

Derivation of the time dependent Gross-Pitaevskii equation for a class of non purely positive potentials

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January 16, 2018

Abstract

We present a microscopic derivation of the time-dependent Gross-Pitaevskii equation starting from an interacting N -particle system of Bosons. We prove convergence of the reduced density matrix corresponding to the exact time evolution to the projector onto the solution of the respective Gross-Pitaevskii equation. Our work extends a previous result by one of us (P.P. [44]) to interaction potentials which need not to be nonnegative, but may have a sufficiently small negative part. One key estimate in our proof is an operator inequality which was first proven by Jun Yin, see [49].

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1 Introduction

The main concern of this work is a generalization of a previous result presented by one of us (P.P. [44]). Specifically, we will analyze the dynamics of a Bose-Einstein condensate in the Gross-Pitaevskii regime for interactions V which need not to be nonnegative, but may have an attractive part.

Let us first define the N -body quantum problem we want to study. The evolution of N interacting bosons is described by a time-dependent wave-function $\Psi_t \in L_s^2(\mathbb{R}^{3N}, \mathbb{C})$, $\|\Psi_t\| = 1$ (throughout this paper norms without index $\|\cdot\|$ always denote the L^2 -norm on the appropriate Hilbert space.). The bosonic N -particle Hilbert space $L_s^2(\mathbb{R}^{3N}, \mathbb{C})$ denotes the set of all $\Psi \in L^2(\mathbb{R}^{3N}, \mathbb{C})$ which are symmetric under pairwise permutations of the variables $x_1, \dots, x_N \in \mathbb{R}^3$. Assuming in addition $\Psi_0 \in H^2(\mathbb{R}^{3N}, \mathbb{C})$, the evolution of Ψ_t is then described by the N -particle Schrödinger equation

$$i\partial_t \Psi_t = H \Psi_t . \quad (1)$$

The time-dependent Hamiltonian H we will study is defined by

$$H = - \sum_{j=1}^N \Delta_j + N^2 \sum_{1 \leq j < k \leq N} V(N(x_j - x_k)) + \sum_{j=1}^N A_t(x_j) . \quad (2)$$

In the following, we assume $A_t \in L^\infty(\mathbb{R}^3, \mathbb{R})$ and $V \in L_c^\infty(\mathbb{R}^3, \mathbb{R})$, V spherically symmetric. We will also use the common notation $V_1(x) = N^2 V(Nx)$. More generally, one can study the properties of Bose gases for a larger class of scaling parameters $0 \leq \beta \leq 1$, setting $V_\beta(x) = N^{-1+3\beta} V(N^\beta x)$. For $0 < \beta \leq 1$ and large particle number N , the potential gets δ -like, which indicates that the mathematical description may become more involved the bigger β is chosen. The so-called Gross-Pitaevskii regime $\beta = 1$ is special, since then the two-particle correlations play a crucial role for the dynamics, see Section 3.1.

We will derive an approximate solution of (1) in the trace class topology of reduced density matrices. Define the one particle reduced density matrix $\gamma_{\Psi_0}^{(1)}$ given by the integral kernel

$$\gamma_{\Psi_0}^{(1)}(x, x') = \int_{\mathbb{R}^{3N-3}} \Psi_0^*(x, x_2, \dots, x_N) \Psi_0(x', x_2, \dots, x_N) d^3 x_2 \dots d^3 x_N . \quad (3)$$

To account for the physical situation of a Bose-Einstein condensate, we assume complete condensation in the limit of large particle number N . This amounts to assume that, for $N \rightarrow \infty$, $\gamma_{\Psi_0}^{(1)} \rightarrow |\varphi_0\rangle\langle\varphi_0|$ in trace norm for some $\varphi_0 \in L^2(\mathbb{R}^3, \mathbb{C})$, $\|\varphi_0\| = 1$. Our main goal is to show the persistence of condensation over time. Let a denote the scattering length of the potential $\frac{1}{2}V$ (see Section 3.1 for the precise definition of a) and let φ_t solve the nonlinear Gross-Pitaevskii equation

$$i\partial_t \varphi_t = (-\Delta + A_t) \varphi_t + 8\pi a |\varphi_t|^2 \varphi_t =: h^{\text{GP}} \varphi_t \quad (4)$$

with initial datum φ_0 (we assume $\varphi_t \in H^2(\mathbb{R}^3, \mathbb{C})$, see below). We then prove that the time evolved reduced density matrix $\gamma_{\Psi_t}^{(1)}$ converges to $|\varphi_t\rangle\langle\varphi_t|$ in trace norm as $N \rightarrow \infty$ with convergence rate of order $N^{-\eta}$ for some $\eta > 0$.

