On SDEs with Partial Damping Inspired by the Navier-Stokes Equations

by

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Dissertation submitted in partial fulfillment of the requirements for the degree of
Doctor of Philosophy in the Department of Mathematics
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Abstract

The solution to the Navier Stokes equations on the 2D torus with stochastic forcing that is white noise in time, coloured in space has a Fourier series representation whose coordinates satisfy a countable system of Stochastic Differential Equations. Inspired by the structure of these equations, we construct a finite system of stochastic differential equations with a similar structure and explore the conditions under which the system has an invariant distribution.

Our main tool to prove existence of invariant distributions are Lyapunov functions, or more generally Lyapunov pairs. In particular, we construct the Lyapunov pairs piecewise over different regions and then use mollifiers to unify these disparate characterisations. We also apply some results from Algebraic Geometry and Matrix Perturbation Theory to study and exploit the geometry of the problem in high dimensions.

The combination of these methods allowed us to prove that a large class of the equations we constructed have invariant distributions. Furthermore we have explicit tail estimates for these invariant distributions.

This dissertation is dedicated my parents, Elizabeth and Desmond Williamson, and my grandparents, Betty and Larry O'Connell and Mary and Alan Williamson. They spent their lives nurturing my enthusiasm for education and this work is, hopefully, a testament to their success.

Contents

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1

Introduction

Consider the 2D Navier-Stokes equation

$$
\begin{aligned}
u_t + (u \cdot \nabla)\, u + \nabla p - \nu \Delta u &= f, \\
\text{div }u &= 0,
\end{aligned}
\tag{1.1}
$$

where p is pressure and ν is viscosity. If $w = \text{curl } u$ we can recover u from w by the divergence condition. On $\mathbb{T}^2 = [-\pi, \pi]^2$ with $f = 0$ w has the Fourier series representation

$$
w(t, x) = \sum_{k \in \mathbb{Z}^2} a_k(t) e^{ik \cdot x}
$$

with reality condition $a_{-k} = \overline{a}_k$. We can then reduce the PDE to the system of ODEs

$$
\frac{da_k}{dt} = -\nu |k|^2 a_k - \sum_{(j,l): j+l=k} \langle j^\perp, l \rangle \left(\frac{1}{|j|^2} - \frac{1}{|l|^2} \right) a_j a_l \tag{1.2}
$$

where $j^\perp$ corresponds to a 90° anti-clockwise rotation in $\mathbb{Z}^2$. Let

$$c_{jl}^k = -\langle j^\perp, l\rangle \left(\frac{1}{|j|^2} - \frac{1}{|l|^2}\right).$$

Then

$$\begin{aligned}
\frac{d}{dt} \parallel u(\cdot, t) \parallel_{L^2}^2 &= \sum_{k\in\mathbb{Z}^2} \frac{d}{dt}|a_k|^2 \\
&= \sum_{k\in\mathbb{Z}^2} \frac{d}{dt} a_k a_{-k} \\
&= -\sum_{k\in\mathbb{Z}^2} \left(a_k \nu| - k|^2 a_{-k} + \nu|k|^2 a_k a_{-k}\right) \\
&\quad + \sum_{k\in\mathbb{Z}^2} \left(\sum_{j+l=k} c_{jl}^k a_j a_l a_{-k} + \sum_{j+l=-k} c_{jl}^{-k} a_j a_l a_k\right) \\
&:= -\sum_{k\in\mathbb{Z}^2} 2\nu|k|^2|a_k|^2 + \sum_{j,k,l\in\mathbb{Z}^3} d_{jkl} a_j a_k a_l
\end{aligned}$$

for some constants d_{jkl} where to avoid ambiguity we sum over unordered triples (j, k, l). See first that $d_{jkl} = 0$ if $j + k + l \neq 0$. If this is the case, then

$$\begin{aligned}
d_{jkl} &= c_{jl}^{-k} + c_{jk}^{-l} + c_{kl}^{-j} \\
&= -\langle j^\perp, l\rangle \left(\frac{1}{|j|^2} - \frac{1}{|l|^2}\right) - \langle j^\perp, k\rangle \left(\frac{1}{|j|^2} - \frac{1}{|k|^2}\right) - \langle k^\perp, l\rangle \left(\frac{1}{|k|^2} - \frac{1}{|l|^2}\right) \\
&= \frac{\langle j^\perp, j\rangle}{|j|^2} - \frac{\langle k^\perp, k\rangle}{|k|^2} - \frac{\langle l^\perp, l\rangle}{|l|^2} \\
&= 0.
\end{aligned} \tag{1.3}$$

Therefore

$$\frac{d}{dt} \parallel u(\cdot, t) \parallel_{L^2} = -\sum_k 2\nu|k|^2|a_k(t)|^2 \leqslant -2\nu \parallel u(\cdot, t) \parallel_{L^2}$$

and so

$$\| u(\cdot, t) \|_{L^2} \leqslant e^{-2\nu t} \| u(\cdot, 0) \|_{L^2} .$$

If, instead of full viscosity, we replace Δ in (1.1) with the operator D given by

$$Da_k(t) = -|k|^2 a_k(t) \mathbb{1}_{k \in K}$$

for some symmetric $K \subset \mathbb{Z}^2$, and let f be white noise in time, coloured in space, i.e.

$$df(t, x) = \sum_{k \in \mathbb{Z}^2} \sigma_k e^{ik \cdot x} dW_k(t)$$

for complex Brownian motions W_k satisfying $\overline{W}_k = W_{-k}$, $\overline{\sigma}_k = \sigma_{-k}$, then control of the energy of the system is no longer a trivial question. We need to study how the effects of dissipation transfer through the coordinates to keep the system from blowing up. Motivated by this question we consider a finite-dimensional analogue to this system.

For simplicity we will consider the case of a real, rather than complex, finite-dimensional analogue of (1.2). Let $X(t)$ be the solution to the SDE

$$dX_i = \left(\sum_{j \leqslant k} c^i_{jk} X_j X_k - b_i X_i \right) dt + \sigma_i dW_i(t) \tag{1.4}$$

where the constants c^i_{jk} satisfy

$$c^i_{jk} + c^j_{ik} + c^k_{ij} = 0. \tag{1.5}$$

This is the real analogue to the condition discovered in (1.3). To reproduce the operator D we can define $b_i = 0$ for $i \leqslant M$ and $b_i > 0$ for $i > M$. Let b, σ be the vector representations of these constants. In Section 3 we will prove, under certain assumptions on $\{c^i_{jk}\}_{i,j,k}$, b and σ, that (1.4) has an invariant distribution.

We will sometimes consider the solution $Y(t)$ of the system

$$\frac{dY_i}{dt} = \sum_{j \leqslant k} c^i_{jk} Y_j Y_k \tag{1.6}$$

which we will refer to as the conserved system as with no dissipation or forcing we have $\frac{d}{dt}|Y(t)| = 0$ for all t. The condition (1.5) indicates that the conserved system can be described as a collection of triples connecting certain triples of coefficients of Y. This motivates some parts of our study. In particular, in Section 2 we will consider the case of $Y(t) \in \mathbb{R}^3$ with only one triple of constants non-zero. In Section 4 we consider a finite list of triples, each of which in turn solely describes the conserved system for a random amount of time. The main results of each endeavour are given at the beginning of their relevant section.

2

A Three Dimensional System

2.1 Summary of Main Results

Here we consider the dynamics induced by a single triple on $X(t) \in \mathbb{R}^3$. Consider the system

$$
\begin{aligned}
dX_1 &= c_1 X_2 X_3 dt + \sigma_1 dW_1, \\
dX_2 &= (c_2 X_1 X_3 - b_2 X_2)\, dt + \sigma_2 dW_2, \\
dX_3 &= (c_3 X_1 X_2 - b_3 X_3)\, dt + \sigma_3 dW_3,
\end{aligned}
\tag{2.1}
$$

where $c_1 + c_2 + c_3 = 0$, $b_2, b_3 > 0$ and the W_i are real independent Brownian motions. Let L be the generator of the system and let $H = c_1 x_2 x_3 \partial_1 + c_2 x_1 x_3 \partial_2 + c_3 x_1 x_2 \partial_3$ be the operator for the conserved component. We will make the following assumption.

Assumption 2.1. *In the system given by (2.1), $c_2 c_3 > 0$ and $\sigma_2^2 + \sigma_3^2 > 0$.*

The main result of this section is the following. Here, as in other chapters, we use l.o.t. to refer to terms that are lower order as $|x| \to \infty$.

Theorem 2.2. *Given Assumption 2.1, the system given by (2.1) has an invariant distribution* μ *that satisfies*

$$\int_{\mathbb{R}^3} e^{a|x|}\mu(dx) < \infty$$

for

$$a < \frac{\min\{b_2, b_3\}\left(|c_2|\sigma_2^2 + |c_3|\sigma_3^2\right)}{|\sigma|^2\left(|c_2|b_2^2 + |c_3|b_3^2\right)} \frac{c_2c_3}{2\sqrt{c_2c_3} + 64|c_2 + c_3|}. \tag{2.2}$$

Furthermore there exists some non-negative function $V(x) = e^{a|x|} + \text{l.o.t.}$ *such that if* $w = 1 + \beta V$ *for some* $\beta > 0$ *and we define the metric*

$$d_w\left(\nu_1, \nu_2\right) = \sup_{|\varphi(x)|\leqslant w(x)} \left(\int_{\mathbb{R}^N} \varphi(z)\left(\nu_1(dz) - \nu_2(dz)\right)\right)$$

on the space of probability measures on $\mathbb{R}^N$*, then there exist constants* $C, \eta > 0$ *such that*

$$d_w\left(\nu_1 P_t, \nu_2 P_t\right) \leqslant Ce^{-\nu t} d_w\left(\nu_1, \nu_2\right)$$

where P_t *is the Markov semigroup associated with* X_t*.*

We will prove Theorem 2.2 by constructing a Lyapunov function on $\mathbb{R}^3$ [9]. Definitions and results related to Lyapunov functions are given in Appendix A.

2.2 Construction of the Lyapunov Function

In the case of dissipation in all three coordinates, it can be shown (see Example 5.1) that the process has an invariant distribution with Gaussian tails. Naively we could take a similar approach here and choose a Lyapunov function that is an increasing function of $|x|$. To this effect let $V_1 = e^{a|x|}$. Then

$$LV_1 \leqslant -a\frac{b_2x_2^2 + b_3x_3^2}{|x|}e^{a|x|} + a^2\frac{|\sigma|^2}{2}e^{a|x|} + \text{l.o.t.}$$

One can see that for $b_2x_2^2 + b_3x_3^2 < a\frac{|\sigma|^2|x|}{2}$ $LV_1 > 0$, and so we need a different function near the x_1-axis. This is the nature of our approach in this section; we will construct different functions, namely V_2 and V_3 in two different regions of $\mathbb{R}^3$ near the x_1-axis where different dynamics are dominant, use twice continuously differentiable monotone functions with range $[0, 1]$ (instead of mollifiers, for reasons that will become clear later), henceforth referred to as pseudo-mollifiers, to make V_2 and V_3 vanish outside of the regions where they cease to be useful, then find constants C, D such that

$$L\left(V_1 + CV_2 + DV_3\right) \leqslant -\epsilon e^{a|x|} + \text{l.o.t.} \tag{2.3}$$

globally for some $\epsilon > 0$.

These functions V_2, V_3 will take the form of scaled approximate expectations of exit times from the given regions, as under nice conditions if τ is the exit time of a process from a region D with infinitesimal generator L and $u(x) = \mathbb{E}_x\tau$, the expectation of τ starting at position x, then $Lu = -1$ in D. To motivate these functions and regions further we must study the behaviour of the system near the x_1-axis.

Assume without loss of generality that $c_2, c_3 > 0$. If $X(t)$ is on or sufficiently close to the x_1-axis then the noise terms are the dominant dynamics. We will consider this region later. Once $X(t)$ has left this region, the dominant dynamics are the conserved dynamics, $(0, c_2X_1X_3, c_3X_1X_2)dt$. One can show under these dynamics that $X_2 \pm \sqrt{\frac{c_2}{c_3}}X_3$ grows exponentially with exponent $\pm X_1\sqrt{c_2c_3}$.

Remark 2.3. This should motivate the stipulation that $c_2c_3 > 0$ in Assumption 2.1, as if $c_2c_3 < 0$ we have elliptic rather than hyperbolic motion in (X_2, X_3). The second condition, that $\sigma_2^2 + \sigma_3^2 > 0$, simply guarantees that $X(t)$ can move off the x_1-axis.

Therefore if we let v be parallel to $\left(0, \sqrt{c_3}, \operatorname{sgn}(x_1)\sqrt{c_2}\right)$ and of norm 1 then

$$H|v \cdot x| = |x_1|\sqrt{c_2c_3}|v \cdot x|.$$

Therefore although it is hard to exactly see the growth of $X_2,\ X_3$, a linear combination of them grows exponentially fast. So if we consider the deterministic exit time τ_2 from the region $\{x : |v \cdot x| \leqslant E\}$ under the conserved dynamics we have

$$\tau_2 = \frac{1}{|x_1|\sqrt{c_2c_3}} \ln\left(\frac{E}{|v \cdot x|}\right). \tag{2.4}$$

With this expression in mind consider the function

$$\begin{aligned} V_2 =& \frac{1}{|x|}e^{a|x|} \ln\left(\frac{B|x|^{\frac{1}{2}}\sqrt{\ln|x|}}{|v \cdot x|}\right) f\left(\frac{|v \cdot x|}{A|x|^{-\frac{1}{2}}\sqrt{\ln|x|}}\right) g\left(\frac{\sqrt{x_2^2+x_3^2}}{B|x|^{\frac{1}{2}}\sqrt{\ln|x|}}\right) \\ :=& W_2 g\left(\frac{\sqrt{x_2^2+x_3^2}}{B|x|^{\frac{1}{2}}\sqrt{\ln|x|}}\right) \end{aligned}$$

where f, g are pseudo-mollifiers satisfying $f(z) = 0$, $g(z) = 1$ for $z \leqslant \frac{1}{2}$, $f(z) = 1$, $g(z) = 0$ for $z \geqslant 1$ and A, B are constants to be chosen later. As we want our Lyapunov function to be large, we scaled the expression in (2.4) by $e^{a|x|}$ and for simplicity replaced $|x_1|\sqrt{c_2c_3}$ with $|x|$, as when we are near the x_1-axis $|x_1| \sim |x|$. As may be expected we cannot rely on this function when $|v \cdot x|$ is too small so we introduce f so that V_2 then vanishes. The function g serves a similar purpose. When $f = 0$ we should expect the noise terms to grow $|v \cdot X(t)|$, and the expectation of the exit time τ_3 from the region $\{x : |v \cdot x| \leqslant E\}$ under only the noise dynamics is given by

$$\mathbb{E}_x\tau_3 = \frac{E - |v \cdot x|^2}{v_2^2\sigma_2^2 + v_3^2\sigma_3^2}.$$

To this effect let

$$\begin{aligned} V_3 =& e^{a|x|}\left(4A^2|x|^{-1}\ln|x| - |v \cdot x|^2\right) h\left(\frac{|v \cdot x|}{A|x|^{-\frac{1}{2}}\sqrt{\ln|x|}}\right) g\left(\frac{\sqrt{x_2^2 + x_3^2}}{B|x|^{\frac{1}{2}}\sqrt{\ln|x|}}\right) \\ :=& W_3 g\left(\frac{\sqrt{x_2^2 + x_3^2}}{B|x|^{\frac{1}{2}}\sqrt{\ln|x|}}\right) \end{aligned}$$

where h is a pseudo-mollifier satisfying $h(z) = 1$ for $z \leqslant 1$, $h(z) = 0$ for $z \geqslant 2$.

The idea is thus. For all $x \in \mathbb{R}^3$, $|x|$ sufficiently large we want (2.3) to hold. We do this by having $LV_i \leqslant -\epsilon_i e^{a|x|}$ for $i = 1, 2, 3$ and some ϵ_i in their respective regions. When either V_2, V_3 ceases to be useful they will vanish, but in the regions where their pseudo-mollifiers are non-constant they may give a positive contribution to the expression in (2.3). In this case the negative contribution from one of the other functions must dominate this positive contribution. This is why the functions f and h are non-constant on non-overlapping intervals, and why we added an $e^{a|x|}$ term to our construction of V_2, V_3.

2.3 Calculations

For now we will assume $g > 0$ and study $L\left(CW_2 + DW_3\right)$, as

$$L\left(CV_2 + DV_3\right) = \left[L\left(CW_2 + DW_3\right)\right]g + \left(CW_2 + DW_3\right)Lg \\ + \sum_{i=1}^{3}\sigma_i^2\left[\frac{d}{dx_i}\left(CW_2 + DW_3\right)\right]\left[\frac{d}{dx_i}g\right].$$

Consider W_2. When $f > 0$,

$$\begin{aligned} LW_2 \leqslant & e^{a|x|}\Bigg[-\sqrt{c_2c_3}f + \frac{|v_2|b_2|x_2| + |v_3|b_3|x_3|}{|x||v\cdot x|}f + \frac{v_2^2\sigma_2^2 + v_3^2\sigma_3^2}{2|x||v\cdot x|^2}f \\ & + \ln\left(\frac{B|x|^{\frac{1}{2}}\sqrt{\ln|x|}}{|v\cdot x|}\right)|f'|\left(\sqrt{c_2c_3} + \frac{|v_2|b_2|x_2| + |v_3|b_3|x_3|}{A|x|^{\frac{1}{2}}\sqrt{\ln|x|}}\right) \\ & + \frac{v_2^2\sigma_2^2 + v_3^2\sigma_3^2}{|x||v\cdot x|}|f'|\frac{1}{A|x|^{-\frac{1}{2}}\sqrt{\ln|x|}} \\ & + \frac{v_2^2\sigma_2^2 + v_3^2\sigma_3^2}{2|x|}\ln\left(\frac{B|x|^{\frac{1}{2}}\sqrt{\ln|x|}}{|v\cdot x|}\right)|f''|\frac{|x|}{A^2\ln|x|}\Bigg] + \text{l.o.t.} \\ \leqslant & e^{a|x|}\Bigg[\left(-\sqrt{c_2c_3} + \frac{2B\sqrt{v_2^2b_2^2 + v_3^2b_3^2}}{A}\right)f \\ & + \left(\sqrt{c_2c_3} + \frac{B\sqrt{v_2^2b_2^2 + v_3^2b_3^2}}{A}\right)\ln|x||f'|\Bigg] + \text{l.o.t.} \end{aligned}$$

Now consider W_3. When $h > 0$,

$$
\begin{aligned}
LW_3 \leqslant & e^{a|x|}\Bigg[-\sqrt{c_3 c_3}|x||v\cdot x|^2 h + 2\left(|v_2|b_2|x_2| + |v_3|b_3|x_3|\right)|v\cdot x|h - \left(v_2^2\sigma_2^2 + v_3^2\sigma_3^2\right)h \\
& + \left(4A^2|x|^{-1}\ln|x| - |v\cdot x|^2\right)|h'|\left(-\sqrt{c_2c_3}|x| + \frac{|v_2|b_2|x_2| + |v_3|b_3|x_3|}{A|x|^{-\frac{1}{2}}\sqrt{\ln|x|}}\right) \\
& + \left(v_2^2\sigma_2^2 + v_3^2\sigma_3^2\right)\left(\frac{3}{2}|h''| + 2|h'|\right)\Bigg] + \text{l.o.t.} \\
\leqslant & e^{a|x|}\Bigg[\left(-\sqrt{c_2c_3}|x||v\cdot x|^2 + 2\sqrt{v_2^2b_2^2 + v_2^2b_3^2}\sqrt{x_2^2 + x_3^2}|v\cdot x| - \left(v_2^2\sigma_2^2 + v_3^2\sigma_3^2\right)\right)h \\
& + \left(4A^2|x|^{-1}\ln|x| - |v\cdot x|^2\right)|h'|\left(-\sqrt{c_2c_3} + \frac{B\sqrt{v_2^2b_2^2 + v_3^2b_3^2}}{A}\right) \\
& + \left(v_2^2\sigma_2^2 + v_3^2\sigma_3^2\right)\left(\frac{3}{2}|h''| + 2|h'|\right)\Bigg].
\end{aligned}
$$

The terms that result from applying H and the dissipative terms to h follow from the fact that $h'(z) \leqslant 0$ for all z. Choose

$$
B < \frac{A\sqrt{c_2c_3}}{4\sqrt{v_2^2b_2^2 + v_3^2b_3^2}} \tag{2.5}
$$

so that this term is negative. Then consider the sum of the first and second terms. Treating them as a quadratic $-pz^2 + qz$ in $z = |v\cdot x|$, we know this quadratic has maximum $\frac{q^2}{4p}$, so that

$$
\begin{aligned}
LW_3 \leqslant & e^{a|x|}\Bigg[\left(\frac{\left(v_2^2 b_2^2 + v_3^2 b_3^2\right)\left(x_2^2 + x_3^2\right)}{|x|\sqrt{c_2 c_3}} - \left(v_2^2 \sigma_2^2 + v_3^2 \sigma_3^2\right) \right) h \\
& + \left(v_2^2 \sigma_2^2 + v_3^2 \sigma_3^2\right) \left(\frac{3}{2}|h''| + 2|h'| \right) \Bigg].
\end{aligned}
$$

Note also that this supremum occurs when

$$
\begin{aligned}
|v \cdot x| = & \frac{\sqrt{v_2^2 b_2^2 + v_3^2 b_3^2}\sqrt{x_2^2 + x_3^2}}{|x|\sqrt{c_2 c_3}} \\
\leqslant & \frac{\sqrt{v_2^2 b_2^2 + v_3^2 b_3^2}}{\sqrt{c_2 c_3}} \frac{B\sqrt{\ln|x|}}{|x|^{\frac{1}{2}}} \\
\leqslant & \frac{A}{4} \frac{\sqrt{\ln|x|}}{|x|^{\frac{1}{2}}}
\end{aligned}
$$

so that the quadratic is decreasing when $|v \cdot x| \geqslant \frac{A\sqrt{\ln|x|}}{2|x|^{\frac{1}{2}}}$.

