This result first appeared in (Bal, 2008, Lemma 2.1) for the case when $m = 4$.

Lemma 2.5.12. *Let* $S_0 = \{s_j\}_{j=1}^{2m} \subset \mathbb{R}^n$ *and let* $q_j : \mathbb{R}^n \times \Omega \to \mathbb{R}$ *be mean-zero stationary random fields satisfying the mixing condition* (2.9) *and with moments of order* $2(2m-1)$. *If* $M = \max_{j=1}^{2m} E|q_j(0)|^{2(2m-1)} < \infty$, *there is a permutation* $\{t_j\}_{j=1}^{2m}$ *of*

S_0 such that

$$\left| E\left[\prod_{j=1}^{2m} q_j(s_j)\right]\right| \leq M^{\frac{m}{2m-1}} \prod_{k=1}^{m-\ell} \mathfrak{f}^{\frac{1}{m-\ell}}\left(\frac{|t_{2k+1}-t_{2k}|}{\kappa}\right), \tag{2.10}$$

where $\ell = 2\lfloor \frac{m}{3} \rfloor$ and $\kappa = 3$. Moreover, this choice of ℓ is optimal when the random fields do not have joint third moment equal to zero:

$$\lim_{(s_{j_1},s_{j_2},s_{j_3})\to 0} E[q_{j_1}(s_{j_1})q_{j_2}(s_{j_2})q_{j_3}(s_{j_3})] = E[q_{j_1}(0)q_{j_2}(0)q_{j_3}(0)] \neq 0, \tag{2.11}$$

for $1 \leq j_1 < j_2 < j_3 \leq 2m$. Finally, we may take $\ell = 0$ if the joint distribution of $\{q_j(s_j)\}_{j=1}^{2m}$ is symmetric about the origin.

Proof. For each $s \in S_0$, let $\nu(s) \in S_0$ be a nearest neighbor of s, i.e. distance$(s, S_0 \setminus \{s\}) = |s - \nu(s)|$. By relabeling the elements of S_0, if necessary, find the largest possible number of disjoint subsets $S_1, S_2, \ldots, S_r$ of S_0 such that $S_k = \{s_k, \nu(s_k)\}$. Let $T_k = \{s \in S_0 \setminus \bigcup_{j=1}^r S_j \,|\, \nu(s) \in S_k\}$ be the set of points whose closest neighbor is in S_k and is not in any of the S_j. By the maximality of r, $\{S_1, S_2, \ldots, S_r, T_1, T_2, \ldots, T_r\}$ forms a partition of S_0, where some of the T_j may be empty. For each $k = 1, 2, \ldots, 2m$, take $\zeta = q_k, \eta = \prod_{j\neq s_k} q_j, S = \{s_k\}$ and $T = S_0 \setminus \{s_k\}$ in (2.9) to obtain that

$$\left| E\left[\prod_{j=1}^{2m} q_j(s_j)\right]\right| \leq \sqrt{E[|\zeta|^2]E[|\eta|^2]}\mathfrak{f}(|s_k - \nu(s_k)|).$$

But $(E[|\zeta|^2])^{2m-1} \leq E[|q_k|^{2(2m-1)}] \leq M$ and $(E[|\eta|^2])^{2m-1} \leq \prod_{j\neq s_k} E[|q_j|^{2(2m-1)}] \leq M^{2m-1}$ by Hölder's Inequality, whence

$$\left| E\left[\prod_{j=1}^{2m} q_j(s_j)\right]\right| \leq M^{\frac{m}{2m-1}}\mathfrak{f}(|s_k - \nu(s_k)|). \tag{2.12}$$

Next, we create a new set S_{r+1} dynamically. This set initially is set to contain exactly one of the points of $S_k, k = 1, 2, \ldots, r$. If t_1', t_2' is a pair of points in T_k we select which point will be in S_{r+1} according to the following rules. There are two possible cases:

1. The closest neighbor (in S_k) to both t'_1 and t'_2 is the same, and $\nu(t'_1) = \nu(t'_2) \in S_k$. Let $t'_3 \in \{t'_1, t'_2\}$ be a point such that $\max\{|t'_1 - \nu(t'_1)|, |t'_2 - \nu(t'_2)|\} \leq |t'_3 - \nu(t'_3)|$. Then, $|t'_1 - t'_2| \leq |t'_1 - \nu(t'_1)| + |t'_2 - \nu(t'_2)| \leq 2|t'_3 - \nu(t'_3)|$. Since $\mathfrak{f}$ is nonincreasing,

$$\mathfrak{f}(|t'_3 - \nu(t'_3)|) \leq \mathfrak{f}\left(\frac{|t'_1 - t'_2|}{2}\right). \tag{2.13}$$

   Add t'_3 to S_{r+1} and delete t'_1 and t'_2 from T_k.

2. The closest neighbor to t'_1 and t'_2 is different. Suppose that $\nu(t'_1) = s_k$ and $\nu(t'_2) = \nu(s_k)$. Let $t'_3 \in \{t'_1, t'_2, s_k\}$ be a point such that $\max\{|t'_1 - \nu(t'_1)|, |t'_2 - \nu(t'_2)|, |s_k - \nu(s_k)|\} \leq |t'_3 - \nu(t'_3)|$. Then, $|t'_1 - t'_2| \leq |t'_1 - \nu(t'_1)| + |s_k - \nu(s_k)| + |\nu(t'_2) - t'_2| \leq |t'_3 - \nu(t'_3)|$, whence

$$\mathfrak{f}(|t'_3 - \nu(t'_3)|) \leq \mathfrak{f}\left(\frac{|t'_1 - t'_2|}{3}\right). \tag{2.14}$$

   Add t'_3 to S_{r+1}, allowing for repetitions in case t'_3 is already in S_{r+1}. Delete t'_1 and t'_2 from T_k.

This algorithm is to be repeated until $|T_k| = 0$ or 1 for all $1 \leq k \leq r$. Let 2ℓ be the number of sets T_k such that $|T_k| = 1$. Observe that $\ell + |S_{r+1}| = m$. Consider inequalities (2.12) with $s_k \in S_{r+1}$ and multiply them. From (2.13) and (2.14) and the fact that $\mathfrak{f}$ is nonincreasing, we obtain (2.10) with $\kappa = 3$. We must now find the minimum value of ℓ that works for all possible configurations S_0. Clearly, the worst case scenario arises when $|T_k| = 1$ for as many k as possible. But since $|S_k| = 2$, ℓ cannot be larger than $\frac{2m}{3}$. Basically, we have three cases:

- $m \equiv 0 \mod 3$. The worst case configuration is realized when $r = \frac{2m}{3}$ and $T_k = 1$ for all $k = 1, 2, \ldots, r$, so that $\ell = r = \frac{2m}{3}$.

- $m \equiv 1 \mod 3$. In this case, $r = \frac{2m+1}{3}$, $T_k = 1$ for $k = 1, 2, \ldots, r-1$, and $T_{2r} = 0$. Thus, $\ell = \frac{r-1}{3} = \frac{2m-2}{3}$.

- $m \equiv 2 \mod 3$. There are two possible scenarios: either $r = \frac{2m-1}{3}$, $T_k = 1$ for $k = 1, 2, \ldots, r-1$, and $T_{2r} = 2$. so that $\ell = r = \frac{2m}{3}$, or $r = \frac{2m+2}{3}$, in which case $T_k = 1$ for $k = 1, 2, \ldots, r-2$, and $T_{2r-1} = T_{2r} = 0$. In both cases, $\ell = \frac{2m-4}{3}$.

Now, let us prove that our choice of ℓ is sharp when $2m = 6$. The general case follows easily by induction. Assume that (2.10) and (2.11) hold with $\ell = 0$. We will arrive to a contradiction. Let $S_1' = \{s_1, s_2, s_3\}$, $S_2' = \{s_4, s_5, s_6\}$ and consider the class of configurations

$$\mathcal{S}_p = \{\{s_1, s_2, s_3, s_4, s_5, s_6\} \subset \mathbb{R}^n \mid \text{diameter}(S_1') <$$

$$\text{diameter}(S_2') < \frac{1}{p}, \text{distance}(S_1', S_2') > p\}.$$

This class is an open set in $\mathbb{R}^{6n}$. From (2.10) with $\ell = 0$ we have

$$|E[q_1(s_1)q_2(s_2)q_3(s_3)q_4(s_4)q_5(s_5)q_6(s_6)]| \leq C\mathfrak{f}^{\frac{1}{3}}\left(\frac{p}{3}\right) \to 0 \text{ as } p \to \infty$$

On the other hand, take $\eta = q_1(s_1)q_2(s_2)q_3(s_3)$, $\zeta = q_4(s_4)q_5(s_5)q_6(s_6)$, $S = S_1$ and $T = S_2$ in (2.9) to see that

$$\left| E\left[\prod_{j=1}^{6} q_j(s_j)\right]\right| \geq |E[q_1(s_1)q_2(s_2)q_3(s_3)]E[q_4(s_4)q_5(s_5)q_6(s_6)]| - M^{\frac{m}{2m-1}}\mathfrak{f}(p), \quad (2.15)$$

which converges to $E[q_1(0)q_2(0)q_3(0)]E[q_4(0)q_5(0)q_6(0)]| > 0$ as $p \to \infty$, a contradiction.

Finally, we have $E[q_{j_1}(s_{j_1})q_{j_2}(s_{j_2})q_{j_3}(s_{j_3})] = 0$ if the joint distribution of the q_j is symmetric so we can reverse inequality (2.15) obtaining

$$|E[q_1(s_1)q_2(s_2)q_3(s_3)q_4(s_4)q_5(s_5)q_6(s_6)]| \leq M^{\frac{m}{2m-1}}\mathfrak{f}(p).$$

Combining this with (2.12) we arrive at (2.10). □

Remark 2.5.13. To simplify notation, we have defined q to be a scalar-valued random process. However, the arguments presented here can be extended to a stationary $\mathbb{R}^n$-valued field with trivial modifications by considering the correlation functions $R_{h,i,j,k}(x) = E[q_{h,i}(y + x)q_{j,k}(y)]$.

2.5.3 Gaussian Processes

Definition 2.5.14. Let X be a Hilbert Space. An X-valued Gaussian Process q is a random process on X such that $\langle q|m\rangle$ is a normal random variable for all $m \in X^*$. The Gaussian Process q on X has a unique Borel Gaussian measure μ such that $\mu(A) = P(X \in A)$ for all Borel sets $A \subset X$.

Theorem 2.5.15 (Kuo (2006))**.** *If q is an X-valued Gaussian process its characteristic function $\varphi_q : X^* \to \mathbb{C}$ is given by*

$$\varphi_q(v) = E_q[e^{i\langle q|v\rangle}] = \exp\{i\langle m|v\rangle - \frac{1}{2}(Qv, v)_{X^*}\}. \tag{2.16}$$

for some $m \in X$ (the mean of w) and a bounded nonnegative-definite self-adjoint operator $Q : X^* \to X^*$ (the covariance operator) of trace class. Moreover, the Riesz Representation Theorem implies that the parameters μ and Q are the unique solutions to the equations $\langle m|v\rangle = E_w\langle w|v\rangle$, and $(Qu, v)_{X^*} = E_w[\langle w - m|u\rangle\langle w - m|v\rangle]$.

In particular, Q is also a Hilbert-Schmidt operator, so if we let $X = H^{-s}(U)$ we can find a kernel $R \in H^{-s}(U) \otimes H^{-s}(U)$ such that

$$(Qu, v)_{H^s(U)} = \langle R, u \otimes v\rangle \text{ for all } u, v \in H^s(U).$$

The function R is often referred to as the covariance function since, when $s = 0$, it is pointwise meaningful in the sense that

$$(Qu)(x) = \int_U R(x, y)\phi(y)\, dy \text{ for all } u \in L^2(U), x \in U,$$

and $R(x, y) = E_w[w(x)w(y)]$ if w has continuous sample paths.

2.5.4 Malliavin Calculus

The goal of this section is to explain how we can define the integral of $f \in H^{-s} \otimes L^2(U)$ with respect to White Noise using Malliavin Calculus techniques. The interested reader may find Nualart (2010); Di Nunno et al. (2009) a useful reference for this material.

The white noise map is an isometry $\dot{W} : L^2(U) \to \mathcal{H} \subset L^2(\Omega)$, where $\mathcal{H}$ represents the subspace of Gaussian random variables. Denote by

$$\mathcal{J} = \left\{ \gamma \in \mathbb{N}_0^{\mathbb{N}} \mid \sum_{k=1}^{\infty} \gamma_k < \infty \right\}$$

the collection of multiindices of finite length. Let $\xi_k = \dot{W}(e_k)$ and, for any multiindex $\gamma \in \mathcal{J}$ define

$$\xi_\gamma = \prod_{k=1}^{\infty} \frac{h_{\gamma_k}(\xi_k)}{\sqrt{\gamma_k!}},$$

where h_j is the jth Hermite polynomial

$$h_j(x) = (-1)^j e^{x^2/2} \frac{d}{dx^j} e^{-x^2/2}.$$

The importance of these random variables is that they form an orthonormal basis of $L^2(\Omega)$. For $F : \mathbb{R}^k \to \mathbb{R}$, the Malliavin derivative of $F(\xi_1, \xi_2 \dots, \xi_k) \in L^2(\Omega)$ is defined as

$$D_y F(\xi_1, \xi_2 \dots, \xi_k) = \sum_{j=1}^{k} \frac{\partial F}{\partial x_j}(\xi_1, \xi_2 \dots, \xi_k) e_j \in L^2(\Omega; L^2(U))$$

We define the Skorokhod integral $\delta f = \int_U f(\cdot, y) \dot{W}(y)\, dy \in L^2(\Omega; H^{-s}(U))$ of $f \in H^{-s} \otimes L^2(U)$ as the dual of the Malliavin derivative, i.e., the unique element of $L^2(\Omega \to H^{-s}(U))$ with the property that

$$E[\delta f(x) \cdot \phi] = E \int_U f(x, y) D_y \phi \, dy \in H^{-s}(U) \text{ for all } \phi \in L^2(\Omega). \tag{2.17}$$

Since the Malliavin derivative is a linear operator, it suffices to check (2.17) for $\phi = \xi_\gamma$. Let ϵ_k be the multiindex with 1 in the kth slot and zero otherwise. It is easy to show (cf. (Nualart, 2010, Section 1.2)) that

$$D_y \xi_\gamma = \sum_{k=1}^{\infty} \sqrt{\gamma_k} \xi_{\gamma - \epsilon_k} e_k \mathbf{1}\{\gamma_k \geq 1\}.$$

Hence, for any $h \in H^{-s}(U)$ we have

$$E \int_U h \otimes e_k D_x \xi_\gamma \, dx = h \sum_{j=1}^{\infty} \sqrt{\gamma_s} \int_U e_k e_s \, dx E[\xi_{\gamma-\epsilon_s} \xi_0] = \begin{cases} h, & \text{for } \gamma = \epsilon_k \\ 0 & \text{for } \gamma \neq \epsilon_k \end{cases},$$

whence it follows that $\delta(h \otimes e_k) = h\xi_k$ by (2.17). In particular, if $f(x,y) = \sum_{k=1}^{\infty} f_k(x) e_k(y)$, the linearity of the Skorokhod operator allows us to get the Wiener chaos decomposition of the Skorokhod integral of f as $\delta f = \sum_{k=1}^{\infty} f_k \xi_k \in L^2(\Omega \otimes H^{-s}(U))$.

2.6 Large Deviation Principle

Throughout this section, X denotes a Hausdorff topological vector space and X^* its topological dual. Here, we state all the basic results we will use in Chapter 5. The reader can consult Dembo and Zeitouni (1998); Hollander (2000); Freidlin and Wentzell (1998) for a more detailed discussion of the results on large deviations presented here. The large deviation principle (LDP for short) characterizes the asymptotic behaviour, as $\varepsilon \to 0$ of a family of probability measures P^ε via upper and lower exponential bounds. We start off this review by recalling some useful definitions.

Definition 2.6.1. The *domain* of a function $f : X \to [-\infty, \infty]$ is the set

$$\mathcal{D}_f = \{x \in X \mid -\infty < f(x) < \infty\}.$$

Definition 2.6.2. The *level set* of $f : X \to [-\infty, \infty]$ at $t \in \mathbb{R}$ is denoted by

$$\Phi_f(t) = \{x \in X \mid f(x) \leq t\}.$$

Definition 2.6.3. A function $f : X \to [-\infty, \infty]$ is said to be *lower semicontinuous* if it satisfies one of the following equivalent definitions:

1. For every $x_0 \in X$ and every $\varepsilon > 0$ there exists a neighborhood U of x_0 such that $f(x) \geq f(x_0) - \varepsilon$ for all $x \in U$. Equivalently, this can be expressed as $\liminf_{x \to x_0} f(x) \geq f(x_0)$.

2. The level sets of f at t, $\Phi_f(t)$, are closed for all $t \in \mathbb{R}$.

Definition 2.6.4. A function $f : X \to [0, \infty]$ is called an *action functional* if $f \not\equiv \infty$ and the sets $\Phi_f(t)$ are compact for all $0 \leq t < \infty$.

Lower semicontinuous functions are important since they attain their minimum on every nonempty compact subset of X. In particular, an action functional is lower semicontinuous. Similarly, an action functional attains its minimum on every nonempty closed subset of X.

Definition 2.6.5. $\lambda(\varepsilon) \geq 0$ is called a *rate function* if $\lim_{\varepsilon \to 0} \lambda(\varepsilon) = \infty$.

Definition 2.6.6. A sequence of Borel probability measures P^ε on X satisfies the Large Deviation Principle (LDP) with rate $\lambda(\varepsilon)$ and action functional S if S is an action functional in the sense of Definition 2.6.4 and

$$-\inf_{x \in A^\circ} S(x) \leq \liminf_{\varepsilon \to 0} \frac{1}{\lambda(\varepsilon)} \ln P^\varepsilon(A) \leq \limsup_{\varepsilon \to 0} \frac{1}{\lambda(\varepsilon)} \ln P^\varepsilon(A) \leq -\inf_{x \in \bar{A}} S(x) \quad (2.18)$$

for all Borel sets $A \subseteq X$, where A° and $\bar{A}$ denote the interior and the closure of A, respectively.

When proving the LDP it is often easier to break (2.18) up into the following two conditions:

$$\text{For any closed set } F \subseteq X, \limsup_{\varepsilon \to 0} \frac{1}{\lambda(\varepsilon)} \ln P^\varepsilon(F) \leq -\inf_{x \in F} I(x). \quad (2.19)$$

$$\text{For any open set } G \subseteq X, \liminf_{\varepsilon \to 0} \frac{1}{\lambda(\varepsilon)} \ln P^\varepsilon(G) \geq -\inf_{x \in G} I(x). \quad (2.20)$$

From a practical point of view, the LDP gives us optimal exponential upper and lower bounds for the probability that the random variable of interest is close to a given realization and, moreover, the bound can be written in terms of the action functional.

We say that a set $A \subseteq X$ is regular if $\inf_{x \in \bar{A}} S(x) = \inf_{x \in A^\circ} S(x)$. It can be shown that (2.18) is equivalent to the condition

$$\lim_{\varepsilon \to 0} \frac{1}{\lambda(\varepsilon)} \ln P^\varepsilon(A) = - \inf_{x \in A} S(x) \tag{2.21}$$

for all Borel regular sets $A \subseteq X$. The usefulness of regular sets lies on the following

Lemma 2.6.7. *Suppose that A is a regular set and V is an open set. If* $\arg\min_{x \in \bar{A}} S(x)$ *belongs to V then*

$$\lim_{\varepsilon \to 0} P^\varepsilon(V \mid A) = \lim_{\varepsilon \to 0} \frac{P^\varepsilon(A \cap V)}{P^\varepsilon(A)} = 1.$$

When proving (2.20), it is convenient to replace the right-hand side of (2.20) simply by $I(x)$, where $x \in G$. In order to prove (2.19), one usually first proves this inequality for compact sets and then shows that most of the probability is concentrated on a compact set on an exponential scale.

Definition 2.6.8. A sequence of Borel probability measures P^ε on X is *exponentially tight* if for every $t > 0$ there exists a compact set $K_t \subset X$ such that $\limsup_{\varepsilon \to 0} \lambda(\varepsilon) \ln P^\varepsilon(X \setminus K_t) < -t$.

Corollary 2.6.9. *Suppose P^ε is a family of exponentially tight measures on X.*

1. *If* (2.19) *holds for all compact sets, then it also holds for all closed sets.*

2. *If* (2.20) *holds for all open sets, then S is an action functional.*

Most of the basic results in large deviation theory have a convex action functional, which plays a major role in the theory. Below we introduce some of the main definitions. The reader should consult Rockafellar (1997) for a deeper exposition of convex analysis.

Definition 2.6.10. A function $f : X \to (-\infty, \infty]$ is said to be *convex* if $f(\alpha x + (1-\alpha)y) \leq \alpha f(x) + (1-\alpha) f(y)$ for all $0 < \alpha < 1$ and all $x, y \in X$. Equivalently, f is convex if for all $x_0 \in \mathcal{D}_f^\circ$ there exists a *subdifferential* $y_0 \in X^*$ such that $f(x) \geq \langle y_0, x - x_0 \rangle + f(x_0)$. We will denote the set of subdifferentials of x_0 by $\partial f(x_0) = \{y_0 \in X^* \,|\, \langle y_0, x_0 \rangle - f(x_0) = \sup_{x \in X}[\langle y_0, x \rangle - f(x)]\}$. The function f is *strictly convex* if the strict inequality holds in the definition of convexity.

Thus, f is convex if for every point of its domain there is a *supporting hyperplane* passing through that point such that the graph of f lies above this hyperplane. In the case when $X = \mathbb{R}^m$, $\mathcal{D}_f$ is a convex set, and f is continuous on $\mathcal{D}_f^\circ$ and Lipchitz continuous on compact subsets of $\mathcal{D}_f^\circ$. The set of subdifferentials of x, $\partial f(x)$, is always a nonempty convex compact set if $x \in \mathcal{D}_f^\circ$. Moreover, if f is differentiable at x then $\partial f(x) = \{\nabla f(x)\}$.

Definition 2.6.11. The Legendre transform of the convex function $f : X \to (\infty, \infty]$ is the function $f^* : X^* \to (\infty, \infty]$ defined by

$$f^*(y) = \sup_{x \in X}[\langle y, x \rangle - f(x)].$$

If we denote our convex function by H, then its Legendre transform will be denoted by L; this notation comes from the Hamilton-Lagrange duality in mechanics. Informally speaking, the Legendre Transform f^* maps the slope of a supporting hyperplane of f to the negative of the $f(x)$-intercept. One of the main properties of the Legendre transform that we will make use of is its biduality: the Legendre transform of L is H.

Theorem 2.6.12 (Duality on $\mathbb{R}^m$). *(Rockafellar, 1997, Section 26) Let $H : \mathbb{R}^m \to (-\infty, \infty]$ be a lower semicontinuous, convex function such that $H \not\equiv \infty$, and let L be its Legendre transform. Then, we have the following properties:*

1. *$L : \mathbb{R}^m \to (-\infty, \infty]$ is a lower semicontinuous, convex function such that $L \not\equiv \infty$.*

2. *$\mathcal{D}_L = \cup_{x\in\mathbb{R}^m} \partial H(x)$. Thus, $y \notin \partial H(x)$ for some $x \in \mathbb{R}^m$ if and only if $L(y) = \infty$.*

3. *If $y \in \partial H(x_0)$ then the supremum in the definition of L is attained at x_0, i.e., $L(y) = x_0 \cdot y - H(x_0)$. If L is strictly convex, then x_0 is unique.*

4. *If $\mathcal{D}_H = \mathbb{R}^m$, L is strict convex on $\mathcal{D}_L^\circ$ if and only if H is differentiable, in which case $y = \nabla H(x_0)$.*

5. *$\partial L(x) = (\partial H)^{-1}(x) = \{y \in \mathbb{R}^m \,|\, x \in \partial H(y)\}$. Hence, $x_0 \in \partial L(y)$ and the Laplace Transform of L is H itself.*

On infinite dimensional spaces, the supremum in the definition of L may not necessarily be attained, however, the duality $H = L^*$ still holds (Dembo and Zeitouni, 1998, Lemma 4.5.8). The next result gives some insight of the Lagrange Transform of functions which grow like powers of $|t|$, and will become useful in the next section.

Lemma 2.6.13. *Let $C > 0$. For each $1 \le p \le \infty$, let $p' = \frac{p}{p-1}$ denote the Holder conjugate of p. Define the function $f_{C,p} : \mathbb{R}^m \to [0, \infty]$ by*

$$f_{C,p}(t) = \frac{C^p}{p}|t|^p, \quad \text{for } 1 \le p < \infty, \text{ and}$$

$$f_{C,\infty}(t) = \begin{cases} 0, & \text{if } |t| \le C \\ \infty, & \text{if } |t| > C. \end{cases}$$

Let $H : \mathbb{R}^m \to (-\infty, \infty]$ be a lower semicontinuous, convex function such that $H \not\equiv \infty$. Then,

(a) $f_{C,p}^*(s) = \frac{1}{p'C^{p'}}|s|^{p'} = f_{1/C,p'}(s)$ *if* $1 < p < \infty$, $f_{C,1}^* = f_{C,\infty}$ *and* $f_{C,\infty}^* = f_{C,1}$.

(b) *If* $\limsup_{|t|\to\infty} \dfrac{H(t)}{|t|^{p'}} \leq \dfrac{C^{p'}}{p'}$ *then* $\liminf_{|s|\to\infty} \dfrac{L(s)}{|s|^p} \geq \dfrac{1}{pC^p}$, *where* $1 < p < \infty$ *and* $0 \leq C < \infty$.

(c) *If* $\liminf_{|t|\to\infty} \dfrac{H(t)}{|t|^{p'}} \geq \dfrac{C^{p'}}{p'}$ *then* $\limsup_{|s|\to\infty} \dfrac{L(s)}{|s|^p} \leq \dfrac{1}{pC^p}$, *where* $1 < p < \infty$ *and* $0 < C \leq \infty$.

(d) $\lim_{|t|\to\infty} \dfrac{H(t)}{|t|^{p'}} = \dfrac{C^{p'}}{p'}$ *if and only if* $\lim_{|s|\to\infty} \dfrac{L(s)}{|s|^p} = \dfrac{1}{pC^p}$, *where* $1 < p < \infty$ *and* $0 \leq C \leq \infty$.

(e) $\limsup_{|t|\to\infty} \dfrac{H(t)}{|t|} \leq C$ *if and only if* $L(s) = \infty$ *for all* $|s| > C$, *where* $0 < C < \infty$.

Proof. (a) Let us consider first the case $1 < p < \infty$. Observe that $f_{C,p}$ is a strictly convex differentiable function with $\mathcal{D}_f = \mathbb{R}^m$. Hence, its Legendre transform is given by $f_{C,p}^*(s) = s \cdot t(s) - \frac{C^p}{p}|t(s)|^p$, where $t(s)$ is the unique solution to the equation $s = \nabla f_{C,p}(t(s)) = C^p|t(s)|^{p-2}t(s)$. It is not hard to check that $t(s) = \left(\frac{|s|}{C^p}\right)^{p'-1}\frac{s}{|s|}$, so that $f_{C,p}^*(s) = \frac{|s|^{p'}}{p'C^{p'}}$.