The rigorous derivation of effective evolution equations has a long history, see e.g. [2, 3, 6, 7, 10, 12, 13, 14, 15, 18, 19, 20, 22, 24, 26, 35, 36, 38, 42, 39, 40, 44, 45, 47] and references

therein. The derivation of the three dimensional time-dependent Gross-Pitaevskii equation for nonnegative potentials was first conducted in [15]. Afterward, this result has been improved by [2, 3, 35, 44]. In the two dimensional case, the correspondent time-dependent Gross-Pitaevskii equation was treated in [18]. Note that in two dimensions, the scaling considered is given by $e^{2N}V(e^N x)$. The ground state properties of dilute Bose gases were treated in [4, 5, 28, 29, 30, 31, 33, 41, 46, 49], see also the monograph [32] and references therein.

As mentioned previously, we will generalize the result presented by one of us (P.P. [44]) to a specific class of interactions V which are not assumed to be nonnegative everywhere. Let us stress that persistence of condensation is not expected for arbitrary V . For strongly attractive potentials, even a small fraction of particles which leave the condensate over time may cluster, subsequently causing the condensate to collapse in finite time. The dynamical collapse of a Bose gas under such circumstances is well known within the physical community and was mathematically treated in [36]. The breakdown of condensation has also been observed in experiments [16]. Consequently, the result we are going to prove can only be valid under certain restrictions on V . The class of potentials we consider is chosen such that V has a repulsive core, i.e. there exists a $r_1 > 0$, such that $V(x) \geq \lambda^+$, for some $\lambda^+ > 0$ and for all $|x| \leq r_1$. This condition prevents clustering of particles. If furthermore the negative part of V fulfills some restrictions (see assumption 2.2), a result by Jun Yin [49] then implies that the Hamiltonian we consider in this note is stable of second kind. The author proves in particular that for such potentials the ground state energy per particle of a dilute, homogeneous Bose gas is at first order given by the well-known formula $4\pi a \rho N$. Among the steps of the proof in [49], it is shown that the Hamiltonian (2) -without external potential A_t - restricted to configurations where at least three particles are close to each other is a nonnegative operator. We will adapt this non-trivial operator inequality in our proof to control the kinetic energy of those particles which leave the condensate, see Lemma 3.23. We like to remark that the assumptions 2.2 on V stated below imply that the scattering length a of the potential $\frac{1}{2}V$ is nonnegative. Consequently, the effective Gross-Pitaevskii dynamics (4) is repulsive, which reflects the fact that the condensate is stable.

The result presented in [49] implies further that there exists an $\epsilon > 0$, such that

$$-\epsilon \sum_{k=1}^N \Delta_k \leq - \sum_{k=1}^N \Delta_k + \sum_{i<j} V_1(x_i - x_j), \quad (5)$$

$$\epsilon \sum_{i<j} |V_1(x_i - x_j)| \leq - \sum_{k=1}^N \Delta_k + \sum_{i<j} V_1(x_i - x_j). \quad (6)$$

The first operator inequality bounds $\|\nabla_1 \Psi_t\|$ uniformly in N , if initially the energy per particle is of order 1. If this were not the case, one cannot expect condensation, see e.g. [36] for a nice discussion. Under the same assumption, the second inequality (6) implies $\|V_1(x_1 - x_2)\Psi_t\| \leq N^{1/2}$, see Lemma 3.21. These two inequalities are crucial in our proof to control the rate of particles which leave the condensate over time and thus to extend the result presented in [44].

2 Main Result

We will bound expressions which are uniformly bounded in N by some (possible time-dependent) constant $C > 0$. We will not distinguish constants appearing in a sequence of estimates,

i.e. in $X \leq CY \leq CZ$ the constants usually differ. We denote by $\langle\langle \cdot, \cdot \rangle\rangle$ the scalar product on $L^2(\mathbb{R}^{3N}, \mathbb{C})$ and by $\langle \cdot, \cdot \rangle$ the scalar product on $L^2(\mathbb{R}^3, \mathbb{C})$. We will use the notation $B_r(x) = \{z \in \mathbb{R}^3 \mid |x - z| < r\}$.