Now assume f is not constant. Then we know that $|v \cdot x| \geqslant \frac{A}{2}|x|^{-\frac{1}{2}}\sqrt{\ln|x|}$ and $h = 1$ is constant so that

$$
LW_3 \leqslant \ln|x| e^{a|x|} \left(-\frac{\sqrt{c_2 c_3} A^2}{4} + AB\sqrt{v_2^2 b_2^2 + v_3^2 b_3^2} \right) + \text{l.o.t.}
$$

Now assume h is not constant. Then we know that $|v \cdot x| \geqslant A|x|^{-\frac{1}{2}}\sqrt{\ln|x|}$, so that

$$
\begin{aligned}
LW_3 \leqslant & e^{a|x|}\Bigg[\ln|x| \left(-\sqrt{c_2 c_3} A^2 + 2AB\sqrt{v_2^2 b_2^2 + v_3^2 b_3^2} \right) h \\
& + \left(v_2^2 \sigma_2^2 + v_3^2 \sigma_3^2\right) \left(\frac{3}{2}|h''| + 2|h'| \right) \Bigg].
\end{aligned}
$$

As h has bounded derivatives, for large $|x|$ the above expression is negative until h is close to 0. As h need only be twice continuously differentiable, we can assume that $h(z) = (2-z)^3$ near $z = 2$. Then, ignoring the $e^{a|x|}$ term for now, applying (2.5) and multiplying the h' term by $A\sqrt{\ln|x|}$, close to $h = 0$ we have

$$\begin{aligned}
&\ln|x|\left(-\sqrt{c_2c_3}A^2 + 2AB\sqrt{v_2^2b_2^2 + v_3^2b_3^2}\right)h + \left(v_2^2\sigma_2^2 + v_3^2\sigma_3^2\right)\left(\frac{3}{2}|h''| + 2|h'|\right)\\
\leqslant& -A^2\frac{\sqrt{c_2c_3}}{2}\ln|x|(2-z)^3 + \left(v_2^2\sigma_2^2 + v_3^2\sigma_3^2\right)\left(9(2-z) + \sqrt{A\ln|x|}6(2-z)^2\right)\\
\leqslant& \frac{1}{A\sqrt{\ln|x|}}\left[\frac{\sqrt{c_2c_3}}{2}\left(A\sqrt{\ln|x|}(2-z)\right)^3 + 6\left(v_2^2\sigma_2^2 + v_3^2\sigma_3^2\right)\left(A\sqrt{\ln|x|}(2-z)\right)^2\right.\\
&\left.+9\left(v_2^2\sigma_2^2 + v_3^2\sigma_3^2\right)\left(A\sqrt{\ln|x|}(2-z)\right)\right].
\end{aligned}$$

The resulting cubic has a supremum independent of $A, |x|$ occurring at $z = 2 - \frac{z_0}{A\sqrt{\ln|x|}}$ for some $z_0 > 0$, and so

$$LW_3 = O\left(\frac{e^{a|x|}}{\sqrt{\ln|x|}}\right).$$

Now consider

$$CW_2 + DW_3.$$

We will deal with the following four situations, where $*$ denotes that the pseudo-mollifier is non-constant.

$$\begin{aligned}
(f,h) =& (1,0),\\
& (0,1),\\
& (*,1),\\
& (1,*).
\end{aligned}$$

In each of the these situations, applying (2.5) and ignoring lower order terms we have

$$L\left(CV_2+DV_3\right)\leqslant e^{a|x|}\left[-C\frac{\sqrt{c_2c_3}}{2}\right]$$

$$L\left(CV_2+DV_3\right)\leqslant e^{a|x|}\Big[-D\left(v_2^2\sigma_2^2+v_3^2\sigma_3^2\right)\Big]+e^{a|x|}D\frac{\left(v_2^2b_2^2+v_3^2b_3^2\right)\left(x_2^2+x_3^2\right)}{|x|\sqrt{c_2c_3}},$$

$$L\left(CV_2+DV_3\right)\leqslant \ln|x|e^{a|x|}\left[C\frac{5\sqrt{c_2c_3}}{4}\sup_z|f'(z)|\right.$$
$$\left.+D\left(-\frac{\sqrt{c_2c_3}A^2}{4}+AB\sqrt{v_2^2b_2^2+v_3^2b_3^2}\right)\right]$$

$$L\left(CV_2+DV_3\right)\leqslant e^{a|x|}\Big[-C\sqrt{c_2c_3}\Big].$$

Our approach is as follows. Fix some small $\epsilon>0$, and let

$$C=\frac{a^2|\sigma|^2(1+\epsilon)}{\sqrt{c_2c_3}},$$

$$D=\frac{a^2|\sigma|^2(1+\epsilon)}{2\left(v_2^2\sigma_2^2+v_3^2\sigma_3^2\right)}.$$

Furthermore, we can choose f such that

$$\sup_z|f'(z)|\leqslant 2+\epsilon.$$

Then, given B, choose A large enough so that

$$C\frac{5}{4}\left(2+\epsilon\right)+D\left(-\frac{\sqrt{c_2c_3}A^2}{4}+AB\sqrt{v_2^2b_2^2+v_3^2b_3^2}\right)\leqslant -1. \tag{2.6}$$

Then we have

$$L(CV_2+DV_3)\leqslant e^{a|x|}\left[-a^2|\sigma|^2\left(\frac{1}{2}+\frac{\epsilon}{2}\right)+\frac{a^2|\sigma|^2(1+\epsilon)}{2\left(v_2^2\sigma_2^2+v_3^2\sigma_3^2\right)}\frac{\left(v_2^2b_2^2+v_3^2b_3^2\right)\left(x_2^2+x_3^2\right)}{|x|\sqrt{c_2c_3}}\right]$$

in all cases. Now consider

$$V = V_1 + CV_2 + DV_3.$$

When $g = 1$ is constant

$$LV \leqslant e^{a|x|}\left[-\frac{a^2|\sigma|^2}{2}\epsilon + \frac{a^2|\sigma|^2(1+\epsilon)}{2\left(v_2^2\sigma_2^2 + v_3^2\sigma_3^2\right)}\frac{\left(v_2^2b_2^2 + v_3^2b_3^2\right)\left(x_2^2 + x_3^2\right)}{|x|\sqrt{c_2c_3}} - a\min\{b_2, b_3\}\frac{x_2^2 + x_3^2}{|x|}\right].$$

Choose a so that

$$a < \frac{2\min\{b_2, b_3\}\sqrt{c_2c_3}\left(v_2^2\sigma_2^2 + v_3^2\sigma_3^2\right)}{(1+\epsilon)|\sigma|^2\left(v_2^2b_2^2 + v_3^2b_3^2\right)} \tag{2.7}$$

and so

$$LV \leqslant -e^{a|x|}\frac{a^2|\sigma|^2}{2}\epsilon.$$

When $g = 0$,

$$LV \leqslant -\ln|x|e^{a|x|}a\min\{b_2, b_3\}B^2 + \text{l.o.t.}$$

Similar to before, we can choose g so that

$$\sup_z |g'(z)| \leqslant 2 + \epsilon.$$

Then when g is non-constant,

$$
\begin{aligned}
LV =& LV_1 + gL\left(CW_2 + DW_3\right) + |Hg|\left(CW_1 + DW_2\right) + \text{l.o.t.} \\
\leqslant & e^{a|x|}\left[\frac{a^2|\sigma|^2}{2}(1-2g) + \frac{x_2^2+x_3^2}{|x|}\left(-a\min\{b_2,b_3\} + ga^2\frac{(1+\epsilon)|\sigma|^2\left(v_2^2b_2^2+v_3^2b_3^2\right)}{2\left(v_2^2\sigma_2^2+v_3^2\sigma_3^2\right)\sqrt{c_2c_3}}\right)\right. \\
&\left. + \frac{(c_2+c_3)\,|x_1x_2x_3|}{B|x|^{\frac{1}{2}}\sqrt{\ln|x|}}|g'|\left(\frac{2a^2|\sigma|^2}{\sqrt{c_2c_3}}\ln|x| + A^2\frac{a^2|\sigma|^2}{v_2^2\sigma^2+v_3^2|\sigma|^2}\ln|x|\right)\right] \\
\leqslant & \ln|x|e^{a|x|}\left[\frac{B^2}{4}\left(-a\min\{b_2,b_3\} + a^2\frac{(1+\epsilon)|\sigma|^2\left(v_2^2b_2^2+v_3^2b_3^2\right)}{2\left(v_2^2\sigma_2^2+v_3^2\sigma_3^2\right)\sqrt{c_2c_3}}\right)\right. \\
&\left. + \frac{c_2+c_3}{2}(2+\epsilon)\left(\frac{2a^2|\sigma|^2}{\sqrt{c_2c_3}} + A^2\frac{a^2|\sigma|^2}{v_2^2\sigma^2+v_3^2|\sigma|^2}\right)\right] \\
=& -\delta\ln|x|e^{a|x|}
\end{aligned}
$$

for some $\delta > 0$ if we choose

$$
a < \frac{\frac{B^2\min\{b_2,b_3\}}{4}}{\frac{B^2(1+\epsilon)|\sigma|^2\left(v_2^2b_2^2+v_3^2b_3^2\right)}{2\left(v_2^2\sigma_2^2+v_3^2\sigma_3^2\right)\sqrt{c_2c_3}} + \frac{c_2+c_3}{2}(2+\epsilon)\left(\frac{2|\sigma|^2}{\sqrt{c_2c_3}} + A^2\frac{|\sigma|^2}{v_2^2\sigma_2^2+v_3^2|\sigma|^2}\right)}. \tag{2.8}
$$

This completes our construction of a Lyapunov function and, together with Proposition A.2 completes our proof of the existence of an invariant measure μ satisfying

$$
\int_{\mathbb{R}^3} e^{a|x|}\mu(dx) < \infty
$$

for any $a > 0$ satisfying (2.7), (2.8). We will now prove the bound (2.2).

See that the bound in (2.8) is an increasing function of both B and A. Furthermore if we satisfy (2.5) by letting

$$B = F \frac{A\sqrt{c_2 c_3}}{4\sqrt{c_2 b_2^2 + c_3 b_3^2}}$$

for some $F < 1$, then let $\epsilon \downarrow 0$, $A \to \infty$ and then let $F \uparrow 1$, (2.8) becomes

$$\begin{aligned} a < & \frac{\frac{\min\{b_2,b_3\}c_2c_3}{64\left(v_2^2 b_2^2 + v_3^2 b_3^2\right)}}{\frac{\sqrt{c_2c_3}|\sigma|^2}{32\left(v_2^2\sigma_2^2+v_3^2\sigma_3^2\right)} + (c_2+c_3)\frac{|\sigma|^2}{v_2^2\sigma_2^2+v_3^2\sigma_3^2}} \\ = & \frac{\min\{b_2, b_3\}\left(v_2^2\sigma_2^2 + v_3^2\sigma_3^2\right)}{|\sigma|^2\left(v_2^2 b_2^2 + v_3^2 b_3^2\right)} \frac{c_2c_3}{2\sqrt{c_2c_3} + 64(c_2+c_3)} \\ = & \frac{\min\{b_2, b_3\}\left(c_3\sigma_2^2 + c_2\sigma_3^2\right)}{|\sigma|^2\left(c_3 b_2^2 + c_2 b_3^2\right)} \frac{c_2c_3}{2\sqrt{c_2c_3} + 64(c_2+c_3)}. \end{aligned}$$

Replacing $c_2 + c_3$ with its lower bound of $2\sqrt{c_2c_3}$ shows that (2.8) is always a more restrictive bound than (2.7), completing the proof of Theorem 2.2.

3

The General Finite Dimensional System

3.1 Summary of Main Results

Here we will tackle the general N dimensional problem introduced in Section 1. For notational convenience, we will replace (1.4), (1.5) and (1.6) with the following. Let $Y(t)$ be a solution of the ODE

$$\frac{dY_i}{dt} = Y^T C^i Y \quad 1 \leqslant i \leqslant N,$$

where the C^i are $N \times N$ symmetric matrices whose entries $(C^i)_{jk} := c^i_{jk}$ satisfy

$$c^i_{jk} + c^j_{ik} + c^k_{ij} = 0,$$

and let $H(y) = \sum_{i=1}^N y^T C^i y \partial_i$ be the generator for Y. Let $X(t)$ be a solution of the SDE

$$dX_i = (X^T C^i X - b_i X_i)dt + \sigma_i dW_i \quad 1 \leqslant i \leqslant N$$

where the W_i are independent Brownian motions in $\mathbb{R}$, $\sigma_i \in \mathbb{R}$ and there exists an $M < N$ such that $b_i > 0 \Leftrightarrow i > M$. Let $L(x) = H(x) - \sum_{i=M+1}^{N} b_i x_i \partial_i + \frac{1}{2}\sum_{i=1}^{N} \sigma_i^2 \partial_i^2$ be the generator for $X(t)$.

We will place the following assumptions on the c_{jk}^i, whose usefulness will be motivated in later sections, but first we have the following definition.

Definition 3.1. *Define*

$$\mathcal{N} = \{y \in \mathbb{R}^N : Y(0) = y \Rightarrow Y_i(t) = 0 \text{ for all } t \in \mathbb{R}, i > M\}.$$

Assumption 3.2.

$$\mathcal{N} = \left\{ y \in \mathbb{R}^M \times \{0\}^{N-M} : \left.\frac{dY_i}{dt}\right|_{t=0} = 0 \text{ for all } i \leqslant N \right\}.$$

Assumption 3.3. *There exist values λ_{min}, $P\sigma_{min}^2 > 0$ such that for all $x \in \mathcal{N}\backslash\{0\}$ the matrix $\left(x^T C^i\right)_{i \leqslant N}$ is diagonalisable and has an eigenvalue $\lambda(x)$ and projection matrix $P(x)$ onto the left-eigenspace such that*

(a) $Re(\lambda(x)) \geqslant \lambda_{min}|x|$,

(b) $\sum_{i=1}^{N} |P(x)\, e_i|^2 \sigma_i^2 \geqslant P\sigma_{min}^2$.

Assumption 3.4. *For any $x \in \mathcal{N}$, with $A_x = (x^T C^i)_{i>M}$, $B_x = (x^T C^i)_{i \leqslant N}$, there exists an $n^* \geqslant 1$ such that*

$$\{z : \max_{i>M} |z_i| = 0\} \bigcap \bigcap_{n=0}^{n^*-1} \mathrm{Ker}(A_x B_x^n)$$

is equal to the tangent space of $\mathcal{N}$ at x.

Theorem 3.5. *Under Assumptions 3.2, 3.3 & 3.4, there exists an invariant distribution μ for the system $X(t)$ such that*

$$\int_{\mathbb{R}^N} \exp\left(a\frac{|x|^{\frac{1}{2}}}{(\ln|x|)^{2^{n^*}+dim(\mathcal{N})+1}}\right)\mu(dx) < \infty$$

for sufficiently small $a > 0$.

Remark 3.6. This result is significantly weaker than that of Section 2. The main reason for this will be given after Lemma 3.12.

3.2 Overview of the Construction of the Lyapunov Function

Although in principle our construction of the Lyapunov function is similar to that in Section 2, there are important differences. First, in Section 2.1 we constructed two functions in different regions near the x_1-axis. One could say that we did this because there was no dissipation on the x_1-axis, but as we will see in the general case the more correct analysis is that there was no movement on the x_1-axis, i.e., if the conserved system Y was given an initial condition on the x_1-axis, it would stay on the x_1-axis for all time. In Section 3.3 we characterise the correct generalisation of this region which we refer to as $\mathcal{N}$, given in Definition 3.1.

One may have noticed that in Section 2.1 the pair $v = (0, \sqrt{|c_3|}, \text{sgn}(x_1)\sqrt{|c_2|})$, $|x_1|\sqrt{c_2 c_3}$ was an eigenpair of the linearisation of the conserved system on the x_1-axis. Here we follow this approach; given a point x near $\mathcal{N}$, we linearise the conserved system at an appropriate point on $\mathcal{N}$ near x, and find an eigenvalue $\lambda(x)$ with $\text{Re}(\lambda(x)) > 0$. If the linearised dynamics are dominant the projection of $X(t)$ onto the eigenspace of $\lambda(x)$ will then grow exponentially fast. In Section 3.3 we will formalise this concept of an eigenpair and in Section 3.4 we will study their properties. Then in Section 3.5 we will then use these eigenpairs to construct functions similar to those in Section 2 near different sections of $\mathcal{N}$. In Section 3.6 we then use pseudo-mollifiers and the functions constructed in Section 3.5 to create a single function defined near $\mathcal{N}$.

Once we have left a certain sized neighbourhood of $\mathcal{N}$ we will find that the functions we constructed in Section 3.5, and by extension the function we defined in Section 3.6 are no longer useful. However, unlike in Section 2.1, once we have left $\mathcal{N}$ we have no guarantee of sizeable dissipation, only of sizeable movement. In particular, we have that one moment of motion (velocity, acceleration or some higher moment) of one X_i, $i > M$, is large. In Section 3.7 we exploit the fact that if one moment of motion is large, the next lowest moment should soon be large, and so on to create a series of functions to cover all regions where a single moment of motion of each X_i, $i > M$ is large, but X_i is not. Much like in previous sections, these functions will be based on appropriate approximate exit times from certain regions.

3.3 Fixed Points of the Conserved System

Consider the following subset of $\mathbb{R}^N$, which will be our main object of study in this section.

Definition 3.1. *Define*

$$\mathcal{N} = \{y \in \mathbb{R}^N : Y(0) = y \Rightarrow Y_i(t) = 0 \textit{ for all } t \in \mathbb{R}, i > M\}.$$

The following result gives a more useful characterisation of $\mathcal{N}$.

Lemma 3.7.

$$\mathcal{N} = \{y \in \mathbb{R}^N : H^n y_i = 0 \textit{ for all } n \geqslant 0, i > M\}.$$

Proof. By definition of the generator of an SDE or ODE, if $|Y(t)| = |Y(0)| = R$,

$$\begin{aligned}\left|\frac{d^n}{dt^n} Y_i(t)\right| &= |H^n(Y(t))Y_i(t)| \\ &\leqslant |R|^{n+1} \left(\max_{i,j,k} |c^i_{jk}|\right)^n \prod_{k=1}^{n} \left(kN^2\right) \\ &\leqslant n!|R|^{n+1} \left(N^2 \max_{i,j,k} |c^i_{jk}|\right)^n,\end{aligned}$$

where in the case of $n = 0$ the product term is equal to 1. This is proved by induction. As $y^T C^i y = \sum_{j,k} c^i_{jk} y_j y_k$, for any $n \geqslant 0$ $H^n y_i$ consists of polynomials of order $n+1$ with coefficients which are the product of n c^i_{jk} terms. When H is applied to these terms the product rule separates each term into up to $n+1$ polynomials (if we apply the product rule and not the chain rule to x_i^k, $k > 1$) of order n which are then multiplied by the sum of up to N^2 quadratics with coefficients c^i_{jk}, giving us up to $(n+1)N^2$ times as many terms as before with coefficients which are the product of $n+1$ c^i_{jk} terms. By induction we get the required inequality. Therefore $Y_i(t) : \mathbb{R} \to \mathbb{R}$ is a real-analytic function for any initial condition $Y(0)$, and so

$$\begin{aligned}&\{y : Y(0) = y \Rightarrow Y_i(t) = Y_i(0) \text{ for all } t \in \mathbb{R},\ i > M\} \\ &= \left\{y : \left.\frac{d^n}{dt^n}\right|_{t=0} Y_i(t) = 0 \text{ for all } n \geqslant 0,\ i > M\right\} \\ &= \{y : H^n y_i = 0 \text{ for all } n \geqslant 0,\ i > M\}.\end{aligned}$$

□

See that, given an initial condition on $\mathcal{N}$, it is not necessarily true that $Y_i(t)$ is constant for $i \leqslant M$; the only guarantee is that $Y(t)$ will not leave $\mathcal{N}$. As having such degenerate movement will prove to be a problem later, we make the following simplifying assumption.

Assumption 3.2.

$$\mathcal{N} = \left\{ y \in \mathbb{R}^M \times \{0\}^{N-M} : \left.\frac{dY_i}{dt}\right|_{t=0} = 0 \text{ for all } i \leqslant N \right\}.$$

Similar to Lemma 3.7, we have the following result.

Lemma 3.8.

$$\mathcal{N} = \left\{ y \in \mathbb{R}^M \times \{0\}^{N-M} : Hy_i = 0 \text{ for all } i \leqslant N \right\}.$$

Although a seemingly weak condition, there are counter-examples; see Example 5.2.

We want to see what happens when $Y(t)$ is near $\mathcal{N}$. In this case by letting $Y = Y - \overline{Y} + \overline{Y}$ where $\overline{Y}$ is a point on $\mathcal{N}$ close to Y,

$$\begin{aligned}
\frac{dY_i}{dt} &= \left(Y - \overline{Y} + \overline{Y}\right)^T C^i \left(Y - \overline{Y} + \overline{Y}\right) \\
&= 2\overline{Y}^T C^i \left(Y - \overline{Y}\right) + \left(Y - \overline{Y}\right)^T C^i \left(Y - \overline{Y}\right) \\
&= 2\overline{Y}^T C^i Y + \left(Y - \overline{Y}\right)^T C^i \left(Y - \overline{Y}\right) \\
\Rightarrow \frac{dY}{dt} &= 2\left(\overline{Y}^T C^i\right)_{i \leqslant N} Y + \left(\left(Y - \overline{Y}\right)^T C^i \left(Y - \overline{Y}\right)\right)_{i \leqslant N}.
\end{aligned} \tag{3.1}$$

The first term is the linearisation of the system at $\overline{Y}$.

In the linear deterministic case $\dot{y} = Ay$, to show exit from a neighbourhood of the nullspace of A one shows the existence of an eigenvalue λ with real positive part and left-eigenvector v of A to show that $|v \cdot y| = |v \cdot (y - \mathrm{Proj}_{N(A)}(y))|$ grows exponentially with rate $\mathrm{Re}(\lambda)$. This motivates our second assumption.

Assumption 3.3. *There exist values λ_{min}, $P\sigma^2_{min} > 0$ such that for all $x \in \mathcal{N}\backslash\{0\}$ the matrix $\left(x^T C^i\right)_{i \leqslant N}$ is diagonalisable and has an eigenvalue $\lambda(x)$ and projection matrix $P(x)$ onto the left-eigenspace such that*

(a) $Re(\lambda(x)) \geqslant \lambda_{min}|x|$,

(b) $\sum_{i=1}^{N} |P(x)\, e_i|^2 \sigma_i^2 \geqslant P\sigma^2_{min}$.

Remark 3.9. This assumption can be seen as the generalisation of Assumption 2.1. That $\left(x^T C^i\right)_{i \leqslant N}$ needs to be diagonalisable is due to a subtlety in existing work in matrix perturbation theory and is justified in Remark B.4.

It is likely clear that there are a number of issues to be overcome here. First, as described this $\overline{Y}$ would be best formalised as a projection of Y onto $\mathcal{N}$, however $\mathcal{N}$ is the solution set to multi-variable polynomials and in general doesn't have a well-defined projection globally. Once this problem has been overcome, we then have a matrix that is a function of a vector x in $\mathbb{R}^N$, so that the eigenpairs will also be functions of x. Solving these problems require a detour into the fields of algebraic geometry and matrix perturbation theory. We leave this work to Appendix B, the main result of which is stated here.

Theorem B.14. *Given any matrix $A(x) \in \mathbb{R}^{n \times n}$ whose entries are Nash functions defined on some semi-algebraic $S \subset \mathbb{R}^N$, there exists a stratification of S into Nash* *manifolds $\{S_j\}_j$, on which each eigenvalue is of constant multiplicity with eigenspace of constant dimension, and each λ_i and the projection matrix P_i onto its eigenspace are Nash.*

Moreover, for any semi-algebraic set $A \subset \mathbb{R}^{2+2n^2}$, if it holds for all $x \in S$ that some pair $(\lambda_i(x), P_i(x)) \in A$, then the above stratification can be refined so that on each S_j, there is some i such that $(\lambda_i(x), P_i(x)) \in A$ for all $x \in S_j$.

The applications of this theorem to our case are thus. First, $\mathcal{N}$ is the intersection of the zeroes of a countable number of polynomials. As the space of polynomials in N variables over the reals is a Noetherian Ring by the Hilbert Basis Theorem, $\mathcal{N}$ is the intersection of the zeroes of a finite number of polynomials and so is an algebraic set. The matrix $\left(x^T C^i\right)_{i \leqslant N}$ is then linear in x. Finally if we let $A_{\alpha,\beta} = \left\{(\lambda, P) \in \mathbb{R}^{2+N^2} : \lambda_1 \geqslant \alpha \text{ and } \sum_{i=1}^N |Pe_i|^2 \sigma_i^2 \geqslant \beta\right\}$ for $\alpha, \beta > 0$ then $A_{\alpha,\beta}$ is a semi-algebraic set. We now apply the work of Appendix B to our particular case, with some additional work.

Theorem 3.10. *Given Assumption 3.3, there exists a stratification of $\mathcal{N}\backslash\{0\}$ into a finite number of Nash manifolds $\{S_j\}_j$ such that $x \in S_j \Rightarrow ax \in S_j$ for all $a > 0$. Moreover for each S_j there is an open neighbourhood U_j, also satisfying $x \in U_j \Rightarrow ax \in U_j$ for all $a > 0$ on which $\rho_j(x) = \operatorname{argmin}_{y \in S_j} |x - y|$, the projection of x onto S_j, is well defined and Nash, and defined on U_j there exist a collection of eigenvalues $\lambda_{i_1}(x), \ldots, \lambda_{i_k}(x)$ for the matrix $\left(\rho_j(x)^T C^i\right)_{i \leqslant N}$ and a projection matrix $P(x)$ onto the*

Minkowski sum of the $\lambda_{i_1}, \ldots, \lambda_{i_k}$-left eigenspaces that is Nash on U_j and such that

$$\min_{m \leqslant k} \mathrm{Re}(\lambda_{i_m}) \geqslant \frac{\lambda_{min}}{2}|x|,$$

$$\sum_{i=1}^{N} |P(x)e_i|^2 \sigma_i^2 \geqslant \frac{P\sigma_{min}^2}{2},$$

$$\left\| \frac{d^k}{dx_i^k} P(x) \right\| \leqslant D_k |x|^{-k}, \quad k \geqslant 0$$

on U_j for some sequence of constants D_k with respect to any matrix norm $\| \cdot \|$.