Next, we compute $f_{C,1}^*$. The inequality $s \cdot t \leq |s|\,|t|$ shows that if $|s| \leq C$ then $f_{C,1}^*(s) = \max_{t\in\mathbb{R}^m}[s \cdot t - C|t|] = 0$, where the maximum is attained at $t = 0$. If $|s| > C$, let $k > 0$ and take $t = \frac{ks}{|s|}$ to see that $f_{C,1}^*(s) \geq s \cdot \frac{ks}{|s|} - C\frac{k|s|}{|s|} = k(|s| - C) \to \infty$ as $k \to \infty$. The identity $f_{C,\infty}^* = f_{C,1}$ follows from the Duality Theorem.

(b) For given $\theta > 0$, there is $N_\theta > 0$ such that $H(t) \leq g_\theta(t)$ for all $t \in \mathbb{R}^m$, where

$$g_\theta(t) = \begin{cases} \frac{(C+\theta)^{p'}}{p'}|N_\theta|^{p'}, & \text{if } |t| \leq N_\theta \\ \frac{(C+\theta)^{p'}}{p'}|t|^{p'}, & \text{if } |t| \geq N_\theta. \end{cases} \tag{2.22}$$

This is because if H was greater than g_θ on $|t| \leq N_\theta$, H would have a maximum in this set, a contradiction of convexity. Observe that

$$L_1(s) = \sup_{|t|\leq N_\theta} \left[s \cdot t - \frac{(C+\theta)^{p'}}{p'} N_\theta^{p'} \right] \leq |s| N_\theta - \frac{(C+\theta)^{p'}}{p'} N_\theta^{p'}, \text{ and}$$

$$L_2(s) = \sup_{|t|> N_\theta} \left[s \cdot t - \frac{(C+\theta)^{p'}}{p'} |t|^{p'} \right] = s \cdot t(s) - \frac{(C+\theta)^{p'}}{p'} |t(s)|^{p'}$$

$$= f^*_{C+\theta,p'}(s) = f_{(C+\theta)^{-1},p}(s)$$

provided that $|t(s)| = \left(\frac{|s|}{C^{p'}}\right)^{p-1} > N_\theta$, or $|s| > N_\theta^{p'-1} C^{p'}$. Thus, for large $|s|$ we have

$$L(s) \geq g^*(s) = \max\{L_1(s), L_2(s)\} = L_2(s) = f_{(C+\theta)^{-1},p}(s) = \frac{|s|^p}{p(C+\theta)^p}$$

The result follows by taking $\theta \to 0$.

(c) For given $\theta > 0$, there is $N_\theta > 0$ such that $H(t) \geq g_\theta(t)$ for all $t \in \mathbb{R}^m$, where

$$g_\theta(t) = \begin{cases} -\kappa \leq N_\theta \\ \frac{(C-\theta)^{p'}}{p'} |t|^{p'}, & \text{if } |t| \geq N_\theta. \end{cases}$$

The existence of κ is due to the lower semicontinuity of H. The proof follows along the same lines of the proof of (b).

(d) is a trivial consequence from (b) and (c).

(e) Assume that $H \leq g_\theta$ with g_θ as in (2.22) and $p' = 1$. Let $k > N_\theta$ and take $t = \frac{ks}{|s|}$ to see that $L(s) \geq g^*(s) \geq s \cdot \frac{ks}{|s|} - (C+\theta)\frac{k|s|}{|s|} = k(|s| - C - \theta)$. The conclusion on L follows by taking $\theta \to 0$ and $k \to \infty$. Finally suppose that $L(s) = \infty$ for $|s| > C$ and $H(s) > -\kappa$ for all $s \in \mathbb{R}^m$. Then,

$$\frac{H(t)}{|t|} = \sup_{s \leq C} \left[\frac{t}{|t|} \cdot s - \frac{L(s)}{|t|} \right] \leq C + \frac{\kappa}{|t|}.$$

The result follows by taking $|t| \to \infty$.

□

We are now ready to state a version of the LDP for weakly dependent sequences, which is in fact a generalization of Cramér's result on sequences of independent and identically distributed random variables.

Theorem 2.6.14 (Ellis-Gärtner)**.** *(Dembo and Zeitouni, 1998, Theorem 2.3.6) Let P^ε be a sequence of Borel probability measures on $\mathbb{R}^m$ such that*

$$H(x) = \lim_{\varepsilon \to 0} \frac{1}{\lambda(\varepsilon)} \ln \int_{\mathbb{R}^m} \exp\{\lambda(\varepsilon) x \cdot y\} \, P^\varepsilon(dy)$$

is differentiable for all $x \in \mathbb{R}^m$. Then H is convex and P^ε satisfies a LDP with rate $\lambda(\varepsilon)$ and convex action functional L, the Legendre Transform of H.

The differentiability condition is required in order for L to be strictly convex and the upper bound (2.19) to hold. We shall need a generalization of the Ellis-Gärtner Theorem to more general topological spaces. Note that once we leave the finite dimensional realm, it is required to check exponential tightness in order for the LDP to hold.

Definition 2.6.15. A function $f : X \to \mathbb{R}$ is Gateaux differentiable if, for every $x, y \in X$, the function $f(x + ty)$ is differentiable at $t = 0$. The Gateaux derivative of f at x in the direction y will be denoted by $D_y f(x)$.

Theorem 2.6.16. *(Dembo and Zeitouni, 1998, Theorem 4.6.14) Suppose P^ε is an exponentially tight sequence of Borel probability measures on a locally convex Hausdorff real topological vector space X. Suppose*

$$Tf = \lim_{\varepsilon \to 0} \frac{1}{\lambda(\varepsilon)} \ln \int_X \exp\{\lambda(\varepsilon) \langle f, \varphi \rangle\} \, P^\varepsilon(d\varphi)$$

is finite for all $f \in X^*$ and is Gateaux differentiable. Then P^ε satisfies a LDP with rate $\lambda(\varepsilon)$ and action functional T^*.

While this abstract Gärtner-Ellis Theorem will be used to obtain the LDP for the oscillatory random field, we shall need results which extend the LDP to more other random fields. Below we present two such propositions, which we will make use of in the subsequent sections.

Lemma 2.6.17. *(Dembo and Zeitouni, 1998, Lemma 4.1.5) Let A be a measurable subset of X such that $P^{\varepsilon}(A) = 1$ for all $\varepsilon > 0$. Suppose that A is equipped with the topology induced by X.*

(a) If A is a closed subset of X and $\{P^{\varepsilon}\}_{\varepsilon>0}$ satisfies the LDP in A with action functional S, then $\{P^{\varepsilon}\}_{\varepsilon>0}$ satisfies the LDP in X with rate function S' such that $S' = S$ on A and $S' = \infty$ on $X \setminus E$.

(b) If $\{P^{\varepsilon}\}_{\varepsilon>0}$ satisfies the LDP in X with rate function S and $\mathcal{D}_S \subset A$, then the same LDP holds in A.

Theorem 2.6.18 (Contraction Principle, 4.2.1 in Dembo and Zeitouni (1998))**.** *Let X and Y be Hausdorff topological spaces and $f : X \to Y$ a continuous function. Suppose that P^{ε} is a family of probability measures satisfying the LDP on X with rate $\lambda(\varepsilon)$ and action functional $S_X : X \to [0, \infty]$. Let $S_Y : Y \to [0, \infty]$ be defined by*

$$S_Y(y) = \inf\{S_X(x) \,|\, x \in X, y = f(x)\},$$

where $\inf \emptyset = \infty$*. Then, the probability measures $P^{\varepsilon} \circ f^{-1}$ on Y satisfy the LDP with rate $\lambda(\varepsilon)$ and action functional S_Y.*

We finish the theoretical discussion with some remarks. The Contraction Principle holds even if f is continuous only on $x \in \mathcal{D}_{S_X}$. More generally, we may only have a continuous function $f : A \to Y$ with A satisfying the conditions of Lemma 2.6.17; the Contraction Principle holds in this case as well.

The Contraction Principle is equivalent to the following more intuitive formulation.

Suppose $\{q^\varepsilon\}_{\varepsilon>0}$ is a collection of random variables on a Hausdorff topological space X satisfying the LDP with action functional S_X and $f : X \to Y$ is continuous. Then, $\{f(q^\varepsilon)\}_{\varepsilon>0}$ also satisfies the LDP on Y with action functional S_Y.

In practice, verifying continuity of a function f directly may be challenging, however, this is implied by sequential continuity provided X is first countable. Examples of first countable spaces are separable metric spaces, topological vector spaces and their weak topologies. Thus, in order to prove continuity it suffices to show that for any sequence x_n converging to $x \in X$, $f(x_n) \to f(x)$ as $n \to \infty$.

3

Main Results

In this chapter we describe and state the main results that will be proved in the next two chapters.

3.1 Central Limit Theorem

3.1.1 CLT for a diffusion in a weakly random media

The first problem we deal with is a linearization of the general case (1.1). Let $a_0 \in \mathcal{A}$ be a uniformly positive-definite matrix and $\beta > 0$. Suppose $\eta^\varepsilon(x) = \eta(\frac{x}{\varepsilon}) \in \mathcal{A}$ is a real-valued stationary random field satisfying the strong mixing condition (2.9). Consider the problem

$$\begin{aligned} \mathcal{L}^\varepsilon u^\varepsilon = -\nabla \cdot ((a_0 + \varepsilon^\alpha \eta^\varepsilon)\nabla u^\varepsilon) + \beta u^\varepsilon = f, &\quad \text{in } U \\ u^\varepsilon = 0, &\quad \text{on } \partial U. \end{aligned} \tag{3.1}$$

For $\alpha > 0$ and $\varepsilon > 0$ small, u^ε can be approximated by u^0, where u^0 solves

$$\begin{aligned} \mathcal{L}^0 u^0 = -\nabla \cdot (a_0 \nabla u^0) + \beta u^0 = f, &\quad \text{in } U \\ u^0 = 0, &\quad \text{on } \partial U. \end{aligned} \tag{3.2}$$

Theorem 3.1.1 (CLT for a diffusion in a weakly random media)**.** *Let $\alpha > \frac{n}{2}$. The law of the random error $\varepsilon^{-\alpha+\frac{n}{2}}(u^\varepsilon - u^0)$ converges on $H^{-s}(U)$ to a Gaussian field $w : \Omega \times \mathbb{R}^n \to \mathbb{R}$ with characteristic function*

$$\varphi(v) = E_w[e^{i\langle w|v\rangle}] = \exp\left\{-\frac{\sigma^2}{2}\|\nabla u_0 \cdot \nabla \mathcal{G} v\|^2_{L^2(U)}\right\}. \tag{3.3}$$

if $s > \frac{n}{2} - 1$. The limiting Gaussian process can be expressed in terms of a Skorokhod integral as $w(x) = \|R\|^{1/2}_{L^1(\mathbb{R})} \int_U [\nabla_y G(x,y) \cdot \nabla u^0(y)]\dot{W}_y \, dy$, where $\dot{W}$ is a White Noise Process.

If $n = 1$, the result can be proved on $C(U)$ and the integral be interpreted as an Ito integral. For $n \geq 2$, $E\|w\|^2_{L^2(U)} = \infty$ due to the singularity of the Green's function, so the convergence result can only be shown on a negative Sobolev space.

3.1.2 CLT for a diffusion in randomly layered media

Let $R = I \times U$, where I denotes the interval $(0,1)$ and U is a open bounded set with smooth boundary. Given any $x \in R$, we will write $x = (x_1, x_2)$, where $x_1 \in I$, $x_2 \in U$.

Let us consider the solutions $u^\varepsilon, u^0 \in H^1_0(R)$ to equations (1.1) and (1.2), respectively, with $f \in H^1(R)$. Here $a(x_1)$ is assumed to be a scalar-valued random field that only depends on x_1.

To simplify the given result, it is convenient to perform the change of variables $x_1 = \frac{x_1}{\sqrt{a^*}}, x_2 = \frac{x_2}{\sqrt{\bar{a}}}$, where $a^* = (E[\frac{1}{a(0)}])^{-1}$ and $\bar{a} = E[a(0)]$. Let $A^\varepsilon(x_1) = \frac{\partial}{\partial x_1}\chi_1^\varepsilon = \frac{a^\varepsilon}{a^*} - 1$ and $B^\varepsilon = \frac{a^\varepsilon}{\bar{a}} - 1$ be mean-zero stationary ergodic random fields satisfying the strong mixing conditions (2.9), with respective correlation functions R_A and R_B.

Theorem 3.1.2 (CLT for a diffusion in randomly layered media)**.** *Under the assumptions stated above, the law of $\frac{1}{\sqrt{\varepsilon}}w^\varepsilon$ converges on $L^2(R)$ to the Gaussian random* *field*

$$\frac{\|R_A\|_{L^1(\mathbb{R})}^{\frac{1}{2}}}{\sqrt{a^*}} \int_0^1 \int_U \frac{\partial u^0}{\partial y_1} G(x,y) \cdot \frac{\partial u^0}{\partial y_1} \, dy_2 \, dW^1_{y_1} + \|R_B\|_{L^1(\mathbb{R})}^{\frac{1}{2}} \int_0^1 \int_U G(x,y) \Delta_{y_2} u^0(y) \, dy_2 \, dW^2_{y_1} \quad (3.4)$$

as $\varepsilon \to 0$*, where* W^1 *and* W^2 *are correlated Brownian motions such that* $dW^1 \, dW^2 = \rho \, dt$*. Here,*

$$\rho = \frac{\int_{\mathbb{R}} E[A(0)B(\tau) + B(0)A(\tau)] \, d\tau}{\left(\int_{\mathbb{R}} E[A(0)A(\tau)] \, d\tau \int_{\mathbb{R}} E[B(0)B(\tau)] \, d\tau\right)^{\frac{1}{2}}}.$$

and G *is the Green's function of* $-\Delta$ *with Dirichlet boundary conditions.*

The proof of this theorem is obtained by analyzing an asymptotic expansion for the terms of a PDE for $z^\varepsilon = u^\varepsilon - u^0 - \chi^\varepsilon \cdot \nabla u^0$. The terms of order $\sqrt{\varepsilon}$ can be shown to characterize the limiting law and the limiting Gaussian process is expressed in terms of an Ito integral.

3.2 Large Deviation Principle

Before stating the main theorems, let us introduce the following terminology on the random fields.

Let $1 \leq p \leq \infty$ be fixed and let p' be its Hölder conjugate. Fix an open bounded set $U \subset \mathbb{R}^n$ with smooth boundary. Given a random field $q^\varepsilon : \mathbb{R}^n \times \Omega \to \mathbb{R}^m$ and $f \in L^1(U; \mathbb{R}^m)$, define the functional

$$Tf = \lim_{\varepsilon \to 0} \varepsilon^n \ln E \exp\left\{\frac{1}{\varepsilon^n} \int_U f_y \cdot q_{y/\varepsilon} \, dy\right\} \quad (3.5)$$

whenever the limit exists. For every $\kappa > 0$ and every measurable function $G : \mathbb{R}^m \to \mathbb{R}$, define the associated functions

$$T_{p,\alpha}(-\kappa) = -\liminf_{\varepsilon \to 0} \varepsilon^n \ln E \exp\left\{-\frac{\kappa}{\varepsilon^n}\|q^\varepsilon\|^\alpha_{L^p(U)}\right\}, \tag{3.6a}$$

$$T_{p,\alpha}(+\kappa) = \limsup_{\varepsilon \to 0} \varepsilon^n \ln E \exp\left\{\frac{\kappa}{\varepsilon^n}\|q^\varepsilon\|^\alpha_{L^p(U)}\right\}, \text{ and}$$

$$T_G = \limsup_{\varepsilon \to 0} \varepsilon^n \ln E \exp\left\{\frac{1}{\varepsilon^n}\int_U G(q_{y/\varepsilon})\, dy\right\}. \tag{3.6b}$$

In particular, when $G(t) = C|t|^p$ we have $T_{C|t|^p} = T_{p,p}(+C)$ for $1 < p < \infty$. We will assume the following conditions:

(A1) There exists a function $H : \mathbb{R}^m \to \mathbb{R} \cup \{\infty\}$ such that

$$Tf = \int_U H(f_y)\, dy \tag{3.7}$$

for all $f : U \to \mathbb{R}^m$ in a dense subset of $L^{p'}(U)$.

(A2) (a) If $p = 1$ there exists a nonnegative increasing convex function $G : \mathbb{R}^m \to \mathbb{R}$ such that

$$\lim_{|t|\to\infty} \frac{G(t)}{|t|} = \infty \quad \text{and} \quad T_G < \infty.$$

(b) If $1 < p < \infty$, there exist $\alpha \geq 1$ and $\kappa_+ > 0$ such that $T_{p,\alpha}(+\kappa_+) < \infty$.

(c) If $p = \infty$, $\|q\|_{L^\infty(U\times\Omega)} < \infty$

(A3) (a) If $p = 1$, $\kappa_H = -\inf_{t\in\mathbb{R}^m} H(t) < \infty$.

(b) If $1 < p < \infty$, there exist $\alpha \geq 1$ and $\kappa_- > 0$ such that $T_{p,\alpha}(-\kappa_-) < \infty$.

(A4) The function H is differentiable.

As usual, L denotes the Legendre Transform of H. These conditions are not too restrictive; section 5.2 provides several examples of widely used random fields satisfying these assumptions. The following results give a LDP in the sense of Definition 2.6.6 on suitable topological spaces.

3.2.1 LDP for a diffusion in random potential

Let $\frac{n}{2} < p \le \infty$. Suppose that u^ε and u^0 solve (1.4) and (1.5), respectively, with Dirichlet boundary conditions. In order to avoid the spectrum of $-\Delta$, we assume that $\sup_{x\in\mathbb{R}^n,\omega\in\Omega} q^\varepsilon(x,\omega) < b$. Recall that $W^{2,p}_w(U)$ is the Sobolev space of functions with p-integrable derivatives up to order 2 with the weak topology.

Theorem 3.2.1 (LDP for a diffusion in random potential)**.** *If $q^\varepsilon : \mathbb{R}^n \times \Omega \to \mathbb{R}$ satisfies conditions (A1)-(A4) then the random field $w^\varepsilon = u^\varepsilon - u^0$ satisfies the LDP on $W^{2,p}_w(U)$ with rate $\frac{1}{\varepsilon^n}$ and action functional*

$$S_3 w = \int_U L\left(\frac{(-\Delta + b)w_y}{u^0_y + w_y}\right) dy \quad \text{if} \quad w \in W^{2,p} \cap W^{1,p}_0(U), \tag{3.8}$$

and $S_3 w = \infty$ otherwise. Here, we take $L(\frac{0}{0}) = 0$. In particular, the Sobolev embedding yields that the LDP also holds on the strong spaces $W^{s,p}(U), s < 2$ and $C^\alpha(\bar{U})$ with $0 \le \alpha < 2 - \frac{n}{p}$.

It will also be shown that S_3 vanishes only if $w \equiv 0$, which means that $P(w^\varepsilon \in A)$ will decay exponentially in $\frac{1}{\varepsilon^n}$ for any closed set A not containing 0, as expected.

3.2.2 LDP for diffusion in randomly layered media

Let us consider the LDP for the random diffusion case(1.1) in layered media as described in Section 3.1.2. As before, define $a^* = (E[\frac{1}{a(0)}])^{-1}$ and $\bar{a} = E[a(0)]$. We remark we will not make use of the change of variables we used in Section 3.1.2.

Theorem 3.2.2 (LDP for random diffusion in layered media)**.** *Suppose that the pair $q^\varepsilon = \left(\frac{a^*}{a(y)} - 1, a(y) - \bar{a}\right)$ satisfies (A1)-(A4) for $p = \infty$ and Lagrangian $L : \mathbb{R}^2 \to \mathbb{R}$. These conditions hold, for example, if the random field a is as in Section 5.2.1 with bounded $\mathfrak{z}$. If $u^\varepsilon, u^0 \in H^1_0(R)$ satisfy (1.1) and (1.2), respectively, then the random* *corrector $w^\varepsilon = u^\varepsilon - u^0$ satisfies the LDP on $L^2(R)$ with rate $\frac{1}{\varepsilon}$ and action functional*

$$S_4 w = \inf \int_U L\left(\frac{a^*}{a(y)} - 1, a(y) - \bar{a}\right) dy. \tag{3.9}$$

Given $w \in H_0^1(R)$, the infimum of S_4 is over all positive-definite matrices $a \in \mathcal{A}$ such that

$$-\frac{\partial}{\partial x_1}\left(a\frac{\partial}{\partial x_1}w\right) - a\Delta_{x_2} w = (a-\bar{a})\Delta_{x_2}u^0 - \frac{\partial}{\partial x_1}\left(a(\frac{a^*}{a} - 1)\frac{\partial}{\partial x_1}u^0\right). \tag{3.10}$$

Here, $\chi_1(x_1) = \int_0^{x_1} \frac{a^*}{a(s)} - 1\, ds$.

In the one-dimensional case, the LDP is proved in Bal et al. (2011) by making use of the explicit solution and it is shown that it suffices to show (A1)-(A4) for the random field $\frac{1}{a^\varepsilon} - 1$.

3.2.3 Reaction-diffusion with random reaction

Here, we state a LDP for the solution u^ε of (1.6). For simplicity, we will assume $b(v, q^\varepsilon)$ is a bounded random field satisfying the Lipschitz conditions (1.7) and the following conditions:

(B1) There exists $H : \mathbb{R}^m \times \mathbb{R}^m \to \mathbb{R}$ such that

$$T'f = \lim_{\varepsilon\to 0} \varepsilon^n \ln E \exp\left\{\frac{1}{\varepsilon^n}\int_U f_y b(v_y, q_{y/\varepsilon})\, dy\right\} = \int_U H(v_y, f_y)\, dy$$

(B4) The function H is continuous in its first argument and differentiable in its second argument.

Conditions (B2) and (B3) are omitted since b is bounded. They are satisfied, for example, if the random field q is as in Section 5.2.1. Define $L(v, s)$ to be the Legendre Transform of $H(v, t)$ in the second variable for v fixed. The next result can be thought of as a nonlinear generalization of Theorem 3.2.1 and the LDP for averaging of ODEs due to (Freidlin and Wentzell, 1998, Chapter 7). For a related work see Freidlin (1985).

Theorem 3.2.3. *Let $0 \leq s < 2$. If u^ε solves (1.6) and q^ε satisfies conditions (B1),(B4) for all step functions v then u^ε satisfies the LDP on $C(0,T;C^s(U))$ with rate $\frac{1}{\varepsilon^n}$ and action functional*

$$S_6 u = \int_U L(u_y, \frac{\partial}{\partial t} u_y - \Delta u_y)\, dy$$

for all $u \in C(0,T;C^s(U))$ satisfying the boundary conditions of u^ε, and $S_6 u = \infty$ otherwise.

By Definition 2.6.6, exponential lower and upper bounds for large deviations are given in terms of an optimization problem. In Section 5.4 simple exponential bounds are derived for the problems shown above for suitable Borel sets.

The basic steps we follow to prove these results are the following:

1. Prove that the sequence of random coefficients q^ε satisfies a LDP on the space $L^p_w(U)$ of p-integrable functions with the weak topology for some $1 \leq p \leq \infty$.

2. Show that the map $F : L^p_w(U) \to X_p$ taking the coefficient q to the solution u (which lives on a suitable Sobolev Space X_p) is continuous. This estimate can often be obtained by proving an apriori estimate on the solution u in terms of $\|q\|_{L^p(U)}$ and using a weak convergence argument. Observe that the continuity is completely independent of the probability distribution of q.

3. The Contraction Principle now can be used to extend the LDP from the space of coefficients $L^p_w(U)$ to the solution space, which can often be described as a suitable Sobolev Space.

4

Central Limit Theorem in Stochastic Homogenization

4.1 Diffusion in a Weakly Random Media Case

In this section we derive a CLT for (3.1) when $\alpha > \frac{n}{2}$. Let u^0 be as in (3.2), and define $w^\varepsilon = u^\varepsilon - u_0$, the error committed by approximating u_0 with u^ε. It is easy to check that $w^\varepsilon = \varepsilon^\alpha \mathcal{G}\nabla \cdot \eta^\varepsilon \nabla u^\varepsilon$, where $\mathcal{G} = (-\nabla \cdot (a_0 \nabla + \beta))^{-1}$ is the Green's operator of u^0 with Dirichlet boundary conditions. Also, let $\mathcal{G}^\varepsilon = (-\nabla \cdot ((a_0 + \varepsilon^\alpha \eta^\varepsilon)\nabla + \beta))^{-1}$ denote the Green's operator of u^ε. By uniform ellipticity, both $\mathcal{G}$ and $\mathcal{G}^\varepsilon$ are uniformly bounded in ε and ω. Note that for every $u, v \in H_0^1(U)$ we have

$$\langle \nabla \cdot \eta^\varepsilon \nabla u | v \rangle = \int_U \eta^\varepsilon \nabla u \cdot \nabla v \, dx \leq \|\eta^\varepsilon\|_{L^\infty(\mathbb{R}^n)} \|u\|_{H_0^1(U)} \|v\|_{H_0^1(U)}$$

$$\therefore \|\nabla \cdot \eta^\varepsilon \nabla u\|_{H^{-1}(U)} \leq \|\eta^\varepsilon\|_{L^\infty(\mathbb{R}^n)} \|u\|_{H_0^1(U)}$$

$$\therefore \|w^\varepsilon\|_{H_0^1(U)} \leq \varepsilon^\alpha \|\mathcal{G}\| \|\nabla \cdot \eta^\varepsilon \nabla u^\varepsilon\|_{H^{-1}(U)} \leq C\varepsilon^\alpha \|\eta^\varepsilon\|_{L^\infty(\mathbb{R}^n)} \|f\|_{H^{-1}(U)}, \tag{4.1}$$

which shows that the error goes to zero strongly in $H_0^1(U)$, as expected. Split w^ε into $w_1^\varepsilon + w_2^\varepsilon$, where $w_1^\varepsilon = \varepsilon^\alpha \mathcal{G}\nabla \cdot \eta^\varepsilon \nabla u_0$ and $w_2^\varepsilon = \varepsilon^\alpha \mathcal{G}\nabla \cdot \eta^\varepsilon \nabla w^\varepsilon = \varepsilon^\alpha \mathcal{G}_\varepsilon \nabla \cdot \eta^\varepsilon \nabla w_1^\varepsilon$. Let's analyze now the contribution of w_1^ε and w_2^ε to the limit as $\varepsilon \to 0$. After replacing u^ε

by u^0 in (4.1) we obtain the strong estimate

$$E\|w_1^\varepsilon\|_{H_0^1(U)}^2 \le C\varepsilon^{2\alpha}\|\eta\|_{L^\infty(U)}\|f\|_{H^{-1}(U)}^2. \tag{4.2}$$

Using integration by parts and Hölder's Inequality we see that

$$E|\langle\phi|w_2^\varepsilon\rangle| = E\left|\varepsilon^\alpha\int_U \eta^\varepsilon\nabla\mathcal{G}\phi\cdot\nabla w^\varepsilon\,dx\right|$$

$$\le \varepsilon^\alpha(E\|\eta^\varepsilon\nabla\mathcal{G}\phi\|_{L^2(U)}^2)^{\frac{1}{2}}(E\|\nabla w^\varepsilon\|_{L^2(U)}^2)^{\frac{1}{2}} \tag{4.3}$$

$$\le C\varepsilon^{2\alpha}\|f\|_{H^{-1}(U)}\|\phi\|_{H^{-1}(U)}. \tag{4.4}$$

By Theorem 2.5.11, the law of $I^\varepsilon = |\langle\varepsilon^{-(\alpha+\frac{n}{2})}w_1^\varepsilon|\phi\rangle = \varepsilon^{-\frac{n}{2}}\int_U \eta^\varepsilon\nabla u_0\cdot\nabla\mathcal{G}\phi\,dx$ converges to $I = \sigma\int_U \nabla u_0\cdot\nabla\mathcal{G}\phi\,dW_y$ (i.e., a Gaussian random variable with mean zero and variance $\sigma^2\|\nabla u_0\cdot\nabla\phi\|_{L^2(U)}^2$) for all $\phi\in H^{-1}(U)$ as $\varepsilon\to 0$.