Define the energy functional $\mathcal{E} : H^2(\mathbb{R}^{3N}, \mathbb{C}) \rightarrow \mathbb{R}$

$$\mathcal{E}(\Psi) = N^{-1} \langle\langle \Psi, H\Psi \rangle\rangle, \quad (7)$$

as well as the Gross-Pitaevskii energy functional $\mathcal{E}^{GP} : H^2(\mathbb{R}^3, \mathbb{C}) \rightarrow \mathbb{R}$

$$\mathcal{E}^{GP}(\varphi) := \langle \nabla \varphi, \nabla \varphi \rangle + \langle \varphi, (A_t + 4\pi a |\varphi|^2) \varphi \rangle = \langle \varphi, (h^{GP} - 4\pi a |\varphi|^2) \varphi \rangle. \quad (8)$$

Next, we will define the class of interaction potentials V we will consider. This class is essentially the one considered in [49], Theorem 2; see also Corollary 1 and Corollary 2 in [49] for a different characterization of the class of potentials V . In this note, we require in addition that the potential changes its sign only once. This facilitates the discussion of the scattering state, see Section 3.1. In principle, one could ease this additional assumption by generalizing the proofs given in Section 3.1

Definition 2.1 Divide \mathbb{R}^3 into cubes C_n , $n \in \mathbb{Z}$ of side length $b_1/\sqrt{3}$; that is $\mathbb{R}^3 = \cup_{n=-\infty}^{\infty} C_n$. Furthermore, assume that $\overset{\circ}{C}_n \cap \overset{\circ}{C}_m = \emptyset$ for $m \neq n$. Define

$$n(b_1, b_2) = \max_{x \in \mathbb{R}^3} \#\{n : C_n \cap B_{b_2}(x) \neq \emptyset\}.$$

Thus, $n(b_1, b_2)$ gives the maximal number of of cubes with side length $b_1/\sqrt{3}$ one needs to cover a sphere with radius b_2 . We remark that $4\sqrt{3}\pi(\frac{b_2}{b_1} - 1)^3 \leq n(b_1, b_2) \leq \frac{\frac{4\pi}{3}(b_1+b_2)^3}{b_1^3 3^{-3/2}} = 4\sqrt{3}\pi(1 + \frac{b_2}{b_1})^3$.

Assumption 2.2 Let $V \in L_c^\infty(\mathbb{R}^3, \mathbb{R})$ spherically symmetric and let $V(x) = V^+(x) - V^-(x)$, where $V^+, V^- \in L_c^\infty(\mathbb{R}^3, \mathbb{R})$ are spherically symmetric, such that $V^+(x), V^-(x) \geq 0$ and the supports of V^+ and V^- are disjoint. Assume that

- (a) For $R > r_2 > 0$, we have $\text{supp}(V^+) = B_{r_2}(0)$ and $\text{supp}(V^-) = B_R(0) \setminus B_{r_2}(0)$.
- (b) There exists $\lambda^+ > 0$ and $r_1 > 0$, such that $V^+(x) \geq \lambda^+$ for all $x \in B_{r_1}(0)$.
- (c) Define $\lambda^- = \|V^-\|_\infty$ as well as $n_1 = n(r_1, R)$ and $n_2 = n(r_1, 3R)$. Define, for $0 < \epsilon < 1$,

$$\mathcal{E}_R(\varphi) = \int_{B_R(0)} \left(|\nabla_x \varphi(x)|^2 + \frac{1}{1-\epsilon} n_1 (2V^+(x) - 4V^-(x)) |\varphi(x)|^2 \right) d^3x. \quad (9)$$

We then assume that for some $0 < \epsilon < 1$

$$\inf_{\varphi \in C^1(\mathbb{R}^3, \mathbb{C}), \varphi(R)=1} (\mathcal{E}_R(\varphi)) \geq 0, \quad (10)$$

$$\lambda^+ > 8n_2 \lambda^-. \quad (11)$$

Remark 2.3 We will use the constants $r_1, r_2, R, \lambda^+, \lambda^-$, as well as n_1, n_2 throughout this paper as defined above.