Proof. Apply Theorem B.14 to the matrix $\left(x^T C^i\right)_{\leqslant N}$ defined on $\mathbb{R}^N$ with the semi-algebraic set $A_{\lambda_{min}, P\sigma_{min}^2}$, and refine the stratification so that $\mathcal{N} \cap \{x : |x| = 1\}$ is a union of strata to get a stratification $\{S_j''\}$. First for each S_j'' of lowest dimension, there exists an eigenvalue of multiplicity k, i.e. there exist $i_1, \ldots, i_k$ such that $\lambda'_{i_1}(x) = \lambda'_{i_2}(x) = \ldots = \lambda'_{i_k}(x) \neq \lambda'_l(x)$ for $l \notin \{i_1, \ldots, i_k\}$ for $x \in S_j''$, with projection matrices $P'_{i_1}(x) = \ldots = P'_{i_k}(x)$ for $x \in S_j''$, such that

$$\mathrm{Re}(\lambda'_{i_1}(x)) \geqslant \lambda_{min},$$

$$\sum_{i=1}^{N} |P'_{i_1}(x)e_i|^2 \sigma_i^2 \geqslant P\sigma_{min}^2$$

on S_j''. Furthermore as S_j'' is of lowest dimension

$$\inf_{x \in S_j''} \min_{l \notin \{i_1, \ldots, i_k\}} |\lambda'_l(x) - \lambda'_{i_1}(x)| = \alpha$$

for some $\alpha > 0$. Then let T_j consist of all $x \in \mathcal{N} \cap \{x : |x| = 1\}$ such that

$$\min_{m \leqslant k} \mathrm{Re}(\lambda'_{i_m}(x)) \geqslant \frac{\lambda_{min}}{2},$$

$$\min_{m \leqslant k} \sum_{i=1}^{N} |P'_{i_m}(x)e_i|^2 \sigma_i^2 \geqslant \frac{P\sigma_{min}^2}{2},$$

$$\min_{l \notin \{i_1, \ldots, i_k\},\, m \leqslant k} |\lambda'_l(x) - \lambda'_{i_m}(x)| \geqslant \frac{\alpha}{2}$$

and note that T_j is a semi-algebraic set that includes an open neighbourhood of S''_j in $\mathcal{N} \cap \{x : |x| = 1\}$. Let $\{V_p\}_p$ consist of all non-empty intersections $T_j \cap S''_n$ for some j, n. Now for each S''_j of next lowest dimension, consider instead $S''_j \backslash \bigcup_p V_p$ with eigenvalues $\lambda'_{i_1}(x) = \ldots = \lambda'_{i_k}(x) \neq \lambda'_l(x)$ for $l \notin \{i_1, \ldots, i_k\}$ on S''_j and projection matrices $P'_{i_1}(x) = \ldots = P'_{i_k}(x)$ on $S''_j \backslash \bigcup_p V_p$ such that

$$\mathrm{Re}(\lambda'_{i_1}(x)) \geqslant \lambda_{min},$$

$$\sum_{i=1}^{N} |P'_{i_1}(x)e_i|^2 \sigma_i^2 \geqslant P\sigma_{min}^2$$

on $S''_j \backslash \bigcup_p V_p$. Then similarly

$$\inf_{x \in S''_j \backslash \bigcup_p V_p} \min_{l \notin \{i_1, \ldots, i_k\}} |\lambda'_l(x) - \lambda'_{i_1}(x)| = \alpha$$

for some $\alpha > 0$ as $S''_j \backslash \bigcup_p V_p$ is separated from the strata of lower dimension, and let

T_j again consist of all $x \in \mathcal{N} \cap \{x : |x| = 1\}$ such that

$$\min_{m \leqslant k} \operatorname{Re}(\lambda'_{i_m}(x)) \geqslant \frac{\lambda_{min}}{2},$$

$$\min_{m \leqslant k} \sum_{i=1}^{N} |P'_{i_m}(x) e_i|^2 \sigma_i^2 \geqslant \frac{P\sigma^2_{min}}{2},$$

$$\min_{l \notin \{i_1, \ldots, i_k,\, m \leqslant k} |\lambda'_l(x) - \lambda'_{i_m}(x)| \geqslant \frac{\alpha}{2}.$$

Again T_j is a semi-algebraic set that includes an open neighbourhood of $S''_j \backslash \bigcup_p V_p$ in $\mathcal{N} \cap \{x : |x| = 1\}$. Add to the collection $\{V_p\}$ all non-empty intersections $T_j \cap S''_n$ for some j, n. Continue this process through S''_j of all but highest dimension, and then add to the collection $\{V_p\}$ all sets of the form $S''_n \backslash \bigcup_j V_p$. Then each semi-algebraic V_p is a subset of some S''_n, and on each V_p either there exists either a single eigenvalue λ_{i_1} with projection matrix $P' = P_{i_1}$, or a collection of eigenvalues $\lambda_{i_1}, \ldots, \lambda_{i_k}$ with total projection matrix $P'(x)$ that satisfy

$$\min_{m \leqslant k} \operatorname{Re}(\lambda_{i_m}(x)) \geqslant \frac{\lambda_{min}}{2},$$

$$\sum_{i=1}^{N} |P'(x) e_i|^2 \sigma_i^2 \geqslant \frac{P\sigma^2_{min}}{2}$$

on V_p as $|P'(x)e_i| \geqslant |P_{i_m}(x)e_i|$ for any i, m, and such that on the boundary of V_p the eigenvalues do not collide with any eigenvalues from outside this collection. Therefore $P'(x)$ is Nash on an open neighbourhood of the closure of V_p in $\mathbb{R}^N$ by Lemma B.3. As V_p is compact this implies that the derivatives of $P'(x)$ are bounded on V_p.

Now consider a stratification of $\mathcal{N} \cap \{x : |x| = 1\}$ so that each V_p is a union of strata to get the stratification $\{S_j'\}_j$. For a fixed S_j' of dimension d and $x \in S_j'$, by Proposition B.9 (b) there are open semi-algebraic $V' \ni x$ in $\mathbb{R}^N$, $U' \subset \mathbb{R}^d$ and a Nash function $f : U' \to V'$ that is a homemorphism onto $S_j' \cap V'$ and such that $Df = \left(\frac{df_i}{dx_k}\right)_{i \leqslant N, k \leqslant d}$ is injective. Then let $U = U' \times (0, \infty)$, let $V = \{ax : a > 0,\ x \in V'\}$, let $S_j = \{ax : a > 0,\ x \in S_j'\}$ and let $g(x, a) = af(x)$, which is Nash and a homeomorphism onto $S_j \cap V$ with continuous inverse $g^{-1}(z) = \left(f^{-1}\left(\frac{z}{|z|}\right), |z|\right)$. Furthermore

$$Dg = \begin{pmatrix} aDf & f \end{pmatrix}$$

which is injective as for any $(y, b) \in \mathbb{R}^{d+1}$, $(Df)\, y,\ fb$ are respectively tangent and normal to the unit sphere. Therefore each S_j is a Nash manifold of dimension $d + 1$ and the collection $\{S_j\}_j$ is a stratification as

$$\begin{aligned} & S_j \cap \operatorname{clos}(S_k) \neq \varnothing \\ \Rightarrow & S_j' \cap \operatorname{clos}(S_k') \neq \varnothing \\ \Rightarrow & S_j' \subset \operatorname{clos}(S_k') \\ \Rightarrow & S_j \subset \operatorname{clos}(S_k). \end{aligned}$$

Let U_j be the neighbourhood of S_j on which the projection ρ_j onto S_j is well-defined and Nash. For any $x \in U_j$, $a > 0$ we can define $\rho_j(ax) = a\rho_j(x)$ so that U_j can be chosen so that $x \in U_j \Rightarrow ax \in U_j$ for any $a > 0$.

Finally, let $\lambda_{i_m}(x) = \lambda'_{i_m}(\rho_j(x))$, $P(x) = P'(\rho_j(x))$ be the eigenvalues and total projection matrix for $\left(\rho_j(x)^T C^i\right)_{i \leqslant N}$ on S_j. Then $\lambda_i(ax) = a\lambda_i(x)$ and $P(ax) = P(x)$

for $a > 0$, and as all derivatives of all orders of projections onto submanifolds of bounded curvature are bounded all derivatives of P are bounded and this, together with $P(ax) = P(x)$ gives

$$\left\| \frac{d^k}{dx_i^k} P(x) \right\| \leqslant D_k |x|^{-k}$$

for some sequence of constants D_k. □

3.4 Further Analysis of the Projection and Eigenpair

Given S_j, further analyses of the projection onto S_j and the eigenpair is necessary to understand their behaviour under the action of the operators H and L. For ease of notation consider some fixed Nash submanifold S with neighbourhood U and collection of eigenvalues $\lambda_1, \ldots, \lambda_k$ for $\left(\rho(x)^T C^i\right)_{i \leqslant N}$ with total projection matrix P as described in Theorem 3.10.

We need show that $P(x)$ has the property we desire. First consider the following result in a more simple setting.

Lemma 3.11. *If $x(t)$ satisfies the system of ODEs $\dot{x} = Ax$ and P is a projection onto the Minkowski sum of the $\lambda_{i_1}, \ldots, \lambda_{i_k}$-left eigenspaces of A with $0 < a \leqslant \mathrm{Re}(\lambda_m) \leqslant b$ for all m, then*

$$\frac{d}{dt}|Px| = \frac{\mathrm{Re}(Px)\mathrm{Re}(PAx) + \mathrm{Im}(Px)\mathrm{Im}(PAx)}{|Px|} = c(x)|Px|$$

for some function c satisfying $a \leqslant c(x) \leqslant b$ for all x.

Proof. The first equality is immediate. Let $v_1, \ldots, v_l$ be an orthogonal basis for the range of P. Then

$$\frac{d}{dt}|Px| = \frac{\sum_{j=1}^{l} \left[\mathrm{Re}(v_j^T x)\mathrm{Re}(v_j^T Ax) + \mathrm{Im}(v_j^T x)\mathrm{Im}(v_j^T Ax)\right]}{|Px|}.$$

For notational convenience consider a fixed $v_j = v = w_1 + \ldots + w_l$ for some eigenvectors $w_1, \ldots, w_l$ with corresponding eigenvalues $\lambda_{i_1}, \ldots, \lambda_{i_l}$. Then

$$\begin{aligned}
&\mathrm{Re}(v^T x)\mathrm{Re}(v^T Ax) + \mathrm{Im}(v^T x)\mathrm{Im}(v^T Ax) \\
=&\left(\sum_j \mathrm{Re}(w_j^T x)\right)\left(\sum_j \mathrm{Re}(w_j^T Ax)\right) + \left(\sum_j \mathrm{Im}(w_j^T x)\right)\left(\sum_j \mathrm{Im}(w_j^T Ax)\right) \\
=&\left(\sum_j \mathrm{Re}(w_j^T x)\right)\left(\sum_j \mathrm{Re}(\lambda_j w_j^T x)\right) + \left(\sum_j \mathrm{Im}(w_j^T x)\right)\left(\sum_j \mathrm{Im}(\lambda_j w_j^T x)\right) \\
=&\left(\sum_j \mathrm{Re}(w_j^T x)\right)\left(\sum_j \mathrm{Re}(\lambda_{i_j})\mathrm{Re}(w_j^T x) - \mathrm{Im}(\lambda_{i_j})\mathrm{Im}(w_j^T x)\right) \\
&+ \left(\sum_j \mathrm{Re}(w_j^T x)\right)\left(\sum_j \mathrm{Re}(\lambda_{i_j})\mathrm{Im}(w_j^T x) + \mathrm{Im}(\lambda_{i_j})\mathrm{Re}(w_j^T x)\right) \\
=&\left(\sum_j \mathrm{Re}(w_j^T x)\right)\left(\sum_j \mathrm{Re}(\lambda_{i_j})\mathrm{Re}(w_j^T x)\right) + \left(\sum_j \mathrm{Re}(w_j^T x)\right)\left(\sum_j \mathrm{Re}(\lambda_{i_j})\mathrm{Re}(w_j^T x)\right) \\
=&p\left(\sum_j \mathrm{Re}(w_j^T x)\right)^2 + q\left(\sum_j \mathrm{Im}(w_j^T x)\right)^2 \\
=&r|v^T x|^2
\end{aligned}$$

for some constants $p, q, r \in [a, b]$. Therefore

$$\frac{d}{dt}|Px| = \frac{\sum_{j=1}^{l} r_j |v_j^T x|^2}{|Px|} = c|Px|$$

for some $c \in [a, b]$ dependent on x. □

Lemma 3.12. *There exists a constant E, dependent only on $\{C^i\}_{i \leqslant N}$ and N, such that*

$$H|P(x)\,x| = c(x)|P(x)\,x| + g(x)$$

for some $g(x)$ satisfying $|g(x)| \leqslant E|x - \rho(x)|^2$ and for some $c(x)$ satisfying $\lambda_{min}|x| \leqslant c(x) \leqslant 2\lambda_{max}|x|$, where

$$\lambda_{max} = \sup_{x \in \mathcal{N}, |x|=1} \max_i \mathrm{Re}(\lambda_i(x)).$$

.

Proof. Similar to (3.1) we have

$$\begin{aligned} H(P(x)\,x) =& P(x)\,Hx + (HP(x))\,x \\ =& P(x)\left[2\left(\rho(x)^T C^i\right)_{i \leqslant N} x + \left((x - \rho(x)^T C^i (x - \rho(x))\right)_{i \leqslant N}\right] + (HP(x))\,x \\ :=& 2P(x)\left(\rho(x)^T C^i\right)_{i \leqslant N} x + h(x). \end{aligned}$$

Then

$$\begin{aligned} H\mathrm{Re}(P(x)\,x) =& 2\mathrm{Re}(P(x)\left(\rho(x)^T C^i\right)_{i \leqslant N} x) + \mathrm{Re}(h(x)), \\ H\mathrm{Im}(P(x)\,x) =& 2\mathrm{Im}(P(x)\left(\rho(x)^T C^i\right)_{i \leqslant N} x) + \mathrm{Im}(h(x)), \end{aligned}$$

so that

$$
\begin{aligned}
&H|P\,(x)\,x| \\
=&\frac{\mathrm{Re}(P\,(x)\,x)H\mathrm{Re}(P\,(x)\,x)+\mathrm{Im}(P\,(x)\,x)H\mathrm{Im}(P\,(x)\,x)}{|P\,(x)\,x|} \\
=&2\frac{\mathrm{Re}(P\,(x)\,x)\mathrm{Re}\left(P(x)\left(\rho(x)^TC^i\right)_{i\leqslant N}x\right)+\mathrm{Im}(P\,(x)\,x)\mathrm{Im}\left(P(x)\left(\rho(x)^TC^i\right)_{i\leqslant N}x\right)}{|P\,(x)\,x|} \\
&+\frac{\mathrm{Re}(P\,(x)\,x)\mathrm{Re}(h(x))+\mathrm{Im}(P\,(x)\,x)\mathrm{Im}(h(x))}{|P\,(x)\,x|} \\
=&c(x)|P\,(x)\,x|+g(x)
\end{aligned}
$$

by Lemma 3.11 where

$$|g(x)|=\left|\frac{\mathrm{Re}(P\,(x)\,x)\mathrm{Re}(h(x))+\mathrm{Im}(P\,(x)\,x)\mathrm{Im}(h(x))}{|P\,(x)\,x|}\right|\leqslant|h(x)|$$

by the Cauchy-Schwarz inequality. To show that $|h(x)|\leqslant E|x-\rho(x)|^2$ for some E it suffices to show that $|(HP(x))x|\leqslant E'|x-\rho(x)|^2$. As $\rho(x)\in S$, $H\rho(x)$ is in the tangent plane of S at $\rho(x)$, and as $P(x)w=0$ for any w in the tangent plane of S including $\rho(x)$ itself,

$$
\begin{aligned}
P(x)\rho(x)&=0 \\
\Rightarrow H(P(x)\rho(x))&=0 \\
\Rightarrow (HP(x))\,\rho(x)+P(x)H\rho(x)&=0 \\
\Rightarrow (HP(x))\,\rho(x)&=0
\end{aligned}
$$

so that

$$|(HP(x))x|=|(HP(x))(x-\rho(x))|\leqslant E'|x-\rho(x)|^2$$

due to the bounds on the derivatives of $P(x)$. □

Remark 3.13. It is the $O\left(|x-\rho(x)|^2\right)$ term, absent in the analogous result in Section 2, which significantly alters our calculations in Section 3.5. This is the main reason that Theorem 3.5 is significantly weaker than Theorem 2.2.

To move easily between derivatives of $|P(x)\,x|$ and $P(x)\,x$, we will prove the following.

Lemma 3.14. *For any C^2 function $f:\mathbb{R}^n\to\mathbb{C}$,*

$$\left|\frac{d}{dx_i}|f(x)|\right| \leqslant \left|\frac{d}{dx_i}f(x)\right|,$$

$$\left|\frac{d^2}{dx_i^2}|f(x)|\right| \leqslant 2\frac{\left|\frac{d}{dx_i}f(x)\right|^2}{|f(x)|} + \left|\frac{d^2}{dx_i^2}f(x)\right|,$$

$$\left|\frac{d}{dx_i}|f(x)|^2\right| \leqslant 2|f(x)|\left|\frac{d}{dx_i}f(x)\right|,$$

$$\frac{d^2}{dx_i^2}|f(x)|^2 = 2\left|\frac{d}{dx_i}f(x)\right|^2 + g(x)$$

where $|g(x)| \leqslant 2|f(x)|\left|\frac{d^2}{dx_i^2}f(x)\right|$.

Proof.

$$\left|\frac{d}{dx_i}|f(x)|\right| = \left|\frac{\mathrm{Re}(f(x))\frac{d}{dx_i}\mathrm{Re}(f(x)) + \mathrm{Im}(f(x))\frac{d}{dx_i}\mathrm{Im}(f(x))}{|f(x)|}\right| \leqslant \left|\frac{d}{dx_i}f(x)\right|$$

by the Cauchy-Schwarz inequality.

$$\begin{aligned}
\left|\frac{d^2}{dx_i^2}|f(x)|\right| =&\left|\frac{d}{dx_i}\frac{\mathrm{Re}(f(x))\frac{d}{dx_i}\mathrm{Re}(f(x))+\mathrm{Im}(f(x))\frac{d}{dx_i}\mathrm{Im}(f(x))}{|f(x)|}\right| \\
=&\left|\frac{\left|\frac{d}{dx_i}f(x)\right|^2}{|f(x)|}+\frac{\mathrm{Re}(f(x))\frac{d^2}{dx_i^2}\mathrm{Re}(f(x))+\mathrm{Im}(f(x))\frac{d^2}{dx_i^2}\mathrm{Im}(f(x))}{|f(x)|}\right. \\
&\left.-\frac{\left[\mathrm{Re}(f(x))\frac{d}{dx}\mathrm{Re}(f(x))+\mathrm{Im}(f(x))\frac{d}{dx}\mathrm{Im}(f(x))\right]^2}{|f(x)|^3}\right| \\
\leqslant&2\frac{\left|\frac{d}{dx_i}f(x)\right|^2}{|f(x)|}+\left|\frac{d^2}{dx_i^2}f(x)\right|
\end{aligned}$$

by the triangle inequality and Cauchy-Schwarz. The third inequality is immediate from the first.

$$\begin{aligned}
\frac{d^2}{dx_i^2}|f(x)|^2 =&\frac{d^2}{dx_i^2}\left[\mathrm{Re}(f(x))^2+\mathrm{Im}(f(x))^2\right] \\
=&2\left(\frac{d}{dx_i}\mathrm{Re}(f(x))\right)^2+2\left(\frac{d}{dx_i}\mathrm{Im}(f(x))\right)^2 \\
&+2\mathrm{Re}(f(x))\frac{d^2}{dx_i^2}\mathrm{Re}(f(x))+2\mathrm{Im}(f(x))\frac{d^2}{dx_i^2}\mathrm{Im}(f(x)) \\
=&2\left|\frac{d}{dx_i}f(x)\right|^2+g(x)
\end{aligned}$$

where

$$g(x)=2\mathrm{Re}(f(x))\frac{d^2}{dx_i^2}\mathrm{Re}(f(x))+2\mathrm{Im}(f(x))\frac{d^2}{dx_i^2}\mathrm{Im}(f(x))$$

$$\Rightarrow |g(x)|\leqslant 2|f(x)|\left|\frac{d^2}{dx_i^2}f(x)\right|.$$

□

Similar to our work in Section 2, we are only interested in $P(x)x$ when $|P(x)x| = o(|x|)$ in $|x|$. For this reason we prove the following corollary.

Corollary 3.15. *If* $|x-\rho(x)| = o(|x|)$ *in* $|x|$,

$$\left|\frac{d}{dx_i}|P(x)x|\right| \leqslant |P(x)e_i| + o(1),$$

$$\left|\frac{d^2}{dx_i^2}|P(x)x|\right| \leqslant 2\frac{|P(x)e_i|^2 + o(1)}{|P(x)x|} + o(1),$$

$$\left|\frac{d}{dx_i}|P(x)x|^2\right| \leqslant 2|P(x)x|\Big(|P(x)e_i| + o(1)\Big),$$

$$\frac{d^2}{dx_i^2}|P(x)x|^2 = 2|P(x)e_i|^2 + o(1).$$

Proof. In this proof let $P_j(x)$ be the j-th column of $P(x)$. Then

$$\begin{aligned}\left|\frac{d}{dx_i}|P(x)x|\right| &\leqslant \left|\sum_j\left(\frac{d}{dx_i}P_j(x)\right)\cdot(x-\rho(x)) + P(x)e_i\right|\\ &\leqslant |P(x)e_i| + o(1).\end{aligned}$$

Then

$$\begin{aligned}\left|\frac{d^2}{dx_i^2}|P(x)x|\right| &\leqslant 2\frac{\left|\frac{d}{dx_i}P(x)x\right|^2}{|P(x)x|} + \left|\frac{d^2}{dx_i^2}P(x)x\right|\\ &\leqslant 2\frac{|P(x)e_i|^2 + o(1)}{|P(x)x|} + o(1).\end{aligned}$$

The third inequality is immediate, as is the fourth. □

As our construction and analysis of our function in Section 3.5 will inevitably be more complex than in Section 2, we will complete most of the calculations now. Here we use the spectral matrix norm

$$\parallel C^i \parallel = \sup_{v:|v|=1} |C^i v| = \sup_{v,w:|v|,|w|=1} |w^T C^i v|.$$

Corollary 3.16. *If* $|x - \rho(x)| = o(|x|)$ *in* $|x|$,

$$\left| H \sqrt{\sum_{i>M} x_i^2} \right| \leqslant 2|x||x - \rho(x)| \left(\sqrt{\sum_{i>M} \parallel C^i \parallel^2} + o(1) \right),$$

$$\left| \sum_{i>M} b_i x_i \frac{d}{dx_i} |P\,(x)\,x| \right| \leqslant \sqrt{\sum_{i>M} x_i^2} \left(\sqrt{\sum_{i>M} |P\,(x)\,e_i|^2 b_i^2} + o(1) \right),$$

$$\left| \sum_{i>M} b_i x_i \frac{d}{dx_i} \sqrt{\sum_{j>M} x_j^2} \right| \leqslant \max_{i>M} b_i \sqrt{\sum_{i>M} x_i^2},$$

$$\left| \sum_i \frac{\sigma_i^2}{2} \frac{d^2}{dx_i^2} \ln |P\,(x)\,x| \right| \leqslant \frac{2 \sum_i \sigma_i^2 |P\,(x)\,e_i|^2 + o(1)}{|P\,(x)\,x|^2} + \frac{o(1)}{|P\,(x)\,x|}$$

$$\sum_i \frac{\sigma_i^2}{2} \frac{d^2}{dx_i^2} |P\,(x)\,x|^2 = \sum_i \sigma_i^2 |P\,(x)\,e_i|^2 + o(1).$$

Proof.

$$\left| H\sqrt{\sum_{i>M} x_i^2} \right| = \left| \frac{\sum_{i>M} x_i x^T C^i x}{\sqrt{\sum_{i>M} x_i^2}} \right|$$

$$= \left| \frac{\sum_{i>M} x_i \left[2\rho(x)^T C^i (x-\rho(x)) + (x-\rho(x))^T C^i (x-\rho(x)) \right]}{\sqrt{\sum_{i>M} x_i^2}} \right|$$

$$\leqslant \left| \frac{2|x||x-\rho(x)| \sum_{i>M} x_i \left[\| C^i \| + o(1) \right]}{\sqrt{\sum_{i>M} x_i^2}} \right|$$

$$\leqslant 2|x||x-\rho(x)| \left(\sqrt{\sum_{i>M} \| C^i \|^2} + o(1) \right).$$

$$\left| \sum_{i>M} b_i x_i \frac{d}{dx_i} |P(x)\, x| \right| \leqslant \left| \sum_{i>M} b_i |x_i| \left(|P(x)\, e_i| + o(1) \right) \right|$$

$$\leqslant \sqrt{\sum_{i>M} x_i^2} \left(\sqrt{\sum_{i>M} b_i^2 |P(x)\, e_i|^2} + o(1) \right).$$

$$\left| \sum_{i>M} b_i x_i \frac{d}{dx_i} \sqrt{\sum_{j>M} x_j^2} \right| = \left| \frac{\sum_{i>M} b_i x_i^2}{\sqrt{\sum_{j>M} x_j^2}} \right|$$

$$\leqslant \max_{i>M} b_i \sqrt{\sum_{i>M} x_i^2}.$$

$$\left| \sum_i \frac{\sigma_i^2}{2} \frac{d^2}{dx_i^2} \ln |P(x)\, x| \right| \leqslant \sum_i \frac{\sigma_i^2}{2} \left[\frac{\left(\frac{d}{dx_i} |P(x)\, x| \right)^2}{|P(x)\, x|^2} + \frac{\left| \frac{d^2}{dx_i^2} |P(x)\, x| \right|}{|P(x)\, x|} \right]$$

$$\leqslant \frac{2\sum_i \sigma_i^2 |P(x)\, e_i|^2 + o(1)}{|P(x)\, x|^2} + \frac{o(1)}{|P(x)\, x|}.$$

$$\sum_i \frac{\sigma_i^2}{2}\frac{d^2}{dx_i^2}|P\left(x\right)x|^2 = \sum_i \sigma_i^2 |P\left(x\right)e_i|^2 + o(1)$$

is immediate. □

Many of the results of this section break down when Assumption 3.2 does not hold. To see a detailed discussion of this, see Example 5.3.