By Theorem 2.5.9, the fluctuations for w^ε are completely characterized by those coming from w_1^ε provided that $\varepsilon^{-(\alpha+\frac{n}{2})}w_2^\varepsilon$ converges to zero in probability. Since $\varepsilon^{-(\alpha+\frac{n}{2})}E|\langle\phi|w_2^\varepsilon\rangle| \le C\varepsilon^{\alpha-\frac{n}{2}}$, this occurs precisely when $\alpha > \frac{n}{2}$.

4.1.1 Convergence in law

Let $\tilde{w}_1^\varepsilon = \varepsilon^{-(\alpha+\frac{n}{2})}w_1^\varepsilon$ be the normalized error. Our goal is to find s such that $\tilde{w}_1^\varepsilon$ converges in distribution to a Gaussian Process in $H^{-s}(U)$. We have already verified condition 1 of Theorem 2.5.7, so it suffices to check (2.8).

Let e_k be an orthonormal sequence of eigenvectors of $\mathcal{L}$ with positive eigenvalues λ_k in increasing order. Since $\{\lambda_j^{-s/2}e_k\}_{k=1}^\infty$ forms an orthonormal basis of eigenvectors for $\mathcal{G}$ on $\mathcal{H}^s(U)$,

$$E\|\tilde{w}_1^\varepsilon\|_{\mathcal{H}^{-s}(U)}^2 = \sum_{k=1}^\infty |\langle\tilde{w}_1^\varepsilon|\lambda_k^{-s/2}e_k\rangle|^2 = \varepsilon^{-n}E\sum_{k=1}^\infty\left|\int_U \eta^\varepsilon\nabla u_0\cdot\nabla\mathcal{G}(\lambda_k^{-s/2}e_k)\,dx\right|^2 \tag{4.5}$$

Extend u^0 and e_k to be 0 outside U. Use Plancherel Equality (4.1.1) to see that

$$E\|\tilde{w}_1^\varepsilon\|^2_{\mathcal{H}^{-s}(U)} = \sum_{k=1}^{\infty} \lambda_k^{-s} \int_{\mathbb{R}^n} \hat{R}(\varepsilon\xi)|\mathcal{F}^{-1}_{x\to\xi}\{\nabla u_0 \cdot \nabla(\lambda_k^{-1} e_k)\}|^2 \, d\xi$$

$$\leq \|\hat{R}\|_{L^\infty(\mathbb{R}^n)} \sum_{k=1}^{\infty} \lambda_k^{-(s+1)} \int_U |\nabla u_0 \cdot \nabla(\lambda_k^{-1/2} e_k)|^2 \, dx$$

Since $\lambda_k^{-1/2} e_k$ forms an orthonormal basis for $H_0^1(U)$, we have

$$E\|\tilde{w}_1^\varepsilon\|^2_{\mathcal{H}^{-s}(U)} \leq C\|\hat{R}\|_{L^\infty(\mathbb{R}^n)} \|\nabla u_0\|_{L^\infty(U)} \sum_{k=1}^{\infty} \lambda_k^{-(s+1)}.$$

By Weyl's Formula (2.7), the series of $\lambda_k^{-(s+1)}$ converges if and only if $s > \frac{n}{2} - 1$. Thus, (2.8) guarantees that the law of $\tilde{w}_1^\varepsilon$ converges on $H^{-s}(U)$ for all $s > \frac{n}{2} - 1$.

Next, we will describe the limiting random field w. Since the finite-dimensional distributions, $\langle \tilde{w}_1^\varepsilon | v \rangle$, are Gaussian with mean 0 and variance $\sigma^2 \|\nabla u_0 \cdot \nabla \mathcal{G} v\|^2_{L^2(U)}$, the random field w has characteristic function (3.3). The covariance operator Q given by

$$(Qu, v)_{H^s(U)} = \int_U \langle \nabla u_0, \nabla \mathcal{G} u \rangle \langle \nabla u_0, \nabla \mathcal{G} v \rangle \, dx \text{ for all } u, v \in H^s(U)$$

satisfies $\mathrm{Tr}(Q) = E\|w\|_{H^{-s}(U)} < \infty$. Hence by Theorem 2.5.15, w is a well-defined Gaussian process on $H^{-s}(U)$.

Let's compute the null space of Q. Note that $u \in \operatorname{Ker} Q$ if and only if $\phi = \mathcal{G} u$ satisfies the Boundary Value Problem

$$\nabla u_0 \cdot \nabla \phi = 0 \qquad \text{in } U$$

$$\phi = 0 \qquad \text{on } \partial U.$$

By uniqueness of first order partial differential operators, it follows that $\phi \equiv 0$, so that Q is a nondegenerate positive-definite operator with $\operatorname{supp} Q = H^{-s}(U)$.

The covariance function is given by

$$R(x, y) = \sigma^2 \int_U \langle \nabla u_0(z), \nabla G(x, z) \rangle \langle \nabla u_0(z), \nabla G(y, z) \rangle \, dz,$$

where G is the Green's function corresponding to $\mathcal{G}$. Evidently, we can only make sense of R in the distributional sense when we apply R to sufficiently smooth functions. Recall that the Dirac delta function $\delta \in H^{-s}(\mathbb{R}^n)$ if and only if $s > \frac{n}{2}$. Since two (distributional) derivatives of the Green's function gives rise to the delta function, it follows that $\frac{\partial G}{\partial x_j}(x, y) \in H^{-s}(U)$ for all $s > \frac{n}{2} - 1$. Therefore, $R \in H^{-s}(U) \otimes H^{-s}(U)$ and the range of s in Theorem 3.1.1 is sharp: there is no measure on $H^{-s}(U)$ with characteristic function (3.3) if $s \leq \frac{n}{2} - 1$.

To finish the proof of Theorem 3.1.1, we need to obtain representation formulas for w. In one-dimension, we have convergence on $H^{1/2}(U)$ and we can in fact extend the convergence to the space of continuous functions. This can be done using the Kolmogorov's Continuity Theorem (see Billingsley (1999)) since, in this case, the Green's function is uniformly Lipschitz continuous . In this case we can even write the limiting Gaussian process as the Ito integral

$$w = \|R\|_{L^1(\mathbb{R}^n)}^{1/2} \int_U \nabla G(x, y) \cdot \nabla u_0 \, dW_y, \tag{4.6}$$

where W is a standard Brownian motion.

In higher dimensions, we can describe w is by understanding the way it acts on test functions

$$\langle w | \psi \rangle = \int_U \nabla u_0 \cdot \nabla \mathcal{G}\psi \, dW(y) \sim \|R\|_{L^1(\mathbb{R}^n)} N\left(0, \int_U |\nabla u_0 \cdot \nabla \mathcal{G}\psi|^2 \, dx\right). \tag{4.7}$$

However, we cannot interpret w in the Ito sense (4.6) since filtrations in the higher dimensional setting are not linear. Instead, we use Malliavin Calculus to define w as a Skorokhod integral.

By Mercer's Theorem we can expand the Green's function as a Fourier series

$$G(x, y) = \sum_{k=1}^{\infty} \frac{1}{\lambda_k} e_k(x) e_k(y) \text{ for all } x, y \in U. \tag{4.8}$$

Convergence of this series holds on $H^{-s} \otimes L^2(U)$, and

$$\|G\|_{H^{-s}\otimes L^2(U)} = \sum_{k=1}^{\infty} \|\lambda_k^{-1} e_k\|^2_{H^{-s}(U)} < \infty$$

if and only if $s > \frac{n}{2} - 2$ by Weyl's formula (2.7), as expected. Therefore, the function

$$f(x, y) = \nabla_y G(x, y) \cdot \nabla u_0(y) = \sum_{k=1}^{\infty} \frac{1}{\lambda_k} [\nabla u_0(y) \cdot \nabla e_k(y)] e_k(x) \text{ for all } x, y \in U \quad (4.9)$$

converges on $H^{-s} \otimes L^2(U), s > \frac{n}{2} - 2$, and its Skorokhod integral is given by

$$\begin{aligned}\delta f(x) &= \|R\|^{1/2}_{L^1(\mathbb{R}^n)} \int_U \nabla G(x, y) \cdot \nabla u_0(y) \, \dot{W}(y) \, dy \\ &= \sum_{k=1}^{\infty} \int_U [\nabla u_0 \cdot \nabla e_k](y) \dot{W}(y) \, dy \frac{1}{\lambda_k} e_k(x) \in L^2(\Omega \otimes H^{-s}(U)).\end{aligned}$$

It is easy to check that $\langle \delta f | \psi \rangle$ has the same distribution as (4.7), so that $w = \delta f$ in law.

To finish the proof of Theorem 3.1.1, we must show tightness (2.8) of w_2^ε. Arguing as in (4.5), we have

$$\begin{aligned}E\|w_2^\varepsilon\|^2_{\mathcal{H}^{-s}(U)} &= \sum_{k=1}^{\infty} |\langle w_2^\varepsilon | \lambda_k^{-s/2} e_k \rangle|^2 \\ &= \varepsilon^\alpha E \sum_{k=1}^{\infty} \left| \int_U \eta^\varepsilon \nabla w_2^\varepsilon \cdot \nabla \mathcal{G}(\lambda_k^{-s/2} e_k) \, dx \right|^2.\end{aligned}$$

By Hölder Inequality, the above reads

$$E\|w_2^\varepsilon\|^2_{\mathcal{H}^{-s}(U)} \leq C\varepsilon^\alpha \sum_{k=1}^{\infty} \lambda_k^{-(s+1)} \|\lambda_k^{-1/2} e_k\|^2_{H_0^1(U)} E\|w^\varepsilon\|^2_{H_0^1(U)} \leq C\varepsilon^{2\alpha} \sum_{k=1}^{\infty} \lambda_k^{-(s+1)} < \infty.$$

By assumption, $\alpha > \frac{n}{2}$, so w_2^ε converges to 0 in probability on $H^{-s}(U)$.

4.1.2 Asymptotic Estimates

This section is devoted to the computation of some asymptotic expansions which may be useful to understand how some pointwise statistics of the random process $\tilde{w}_\varepsilon$ diverge as $\varepsilon \to 0$, as expected. Firstly, we will need a generalization of Plancherel's Equality when the integrands are not square-integrable.

Lemma 4.1.1. *If $u, v \in L^p(\mathbb{R}^n)$, where $1 < p \leq 2$ then*

$$\int_{\mathbb{R}^n} u\hat{v}\, dx = \int_{\mathbb{R}^n} \hat{u}v\, dx.$$

Furthermore, if $\hat{u} \in L^p(\mathbb{R}^n)$ then

$$\int_{\mathbb{R}^n} uv\, dx = \int_{\mathbb{R}^n} \check{u}\hat{v}\, dx.$$

Proof. The result is true if $u \in L^p(\mathbb{R}^n) \subset \mathcal{S}'(\mathbb{R}^n)$ and $v \in \mathcal{S}(\mathbb{R}^n)$, where $\mathcal{S}(\mathbb{R}^n)$ is the Schwartz space of rapidly decreasing functions on $\mathbb{R}^n$. Note that Young's Inequality gives that $\hat{u} \in L^{p'}(\mathbb{R}^n)$, $\frac{1}{p} + \frac{1}{p'} = 1$. The proof follows from the density of $\mathcal{S}(\mathbb{R}^n)$ in $L^p(\mathbb{R}^n)$. □

We will use this result as follows. Let $N(x,y) = N(y-x) = \frac{1}{c_n |y-x|^{n-2}}$, if $n \geq 3$, and $N(x,y) = \frac{1}{c_n} \ln(|x-y|)$, if $n = 2$, be the Newtonian potential, where c_n is the hypervolume of the unit sphere in $\mathbb{R}^n$. We will take $u(y) = \frac{1}{\varepsilon^n} R\left(\frac{y-z}{\varepsilon}\right)$, and $v(y) = N(y)$. We will assume here that $R \in \mathcal{S}$, so that Plancherel's Equality can be applied. Recall that

$$E|\tilde{w}_1^\varepsilon(x)|^2 = \int_U \int_U \frac{1}{\varepsilon^n} R\left(\frac{y-z}{\varepsilon}\right) \nabla_y G(x,y) \cdot \nabla u_0(y) \nabla_z G(x,z) \cdot \nabla u_0(z)\, dy\, dz$$

Assume $n \geq 3$. Let $\kappa_j(x,y) = \frac{\partial u_0}{\partial y_j}(y)[\kappa(x,y) - \frac{1}{n-2} \frac{|y-x|^2}{y_j - x_j} \frac{\partial \kappa}{\partial y_j}(x,y)]$, where

$$G(x,y) = \frac{\kappa(x,y)}{|x-y|^{n-2}}$$

and κ is a continuously differentiable function. Clearly, $\kappa_j \in C(\bar{U} \times \bar{U})$ is Hölder continuous on each variable, and $\frac{\partial G}{\partial y_j}\frac{\partial u_0}{\partial y_j} = \kappa_j \frac{\partial N}{\partial y_j}$. Then,

$$E|\tilde{w}_1^\varepsilon(x)|^2 = \sum_{j,k=1}^{n} \int_U \int_U \frac{1}{\varepsilon^n} R\left(\frac{y-z}{\varepsilon}\right) \kappa_j(x,y) \frac{\partial N}{\partial y_j}(x,y) \kappa_k(x,z) \frac{\partial N}{\partial z_k}(x,z)\, dy\, dz. \tag{4.10}$$

It will be shown that this last expression is asymptotic to

$$E|\tilde{w}_1^\varepsilon(x)|^2 \sim (2\pi)^{\frac{n}{2}} \sum_{j,k=1}^{n} \kappa_j(x,x)\kappa_k(x,x) \int_{\mathbb{R}^n} \hat{R}(\varepsilon\xi) |\widehat{\frac{\partial N}{\partial y_j}}|^2\, d\xi \tag{4.11}$$

$$\therefore E|\tilde{w}_1^\varepsilon(x)|^2 \sim \varepsilon^{-n+2} (2\pi)^{\frac{n}{2}} \sum_{j,k=1}^{n} \kappa_j(x,x)\kappa_k(x,x) \int_{\mathbb{R}^n} \hat{R}(\xi) \frac{\xi_j^2}{|\xi|^4}\, d\xi,$$

where the latter equality follows from $\hat{N}(\xi) = |\xi|^{-2}$. All we have to do is to argue that we can replace the y dependent function $\kappa_j(x,y)$ with the constant $\kappa(x,x)$, extend the domain of integration from $U \times U$ to $\mathbb{R}^n \times \mathbb{R}^n$, and use Parseval's Equality. Since κ_j is Hölder continuous, say with exponent r, a computation gives

$$\left| \int_U \int_U \frac{1}{\varepsilon^n} R\left(\frac{y-z}{\varepsilon}\right) [\kappa_j(x,y) - \kappa_j(x,x)] \frac{\partial N}{\partial y_j}(x,y) \kappa_k(x,z) \frac{\partial N}{\partial z_k}(x,z)\, dy\, dz \right| \leq$$

$$C \int_U \int_U \frac{1}{\varepsilon^n} R\left(\frac{y-z}{\varepsilon}\right) \frac{dy\, dz}{|y-x|^{n-1-r}|z-x|^{n-1}}$$

$$\leq C \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} R(y) \frac{dy\, dz}{|\varepsilon y + z|^{n-1-r}|z|^{n-1}}, \tag{4.12}$$

which is finite since the Green's function is square-integrable at infinity and integrable near 0. Hence, after applying Parseval's Equality as usual we get that the last expression is

$$C \int_{\mathbb{R}^n} \hat{R}(\varepsilon\xi) \mathcal{F}^{-1}\{|y|^{-n+1+r}\} \overline{\mathcal{F}^{-1}\{|y|^{-n+1}\}}\, dx,$$

which provides a lower order term to (4.11) since the order of one of the singularities is smaller than $n-1$. This proves the first part of our assertion. The second part follows easily from an estimate similar to (4.12) and the Dominated Convergence Theorem.

Note that the proof we just gave does not work for the case when $n = 2$, mainly because the terms of the form $\frac{\partial N}{\partial y_j}$ are not square integrable neither near 0 nor at infinity. Thus, we must proceed in a different way to obtain the asymptotic result. In this case, we will prove that expression (4.10) is of order $\ln(\varepsilon)$. Arguing as before, we get that the main contribution to the integral (4.10) as $\varepsilon \to 0$ comes from taking y and z close to x:

$$E|\tilde{w}_1^\varepsilon(x)|^2 \sim \sum_{j,k=1}^{n} \kappa_j(x,x)\kappa_k(x,x) \int_{B(x,r)} \int_{\mathbb{R}^n} \frac{1}{\varepsilon^n} R\left(\frac{y-z}{\varepsilon}\right) \frac{\partial N}{\partial y_j}(x,y) \frac{\partial N}{\partial z_k}(x,z)\, dy\, dz.$$

Here, $B(x,r)$ represents the ball in $\mathbb{R}^n$ centered at x with radius r for sufficiently small r. Note that we can integrate over all $y \in \mathbb{R}^n$ by the Dominated Convergence Theorem since $R \in L^1(\mathbb{R}^n)$ and $\frac{\partial N}{\partial y_j}$ decreases to zero for large y. Next, use Plancherel's Equality to obtain

$$E|\tilde{w}_1^\varepsilon(x)|^2 \sim \frac{1}{(2\pi)^{\frac{n}{2}}} \sum_{j,k=1}^{n} \kappa_j(x,x)\kappa_k(x,x) \int_{B(0,r)} \int_{\mathbb{R}^n} \hat{R}(\varepsilon\xi) \frac{i\xi_j}{|\xi|^2} e^{-i\xi\cdot z} \frac{\partial N}{\partial y_k}(z)\, d\xi\, dz$$

$$E|\tilde{w}_1^\varepsilon(x)|^2 \sim \frac{1}{\varepsilon(2\pi)^{\frac{n}{2}}} \sum_{j,k=1}^{n} \kappa_j(x,x)\kappa_k(x,x) \int_{B(0,r)} \int_{\mathbb{R}^n} \hat{R}(\xi) \frac{i\xi_j}{|\xi|^2} e^{-i\xi\cdot z/\varepsilon} \frac{\partial N}{\partial y_k}(z)\, d\xi\, dz$$

$$E|\tilde{w}_1^\varepsilon(x)|^2 \sim \frac{1}{(2\pi)^{\frac{n}{2}}} \sum_{j,k=1}^{n} \kappa_j(x,x)\kappa_k(x,x) \int_{B(0,\frac{r}{\varepsilon})} \int_{\mathbb{R}^n} \hat{R}(\xi) \frac{i\xi_j}{|\xi|^2} e^{-i\xi\cdot z} \frac{\partial N}{\partial y_k}(z)\, d\xi\, dz$$

Now we want to replace $\hat{R}(\xi)$ with $\hat{R}(0)$. To this end we assume $\hat{R}$ is Hölder continuous with exponent s. This contribution again provides a lower order term. Thus,

$$E|\tilde{w}_1^\varepsilon(x)|^2 \sim \frac{\|R\|_{L^1(\mathbb{R}^n)}}{(2\pi)^{\frac{n}{2}}} \sum_{j,k=1}^{n} \kappa_j(x,x)\kappa_k(x,x) \int_{B(0,\frac{r}{\varepsilon})} \frac{\partial N}{\partial y_j}(z)\frac{\partial N}{\partial y_k}(z)\,dz$$

$$E|\tilde{w}_1^\varepsilon(x)|^2 \sim \frac{\|R\|_{L^1(\mathbb{R}^n)}}{(2\pi)^{\frac{n}{2}}} \sum_{j,k=1}^{n} \kappa_j(x,x)\kappa_k(x,x) \int_{B(0,\frac{r}{\varepsilon})} \frac{y_k y_j}{c_n^2|y|^4}\,dz.$$

Note that $y_k y_j$ is radially symmetric, so that the integral is exactly zero when $k \neq j$. Hence,

$$\begin{aligned} E|\tilde{w}_1^\varepsilon(x)|^2 &\sim \frac{\|R\|_{L^1(\mathbb{R}^n)}}{(2\pi)^{\frac{n}{2}}} \sum_{k=1}^{n} |\kappa_j(x,x)|^2 \int_{B(0,\frac{r}{\varepsilon})} \frac{1}{c_n^2|y|^2}\,dz \\ &\sim \frac{|\ln\varepsilon|}{c_2}\frac{\|R\|_{L^1(\mathbb{R}^n)}}{(2\pi)^{\frac{n}{2}}} \sum_{k=1}^{n} |\kappa_j(x,x)|^2. \end{aligned}$$

If $\alpha \leq \frac{n}{2}$, then w_2^ε is not negligible anymore and w^ε would have to be expanded further in order to obtain the CLT. For example if $\frac{n}{4} < \alpha \leq \frac{n}{2}$, consider the following Neumann series

$$w^\varepsilon = \varepsilon^\alpha \mathcal{G}\nabla\cdot\eta^\varepsilon\nabla u_0 + \varepsilon^{2\alpha}\mathcal{G}\nabla\cdot\eta^\varepsilon\nabla\mathcal{G}\nabla\cdot\eta^\varepsilon\nabla u_0 + \varepsilon^{2\alpha}\mathcal{G}\nabla\cdot\eta^\varepsilon\nabla\mathcal{G}\nabla\cdot\eta^\varepsilon\nabla w^\varepsilon.$$

If $w_1^\varepsilon, w_2^\varepsilon, w_3^\varepsilon$ are the three terms in the right-hand side, $\varepsilon^{-(\alpha+\frac{n}{2})}w_3^\varepsilon$ can be shown to converge to zero weakly in probability, so that w_1^ε and w_2^ε both contribute to the limit in law. In fact, the oscillatory integral w_1^ε converges to the Gaussian random field while w_2^ε becomes a second-order corrector.

4.2 Diffusion in Randomly Layered Media

In this section a Central Limit result will be derived for the random error $z^\varepsilon = u^\varepsilon - u^0 - \chi^\varepsilon\cdot\nabla u^0$ of the random diffusion problem (1.1) in the case of layered media with Dirichlet boundary conditions. This limit will then be used to analyze the fluctuations of $w^\varepsilon = u^\varepsilon - u^0$. As described in Subsection **??**, this corresponds to the case where a is an scalar-valued random field depending only on x_1, which satisfies the strong mixing condition (2.9). We will also assume that $f \in H^1(R)$.

Let $R = I \times U$, where I is the interval $(0,1)$ and U is a open bounded set with smooth boundary. Assume $f \in C^1(R)$. Given any $x \in R$, we will write $x = (x_1, x_2) = (x_1', x_2', \ldots, x_n')$, where $x_1 = x_1' \in I$, $x_2 = (x_2', \ldots, x_n') \in U$.

Recall that χ is the unique random solution to the cell problem (**??**) such that $\chi(0) = 0$ and $\nabla\chi$ is stationary. In this case, only the first coordinate of χ is nonzero and $\chi_1(y) = \int_0^y [\frac{a^*}{a(s)} - 1]\, ds$. Here, $a^* = (E[\frac{1}{a(0)}])^{-1}$. The matrix $a((\nabla\chi)^T + I)$ turns out to be diagonal with the entry $(1,1)$ being a^* and the rest of the diagonal entries equal a. As usual we define $a^\varepsilon(x_1) = a(\frac{x_1}{\varepsilon})$ and $\chi^\varepsilon(x_1) = \varepsilon\chi(\frac{x_1}{\varepsilon})$. An algebraic computation shows that z^ε satisfies the PDE

$$-\nabla \cdot (a^\varepsilon \nabla z^\varepsilon) = \mathrm{Tr}([a^\varepsilon((\nabla\chi^\varepsilon)^T + I) - a^0]D^2u^0) + \nabla \cdot (a^\varepsilon D^2 u^0 \chi^\varepsilon),$$

which reduces in the layered media case to

$$-\frac{\partial}{\partial x_1}\left(a^\varepsilon \frac{\partial}{\partial x_1} z^\varepsilon\right) - a^\varepsilon \Delta_{x_2} z^\varepsilon = \\ (a^\varepsilon - \bar{a})\Delta_{x_2}u^0 + \frac{\partial}{\partial x_1}\left(a^\varepsilon \chi_1^\varepsilon \frac{\partial^2}{\partial x_1^2}u^0\right) + a^\varepsilon \chi_1^\varepsilon \frac{\partial}{\partial x_1}\Delta_{x_2}u^0, \quad (4.13)$$

where $\bar{a} = E[a(0)]$. Let use a similarity reduction by means of the change of variables $x_1 = \frac{x_1}{\sqrt{a^*}}, x_2 = \frac{x_2}{\sqrt{\bar{a}}}$. Under these transformations, (4.13) reads

$$-\frac{\partial}{\partial x_1}\left(\frac{a^\varepsilon}{a^*}\frac{\partial}{\partial x_1} z^\varepsilon\right) - \frac{a^\varepsilon}{\bar{a}}\Delta_{x_2} z^\varepsilon = \\ \left(\frac{a^\varepsilon}{\bar{a}} - 1\right)\Delta_{x_2}u^0 + \frac{\partial}{\partial x_1}\left(\frac{a^\varepsilon}{a^*}\frac{1}{\sqrt{a^*}}\chi_1^\varepsilon \frac{\partial^2}{\partial x_1^2}u^0\right) + \frac{a^\varepsilon}{\bar{a}}\frac{1}{\sqrt{a^*}}\chi_1^\varepsilon \frac{\partial}{\partial x_1}\Delta_{x_2}u^0.$$

Let $A^\varepsilon(x_1) = \frac{\partial}{\partial x_1}\chi_1^\varepsilon = \frac{a^\varepsilon}{a^*} - 1$ and $B^\varepsilon = \frac{a^\varepsilon}{\bar{a}} - 1$ be mean-zero stationary ergodic random fields. Integrate the terms of this equation from 0 to x_1, multiply by $\frac{a^*}{a^\varepsilon}$, differentiate with respect of x_1 and rearrange terms to obtain the equation

$$-\Delta z^\varepsilon = I^\varepsilon + J^\varepsilon + R^\varepsilon, \text{ where} \tag{4.14}$$

$$I^\varepsilon = \frac{\partial}{\partial x_1}\left(\frac{1}{\sqrt{a^*}}\chi_1^\varepsilon \frac{\partial^2}{\partial x_1^2}u^0\right) + \frac{1}{\sqrt{a^*}}\chi_1^\varepsilon \frac{\partial}{\partial x_1}\Delta_{x_2}u^0 + B^\varepsilon \Delta_{x_2}u^0,$$

$$J^\varepsilon = \frac{\partial}{\partial x_1}\left(A^\varepsilon \int_0^{x_1} \frac{a^\varepsilon}{\bar{a}}\Delta_{x_2} z^\varepsilon \, dr\right) + B^\varepsilon z^\varepsilon, \text{ and}$$

$$R^\varepsilon = \frac{\partial}{\partial x_1}\left(A^\varepsilon \int_0^{x_1} (B^\varepsilon + 1)\frac{1}{\sqrt{a^*}}\chi_1^\varepsilon \frac{\partial}{\partial x_1}\Delta_{x_2}u^0 \, dr\right) + B^\varepsilon \frac{1}{\sqrt{a^*}}\chi_1^\varepsilon \frac{\partial}{\partial x_1}\Delta_{x_2}u^0 \, dr +$$

$$\frac{\partial}{\partial x_1}\left(A^\varepsilon \int_0^{x_1} B^\varepsilon \Delta_{x_2}u^0 \, dr\right) - K^\varepsilon \frac{\partial}{\partial x_1}A^\varepsilon.$$

Here, $K^\varepsilon(x_2) = \frac{a^\varepsilon}{a^*}\frac{\partial}{\partial x_1}z^\varepsilon|_{x_1=0}$ is an integration constant. Since A^ε and B^ε are functions of a^ε, they also satisfy the strong mixing conditions. Let $R_A, R_B \in L^1 \cap L^\infty(\mathbb{R})$ denote the correlation functions of A and B, respectively. In Section 5.4.2 we will see that the map taking $(A^\varepsilon, B^\varepsilon)$ to w^ε is continuous, whence these random fields determine the limit in law of $\frac{1}{\sqrt{\varepsilon}}z^\varepsilon$.