Remark 2.4 Condition (10) implies $a \geq 0$, see Theorem C.1.,(C.8.) in [32]. Assumption 2.2 implies that there exists $\epsilon > 0, \mu > 0$ such that

$$-\sum_{k=1}^N \Delta_k + \sum_{i < j=1}^N (V_1^+(x_i - x_j) - (1 + \epsilon)V_1^-(x_i - x_j)) \geq 0, \quad (12)$$

$$-(1 - \mu) \sum_{k=1}^N \Delta_k + \sum_{i < j=1}^N V_1(x_i - x_j) \geq 0, \quad (13)$$

see Lemma 3.13 and Corollary 3.19. The operator inequality (12) can only hold for $a \geq 0$, see [48] and is thus in accordance with Condition (10). Thus, although the potential V may have an attractive part V^- , the effective Gross-Pitaevskii equation (4) is repulsive.

It also follows from assumption 2.2 (c)

$$-\Delta + \frac{1}{2}V \geq 0. \quad (14)$$

We now state the main Theorem:

Theorem 2.5 Let $\Psi_0 \in L_s^2(\mathbb{R}^{3N}, \mathbb{C}) \cap H^2(\mathbb{R}^{3N}, \mathbb{C})$ with $\|\Psi_0\| = 1$. Let $\varphi_0 \in H^2(\mathbb{R}^3, \mathbb{C})$ with $\|\varphi_0\| = 1$. Let $\lim_{N \rightarrow \infty} \text{Tr}|\gamma_{\Psi_0}^{(1)} - |\varphi_0\rangle\langle\varphi_0|| = 0$, as well as $\lim_{N \rightarrow \infty} \mathcal{E}(\Psi_0) = \mathcal{E}^{GP}(\varphi_0)$. Let Ψ_t the unique solution to $i\partial_t \Psi_t = H\Psi_t$ with initial datum Ψ_0 and assume that V fulfills assumption 2.2. Let φ_t the unique solution to $i\partial_t \varphi_t = h^{GP}\varphi_t$ with initial datum φ_0 and assume $\varphi_t \in H^2(\mathbb{R}^3, \mathbb{C})$. Let the external potential A_t fulfill $A_t, \dot{A}_t \in L^\infty(\mathbb{R}^3, \mathbb{R})$ for all $t \in \mathbb{R}$. Then,

(a) for any $t > 0$

$$\lim_{N \rightarrow \infty} \mu_1^{\Psi_t} = |\varphi_t\rangle\langle\varphi_t| \quad (15)$$

in operator norm.

(b) if $\int_0^\infty (\|\varphi_s\|_\infty + \|\nabla\varphi_s\|_{6,loc} + \|\dot{A}_s\|_\infty) ds < \infty$ where $\|\cdot\|_{6,loc} : L^2(\mathbb{R}^3, \mathbb{C}) \rightarrow \mathbb{R}^+$ is the “local L^6 -norm” given by

$$\|\varphi\|_{6,loc} := \sup_{x \in \mathbb{R}^3} \|\mathbf{1}_{|\cdot-x| \leq 1} \varphi\|_6,$$

then the convergence (15) is uniform in $t > 0$.

Remark 2.6 (a) Note that convergence of μ_1^{Ψ} to $|\varphi\rangle\langle\varphi|$ in operator norm is equivalent to convergence in trace norm, since $|\varphi\rangle\langle\varphi|$ is a rank one projection [47]. Other equivalent definitions of asymptotic 100% condensation can be found in [34].

(b) For potentials V which satisfy assumption 2.2, convergence of $\mathcal{E}(\Psi^{gs}) - \mathcal{E}^{GP}(\varphi^{gs}) \rightarrow 0$ was shown in [49] for homogeneous gases.

(c) By Sobolev’s inequality, it follows that $\|\nabla\varphi_s\|_{6,loc} \leq \|\nabla\varphi_s\|_6 \leq \|\Delta\varphi\|$. Thus $\|\nabla\varphi_s\|_{6,loc}$ can be bounded controlling $\langle\varphi_s, (h^{GP})^2 \varphi_s\rangle$ sufficiently well.