3.5 Constructing Functions Near $\mathcal{N}$

We first need to construct a function near each S_j. This work will be similar to that in Section 2, with the term $|v \cdot x|$ replaced with $|P\left(x\right)x|$, and as such we will not re-motivate its construction. However for now we will ignore scaling factors and some pseudo-mollifiers, instead adding them in Section 3.6.

Our main result in this section is as follows.

Lemma 3.17. *On each region $U_j \cap \{x : |x - \rho_j(x)| \leqslant B|x|^{\frac{1}{4}}\}$ for some $B > 0$, there exists a function W_j and a constant K, dependent only on $P\sigma_{min}^2, \lambda_{min}, \lambda_{max}$ such that*

$$LW_j \leqslant -1 + \text{l.o.t.},$$

$$|W_j| \leqslant K\frac{\ln|x|}{|x|} + \text{l.o.t.},$$

$$\left|\frac{d}{dx_i}W_j\right| \leqslant K\frac{\sqrt{\ln|x|}}{|x|^{\frac{1}{2}}} + \text{l.o.t.}$$

Again for notational convenience consider some fixed Nash submanifold S with neighbourhood U and collection of eigenvalues $\lambda_1, \ldots, \lambda_k$ for $\left(\rho(x)^T C^i\right)_{i\leqslant N}$ with total projection matrix P as described in Theorem 3.10. We assume throughout that $|x-\rho(x)| \leqslant B|x|^{\frac{1}{4}}$ and note that

$$|P\,(x)\,x| = |P\,(x)\,(x-\rho(x)| \leqslant |x-\rho(x)|,$$

$$\sqrt{\sum_{i>M} x_i^2} \leqslant |x-\rho(x)|.$$

Let

$$V_2 = \frac{1}{|x|} \ln\left(\frac{B|x|^{\frac{1}{4}}}{|P\,(x)\,x|}\right) f\left(\frac{|P\,(x)\,x|}{A|x|^{-\frac{1}{2}}\sqrt{\ln|x|}}\right)$$

where f is a pseudo-mollifier satisfying $f\,(z) = 0$ for $z \leqslant \frac{1}{2}$, $f(z) = 1$ for $z \geqslant 1$. When $f > 0$,

$$\begin{aligned}
LV_2 \leqslant & -\lambda_{min} f + \frac{E|x-\rho(x)|^2}{|x||P\,(x)\,x|} + \frac{|x-\rho(x)|\sqrt{\sum_{i>M}|P\,(x)\,e_i|^2 b_i^2}}{|x||P\,(x)\,x|} f \\
& + \frac{2\sum_{i>M}\sigma_i^2|P\,(x)\,e_i|^2}{|x||P\,(x)\,x|^2} f + \left(2\lambda_{max} + \frac{E|x-\rho(x)|^2}{A|x|^{\frac{1}{2}}\sqrt{\ln|x|}}\right)\ln\left(\frac{B|x|^{\frac{1}{4}}}{|P\,(x)\,x|}\right)|f'| \\
& + \frac{|x-\rho(x)|\sqrt{\sum_{i>M}|P\,(x)\,e_i|^2 b_i^2}}{A|x|^{\frac{1}{2}}\sqrt{\ln|x|}}\ln\left(\frac{B|x|^{\frac{1}{4}}}{|P\,(x)\,x|}\right)|f'| \\
& + \frac{\sum_i \sigma_i^2|P\,(x)\,e_i|^2}{2|x||P\,(x)\,x|}|f'|\frac{1}{A|x|^{-\frac{1}{2}}\sqrt{\ln|x|}} \\
& + \frac{\sum_i \sigma_i^2|P\,(x)\,e_i|^2}{2|x|}\ln\left(\frac{B|x|^{\frac{1}{4}}}{|P\,(x)\,x|}\right)|f''|\frac{|x|}{A^2\ln|x|} + \text{l.o.t.} \\
\leqslant & -\lambda_{min} f + \frac{3\lambda_{max}}{2}\ln|x||f'| + \text{l.o.t.}
\end{aligned}$$

Let

$$V_3 = \left(4A^2|x|^{-1}\ln|x| - |P(x)\,x|^2\right) h\left(\frac{|P(x)\,x|}{A|x|^{-\frac{1}{2}}\sqrt{\ln|x|}}\right)$$

where h is a pseudo-mollifier satisfying $h(z) = 1$ for $z \leqslant 1$, $h(z) = 0$ for $z \geqslant 2$ with $h(z) = (2-z)^3$ for z sufficiently close to 2. Then

$$\begin{aligned} LV_3 \leqslant & -2\lambda_{min}|x||P(x)\,x|^2 h + 2\left(E|x-\rho(x)|^2 \right. \\ & \left. +|x-\rho(x)|\sqrt{\sum_{i>M}|P(x)e_i|^2 b_i^2}\right)|P(x)\,x|h - P\sigma_{min}^2 h \\ & + \left(4A^2|x|^{-1}\ln|x| - |P(x)\,x|^2\right)|h'(z)|\left(-\frac{\lambda_{min}}{2}|x| \right. \\ & \left. + \frac{E|x-\rho(x)|^2 + |x-\rho(x)|\sqrt{\sum_{i>M}|P(x)e_i|^2 b_i^2}}{A|x|^{-\frac{1}{2}}\sqrt{\ln|x|}}\right) \\ & + \sum_i \sigma_i^2 |P(x)\,e_i|^2\left(\frac{3}{2}|h''| + 2|h'|\right) + \text{l.o.t.} \\ \leqslant & \left(-2\lambda_{min}|x||P(x)\,x|^2 + 2EB^2|x|^{\frac{1}{2}}|P(x)\,x| - P\sigma_{min}^2\right) h \\ & + \sum_i \sigma_i^2 |P(x)\,e_i|^2\left(\frac{3}{2}|h''| + 2|h'|\right) + \text{l.o.t.} \end{aligned}$$

Maximising over the quadratic in $|P(x)\,x|$ then gives us

$$LV_3 \leqslant \left(\frac{E^2B^4}{2\lambda_{min}} - P\sigma_{min}^2\right) h + \sum_i \sigma_i^2|P(x)\,e_i|^2\left(\frac{3}{2}|h''| + 2|h'|\right) + \text{l.o.t.},$$

so assume

$$B < \left(\frac{\lambda_{min}P\sigma_{min}^2}{E^2}\right)^4 \tag{3.2}$$

so that

$$LV_3 \leqslant -\frac{P\sigma_{min}^2}{2}h + \sum_i \sigma_i^2 |P(x)\, e_i|^2 \left(\frac{3}{2}|h''| + 2|h'|\right) + \text{l.o.t.}.$$

Note also that this supremum occurs when

$$\begin{aligned} |P(x)\, x| =& \frac{EB^2|x|^{\frac{1}{2}}}{2|x|\lambda_{min}} \\ =& O\left(|x|^{-\frac{1}{2}}\right) \end{aligned}$$

so that the quadratic is decreasing when $|P(x)\, x| \geqslant \frac{A\sqrt{\ln|x|}}{2|x|^{\frac{1}{2}}}$.

Now assume f is not constant. Then we know that $|P(x)\, x| \geqslant \frac{A}{2}|x|^{-\frac{1}{2}}\sqrt{\ln|x|}$ and $h = 1$ is constant so that

$$LV_3 \leqslant -\frac{\lambda_{min}}{2}A^2 \ln|x| + \text{l.o.t.}$$

Now assume h is not constant. Then we know that $|P(x)\, x| \geqslant A|x|^{-\frac{1}{2}}\sqrt{\ln|x|}$, so that

$$LV_3 \leqslant -\lambda_{min}A^2 \ln|x| h + \sum_i \sigma_i^2 |P(x)\, e_i|^2 \left(\frac{3}{2}|h''| + 2|h'|\right) + \text{l.o.t.}.$$

For large $|x|$ this term is negative until h is very small, and as $h(z) = (2-z)^3$ near

$z = 2$, multiplying the h' term by $A\sqrt{\ln|x|}$ gives

$$
\begin{aligned}
&-\lambda_{min}A^2 \ln|x|h + \sum_i \sigma_i^2 |P(x)\, e_i|^2 \left(\frac{3}{2}|h''| + 2|h'|\right) \\
\leqslant &\frac{1}{A\sqrt{\ln|x|}}\left[-\lambda_{min}\left(A\sqrt{\ln|x|}(2-z)\right)^3 + 6\sum_i \sigma_i^2 |P(x)\, e_i|^2 \left(A\sqrt{\ln|x|}(2-z)\right)^2\right. \\
&\left.+9\sum_i \sigma_i^2 |P(x)\, e_i|^2 \left(A\sqrt{\ln|x|}(2-z)\right)\right] \\
= &O\left(\frac{1}{\sqrt{\ln|x|}}\right).
\end{aligned}
$$

Now consider

$$CV_2 + DV_3.$$

We will deal with the following four situations, where $*$ denotes that the pseudomollifier is non-constant.

$$
\begin{aligned}
(f, h) = &(1, 0), \\
&(0, 1), \\
&(*, 1), \\
&(1, *).
\end{aligned}
$$

In each of the these situations we have, ignoring lower order terms,

$$L(CV_2 + DV_3) \leqslant -C\lambda_{min},$$

$$L(CV_2 + DV_3) \leqslant -\frac{DP\sigma_{min}^2}{2},$$

$$L(CV_2 + DV_3) \leqslant \ln|x| \left[\frac{3\lambda_{max}}{2}C - DA^2\lambda_{min}\right],$$

$$L(CV_2 + DV_3) \leqslant -C\lambda_{min}.$$

Therefore choosing $C = \frac{1}{\lambda_{min}}$, $D = \frac{2}{P\sigma_{min}^2}$, and

$$A > \sqrt{\frac{3\lambda_{max}C}{2\lambda_{min}D}},$$

and letting $W_j = CV_2 + DV_3$ on each U_j, $LW_j \leqslant -1$ on $U_j \cap \left\{x : |x - \rho(x)| \leqslant B|x|^{\frac{1}{4}}\right\}$. The other properties of W_j are easily checked and so we have proved Lemma 3.17.

3.6 Constructing a Single Function Near $\mathcal{N}$

Our idea from here is as follows. If we construct a function W_j near some S_j, it ceases to be useful when $|x - \rho_j(x)|$ is large, as we proved in Section 3.5. However $|x - \rho_j(x)|$ being large does not guarantee that $d(x, \mathcal{N})$ is large; we could just be near another S_k for some k. Therefore we will use the functions constructed in Section 3.5 to create a single function defined globally and useful near $\mathcal{N}$. Our main result here is as follows.

Lemma 3.18. *There exists a function $\widetilde{W}$, defined on all of $\mathbb{R}^N$, and a collection of* *constants* B_j, $\overline{K}$ *such that*

$$L\widetilde{W} \leqslant \begin{cases} -1 & \text{if } |x-\rho_j(x)| \leqslant \frac{B_j|x|^{\frac{1}{4}}}{2(\ln|x|)^{dim(S_j)}} \text{ for some } j \\ \overline{K}\ln|x| & \text{if } |x-\rho_j(x)| \in \left[\frac{B_j|x|^{\frac{1}{4}}}{2(\ln|x|)^{dim(S_j)}}, \frac{B_j|x|^{\frac{1}{4}}}{(\ln|x|)^{dim(S_j)}}\right] \text{ for some } j \end{cases}$$

$$\widetilde{W} = 0 \text{ if } |x-\rho_j(x)| > \frac{B_j|x|^{\frac{1}{4}}}{(\ln|x|)^{dim(S_j)}} \text{ for all } j,$$

$$|\widetilde{W}| \leqslant \overline{K}\frac{\ln|x|}{|x|},$$

$$\left|\frac{d}{dx_i}\widetilde{W}\right| \leqslant \overline{K}\frac{\sqrt{\ln|x|}}{|x|^{\frac{1}{2}}}.$$

The reason why we choose the bounds

$$|x-\rho_j(x)| \sim \frac{|x|^{\frac{1}{4}}}{(\ln|x|)^{\dim(S_j)}}$$

rather than the bound

$$|x-\rho_j(x)| \sim |x|^{\frac{1}{4}}$$

that we assumed in Section 3.5 will be explained in Remark 3.20.

For each j, let $E_j = \{i : S_i \subset \operatorname{clos}(S_j)\}$ be the indices of the lower-dimensional neighbours of S_j, let

$$U'_j = U_j \cap \left\{x : |x-\rho_j(x) \leqslant \frac{B_j|x|^{\frac{1}{4}}}{(\ln|x|)^{\dim(S_j)}}\right\},$$

and let

$$\overline{W}_j(x) = \begin{cases} W_j(x) g\left(\frac{|x-\rho_j(x)|}{B_j \frac{|x|^{\frac{1}{4}}}{(\ln|x|)^{\dim(S_j)}}}\right) \prod_{i\in E_j} f\left(\frac{|x-\rho_i(x)|}{B_i \frac{|x|^{\frac{1}{4}}}{(\ln|x|)^{\dim(S_i)}}}\right) & x \in U'_j \\ 0 & \text{otherwise} \end{cases}$$

where f and g are pseudo-mollifiers satisfying $f\left(\frac{1}{4}\right) = 0$, $f\left(\frac{1}{2}\right) = 1$, $g\left(\frac{1}{2}\right) = 1$, $g(1) = 0$. For notational convenience let

$$f_i = f\left(\frac{|x-\rho_i(x)|}{B_i \frac{|x|^{\frac{1}{4}}}{(\ln|x|)^{\dim(S_i)}}}\right),$$

and on the region where $\rho_i(x)$ is ill-defined or not Nash we let $f_i = 1$.

Some explanation is required. Given some S_j with function W_j, when $|x-\rho_j(x)|$ is large W_j ceases to be useful and so we add the pseudo-mollifier g so that the function then vanishes. Similarly, $W_j(x)$ is not well-defined on ∂U_j, the boundary of U_j, in particular when the projection onto S_j is no longer well-defined. If we are close to this region, which can be informally characterised by $\rho_j(x)$ being close to ∂S_j, then by the nature of the stratification $\rho_j(x)$ is close to some other submanifold S_k of lower dimension. Therefore by making W_j vanish near the lower-dimensional neighbours of S_j the resulting function $\overline{W}_j$ can be made twice continuously differentiable globally.

We first need show that $\overline{W}_j$ is C^2 for large enough $|x|$. By the nature of f_i, g we need only check that $\overline{W}_j$ is C^2 where f_i or g cease to be well-defined. If $\{x_n\}_n$ is a sequence in U'_j converging to $x \in \partial U'_j$ with $\rho_j(x)$ not well-defined, then it must hold that $\rho_j(x_n)$ converges to some $y \in \partial S_j$. It follows from the nature of the Nash stratification that $y \in S_i$ for some $i \in E_j$ and so for large enough $|x|$ we are close enough to S_i that $\rho_i(x)$ is well-defined and Nash. If $g = 0$ then $\overline{W}_j$ is C^2 trivially, however if $g > 0$, then for $|x|$ large enough $f_i = 0$, so that $\overline{W}_j = 0$ is C^2 at x. We now adopt a similar strategy to prove smoothness on ∂U_i for i in $i \in E_j$. If $\{x_n\}_n$ is a sequence in U_i' converging to $x \in \partial U_i'$ with $\rho_i(x)$ not well-defined, then it must hold that $\rho_i(x_n)$ converges to some $y \in S_k$ for some $k \in E_i$ and so for large enough $|x|$ $\rho_k(x)$ is well-defined and Nash. If $f_i = 0$ then $\overline{W}_j$ is C^2 trivially, however if $f_i > 0$, then for $|x|$ large enough $f_k = 0$, so that $\overline{W}_j = 0$ is C^2 at x.

Remark 3.19. Although notationally heavy the concepts here are relatively straightforward. Consider the case of S_2 the open square in three dimensional space, with S_{11}, S_{12} its two one-dimensional open edges near a corner S_0. Imagine a small thickening around each submanifold that is significantly larger for those of lower dimension and such that the projection onto these submanifolds is well-defined; call these sets U_2', U_{11}', U_{12}', U_0'. Let $\overline{W}_2$ be the pseudo-mollified version of the function W_2 defined on U_2. As x approaches the boundary of any U_2, U_1^1, U_1^2, U_0 $\overline{W}_2$ vanishes. U_{11}', U_{12}', U_0' cover ∂S_2 and the boundary of U_2' near the boundary of S_2, so that $\overline{W}_2$ is C^2 globally. Similar arguments can be made in this setting for the functions $\overline{W}_{11}$, $\overline{W}_{12}$, $\overline{W}_0$.

First note that, for any ρ_i with $i \in E_j$ and $\rho_j(x)$ well-defined,

$$
\begin{aligned}
|H|x-\rho_i(x)|| &= \left| \frac{\sum_{i=1}^N x^T C^i x (x_i - \rho_i(x)) \frac{d}{dx_i}(x_i - \rho_i(x))}{|x-\rho_i(x)|} \right| \\
&\leqslant \sqrt{\sum_{i=1}^N (x^T C^i x)^2} \\
&= \sqrt{\sum_{i=1}^N \left(2\rho_j(x)^T C^i (x-\rho_j(x)) + (x-\rho_j(x))^T C^i (x-\rho_j(x))\right)^2} \\
&\leqslant 2|x||x-\rho_j(x)| \sqrt{\sum_{i=1}^N \| C^i \|^2} + \text{l.o.t.}
\end{aligned}
$$

Henceforth we define

$$
\| C \| = \sqrt{\sum_{i=1}^N \| C^i \|^2}.
$$

One can see that in the region U_j', where K is the constant given in Lemma 3.17,

$$
\begin{aligned}
L\overline{W}_j \leqslant & (LW_j) g \prod_{i \in E_j} f_i + K \frac{\ln|x|}{|x|}(Lg) \prod_{i \in E_j} f_i + K \frac{\ln|x|}{|x|} \sum_{k \in E_j} (Lf_k) g \prod_{i \in E_j,\, i \neq k} f_i \\
\leqslant & - g \prod_{i \in E_j} f_i + 2K \| C \| \, |g'(z)| \ln|x| \\
& + \sum_{k \in E_j} 2K \frac{B_j}{B_k} (\ln|x|)^{1+\dim(S_k)-\dim(S_j)} \| C \| \, |f_k'| g \prod_{i \in E_j,\, i \neq k} f_i + \text{l.o.t.}
\end{aligned}
$$

Furthermore, if $x \in U_l'$ for some l with $\dim(S_l) > \dim(S_j)$, then

$$
L\overline{W}_j \leqslant -g \prod_{i \in E_j} f_i + \frac{B_j}{B_l} 2 \| C \| \, |g'(z)| \prod_{i \in E_j} f_i + \text{l.o.t.}
$$

Remark 3.20. At this point it should be more clear why we chose the pseudo-mollifiers to change around the region

$$|x - \rho_i(x)| \sim \frac{|x|^{\frac{1}{4}}}{(\ln|x|)^{\dim(S_i)}},$$

rather than around $|x - \rho_i(x)| \sim |x|^{\frac{1}{4}}$. The functions W_j constructed in Section 3.5 satsify $LW_j \leqslant -1$ but $W_j \sim \frac{\ln|x|}{|x|}$, not $\frac{1}{|x|}$.

Now define $J_d = \{j : \dim(S_j) = d\}$, $\widetilde{W}_d = \sum_{j \in J_d} \overline{W}_j$, $T_d = \bigcup_{j \in J_d} S_j$, and commonly define $B_j = G_d$ if $\dim(S_j) = d$ for some constants G_d to be defined later with the assumption that G_d is a decreasing function of d. Then let

$$Q_d(x) = \frac{|d(x, T_d)|}{\frac{G_d|x|^{\frac{1}{4}}}{(\ln|x|)^d}}.$$

Note that

$$\bigcup_{j \in J_d} U_j' = \{x : Q_d(x) \leqslant 1\} \bigcap \bigcup_{j \in J_d} U_j.$$

Then

$$L\widetilde{W}_d \leqslant \begin{cases} \sum_{d'<d} \frac{|J_{d'}|G_d}{G_{d'}} 2K \parallel C \parallel \sup_z |f'(z)| & Q_d \leqslant \frac{1}{2}, Q_{d'} \in \left(\frac{1}{4}, \frac{1}{2}\right) \text{ for some } d' < d, \\ -1 & Q_d \leqslant \frac{1}{2},\ Q_{d'} > \frac{1}{2} \text{ for all } d' < d, \\ 2K \parallel C \parallel \ln|x| \sup_z |g'(z)| & Q_d \in \left(\frac{1}{2}, 1\right),\ Q_{d'} > \frac{1}{2} \text{ for all } d' > d, \\ \frac{G_{d+1}}{G_d} 2K \parallel C \parallel \sup_z |g'(z)| & Q_d \in \left(\frac{1}{2}, 1\right),\ Q_{d'} \leqslant \frac{1}{2} \text{ for some } d' > d, \\ 0 & \text{otherwise.} \end{cases}$$

$$\leqslant \begin{cases} |J| \frac{G_d}{G_{d-1}} 10K \parallel C \parallel & Q_d \leqslant \frac{1}{2}, Q_{d'} \in \left(\frac{1}{4}, \frac{1}{2}\right) \text{ for some } d' < d, \\ -1 & Q_d \leqslant \frac{1}{2},\ Q_{d'} > \frac{1}{2} \text{ for all } d' < d, \\ 10K \parallel C \parallel \ln|x| & Q_d \in \left(\frac{1}{2}, 1\right),\ Q_{d'} > \frac{1}{2} \text{ for all } d' > d, \\ \frac{G_{d+1}}{G_d} 10K \parallel C \parallel & Q_d \in \left(\frac{1}{2}, 1\right),\ Q_{d'} \leqslant \frac{1}{2} \text{ for some } d' > d, \\ 0 & \text{otherwise.} \end{cases}$$

where $J = \bigcup_d J_d$ so that $|J|$ is the number of submanifolds in the stratification of $\mathcal{N}$, and we assume that $\sup_z |f'(z)|, \sup_z |g'(z)| \leqslant 5$. Of course sharper bounds are possible in both cases, however as we have complete freedom over G_d it is unimportant. In the case of $d = 1$ we take the second line to be true rather than the first. If we then let $G_d = \alpha^d$ for some α such that

$$10|J|K \parallel C \parallel \alpha < \frac{1}{2},$$

then we have that

$$L\sum_d \widetilde{W}_d \leqslant \begin{cases} -\frac{1}{2} & Q_d \leqslant \frac{1}{2} \text{ for some } d, \\ 10K \parallel C \parallel \ln|x| & \text{otherwise.} \end{cases}$$

Letting

$$\widetilde{W} = 2\sum_d \widetilde{W}_d$$

completes the proof of Lemma 3.18, as the remaining properties of $\widetilde{W}$ follow trivially from Lemma 3.17.

3.7 Guaranteeing Motion in X_i, $i > M$

3.7.1 Motivation of the Approach and Main Result

The main result of this section is as follows, where the integer n^* is that given in Assumption 3.4.

Lemma 3.21. *Given any ϵ, B, there exists some function $\overline{W}$ and some α_0 such that, in the region where*

$$\sqrt{\sum_{i>M} x_i^2} \leqslant \alpha_0 \frac{B|x|^{\frac{1}{4}}}{(\ln|x|)^{2n*+dim(\mathcal{N})}},$$

$$d(x,\mathcal{N}) \geqslant \frac{B|x|^{\frac{1}{4}}}{(\ln|x|)^{dim(\mathcal{N})}},$$

we have

$$L\overline{W} \leqslant -1,$$

$$\left|\overline{W}\right| \leqslant \frac{\epsilon}{|x|\ln|x|},$$

$$\left|\frac{d}{dx_i}\overline{W}\right| = o\left(\frac{1}{|x|}\right).$$

Lemma 3.18 gives a function useful in an $O\left(\frac{|x|^{\frac{1}{4}}}{(\ln|x|)^{\dim(\mathcal{N})}}\right)$ neighbourhood of $\mathcal{N}$. Once we have left these regions we would like to be able to rely on quick movement of X_i for $i > M$ to guarantee that we quickly enter regions where one such X_i is large. Recalling the definition of $\mathcal{N}$, if we let $Y(0) = x \notin \mathcal{N}$ then we can only guarantee that one moment of $\frac{d^n}{dt^n}\big|_{t=0} Y_i(t) = H^n x_i$ is non-zero. The idea in this section is that a large $H^n x_i$ will cause fast growth of $H^{n-1}x_i$, which will in turn cause fast growth of $H^{n-2}x_i$, cascading all the way down to fast growth of $H^0 x_i = x_i$. Howevewr we need to formalise what we mean by large. To do this it will help to get some idea of what $H^n x_i$ looks like, beyond the rough bounds we had from the proof of Lemma 3.7. We know that $H^n x_i$ is an $n+1$-degree polynomial, and if we replace x with $x - y + y$ where y satisfies $d(x, \mathcal{N}) = d(x, y)$ which exists even if it is not unique, that $H^n x_i \leqslant E|x|^n|x-y| = E|x|^n d(x, \mathcal{N})$ for some constant E dependent only on c^i_{jk}, N and n.