Let $\mathcal{G} = (-\Delta)^{-1}$ be the Green's operator of $-\Delta$ on R with Dirichlet Boundary Conditions. We will show that $\mathcal{G}I^\varepsilon$ is of order $\varepsilon^{\frac{1}{2}}$ and $\mathcal{G}J^\varepsilon, \mathcal{G}R^\varepsilon$ are lower order terms, so that the fluctuations are determined exclusively by $\mathcal{G}I^\varepsilon$.

Let us now make a small digression before continuing with the analysis of (4.14).

4.2.1 Analysis of the Laplacian

Here we analyze the spectral decomposition of $-\Delta = -\frac{\partial^2}{\partial x_1^2} - \Delta_{x_2}$ with Dirichlet Boundary Conditions. Due to the cylindrical form of $R = I \times U$, the operators $-\frac{\partial}{\partial x_1}$ and $-\Delta_{x_2}$ have Dirichlet Boundary Conditions on $\{0, 1\}$ and ∂U, respectively.

Since $-\Delta_{x_2}$ has compact resolvent, there exists an orthonormal basis of eigenfunctions $\phi_k \in L^2(U)$ with corresponding positive eigenvalues λ_k, which we can order in an increasing fashion, satisfying $\lim_{k\to\infty} \lambda_k = \infty$ and the boundary value problem

$$-\Delta_{x_2}\phi_k = \lambda_k \phi_k \text{ with } \phi_k \in H_0^1(U). \tag{4.15}$$

More generally, $-\Delta_{x_2}$ is also self-adjoint with compact resolvent on $H_0^1(U)$, and

$$(\phi_j, \phi_k)_{H_0^1(U)} = \int_U \nabla\phi_j \cdot \nabla\phi_k \, dx = (\phi_j, -\Delta_{x_2}\phi_k)_{L^2(U)} = \lambda_j \delta_{j,k}.$$

Thus, $\{\lambda_k^{1/2}\phi_k\}_{k=1}^{\infty}$ forms a sequence of orthonormal eigenfunctions for $-\Delta_{x_2}$ on $H_0^1(U)$ with corresponding eigenvalues $\lambda_k^{1/2}$.

An analogous result holds for $-\frac{\partial}{\partial x_1}$ on I: there is a sequence of eigenvalues/vectors $\{(\mu_j, \psi_j)\}_{j=1}^{\infty}$ of the problem

$$-\frac{d^2}{dx_1^2}\psi_j = \mu_j \psi_j \text{ with } \psi_j \in \mathcal{V}_1 \tag{4.16}$$

with Dirichlet Boundary Conditions such that $\{\mu_j^{1/2}\psi_j\}_{j=1}^{\infty}$ forms an orthonormal basis for $H_0^1(I)$. In fact, in this case we have $\psi_j = \sin(2\pi j)$ and $\mu_j = (2\pi j)^2$.

It will also be assumed that the eigenvalues are ordered increasingly and diverge to infinity. These results can easily be combined to yield the spectral result for $-\Delta$ on R: the sequences $\{\mu_j + \lambda_k\}_{j,k=1}^{\infty}$ and $\{\psi_j\phi_k\}_{j,k=1}^{\infty}$ comprise all of the eigenvalues/functions for the problem

$$-\Delta_x \psi_j \phi_k = (\mu_j + \lambda_k)\psi_j\phi_k \text{ with } \psi_j \in \mathcal{V}_1, \phi_k \in \mathcal{V}_2, \tag{4.17}$$

and $\{(\mu_j + \lambda_k)^{1/2}\psi_j\phi_k\}_{j,k=1}^{\infty}$ forms an orthonormal basis for $H_0^1(R)$. Thus, if $v \in H_0^1(R)$, there is a sequence $\{v_{j,k}\}_{j,k=1}^{\infty}$ of scalars such that

$$v(x) = \sum_{j,k=1}^{\infty} v_{j,k}\psi(x_1)\phi(x_2) = \sum_{k=1}^{\infty} v_k(x_1)\phi(x_2) \tag{4.18}$$

converges in $H_0^1(R)$. Here, $v_k(x_1) = \sum_{j=1}^{\infty} v_{j,k} = \int_R v\phi_k\, dx$ and $v_{j,k} = \int_R v\psi_j\phi_k\, dx$. Moreover, we have the following type of Bessel's Equality

$$\|v\|^2_{H_0^1(R)} = \int_R |\nabla v|^2\, dx = -\int_R v\, \frac{\partial^2}{\partial x_1^2} v\, dx - \int_R v\, \Delta_{x_2} v\, dx$$

$$= \sum_{j,k=1}^{\infty} (\mu_j + \lambda_k)|v_{j,k}|^2. \tag{4.19}$$

The interpretation of this equality is that the $H_0^1(U)$-norm of v requires one derivative in the μ_j direction or one derivative in the λ_k direction. In particular,

$$\lambda_k \sum_{k=1}^{\infty} \|v_k\|^2_{L^2(U)} \le \|v\|^2_{H_0^1(R)}.$$

By Duality, (4.19) gives the following representation for $H^{-1}(R)$.

$$\|v\|^2_{H^{-1}(R)} = \sum_{j,k=1}^{\infty} \frac{1}{\mu_j + \lambda_k} |v_{j,k}|^2. \tag{4.20}$$

4.2.2 Convergence in Law

Lemma 4.2.1. *Let $g_1, g_2 \in L^2(R)$. Then*

$$E\left|\int_0^1 \int_U g_1(y_1,y_2)A^\varepsilon(y_1) \int_0^1 B^\varepsilon(r)g_2(r,y_2)\, dr\, dy_2\, dy_1\right|^2$$

$$\le C\varepsilon^2 \left|\int_U \left(\int_0^1 |g_1(y_1,y_2)|^2\, dy_1\right)^{\frac{1}{2}} dy_2 \int_U \left(\int_0^1 |g_2(y_1,y_2)|^2\, dy_1\right)^{\frac{1}{2}} dy_2\right|^2.$$

Proof. The proof follows the ideas of Bal (2008). Expanding the integrals

$$E\left|\int_0^1 \int_U g_1(y_1,y_2)A^\varepsilon(y_1) \int_0^{y_1} B^\varepsilon(r)g_2(r,y_2)\, dr\, dy_2\, dy_1\right|^2 = \int_U \int_U \int_0^1 \int_0^1 \int_0^1 \int_0^1$$

$$E[A^\varepsilon(y_1)A^\varepsilon(\tilde{y}_1)B^\varepsilon(r)B^\varepsilon(\tilde{r})]g_1(y)g_1(\tilde{y})g_2(r,y_2)g_2(\tilde{r},\tilde{y}_2)\, dr\, d\tilde{r}\, dy_1\, d\tilde{y}_1\, dy_2\, d\tilde{y}_2$$

By Lemma 2.5.12, the expression above is bounded by a sum of terms of the form

$$\leq C \int_U \int_U \int_0^1 \int_0^1 \mathfrak{f}^{\frac{1}{2}}\left(\frac{|t_1 - t_2|}{3\varepsilon}\right) H_1(t_1)H_2(t_2)\,dt_1\,dt_2$$

$$\int_0^1 \int_0^1 \mathfrak{f}^{\frac{1}{2}}\left(\frac{|t_3 - t_4|}{3\varepsilon}\right) H_3(t_3)H_4(t_4)\,dt_3\,dt_4\,dy_2\,d\tilde{y}_2,$$

where (t_1, t_2, t_3, t_4) is a permutation of $(y_1, \tilde{y}_1, r, \tilde{r})$ and the $H's$ are a permutation of the functions g. Extend the functions H_j to be zero outside R. Use Plancherel's Inequality (4.1.1) as in the proof of Lemma 2.5.11 and Hölder's Inequality to see that the variance is bounded by a sum of terms of the form

$$C\varepsilon^2 \int_U \int_U \int_{\mathbb{R}} \widehat{\mathfrak{f}^{\frac{1}{2}}}(\varepsilon\xi)\widehat{H_1}(\xi)\overline{\widehat{H_2}(\xi)}\,d\xi \int_{\mathbb{R}} \widehat{\mathfrak{f}^{\frac{1}{2}}}(\varepsilon\xi)\widehat{H_1}(\xi)\overline{\widehat{H_2}(\xi)}\,d\xi\,dy_2\,d\tilde{y}_2 \leq$$

$$C\varepsilon^2 \int_{U^2} \left(\prod_{j=1}^4 \int_{\mathbb{R}} |\widehat{\mathfrak{f}^{\frac{1}{2}}}(\varepsilon\xi)|\,|\widehat{H_j}(\xi)|^2\right)^{\frac{1}{2}} dy_2\,d\tilde{y}_2.$$

This concludes the proof. □

Observe that if $x'_j \in x_2$ then $\frac{\partial}{\partial x'_j} z^\varepsilon$ satisfies the equation (4.13) with u^0 replaced by $\frac{\partial}{\partial x'_j} u^0$ and boundary condition $\frac{\partial}{\partial x'_j} z^\varepsilon = -\frac{\partial}{\partial x'_j}(u^\varepsilon - u^0) - \chi_1^\varepsilon \cdot \nabla(\frac{\partial}{\partial x'_j} u^0)$ on ∂R. Since $u^0 \in H^3(U)$ by the elliptic regularity theory, $\frac{\partial}{\partial x'_j} z^\varepsilon \in H^1(R)$, and the proof of the homogenization result (see also Lemma (5.4.2)) shows that $\frac{\partial}{\partial x'_j} z^\varepsilon \to 0$ in $H^1(R)$.

Lemma 4.2.2. $\frac{1}{\sqrt{\varepsilon}}\mathcal{G}R^\varepsilon$ *converges to* 0 *in probability in* $L^2(R)$ *as* $\varepsilon \to 0$.

Proof. Denote the terms of R^ε by $R_j^\varepsilon, j = 1, 2, 3, 4$. We start off by verifying Condition 1 of Corollary 2.5.7 for $\mathcal{G}R_1^\varepsilon$. Using Integration by Parts and Fubini Theorem we get

$$E\left|\int_R \mathcal{G}[R_1^\varepsilon]\psi_j\phi_k\,dx\right|^2 = E\left|\int_R A^\varepsilon \int_0^{x_1} (B^\varepsilon + 1)\frac{1}{\sqrt{a^*}}\chi_1^\varepsilon \frac{\partial}{\partial r} u^0\,dr\,\Delta_{x_2}\frac{\partial}{\partial x_1}\mathcal{G}\psi_j\phi_k\,dx\right|^2$$

Since $(\mu_j + \lambda_k)\mathcal{G}[\psi_j(x_1)\phi_k(x_2)] = \psi_j(x_1)\phi_k(x_2)$ and $\frac{d}{dx_1}\psi_j(x_1) = \mu_j^{1/2}\psi_j'(x_1)$ the expression above simplifies to

$$\frac{\mu_j \lambda_k^2}{\sqrt{a^*}(\mu_j + \lambda_k)^2} E \left| \int_R A^\varepsilon \int_0^{x_1} (B^\varepsilon + 1)\chi_1^\varepsilon \frac{\partial}{\partial r} u^0 \, dr \, \psi_j \phi_k \, dx \right|^2 .$$

Now use Fubini Theorem and Hölder's Inequality to obtain the upper bound

$$\frac{\mu_j \lambda_k^2}{\sqrt{a^*}(\mu_j + \lambda_k)^2} E \left| \int_0^1 (B^\varepsilon + 1) \frac{\partial}{\partial r} u_k^0 \int_r^1 A^\varepsilon \psi_j \, dx_1 \int_0^r A^\varepsilon \, d\tilde{x}_1 \, dr \right|^2 \leq$$

$$C \frac{\mu_j \lambda_k^2}{\sqrt{a^*}(\mu_j + \lambda_k)^2} \left\| \frac{\partial}{\partial r} u_k^0 \right\|_{L^2(I)}^2 \int_0^1 E \left| \int_r^1 A^\varepsilon \psi_j \, dx_1 \int_0^r A^\varepsilon \, d\tilde{x}_1 \right|^2 dr \quad (4.21)$$

By Lemma 4.2.1, the expected value on the expression above is bounded by $C\varepsilon^2$, which proves Condition 1. To prove the tightness condition (2.8), it suffices to show that the sum over j, k of (4.21) is $\ll C\varepsilon$, where C is independent of ε. Let $\frac{3}{2} < s < 2$. Then, (4.21) is less than

$$C\varepsilon^2 \sum_{j,k=1}^{\infty} \frac{\mu_j}{(\mu_j + \lambda_k)^s} \frac{\lambda_k^2}{(\mu_j + \lambda_k)^{2-s}} \left\| \frac{\partial}{\partial x_1} u_k^0 \right\|_{L^2(I)}^2 \leq C\varepsilon^2 \sum_{j=1}^{\infty} \mu_j^{1-s} \sum_{k=1}^{\infty} \lambda_k^s \left\| \frac{\partial}{\partial x_1} u_k^0 \right\|_{L^2(I)}^2 .$$

Since $\mu_j = 2\pi j^2$, the sum over j is finite. By definition of the fractional powers of $-\Delta_{x_2}$, the sum over k equals $\left\| \frac{\partial}{\partial s} \Delta_{x^2}^{r/2} u^0 \right\|_{L^2(I)}^2 \leq C\|u^0\|_{H^2(R)}^2 \leq C\|f\|_{L^2(R)}^2$. This proves tightness for $\mathcal{G}R_1^\varepsilon$. The proofs for R_2^ε and R_3^ε are quite similar and are omitted. For $\mathcal{G}R_4^\varepsilon$, the same steps show that

$$E \left| \int_R \mathcal{G}[R_4^\varepsilon] \psi_j \phi_k \, dx \right|^2 = \frac{\mu_j}{(\mu_j + \lambda_k)^2} |K_k^\varepsilon|^2 E \left| \int_0^1 A^\varepsilon \psi_j' \, dx_1 \right|^2 , \quad (4.22)$$

where $K_k^\varepsilon = \int_U K^\varepsilon(x_2)\phi_k(x_2)\, dx_2$. Theorem 2.5.11 shows that the expected value is of order ε, and K_k^ε vanishes in the limit as $\varepsilon \to 0$ by the Homogenization Theorem. This proves Condition 1 of (2.5.7). To prove tightness (2.8), observe that for $\frac{3}{2} < s < 2$, the sum over j, k of (4.22) is bounded by

$$C\varepsilon \sum_{j=1}^{\infty} \mu_j^{1-s} \sum_{k=1}^{\infty} \frac{K_k^\varepsilon}{\lambda_k^{2-s}} \leq C\varepsilon \|K^\varepsilon\|_{H^{-(2-s)}(U)}.$$

This proves tightness and completes the proof of the lemma. □

Next, we analyze J_1^ε, the first term of J^ε. The key steps are similar as before. Using Bessel's Equality, Integration by Parts and Fubini Theorem we see that

$$E\,\|\mathcal{G}J_1^\varepsilon\|_{L^2(R)}^2 = E\left|\int_R A^\varepsilon(x_1) \int_0^{x_1} \frac{a^\varepsilon(r)}{\bar{a}} z^\varepsilon(r, x_2)\, dr\, \Delta_{x_2} \frac{\partial}{\partial x_1} \mathcal{G}\left[\psi_j(x_1)\phi_k(x_2)\right]\, dx\right|^2$$

$$= E\frac{\mu_j + \lambda_k^2}{(\mu_j + \lambda_k)^2} \left|\int_0^1 \frac{a^\varepsilon(r)}{\bar{a}} z_k^\varepsilon(r) \int_r^1 A^\varepsilon(x_1)\, \psi_j'(x_1)\, dx_1\, dr\right|^2$$

An application of Hölder's Inequality and Theorem 2.5.11 implies that the last expression is bounded by

$$C \sum_{j,k=1}^{\infty} \frac{\mu_j + \lambda_k^2}{(\mu_j + \lambda_k)^2} \|z_k^\varepsilon\|_{L^\infty(\Omega)\otimes L^2(I)}^2 \int_0^1 E\left|\int_r^1 A^\varepsilon(x_1)\, \psi_j'(x_1)\, dx_1\right|^2 dr \leq$$

$$C\varepsilon \|\hat{R}_A\|_{L^\infty(\mathbb{R})} \sum_{j=1}^{\infty} \mu_j^{1-s} \sum_{k=1}^{\infty} \lambda_k^s \|z_k^\varepsilon\|_{L^\infty(\Omega)\otimes L^2(I)}^2 \leq \|\Delta_{x_2}^{s/2} z^\varepsilon\|_{L^\infty(\Omega)\otimes L^2(R)}^2,$$

where $\frac{3}{2} < s < 2$, and (2.8) holds. We may recast $\mathcal{G}I^\varepsilon$ as

$$\mathcal{G}I^\varepsilon = \int_0^1 \frac{A^\varepsilon(y_1)}{\sqrt{a^*}} \left[G(x, y)\frac{\partial^2 u^0}{\partial y_1^2}(y) + \int_r^1 G(x, r, y_2)\frac{\partial f}{\partial y_1}(r, y_2)\, dr\right] + B^\varepsilon(y_1) G(x, y) \Delta_{y_2} u^0(y)\, dy.$$

By Theorem 2.5.11, for fixed x, $\frac{1}{\sqrt{\varepsilon}}\mathcal{G}I^\varepsilon$ converges in law to a mean zero normal random variable with variance $\sigma^2(y) = E\int_{\mathbb{R}} H(y, r)H(y, 0)\, dr$, where

$$H(y, r) = \frac{A(y_1)}{\sqrt{a^*}} \left[G(x, y)\frac{\partial^2 u^0}{\partial y_1^2}(y) + \int_r^1 G(x, r, y_2)\frac{\partial f}{\partial y_1}(r, y_2)\, dr\right] + B(y_1) G(x, y) \Delta_{y_2} u^0(y)\, dy.$$

It is easy to show tightness in $L^2(U)$ as we did above, so Theorem 2.5.11 guarantees that $\frac{1}{\sqrt{\varepsilon}}\mathcal{G}I^\varepsilon$ converges in law on $L^2(U)$ to $\int_0^1 \int_U \sigma(y)\,dy_2\,dW_{y_1}$, where W_{y_1} is a standard Brownian motion. An alternative simpler representation is given by the following

Lemma 4.2.3. *The law of $\frac{1}{\sqrt{\varepsilon}}\mathcal{G}I^\varepsilon$ converges on $L^2(R)$ to*

$$\frac{\|R_A\|_{L^1(\mathbb{R})}^{\frac{1}{2}}}{\sqrt{a^*}}\int_0^1 \int_U \left[G(x,y)\frac{\partial^2 u^0}{\partial y_1^2}(y) + \int_r^1 G(x,r,y_2)\frac{\partial f}{\partial y_1}(r,y_2)\,dr\right] dy_2\,dW_{y_1}^1 +$$
$$\|R_B\|_{L^1(\mathbb{R})}^{\frac{1}{2}}\int_0^1 \int_U G(x,y)\Delta_{y_2}u^0(y)\,dy_2\,dW_{y_1}^2 \quad (4.23)$$

as $\varepsilon \to 0$, where W^1 and W^2 are correlated Brownian motions such that $dW^1\,dW^2 = \rho\,dt$. Here,

$$\rho = \frac{\int_{\mathbb{R}} E[A(0)B(\tau) + B(0)A(\tau)]\,d\tau}{\left(\int_{\mathbb{R}} E[A(0)A(\tau)]\,d\tau \int_{\mathbb{R}} E[B(0)B(\tau)]\,d\tau\right)^{\frac{1}{2}}}.$$

Given our results so far, Theorem 2.5.9 implies that the limiting law of $\frac{1}{\sqrt{\varepsilon}}z^\varepsilon$ is the same as the limiting law of $\frac{1}{\sqrt{\varepsilon}}z_1^\varepsilon$, where $z_1^\varepsilon = \mathcal{G}[B^\varepsilon z_1^\varepsilon] + \mathcal{G}[I^\varepsilon]$. Iterating this equation once produces

$$z_1^\varepsilon = \mathcal{G}[B^\varepsilon\mathcal{G}[B^\varepsilon z_1^\varepsilon]] + \mathcal{G}[B^\varepsilon\mathcal{G}[I^\varepsilon]] + \mathcal{G}[I^\varepsilon].$$

We will show that the first two terms are order $\ll \sqrt{\varepsilon}$, meaning that the law of z_1^ε converges to (4.23). We will work out the first term J_2^ε; the second one is easier and is left to the reader. Integration by parts and Fubini Theorem gives

$$E\,\|\mathcal{G}J_2^\varepsilon\|_{L^2(R)}^2 = E\left|\int_R B^\varepsilon(x_1)\int_R G(x,y)B^\varepsilon(y)z^\varepsilon(y)\,dy\,\Delta_{x_2}\Delta_{x_2}\mathcal{G}\left[\psi_j(x_1)\phi_k(x_2)\right]\,dx\right|^2$$
$$= \frac{\lambda_k^4}{(\mu_j+\lambda_k)^2}E\left|\int_R B^\varepsilon(y_1)z^\varepsilon(y)\int_R B^\varepsilon(x_1)G(x,y)\psi_j(x_1)\phi_k(x_2)\,dx\,dy\right|^2$$

By Mercer's Theorem we can write $G(x,y) = \sum_{k=1}^{\infty} G_k(x_1,y_1)\phi_k(x_2)\phi_k(y_2)$ where $G_k(x_1,y_1)$ is the Green's function of the operator $-\frac{d^2}{dx_1^2}+\lambda_k$ with homogeneous Dirichlet boundary conditions. Thus,

$$E\,\|\mathcal{G}J_2^\varepsilon\|^2_{L^2(R)} = \frac{\lambda_k^4}{(\mu_j+\lambda_k)^2} E \left| \int_0^1 B^\varepsilon(y_1) z_k^\varepsilon(y) \int_0^1 B^\varepsilon(x_1) G_k(x_1,y_1)\psi_j(x_1)\,dx_1\,dy_1 \right|^2 .$$

Now use Hölder's Inequality to see that

$$E\,\|\mathcal{G}J_2^\varepsilon\|^2_{L^2(R)} \le \frac{C\lambda_k^4}{(\mu_j+\lambda_k)^2} \|z_k^\varepsilon\|^2_{L^2(I)} E \int_0^1 \int_0^1 \left| B^\varepsilon(x_1) G_k(x_1,y_1)\psi_j(x_1)\,dx_1 \right|^2 dy_1.$$

Since $G_k(x_1,y_1) \le \frac{C}{\sqrt{\lambda_k}} \exp\{-\sqrt{\lambda_k}|x_1-y_1|\}$ and ψ is bounded, Theorem 2.5.11 gives

$$E\,\|\mathcal{G}J_2^\varepsilon\|^2_{L^2(R)} \le C\varepsilon \frac{\lambda_k^3}{(\mu_j+\lambda_k)^2} \|z_k^\varepsilon\|^2_{L^2(I)} \le C\varepsilon \frac{1}{\mu^s} \lambda_k^{1+s} \|z_k^\varepsilon\|^2_{L^2(I)}.$$

For $\frac{1}{2} < s < 2$, the sum over i,j of the expression above is finite and bounded by $C\varepsilon \|\Delta_{x_2}^{(1+s)/2} z_k^\varepsilon\|^2_{L^2(I)} \ll \varepsilon$, which shows (2.8). Thus, the law of $\frac{1}{\sqrt{\varepsilon}} z^\varepsilon$ converges on $L^2(R)$ to (4.23). To obtain the limit of $w^\varepsilon = u^\varepsilon - u^0$, observe that

$$\begin{aligned}
\Delta(z^\varepsilon - w^\varepsilon) &= \Delta\Big(\frac{1}{\sqrt{a^*}} \chi_1^\varepsilon \frac{\partial}{\partial x_1} u^0\Big) \\
&= \frac{1}{\sqrt{a^*}} \chi_1^\varepsilon \frac{\partial}{\partial x_1} \Delta_{x_2} u^0 + \frac{1}{\sqrt{a^*}} \frac{\partial}{\partial x_1}\Big(\chi_1^\varepsilon \frac{\partial^2}{\partial x_1^2} u^0\Big) + \frac{1}{\sqrt{a^*}} \frac{\partial}{\partial x_1}\Big(A^\varepsilon \frac{\partial}{\partial x_1} u^0\Big).
\end{aligned}$$

Comparing this expression with the oscillatory integral I^ε in (4.14) we see that $w^\varepsilon = -\frac{1}{\sqrt{a^*}} \frac{\partial}{\partial x_1}(A^\varepsilon \frac{\partial}{\partial x_1} u^0) + B^\varepsilon \Delta_{x_2} u^0 + o(\sqrt{\varepsilon})$, from where theorem 3.1.2 immediately follows.

5

Large Deviation Principle in Homogenization

5.1 LDP for Oscillatory Integrals

Here will investigate conditions under which large deviations for the sequence of oscillatory random fields q^ε exist. This will follow from the finiteness of some exponential moments together with other mild technical conditions. Fix $1 \leq p \leq \infty$. Denote by $L^p_w(U)$ the topological space of p-integrable functions with values in $\mathbb{R}^m$ with the weak topology if $1 \leq p < \infty$, or the weak-* topology if $p = \infty$. Recall that a sequence of functions $f^k : U \to \mathbb{R}^m$ converges to f in $L^p_w(U)$ as $k \to \infty$ if

$$\lim_{k\to\infty} \int_U (f^k_y - f_y) \cdot g_y \, dy = 0$$

for all $g \in L^{p'}(U)$, where p' is the Hölder conjugate of p $\left(\frac{1}{p} + \frac{1}{p'} = 1, \frac{1}{\infty} = 0\right)$. Throughout the remainder of this chapter, unless noted otherwise, it will be assumed that the random field q^ε satisfies assumptions (A1)-(A4) stated in Chapter 3. We will see that (A1)-(A4) imply the LDP for q^ε in the sense of (2.18) on $X = L^p_w(U)$.