On the other hand, $\|\nabla\varphi_s\|_{6,loc} \leq \|\nabla\varphi_s\|_\infty$. Since we are in the defocussing regime one expects, after the potential is turned off, that $\|\varphi\|_\infty$ and $\|\nabla\varphi\|_\infty$ decay like $t^{-3/2}$. Whenever this is the case $\int_0^\infty (\|\varphi_s\|_\infty + \|\nabla\varphi_s\|_{6,loc} + \|\dot{A}_s\|_\infty) ds < \infty$ and we get convergence uniformly in t .

- (d) *Existence of solutions of the Gross-Pitaevskii equation is well understood. The condition $\varphi_t \in H^2(\mathbb{R}^3, \mathbb{C})$ can be proven for a large class of external potentials, assuming sufficient regularity of the initial datum φ_0 , see e.g. [9].*
- (e) *The proof of Theorem 2.5 implies that the rate of convergence is of order $N^{-\delta}$ for some $\delta > 0$, assuming that $|\gamma_{\Psi_0}^{(1)} - |\varphi_0\rangle\langle\varphi_0|| \leq CN^{-2\delta}$, as well as assuming that the convergence rate of $\lim_{N \rightarrow \infty} \mathcal{E}(\Psi_0) = \mathcal{E}^{GP}(\varphi_0)$ to be least of order $N^{-2\delta}$.*
- (f) *The Theorem can straightforwardly be adapted to the two-dimensional case. There, one considers the scaling $V_N(x) = e^{2N}V(e^N x)$, for $V \in L_c^\infty(\mathbb{R}^2, \mathbb{R})$ spherically symmetric, see [18]. Note that due to the different scaling behavior of the potential, most of the respective bounds given below read differently in two dimensions. In this note, we are mainly concerned with the three-dimensional case. However, we will also give the respective proofs of certain Lemmata for the two-dimensional system in cases where some nontrivial modifications are needed.*

3 Proof of Theorem 2.5

The method our proof relies on is explained in details in [45]. Heuristically speaking it is based on the idea of counting for each time t the relative number of those particles which are not in the state φ_t and estimating the time derivative of that value. In this note we will only focus on the modifications one needs to perform in order to generalize the result of [44] to more general interactions V . We will therefore often omit large parts of existing proofs and refer the reader to [44] for the detailed steps and motivations.

First, we will recall some important definitions we will need during the proof.

Definition 3.1 *Let $\varphi \in L^2(\mathbb{R}^3, \mathbb{C})$.*

- (a) *For any $1 \leq j \leq N$ the projectors $p_j^\varphi : L^2(\mathbb{R}^{3N}, \mathbb{C}) \rightarrow L^2(\mathbb{R}^{3N}, \mathbb{C})$ and $q_j^\varphi : L^2(\mathbb{R}^{3N}, \mathbb{C}) \rightarrow L^2(\mathbb{R}^{3N}, \mathbb{C})$ are defined by*

$$p_j^\varphi \Psi = \varphi(x_j) \int \varphi^*(\tilde{x}_j) \Psi(x_1, \dots, \tilde{x}_j, \dots, x_N) d^3 \tilde{x}_j \quad \forall \Psi \in L^2(\mathbb{R}^{3N}, \mathbb{C})$$

and $q_j^\varphi = 1 - p_j^\varphi$.

We also use the bra-ket notation $p_j^\varphi = |\varphi(x_j)\rangle\langle\varphi(x_j)|$. For better readability, we will sometimes use the notation p_j, q_j .

- (b) *For any $0 \leq k \leq N$ we define the set*

$$\mathcal{S}_k := \{(s_1, s_2, \dots, s_N) \in \{0, 1\}^N ; \sum_{j=1}^N s_j = k\}$$

and the orthogonal projector P_k^φ acting on $L^2(\mathbb{R}^{3N}, \mathbb{C})$ as

$$P_k^\varphi := \sum_{\vec{a} \in \mathcal{S}_k} \prod_{j=1}^N (p_j^\varphi)^{1-s_j} (q_j^\varphi)^{s_j} .$$

For negative k and $k > N$ we set $P_k^\varphi := 0$.

(c) For any function $m : \mathbb{N}_0 \rightarrow \mathbb{R}_0^+$ we define the operator $\widehat{m}^\varphi : L^2(\mathbb{R}^{3N}, \mathbb{C}) \rightarrow L^2(\mathbb{R}^{3N}, \mathbb{C})$ as

$$\widehat{m}^\varphi := \sum_{j=0}^N m(j) P_j^\varphi. \quad (16)$$

We furthermore define \widehat{n}^φ with $n(k) = \sqrt{\frac{k}{N}}$.