3.7.2 Consequences of Assumption 3.4

For reasons that will become clear later, for every $x \notin \mathcal{N}$ yet still in the subspace $\{x : \max_{i>M} |x_i| = 0\}$ we want at least one of the $H^n x_i$ to be $O\left(|x|^n d(x, \mathcal{N})\right)$. More formally we want there to exist an $\alpha > 0$ such that

$$\inf_{x \notin \mathcal{N}} \sup_{n \geqslant 0} \max_{i>M} \frac{|H^n x_i|}{|x|^n d(x, \mathcal{N})} \geqslant \alpha.$$

Assumption 3.4 gives a necessary and sufficient condition for this to be the case.

Assumption 3.4. *For any $x \in \mathcal{N}$, with $A_x = (x^T C^i)_{i>M}$, $B_x = (x^T C^i)_{i \leqslant N}$, there exists an $n^* \geqslant 1$ such that*

$$\{z : \max_{i>M} |z_i| = 0\} \bigcap \bigcap_{n=0}^{n^*-1} \operatorname{Ker}(A_x B_x^n)$$

is equal to the tangent space of $\mathcal{N}$ at x.

Lemma 3.22. *Assumption 3.4 holds if and only if*

$$\inf_{x \notin \mathcal{N}} \sup_{n \leqslant n^*} \max_{i>M} \frac{|H^n x_i|}{|x|^n d(x, \mathcal{N})} \geqslant \alpha$$

for some $\alpha > 0$.

To see why Assumption 3.4 is useful, note that we are saying for all initial conditions $Y(0) = x$ away from $\mathcal{N}$ where $|x|$ is large, at least one of $\frac{d^n}{dt^n}\big|_{t=0} Y_i(t)$ has as large an order as we could expect it to have. To see a counter-example to Assumption 3.4, see Example 5.4.

Proof of Lemma 3.22. We first prove that for all $n \geqslant 2$, $i > M$,

$$H^n x_i = 2^{n-1} \sum_{j_1=1}^{N} \cdots \sum_{j_{n-1}=1}^{N} (C^i x)_{j_1} (C^{j_1} x)_{j_2} \dots (C^{j_{n-2}} x)_{j_{n-1}} x^T C^{j_{n-1}} x + R_n(x).$$

where $R_n(x)$ is a sum of terms of the form $f(x)(x^T C^j x)(x^T C^k x)$ where $f(x) = O\left(|x|^{n-2}\right)$ and $(C^i x)_j$ is the j-th term of the vector $C^i x$. See that for $n = 2$,

$$H(x^T C^i x) = \sum_{j=1}^{N} (x^T C^i + C^i x)_j x^T C^j x = 2 \sum_{j=1}^{N} (C^i x)_j x^T C^j x.$$

Now assume that the first formula holds for some n and apply H to the given formula for $H^n x^T C^i x$. Applying H to any element of $R_n(x)$ will give more elements of the form $f(x)(x^T C^j x)(x^T C^k x)$ where now $f(x) = O(|x|^{n-1})$ and so they can be considered part of $R_{n+1}(x)$. Applying H to the other term involves the product rule, and as one can see that $H(C^{j_m} x)_{j_{m+1}} = \sum_{k=1}^{n} c^{j_m}_{j_{m+1} k} x^T C^k x$ the only term that arises from applying the product rule that can't be considered a term in R_{n+1} is

$$2^{n-1} \sum_{j_1=1}^{N} \cdots \sum_{j_{n-1}=1}^{N} (C^i x)_{j_1} (C^{j_1} x)_{j_2} \dots (C^{j_{n-2}} x)_{j_{n-1}} H x^T C^{j_{n-1}} x$$

$$= 2^n \sum_{j_1=1}^{N} \cdots \sum_{j_{n-1}=1}^{N} (C^i x)_{j_1} (C^{j_1} x)_{j_2} \dots (C^{j_{n-2}} x)_{j_{n-1}} \sum_{j_n=1}^{N} (C^{j_{n-1}} x)_{j_n} x^T C^{j_n} x.$$

Then for all $n \geqslant 2$ we can now see that $\frac{d}{dx_j} H^n x_i$, evaluated at a point $x \in \mathcal{N}$ is equal to

$$
\begin{aligned}
&2^{n-1} \sum_{j_1=1}^{N} \cdots \sum_{j_{n-1}=1}^{N} (C^i x)_{j_1} (C^{j_1} x)_{j_2} \ldots (C^{j_{n-2}} x)_{j_{n-1}} 2 \left(C^{j_{n-1}} x\right)_j \\
=&2^n \sum_{j_1=1}^{N} \cdots \sum_{j_{n-1}=1}^{N} (A_x)_{i-M, j_1} (B_x)_{j_1, j_2} \ldots (B_x)_{j_{n-1}, j} \\
=&2^n \left(A_x B_x^{n-1}\right)_j .
\end{aligned}
$$

For $n = 1$,

$$
\frac{d}{dx_j} x^T C^i x = 2 \left(x^T C^i\right)_j = (A_x)_j .
$$

Therefore a linear approximation to the vector $\left(H^n x_{i+M}\right)_i$ at $x + y$ for $x \in \mathcal{N}$ and y small is given by $A_x B_x^{n-1} y$, completing the proof. □

3.7.3 Construction of a Function Away From $\mathcal{N}$

We will now construct a function in the region where

$$
\max_{i > M} |x_i| \leqslant \alpha_0 \frac{B|x|^{\frac{1}{4}}}{(\ln |x|)^{2n* + \dim(\mathcal{N})}},
$$

$$
d(x, \mathcal{N}) \geqslant \frac{B|x|^{\frac{1}{4}}}{(\ln |x|)^{\dim(\mathcal{N})}}
$$

for some constant B. Let

$$
g_n = \alpha_n \frac{|x|^n d(x)}{(\ln |x|)^{2n* - n}}
$$

for some $\alpha_n \leqslant \alpha$ to be decided later where $d(x)$ is a C^2 approximation to the distance function defined as follows. First note that $d_j(x) = \sqrt{1 + d(x, S_j)^2}$ is a C^2 function on each U_j. Then let

$$\widetilde{d}_j = d_j(x) \prod_{i \in E_j} f\left(x - \rho_i(x)\right)$$

for a pseudomollifier f satisfying $f(1) = 1$, $f(0) = 0$, where $f_i = 1$ if $\rho_i(x)$ is not well-defined or Nash and E_j are the lower-dimensional neighbours of S_j. Similar to the construction of $\overline{W}_j$ in Section 3.6, each $\widetilde{d}_j$ is C^2 on the set $\{x : d(x, S_j) \leqslant \epsilon|x|\}$ for some $\epsilon > 0$ and moreover has bounded derivatives. Therefore the function

$$\widetilde{d}(x) = -\ln\left(\sum_j e^{-\widetilde{d}_j(x)}\right)$$

is C^2 with bounded derivatives on the set $\{x : d(x, \mathcal{N}) \leqslant \epsilon|x|\}$ for some $\epsilon > 0$, and satisfies

$$\min_j \widetilde{d}_j(x) - \ln|J| \leqslant \widetilde{d}(x) \leqslant \min_j \widetilde{d}_j(x)$$

where $|J|$ is the number of submanifolds in the stratification of $\mathcal{N}$. Finally, as

$$d(x, \mathcal{N}) = \min_j \widetilde{d}_j(x)$$

when $d(x, \mathcal{N}) \geqslant 1$, $|\widetilde{d}(x) - d(x, \mathcal{N})|$ is bounded on the set $\{x : d(x, \mathcal{N}) \leqslant \epsilon|x|\}$. Then applying the Stone-Weierstrass Theorem to $d(x, \mathcal{N})$ on $\{x : |x| \leqslant 1, d(x, \mathcal{N}) \geqslant \epsilon'\}$ for some $\epsilon' < \epsilon$ to get a function $\overline{d}$ differing from $d(x, \mathcal{N})$ by no more than δ, let $\hat{d}(x) = |x|\overline{d}\left(\frac{x}{|x|}\right)$, and let

$$d(x) = \hat{d}(x) f\left(\frac{\widetilde{d}(x) - \epsilon'|x|}{\epsilon|x| - \epsilon'|x|}\right) + \widetilde{d}(x)\left(1 - f\left(\frac{\widetilde{d}(x) - \epsilon'|x|}{\epsilon|x| - \epsilon'|x|}\right)\right).$$

Then $|d(x) - d(x, \mathcal{N})| \leqslant A + \delta d(x, \mathcal{N})$ for some $A > 0$ where δ can be chosen as small as possible, which implies that $1 - \delta \leqslant \frac{d(x)}{d(x,\mathcal{N})} \leqslant 1 + \delta$ in the region where $d(x, \mathcal{N}) >> 1$ which is sufficient for our purposes.

The function g_n is supposed to represent the value at which we consider $H^n x_i$ to be large enough to cause $H^{n-1}x_i$ to change rapidly without changing too quickly itself due to more dominant dynamics. Remember that $|H^n x_i| \leqslant E|x|^n d(x, \mathcal{N})$ for all x where E is some constant dependent on c^i_{jk}, N and n. However as we will only consider $H^n x_i$ for $n \leqslant n^*$ we can actually choose E independent of n. In the case of $n = 0$ g_0 is when x_i is itself large enough that we can expect sizeable dissipation. An approximate exit time from the region $\{x : |H^n x_i| \leqslant g_n\}$ under the H dynamics would be given by

$$\tau_{i,n} = \frac{g_n - \operatorname{sgn}(H^{n+1}x_i)H^n x_i}{|H^{n+1}x_i|}.$$

However rather than trying to deal with $\min_i \tau_{i,n}$ or some approximation thereof we will use the function

$$W_n = \frac{-\sum_{i>M} \frac{H^{n+1}x_i}{g_{n+1}(x)} H^n x_i}{\sqrt{\sum_{i>M} \left(H^{n+1}x_i\right)^2}}.$$

In short, the $\frac{H^{n+1}x_i}{g_{n+1}}$ term replaces a more intuitive sgn function and greatly simplifies calculations, and as g_n is unimportant is is ignored. We will only consider W_n in the region where $\sqrt{\sum_{i>M} \left(H^{n+1}x_i\right)^2} > \frac{g_{n+1}(x)}{4}$ yet $\sqrt{\sum_{i>M} \left(H^m x_i\right)^2} \leqslant g_m(x)$ for all $m \leqslant n$.

See that

$$
\begin{aligned}
Hg_n &= \alpha_n |x|^n \sum_{i=1}^{N} x^T C^i x \frac{d}{dx_i} d(x) \\
&\leqslant 2\alpha_n |x|^{n+1} d(x) \sum_{i=1}^{n} \parallel C^i \parallel^2 \\
&= 2|x| g_n \parallel C \parallel^2 .
\end{aligned}
$$

Then by numerous applications of Cauchy-Schwarz,

$$
\begin{aligned}
HW_n = &- \frac{\sum_{i>M} \frac{H^{n+1}x_i}{g_{n+1}(x)} H^{n+1}x_i}{\sqrt{\sum_{i>M} \left(H^{n+1}x_i\right)^2}} \\
&+ \frac{\sum_{i>M} \frac{\left|H^{n+2}x_i\right|}{g_{n+1}(x)} \left|H^n x_i\right| + \frac{\left|H^{n+1}\right|}{g_{n+1}^2} \left|H^n x_i\right| 2 \parallel C \parallel |x| g_{n+1}}{\sqrt{\sum_{i>M} \left(H^{n+1}x_i\right)^2}} \\
&+ \frac{\sum_{i>M} \frac{\left|H^{n+1}x_i\right|}{g_{n+1}(x)} \left|H^n x_i\right|}{\left(\sum_{i>M} \left(H^{n+1}x_i\right)^2\right)^{\frac{3}{2}}} \sum_{i>M} \left|H^{n+1}x_i\right| \left|H^{n+2}x_i\right| \\
&\leqslant -\frac{1}{4} + \frac{4E\sqrt{N-M}|x|^{n+2} d(x) g_n}{g_{n+1}^2} + 2\sqrt{N-M} \parallel C \parallel^2 \frac{|x| g_n}{g_{n+1}} \\
&+ \frac{4\sqrt{N-M} E g_n |x|^{n+2} d(x)}{g_{n+1}^2} \\
\leqslant &- \frac{1}{4} + \frac{8\alpha_n E\sqrt{N-M}}{\alpha_{n+1}^2}.
\end{aligned}
$$

As $\frac{d}{dx_j} H^k x_i \leqslant E|x|^k$ for all j, k, the dissipation and noise terms provide lower order contributions to LW_n.

Finally, we need to add to pseudo-mollifiers to W_n so that it vanishes when H_n grows large enough and when H_{n+1} becomes too small. Therefore let

$$\overline{W}_n = f\left(\frac{\sqrt{\sum_{i>M}\left(H^{n+1}x_i\right)^2}}{g_{n+1}}\right)\prod_{k=1}^{n} h\left(\frac{\sqrt{\sum_{i>M}\left(H^{k}x_i\right)^2}}{g_k}\right)W_n$$

for pseudo-mollifiers f satisfying $f\left(\frac{1}{4}\right) = 0$, $f\left(\frac{1}{2}\right) = 1$, $h\left(\frac{1}{2}\right) = 1$, $h\left(1\right) = 0$, and where in the case of $n = 0$ the empty product is equal to 1. Then

$$L\overline{W}_n \leqslant \left(-\frac{1}{4} + \frac{8\alpha_n E\sqrt{N-M}}{\alpha_{n+1}^2}\right) f\prod_{k=1}^{n} h_k + \frac{g_n}{g_{n+1}}|f'|\prod_{k=1}^{n} h_k \frac{\sum_{i>M}\left|H^{n+1}x_i\right|\left|H^{n+2}x_i\right|}{g_{n+1}\sqrt{\sum_{i>M}\left(H^{n+1}x_i\right)^2}}$$

$$+\sum_k \frac{g_n}{g_{n+1}} f|h'|\prod_{l\neq k} h_l \frac{\sum_{i>M}\left|H^{k}x_i\right|\left|H^{k+1}x_i\right|}{g_k\sqrt{\sum_{i>M}\left(H^{k}x_i\right)^2}}$$

$$\leqslant \left(-\frac{1}{4} + \frac{8\alpha_n E\sqrt{N-M}}{\alpha_{n+1}^2}\right) f\prod_{k} h_k + \frac{\alpha_n}{\alpha_{n+1}}|f'|\prod_{k} h_k + f|h_n'|\prod_{k=1}^{n-1} h.$$

Then let

$$\overline{W} = \sum_{n=0}^{n^*-1} \beta_n \overline{W}_n.$$

If $h_k' \neq 0$, then $f_k = 1$ is constant and vice versa. Then

$$L\overline{W} \leqslant \sum_{n=1}^{n^*-1} \mathbb{1}_{h_n' \neq 0} \prod_{k=1}^{n-1} h_k \left(\beta_n f_{n+1} \sup_z |h'(z)| + \beta_{n-1} \left(-\frac{1}{4} + \frac{8E\sqrt{N-M}\alpha_{n-1}}{\alpha_n^2} \right) \right)$$

$$+ \sum_{n=1}^{n^*-1} \mathbb{1}_{f_n' \neq 0} \prod_{k=1}^{n-1} h_k \left(\frac{\beta_{n-1}\alpha_{n-1}}{\alpha_n} \sup_z |f'(z)| \right.$$

$$\left. + \sum_{k=n}^{n^*-1} f_{k+1} \prod_{m=n+1}^{k} h_m \beta_k \left(-\frac{1}{4} + \frac{8E\sqrt{N-M}\alpha_k}{\alpha_{k+1}^2} \right) \right)$$

$$+ \sum_{n=1}^{n^*} \mathbb{1}_{\text{All pseudo-mollifiers constant, } f_n=1, h_l=1 \text{ for all } l<n} \left(-\frac{\beta_n}{4} \right).$$

Given any $\beta_{n^*-1}, \alpha_{n^*} \leqslant \alpha$, we first note that for any $\epsilon > 0$ we can fix $\alpha_{n^*-1} \leqslant \alpha$ so that

$$\frac{\beta_{n^*-1}\alpha_{n^*-1}}{\alpha_{n^*}} \leqslant \epsilon. \tag{3.3}$$

This stipulation is unique to α_{n^*-1} and gives guarantees that $W \leqslant \frac{\epsilon}{|x|\ln|x|} + \text{l.o.t.}$ Then we can choose α_{n^*-1} small enough so that

$$\frac{8\alpha_{n^*-1}E\sqrt{N-M}}{\alpha_{n^*}^2} \leqslant \frac{1}{8}.$$

Then we can choose $\beta_{n^*-2} \geqslant 4$ so that

$$\beta_{n^*-1} \sup_z |h'(z)|E - \frac{\beta_{n^*-2}}{8} \leqslant -1.$$

Then we can choose $\alpha_{n^*-2} \leqslant \alpha$ small enough so that

$$\beta_{n^*-2}\frac{\alpha_{n^*-2}\sup_z |f'(z)|}{\alpha_{n^*-1}^2} - \frac{\beta_{n^*-1}}{8} \leqslant -1,$$

$$\frac{8\alpha_{n^*-2}E\sqrt{N-M}}{\alpha_{n^*-1}^2} \leqslant \frac{1}{8}.$$

Continuing by induction, given $\alpha_n \leqslant \alpha$ small enough so that

$$\frac{8\alpha_n E\sqrt{N-M}}{\alpha_{n+1}^2} \leqslant \frac{1}{8}.$$

Then we can choose $\beta_{n-1} \geqslant 4$ so that

$$\beta_n \sup_z |h'(z)| E - \frac{\beta_{n-1}}{8} \leqslant -1.$$

Then we can choose $\alpha_{n-1} \leqslant \alpha$ small enough so that

$$\beta_{n-1}\frac{\alpha_{n-1}\sup_z |f'(z)|}{\alpha_n^2} - \min_{k\geqslant n}\frac{\beta_k}{8} \leqslant -1,$$

down to β_0, α_0. Then

$$\begin{aligned} L\overline{W} \leqslant &- \sum_{n=1}^{n^*-1} \mathbb{1}_{h_n'\neq 0}\prod_{k=1}^{n-1} h_k \\ &- \sum_{n=1}^{n^*-1} \mathbb{1}_{f_n'\neq 0}\prod_{k=1}^{n-1} h_k \left(\frac{\beta_{n-1}\alpha_{n-1}}{\alpha_n}\sup_z |f'(z)| - \frac{\min_{k\geqslant n}\beta_k}{8}\sum_{k=n}^{n^*-1} f_{k+1}\prod_{m=n+1}^{k} h_m\right) \\ &- \sum_{n=1}^{n^*} \mathbb{1}_{\text{All pseudo-mollifiers constant, } f_n=1, h_l=1 \text{ for all } l<n}. \end{aligned}$$

We claim that the sum

$$\sum_{k=n}^{n^*-1} f_{k+1}\prod_{m=n+1}^{k} h_m \geqslant 1$$

if $h_k > 0$ for all $k \leqslant n-1$. Assuming $h_k > 0$ for all $k \leqslant n-1$, either $f_{n+1} = 1$, in which case we are done, or $h_{n+1} = 1$. Then if $f_{n+2} = 1$ we are again done, so assume $h_{n+2} = 1$ instead. Continuing like this, the only way this sum can be less than 1 is if

$h_k = 1$ for all $k \leqslant n^* - 1$. But then $f_{n*} = 1$ by Lemma 3.22 as each $\alpha_n \leqslant \alpha$ and so the sum is still greater than or equal to 1. So we have that

$$L\overline{W} \leqslant -1,$$

completing the proof of Lemma 3.21.

3.8 Constructing the Lyapunov Function

First let $\widetilde{W}$ be the function obtained from Lemma 3.18 with associated constants B_j, $\overline{K}$, and let $B = \min_j B_j$. Then for some $\epsilon > 0$, and B, obtain the function $\overline{W}$ from Lemma 3.21 with associated constant α_0. Then consider the function

$$V_2 = P\widetilde{W} f\left(\frac{d(x)}{\frac{B|x|^{\frac{1}{4}}}{(\ln|x|)^{\dim(\mathcal{N})}}}\right) + Q\ln|x|\overline{W} g\left(\frac{d(x)}{\frac{B|x|^{\frac{1}{4}}}{(\ln|x|)^{\dim(\mathcal{N})}}}\right)$$

where f, g are pseudo-mollifiers satisfying $f\left(\frac{1}{2}\right) = 1$, $f(1) = 0$, $g\left(\frac{1}{4}\right) = 0$, $g\left(\frac{1}{2}\right) = 1$ and P, Q are constants. Then

$$\begin{aligned} LV_2 \leqslant & \mathbb{1}_{f'=g'=0}\left(-Pf - Q\ln|x|g\right) + \ln|x|\mathbb{1}_{f'\neq 0}\left(P\sup_z |f'(z)| 2 \parallel C \parallel \overline{K} - Q\right) \\ & + \mathbb{1}_{g'\neq 0}\left(-P + \frac{\epsilon}{Q} 2 \parallel C \parallel \sup_z |g'(z)|\right). \end{aligned}$$

Given $P = 2$, choose Q so that

$$P\sup_z |f'(z)| 2 \parallel C \parallel \overline{K} - Q = -1.$$

Then choose ϵ so that

$$-2 + \frac{\epsilon}{Q} 2 \parallel C \parallel \sup_z |g'(z)| = -1.$$

Therefore

$$LV_2 \leqslant -1$$

in the region where

$$\sqrt{\sum_{i>M} x_i^2} \leqslant \frac{\alpha_0 B|x|^{\frac{1}{4}}}{(\ln|x|)^{2n^*+\dim(\mathcal{N})}}.$$

Finally let

$$p(z) = \frac{z^{\frac{1}{2}}}{(\ln z)^{2n^*+\dim(\mathcal{N})+1}}$$

and let

$$V = e^{ap(|x|)} + |\sigma|^2 \left(ap'(|x|)\right)^2 e^{ap(|x|)} V_2 h\left(\frac{\sqrt{\sum_{i>M} x_i^2}}{\frac{\alpha_0 B|x|^{\frac{1}{4}}}{(\ln|x|)^{2n^*+\dim(\mathcal{N})}}}\right)$$

where h is some pseudo-mollifier satisfying $h\left(\frac{1}{2}\right) = 1$, $h(1) = 0$. Then when $h = 0$

$$\begin{aligned} LV &\leqslant \left[\frac{|\sigma|^2}{2}\left(\frac{a}{2|x|^{\frac{1}{2}}(\ln|x|)^{2n^*+\dim(\mathcal{N})+1}}\right)^2 - \frac{a\sum_{i>M} b_i x_i^2}{2|x|^{\frac{3}{2}}(\ln|x|)^{2n^*+\dim(\mathcal{N})+1}}\right] e^{ap(|x|)} \\ &\leqslant \left[\frac{a^2\sigma|^2}{8|x|(\ln|x|)^{2n^*+1+2\dim(\mathcal{N})+2}} - \frac{a\alpha_0 B \min_{i>M} b_i}{2|x|(\ln|x|)^{2n^*+1+2\dim(\mathcal{N})+1}}\right] e^{ap(|x|)} \\ &= -\frac{a\alpha_0 B \min_{i>M} b_i}{2|x|(\ln|x|)^{2n^*+1+2\dim(\mathcal{N})+1}} e^{ap(|x|)}. \end{aligned}$$

When $h = 1$ is constant

$$LV \leqslant \left[\frac{|\sigma|^2}{2} \left(\frac{a}{2|x|^{\frac{1}{2}} (\ln|x|)^{2^{n^*}+\dim(\mathcal{N})+1}} \right)^2 \right.$$

$$\left. -|\sigma|^2 \left(\frac{a}{2|x|^{\frac{1}{2}} (\ln|x|)^{2^{n^*}+\dim(\mathcal{N})+1}} \right)^2 \right] e^{ap(|x|)}$$

$$= -\frac{|\sigma|^2}{2} \left(\frac{a}{2|x|^{\frac{1}{2}} (\ln|x|)^{2^{n^*}+\dim(\mathcal{N})+1}} \right)^2 e^{ap(|x|)}.$$

When $h' \neq 0$

$$LV \leqslant \left[2\overline{K}P \parallel C \parallel \sup_z |h'(z)| \ln|x| |\sigma|^2 \left(\frac{a}{2|x|^{\frac{1}{2}} (\ln|x|)^{2^{n^*}+\dim(\mathcal{N})+1}} \right)^2 \right.$$

$$\left. - \frac{a \sum_{i>M} b_i x_i^2}{2|x|^{\frac{3}{2}} (\ln|x|)^{2^{n^*}+\dim(\mathcal{N})+1}} \right] e^{ap(|x|)}$$

$$\leqslant \left[2\overline{K}P \parallel C \parallel \sup_z |h'(z)|\sigma|^2 \frac{a^2}{4|x| (\ln|x|)^{2^{n*+1}+2\dim(\mathcal{N})+1}} \right.$$

$$\left. - \frac{a \min_{i>M} b_i \alpha_0 B}{4|x| (\ln|x|)^{2^{n*+1}+2\dim(\mathcal{N})+1}} \right] e^{ap(|x|)}.$$

So choose

$$a < \frac{\min_{i>M} b_i \alpha_0 B}{8} \overline{K}P \parallel C \parallel \sup_z |h'(z)|\sigma|^2,$$

giving the Lyapunov pair (V, W) with

$$W = \frac{1}{|x| (\ln|x|)^{2^{n*+1}+2\dim(\mathcal{N})+1}} e^{a \frac{|x|^{\frac{1}{2}}}{(\ln|x|)^{2^{n^*}+\dim(\mathcal{N})+1}}}.$$

This completes the proof of Theorem 3.5.