Before moving on to the proof of the LDP, let us pause to state some remarks and results whose proofs are found below.

Condition (A2) is an absolutely necessary ingredient in the proof of the LDP since it guarantees the sequence q^ε is exponentially tight.

Moreover, if q^ε satisfies the LDP and (A2) holds, then (A1) must hold as well by Varadhan's Lemma (Dembo and Zeitouni, 1998, Section 4.3). In view of Theorem 2.6.16, (A1) is a very natural condition to impose for the LDP to hold. In particular, this condition provides a representation formula for H; indeed, after taking $f(t) \equiv t$ in (3.5) we obtain

$$H(t) = \lim_{\varepsilon \to 0} \frac{1}{|U/\varepsilon|} \ln E \exp \left\{ \int_{U/\varepsilon} t \cdot q_y \, dy \right\}, \tag{5.1}$$

where $U/\varepsilon = \{\frac{x}{\varepsilon} \,|\, x \in U\}$. Also, observe that $H(0) = 0$.

Other important consequences of (A2) are that $\mathcal{D}_H = \mathbb{R}^m$, and that it yields the upper bound (2.19).

Conditions (A3) and (A4) are technical conditions necessary to ensure the lower bound (2.20). Observe that (A2)-(A3) are stronger assumptions for larger p, so we may think of p as a measure of regularity of q in terms of the finiteness of its exponential moments.

Let us just remark that the last condition (A4) is inherited from the Gärtner-Ellis Theorem. Indeed, by the Duality Theorem (A4) implies that the Legendre transform L of H is strictly convex, which guarantees that the convex function $a \cdot s - L(s)$ has a *unique* minimum. (Dembo and Zeitouni, 1998, Section 2.3) provides nontrivial examples on how the Gärtner-Ellis Theorem fails to provide an optimal lower bound should (A4) is not satisfied and additional work must be done in order to prove the LDP.

Since $T_{p,\alpha_1} < T_{p,\alpha_2}$ for $1 \leq \alpha_1 < \alpha_2$, it suffices to take $\alpha = 1$ in (A2)-(A3). In practice, however, it is much easier to verify them by taking $\alpha = p$. Important examples of random fields satisfying (A1)-(A4) will be studied in the next section.

The next theorem is an important consequence of (A2)-(A3) and it provides a rate of growth of $T_{p,1}$.

Lemma 5.1.1. *Suppose* $1 < p \leq \infty$. *There exists a constant* $\kappa_H > 0$ *such that* $0 \leq T_{p,1}(r) \leq \kappa_H(|r|^{p'} + |U|)$ *for all* $r \in \mathbb{R}$.

Proof. Clearly, we can take $\kappa_H = \|q\|_{L^\infty(\Omega\times\mathbb{R}^n)}$ if $p = \infty$, so suppose $1 < p < \infty$. Observe that $0 \leq T_{p,1}(\pm r) \leq T_{p,1}(\pm\kappa_\pm) < \infty$ for all $0 \leq r \leq \kappa_\pm$. If $r > \kappa_\pm$ we have

$$\begin{aligned}
\frac{r^p}{\varepsilon^{np}}\|q^\varepsilon\|^p_{L^p(U)} &= \varepsilon^{-np}\left(\frac{r}{\kappa_\pm}\right)^{p'p}\kappa_\pm^p \int_U |q_{y/\varepsilon}|^p \left(\frac{\kappa_\pm}{r}\right)^{p'} dy \\
&= \varepsilon^{-np}\left(\frac{r}{\kappa_\pm}\right)^{p'p}\kappa_\pm^p \int_{\left(\frac{\kappa_\pm}{r}\right)^{\frac{p'}{n}}U} \left|q\left(\left(\frac{r}{\kappa_\pm}\right)^{p'/n}\frac{y}{\varepsilon}\right)\right|^p dy \\
&= \tilde{\varepsilon}^{-np}\kappa_\pm^p \int_{\left(\frac{\kappa_\pm}{r}\right)^{\frac{p'}{n}}U} |q_{y/\tilde{\varepsilon}}|^p \, dy,
\end{aligned}$$

where $\tilde{\varepsilon} = \left(\frac{\kappa_\pm}{r}\right)^{p'/n}\varepsilon$ and $cU = \{cx \,|\, x \in U\}$ for $c > 0$. Thus, both

$$T_{p,1}(+r) = \limsup_{\tilde{\varepsilon}\to 0}\left(\frac{r}{\kappa_+}\right)^{p'}\tilde{\varepsilon}^n \ln E \exp\left\{\frac{\kappa_+}{\tilde{\varepsilon}^n}\|q^{\tilde{\varepsilon}}\|_{L^p\left(\left(\frac{\kappa_+}{r}\right)^{p'/n}U\right)}\right\} \leq \left(\frac{r}{\kappa_+}\right)^{p'} T_{+\kappa_+},$$

$$T_{p,1}(-r) = -\liminf_{\tilde{\varepsilon}\to 0}\left(\frac{r}{\kappa_-}\right)^{p'}\tilde{\varepsilon}^n \ln E \exp\left\{-\frac{\kappa_-}{\tilde{\varepsilon}^n}\|q^{\tilde{\varepsilon}}\|_{L^p\left(\left(\frac{\kappa_-}{r}\right)^{p'/n}U\right)}\right\} \leq \left(\frac{r}{\kappa_-}\right)^{p'} T_{-\kappa_-}$$

are finite and we can take $\kappa_H = \max\left\{\frac{T_{-\kappa_-}}{\kappa_-^{p'}}, \frac{T_{+\kappa_+}}{\kappa_+^{p'}}, \frac{\kappa_-}{|U|}, \frac{\kappa_+}{|U|}\right\}$. □

Apply Hölder's Inequality to the representation formula (5.1) and use Lemma 5.1.1 to obtain the following upper and lower bounds for H when $1 < p \leq \infty$:

$$H(t) \leq \frac{1}{|U|}T_{p,1}(|U|^{\frac{1}{p'}}|t|) \leq \kappa_H(|t|^{p'} + 1) \tag{5.2a}$$

$$H(t) \geq -\frac{1}{|U|}T_{p,1}(-|U|^{\frac{1}{p'}}|t|) \geq -\kappa_H(|t|^{p'} + 1) \tag{5.2b}$$

When $p = 1$, bounds for H can also be obtained from the fact that $G(t) \gg |t|$ as $|t| \to \infty$. Indeed, for every $|t| > 0$ there is $K_{|t|} > 0$ such that $|t||q_y| \leq G(q_y) + K_{|t|}$.

A straightforward estimation produces

$$H(t) \leq \limsup_{\varepsilon\to 0} \frac{1}{|U/\varepsilon|} \ln E \exp\left\{\int_{U/\varepsilon} [G(q_y) + K_{|t|}]\, dy\right\} \leq T_G + K_{|t|} < \infty.$$

In particular, $\mathcal{D}_H = \mathbb{R}^m$ for all $1 \leq p \leq \infty$. These estimates together with the Dominated Convergence Theorem yield the following

Lemma 5.1.2. *If T is defined as in* (3.5)*, then Tf is well-defined and finite for all $f \in L^{p'}(U), 1 \leq p \leq \infty$.*

One last important property is the convexity of H and $T_{p,\alpha}$ which follows from elementary properties of the exponential and logarithmic functions and Hölder Inequality. Indeed, for any $s, t \in \mathbb{R}^m$ and $0 < \alpha < 1$ we have

$$H(\alpha s + (1-\alpha)t) = \lim_{\varepsilon\to\ 0} \frac{1}{|U/\varepsilon|} \ln E \exp\left\{\int_{U/\varepsilon} (\alpha s + (1-\alpha)t)\cdot q_y\, dy\right\}$$

$$= \lim_{\varepsilon\to\ 0} \frac{1}{|U/\varepsilon|} \ln E\left[\left(\exp\left\{\int_{U/\varepsilon} s\cdot q_y\, dy\right\}\right)^{\alpha}\left(\exp\left\{\int_{U/\varepsilon} t\cdot q_y\, dy\right\}\right)^{1-\alpha}\right]$$

$$\leq \lim_{\varepsilon\to\ 0} \frac{1}{|U/\varepsilon|} \ln\left[\left(E\exp\left\{\int_{U/\varepsilon} s\cdot q_y\, dy\right\}\right)^{\alpha}\left(E\exp\left\{\int_{U/\varepsilon} t\cdot q_y\, dy\right\}\right)^{1-\alpha}\right]$$

$$= \lim_{\varepsilon\to\ 0} \frac{\alpha}{|U/\varepsilon|} \ln E\exp\left\{\int_{U/\varepsilon} s\cdot q_y\, dy\right\} + \lim_{\varepsilon\to\ 0} \frac{1-\alpha}{|U/\varepsilon|} \ln E\exp\left\{\int_{U/\varepsilon} t\cdot q_y\, dy\right\}$$

$$\therefore H(\alpha s + (1-\alpha)t) \leq \alpha H(s) + (1-\alpha)H(t).$$

A similar computation shows that $T_{p,\alpha}$ is convex on the real line.
Now, let L be the Legendre Transform of H. It follows from the Duality Theorem and (A4) that L is strictly convex, lower semicontinuous, and $L \not\equiv \infty$. L is also nonnegative since $H(0) = 0$.

Lemma 5.1.3 (Bounds on L)**.**

(a) If $p = \infty$, then $\mathcal{D}_L$ is compact, that is, L is finite only inside a compact set.

(b) If $1 < p < \infty$, there exist a constant $\kappa_L > 0$ such that $L(s) \geq \kappa_L(|s|^p - 1)$ for all $s \in \mathbb{R}^m$.

(c) If $p = 1$, for every $N > 0$ we can find $\kappa_N > 0$ such that $L(s) \geq N|s| - \kappa_N$ for all $s \in \mathbb{R}^m$.

Proof. The cases $1 < p \leq \infty$ are easy consequences of (5.2a) and Lemma 2.6.13. So suppose $p = 1$. Fix $|s_0| = 1$. Let $t_0 \in \partial L(s_0)$ and $N > 2|t_0|$. Since $H(t)$ is finite and convex for all $t \in \mathbb{R}^m$, the function $a \to L(as_0)$ is also finite and convex and lemma 2.6.13 yields $\limsup_{a\to\infty} \dfrac{L(as_0)}{a} = \infty$. Since L is convex, we can find $a_0 > 0$ such that $L(as_0) > 2aN$ for $a > a_0$. Observe that if $|s| = 1$ then $|s - s_0| \leq 2$ and $|t_0 \cdot (s - s_0)| \leq N$, whence the the convexity of L yields

$$L(as) \geq at_0 \cdot (s - s_0) + L(as_0) > a[t_0 \cdot (s - s_0) + 2N] \geq aN$$

for all $|s| = 1$ and $a > a_0$. Therefore, $\kappa_N = \sup_{s\in\mathbb{R}^m}[N|s| - L(s)] < \infty$, and the proof is complete. □

Before we prove the important result of this section, we state the following classical compactness results. We refer to Diestel and Uhl (1977) for a proof.

Theorem 5.1.4 (Banach-Alaoglu)**.** *Let $1 < p \leq \infty$. $A \subset L^p(U)$ is compact in $L^p_w(U)$ if and only if A is bounded in $L^p(U)$, i.e., $\sup_{f\in A} \|f\|_{L^p(U)} < \infty$.*

Theorem 5.1.5 (Dunford-Pettis)**.** *$A \subset L^1(U)$ is compact in $L^1_w(U)$ if and only if A is bounded in $L^1(U)$ and uniformly integrable. Boundedness means $\sup_{f\in A} \|f\|_{L^1(U)} < \infty$. Uniform integrability means that*

$$\lim_{N\to\infty} \sup_{f\in A} \int_{|f(x)|\geq N} |f(x)|\, dx = 0.$$

Theorem 5.1.6 (de la Vallée-Poussin). *$A \subset L^1(U)$ is uniformly integrable if and only if there is a nonnegative increasing convex function G satsifying*

$$\lim_{|t|\to\infty} \frac{G(t)}{|t|} = \infty \qquad \text{and} \qquad \sup_{f\in A} \int_U G(|f(x)|)\,dx < \infty.$$

Theorem 5.1.7. *Let the $\mathbb{R}^m$-valued random field q satisfy assumptions (A1)-(A4). Then, q^ε satisfies the LDP on $L^p_w(U)$ with rate $\frac{1}{\varepsilon^n}$ and action functional*

$$S\varphi = \int_U L(\varphi_y)\,dy. \tag{5.3}$$

Proof. We must verify the conditions of Theorem 2.6.16. Exponential tightness follows directly from the (A2), the compactness results stated above and the exponential Chebyshev Inequality. The case $p = \infty$ is obvious. When $1 < p < \infty$, the Banach-Alaoglu Theorem the set $K_C = \{\varphi \in L^p(U) \mid \|\varphi\|_{L^p(U)} \le C\}$ is compact in $L^p_w(U)$ for each $C > 0$, so that

$$\begin{aligned}\limsup_{\varepsilon\to 0} \varepsilon^n \ln P(\|q^\varepsilon\|_{L^p(U)} > C) &\le -C\kappa_+ + \limsup_{\varepsilon\to 0} \varepsilon^n \ln E \exp\left\{\frac{\kappa_+}{\varepsilon^n}\|q^\varepsilon\|_{L^p(U)}\right\} \\ &= -C\kappa_+ + T_{p,1}(+\kappa_+).\end{aligned}$$

When $p = 1$, the same argument works for $K_C = \{\varphi \in L^1(U) \mid \int_U G(|\varphi_y|)\,dy \le C\}$. Next, let T^* is the Legendre Transform of T. In general, the computation of T^* involves solving an infinite dimensional convex optimization problem, however, the integral form of the limit in (3.7) allows us to reduce the problem to m dimensions. More specifically, we will check that $T^*\varphi = \int_U L(\varphi_y)\,dy$ for all $\varphi \in L^p(U)$. The argument is standard and is adapted from (Baldi, 1988, Lemma 2.7) Note that $H(0) = 0$,

$T^*\varphi \geq 0$ for all $\varphi \in L^p(U)$, so that

$$\begin{aligned} T^*\varphi &= \sup_{f \in L^{p'}(U)} \int_U [f_y \cdot \varphi_y - H(f_y)]\, dy \\ &\leq \int_U \sup_{f \in L^{p'}(U)} [f_y \cdot \varphi_y - H(f_y)]\, dy \\ &\leq \int_U \sup_{f:U \to \mathbb{R}^m} [f_y \cdot \varphi_y - H(f_y)]\, dy \\ &= \int_U L(\varphi_y)\, dy. \end{aligned}$$

For each $s \in \mathbb{R}^m$, let $I_s = \{t \in \mathbb{R}^m \,|\, s \cdot t - H(t) \geq 0\}$. Since $t \to s \cdot t - H(t)$ is a concave function which vanishes at $t = 0$, I_s is a closed convex set containing 0. Hence, for every $i, j \in \mathbb{N}$ we can find measurable functions $t_i, t_{i,j}$ such that

$$s \cdot t_i(s) - H(t_i(s)) \geq \begin{cases} \max\{L(s) - \frac{1}{i}, 0\}, & \text{if } L(s) < \infty \\ i, & \text{if } L(s) = \infty \end{cases},$$

$$t_{i,j}(s) = \begin{cases} t_i(s), & \text{if } |t_i(s)| \leq j \\ j\frac{t_i(s)}{|t_i(s)|}, & \text{if } |t_i(s)| > j \end{cases},$$

and $t_{i,j}(s) \in I_s$ for all $s \in \mathbb{R}^m$. For $\varphi \in L^p(U)$, $f^{i,j} = t_{i,j} \circ \varphi \in L^\infty(U)$ and $f^{i,j}_y \cdot \varphi_y - H(f^{i,j}_y) \geq 0$ for all $y \in U$, so that

$$T^*\varphi = \sup_{f \in L^{p'}(U)} \int_U [f_y \cdot \varphi_y - H(f_y)]\, dy \geq \liminf_{i,j \to \infty} \int_U [f^{i,j}_y \cdot \varphi_y - H(f^{i,j}_y)]\, dy.$$

Letting $j \to \infty$, by Fatou's Lemma

$$T^*\varphi \geq \int_U \liminf_{i \to \infty} [t_i(\varphi_y) \cdot \varphi_y - H(t_i(\varphi_y))]\, dy. \tag{5.4}$$

Now, let $A_\varphi = \{y \in U \,|\, L(\varphi_y) = \infty\}$ and let $|A_\varphi|$ denote the Lebesgue measure of A_φ. If $|A\varphi| = 0$, then (5.4) gives

$$T^*\varphi \geq \int_U L(\varphi_y)\, dy - \frac{|U|}{i};$$

otherwise, $T^*\varphi \geq i|A_\varphi|$. Since i is arbitrary, this proves the representation formula for T^*.

Finally, we check Gateaux differentiability of T. Consider first the case $1 < p < \infty$. Let $f, g \in L^{p'}(U)$ and $y \in U$. Since $\mathcal{D}_H = \mathbb{R}^m$, by the Duality of the Legendre Transform the function $\varphi^f = \nabla H \circ f$ is the unique solution to the equation

$$H(f_y) = \sup_{s\in\mathbb{R}^m} [s \cdot f_y - L(s)] = \varphi^f_y \cdot f_y - L(\varphi^f_y), \tag{5.5}$$

and $f_y \in \partial L(\varphi^f_y)$. Integrate (5.5) over U and use Young's Inequality together with the bounds (5.2) and Lemma 5.1.3 to see that

$$-\kappa_H \int_U (|f_y|^{p'} + 1)\, dy \leq \frac{1}{\theta^{p'} p'} \int_U |f_y|^{p'}\, dy + \frac{\theta^p}{p} \int_U |\varphi^f_y|^p\, dy - \kappa_L \int_U (|\varphi^f_y|^p - 1)\, dy$$

holds for all $\theta > 0$. By taking θ sufficiently small we can find constants $\kappa_1, \kappa_2 > 0$, depending only on p, κ_H, κ_L and $|U|$, such that $\|\varphi^f\|_{L^p(U)} \leq \kappa_1 \|f\|_{L^{p'}(U)} + \kappa_2$. An application of the Dominated Convergence Theorem shows that

$$D_g T(f) = \int_U g_y \cdot \nabla H(f_y)\, dy \tag{5.6}$$

and $|D_g T(f)| \leq \|g\|_{L^{p'}(U)}(\kappa_1 \|f\|_{L^{p'}(U)} + \kappa_2)$. This concludes the proof that T is differentiable when $1 < p < \infty$.

If $p = \infty$, H grows at most linearly, and ∇H is uniformly bounded. Hence, (5.6) is well-defined and $|D_g T(f)| \leq \|g\|_{L^1(U)} \|\nabla H\|_{L^\infty(U)}$.

Lastly, suppose that $p = 1$ and take $f, g \in L^\infty(U)$. Integrate (5.5) over U, use the lower bound on H and Lemma 5.1.3 to see that

$$-\kappa_H |U| \leq (\|f\|_{L^\infty(U)} - N)\|\varphi^f\|_{L^1(U)} + \kappa_N |U|.$$

In particular, if we pick $N > \|f\|_{L^\infty(U)}$, we obtain that $\|\varphi^f\|_{L^1(U)}$ is bounded. The Dominated Convergence Theorem can be applied again to show that the Gateaux derivative (5.6) is well-defined, and

$$|D_g T(f)| \leq |U| \frac{(\kappa_N + \kappa_H)\|g\|_{L^\infty(U)}}{(N - \|f\|_{L^\infty(U)})}.$$

This completes the proof. $\square$

Lemma 5.1.8. *The action functional S in* (5.3) *vanishes only on the constant trajectory $\varphi \equiv E[q_0]$.*

Proof. It suffices to show that $L(s) = 0$ only when $s = E[q_0]$. Use Jensen's Inequality $E[X] \leq \ln E[e^X]$ with $X = \int_{U/\varepsilon} t \cdot q_y \, dy$ to get that

$$H(t) \geq \lim_{\varepsilon \to 0} \frac{1}{|U/\varepsilon|} E\left\{\int_{U/\varepsilon} t \cdot q_y \, dy\right\} = t \cdot E[q_0]. \tag{5.7}$$

Consequently, $0 \leq L(s) = \sup_{t \in \mathbb{R}^m} [s \cdot t - H(t)] \leq \sup_{t \in \mathbb{R}^m} [(s - E[q_0]) \cdot t]$, so that $L(E[q_0]) = 0$. It follows from the strict convexity of L that $s = E[q_0]$ is the only point at which $L(s) = 0$, and the lemma follows. $\square$

In practice, we are interested in the large deviations of probabilities of sets of the form

$$A_c(\delta) = \left\{\varphi \in L^p(U) \mid \left|\int_U f_y \cdot (\varphi_y - E[q_0]) \, dy - c\right| < \delta\right\} \tag{5.8}$$

for a fixed $f \in L^p(U)$. By (2.19) and Theorem 5.1.7, if $c \neq 0$, $0 < \theta < \inf_{\varphi \in A_c(\delta)} S\varphi$ and $\delta > 0$ we can find $\varepsilon_0(c, \delta, \theta) > 0$ such that $P(q^\varepsilon \in A_c(\delta)) < e^{-\frac{\theta}{\varepsilon^n}}$ for all $0 < \varepsilon < \varepsilon_0$. Similarly, if $0 < \theta < \inf_{\varphi \in A_0(\delta)^c} S\varphi$ and $\delta > 0$ by (2.20) we can find $\varepsilon_0(\delta, \theta) > 0$ such that $P(q^\varepsilon \in A_0(\delta)^c) < e^{-\frac{\theta}{\varepsilon^n}}$ for all $0 < \varepsilon < \varepsilon_0$.

In general, the exact computation of deviation probabilities may be a hard task, however, in some situations we may be able to get closed formulas. As an example, suppose that $1 < p < \infty$ and $L(s) \sim C|s|^p$ as $s \to \infty$ (equivalently, $H(t) \sim \tilde{C}|t|^{p'}$ as $t \to \infty$) for some constants $C, \tilde{C} > 0$. Note that if $f \in L^{p'}(U)$ and $r > 1$ the

Dominated Convergence Theorem guarantees the existence of $\theta(r) > 0$ such that

$$\begin{aligned}\inf_{\varphi\in A_0(\delta)^c}\int_U L(\varphi_y)\,dy &\geq \inf_{\varphi\in A_0(r\delta)^c}\int_U L(\varphi_y)\,dy\\ &= \inf_{\varphi\in A_0(\delta)^c}\int_U L(r\varphi_y)\,dy\\ &\geq \inf_{\varphi\in A_0(\delta)^c}\int_U L(\varphi_y)\,dy - \theta\end{aligned}$$

with $\theta \to 0$ as $r \to 1$. Thus, $A_0(\delta)^c$ is a regular Borel set. In particular, if $L(s) = C|s|^p$ we can compute the limit (2.21) exactly. Indeed, if $\varphi \in A_0(\delta)^c$ then Holder's Inequality implies

$$\frac{C\delta^p}{\|f\|^p_{L^{p'}(U)}} \leq C\frac{\left|\int_U f_y\cdot\varphi_y\,dy\right|^p}{\|f\|^p_{L^{p'}(U)}} \leq \int_U L(\varphi_y)\,dy.$$

Since equality in Holder's Inequality holds if and only if there is $\alpha \in \mathbb{R}$ such that $\varphi_y = \alpha|f_y|^{p'-2}f_y$ for a.e. $y \in U$, (2.21) becomes

$$\lim_{\varepsilon\to 0}\varepsilon^n \ln P(A_0(\delta)^c) = \inf_{\varphi\in A_0(\delta)^c}\int_U L(\varphi_y)\,dy = -\frac{C\delta^p}{\|f\|^p_{L^{p'}(U)}},$$

where the infimum is attained at $\varphi = \pm\frac{\delta}{\|f\|^2_{L^{p'}(U)}}f$. Thus,

$$\lim_{\varepsilon\to 0} P(|\|f\|^2_{L^{p'}(U)}q^\varepsilon - \delta f| < \theta \quad \text{or} \quad |\|f\|^2_{L^{p'}(U)}q^\varepsilon + \delta f| < \theta \quad | \quad q^\varepsilon \in A_0(\delta)^c) = 1$$

for all $\theta > 0$.

It is often desired to know whether one can lift the topological space under which the LDP was formulated. The corollary to the following result gives criteria under which no such improvement can be done.

Lemma 5.1.9. *Take $1 < p < \infty$. Let $H \not\equiv \infty$ be a lower semicontinuous, convex function, let L be its Legendre Transform, and consider the functional S given by* (5.3). *If any one of the equivalent conditions (see Lemma 2.6.13)*

$$\lim_{s\to\infty}\frac{L(s)}{|s|^p} = 0 \quad \text{or} \quad \lim_{t\to\infty}\frac{H(t)}{|t|^{p'}} = \infty \tag{5.9}$$

holds, then $\Phi_S(t)$ *is not compact in* $L^p_w(U)$ *for any* $t > 0$.

Proof. In order to prove the first statement, by the Banach-Alaoglu Theorem it suffices to construct a sequence $\varphi^k \in L^p(U)$ such that $\varphi^k \in \Phi_S(t)$ but $\lim_{k\to\infty} \|\varphi^k\|_{L^p(U)} = \infty$. Without loss of generality, we may assume that $L(0) = 0$. Let $\alpha_n = \frac{2\pi^{\frac{n}{2}}}{n\Gamma(\frac{n}{2})}$ denote the hypervolume of the unit ball in $\mathbb{R}^n$. Fix $y_0 \in U$. For every positive integer k we can find $N_k > 0$ large enough such that

$$r_k = \left(\frac{kt}{N_k^p \alpha^n}\right)^{\frac{1}{n}} < \text{distance}(y_0, \partial U)$$

and $L(s) \le \frac{1}{k}|s|^p$ for $|s| > N_k$. Then, the functions $\varphi^k_y = N_k \mathbf{1}(|y - y_0| < r_k)$ satisfy

$$\int_U L(\varphi^k_y)\, dy = \alpha_n r_k^n L(N_k) \le t, \text{ and}$$

$$\int_U |\varphi^k_y|^p\, dy = \alpha_n r_k^n N_k^p = kt.$$

This conclude the proof. ∎

Corollary 5.1.10. *Let* q^ε *be a sequence of random fields satisfying the LDP on* $L^{\tilde p}_w(U), 1 \le \tilde p < \infty$, *with action functional* S *as in* (5.3). *If there exists* $\tilde p < p < \infty$ *such that* (5.9) *holds, then* q^ε *does not satisfy the LDP on* $L^p_w(U)$.