Definition 3.2 For any $1 \leq j \neq k \leq N$, let

$$a_{j,k} := \{(x_1, x_2, \dots, x_N) \in \mathbb{R}^{3N} : |x_j - x_k| < N^{-26/27}\}, \quad (17)$$

$$\overline{\mathcal{A}}_j := \bigcup_{k \neq j} a_{j,k} \quad \mathcal{A}_j := \mathbb{R}^{3N} \setminus \overline{\mathcal{A}}_j \quad \overline{\mathcal{B}}_j := \bigcup_{k,l \neq j} a_{k,l} \quad \mathcal{B}_j := \mathbb{R}^{3N} \setminus \overline{\mathcal{B}}_j. \quad (18)$$

(In two dimensions, the sets \mathcal{A}_j and \mathcal{B}_j are defined differently, see [18].) Furthermore, define for any set $A \subset \mathbb{R}^{3N}$ the operator $\mathbb{1}_A : L^2(\mathbb{R}^{3N}, \mathbb{C}) \rightarrow L^2(\mathbb{R}^{3N}, \mathbb{C})$ as the projection onto the set A .

Many Lemmata which were proven in [44] are valid for generic interaction potentials V and need not to be modified. In the following, we will state a general criteria under which assumptions on Ψ_t Theorem 2.5 is valid (see (b),(c) and (d) below). Subsequently, we prove that these assumptions are valid if the potential V fulfills assumption 2.2.

Lemma 3.3 Let $\Psi_0 \in L_s^2(\mathbb{R}^{3N}, \mathbb{C}) \cap H^2(\mathbb{R}^{3N}, \mathbb{C})$ with $\|\Psi_0\| = 1$. Let $\varphi_0 \in H^2(\mathbb{R}^3, \mathbb{C})$ with $\|\varphi_0\| = 1$. Let $\lim_{N \rightarrow \infty} \gamma_{\Psi_0}^{(1)} = |\varphi_0\rangle\langle\varphi_0|$ in trace norm as well as $\lim_{N \rightarrow \infty} \mathcal{E}(\Psi_0) = \mathcal{E}^{GP}(\varphi_0)$. Let Ψ_t the unique solution to $i\partial_t \Psi_t = H\Psi_t$ with initial datum Ψ_0 and assume $V \in L_c^\infty(\mathbb{R}^3, \mathbb{R})$ spherically symmetric. Let φ_t the unique solution to $i\partial_t \varphi_t = h^{GP} \varphi_t$ with initial datum φ_0 . Assume $A_t, \dot{A}_t \in L^\infty(\mathbb{R}^3, \mathbb{R})$. If,

(a)

$$\varphi_t \in H^2(\mathbb{R}^3, \mathbb{C}). \quad (19)$$

(b)

$$\|V_1(x_1 - x_2)\Psi_t\| \leq CN^{1/2}. \quad (20)$$

(c)

$$\|\nabla_1 \Psi_t\| \leq C. \quad (21)$$

(d) for some $\eta > 0$, the following inequality holds:

$$\|\mathbb{1}_{\mathcal{A}_1} \nabla_1 q_1^{\varphi_t} \Psi_t\|^2 + \|\mathbb{1}_{\overline{\mathcal{B}}_1} \nabla_1 \Psi_t\|^2 \leq C (\langle \Psi_t, \widehat{n}^{\varphi_t} \Psi_t \rangle + N^{-\eta}) + |\mathcal{E}(\Psi_t) - \mathcal{E}^{GP}(\varphi_t)|. \quad (22)$$

(e)

$$V \text{ is chosen such that Lemma 3.9 is fulfilled.} \quad (23)$$

Then, for any $t > 0$

$$\lim_{N \rightarrow \infty} \gamma_{\Psi_t}^{(1)} = |\varphi_t\rangle\langle\varphi_t| \quad (24)$$

in trace norm.