4

Operator Splitting

4.1 Introduction

In this section we consider some sort of compromise between the simplicity of a single triple in Section 2, and the relative chaos of the general finite dimensional system given in Section 3. We consider a finite dimensional system with multiple triples of interactions. Then, instead of the drift term of the SDE being made up of the linear combination of the triples, we choose each triple to be the entirety of the drift term for a random amount of time before replacing it with the next triple in the list. Once we have cycled through all triples we apply a dissipative term to a strict subset of the coordinates for a random amount of time, before repeating the entire process. Each $dX_i(t)$ also has a $\sigma_i dW_i(t)$ term, active for all time. Without loss of generality we only consider triples have exactly one of the $c^i_{jk} = 0$, as a drift term associated with a general triple can be seen as the sum of two terms associated with triples of this type.

4.2 The System, and Summary of Main Results

First fix the sequences

$$S_n = (i_n, j_n, k_n) \in \{1, \ldots, N\}^3,\ a_n \in \mathbb{R}$$

for $n \geqslant 1$ satisfying, for some $n^* \in \mathbb{N}$, $S_{n+n^*} = S_n$, $a_{n+n^*} = a_n$ and $a_n = 0 \Leftrightarrow n = 0 \pmod{n^*}$ for some $k \in \mathbb{N}$, and $i_n \neq j_n \neq k_n \neq i_n$. Then define the sequence of vector fields

$$B_n : \mathbb{R}^N \to \mathbb{R}^N : x \mapsto -a_n x_{i_n} x_{k_n} e_{j_n} + a_n x_{i_n} x_{j_n} e_{k_n}.$$

Let $I \subset \{1, \ldots, N\}$ and define the vector field

$$D_I : \mathbb{R}^N \to \mathbb{N} : x \mapsto -\sum_{i=1}^{N} 1_{i \in I} x_i.$$

Let $\{\tau_n\}_{n=0}^{\infty}$ be a sequence of random variables satisfying $\tau_0 = 0$, $\tau_n - \tau_{n-1}$ i.i.d. on $[0, \infty)$ such that $\mathbb{P}[\tau_n - \tau_{n-1} > 0] > 0$, $\mathbb{E}(\tau_n - \tau_{n-1}) < \infty$, and let

$$\tau^{-1} : \mathbb{R} \to \mathbb{N} : t \mapsto \operatorname{argmin}_n \{n : t < \tau_n\}$$

be the index of the first τ_n after time t. Let $W_t = (W_t^1, \ldots, W_t^N)$ be an N-dimensional Brownian motion independent of the sequence $\{\tau_n\}_{n=1}^{\infty}$ and then let $x(t)$ be the stochastic process in $\mathbb{R}^N$ satisfying

$$dx(t) = \left(B_{\tau^{-1}(t)}(x_t) - \mathbb{1}_{\tau^{-1}(t)=0 \bmod n^*} D_I(x(t))\right) dt + dW_t.$$

Remark 4.1. The process can be thought of as follows. For all $t \geqslant 0$ $x(t)$ is affected by the Brownian motion terms. Between random times τ_n and τ_{n+1} the coordinates $(x_{i_n}, x_{j_n}, x_{k_n})$ also have drift terms $(0, -a_n x_{i_n} x_{k_n}, a_n x_{i_n} x_{j_n})$, and after $n^* - 1$ such triples there is dissipation in x_i for $i \in I$ for a random amount of time, after which the process repeats. Some examples of such processes are given in Section 5.

The $B_{\tau^{-1}(t)}$ term serves to move energy between coordinates. In order to have control on the energy of the system over large time it must be possible for energy in any coordinate to be moved to coordinates with dissipation. To this end we have the following assumption.

Assumption 4.2. *Let $G = (\{1, \ldots, N\}, E)$ be the undirected graph with $(j, k) \in E \Leftrightarrow$ for some $n \neq 0$ (mod n^*), and some $i \in \{1, \ldots, N\}$, $(i, j, k) = S_n$ or $(i, k, j) = S_n$. Assume every connected component of the graph G contains an element of I.*

The assumption is somewhat intuitive: For fixed triple S_n and ignoring noise, $x_{j_n}^2(t) + x_{k_n}^2(t)$ is constant, and $(x_{j_n}(t), x_{k_n}(t))$ in $\mathbb{R}^2$ has angular speed $|a_n x_{i_n}(t)|$. Then energy can be transferred quickly between x_{j_n} and x_{k_n}. In fact we can prove that Assumption 4.2 is necessary.

Lemma 4.3. *If Assumption 4.2 is false then* $\lim_{t\to\infty} \mathbb{E}\left[|x(t)|^2\right] = \infty$.

Proof. Let there be no path from j to any $i \in I$, and let J be the connected component of G containing j. Then for all $n \geqslant 0$ $j_n, k_n \in J$ or $j_n, k_n \in J^c$ and so B_{S_n} transfers energy entirely between elements of J or elements of J^c. Therefore for all $t \geqslant 0$

$$\begin{aligned} d\left(\sum_{j\in J} x_j^2\right) &= 2\sum_{j\in J} x_j dW_t^i + |J|dt \\ \Rightarrow \mathbb{E}\left[|x(t)|^2\right] \geqslant \mathbb{E}\left[\sum_{j\in J} x_j^2(t)\right] &= \sum_{j\in J} x_j^2(0) + |J|t \\ &\to \infty \text{ as } t \to \infty. \end{aligned}$$

□

Our main result is as follows.

Theorem 4.4. *Given Assumption 4.2, the system* $\{x(t)\}_{t\geqslant 0}$ *satisfies*

$$\limsup_{t\to\infty} \mathbb{E}|x(t)|^2 < \infty$$

and for all $n \geqslant 0$ *the discrete-time process* $\{x(\tau_{n+kn*})\}_{k\geqslant 0}$ *has a unique invariant distribution on* $\mathbb{R}$.

As preparation for the proof of Theorem 4.4 we consider the behaviour of a single triple.

4.3 A Single Triple

Let $x(t) \in \mathbb{R}^3$ satisfy

$$\begin{aligned} dx_1 &= dW_t^1, \\ dx_2 &= -ax_1x_3dt + dW_t^2, \\ dx_3 &= ax_1x_2dt + dW_t^3. \end{aligned}$$

Without loss of generality assume $a > 0$. Let $R = \sqrt{x_2^2(0) + x_3^2(0)}$ and let

$$(x_1, X_2^{(R)}, X_3^{(R)}) = \left(x_1, \frac{x_2}{R}, \frac{x_3}{R}\right)$$

so that

$$\begin{aligned} dx_1 &= dW_t^1, \\ dX_2^{(R)} &= -ax_1X_3^{(R)}dt + \frac{1}{R}dW_t^2, \\ dX_3^{(R)} &= ax_1X_2^{(R)}dt + \frac{1}{R}dW_t^3. \end{aligned}$$

As $R \to \infty$ we get the natural limit process

$$\begin{aligned} dx_1 &= dW_t^1, \\ dX_2 &= -ax_1X_3dt, \\ dX_3 &= ax_1X_2dt. \end{aligned}$$

The main result is this section is as follows.

Lemma 4.5. *There exists a constant κ and a function $g(R)$, dependent only on the distribution of τ, satisfying $\kappa \in (0,1)$, $g > 0$, g decreasing, $\lim_{R\to\infty} \frac{g(R)}{R^2} = 0$ such that*

$$\liminf_{R\to\infty} \mathbb{E}X_2^2(\tau), \liminf_{R\to\infty} \mathbb{E}X_3^2(\tau) \geqslant \kappa R^2 - g(R).$$

Remark 4.6. The idea behind this result is intuitive. When R is large (x_2, x_3) moves on a circle of radius R with angular speed $a|x_1|$. Even if $|x_1|$ is small, its Brownian motion will make it large enough so that (x_2, x_3) will move at least a small percentage of the way around the circle, and so both $|x_2(\tau)|, |x_3(\tau)| \geqslant \epsilon R$ for some $\epsilon > 0$ with high probability and in expectation.

See that $X_2^2(t) + X_3^2(t) = 1$ for all t, so that if $\tan\theta = \frac{X_3(0)}{X_2(0)}$ we have

$$X_2(t) = \cos\left(\theta + \int_0^t ax_1(s)ds\right),$$

$$X_3(t) = \sin\left(\theta + \int_0^t ax_1(s)ds\right).$$

See that

$$\theta + \int_0^t ax_1(s)ds = \theta + x_1(0)t + a\int_0^t W_s^1 ds$$

$$\sim \mathcal{N}\left(\theta + ax_1(0)t, \frac{a^2t^2}{2}\right).$$

Therefore for any fixed $t > 0,\ x_1(0) \in \mathbb{R}$,

$$\theta + \int_0^t ax_1(s)ds$$

has a continuous density that is positive everywhere, and therefore so do $X_2(t),\ X_3(t)$. If we let $Z(t) \sim \mathcal{N}\left(\theta + axt, \frac{a^2t^2}{2}\right)$ and let

$$p_{\theta,x,\epsilon,t} = \mathbb{P}\left[\sin Z(t) \notin [-\epsilon, \epsilon]\right],$$

then for any $\epsilon > 0,\ t > 0,\ \theta, x \in \mathbb{R}$ we have $p_{\theta,x,\epsilon,t} > 0$ and furthermore, as

$$p_{\theta,x,\epsilon,t} = p_{\theta+2\pi,x,\epsilon,t} = p_{\theta,x+\frac{2\pi}{at},\epsilon,t},$$

for fixed ϵ, t we have that

$$p_{\epsilon,t} := \inf_{\theta,x} p_{\theta,x,\epsilon,t} > 0.$$

This gives us the following result.

Lemma 4.7. *There exists a* $p_{\epsilon,t} \in (0,1)$ *over all* $\epsilon > 0$, $t > 0$ *such that*

$$\mathbb{P}\left[|X_3(t)| > \epsilon\right] \geqslant p_{\epsilon,t}$$

for all initial conditions $(x_1(0), X_2(0), X_3(0))$.

We next show that X and $X^{(R)}$ are close with high probability and in expectation on any finite time interval.

Lemma 4.8. *For any* $T > 0$, $R \geqslant 1$, *given* $X(0) = X^{(R)}(0)$ *we have for any* $\alpha, \beta > 0$ *that*

$$\sup_{t\in[0,T]} \left\{ \left(X_2^{(R)}(t) - X_2(t)\right)^2 + \left(X_3^{(R)}(t) - X_3(t)\right)^2 \right\} \leqslant 2\left(\frac{T}{R^2} + \beta\right) \exp\left(\frac{2\alpha T}{R^2}\right).$$

with probability $\geqslant 1 - 2e^{-\alpha\beta}$.

Proof. Fix $T > 0, R \geqslant 1$. It is easy to see that $X^{(R)}$, X have unique strong solutions on the interval $[0,T]$. Using Itô's formula we have that

$$\begin{aligned}\left(X_2^{(R)}(t) - X_2(t)\right)^2 + \left(X_3^{(R)}(t) - X_3(t)\right)^2 &= \int_0^t \frac{2}{R}\left(X_2^{(R)}(s) - X_2(s)\right) dW_s^2 \\ &\quad + \int_0^t \frac{2}{R}\left(X_3^{(R)}(s) - X_3(s)\right) dW_s^3 \\ &\quad + \int_0^t \frac{2}{R^2} ds.\end{aligned}$$

We now apply the exponential martingale inequality to the martingales

$$M_2(t) = \int_0^t \frac{2}{R}\left(X_2^{(R)}(s) - X_2(s)\right) dW_s^2,$$

$$M_3(t) = \int_0^t \frac{2}{R}\left(X_3^{(R)}(s) - X_3(s)\right) dW_s^3,$$

with quadratic variations

$$\langle M_2\rangle_t = \int_0^t \frac{4}{R}\left(X_2^{(R)}(s) - X_2(s)\right)^2 ds,$$

$$\langle M_3\rangle_t = \int_0^t \frac{4}{R}\left(X_3^{(R)}(s) - X_3(s)\right)^2 ds.$$

As (X_2, X_3) stays in the unit ball in $\mathbb{R}^2$, one can easily check that the Novikov condition is satisfies for each $M_i(t)$ for all times $t \geqslant 0$. This assures us that

$$Z_{i,\delta} = \exp\left(\delta M_i(t) - \frac{\delta^2}{2}\langle M_i\rangle_t\right)$$

is a martingale for any $\delta > 0$ with $Z_{i,\delta}(0) = 1$. So for $i = 2, 3$ define the events

$$E^i_{\alpha,\beta} = \left\{\sup_{t\in[0,T]}\left(M_i(t) - \frac{\alpha}{2}\langle M_i\rangle_t\right) \leqslant \beta\right\}$$

for $i = 2, 3$. On the event $E^2_{\alpha,\beta} \cap E^3_{\alpha,\beta}$ we have that

$$\sup_{t\in[0,T]}\left\{\left(X_2^{(R)}(t) - X_2(t)\right)^2 + \left(X_3^{(R)}(t) - X_3(t)\right)^2\right\}$$

$$\leqslant \frac{2a}{R^2}\int_0^T \left(X_2^{(R)}(s) - X_2(s)\right)^2 + \left(X_3^{(R)}(s) - X_3(s)\right)^2 ds + 2\left(\frac{T}{R^2} + \beta\right).$$

Applying Gronwall's inequality on

$$Y(T) = \sup_{t\in[0,T]}\left\{\left(X_2^{(R)}(t) - X_2(t)\right)^2 + \left(X_3^{(R)}(t) - X_3(t)\right)^2\right\}$$

then gives

$$\sup_{t\in[0,T]}\left\{\left(X_2^{(R)}(t) - X_2(t)\right)^2 + \left(X_3^{(R)}(t) - X_3(t)\right)^2\right\} \leqslant 2\left(\frac{T}{R^2} + \beta\right)\exp\left(\frac{2\alpha T}{R^2}\right).$$

As $\mathbb{P}[E^i_{\alpha,\beta}] \geqslant 1 - e^{-\alpha\beta}$ we have that $\mathbb{P}[E^2_{\alpha,\beta} \cap E^3_{\alpha,\beta}] \geqslant 1 - 2e^{-\alpha\beta}$, giving the required result. □

From Lemmas 4.7 and 4.8 we can get bounds on $\mathbb{E}x_2^2(\tau), \mathbb{E}x_3^2(\tau)$. Fix some $\epsilon > 0$, $0 < \lambda < 1$, and given τ, fix some $\alpha_\tau, \beta_\tau > 0$, and assume that

$$|X_3(\tau)| \geqslant \sqrt{\epsilon} + \frac{1}{R^\lambda},$$

$$\left[\left(X_2^{(R)}(\tau) - X(\tau)\right)^2 + \left(X_3^{(R)}(\tau) - X_3(\tau)\right)^2\right] \leqslant 2\left(\frac{\tau}{R^2} + \beta_\tau\right)\exp\left(\frac{2\alpha_r t\tau}{R^2}\right).$$

This happens with probability

$$\geqslant 1 - \min\{2e^{-\alpha_\tau\beta_\tau}, 1\} - (1 - p_{\sqrt{\epsilon}+\frac{1}{R^\lambda},\tau}) = p_{\sqrt{\epsilon}+\frac{1}{R^\lambda},\tau} - \min\{2e^{-\alpha_\tau\beta_\tau}, 1\},$$

and it implies

$$\begin{aligned}|x_3(\tau)| &= R\left|X_3^{(R)}(\tau)\right| \\ &\geqslant R\,|X_3(\tau)| - R\left|X_3^{(R)}(\tau) - X_3(\tau)\right| \\ &\geqslant R\left[\sqrt{\epsilon} + \frac{1}{R^\lambda} - 2\left(\frac{\tau}{R^2} + \beta_\tau\right)\exp\left(\frac{2\alpha_\tau\tau}{R^2}\right)\right]\end{aligned}$$

and similarly for $x_2(\tau)$. We want this term to be independent of τ. Therefore let $\beta_\tau = \frac{\tau}{R^\gamma}$ for some $2\lambda < \gamma < 2$, $\alpha_\tau = \frac{R^2}{2\tau}\ln\left(\frac{R^{\gamma-\lambda}}{4\tau}\right)$ so that for $R \geqslant 1$

$$2\left(\frac{\tau}{R^2} + \beta\right)\exp\left(\frac{2\alpha\tau}{R^2}\right) \leqslant \frac{1}{R^\lambda}$$

and so this implies

$$|x_3(\tau)| \geqslant R\sqrt{\epsilon}.$$

See then by Fubini's Theorem that

$$\begin{aligned}\mathbb{E}x_3^2(\tau) =&\mathbb{E}\int_0^\infty \mathbb{P}\left[|x_3(\tau)| \geqslant \sqrt{x}\middle|\,\tau\right]dx\\ \geqslant&R^2\mathbb{E}\int_0^1 \mathbb{P}\left[|x_3(\tau)| \geqslant R\sqrt{\epsilon}\middle|\,\tau\right]d\epsilon\\ \geqslant&R^2\mathbb{E}\int_0^1 p_{\sqrt{\epsilon}+R^{-\gamma},\tau}d\epsilon - \mathbb{E}\min\left\{2e^{-\alpha_\tau\beta_{rt}},1\right\}\\ =&R^2\mathbb{E}\int_0^1 p_{\sqrt{\epsilon}+R^{-\gamma},\tau}d\epsilon - \mathbb{E}\min\left\{2\left(\frac{4\tau}{R^{\gamma-\lambda}}\right)^{\frac{R^{2-\gamma}}{2}},1\right\}.\end{aligned}$$

See that

$$\mathbb{E}\min\left\{2e^{-\alpha_\tau\beta_\tau},1\right\} = \mathbb{E}\min\left\{2\left(\frac{4\tau}{R^{\gamma-\lambda}}\right)^{\frac{R^{2-\gamma}}{2}},1\right\}.$$

Fix some monotonic $f(R)\to\infty$ as $R\to\infty$, and see that

$$\mathbb{E}\min\left\{2\left(\frac{4\tau}{R^{\gamma-\lambda}}\right)^{\frac{R^{2-\gamma}}{2}},1\right\} \leqslant \mathbb{P}[\tau > f(R)] + 2\left(\frac{4f(R)}{R^{\gamma-\lambda}}\right)^{\frac{R^{2-\gamma}}{2}}.$$

It's hard to see an optimal $f(R)$ over all R and all distributions, yet choosing

$$f(R) = \frac{R^{\gamma-2\lambda}}{4}$$

gives

$$\mathbb{E}\min\left\{2\left(\frac{4\tau}{R^{\gamma-\lambda}}\right)^{\frac{R^{2-\gamma}}{2}},1\right\} \leqslant \mathbb{P}\left[\tau > \frac{R^{\gamma-2\lambda}}{4}\right] + 2R^{-\frac{\lambda R^{2-\gamma}}{2}} \to 0$$

as $R \to \infty$.

Therefore

$$\liminf_{R \to \infty} \frac{\mathbb{E}x_3^2(\tau)}{R^2} \geqslant \mathbb{E} \int_0^1 p_{\sqrt{\epsilon},\tau} d\epsilon > 0.$$

Identical limits hold for x_2, proving Lemma 4.5.

Remark 4.9. Note that this constant κ, like $p_{\epsilon,t}$, is dependent on a. However as there are only a finite number of values a_n can take, we can consider κ to be the minimum over all possible values of a_n.

4.4 The Entire Process

We now consider the entire process. Our approach is as follows. For each coordinate x_j with $j \notin I$ we consider a particular path $x_j = x_0 \to x_{j_1} \to \ldots \to x_{j_m} = x_i$ with $i \in I$, and for some $k \in \mathbb{N}$ we get a lower bound on $\mathbb{E}\left[x_i^2\left(\tau_{kn^*-1}\right)\right]$ in terms of $x_j^2(0)$. In this way, given a large amount of energy initially in x_j we can prove in expectation that a certain proportion of this energy is eventually moved to some x_i, $i \in I$, and so is removed from the system.

For any x_{j_0} with $j_0 \notin I$ we define the optimal path for x_{j_0} in the following way. Every edge (p, q) in the graph G as described in Assumption 4.2 can be associated with the subset of $\mathbb{N}$, here denoted $E_{p,q}$, containing elements n such that $(k, p, q) = S_n$ or $(k, q, p) = S_n$ for some k and $n \neq 0 \bmod n^*$, i.e. representing the time intervals on which x_p and x_q are interacting. If $(j_0, j_1), (j_1, j_2) \ldots (j_m, i)$ represents a path from j_0

to i, define $a_0 = 0$ and $a_k = \text{argmin}\{n \in E_{j_{k-1},j_k} : n > a_{k-1}\}$ for $1 \leqslant k \leqslant m-1$, then we define the time length of this path as a_m. Note that a_m represents the number of random time intervals $[\tau_{n-1}, \tau_n]$ needed to transfer energy from x_j to x_i along this path. The optimal path between x_{j_0} and x_i for $i \in I$ is then the path with the smallest time length, where if there is no path between x_{j_0} and x_i we define the time length as $+\infty$. The optimal path for x_{j_0} is then the path with the smallest time length over all x_i with $i \in I$, and we define this time length as T_{j_0}. By Assumption 4.2 $T_{j_0} < \infty$ for all j_0.