Proof. We argue by contradiction. So suppose that q^ε also satisfies the LDP on $L^p_w(U)$ with action functional S_0. An application of the Contraction Principle and Lemma 2.6.17 to the identity operator from $L^p_w(U)$ to $L^{\tilde p}_w(U)$, shows that

$$S\varphi = \begin{cases} S_0\varphi, & \text{if } \varphi \in L^p(U) \\ \infty, & \text{if } \varphi \in L^{\tilde p}_w(U) \setminus L^p(U) \end{cases}.$$

However, S does not have compact level sets in $L^p_w(U)$ by Lemma 5.1.9, a contradiction. ∎

5.2 Examples

In this section we describe some important random fields which arise in applications and satisfy the LDP on $L^p_w(U)$ for some $1 \leq p \leq \infty$.

5.2.1 Independent and identically distributed random variables

Let $\{\mathfrak{z}_j\}_{j\in\mathbb{Z}^n}$ be independent and identically distributed random variables with values in $\mathbb{R}^m$. Denote by Q_j the unit cube in $\mathbb{R}^n$ centered at $j \in \mathbb{Z}^n$ and let the random field be defined by

$$q_y = \sum_{j\in\mathbb{Z}^n} \mathfrak{z}_j \mathbf{1}_{Q_j}(y).$$

This example corresponds to the classical problem of Cramér in Large Deviation Theory and is considered in the literature as the prototype of a stationary random field. In Bal et al. (2011), this choice of q was used to obtain a *pointwise* LDP for the homogenization of (1.1) for $n = 1$. Here we aim to extend that work to $L^p(U)$. Let $H(t) = \ln E \exp\{t \cdot \mathfrak{z}_j\}$, and assume that

1. $H_G = \ln E \exp\{G(|\mathfrak{z}_j|)\}$ if $p = 1$, where G satisfies the conditions in (A2),
2. $H_p(\kappa) = \ln E \exp\{\kappa|\mathfrak{z}_j|^p)\} < \infty$ if $1 < p < \infty$, and
3. $\|\mathfrak{z}_j\|_{L^\infty(\Omega)} < \infty$ if $p = \infty$.

It will be shown that q^ε satisfies (A1)-(A4). Let $U \subset \mathbb{R}^n$ be an open, bounded set. Define the sets of subindices

$$\mathcal{I}^\varepsilon = \{j \in \mathbb{Z}^n \,|\, Q_j \subset U/\varepsilon\} \text{ and } \mathcal{J}^\varepsilon = \{j \in \mathbb{Z}^n \,|\, Q_j \cap (U/\varepsilon) \neq \emptyset \text{ but } j \notin \mathcal{I}^\varepsilon\}.$$

For each $j \in \mathcal{J}^\varepsilon \cup \mathcal{I}^\varepsilon$, let $Q^\varepsilon_j = Q_j \cap (U/\varepsilon)$. Clearly, $Q^\varepsilon_j = Q_j$ if $j \in \mathcal{I}^\varepsilon$. Observe that

$$|\mathcal{J}^\varepsilon| \leq |\{y \in \mathbb{R}^n \mid \text{distance}(y, \partial(U/\varepsilon)) < \sqrt{n}\}| \leq 2\sqrt{n}|\partial(U/\varepsilon)| = \frac{2\sqrt{n}|\partial U|}{\varepsilon^{n-1}}, \quad (5.10)$$

and $|U| - 2\sqrt{n}|\partial U|\varepsilon \le \varepsilon^n(|\partial(U/\varepsilon)| - |U^\varepsilon|) = \varepsilon^n|\mathcal{I}^\varepsilon| \le |U|$, so that $\lim_{\varepsilon\to 0}\varepsilon^n|\mathcal{I}^\varepsilon| = |U|$. Given $f \in L^\infty(\bar{U})$, define the step function

$$h_y^\varepsilon = \int_{Q_j^\varepsilon} f_{\varepsilon\tilde{y}}\, d\tilde{y} = \frac{1}{\varepsilon^n}\int_{\varepsilon Q_j^\varepsilon} f_{\tilde{y}}\, d\tilde{y} \text{ for } y \in \varepsilon Q_j^\varepsilon, j \in \mathcal{J}^\varepsilon \cup \mathcal{I}^\varepsilon.$$

Using the independence of the sequence $\{\mathfrak{z}_j\}_{j\in\mathbb{Z}^n}$ we get

$$\begin{aligned} Tf &= \lim_{\varepsilon\to 0}\varepsilon^n \ln E\exp\left\{\int_{U/\varepsilon} f_{\varepsilon y}\cdot q_y\, dy\right\} \\ &= \lim_{\varepsilon\to 0}\varepsilon^n\sum_{j\in\mathcal{I}^\varepsilon} H(h_{\varepsilon j}^\varepsilon) + \lim_{\varepsilon\to 0}\varepsilon^n\sum_{j\in\mathcal{J}^\varepsilon} H(h_{\varepsilon j}^\varepsilon). \end{aligned} \tag{5.11}$$

Let $\bar{H}_f = \max_{|t|\le\|f\|_{L^\infty(U)}} |H(t)| < \infty$. The last term of (5.11) vanishes since

$$\varepsilon^n\sum_{j\in\mathcal{J}^\varepsilon}|H(h_{\varepsilon j}^\varepsilon)| \le \varepsilon^n\bar{H}_f|\mathcal{J}^\varepsilon| \le 2\sqrt{n}\bar{H}_f|\partial U|\varepsilon,$$

where the last inequality follows from (5.10). A similar computation yields

$$\sum_{j\in\mathcal{J}^\varepsilon}|H(h_{\varepsilon j}^\varepsilon)|\,|\varepsilon Q_j^\varepsilon| \le \varepsilon^n\bar{H}_f\,|\mathcal{J}^\varepsilon| \le 2\sqrt{n}\bar{H}_f\,|\partial U|\varepsilon.$$

By the Lebesgue Differentiation Theorem, h_y^ε converges to f_y for almost every $y \in U$ as $\varepsilon \to 0$, and $\|H\circ h^\varepsilon\|_{L^\infty(U)} \le \bar{H}_f$. Thus, the Dominated Convergence Theorem applies and (5.11) yields

$$Tf = \lim_{\varepsilon\to 0}\sum_{j\in\mathcal{I}^\varepsilon\cup\mathcal{J}^\varepsilon} H(h_{\varepsilon j}^\varepsilon)|\varepsilon Q_j^\varepsilon| = \lim_{\varepsilon\to 0}\int_U H(h_{\varepsilon y}^\varepsilon)\, dy = \int_U H(f_y)\, dy.$$

This establishes (A1). Next, observe that if $1 < p < \infty$,

$$T_{p,p}(\kappa) = \lim_{\varepsilon\to 0}\varepsilon^n \ln E\exp\{\kappa\|q\|_{L^p(U/\varepsilon)}^p\} = \lim_{\varepsilon\to 0}\varepsilon^n\sum_{j\in\mathcal{I}^\varepsilon\cup\mathcal{J}^\varepsilon} H_p(\kappa|Q_j^\varepsilon|) = T_{|q|^p}g,$$

where $T_{|q|^p}$ represents the functional (3.5) corresponding to the random field $|q|^p$, and $g \equiv \kappa$. Thus, from (A1) it follows that $T_{p,p}(\kappa) = |U|H_p(\kappa)$, establishing (A2).

The other cases of p and (A3) are established in a similar way. Lastly, to prove (A4) note that by our assumption of finite pth exponential moments, the Dominated Convergence Theorem implies the differentiability of H and

$$\frac{\partial}{\partial t_k} H(t) = \frac{E[\mathfrak{z}_j^k \exp\{t \cdot \mathfrak{z}_j\}]}{E[\exp\{t \cdot \mathfrak{z}_j\}]},$$

where $\mathfrak{z}_j^k$ is the kth coordinate of $\mathfrak{z}_j$; in fact, H is infinitely differentiable.

One case of interest is when $\mathfrak{z}_j \sim N(a, \Sigma)$, in which case $H(t) = a \cdot t + \frac{1}{2}(\Sigma t, t)$, $L(s) = \frac{1}{2}(\Sigma^{-1}(s-a), s-a)$, and the LDP holds on $L^2_w(U)$. Another case is when $\mathfrak{z}_j \sim \text{Poisson}(\lambda)$, where $H(t) = \lambda(e^t - 1)$, $L(s) = s\ln(\frac{s}{\lambda}) - s + \lambda$, $G(s) = |s|\ln(|s|)$ and $p = 1$.

5.2.2 Moving Average Process

Denote by Q_j the unit cube in $\mathbb{R}^n$ centered at $j \in \mathbb{Z}^n$. A very useful way to generate long-range dependencies is to consider an infinite moving average (MA) random field

$$q_y = \sum_{j,k \in \mathbb{Z}^n} g_{k-j} \mathfrak{z}_j \mathbf{1}_{Q_k}(y).$$

Here, $\{g_j\}_{j \in \mathbb{Z}^n}$ is an absolutely summable real-valued sequence and $\{\mathfrak{z}_j\}_{j \in \mathbb{Z}^n}$ is a sequence of independent and identically distributed random variables. This model was also considered in Bal et al. (2011) to introduce correlations to the random media. Let $H(t) = \ln E \exp\{\|g\|_{\ell^1(\mathbb{Z}^n)} t \cdot \mathfrak{z}_j\}$, and assume the same regularity conditions of the moment generating function of $\mathfrak{z}_j$ from Section 5.2.1. It will be shown that q^ε satisfies (A1)-(A4) on $L^p_w(U)$. Let us prove (A1). Fix $f \in L^\infty(U)$. Using the independence of the sequence $\{\mathfrak{z}_j\}_{j \in \mathbb{Z}^n}$ we obtain

$$\begin{aligned} Tf &= \lim_{\varepsilon \to 0} \varepsilon^n \ln E \exp\left\{\frac{1}{\varepsilon^n} \int_U f_y \cdot q_{y/\varepsilon}\, dy\right\} \\ &= \lim_{\varepsilon \to 0} \varepsilon^n \sum_{j \in \mathbb{Z}^n} H\left(\frac{1}{\varepsilon^n} \sum_{k \in \mathcal{I}} \frac{g_{j-k}}{\|g\|_{\ell^1(\mathbb{Z}^n)}} \int_U \mathbf{1}_{Q_k}\left(\frac{y}{\varepsilon}\right) f_y\, dy\right). \end{aligned}$$

By defining the kernel $\eta(x,y) = \sum_{j,k\in\mathbb{Z}^n} \frac{g_{j-k}}{\|g\|_{\ell^1(\mathbb{Z}^n)}} \mathbf{1}_{Q_j}(x)\mathbf{1}_{Q_k}(y)$, the line above can be rewritten as

$$\begin{aligned} Tf &= \lim_{\varepsilon\to 0} \varepsilon^n \int_{\mathbb{R}^n} H\left(\int_U \frac{1}{\varepsilon^n}\eta\left(x, \frac{y}{\varepsilon}\right) f_y \, dy\right) dx \\ &= \lim_{\varepsilon\to 0} \int_{\mathbb{R}^n} H\left(\int_U \frac{1}{\varepsilon^n}\eta\left(\frac{x}{\varepsilon}, \frac{y}{\varepsilon}\right) f_y \, dy\right) dx. \end{aligned}$$

Since η satisfies

$$\int_{\mathbb{R}^n} \eta(x,y)\, dy = 1 \text{ and } \|\eta\|_{L^\infty(\mathbb{R}^n)} \le 1, \tag{5.12}$$

we expect that $\frac{1}{\varepsilon^n}\eta\left(\frac{x}{\varepsilon}, \frac{y}{\varepsilon}\right)$ is an approximation to the identity. Indeed, if $|x-y| > r, \frac{x}{\varepsilon} \in Q_j$ and $\frac{y}{\varepsilon} \in Q_k$ we have that $|\frac{x}{\varepsilon} - j| < \sqrt{n}$, $|\frac{y}{\varepsilon} - k| < \sqrt{n}$ and $|k-j| > \frac{r}{\varepsilon} - 2\sqrt{n}$. Therefore, for any $\theta > 0$ we can find $\varepsilon > 0$ small enough such that

$$\left|\int_{|y-x|>r} \frac{1}{\varepsilon^n}\eta\left(\frac{x}{\varepsilon}, \frac{y}{\varepsilon}\right) f_y \, dy\right| \le \|f\|_{L^\infty(U)} \sum_{|s|>\frac{r}{\varepsilon}-2\sqrt{n}} |g_s| < \theta. \tag{5.13}$$

Using (5.12), (5.13) repeatedly and the Lebesgue Differentiation Theorem gives

$$\begin{aligned} \left|\int_U \frac{1}{\varepsilon^n}\eta\left(\frac{x}{\varepsilon}, \frac{y}{\varepsilon}\right) f_y \, dy - f_x \mathbf{1}_U(x)\right| &\le \left|\int_U \frac{1}{\varepsilon^n}\eta\left(\frac{x}{\varepsilon}, \frac{y}{\varepsilon}\right) (f_y - f_x)\, dy\right| + \theta \\ &\le \left|\int_{|y-x|<r} \frac{1}{\varepsilon^n}\eta\left(\frac{x}{\varepsilon}, \frac{y}{\varepsilon}\right) (f_y - f_x)\, dy\right| + 2\theta \\ &\le \frac{1}{\varepsilon^n}\int_{|y-x|<r} |f_y - f_x|\, dy + 2\theta < 3\theta \end{aligned}$$

for almost every $x \in \mathbb{R}^n$ and sufficiently small $\varepsilon > 0$. A simple application of the Dominated Convergence Theorem proves (A1).

Next, we prove (A2) when $1 < p < \infty$. The other cases can be shown in a similar way. If $\mathcal{I}^\varepsilon$, $\mathcal{J}^\varepsilon$ and Q_j^ε are as in Section 5.2.1, then

$$\|q^\varepsilon\|_{L^p(U)}^p = \varepsilon^n \|g\|_{\ell^1(\mathbb{Z}^n)}^p \sum_{k\in\mathcal{I}^\varepsilon\cup\mathcal{J}^\varepsilon} |Q_k^\varepsilon| \left|\sum_{j\in\mathbb{Z}^n} \frac{g_{k-j}}{\|g\|_{\ell^1(\mathbb{Z}^n)}} \mathfrak{z}_j\right|^p.$$

Since the power function $|x|^p$ is convex and $\sum_{j\in\mathbb{Z}^n} \frac{g_j}{\|g\|_{\ell^1(\mathbb{Z}^n)}} = 1$, Jensen's Inequality gives

$$\|q^\varepsilon\|^p_{L^p(U)} \le \varepsilon^n \|g\|^p_{\ell^1(\mathbb{Z}^n)} \sum_{k\in\mathcal{I}^\varepsilon\cup\mathcal{J}^\varepsilon} |Q^\varepsilon_k| \sum_{j\in\mathbb{Z}^n} \frac{|g_{k-j}|}{\|g\|_{\ell^1(\mathbb{Z}^n)}} |\mathfrak{z}_j|^p$$

Use the independence of $\{\mathfrak{z}_j\}_{j\in\mathbb{Z}^n}$ and the definition of η to compute

$$\begin{aligned} T_{p,p}(\kappa) &\le \limsup_{\varepsilon\to\infty} \varepsilon^n \sum_{j\in\mathbb{Z}^n} H_p\left(\kappa\|g\|^p_{\ell^1(\mathbb{Z}^n)} \sum_{k\in\mathcal{I}^\varepsilon\cup\mathcal{J}^\varepsilon} |Q^\varepsilon_k| \frac{|g_{k-j}|}{\|g\|_{\ell^1(\mathbb{Z}^n)}}\right) \\ &= \limsup_{\varepsilon\to\infty} \int_{\mathbb{R}^n} H_p\left(\kappa\|g\|^p_{\ell^1(\mathbb{Z}^n)} \int_U \frac{1}{\varepsilon^n}\eta\left(\frac{x}{\varepsilon},\frac{y}{\varepsilon}\right) dy\right) dx \\ &= |U| H_p(\kappa\|g\|^p_{\ell^1(\mathbb{Z}^n)}) < \infty. \end{aligned}$$

(A3) and (A4) can be proved similarly, and this completes the proof.

5.2.3 Stationary Gaussian Processes

In order to avoid cumbersome notation we will consider a stationary Gaussian process q with real-valued sample paths; the general case can be obtained in a similar way. By taking $X = L^2(U)$ and $V \subset\subset \mathbb{R}^m$ in (2.16) the random field q has logarithmic moment generating function

$$\ln E \exp\left\{\int_V f_y\, q_y\, dy\right\} = a\int_V f_y\, dy + \frac{1}{2}\int_V\int_V R(x-y) f_x f_y\, dx\, dy \qquad (5.14)$$

for all $f \in L^2(V)$. Here, $a \in \mathbb{R}$ is the constant mean and $R(y) = E[q_y q_0] - a^2$ is the covariance kernel.

Short-Range Correlations

First, let us consider the case where q has short-range correlations, i.e. $R \in L^1 \cap L^\infty(\mathbb{R}^n)$. It will be shown that q^e satisfies the LDP on $L^2_w(U)$. Pick $f \in L^2(U)$ and extend it to be 0 outside U. Using (5.14), the functional T can be written in terms of a and R as follows

$$\begin{aligned} Tf &= \lim_{\varepsilon\to 0} \varepsilon^n \ln E \exp\left\{ \int_{U/\varepsilon} f_{\varepsilon y}\, q_y\, dy \right\} \\ &= a \lim_{\varepsilon\to 0} \varepsilon^n \int_{U/\varepsilon} f_{\varepsilon y}\, dy + \frac{1}{2} \lim_{\varepsilon\to 0} \varepsilon^n \int_{U/\varepsilon} \int_{U/\varepsilon} R(x-y) f_{\varepsilon x} f_{\varepsilon y}\, dx\, dy \\ &= a \int_U f_y\, dy + \frac{1}{2} \lim_{\varepsilon\to 0} \int_U \int_U \frac{1}{\varepsilon^n} R\left(\frac{x-y}{\varepsilon}\right) f_x f_y\, dx\, dy \end{aligned}$$

By Bochner Theorem the Fourier Transform of R (defined in (2.1)), $\hat{R}$, is nonnegative and belongs to $L^1 \cap L^\infty(\mathbb{R}^n)$. Since $\frac{1}{\varepsilon^n} R\left(\frac{x-y}{\varepsilon}\right)$ is an approximation to the identity, we may apply Plancherel's Equality (4.1.1) to obtain the limit of Tf. Indeed, extend f to be 0 outside U so that

$$\begin{aligned} Tf &= a \int_U f_y\, dy + \frac{(2\pi)^{\frac{n}{2}}}{2} \lim_{\varepsilon\to 0} \int_{\mathbb{R}^n} \hat{R}(\varepsilon\xi) |\hat{f}_\xi|^2\, d\xi \\ &= a \int_U f_y\, dy + \|R\|_{L^1(\mathbb{R}^n)} \|f\|^2_{L^2(U)} \end{aligned}$$

by the Dominated Convergence Theorem. Hence, (A1) holds with $H(t) = at + \frac{1}{2}\|R\|_{L^1(\mathbb{R}^n)} |t|^2$ and $L(s) = \frac{1}{2}\|R\|^{-1}_{L^1(\mathbb{R}^n)} |s-a|^2$.

(A4) is obviously true, so now we focus on verifying (A2)-(A3). By considering the mean zero process $\tilde{q} = q - a$, we may assume without loss of generality that $a = 0$. Define the covariance operator of q on $L^2(U/\varepsilon)$ by $(Q^\varepsilon f)(x) = \int_{U/\varepsilon} R(x-y) f(y)\, dy$. In order for q to have locally square integrable sample paths, it is necessary and sufficient that Q^ε be a symmetric, nonnegative definite operator of trace class on $L^2(V)$ (see Kuo (2006)). Let $\{(\lambda_k^\varepsilon, f_k^\varepsilon)\}_{k=1}^\infty$ be the sequence of pairs of eigenvalues/eigenvectors of Q^ε, ordered so that $\lambda_1^\varepsilon \geq \lambda_2^\varepsilon \geq \cdots$ and $\lim_{k\to\infty} \lambda_k^\varepsilon = 0$. These eigenvectors can be chosen to form an orthonormal basis of $L^2(U/\varepsilon)$.

The main ingredient to proving exponential tightness of q^ε is to show that $\bar{\lambda} = \sup_{0<\varepsilon<1} \lambda_1^\varepsilon < \infty$. Indeed, by Plancherel's Identity, for any $f \in L^2(U/\varepsilon)$ we extend it to be 0 outside U and

$$\int_{U/\varepsilon}\int_{U/\varepsilon} R(x-y)f(x)f(y)\,dx\,dy = \int_{U/\varepsilon}\int_{\mathbb{R}^n} e^{-iy\cdot\xi}\hat{R}(\xi)\bar{\hat{f}}_\xi f_y\,d\xi\,dy$$

$$= \int_{\mathbb{R}^n} (2\pi)^{\frac{n}{2}}\hat{R}(\xi)|\hat{f}(\xi)|^2\,d\xi \tag{5.15}$$

$$\leq (2\pi)^{\frac{n}{2}}\|\hat{R}\|_{L^\infty(\mathbb{R}^n)}\|f\|^2_{L^2(U/\varepsilon)}.$$

In particular, by taking $f = f_k^\varepsilon$ above, it follows from Rayleigh's Variational Theorem that $\bar{\lambda} \leq (2\pi)^{\frac{n}{2}}\|\hat{R}\|_{L^\infty(\mathbb{R}^n)} < \infty$. Moreover, since

$$E\left[\int_{U/\varepsilon} f_k^\varepsilon(x)q(x)\,dx \int_{U/\varepsilon} f_j^\varepsilon(y)q(y)\,dy\right] = \int_{U/\varepsilon}\int_{U/\varepsilon} R(x-y)f_k^\varepsilon(x)f_j^\varepsilon(y)\,dx\,dy$$

$$= \lambda_k^\varepsilon \int_{U/\varepsilon} f_k^\varepsilon(y)f_j^\varepsilon(y)\,dy = \lambda_k^\varepsilon\delta_{jk},$$

the eigenvectors f_k^ε provide the spectral decomposition for

$$q = \sum_{k=1}^\infty \left[\int_{U/\varepsilon} f_k^\varepsilon(y)q(y)\,dy\right] f_k^\varepsilon = \sum_{k=1}^\infty \sqrt{\lambda_k^\varepsilon}Z_k f_k^\varepsilon$$

on $L^2(U)$. Here, $Z_k = \frac{1}{\sqrt{\lambda_k^\varepsilon}}\int_{U/\varepsilon} f_k^\varepsilon(y)q(y)\,dy$ is a sequence of independent standard normal random variables. For any $\kappa < (2\bar{\lambda})^{-1}$, with $\bar{\lambda} = \sup_{0<\varepsilon<1}\lambda_1^\varepsilon < \infty$. the spectral decomposition of q gives

$$E\exp\left\{\frac{\kappa}{\varepsilon^n}\|q^\varepsilon\|^2_{L^2(U)}\right\} = E\exp\{\kappa\|q\|^2_{L^2(U/\varepsilon)}\}$$

$$= E\exp\left\{\kappa\sum_{k=1}^\infty\left|\int_{U/\varepsilon} f_k^\varepsilon(y)q(y)\,dy\right|^2\right\}$$

$$= \prod_{k=1}^\infty E\exp\left\{\kappa\lambda_k^\varepsilon|Z_k|^2\right\}$$

$$= \prod_{k=1}^\infty \frac{1}{\sqrt{1-2\kappa\lambda_k^\varepsilon}}.$$

We proceed to estimate this quantity. The stationarity of the Gaussian process allows us to conclude that the trace is of order ε^{-n}, as the following computations show:

$$\begin{aligned}
\varepsilon^n \operatorname{Tr}(Q^\varepsilon) &= \varepsilon^n \sum_{k=1}^{\infty} \lambda_k^\varepsilon = \varepsilon^n \sum_{k=1}^{\infty} \lambda_k^\varepsilon \int_{U/\varepsilon} |f_k^\varepsilon(y)|^2\, dy \\
&= \varepsilon^n \sum_{k=1}^{\infty} \int_{U/\varepsilon} \int_{U/\varepsilon} R(x-y) f_k^\varepsilon(x) f_k^\varepsilon(y)\, dx\, dy \\
&= \varepsilon^n E \sum_{k=1}^{\infty} \left| \int_{U/\varepsilon} f_k^\varepsilon(y) q(y)\, dx \right|^2 \\
&= \varepsilon^n E \int_{U/\varepsilon} |q(y)|^2\, dy = |U| R(0).
\end{aligned}$$

Given a real number b, and any x between 0 and b, the concavity of the logarithmic function guarantees the inequality $\frac{\ln(b+1)}{b} x \leq \ln(1+x) \leq x$ holds. Thus, by taking $b = \pm 2\kappa\lambda_1^\varepsilon$, $\kappa < (2\bar{\lambda})^{-1}$, we get

$$T_{-\kappa,2} = \limsup_{\varepsilon\to 0} \frac{\varepsilon^n}{2} \sum_{k=1}^{\infty} \ln(1 + 2\kappa\lambda_k^\varepsilon) \leq \limsup_{\varepsilon\to 0} \kappa\varepsilon^n \sum_{k=1}^{\infty} \lambda_k^\varepsilon = \kappa |U| R(0), \text{ and}$$

$$\begin{aligned}
T_{+\kappa,2} &= -\liminf_{\varepsilon\to 0} \frac{\varepsilon^n}{2} \sum_{k=1}^{\infty} \ln(1 - 2\kappa\lambda_k^\varepsilon) \\
&\leq -\liminf_{\varepsilon\to 0} \frac{\varepsilon^n \ln(1 - 2\kappa\lambda_1^\varepsilon)}{2\lambda_1^\varepsilon} \sum_{k=1}^{\infty} \lambda_k^\varepsilon \\
&= -|U| R(0) \kappa \liminf_{\varepsilon\to 0} \frac{\ln(1 - 2\kappa\lambda_1^\varepsilon)}{2\kappa\lambda_1^\varepsilon} \\
&\leq -\kappa |U| R(0) \inf_{0 < x \leq 2\kappa\bar{\lambda}} \frac{\ln(1-x)}{x} \\
&= -\frac{|U| R(0) \ln(1 - 2\kappa\bar{\lambda})}{2\bar{\lambda}}.
\end{aligned}$$

This finishes the proof of (A2)-(A4) and completes the case with short-range correlations.