If $L = |x(0)|$ it must hold that $|X_{j_0}(0)| \geqslant \frac{L}{\sqrt{N}}$ for some j_0. Assume for now that $j_0 \notin I$. Let T_{j_0} be the time length of the optimal path $(j_0, j_1), (j_1, j_2), \ldots, (j_m, i)$ with associated a_k for $0 \leqslant k \leqslant m$. First see that $a_{k+1} < a_k + n^*$, and let

$$M_k = \left| \bigcup_l E_{k,l} \bigcap \{a_k + 1, a_k + 2, \ldots, a_{k+1} - 1\} \right|$$

be the number of interactions that x_{j_k} is involved in between when it receives the energy from $x_{j_{k-1}}$ (or in the case of $k = 0$, from time 0), and when it transfers this energy to $x_{j_{k+1}}$. If x_j is not involved in the interaction between times τ_n and τ_{n+1}, then

$$\mathbb{E}\left[x_j^2(\tau_{n+1}) \middle| x_j^2(\tau_n) \right] = x_j^2(\tau_n) + \mathbb{E}\left[\tau_{n+1} - \tau_n\right].$$

If it is, then with no knowledge of the coordinate it is interacting with all we can say is

$$\mathbb{E}\left[x_j^2(\tau_{n+1}) \middle| x_j^2(\tau_n) \right] \geqslant \kappa x_j^2(\tau_n) + g(x_j^2(\tau_n)).$$

One can then see that if we want a guaranteed lower bound on $\mathbb{E}x_{j_1}^2(\tau_{a_1})$, we must assume that x_{j_0} is involved in every interaction S_n for $n \leqslant a_1 - 1$, so that

$$\mathbb{E}x_{j_0}^2(\tau_{a_0}) \geqslant \kappa^{a_0-1}\frac{L^2}{\sqrt{N}} - \sum_{i=0}^{a_0-1} g(L)^i \kappa^{a_0-i}.$$

Then we have

$$\mathbb{E}x_{j_1}^2(\tau_{a_1}) \geqslant \kappa^{a_1}\frac{L^2}{\sqrt{N}} - \sum_{k=0}^{a_1} g(L)^k \kappa^{a_1-k}.$$

Similarly, if l is the smallest integer such that $a_m < ln^*$ we can show that

$$\mathbb{E}x_i^2(\tau_{ln^*-1}) \geqslant \kappa^{ln^*-1}\frac{L^2}{\sqrt{N}} - \sum_{k=1}^{ln^*-1} g(L)^k \kappa^{ln^*-1-k}.$$

Between times τ_{ln^*-1} and τ_{ln^*} x_i behaves like an Ornstein Uhlenbeck process and so

$$\begin{aligned}
\mathbb{E}|x(\tau_{ln^*})|^2 \leqslant L^2 + N\mathbb{E}\tau_{ln^*} - \int_{\tau_{ln^*-1}}^{\tau_{ln^*}} \mathbb{E}x_i^2(t)dt \\
=& L^2 + N\mathbb{E}\tau_{ln^*} - \left(\mathbb{E}x_i^2(\tau_{ln^*-1}) - \frac{1}{2}\right)\frac{1-\mathbb{E}e^{-2(\tau_{ln^*}-\tau_{ln^*-1})}}{2} \\
&+ \frac{\mathbb{E}\left[\tau_{ln^*} - \tau_{ln^*-1}\right]}{2} \\
\leqslant& L^2\left(1 - \frac{\kappa^{ln^*-1}}{\sqrt{N}}\frac{1-\mathbb{E}e^{-2\tau_1}}{2}\right) + h(L)
\end{aligned}$$

for some monotonic function $h(L) > 0$ that satifies $\lim_{L\to\infty}\frac{h(L)}{L^2} = 0$. As

$$\frac{\kappa^{ln^*-1}}{\sqrt{N}}\frac{1-\mathbb{E}e^{-2\tau_1}}{2} > 0$$

we have that there exists an $\alpha \in (0,1)$ such that

$$\mathbb{E}|x(\tau_{ln^*})|^2 \leqslant \alpha|x(0)|^2$$

for large enough $|x(0)|$. This proves that the sequence $\{\mathbb{E}|x(\tau_{mln^*})|^2\}_{m=1}^{\infty}$ is bounded. Now fix some $t > 0$, and sequence $\{\tau_n\}$. There exists some m such that $mln^* \leqslant t < (m+1)ln^*$, and so

$$\begin{aligned}\mathbb{E}\left[|x(t)|^2\middle|\{\tau_n\}\right] &= \mathbb{E}\left[|x(\tau_{mln^*})|^2\middle|\{\tau_n\}\right] + N(t-\tau_{mln^*})\\ &\leqslant \mathbb{E}\left[|x(\tau_{mln^*})|^2\middle|\{\tau_n\}\right] + N(\tau_{(m+1)ln^*}-\tau_{mln^*})\\ \Rightarrow \mathbb{E}|x(t)|^2 &\leqslant \mathbb{E}|x(\tau_{mln^*})|^2 + Nln^*\mathbb{E}\tau_1.\end{aligned}$$

This proves that

$$\limsup_{t\to\infty} \mathbb{E}|x(t)|^2 < \infty,$$

proving Theorem 4.4.

Remark 4.10. One may ask why we did not try to get tail estimates for the invariant distribution of the discrete time process $\{x(\tau_{n+kn^*})\}_k$. In short, tail estimates are limited by the distribution of the random times $\tau_{n+1} - \tau_n$ and not the structure of the process itself, making the endeavour less interesting.

5

Examples

5.1 Examples for Chapter 2

Example 5.1 (Dissipation in all coordinates)**.** Consider the system

$$\begin{aligned} dX_1 &= (c_1 X_2 X_3 - b_1 X_2)\, dt + \sigma_1 dW_1(t), \\ dX_2 &= (c_2 X_1 X_3 - b_2 X_2)\, dt + \sigma_2 dW_2(t), \\ dX_3 &= (c_3 X_1 X_2 - b_3 X_3)\, dt + \sigma_3 dW_3(t), \end{aligned} \tag{5.1}$$

with infinitesimal generator L, where $b_i > 0$ for all i, and let $V = e^{a|x|^2}$ for some $a > 0$. Then

$$\begin{aligned} LV =& e^{a|x|^2} \left[-2a \sum_i b_i x_i^2 + \sum_i \sigma_i^2 \left(a + 2a^2 x_i^2 \right) \right] \\ \leqslant& e^{a|x|^2} \left[|x|^2 \left(-2a \min_i b_i + 2a^2 \max_i \sigma_i^2 \right) + a|\sigma|^2 \right]. \end{aligned}$$

If we choose $a < \frac{\min_i b_i}{\max_i \sigma_i^2}$ and let $\epsilon = -2a \min_i b_i + 2a^2 \max_i \sigma_i^2 > 0$ then

$$\begin{aligned} LV &\leqslant \left(-\epsilon|x|^2 + a|\sigma|^2\right) e^{a|x|^2} \\ &\leqslant -\epsilon' V \end{aligned}$$

for any $\epsilon' < \epsilon$ and sufficiently large $|x|$. Then we have that $X(t)$ has an invariant distribution μ satisfying

$$\int_{\mathbb{R}^3} e^{a|x|^2} \mu(dx) < \infty.$$

Similar, albeit weaker results can be obtained by showing that many increasing functions of $|x|$ with compact level sets, such as $|x|^q$ or $e^{a|x|}$ for $q > 0$ are Lyapunov functions for the process $X(t)$.

5.2 Examples for Chapter 3

Example 5.2 (Counter-example to Assumption 3.2)**.** The most trivial counter-example to Assumption 3.2 is that where $c^i_{jk} = 0$ if $i > M$ and $j, k \leqslant M$ as then $\mathcal{N}$ is the $y_1 \ldots y_M$ plane and by setting $c^i_{jk} \neq 0$ for some $i, j, k \leqslant M$ Assumption 3.2 can fail.

To see a non-trivial case, consider the system with $N = 5$, $M = 3$. Let $c^4_{12} = c^5_{12} = c^4_{13} = c^5_{13} = 1$, and for all other $j, k \leqslant 3$ let $c^4_{jk} = c^5_{jk} = 0$. As $\mathcal{N}$ depends only on these terms we get that $y_4,\ y_5,\ Hy_4,\ Hy_5 = 0$ if $y_4, y_5 = 0$ and if $y_1 = 0$ or $y_2 = -y_3$. To see if higher moments $H^n y_i = 0$ for $i = 4, 5$ we need to know the motion of the other coordinates. If $c^2_{13} = c^3_{12} = 1$, $c^1_{23} = -2$ and $c^i_{jk} = 0$ for all other $i, j, k \leqslant 3$, then $\frac{d}{dt}(Y_2(t)+Y_3(t)) = Y_1(t)(Y_2(t)+Y_3(t))$ so that if $Y_2(0)+Y_3(0) = 0$ then $Y_2(t)+Y_3(t) = 0$

for all t. However $Y_1(0) = 0 \nRightarrow Y_1(t) = 0$ for all t. Therefore if $Y_2(0) + Y_3(0) = 0$, $\frac{d}{dt}Y_i(t) = 0$ for all t, $i = 4, 5$ so that $\mathcal{N} = \{y : y_4 = y_5 = 0 \text{ and } y_2 = -y_3\}$ and on $\mathcal{N} \cap \{y : y_i \neq 0 \text{ for at least two } i \in \{1,2,3\}\}$ $\frac{d}{dt}Y(t) \neq 0$ for any t.

Example 5.3 (Counter-example to Lemma 3.12 and subsequent results)**.** Consider the case where for some $1 \leqslant d \leqslant M$, $c^i_{jk} = 0$ if $i > M$ and $j, k \leqslant d$, or if $i \leqslant M$, $j, k > M$, so that $\mathcal{N}$ is the $y_1 \dots y_d$ plane. Now let $c^i_{jk} \neq 0$ for at least one $i, j, k \leqslant d$ so that Assumption 3.2 is false. Then the matrix

$$\left(\rho(x)C^i\right)_{i \leqslant N} = \begin{pmatrix} A & \vdots & 0 \\ \cdots & & \cdots \\ 0 & \vdots & D \end{pmatrix}$$

for some matrices A, D with A a $d \times d$ matrix, and assume for simplicity that all eigenspaces are of dimension 1. Then $\lambda(x), (v(x), w(x))$ is a left-eigenpair for $(\rho(x)C^i)_{i \leqslant N}$ iff $\lambda(x), v(x)$ is an eigenpair for A or $\lambda(x), w(x)$ is an eigenpair for D. In the former case $v(x)^T(x - \rho(x)) = 0$ uniformly, whereas previously we never had to worry that $v(x) \cdot x = v(x) \cdot (x - \rho(x))$ could be 0 uniformly. Furthermore even in the latter case

$$Hv(x) \cdot (x - \rho(x)) = (Hv(x)) \cdot (x - \rho(x)) + v(x) \cdot (H(x - \rho(x)),$$

and even if we still have

$$H(x - \rho(x)) = \left(\rho^T C^i\right)_{i \leqslant N} (x - \rho(x)) + O(|x - \rho(x)|^2),$$

we have that each term of $Hv(x)$ is potentially $O(|x|)$ so that

$$Hv(x) \cdot (x - \rho(x)) = O\left(|x||x - \rho(x)|\right) + \lambda(x)\, v(x) \cdot (x - \rho(x)) + O\left(|x - \rho(x)|^2\right).$$

As $\lambda(x)\, v(x) \cdot (x - \rho(x))$ is only potentially $O(|x||x - \rho(x)|)$, we have no clear dominance of the desired term. In short this behaviour can be characterised as such: the reason that the norm of the projection of $X(t)$ onto the eigenspace is growing exponentially fast is changing at the same rate of the growth itself.

To show that this can indeed be the case, let $N = 4$, $M = 2$, and let $c^4_{jk} = 0$ for all $j, k \leqslant 2$. Then $\mathcal{N}$ is the $y_1 y_2$ plane. Let $c^1_{12}, c^2_{11} \neq 0$ so that Assumption 3.2 does not hold. Then

$$D = y_1 \begin{pmatrix} c^3_{13} & c^3_{14} \\ c^4_{13} & c^4_{14} \end{pmatrix} + y_2 \begin{pmatrix} c^3_{23} & c^3_{24} \\ c^4_{23} & c^4_{24} \end{pmatrix}.$$

Note that by appropriately altering c^i_{jk} for $i \leqslant 2$, $j, k \geqslant 3$ we have full freedom in the coefficients defining D. Let

$$D = y_1 \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} + y_2 \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}$$

with non-constant eigenvectors

$$w(y) = \begin{pmatrix} 2y_1 - 2y_2 \\ -2y_2 \mp \sqrt{y_1^2 - 20 y_1 y_2 - 8 y_2^2} \end{pmatrix}.$$

Let $v(y) = \frac{w(y)}{|w(y)|}$. As H at $(y_1, y_2, 0, 0)$ is defined by c^2_{11}, c^1_{12}, one could not expect $Hv(y) = 0$ in general, and as a result $Hv(y) = O(|y|)$. Therefore for general $y \in \mathbb{R}^4$ we have $Hv(y)(y - \rho(y)) = O(|x||y - \rho(y)|)$.

Example 5.4 (Counter-example to Assumption 3.4)**.** Consider the case of $c^i_{jk} \neq 0$ only for $i, j, k > M$. Then $\mathcal{N}$ is the $y_1 \dots y_M$ axis, and $H^n y_i = O\left(d(x, \mathcal{N})^{2n}\right)$ as each $H^n y_i$ is a polynomial only in y_j, $j > M$.

5.3 Examples for Chapter 4

Example 5.5 (The chain)**.** Consider the following system on $x \in \mathbb{R}^N$. Let the sequence of random times $\{\tau_n\}_{n\geqslant 0}$ satisfy $\tau_{n+1} - \tau_n \sim \text{Exp}(1)$ independent and $\tau_0 = 0$. For $t \in [\tau_{n-1}, \tau_n)$ with $n \neq 0 \pmod N$, the system satisfies

$$\begin{aligned} dx_n &= -x_{n-1}x_{n+1}dt + dW_t^n, \\ dx_{n+1} &= x_{n-1}x_n dt + dW_t^{n+1}, \\ dx_j &= dW_t^j \text{ for } j \neq n,\ n+1, \end{aligned}$$

where in the case of $n = 1$ x_0 is replaced with x_N, and for $t \in [\tau_{n-1}, \tau_n)$ with $n = 0 \pmod N$, the system satisfies

$$\begin{aligned} dx_N &= -x_N dt + dW_t^n, \\ dx_j &= dW_t^j \text{ for } j \neq N. \end{aligned}$$

In short, for $n = 1, \ldots, N-1$, between times τ_{n-1} and τ_n, (x_n, x_{n+1}) moves anti-clockwise on a circle in the $x_n x_{n+1}$ plane with angular velocity x_{n-1} (if we ignore the Brownian terms in x_n, x_{n+1}), while all other coordinates behave as Brownian motions. Then between times τ_{N-1} and τ_N x_N behaves as an Ornstein-Uhlenbeck process, while again all other terms behave as Brownian motions. In order for $|x(t)|^2$ to remain controlled over time, one needs energy in all coordinates to eventually transfer to x_N over time.

Example 5.6 (The reversed chain)**.** Here we consider an identical example to Example 5.5, except that in this case, between times τ_{n-1} and τ_n for $n = 0 \pmod N$ we have dissipation in x_1. Also assume that $x(0) = (0, 0, \ldots, 0, L)$ for some large L. We should expect the system to behave in the following way. Between times 0 and τ_{N-2} the energy in x_N is affected only by Brownian motion, and so $x_N(\tau_{N-2}) = L + O(1)$. The other coordinates transfer energy between themselves, but this energy must have come from their Brownian motion terms, and so we expect $x_j(\tau_{N-2}) = O(1)$ for $j \neq N$. By time τ_{N-1} not much has changed except now we should expect $x_{N-1}(\tau_{N-1}) = O(L)$. As x_1 is small by comparison, no sizeable energy is removed between times τ_{N-1} and τ_N, and one could even prove $\mathbb{E}|x(\tau_N)|^2 > L^2$. However by repeating this argument, we would expect $x_{N-2}(\tau_{2N-1}) = O(L)$, then $x_{N-3}(\tau_{3N-1}) = O(L)$, until we have $x_1(\tau_{(N-1)N-1}) = O(L)$, which would imply that $\mathbb{E}|x(\tau_{(N-1)N})|^2 \leqslant \kappa L$ for some $\kappa < 1$. Therefore, we should expect to lose a sizeable amount of energy by time τ_{kN} for some k, but not necessarily by time τ_N.

Example 5.7 (Counter-example to Assumption 4.2)**.** In Examples 5.5 and 5.6, between times 0 and τ_1 we can say that we paired x_2 and x_3, powered by x_1, then we paired x_3 and x_4, powered by x_2, and so on. Here, to avoid complex notation we will use this language. First we pair x_1 and x_3, powered by x_{N-1}, then we pair x_2 and x_4, powered by x_N, then we pair x_3 and x_5, powered by x_1, and so on up until we pair x_{N-2} and x_N, powered by x_{N-4}. We then have dissipation in x_1, and the cycle starts again. Therefore energy in x_j can be transferred to x_{j-2} (if $j \geqslant 3$) and to x_{j+2} (if $j \leqslant N-2$), and from there to $x_{j\pm4}$, etc. But energy cannot be transferred between x_j with j even and x_k with k odd, and so in particular energy cannot be transferred from x_j with j even to x_1. We can instead show that

$$\mathbb{E}\left[\sum_{j=1}^{\lfloor \frac{N}{2} \rfloor} x_{2j}^2(t)\right] \to \infty$$

as $t \to \infty$.

6

Conclusions

Our original aim when undertaking this project was to prove the existence of an invariant distribution for the solution to the stochastic Navier-Stokes equation as described in Section 1. By considering the Fourier series of the solution we reduced this to the problem of finding an invariant distribution of a stochastic differential equation on infinite dimensional Euclidean space. With this in mind we decided to first prove a similar statement for a general stochastic differential equations of the same form first on three and then on general finite dimensional Euclidean space, believing that with every step the method of proof would become more involved but would be ultimately similar in structure. Although this was the case as we advanced from three dimensions to general finite dimensions, to recreate this proof in infinite dimensions would require significant generalisations of not only our results in Appendix B but the results stated from other works, many of which may break down in infinite dimensions.

Despite this, however, we believe the finite dimensional result alone is significant. Furthermore it serves as a successful example of the use of mollifiers to construct a Lyapunov function for a process that behaves qualitatively differently on different regions of space, and as an application of matrix perturbation theory in multiple dimensions.

Appendix A

Lyapunov Functions

Here we list the results related to Lyapunov functions that we use throughout this work. These are not the most general results or definitions, but they suffice for our purposes. We will not provide proofs for any of these results; they can be found in [9].

Definition A.1. *Let V, $W : \mathbb{R}^N \to [0, \infty)$ be twice continuously differentiable and continuous functions respectively. We call (V, W) a Lyapunov pair corresponding to the Markov process $\{X_t\}_{t \geqslant 0}$ with infinitesimal generator L if V, W have compact level sets and there exist constants a, b such that*

$$LV \leqslant -aW + b$$

for all $x \in \mathbb{R}^N$. If $V = W$ we call V a Lyapunov function.

Note that Lyapunov pairs need only be defined outside of a compact set, and the condition

$$LV \leqslant -aW + b$$

for all $x \in \mathbb{R}^N$ holds if

$$LV \leqslant -cW + U$$

for some function $U(x) \ll W(x)$ as $|x| \to \infty$.

Proposition A.2. *If X_t has a Lyapunov pair V, W then*

(a) X_t is non-explosive; that it if $\tau_n = \inf\{t : |X_t| > n\}$ and $\tau_\infty = \lim_{n\to\infty} \tau_n$,

$$\mathbb{P}_{x_0}\left[\tau_\infty = \infty\right] = 1$$

for all initial conditions $x_0 \in \mathbb{R}^N$. In particular, for all initial conditions x_0, X_t is well-defined for all $t \geqslant 0$ almost surely.

(b) X_t has an invariant probability measure μ satisfying

$$\int_{\mathbb{R}^N} W(x)\mu(dx) < \infty.$$

Furthermore if X_t has a uniformly elliptic diffusion matrix this invariant measure is unique, ergodic and has a smooth and everywhere positive density with respect to Lebesgue measure on $\mathbb{R}^N$.

(c) If $V = W$, $w = 1 + \beta W$ for some $\beta > 0$ and we define the distance function

$$d_w\left(\nu_1, \nu_2\right) = \sup_{g:|g| \leqslant w} \int_{\mathbb{R}^N} g(x)\left(\nu_1 - \nu_2\right)(dx)$$

on the space of probability measures ν *on* $\mathbb{R}^N$ *such that* $\int_{\mathbb{R}^N} w(x)\nu(dx) < \infty$, *then there exist constants* $C, \eta > 0$ *such that*

$$d_w\left(P_t\nu_1, P_t\nu_2\right) \leqslant Ce^{-\eta t} d_w\left(\nu_1, \nu_2\right)$$

where P_t *is the Markov semigroup associated with* X_t.

Appendix B

Real Algebraic Geometry and Matrix Perturbation Theory

Given a matrix $A(\epsilon)$, or more generally a linear operator as a function of $\epsilon \in \mathbb{R}$ or $\mathbb{C}$, properties such as its eigenvalues and eigenvectors have long been of interest in mathematics [1,3,5], mechanics [6,7] and computer science [2]. In particular, it is known that if $A(\epsilon)$ is an analytic function of $\epsilon \in \mathbb{C}$ the eigenvalues and eigenvectors of $A(\epsilon)$ change analytically in regions where they are of constant multiplicity and have potential algebraic singularities at points where multiplicity changes [1,5]. Non-analytic perturbations are known to result in discontinuous eigenvectors.

Example B.1. [1] Consider the matrix

$$A(\epsilon) = e^{-\frac{1}{\epsilon^2}} \begin{pmatrix} \cos\left(\frac{2}{\epsilon}\right) & \sin\left(\frac{2}{\epsilon}\right) \\ \sin\left(\frac{2}{\epsilon}\right) & -\cos\left(\frac{2}{\epsilon}\right) \end{pmatrix}, \quad A(0) = 0$$

with $\epsilon \in \mathbb{R}$. This matrix has smooth coefficients with smooth eigenvalues $\lambda(\epsilon) =$

$\pm e^{-\frac{1}{\epsilon^2}}$, $\lambda(0) = 0$, yet its normalised eigenvectors

$$v(\epsilon) = \begin{pmatrix} \frac{\sin\left(\frac{2}{\epsilon}\right)}{\sqrt{2(1\mp\cos\left(\frac{2}{\epsilon}\right)}} \\ \pm\sqrt{\frac{1\mp\cos\left(\frac{2}{\epsilon}\right)}{2}} \end{pmatrix},$$

smooth for all $\epsilon \neq 0$, have no limit as $\epsilon \to 0$.

A considerable obstacle to extending known results about analytic perturbation of eigenvalues to the multi-parameter case is knowledge of the structure of the sets where the eigenvalues change multiplicity, as they are no longer isolated points in $\mathbb{C}$.

Example B.2. The matrix

$$A(x,y) = \begin{pmatrix} 0 & x \\ y & 0 \end{pmatrix}$$

has eigenvalues $\lambda(x,y) = \pm\sqrt{xy}$ and eigenvectors $v(x,y) = (\pm\sqrt{x}, \sqrt{y})^T$. Multiplicity of the eigenvalues changes on the x and y axes, on which again the eigenvalues are analytic (as a function of one variable) away from $(x,y) = (0,0)$.

In the general case, as eigenvectors are not uniquely defined we often instead consider the projection matrix onto the eigenspace or generalised eigenspace. Furthermore, given any total ordering $\prec$ on $\mathbb{C}$ indices can be chosen so that $\lambda_i(x) \prec \lambda_j(x) \Leftrightarrow i < j$, so that this non-uniqueness problem is also solved. Henceforth we follow both of these conventions.

If we were to consider only local behaviour near a change in multiplicity of the eigenvalues we have the following result.

Lemma B.3. *Let $A(x)$ be an $N \times N$ analytic matrix function of $x \in \mathbb{R}^n$ with continuous eigenvalues with multiplicity $\{\lambda_i(x)\}_{i=1}^N$ such that for some $1 \leqslant k \leqslant N$ $\lambda_1(0) = \lambda_2(0) = \ldots = \lambda_k(0) \neq \lambda_l(0)$ for $l > k$. Then if $P(x)$ is the projection matrix onto the Minkowski sum of the generalised eigenspaces of $\lambda_1(x), \ldots, \lambda_k(x)$, $P(x)$ is analytic in an open neighbourhood of* 0.

Proof. We first prove this in the case of $A(x)$ a function of $x \in \mathbb{R}$. We define the resolvent of $A(x)$ as

$$R(x, \gamma) = (A(x) - \gamma I_N)^{-1}.$$

For fixed x the singularities of $R(x, \cdot)$ are exactly the eigenvalues of $A(x)$. Consider the Laurent series of $R(x, \cdot)$ centred at such an eigenvalue $\lambda_i(x)$, and let $Q_i(x)$ denote the negative of the coefficient of the $(\gamma - \lambda_i(x))^{-1}$ term. $Q_i(x)$ is a projection onto the generalised $\lambda_i(x)$-eigenspace, which is in turn contained in the nullspace of Q_j for $j \neq i$ [1][Chapter 1, Section 5.3]. Then although each matrix $Q_i(x)$, $i \leqslant k$ may have singularities at $x = 0$, $\sum_{i=1}^k Q_i(x)$ is analytic in a neighbourhood of $x = 0$ [1][Chapter 2, Section 1.4]. Therefore if $v_1(0), \ldots, v_l(0)$ is a basis for the generalised $\lambda_1(0)$-eigenspace,

$$\left\{ \sum_{i=1}^k Q_i(x) v_j(0) \right\}_{j=1}^l$$

is an analytic basis for the Minkowski sum of the generalised eigenspaces of $\lambda_1(x)$,...,$\lambda_k(x)$ for sufficiently small x. Then as dot products and the Gram-Schmidt algorithm are analytic functions of their inputs, we can use this analytic basis to construct an analytic projection matrix $P(x)$.

To extend this to the multivariate case, it is known by Boman's Theorem [10] that as $P(x(t))$ is smooth at $t = 0$ for all smooth paths $x(t)$ satisfying $x(0) = 0$, that $P(x)$ is smooth and therefore has a multivariate power series. As $P(tx_0)$ is also analytic for all x_0 in the unit sphere in $\mathbb{R}^n$ this power series converges for $|t| < R_{x_0}$ and is equal to $\mathcal{P}(x)$ whenever it converges. This implies that the power series converges for all $|x| < R = \inf_{|x_0|=1} R_{x_0} > 0$ and $R > 0$ by compactness. $\square$

Remark B.4. Note first that these matrices Q_i are not orthogonal projection matrices, so that they differ from the projection matrices P_i that we will construct in Theorem 3.10. Secondly, to our knowledge there is no analogous result allowing us to construct an analytic basis for the Minkowski sum of the eigenspaces near a change in multiplicity rather than the Minkowski sum of the generalised eigenspaces. This is why we stipulate in Assumption 3.3 that the matrix $\left(x^T C^i\right)_{i \leqslant N}$ is diagonalisable.