Long-Range Correlations

Here, we assume that the spectral density $\hat{R}$ is continuous and positive everywhere except at 0, where

$$\hat{R}(\xi) \sim |\xi|^{\alpha-n} L\left(\frac{1}{|\xi|}\right) b\left(\frac{\xi}{|\xi|}\right) \text{ as } \xi \to 0 \tag{5.16}$$

Here, $0 < \alpha < n$, L is a slowly varying function and b is positive and continuous on the unit sphere in $\mathbb{R}^n$. A slowly varying function is a positive monotonic function such that $L(nx) \sim L(x)$ as $n \to \infty$ for all $x > 0$. A random field satisfying condition (5.16) is said to have *long-range correlations.* By inverting the Fourier Tranform in n dimensions using the Riesz Tranform, Wainger (1965) proved that having long-range correlations is equivalent to the correlations decreasing like

$$R(x) \sim |x|^{-\alpha} \tilde{L}\left(\frac{1}{|\xi|}\right) \tilde{b}\left(\frac{\xi}{|\xi|}\right) \text{ as } \xi \to 0.$$

Observe that R is not integrable near infinity, so Theorem 5.1.7 does not apply. Instead, Theorem 2.6.16 will be used directly to show that q^ε also satisfies the LDP but with rate $\frac{1}{\varepsilon^\alpha}$. Given $f \in L^2(U)$, extend it to be 0 outside U and compute Tf uding Plancherel's Identity (4.1.1) as we did in the previous case (5.15) using (5.14):

$$Tf = \lim_{\varepsilon \to 0} \varepsilon^\alpha \ln E \exp\left\{\frac{1}{\varepsilon^\alpha} \int_U f_y q_{y/\varepsilon}\, dy\right\}$$

$$Tf = a \int_U f_y\, dy + \frac{1}{2} \lim_{\varepsilon \to 0} \int_U \int_U \frac{1}{\varepsilon^\alpha} R\left(\frac{x-y}{\varepsilon}\right) f_x f_y\, dx\, dy$$

$$Tf = a \int_U f_y\, dy + \frac{(2\pi)^{\frac{n}{2}}}{2} \lim_{\varepsilon \to 0} \int_{\mathbb{R}^n} \varepsilon^{n-\alpha} \hat{R}(\varepsilon\xi) |\hat{f}_\xi|^2\, d\xi$$

Since $\hat{R}$ is bounded away from the origin and $\hat{f} \in L^2(\mathbb{R}^n)$, the integral over $\{|\xi| > \frac{1}{\varepsilon}\}$ of $\varepsilon^{n-\alpha}\hat{R}(\varepsilon\xi)|\hat{f}_\xi|^2$ vanishes as $\varepsilon \to 0$. Hence, the contribution to the integral comes asymptotically from $\{|\xi| \leq \frac{1}{\varepsilon}\}$ and (5.16). Indeed, $\hat{R}$ is integrable near 0, so we may integrate $\varepsilon^{n-\alpha}\hat{R}(\varepsilon\xi)|\hat{f}_\xi|^2$ over $\{|\xi| \le \frac{1}{\varepsilon}\}$ to obtain

$$Tf = a\int_U f_y\,dy + \frac{(2\pi)^{\frac{n}{2}}}{2}\int_{\mathbb{R}^n}|\xi|^{\alpha-n}L\left(\frac{1}{|\xi|}\right)b\left(\frac{\xi}{|\xi|}\right)|\hat{f}_\xi|^2\,d\xi.$$

As noted in the earlier case, exponential tightness will follow if we can show $\bar{\lambda} < \infty$. Given $f \in L^2(U/\varepsilon)$ and $0 < \delta < \|f\|_{L^2(U/\varepsilon)}$ we can find $r > 0$ sufficiently small such that

$$|\hat{f}(\xi)| < |\hat{f}(0)| + \delta = \|f\|_{L^1(U)} + \delta \le 2\|f\|_{L^2(U/\varepsilon)} \text{ if } |\xi| < r \text{ and}$$

$$\hat{R}(\xi) < (1+\delta)|\xi|^{\alpha-n}L\left(\frac{1}{|\xi|}\right)b\left(\frac{\xi}{|\xi|}\right) \text{ if } |\xi| < r.$$

Split (5.15) into the integrals over $\{|\xi| < r\}$ and $\{|\xi| \ge r\}$ and apply the estimates above to see that

$$\left|\int_{U/\varepsilon}\int_{U/\varepsilon} R(x-y)f_x f_y\,dx\,dy\right| \le$$

$$(2\pi)^{\frac{n}{2}}\left[\frac{4r^\alpha}{\alpha}(1+\delta)\omega_n\|L\|_{L^\infty(\mathbb{R}^+)}\|b\|_{L^\infty(\mathbb{S}^{n-1})} + \sup_{|\xi|\ge r}\hat{R}(\xi)\right]\|f\|^2_{L^2(U/\varepsilon)}$$

where ω_n is the surphace area of the unit sphere in $\mathbb{R}^n$. The finiteness of $\bar{\lambda}$ follows from Rayleigh's Variational Theorem.

It is easy to check that T is Gateaux differentiable and, for any $f, g \in L^2(U)$, we have

$$D_gTf = a\int_U g_y\,dy + (2\pi)^{\frac{n}{2}}\int_{\mathbb{R}^n}|\xi|^{\alpha-n}L\left(\frac{1}{|\xi|}\right)b\left(\frac{\xi}{|\xi|}\right)\Re(\hat{f}_\xi \cdot \bar{\hat{g}}_\xi)\,d\xi.$$

Therefore, q^ε satisfies the LDP on $L^2_w(U)$ with rate ε^α and action functional T^*. Finally we proceed to compute T^*. By the definition of T and Plancherel's Identity, we have

$$T^*\varphi = \sup_{f\in L^2(U)}\int_{\mathbb{R}^n}\left[(\hat{\varphi}_\xi - a(\widehat{\mathbf{1}_U})_\xi)\bar{\hat{f}}_\xi - \frac{(2\pi)^{\frac{n}{2}}}{2}|\xi|^{\alpha-n}L\left(\frac{1}{|\xi|}\right)b\left(\frac{\xi}{|\xi|}\right)|\hat{f}_\xi|^2\right]d\xi.$$

Since T^* is a real valued functional, we can rewrite it as

$$T^*\varphi = \sup_{f\in L^2(U)} \int_{\mathbb{R}^n} \left[\Re[(\hat{\varphi}_\xi - a(\widehat{\mathbf{1}_U})_\xi)\bar{\hat{f}}_\xi] - \frac{(2\pi)^{\frac{n}{2}}}{2}|\xi|^{\alpha-n} L\left(\frac{1}{|\xi|}\right) b\left(\frac{\xi}{|\xi|}\right) |\hat{f}_\xi|^2\right] d\xi.$$

Here, we assumed both f and φ are extended to zero outside of U. The integrand is a quadratic function in $\hat{f}$ and it attains its maximum at $\hat{f}_\xi = \frac{\hat{\varphi}_\xi - a(\widehat{\mathbf{1}_U})_\xi}{(2\pi)^{\frac{n}{2}} L\left(\frac{1}{|\xi|}\right) b\left(\frac{\xi}{|\xi|}\right)}|\xi|^{n-\alpha}$. Arguing as in (5.4) in the proof of Theorem 5.1.7, we see that the supremum (which could be ∞) is attained at this function and

$$T^*\varphi = \int_{\mathbb{R}^n} \frac{|\hat{\varphi}_\xi - a(\widehat{\mathbf{1}_U})_\xi|^2}{2(2\pi)^{\frac{n}{2}} L\left(\frac{1}{|\xi|}\right) b\left(\frac{\xi}{|\xi|}\right)}|\xi|^{n-\alpha}\, d\xi.$$

As a final remark, Corollary 5.1.10 implies that the stationary Gaussian random fields do not satisfy the LDP on $L^p_w(U)$ for $p > 2$.

5.2.4 Poisson Point Process

Below we introduce the basic concepts we shall need in our work, we refer to the reader to (Bal and Jing, 2010, Section 3.2) and Cox and Isham (1980) for an account of the main properties of these fields. A point process is a countable random subset $\mathfrak{Z} = \{\mathfrak{z}_j\}_{j=1}^\infty \subset \mathbb{R}^n$. For any Borel $A \subset \mathbb{R}^n$, let N_A denote the cardinality of $A \cap \mathfrak{Z}$. A stationary Poisson Point Process (PPP) with intensity ν, denoted by $\mathfrak{Z}_\nu = \{\mathfrak{z}_j\}_{j=1}^\infty$, is a point process satisfying the following two conditions:

(a) N_A is a Poisson random variable with mean $\nu|A|$, i.e.,

$$P(N_A = k) = e^{-\nu|A|}\frac{(\nu|A|)^k}{k!}.$$

Here, $|A|$ denotes the Lebesgue measure of A.

(b) If $A_1, A_2, \ldots, A_r$ are disjoint Borel sets then $N_{A_1}, N_{A_2}, \ldots, N_{A_r}$ are independent.

Let ψ be a uniformly bounded function with compact support. If

$$q_y = \sum_{j=1}^{\infty} \psi(y - \mathfrak{z}_j),$$

we will see that the sequence of oscillatory fields q^ε satisfies (A1)-(A4) for $p = 1$. Let us start by verifying that $H(t) = \nu[\exp\{\|\psi\|_{L^1(\mathbb{R}^n)} t\} - 1]$ so that $L(s) = \frac{s}{\|\psi\|_{L^1(\mathbb{R}^n)}} \ln\left(\frac{s}{\nu\|\psi\|_{L^1(\mathbb{R}^n)}}\right) - \frac{s}{\|\psi\|_{L^1(\mathbb{R}^n)}} + \nu$. Let Ψ denote the support of ψ. Recall that for $f \in L^\infty(U)$, the moment generating function of q^ε is given by

$$m.g.f. = E \exp\left\{\frac{1}{\varepsilon^n}\int_U f_y \sum_{j=1}^{\infty} \psi\left(\frac{y}{\varepsilon} - \mathfrak{z}_j\right)\, dy\right\}$$

Since $y \in U$ in the integral above, the only values of $\mathfrak{z}_j$ which are taken into account are in the set $U/\varepsilon - \Psi$. Thus, we can use the Law of Total Probability by conditioning on the set $N(U/\varepsilon - \Psi)$ to see that the moment generating function of q^ε equals

$$\sum_{k=0}^{\infty} E\left[\exp\left\{\frac{1}{\varepsilon^n}\int_U f_y \sum_{j=1}^{k} \psi\left(\frac{y}{\varepsilon} - \mathfrak{z}'_j\right)\, dy\right\} \mid N(U/\varepsilon - \Psi) = k\right] \cdot$$
$$P(N(U/\varepsilon - \Psi) = k)$$

It is well known (Bal and Jing, 2010, page 11) that if A is an open bounded subset of $\mathbb{R}^n$ then, conditioned on $N_A = k$, the k points of $\mathfrak{Z}_\nu \cap A = \{X'_1, X'_2, \ldots, X'_k\}$ are independent and distributed according to the uniform distribution on A. Thus, the last expression simplifies to

$$\sum_{k=0}^{\infty}\left[\frac{1}{|U/\varepsilon - \Psi|}\int_{U/\varepsilon-\Psi} \exp\left\{\frac{1}{\varepsilon^n}\int_U f_y \psi\left(\frac{y}{\varepsilon} - x\right)\, dy\right\}\, dx\right]^k \cdot$$
$$e^{-\nu|U/\varepsilon - \Psi|}\frac{(\nu|U/\varepsilon - \Psi|)^k}{k!}$$
$$= e^{-\nu|U/\varepsilon-\Psi|}\sum_{k=0}^{\infty}\frac{1}{k!}\left[\nu\int_{U/\varepsilon-\Psi} \exp\left\{\frac{1}{\varepsilon^n}\int_U f_y \psi\left(\frac{y}{\varepsilon} - x\right)\, dy\right\}\, dx\right]^k.$$

Observe that this is the power series of the exponential function. Perform the change of variable $x \to \frac{x}{\varepsilon}$ to obtain

$$m.g.f. = \exp\left\{\frac{\nu}{\varepsilon^n}\int_{U-\varepsilon\Psi}\left[\exp\left\{\int_U f_y \frac{1}{\varepsilon^n}\psi\left(\frac{y-x}{\varepsilon}\right)\,dy\right\}-1\right]\,dx\right\}.$$

Since $\frac{1}{\|\psi\|_{L^1(\mathbb{R}^n)}\varepsilon^n}\psi\left(\frac{y-x}{\varepsilon}\right)$ is an approximation to the identity, the Dominated Convergence Theorem and (3.5) give the functional

$$Tf = \nu\int_U\left[\exp\left\{\|\psi\|_{L^1(\mathbb{R}^n)}f_x\right\}-1\right]\,dx,$$

showing (A1). We will show (A2) with $G(t) = \frac{\alpha}{\|\psi\|_{L^1(\mathbb{R}^n)}}|t|\ln(|t|), 0 < \alpha < 1$, where we assume $0\cdot\ln(0) = 0$. By conditioning with respect to $N(U/\varepsilon - \Psi)$ and using the uniform boundedness of ψ we estimate the moment generating function of $G(q^\varepsilon)$ as we did with Tf:

$$E\exp\left\{\frac{\alpha}{\varepsilon^n\|\psi\|_{L^1(\mathbb{R}^n)}}\int_U |q_{y/\varepsilon}|\ln(|q_{y/\varepsilon}|)\,dy\right\} \le$$

$$\sum_{k=1}^{\infty} E\left[\exp\left\{\frac{\alpha[\ln k + \ln\|\psi\|_{L^\infty(\mathbb{R}^n)}]}{\varepsilon^n\|\psi\|_{L^1(\mathbb{R}^n)}}\int_U\sum_{j=1}^{k}|\psi|\left(\frac{y}{\varepsilon}-\mathfrak{z}'_j\right)\,dy\right\}\mid N(U/\varepsilon-\Psi)=k\right]\cdot$$

$$P(N(U/\varepsilon-\Psi)=k)+P(N(U/\varepsilon-\Psi)=0)$$

$$=\sum_{k=0}^{\infty}\exp\{\alpha k[\ln k+\ln\|\psi\|_{L^\infty(\mathbb{R}^n)}]\}e^{-\nu|U/\varepsilon-\Psi|}\frac{(\nu|U/\varepsilon-\Psi|)^k}{k!}$$

$$= e^{-\frac{\nu}{\varepsilon^n}|U-\varepsilon\Psi|}\sum_{k=0}^{\infty}\frac{1}{k!}\left[\frac{\nu}{\varepsilon^n}|U-\varepsilon\Psi|\exp\{\alpha[\ln k+\ln\|\psi\|_{L^\infty(\mathbb{R}^n)}]\}\right]^k$$

We must show that the series above grows no faster than $e^{\frac{\kappa}{\varepsilon^n}}$. Since $k!$ grows like k^k times lower order terms by the Stirling approximation, (A2) is equivalent to showing that the series $\sum_{k=0}^{\infty}\exp\{k\ln(\frac{C}{\varepsilon^n})-(1-\alpha)k\ln k\}$ grows exponentially fast in $\frac{1}{\varepsilon^n}$, where C is a constant. We take $x = \ln(\frac{C}{\varepsilon^n})$ and estimate the series by analyzing the corresponding integral $\int_0^\infty \exp\{xt - (1-\alpha)t\ln t\}\,dt$ as $x \to \infty$ using Laplace's Lemma.

Let $y = \exp\{\frac{x}{1-\alpha} - 1\} = e^{-1}(\frac{C}{\varepsilon^n})^{\frac{1}{1-\alpha}}$ and perform the change of variables $t = ys$ to see that this last integral equals $I(y) = y\int_{1/y}^\infty \exp\{(1-\alpha)ys(1-\ln(s))\}\,ds$. The maximum in s of the integrand is attained at $s = 1$, whence Laplace's Lemma (Bender and Orszag, 1978, page 267) yields that $I(y) \sim \sqrt{\frac{2\pi y}{\beta}}e^{\beta y}$. By writing y in terms of ε, we see that I grows at most exponentially in $\frac{1}{\varepsilon^n}$, which shows that $T_G < \infty$.

(A3) and (A4) are trivial, so we conclude that q^ε satisfies the LDP on $L^1_w(U)$. Moreover, Corollary 5.1.10 guarantees that the LDP does not hold for $p > 1$.

5.2.5 Bounded Stationary Process

In some applications related to homogenization theory, it may be required that q be uniformly bounded. Let q^ε be a random field satisfying (A1)-(A4) for some $1 < p < \infty$ and let $V \subset \mathbb{R}^m$ be an open bounded set. Suppose that $b : \mathbb{R}^m \to V$ is a C^1-diffeomorphism with a uniformly bounded gradient. We will show that the transformation $F : L^p_w(U) \to L^\infty_w(U)$ defined by $F\varphi = b \circ \varphi$ is continuous so the Contraction Principle guarantees $b(q^\varepsilon)$ satisfies the LDP on $L^\infty_w(U)$ with rate $\frac{1}{\varepsilon^n}$ and action functional

$$S\varphi = \int_U (L \circ b^{-1})(\varphi_y)\,dy \text{ if } \operatorname{Range}\varphi \subset V$$

and $S\varphi = \infty$ otherwise. We emphasize that S (or $L \circ b^{-1}$) may not necessarily be convex functions.

Now we move on to the proof of continuity. Let $\{\varphi^k\}_{k=1}^\infty$ be a sequence of functions converging to φ in $L^p_w(U)$, and pick $f \in L^1(U)$. Fix $\theta > 0$. Recall that $\Phi_f(t) = \{y \in U \mid |f_y| \leq t\}$ denotes the level sets of f. Pick t large enough so that $\int_{U\setminus\Phi_f(t)} |f_y|\,dy \leq$

$\frac{\theta}{4\|b\|_{L^\infty(U)}}$. Use the Mean Value Theorem to compute

$$\left|\int_U f_y \cdot [b(\varphi_y^k) - b(\varphi_y)]\, dy\right| \le \left|\int_{\Phi_f(t)} f_y \cdot [b(\varphi_y^k) - b(\varphi_y)]\, dy\right| + \int_{U\setminus\Phi_f(t)} |f_y|\, |b(\varphi_y^k) - b(\varphi_y)|\, dy$$

$$\le \left|\int_{\Phi_f(t)} \int_0^1 f_y \cdot [\nabla b(r\varphi_y^k + (1-r)\varphi_y)\,(\varphi_y^k - \varphi_y)]\, dr\, dy\right| + \frac{\theta}{2}$$

$$= \left|\int_{\Phi_f(t)} B_y f_y \cdot (\varphi_y^k - \varphi_y)\, dy\right| + \frac{\theta}{2} < \theta,$$

since $B_y = \int_0^1 [\nabla b(r\varphi_y^k + (1-r)\varphi_y)]^*\, dr$ is a matrix-valued function whose entries are uniformly bounded. Thus, $Bf\mathbf{1}_{\Phi_f(t)} \in L^\infty(U) \subset L^{p'}(U)$ and $F(\varphi^k)$ converges to $F(\varphi)$ in $L^\infty_w(U)$.

Remark 5.2.1. If $V \subset \mathbb{R}^m$ is only an open set and b is as in the last example, then the same proof shows that the LDP holds on $L^p_w(U)$ with the same rate and action functional. Furthermore, since $|b(s)| \le \|\nabla b\|_{L^\infty(\mathbb{R}^m)}|s|$, the family of random fields $b \circ q^\varepsilon$ satisfies (A2) and (A3), and $L \circ b^{-1}$ is lower bounded by $\kappa_L \|\nabla b\|^{-p}_{L^\infty(\mathbb{R}^m)}|s|^p$.

In the rest of this article it will be assumed that q^ε satisfies (A1)-(A4).

5.3 LDP for oscillatory integrals

In this section we find a LDP for "oscillatory integrals" of the form

$$v^\varepsilon(x) = \int_U G(x,y) q^\varepsilon_y\, dy,$$

where G is a kernel on $U \times U$ satisfying certain smoothness properties. It is usually the case that the smoother G is, the larger the space on which we can prove the LDP.

Theorem 5.3.1. *Let $G : U \times U \to \mathbb{R}^m$ be a measurable kernel such that*

1. *For each $x \in U$, the partial derivative $\partial_x^\alpha G(x,y)$ exists for almost every $y \in U$ and all $|\alpha| \leq r$.*

2. *$G \in C^r(\bar{U}; L^{p'}(U))$, i.e., $\|G\|_{C^r(\bar{U};L^{p'}(U))} = \sum_{|\alpha|\leq r} \sup_{x\in U} \left(\int_U |\partial_x^\alpha G(x,y)|^{p'} \, dy \right)^{\frac{1}{p'}} < \infty$.*

3. *For every $x_0 \in U$ and each $|\alpha| \leq r$,*

$$\lim_{x\to x_0} \int_U |\partial_x^\alpha G(x,y) - \partial_x^\alpha G(x_0,y)|^{p'} \, dy = 0.$$

Then, the random fields $v^\varepsilon(x) = \int_U G(x,y) q_y^\varepsilon \, dy$ satisfy a LDP on $C^r(\bar{U})$ with rate $\frac{1}{\varepsilon^n}$ and action functional

$$S_1 v = \inf_{\{\varphi \in L^p(U) \,|\, v(x) = \int_U G(x,y)\varphi_y \, dy\}} \int_U L(\varphi_y) \, dy. \tag{5.17}$$

Observe that if condition 2 holds and the limit $\lim_{x\to x_0} \partial_x^\alpha G(x,y) = \partial_x^\alpha G(x_0,y)$ exists and is finite for a.e. $y \in U$, then condition 3 is satisfied by the Dominated Convergence Theorem.

Proof. Consider $\mathcal{G} : L_w^p(U) \to C^r(\bar{U})$ defined by $(\mathcal{G}\varphi)_x = \int_U G(x,y)\varphi_y \, dy$. Firstly, we check that the range of $\mathcal{G}$ is actually $C^r(\bar{U})$. Conditions 1 and 2, together with the Dominated Convergence Theorem, imply that derivatives of all orders $|\alpha| < r$ exist and $\partial_x^\alpha (\mathcal{G}\varphi)_x = \int_U \partial_x^\alpha G(x,y)\varphi_y \, dy$. Condition 3 guarantees the continuity of the derivatives $\partial^\alpha \mathcal{G}\varphi$ for all $|\alpha| \leq r$ when $\varphi \in L_w^p(U)$.
By the Contraction Principle, we just have to show that $\mathcal{G}$ is a (sequentially) continuous operator. Fix $\theta > 0$ and $|\alpha| < r$. Let $\{\varphi^k\}_{k=1}^\infty$ be a sequence of functions converging to φ in $L_w^p(U)$, and let $M = \sup_{k=1}^\infty \|\varphi^k\|_{L^p(U)} < \infty$. For each $x_0 \in \bar{U}$,

set $A_{x_0} = \{x \in \bar{U} \mid \int_U |\partial_x^\alpha G(x,y) - \partial_x^\alpha G(x_0,y)|^{p'}\, dy < (\frac{\theta}{4M})^{p'}\}$. This can be done by Conditions 2 and 3. Select a finite subcover $\{A_j^\circ\}_{j\in\mathcal{J}} \subset \{A_x^\circ\}_{x\in\bar{U}}$, and let $j(x) = \{j \in \mathcal{J} \mid x \in A_j^\circ\}$. Let $K > 0$ be large enough so that

$$\left| \int_U \partial_j^\alpha G(j,y)\, (\varphi_y^k - \varphi_y)\, dy \right| < \frac{\theta}{2}$$

for all $j \in \mathcal{J}$ and all $k \geq K$. Then,

$$\begin{aligned} \left| \int_U \partial_x^\alpha G(x,y)\, (\varphi_y^k - \varphi_y)\, dy \right| &\leq \left| \int_U \partial_{j(x)}^\alpha G(j(x),y)\, (\varphi_y^k - \varphi_y)\, dy \right| \\ &\quad + 2M \left(\int_U |\partial_x^\alpha G(x,y) - \partial_{j(x)}^\alpha G(j(x),y)|^{p'}\, dy \right)^{\frac{1}{p'}} \\ &< \theta \end{aligned}$$

uniformly in $x \in \bar{U}$, for all $k \geq K$, and we are done. □

The following theorem is a slight generalization of Theorem 5.3.1.

Theorem 5.3.2. *Let $G : U \times U \to \mathbb{R}^m$ be a kernel such that*

1. *For each $x \in U$, the partial derivative $\partial_x^\alpha G(x,y)$ exists for almost every $y \in U$ and all $|\alpha| \leq r$.*

2. *There is $0 < s < 1$ such that $G \in C^{r+s}(\bar{U}; L^{p'}(U))$, i.e.,*

$$\|G\|_{C^{r+s}(\bar{U};L^{p'}(U))} = \|G\|_{C^{r}(\bar{U};L^{p'}(U))} + \sum_{|\alpha|=r} \sup_{x_1\neq x_2} \left(\int_U \frac{|\partial_x^\alpha G(x_1,y) - \partial_x^\alpha G(x_2,y)|^{p'}}{|x_1 - x_2|^{sp'}}\, dy \right)^{\frac{1}{p'}} < \infty.$$

Then, the random fields $v^\varepsilon(x) = \int_U G(x,y) q_y^\varepsilon\, dy$ satisfy the LDP on $C^{r+t}(\bar{U})$ with rate $\frac{1}{\varepsilon^n}$ and action functional (5.17) for any $0 < t < s$.

Proof. By Theorem 5.3.1 the LDP holds on $C^r(\bar{U})$, so we may assume without loss of generality that $r = 0$. Consider the operator $\mathcal{G} : L^p_w(U) \to C^t(\bar{U})$ defined by $(\mathcal{G}\varphi)_x = \int_U G(x,y)\varphi_y \, dy$. Let $\{\varphi^k\}_{k=1}^\infty$ be a sequence of functions converging to φ in $L^p_w(U)$. The conditions of the theorem, together with the Dominated Convergence Theorem, imply that Range$(\mathcal{G}) = C^s(\bar{U})$ and $\mathcal{G}(\varphi^k - \varphi)$ is uniformly bounded in $C^s(\bar{U})$. Since $C^t(\bar{U})$ is compactly embedded in $C^s(\bar{U})$, the sequence $\mathcal{G}\varphi^k$ converges strongly to $\mathcal{G}\varphi$ in $C^t(\bar{U})$. The result follows from the Contraction Principle. □

Theorem 5.3.3. *Let $G : U \times U \to \mathbb{R}^m$ be a jointly measurable kernel such that $G \in L^r(U; L^{p'}(U))$, i.e.,*

$$\|G\|^r_{L^r(U;L^{p'}(U))} = \int_U \left(\int_U |G(x,y)|^{p'} \, dy \right)^{\frac{r}{p'}} dx < \infty.$$

Then, the random fields $v^\varepsilon(x) = \int_U G(x,y) q^\varepsilon_y \, dy$ satisfy a LDP on $L^r(U)$ with rate $\frac{1}{\varepsilon^n}$ and action functional (5.17)*.*

Proof. The proof follows along the lines of Theorem 5.3.1. Let $\mathcal{G}$, φ^k, φ and M be as in Theorem 5.3.1. That $\mathcal{G}\varphi \in L^r(U)$ for $\varphi \in L^p(U)$ follows easily from the Dominated Convergence Theorem. Moreover, $(\mathcal{G}\varphi^k)_x$ converges to $(\mathcal{G}\varphi)_x$ for all x and $\mathcal{G}\varphi^k$ is dominated by $M\|G(x,\cdot)\|_{L^{p'}(U)} \in L^r(U)$, so that $(\mathcal{G}\varphi^k)_x$ converges to $(\mathcal{G}\varphi)_x$ in $L^r(U)$. The proof follows from the Contraction Principle. □

5.4 Applications to Homogenization and Averaging

Here we give several examples of PDEs for which the LDP can be found when the coefficients are random and rapidly oscillating. Given the results of the previous chapters, the reader should note that the only hard work here is to prove apriori estimates of appropriate norms of u, the solution to a PDE, with respect to the $L^p(U)$ norm of the random coefficients. For the sake of clarity we will assume throughout that the PDEs have homogeneous Dirichlet boundary conditions, however, the proofs carry over to more general boundary conditions.

5.4.1 Diffusion in a random potential

Here we assume that $\frac{n}{2} < p \leq \infty$. Suppose that $\varphi \in L^p(U)$ satisfies $\|\varphi\|_{L^p(U)} \leq M$ and $\sup_{x \in U} \varphi(x) < b$, so as to avoid the spectrum of $\mathcal{L} = -\Delta + b$.

Lemma 5.4.1. *For any $f \in L^p(U)$ there is a unique solution $u \in W^{2,p} \cap W_0^{1,p}(U)$ to the equation*

$$\mathcal{L}u = \varphi u + f \tag{5.18}$$

which satisfies the estimate

$$\|u\|_{W^{2,p}(U)} \leq C_M \|f\|_{L^p(U)}, \tag{5.19}$$

where C_M depends only on $n, p, \mathrm{diameter}(U)$ and M.

Proof. Suppose first that $\varphi \in L^\infty(U)$. By Lemma 2.3.1, there is a unique solution $u \in W^{2,p} \cap W_0^{1,p}(U) \subset\subset C^\alpha(U)$ to the equation (5.18). Here, we can take $\alpha < 2 - \frac{n}{p}$ by the Sobolev Embedding. Moreover, by (2.4) and Hölder inequality, u satisfies the estimate

$$\|u\|_{W^{2,p} \cap W_0^{1,p}(U)} \leq C(M\|u\|_{L^\infty(U)} + \|f\|_{L^p(U)}),$$

for C independent of φ, u and f.

To prove (5.19), we argue by contradiction. Suppose that there exist sequences $u^k \in W^{2,p} \cap W_0^{1,p}(U)$, $f^k \in L^{p_1}(U)$ and $\varphi^k \in L^\infty(U)$ satisfying $\|u^k\|_{C(U)} = 1$, $\|f^k\|_{L^p(U)} \to 0$ and $\|\varphi^k\|_{L^p(U)} \leq M$. Find subsequences, which we relabel as u^k, f^k, φ^k, such that

$u^k \to u$ in $W^{2,p}_w \cap C(U)$, and $\varphi^k \to \varphi$ in $L^p_w(U)$. Therefore,

$$\int_U \mathcal{L}u^k \, g \, dx \to \int_U \mathcal{L}u \, g \, dx$$

$$\int_U \varphi^k u^k g \, dx \to \int_U \varphi u g \, dx, \text{ and}$$

$$\int_U f^k g \, dx \to 0$$

for all $g \in L^{p'}(U)$, so that u satisfies $\int_U \mathcal{L}u \, g \, dx = \int_U \varphi u g \, dx$. Since $g \in L^{p'}(U)$ is arbitrary we have that $\mathcal{L}u = \varphi u$ a.e. Hence, $u \equiv 0$ by the uniqueness assertion, which contradicts $\|u\|_{C(U)} = 1$.

Next, we prove the result for arbitrary $\varphi \in L^p(U)$. To do this, choose a sequence $\{\varphi^k\}_{k=1}^{\infty} \subset L^\infty(U)$ which converges to φ in $L^p(U)$. Let $u^k \in W^{2,p} \cap W^{1,p}_0(U)$ be the solution to $\mathcal{L}u^k = \varphi^k u^k + f$. By (5.19), the sequence u^k is uniformly bounded in $W^{2,p} \cap W^{1,p}_0(U)$ so it has a convergent subsequence in $W^{2,p}_{w,X} \cap C(U)$. The argument of the last paragraph shows that the limit u solves (5.18) and satisfies (5.19). Finally, if u_1 and u_2 are two distinct solutions of (5.18), $u_1 - u_2$ solves (5.18) with $f \equiv 0$ and (5.19) implies that $u_1 \equiv u_2$. $\square$

Let q be a mean zero stationary random field with sample paths in $L^p(U)$ with $\sup_{x \in U} q(x) < b$ a.s. Then, there is a unique solution $u^\varepsilon \in W^{2,p} \cap W^{1,p}_0(U)$ to $\mathcal{L}u^\varepsilon - q^\varepsilon u^\varepsilon = f$ in U. If q is ergodic, it was shown by Bal (2008) that u^ε converges to u^0 in $L^2(\Omega; W^{1,p}_0(U))$, where $u^0 \in W^{2,p} \cap W^{1,p}_0(U)$ is the solution to the problem $\mathcal{L}u^0 = f$. Thus, we expect $w^\varepsilon = u^\varepsilon - u^0$ to satisfy the LDP if q^ε satisfies (A1)-(A4). Observe that $w^\varepsilon \in H^1_0(U)$ satisfies the homogeneous BVP

$$\mathcal{L}w^\varepsilon = q^\varepsilon (u^0 + w^\varepsilon) \tag{5.20}$$

Proof of Theorem 3.2.1. Pick a sequence $\varphi^k \to \varphi$ in $L^p_w(U)$ satisfying $\sup_{x \in U} \varphi(x) < b$ a.s. and let $v^k \in W^{2,p} \cap W^{1,p}_0(U)$ solve $\mathcal{L}v^k = \varphi^k (u^0 + v^k)$. We claim that $v^k \to v$ in

$W_w^{2,p}$, so that the mapping $\varphi \to w$ is (sequentially) continuous. Since the sequence φ^k is uniformly bounded in $L^p(U)$ by some $M > 0$, (5.19) gives

$$\|v^k\|_{W^{2,p}\cap W_0^{1,p}(U)} \leq MC_M \|u^0\|_{L^p(U)}.$$

Hence, there is a subsequence v^{k_j} which converges in $W_w^{2,p} \cap W_0^{1,p}(U)$ to some v. By integrating the equation for v^k against any $g \in L^{p'}(U)$ and taking $k \to \infty$ as we did earlier we conclude that v is the unique solution to $\mathcal{L}v = \varphi(u^0 + v)$. Therefore, the original sequence v^k converges to v, proving the claim.

By the Contraction Principle, u^ε satisfies the LDP on $W_w^{2,p}(U)$ with action functional

$$\begin{aligned} S_2 v &= \inf_{\{\psi \in L^p(U)\,|\,\mathcal{L}w = \psi(u^0+w)\}} \int_U L(\psi_y)\,dy \\ &= \int_U \inf_{\{\psi \in L^p(U)\,|\,\mathcal{L}w = \psi(u^0+w)\}} L(\psi_y)\,dy \\ &= \int_{U\setminus A} L\left(\frac{\mathcal{L}v_y}{f_y}\right) dy + \int_A \inf_{\psi \in L^p(U)} L(\psi_y)\,dy, \end{aligned}$$

where $A = \{y \in U \,|\, u_y^0 + v_y = 0\}$. The argument used in the proof of Theorem 5.1.7 allowed us to interchange the infimum and the integral. If $y \in A$ and $\mathcal{L}v_y \neq 0$ then $\mathcal{L}v_y = \infty$. On the other hand, if $y \in A$ and $\mathcal{L}v_y = 0$ then $L(\psi_y)$ is clearly minimized when $\psi_y = E[q(0)]$, in which case $L(\psi_y) = 0$. This proves (3.8) and completes the proof of the theorem. □

Since $L(s) = 0$ and L is nonnegative, $S_3 w = 0$ if and only if $w \equiv 0$. Hence, it follows from (2.20) that for every $\delta > 0$, $\gamma > 0$ and $w \in W^{2,p} \cap W_0^{1,p}$ there is $\varepsilon_0 > 0$ such that

$$P\left(\|u^\varepsilon - u^0 - w\|_{W_0^{1,p}(U)} < \delta\right) \geq e^{-\frac{1}{\varepsilon^n}[S_3 w - \gamma]}$$

for all $0 < \varepsilon < \varepsilon_0$. While the right-hand side can be computed relatively easily, for the upper bounds (2.19), $\inf_{w\in\bar{A}} S_3 w$ must be estimated in general using optimal control techniques.

In the case when $p = 2$ and $A = \{w \in H_0^1(U) \,|\, \|w\|_{H_0^1(U)} \geq \delta\}$, however, a simple upper bound is available. By Lemma (5.1.3), there is $\kappa_0 > 0$ such that

$$L(s) \geq \kappa |s|^2 \tag{5.21}$$

for large $|s|$. Since, q is a mean zero random field, $L(0) = 0$ and $L(0) = 0$. By strict convexity of L, we can find $\kappa > 0$ such that (5.21) holds for small $|s|$; whence inequality (5.21) is true for all $s \in \mathbb{R}$.

Multiply (5.20) by w^ε, integrate and use the fact that $b - q^\varepsilon > 0$ to see that $\|w^\varepsilon\|_{H_0^1(U)} \leq C\|u^0\|_{L^\infty(U)}\|q^\varepsilon\|_{L^2(U)}$, where C is independent of ε. Using this apriori estimate and (5.21), we compute

$$\begin{aligned}
\inf_{\|w\|_{H_0^1(U)} \geq \delta} S_3 w &= \inf_{\substack{\|w\|_{H_0^1(U)} \geq \delta \\ \mathcal{L}w = \varphi(u^0 + w)}} \int_U L(\varphi_y)\, dy \\
&\geq \inf_{\substack{\|w\|_{H_0^1(U)} \geq \delta \\ \mathcal{L}w = \varphi(u^0 + w)}} \kappa \|\varphi\|^2_{L^2(U)} \\
&\geq \inf_{\|w\|_{H_0^1(U)} \geq \delta} \frac{\kappa \|w\|^2_{H_0^1(U)}}{C^2 \|u^0\|^2_{L^\infty(U)}} \\
&= \frac{\kappa \delta^2}{C^2 \|u^0\|^2_{L^\infty(U)}}.
\end{aligned}$$

From (2.19) we conclude that for every $\delta, \gamma > 0$ there is $\varepsilon_0 > 0$ such that

$$P(\|w^\varepsilon\|_{H_0^1(U)} \geq \delta) < \exp\left\{ -\frac{1}{\varepsilon^n} \left(\frac{\kappa \delta^2}{C^2 \|u^0\|^2_{L^\infty(U)}} - \gamma \right) \right\} \tag{5.22}$$

for all $0 < \varepsilon < \varepsilon_0$.

Finally, we remark that our LDP works as long as the random field is upper bounded so as to avoid the spectrum of $\mathcal{L}$, however, it does not have to be lower bounded as long as its exponential p moments are uniformly bounded.

5.4.2 Random diffusion

In this subsection we study the large deviations for $u^\varepsilon \in H_0^1(U)$ solving the problem (1.1)in the randomly layered media case, where as usual, we set $a^\varepsilon(x) = a(\frac{x}{\varepsilon})$ and $f \in L^2(U)$.

Define $a^* = (E[\frac{1}{a(0)}])^{-1}$ and $\bar{a} = E[a(0)]$ and consider the random fields $A^\varepsilon(x_1) = \frac{a^*}{a^\varepsilon} - 1$ and $B^\varepsilon = a^\varepsilon - \bar{a}$. The next lemma is a homogenization result expressed in terms of weak convergence whose proof is essentially the same as in the periodic case with minor modifications. For consistency, we sketch the ideas of the proof whose details can be found the literature such as (Bensoussan et al., 2011, Sections 1.3,1.5) or (Pavliotis and Stuart, 2000, Chapter 19).

Lemma 5.4.2. *Let $b^k \in \mathcal{A}$ be a sequence of bounded random fields $a_{min} \leq b^k \leq a_{max}$. Suppose that the sequence $\{\left(\frac{a^*}{b^k} - 1, b^k - \bar{a}\right)\}_{k=1}^\infty \subset \mathbb{R}^2$ converges to (A, B) in $L_w^2(R)$ weakly. Let $v^k, u^0 \in H_0^1(R)$ solve $-\nabla \cdot b^k \nabla v^k = f$ and $-a^* \frac{\partial^2 u^0}{\partial x_1^2} - \bar{a}\Delta_{x_2} u^0 = f$, respectively. Define $z^k = v^k - u^0 - (\psi_1^k - \psi_1) \cdot \frac{\partial u^0}{\partial x_1}$, where $\psi_1^k(x_1) = \int_0^{x_1} \frac{a^*}{b^k(s)} - 1\, ds$ and $\psi_1(x_1) = \int_0^{x_1} A(s)\, ds$. Then, $z^k - z^0 - (\psi_1^k - \psi_1) \cdot \frac{\partial z^0}{\partial x_1}$ vanishes in $H^1(R)$, where $z^0 \in H_0^1(R)$ solves*

$$-a^* \frac{\partial^2}{\partial x_1^2} z^0 - \bar{a}\Delta_{x_2} z^0 = B\Delta_{x_2} u^0 \tag{5.23}$$

Proof. Define the sequences $A^k(x_1) = \frac{a^*}{b^k(x_1)} - 1 - A$, and $B^k(x_1) = b^k(x_1) - \bar{a} - B$. Then, A^k and B^k converge to 0 weakly in $L_w^2(R)$ and $\psi_1^k - \psi_1$ converges to 0 strongly in $L^2(R)$. By rearranging terms of equation (4.13), a simple computation shows that the error z^k satisfies the PDE

$$\begin{aligned} -\nabla \cdot (b^k \nabla z^k) &= B\Delta_{x_2} u^0 + \\ & B^k \Delta_{x_2} u^0 + \frac{\partial}{\partial x_1}\left(b^k(\psi_1^k - \psi)\frac{\partial^2}{\partial x_1^2} u^0\right) + b^k(\psi_1^k - \psi_1)\frac{\partial}{\partial x_1}\Delta_{x_2} u^0. \end{aligned} \tag{5.24}$$

Multiply each term of (5.24) by vz^k, where $v \in H_0^1(R)$ is a test function, integrate over R and integrate by parts. By the definition of the sequences, A^k and B^k, the second line of (5.24) vanishes weakly in the limit as $k \to \infty$, so we may disregard it. Observe that $\psi_1^k - \psi_1^0$ vanishes in $L^2(\partial R)$ as $k \to \infty$. By selecting a suitable test function v (cf. Evans (2010); Bensoussan et al. (2011)), we can show that for large k, $z^k - z^0$ can be approximated in $H^1(R)$ by the solution $\tilde{z}^k \in H_0^1(R)$ to the PDE

$$-\nabla \cdot (b^k \nabla \tilde{z}^k) = B\Delta_{x_2} u^0 \tag{5.25}$$

But the homogenization theory shows that $\tilde{z}^k - z^0 - (\psi_1^k - \psi_1) \cdot \frac{\partial z^0}{\partial x_1}$ converges to 0 in $H^1(U)$, and we are done. □

Since $\psi_1^k - \psi_1^0$ converges to 0 in $L^2(R)$, the proposition above shows that $v^k - u^0$ converges to z^0 in $L^2(R)$, whence we have proved the following

Corollary 5.4.3. *The map $F : L_w^2(R) \to L^2(R)$ which takes the coefficients $(A^\varepsilon, B^\varepsilon)$ to the homogenization error $u^\varepsilon - u^0$ is continuous.*

A simple computation shows that $w^\varepsilon = u^\varepsilon - u^0 \in H_0^1(U)$ solves equation (3.10), whence the Contraction Principle yields now Theorem 3.2.2. Here, the reader should interpret b^k, ψ^k and v^k as *rare realizations* of $a^\varepsilon, \chi^\varepsilon$ and u^ε. Fix a function $a_{min} < b(x_1) < a_{max}$ and $\varepsilon > 0$ small, and assume that the random pair $\{(A^\varepsilon(\omega), B^\varepsilon(\omega)))\}$ is close to the unlikely outcome $(A, B) = (\frac{a^*}{b} - 1, b - \bar{a})$ in $L^2(R)$. Then, the random error $w^\varepsilon = u^\varepsilon - u^0$ is close to z^0 in $L^2(R)$, which solves (5.23). In fact, (2.20) yields that for every $\delta, \gamma > 0$ we have

$$P(\|w^\varepsilon - z^0\|_{L^2(U)} < \delta) \geq \exp\left\{\frac{1}{\varepsilon^n}\left[\int_U L\left(\frac{a^*}{b(y)} - 1, b(y) - \bar{a}\right) dy - \gamma\right]\right\}$$

provided $\varepsilon > 0$ is small enough.

Now, we discuss some of the main issues of our main result. In order to compute the upper bound (2.19), a convex optimization must be performed in order to compute the action functional. However, for certain special sets such as $A = \{\|w^\varepsilon\|_{L^2(U)} > \delta\}$, an upper bound can be computed. Indeed, from the equation (4.13) we have $\|w^\varepsilon\|_{L^2(U)} \leq C\|D^2 u^0\|_{L^2(U)}$, where C only depends on $\|A^\varepsilon\|_{L^2(U)}$ and $\|B^\varepsilon\|_{L^2(U)}$. The same computations used to derive (5.22) show that for every $\delta, \gamma > 0$,

$$P(\|w^\varepsilon\|_{L^2(U)} \geq \delta) < \exp\left\{-\frac{1}{\varepsilon^n}\left(\frac{\kappa\delta^2}{C^2\|D^2u^0\|^2_{L^2(U)}} - \gamma\right)\right\}$$

for sufficiently small ε. In Section 4.2 a Central Limit Result is obtained for this layered media case. Observe that while the CLT was obtained by truncating an asymptotic expansion of $u^\varepsilon - u^0$ and analyzing the fluctuations of the leading order terms, the large deviations for $u^\varepsilon - u^0$ depend on all the terms in (4.14).

5.4.3 Reaction-diffusion Equation in Random Media

Here we prove the LDP for the solution u^ε to the nonlinear BVP (1.6). For the sake of simplicity, we consider the case where b is bounded and uniformly Lipschitz continuous, but this assumption can be relaxed. Let $0 < s < 2$ and denote by Y the space of functions $C(0,T;C^s(U))$ satisfying the same boundary conditions as u^ε. By the compactness results, we can find $1 \leq p < \infty$ such that the space $C^a(0,T;W^{s+\delta,p}(U))$ with the weak topology compactly contains Y (with the strong topology). By taking $f = b(u,\varphi)$ in (2.5) and (2.6) we have

$$\|u^\varepsilon\|_{L^p(0,T;W^{2,p}(U))} + \|\frac{\partial}{\partial t}u^\varepsilon\|_{L^p(0,T;L^p(U))} + \|u^\varepsilon\|_{C^a(0,T;W^{s,p}(U))}$$
$$\leq C(\text{diameter}(U), n)[\|f\|_{L^\infty(U)} + \|g\|_{C^a(U)}].$$

The standard limiting argument used in Section 5.4.1 easily shows

Lemma 5.4.4. *Consider the solution v^k of the quasilinear problem*

$$\frac{\partial}{\partial t}v^k = \Delta v^k + b(v^k, \varphi^k) \qquad \textit{in } U$$

with the same boundary conditions as u^ε. Suppose that

$$\lim_{k\to\infty}\int_U [b(v_y,\varphi_y^k)-\bar{b}(v_y)]f_y\,dy = 0$$

for all step functions v. Then, v^k converges to v^0 in Y as $k\to\infty$, where v^0 solves

$$\frac{\partial}{\partial t}v^0 = \Delta v^0 + \bar{b}(v^0,\varphi) \qquad \text{in } U.$$

In particular, if $b(v,q(x))$ is ergodic for all step functions v, then Lemma 5.4.4 proves the convergence of u^ε to the solution u^0 of the problem (1.10).

The strategy is to prove first the LDP for the approximate problem (1.8) and use this result to obtain the desired probability bounds on u^ε. Suppose that q^ε satisfies conditions (B1) and (B4) stated earlier in Chapter 3. Here, (B2) and (B3) are trivial by our assumption that b is uniformly bounded. Clearly, these conditions imply the LDP for $b(v,q_{y/\varepsilon})$ on $L^p_w(U)$ for fixed v. The following lemma follows immediately from Lemma 5.4.4 and the Contraction Principle. Here, $L(v,s)$ is the Legendre Transform of H in the second component for fixed v.

Lemma 5.4.5. *For every $v\in L^2(0,T;C(U))$, u_v^ε satisfies the LDP on Y with rate $\frac{1}{\varepsilon^n}$ and action functional*

$$S_5(u;v) = \int_U L(v_y,\frac{\partial}{\partial t}u_y - \Delta u_y)\,dy$$

for all $u\in Y$.

Now we proceed to the proof of the main result of this section.

Proof of Theorem 3.2.3. The proof is a standard approximation result and uses ideas of the proof of (Freidlin and Wentzell, 1998, Theorem 7.2). Observe that $w_v^\varepsilon = u^\varepsilon - u_v^\varepsilon$ satisfies the PDE

$$\frac{\partial}{\partial t}w_v^\varepsilon = \Delta w_v^\varepsilon + b(u^\varepsilon,q(\frac{x}{\varepsilon})) - b(v,q(\frac{x}{\varepsilon})) \qquad \text{in } U$$

with homogeneous boundary conditions. The Lipschitz constant on b gives

$$\begin{aligned}\|u_v^\varepsilon - v\|_Y &\leq \|w_v^\varepsilon\|_Y + \|u^\varepsilon - v\|_Y \\ &\leq C\|b(u^\varepsilon, q^\varepsilon) - b(v, q^\varepsilon)\|_{L^\infty(0,T;L^\infty(U))} + \|u^\varepsilon - v\|_Y \\ &\leq (CK+1)\|u^\varepsilon - v\|_Y\end{aligned}$$

On the other hand, in (Lieberman, 2005, Section 8.2) it is shown using the Banach fixed point Theorem that $\|u^\varepsilon - v\|_Y \leq C(K, \|b\|_{L^\infty(U)})\|u_v^\varepsilon - v\|_Y$. This is intuitively true since if u_v^ε is close to v then the fixed point u^ε must also be close to v.

Now we proceed to prove (2.20). Given an open set $A \subset Y$, pick a small ball $B(v,\delta) \subset Y$ and $\gamma > 0$. Then, by the estimates above,

$$\begin{aligned}P(u^\varepsilon \in A) &\geq P(\|u^\varepsilon - v\|_Y < \delta) \\ &\geq P(\|u_v^\varepsilon - v\|_Y < \delta') \\ &\geq \exp\{-\frac{1}{\varepsilon^n}(S_5(v;v) - \gamma)\}\end{aligned}$$

for small $e > 0$. Taking the infimum over all $v \in A$ yields (2.20). Since $Y \subset\subset C^a(0,T;W^{2,r}(U))$ and $\|u^\varepsilon\|_{C^a(0,T;W^{2,r}(U))} \leq \|b\|_{L^\infty(0,T;L^\infty(U))}$, the sequence u^ε is exponentially tight. By Corollary 2.6.9, it suffices to prove (2.19), for a compact $A \subset Y$. Let $\{B(v_j, \frac{\delta}{CK+1})\}_{j=1}^N$ be a finite subcover of A. Then,

$$\begin{aligned}P(u^\varepsilon \in A) &\leq \sum_{j=1}^N P(\|u^\varepsilon - v_k\|_Y \leq \frac{\delta}{CK+1}) \\ &\leq \sum_{j=1}^N P(\|u_{v_k}^\varepsilon - v_k\|_Y \leq \delta) \\ &\leq \sum_{j=1}^N \exp\{-\frac{1}{\varepsilon^n}(\inf_{|u-v_j|<\delta} S_5(u;v_j) - \gamma)\}\end{aligned}$$

Use the inequality $\sum_k e^{a_k} \leq \exp\{\max_k a_k + \ln N\}$ to see that

$$P(u^\varepsilon \in A) \leq \exp\{-\frac{1}{\varepsilon^n}(\inf_{u \in \bigcup_{j=1}^N B(v_k,\delta)} S_5(u;v_j) - \gamma')\}$$

for $\varepsilon > 0$ small. Taking $\delta \to 0$ yields (2.19) and we are done. □

A trivial modification of Lemma 5.1.8 shows that $L(v, v)$ vanishes only when $v = u^0$. This allows to compute large deviations of u^ε about u^0.

Now we discuss some generalizations. First, the condition of Lipschitz continuous can be replaced by locally Lipschitz continuous at the expense of global solvability in Y only for small $T > 0$; the same proof holds in this case as well. For large T the (viscosity) solution may cease to be continuous, so we should only expect estimates on $L^p(0, T; W^{1,p}(U))$ weakly.

Secondly, for simplicity we have assumed that q is only rapidly oscillatory in x. Should the random field depend also on $\frac{t}{\varepsilon}$, the same proof goes through with minor changes, except that we must replace n with $n + 1$ and the action functionals will also contain an integral with respect to t.

Thirdly, we have assumed that the Lipschitz condition is independent of q^ε. If $|b(v_1, \varphi) - b(v_2, \varphi)| \leq K|\varphi||v_1 - v_2|$ and $\frac{n}{2} < p \leq \infty$ and we drop the uniform boundedness of b, the large deviation principle also holds on $L^p(0, T; W^{2,p}(U))$ with its weak topology. Strong convergence may not be possible since the solutions are in general not Hölder continuous.

In the special case where $b(v, \varphi) = \varphi b(v)$ we have, by definition, $H(v, f) = H(b(v)f)$, so that $L(v, \varphi) = L(\frac{f}{b(v)})$. A particular example is given in Section 5.4.1, where an upper bound probability is explicitly computed.

6

Conclusion and Future Directions

This work obtained limit theorems for problems arising in stochastic homogenization. The first part of this thesis studied the fluctuations for a particular case of (1.4), however, the techniques used here are not directly applicable for the general case. We have also not analyzed the effect of long-range correlations in the Central Limit Theorem where we expect the CLT to hold with a rate depending on the decay of the correlation functions. We remark that the same result holds for other boundary conditions on which the BVP is well-defined (with obvious modifications), and also in the case of the BVP for the whole domain, where an additional analysis of the rate of decay of the Green's function must be performed.

For the second half of the thesis, as far as the author is concerned, we provide the first large deviation results in stochastic homogenization and averaging in more than one dimension. From the examples presented here it is apparent that the method works equally well for nonsingular problems such as (1.4) as well as nonlinear problems such as (1.6). Moreover, the derivation of the large deviation results does not depend on the particular choice of the random field itself but rather on its regularity (as measured by p). The LDP was also shown to hold in more general settings such as Gaussian fields with long-range correlations.

The homogenization results can also be extended to the whole domain in the case when the random field is uniformly bounded. Since the random fields satisfying conditions (A1)-(A4) only have locally L^p sample paths, an extension to the unbounded case would depend on the rate of decay of the Green's function.

This thesis is essentially the obligatory first step required to analyze other more interesting open problems which would complement this work. A first example is the computation of the LDP in homogenization or averaging for a SDE or SPDE in random media, in which case there are essentially two sources of randomness. Questions on stability and behavior of a system on a large time interval are intimately connected with the analysis of the large deviations of the system (cf. Freidlin and Wentzell (1998)).

Another important application of this problems arise in uncertainty quantification, where one is interested in studying rare events of the stochastic system. In particular, it is desired to obtain good Monte Carlo estimators to calculate transition probabilities for dynamical systems with multiple scales. Some work in the case of periodic media has been done recently by Dupuis et al. (2012), but the case of random media remains open. Other further directions worth exploring, also in connection with uncertainty quantification, include hypothesis testing, calibration of parameters and stochastic control.

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Biography

Esteban Alejandro Chávez Casillas was born on December 1st, 1982 in México City, México DF. After completing his bachelor degree from Universidad Anáhuac del Norte with a major in Actuarial Sciences in 2005, he got a masters' degree in mathematics from the University of Louisville, KY in 2007.