Of course if we allow $A(x)$ to be a function of $x \in U$ a subset of $\mathbb{R}^N$ or $\mathbb{C}^N$, there exists a partition of U into sets S_i such that for $x \in S_i$ each eigenvalue is of constant multiplicity. However, as many of these sets will not be open, even defining analyticity on these sets is a non-trivial task. In order to make any meaningful statements about the behaviour of the eigenvalues on these sets we need to understand the structure of these sets. To answer these questions we will use tools and results from real algebraic geometry, as when the given matrix $A(x)$ is polynomial rather than analytic in $x \in \mathbb{R}^N$ the appropriate partition of $\mathbb{R}^N$ necessary happens to be constructed from semi-algebraic sets.

The following definitions and results from real algebraic geometry are not the most general but are consistent with the literature in the case of Euclidean space.

Definition B.5. *(a) $S \subset \mathbb{R}^N$ is said to be semi-algebraic if it can be represented as the finite union of sets of the form*

$$\{x \in \mathbb{R}^N : f_1(x) = 0, \dots f_l(x) = 0, g_1(x) > 0, \dots, g_m(x) > 0\}$$

where $f_1(x), \dots, f_l(x), g_1(x), \dots g_m(x)$ are polynomials in x with real coefficients. If S is the finite union of sets of the form

$$\{x \in \mathbb{R}^N : f_1(x) = 0, \dots f_l(x) = 0\}$$

we say that S is an algebraic set.

(b) A function $f : S \to T$ from a semi-algebraic set $S \subset \mathbb{R}^M$ to a semi-algebraic set $T \subset \mathbb{R}^N$ is semi-algebraic if its graph in $\mathbb{R}^{M+N}$ is a semi-algebraic set.

Semi-algebraic sets are closed under finite union, intersection and complementation. We will also use the following basic results about semi-algebraic sets throughout this section.

Proposition B.6. *(a) If $S \subset \mathbb{R}^{N+1}$ is semi-algebraic and*

$$\Pi : \mathbb{R}^{N+1} \to \mathbb{R}^N : (x_1, \dots, x_N, x_{N+1}) \mapsto (x_1, \dots, x_N)$$

is a projection then $\Pi(S) \subset \mathbb{R}^N$ is semi-algebraic.

(b) A set of the form

$$\{x \in \mathbb{R}^N : f_1(x) = 0, \dots f_l(x) = 0, g_1(x) > 0, \dots, g_m(x) > 0\}$$

for semi-algebraic $f_j, g_j : \mathbb{R}^N \to \mathbb{R}$ is semi-algebraic.

(c) If $f : S \to T$ is semi-algebraic and $A \subset S$, $B \subset T$ are semi-algebraic then $f(A)$ and $f^{-1}(B)$ are semi-algebraic.

(d) For any semi-algebraic A, the set of semi-algebraic functions defined on A form a ring.

(e) Every semi-algebraic $B \subset \mathbb{R}^n$ is the disjoint union of a finite number of connected semi-algebraic sets $B_1, \ldots, B_m$.

Proof. (a) [4][Theorem 2.2.1].

(b) If $f_j, g_j : \mathbb{R}^N \to \mathbb{R}$ are semi-algebraic, the set

$$\{x : f_j(x) = 0\},\ (\text{resp. } \{x : g_j(x) > 0\})$$

is equal to the projection from $\mathbb{R}^n \times \mathbb{R}$ to $\mathbb{R}^n$ of the intersection of the graph of f_j (resp. g_j) with the semi-algebraic set

$$\left\{y \in \mathbb{R}^{N+1} : y_{N+1} = 0\right\}\ (\text{resp. } \left\{y \in \mathbb{R}^{N+1} : y_{N+1} > 0\right\})$$

and so is semi-algebraic.

(c) [4][Proposition 2.2.7].

(d) [4][Proposition 2.2.6].

(e) [4][Theorem 2.4.5].

□

Real algebraic geometry also has its own version of the Implicit Function Theorem.

Proposition B.7 (Implicit Function Theorem)**.** *Let* $(x_0, y_0) \in \mathbb{R}^{m+n}$ *and let* $f_1, \ldots, f_n$ *be* $k \geqslant 1$ *times continuously differentiable semi-algebraic functions defined on an open neighbourhood of* (x_0, y_0) *such that* $f_1(x_0, y_0) = \ldots = f_n(x_0, y_0) = 0$ *and the matrix*

$$\left(\frac{df_i}{dy_j}(x_0, y_0)\right)_{ij}$$

is invertible. Then there exists an open semi-algebraic neighbourhood U *(resp.* V*) of* x_0 *(resp.* y_0*) in* $\mathbb{R}^m$ *(resp.* $\mathbb{R}^n$*) and a function* $\varphi : U \to V$ *with all partial derivatives up to order* k *existing and continuous such that* $\varphi(x_0) = y_0$ *and*

$$f_1(x, y) = \ldots = f_n(x, y) = 0 \Leftrightarrow y = \varphi(x)$$

for every $(x, y) \in U \times V$*.*

Proof. [4][Corollary 2.9.8]. □

A fundamental object in real algebraic geometry is a Nash manifold.

Definition B.8. *(a) A semi-algebraic function* $f : U \to \mathbb{R}$ *for open semi-algebraic* $U \subset \mathbb{R}^N$ *is a Nash function if it is an analytic function that can be represented as the solution of* $p_n(x)f(x)^n + p_{n-1}(x)f(x)^{n-1} + \ldots + p_0(x) = 0$ *for some* $n > 0$ *where each* $p_k(x)$ *are polynomials in* x *with real coefficients.* $f : U \to \mathbb{R}^M$ *is Nash if each of its components* $f_i : U \to \mathbb{R}$ *is Nash.*

(b) A semi-algebraic set $M \subset \mathbb{R}^N$ *is said to be a Nash submanifold of* $\mathbb{R}^N$ *of dimension* d *if for every* $x \in M$ *there exists a neighbourhood* $U \subset \mathbb{R}^N$ *of* x*, a neighbourhood* $U' \subset \mathbb{R}^N$ *of 0 and a Nash diffeomorphism* $f : U \to U'$ *such that* $f(x) = 0$ *and* $f(M \cap U) = (\mathbb{R}^d \times \{0\}) \cap U'$*.*

A necessary and sufficient condition for a function defined on an open semi-algebraic subset of $\mathbb{R}^N$ to be Nash is for it to be smooth and semi-algebraic [4][Proposition 8.1.8]. Therefore the Implicit Function Theorem also applies to the case of Nash functions; if each f_i is Nash, the corresponding implicit function φ is Nash. Note also that this definition of Nash submanifold coincides with the classical definitions [8] of C^k, smooth or analytic submanifolds of Euclidean space.

The following is a useful alternative characterisation of Nash manifolds, analogous to similar results about smooth or analytic submanifolds of $\mathbb{R}^N$.

Lemma B.9. *The following are equivalent for $S \subset \mathbb{R}^N$.*

(a) S is a Nash manifold of dimension d.

(b) For every $x \in S$ there exists open semi-algebraic $V \ni x$ in $\mathbb{R}^N$, $U \subset \mathbb{R}^d$ and a Nash function $f : U \to V$ that is a homeomorphism onto $S \cap V$ and $Df = \left(\frac{df_i}{dy_j}\right)_{i \leqslant M, j \leqslant d}$ is injective for all $y \in U$.

(c) For every $x \in S$ there exists open semi-algebraic $B \ni x$ in $\mathbb{R}^N$, $A \subset \mathbb{R}^d$, permutation σ on $\{1, \ldots, N\}$ and Nash function $g : A \to \mathbb{R}^{N-d}$ such that

$$S \cap B = \{z \in \mathbb{R}^N : (z_{\sigma(d+1)}, \ldots, z_{\sigma(N)}) = g(z_{\sigma(1)}, \ldots, z_{\sigma(d)})\}$$

Proof. That $(a) \Rightarrow (b)$ is immediate. To see that $(b) \Rightarrow (c)$, fix $x \in S$ with given $f : U \to V$ and let σ be such that columns $\sigma(1), \ldots, \sigma(d)$ of Df form an injective matrix at $y = f^{-1}(x)$. If $\pi : \mathbb{R}^N \to \mathbb{R}^d : (z_1, \ldots, z_N) \mapsto (z_{\sigma(1)}, \ldots, z_{\sigma(d)})$ then $D(\pi \circ f)$ is an isomorphism, so by the inverse function theorem for semi-algebraic functions

[4][Proposition 2.9.7] there are open semi-algebraic $B' \subset U$, $A \subset \pi(V)$ such that $(\pi \circ f|_{B'})^{-1} : A \to B'$ exists and is Nash. By letting g be the $\sigma(d+1), \ldots, \sigma(N)$ components of $f \circ (\pi \circ f|_{B'})^{-1}$ with B an open neighbourhood of $f(B')$ such that $S \cap B = f(B')$ (c) is proved.

To show that $(c) \Rightarrow (a)$, fix $x \in S$ with $g : A \to \mathbb{R}^{N-d}$. Then

$$
\begin{aligned}
&f(x_1, \ldots, x_N) \\
=&(x_{\sigma(1)}, \ldots, x_{\sigma(d)}, x_{\sigma(d+1)} - g_1(x_{\sigma(1)}, \ldots, x_{\sigma(d)}), \ldots, x_{\sigma(M)} - g_{N-d}(x_{\sigma(1)}, \ldots, x_{\sigma(d)})),
\end{aligned}
$$

with

$$
\begin{aligned}
&f^{-1}(y_1, \ldots, y_N) \\
=&\pi^{-1}(y_1, \ldots, y_d, y_{d+1} + g_1(y_1, \ldots, y_d), \ldots, y_N + g_{N-d}(y_1, \ldots, y_d))
\end{aligned}
$$

is a Nash diffeomorphism defined on an open neighbourhood of $x \in \mathbb{R}^N$ with $f_{d+1}(x) = \ldots = f_N(x) = 0$ if and only if $x \in S \cap B$. □

The centrality of Nash manifolds to real algebraic geometry is elucidated in the following result.

Proposition B.10. *Every semi-algebraic $G \subset \mathbb{R}^N$ with a finite collection $\{F_\alpha\}_\alpha$ of semi-algebraic subsets of G can be stratified into a finite number of disjoint connected Nash manifolds G_i such that $G_i \cap clos(G_j) \neq \varnothing \Rightarrow G_i \subset clos(G_j)$ and $dim(G_i) < dim(G_j)$, and such that each F_α is a union of some strata of this stratification.*

Proof. [4][Proposition 9.1.8]. □

Finally we have the following result relating a Nash manifold to its ambient space.

Proposition B.11. *If $S \subset \mathbb{R}^N$ is a Nash manifold, there exists an open semi-algebraic neighbourhood U of S such that the projection $\rho(x) = argmin_{y \in S} d(x, y)$ is well-defined and Nash.*

Proof. [4][Corollary 8.9.5]. □

We previously stated the problem in even defining analyticity of eigenvalues and eigenvectors on non-open regions. We first address this.

Definition B.12. *Let $S \subset \mathbb{R}^N$ be a non-open Nash manifold. We say that $f : S \to \mathbb{R}$ is a Nash function on S if there exists an open semi-algebraic neighbourhood U of S and a Nash function $\tilde{f} : U \to \mathbb{R}$ such that $\tilde{f}\Big|_S = f$.*

As we are exploiting results in real algebraic geometry, we will identify complex numbers as elements of $\mathbb{R}^2$ rather than $\mathbb{C}$, so that we say that a function $f : \mathbb{R}^N \to \mathbb{C}^k$ is Nash if the corresponding function from $\mathbb{R}^N$ to $\mathbb{R}^{2k}$ is Nash.

Our main results are given below.

Theorem B.13. *Given a polynomial*

$$p_0(x) + p_1(x)\lambda + \ldots + p_n(x)\lambda^n,$$

where each p_i are Nash functions defined on a semi-algebraic $S \subset \mathbb{R}^N$, there exists a stratification of S, $\{S_j\}$, on which each $\lambda_i : S_j \to \mathbb{R}^2$ is a Nash function of x on each S_j.

Theorem B.14. *Given any matrix $A(x) \in \mathbb{R}^{n\times n}$ whose entries are Nash functions defined on some semi-algebraic $S \subset \mathbb{R}^N$, there exists a stratification of S into Nash manifolds $\{S_j\}_j$, on which each eigenvalue is of constant multiplicity with eigenspace of constant dimension, and each λ_i and the projection matrix P_i onto its eigenspace are Nash.*

Moreover, for any semi-algebraic set $A \subset \mathbb{R}^{2+2n^2}$, if it holds for all $x \in S$ that some pair $(\lambda_i(x), P_i(x)) \in A$, then the above stratification can be refined so that on each S_j, there is some i such that $(\lambda_i(x), P_i(x)) \in A$ for all $x \in S_j$.

Our motivation to prove the final statement is related to Assumption 3.3; however here we will not discuss our ultimate motivations and instead restrict ourselves to proving results related to real algebraic geometry and matrix perturbation theory.

We will first characterise the regions where the roots of such a polynomial are of constant multiplicity.

Lemma B.15. *Given a polynomial $f(p,\lambda) = p_0 + p_1\lambda + \ldots + p_n\lambda^n$, the set*

$$A_k = \{p \in \mathbb{R}^{n+1} : f(p,\lambda) \text{ has } k \text{ distinct roots}\}$$

is a semi-algebraic subset of $\mathbb{R}^{n+1}$ for each $k \in \{1,\ldots,n,\infty\}$.

Proof. First see that $A_\infty = \{0\}$. For some $1 \leqslant d \leqslant n$, assume throughout that $p_{d+1} = p_{d+2} = \ldots + p_n = 0$, $p_d \neq 0$, so that $f(p,x)$ is of degree d. Let $g =$ degree of $GCD\left(f(p,\lambda), \frac{df}{d\lambda}(p,\lambda)\right)$, the number of repeated roots of $f(p,\lambda)$. Then $f(p,\lambda)$ had $d-g$ distinct roots and the degree of $LCM\left(f(p,\lambda), \frac{df}{d\lambda}(p,\lambda)\right) = 2d-g-1$.

Let $q(p,\lambda), r(p,\lambda), s(p,\lambda)$ be polynomials of degree g, $d-g$ and $d-g-1$ respectively such that

$$GCD\left(f(p,\lambda), \frac{df}{d\lambda}(p,\lambda)\right) = q(p,\lambda),$$

$$f(p,\lambda) = r(p,\lambda)q(p,\lambda),$$

$$\frac{df}{d\lambda}(p,\lambda) = s(p,\lambda)q(p,\lambda).$$

Then

$$f(p,\lambda)s(p,\lambda) - \frac{df}{d\lambda}(p,\lambda)r(p,\lambda) = 0.$$

The condition that $f(p,\lambda)$ has at most k distinct roots is equivalent to the condition $d-g \leqslant k$, which is equivalent to the condition that there are polynomials $s^*(x,\lambda) = x_0 + x_1\lambda + \ldots + x_{k-1}\lambda^{k-1}$, $r^*(x,\lambda) = x_k + x_{k+1}\lambda + \ldots + x_{2k-1}\lambda^k$ such that

$$f(p,\lambda)s^*(x,\lambda) - \frac{df}{d\lambda}(p,\lambda)r^*(x,\lambda) = 0.$$

As we can write

$$f(p,\lambda)s^*(x,\lambda) - \frac{df}{d\lambda}(p,\lambda)r^*(x,\lambda) := y_0(x,p) + y_1(x,p)\lambda + \ldots + y_{d+k-1}(x,p)\lambda^{d+k-1},$$

this condition is equivalent to the condition that the system

$$y_1(p,x) = 0, \ldots, y_{d+k-1}(p,x) = 0 \tag{B.1}$$

has non-zero solutions in x. As each $y_i(p,x)$ is linear in x this is equivalent to the the vanishing of all $(2k+1)$-minors of the matrix associated with the linear map $x \mapsto (y_1(p,x), \ldots y_{d+k-1}(p,x))$. As each of these minors is a polynomial in p, the set

$$M_{d,k} = \{p \in \mathbb{R}^{n+1} : f \text{ is of degree } d \text{ and there at most } k \text{ distinct roots}\}$$

is a semi-algebraic set. As

$$A_k = \bigcup_{d \geqslant k} M_{d,k} \backslash M_{d,k-1},$$

each A_k is a semi-algebraic set for each $k \geqslant 1$. □

Corollary B.16. *Given a polynomial* $f(p(x), \lambda) = p_0(x) + p_1(x)\lambda + \ldots + p_n(x)\lambda^n$, *where each* p_i *is a semi-algebraic function of* x *defined on some common semi-algebraic set* $S \subset \mathbb{R}^m$, *the set* $A_k = \{x \in S : f(p(x), \lambda)$ *has* k *distinct roots*$\}$ *is a semi-algebraic subset of* $\mathbb{R}^m$ *for each* $k \in \{1, \ldots, n\}$.

Proof. Let $B_k = \{p \in \mathbb{R}^{n+1} : f(p, \lambda)$ has k distinct roots$\}$. Then $A_k = p^{-1}(B_k)$ is semi-algebraic. □

We are now ready to prove Theorem B.13.

Proof of Theorem B.13. Let G be the multi-graph of the roots, which is a semi-algebraic subset of $\mathbb{R}^{n+3}$ as it is the set of solutions (x, a, b) to

$$p_0(x) + p_1(x)(a + ib) + \ldots + p_n(x)(a + ib)^n = q(x, a, b) + ir(x, a, b) = 0$$

for some Nash $q, r : \mathbb{R}^{n+3} \to \mathbb{R}$. Let G_i be the graph of λ_i, and let

$$f(p, \lambda) = p_0 + p_1\lambda + \ldots + p_n\lambda^n.$$

Let $A_k = \{p \in \mathbb{R}^{n+1} : f(p, \lambda)$ has k distinct roots$\}$, and let $\{A_{kl}\}_l$ be the connected components of A_k. For each l, let $\{B_{klm}\}_m$ be the semi-algebraic connected components of $B_{kl} = (A_{kl} \times \mathbb{R}^2) \cap G$, the multi-graph restricted to A_{kl}. B_{kl} can also be decomposed into $\{C_{kli}\}_i$, the graphs of the distinct $\lambda_i(p)$ restricted to A_{kl}. These C_{kli} are also connected as each λ_i is distinct and continuous on the connected set

A_{kl}. Therefore it must follow that each $C_{kli} = B_{klm}$ for some m, so that C_{kli} is semi-algebraic and so each $\lambda_i(p)$ is a semi-algebraic function on each A_{kl}. Defining $E_{kl} = p^{-1}(A_{kl}) = \{x \in S : p(x) \in A_{kl}\}$, each $\lambda_i(x)$ is a semi-algebraic function on each E_{kl}.

Now let $\{E_{klm}\}_{k,l,m}$ be a Nash stratification of S such that $\cup_m E_{klm} = E_{kl}$ for each k, l. Fix some E_{klm} and some $x \in E_{klm}$, and let $g : U \to U'$ be the Nash diffeomorphism described in Definition B.8 (b). Then the polynomial

$$p_0\left(g^{-1}\big|_{\mathbb{R}^d\times\{0\}}(y)\right) + p_1\left(g^{-1}\big|_{\mathbb{R}^d\times\{0\}}(y)\right)\lambda + \ldots + p_n\left(g^{-1}\big|_{\mathbb{R}^d\times\{0\}}(y)\right)\lambda^n$$

has smooth roots $\lambda_i \circ g^{-1}(y)$ by applying the Smooth Implicit Function Theorem to the correct derivative of the above polynomial with respect to λ.

Then let ρ be the Nash projection onto E_{klm} defined on U. Then the function $\widetilde{\lambda}_i(x) = \lambda \circ g^{-1} \circ g \circ \rho(x)$, defined on U, is Nash and agrees with λ_i on E_{klm}. $\square$

Remark B.17. One could recreate much of the spirit of the above work in the case of each $p_l(z)$ being complex-valued functions defined on some subset of $\mathbb{C}^N$ such that $p_l(z) = p_l(x, y) = u_l(x, y) + iv(x, y)$ where u, v are Nash functions defined on some open semi-algebraic $S \subset \mathbb{R}^{2N}$. However the concluding result, that $\lambda_k(x, y)$ is a Nash function of x, y, is not enough to prove that $\lambda_k(z)$ is an analytic function of z as one cannot recover any information about the Cauchy-Riemann equations for λ_k. It is for this reason that we restrict our result to the case of polynomials with real coefficients.

Remark B.18. It may also be possible to recreate much of this work in the case of the p_i being real analytic functions and relying on analytic submanifolds. However this would result in a stratification into a countable number of submanifolds which is not useful for our purposes.

We will now prove the following.

Lemma B.19. *For any matrix $A(x) \in \mathbb{R}^{n\times n}$ whose entries are Nash functions defined on some semi-algebraic $S \subset \mathbb{R}^N$, such that dim$(N(A(x))) \geqslant 1$ is constant on S, one can locally define an orthonormal basis for $N(A(x))$. Furthermore the projection matrix $P(x)$ onto $N(A(x))$ is Nash.*

Proof. Let $k = \dim(N(A(x))) \geqslant 1$, and given some $x_0 \in S$ let $v_1(x_0), \ldots, v_k(x_0)$ be any basis for $N(A)$. There exist $n-k$ linearly independent rows of $A(x_0)$, which are also linearly independent in a neighbourhood U of x_0. On U define $B(x)$ as the matrix with these rows. Now consider the linear transformation

$$w \mapsto \begin{pmatrix} B(x) \\ v_1(x_0) \\ v_2(x_0) \\ \vdots \\ v_k(x_0) \end{pmatrix} w.$$

Note that this matrix is invertible. Then by Proposition B.7 there exists a Nash continuation of each $v_1, \ldots, v_k$ in some open $V \subset U$ such that the norm of each v_i is bounded away from 0. As the function $g(x) = \frac{1}{\sqrt{x}}$, defined on $x > 0$, is smooth and has graph given by $y^2x - 1 = 0$, $y > 0$ it is Nash. Thus normalising each vector preserves their Nash property and as Gram-Schmidt is then a polynomial operation we have an orthonormal Nash basis $v_1(x), \ldots, v_k(x)$ defined on V. This implies that the projection matrix $P(x) = \sum_i v_i(x)v_i(x)^T$ (viewing the v_i as column vectors) is Nash on V, and as P is unique it can be extended as a Nash function to all of S.

□

We are now ready to prove Theorem $B.14$.

Proof of Theorem B.14. From Theorem B.13, we can immediately stratify S into Nash $\{T_a\}$ such that each eigenvalue is Nash and of constant multiplicity on each T_a. Fix T_a, and note that the set

$$B_{iak} = \{x \in T_a : \dim\left(N\left(A(x) - \lambda_i(x)I_n\right)\right) = k\}$$

is a semi-algebraic set as it is equivalent to the vanishing of all $(n-k+1)$-minors of the Nash $A(x) - \lambda_i(x)I_n$ and the non-vanishing of at least one $(n-k)$-minor. As each collection $\{B_{iak}\}_k$ is a partition of T_a, we can construct a partition

$$\{B_{1ak_1} \cap B_{2ak_2} \cap \ldots \cap B_{nak_n}\}_{a \in \{1,\ldots,n\},(k_i) \in \{1,\ldots,n\}^n}$$

of S. If we stratify S into $\{S_j\}$ such that each element of this partition is a union of some $\{S_j\}$, then on each $\{S_j\}$ each eigenvalue is of constant multiplicity and is Nash, and each eigenspace is of constant dimension. Applying Lemma B.19 immediately gives the Nash projection matrix $P_i(x)$ onto the eigenspace of λ_i defined on S_j.

Now let $A \subset \mathbb{R}^{2+2n^2}$ be a semi-algebraic set with the given assumptions. Then let $D_i = \{x : (\lambda_i(x), P_i(x)) \in A\}$, and refine the stratification so that each D_i is the union of some strata S_j. As the D_i cover S this completes the proof. □

Remark B.20. The fact that we cannot make more impressive statements about the eigenvectors stems from similar problems encountered by others, namely that they are not uniquely defined and are difficult to separate from generalised eigenvectors [1]. Moreover exploiting the projection may result in vectors that vanish. This was dealt with in [5] by exploiting the Smith-Normal form of matrices, which is difficult to do in the multivariate case, and in [1] by assuming that the domain in which the eigenvalues are of constant multiplicity is simply-connected and constructing an analytic invertible matrix $U(x)$ that gave a bijection between the generalised eigenspace at ϵ and the generalised eigenspace at some anchor point x_0. $U(x)$ was given as the solution of a matrix-valued ODE whose existence, uniqueness and analyticity was justified using Picard iterates.

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Biography

Brendan Williamson graduated from Dublin City University in 2013 with a degree in Actuarial Mathematics, before earning his PhD at Duke University in 2019 in Mathematics. In June 2019 he will begin a position as a Quantitative Assistant Trader at Susquehanna International Group in Dublin, Ireland.