Remark 3.4 *It has been shown in [18, 44] that the conditions (20), (21),(22) and (23) are fulfilled for nonnegative potentials $V \in L_c^\infty(\mathbb{R}^d, \mathbb{R})$, with $d = 2, 3$ ¹. Conditions (20)-(22) are essentially those conditions which are non-trivial to prove and also lead to the class of potentials 2.2 we consider in this note.*

We furthermore like to remark that our proof of Lemma 3.9 uses the assumption that V changes its sign only once and that V is positive around the origin. As mentioned, we expect Lemma 3.9 to be valid for a larger class of potentials than those defined in assumption 2.2.

Proof: We like to recall the scheme of the proof of the equivalent of Theorem 2.5 for non-negative potentials. The proof presented in [44] can be seen as a two-step argument. First, it is shown in Section 6.2.2. in [44] that the convergence (24) *generally* follows, if certain functionals $\gamma_x(\Psi_t, \varphi_t)$, with $x \in \{a, b, c, d, e, f\}$, can be bounded sufficiently well, that is $|\gamma_x(\Psi_t, \varphi_t)| \leq CN^{-\delta}$, $\delta > 0$ ². The exact definition of these functionals can be found in Definition 6.2. and Definition 6.3. in [44].

It is then proven in Lemma A.1. in [44] that the bound $|\gamma_x(\Psi_t, \varphi_t)| \leq CN^{-\delta}$, $x \in \{a, b, c, d, e\}$ is valid for nonnegative potentials $V \in L_c^\infty(\mathbb{R}^3, \mathbb{C})$.

In the following, we will show that the estimates $|\gamma_x(\Psi_t, \varphi_t)| \leq CN^{-\delta}$ given in [44] remain valid under the conditions (19)-(23). Note that we will not restate the estimates given in [44], but only focus on the modifications one needs to perform.

The bound of $|\gamma_a(\Psi_t, \varphi_t)| \leq CN^{-\delta}$ directly follows from $\dot{A}_t \in L^\infty(\mathbb{R}^3, \mathbb{R})$, see Lemma A.1. in [44]. The required bound of $|\gamma_b(\Psi_t, \varphi_t)|$ is derived in Lemma A.4., pp.31-37 in [44]. Following the estimates given in [44], it can be verified line-by-line that the given bounds are valid, if conditions (19)-(23) and $A_t \in L^\infty(\mathbb{R}^3, \mathbb{R})$ hold. Furthermore, it can be verified that the functionals γ_c and γ_e can be controlled using conditions (19)-(23), see Lemma A.1. and pp.38-42 in [44]. The estimate for γ_f is valid under conditions (19) and (23) and can be found in p. 34 in [35] and p. 53 in [18].

In two dimensions, γ_d can be bounded, using conditions (19)-(23), see pp.50-52 [18] (we like to recall that the N -dependent bounds given in Lemma 3.9 read slightly different in two dimensions).

In three dimensions, the functional γ_d can be bounded, using in addition the following estimate: Let $m^\alpha(k) = m(k) - m(k + 1)$, where, for some $\xi > 0$,

$$m(k) = \begin{cases} \sqrt{k/N}, & \text{for } k \geq N^{1-2\xi}, \\ 1/2(N^{-1+\xi}k + N^{-\xi}), & \text{else.} \end{cases}$$

¹ Condition (20) reads $\|e^{2N}V(e^N(x_1 - x_2))\Psi_t\| \leq Ce^N N^{-1/2}$ in two dimensions, see Lemma 7.8 in [18]. Furthermore, for the two-dimensional system, we need higher regularity of φ_t . There, condition (19) reads $\varphi_t \in H^3(\mathbb{R}^2, \mathbb{C})$.

²The functional called $\gamma_f(\Psi_t, \varphi_t)$ is actually missing in [44]. The definition of this functional can be found in in equation (6.10) [35] and in p. 32 [18]. In these papers, it is furthermore shown that the respective bound $|\gamma_f(\Psi_t, \varphi_t)| \leq CN^{-\delta}$, $\delta > 0$ holds, assuming $V \in L_c^\infty(\mathbb{R}^d, \mathbb{R})$ to be nonnegative.

We control

$$N^3 \left| \langle \langle \Psi_t, \mathbf{1}_{\overline{\mathcal{B}}_1} g_{\beta_{1,1}}(x_1 - x_3) V_1(x_1 - x_2) \widehat{m}^{a^{\varphi_t}} p_1^{\varphi_t} \mathbf{1}_{\mathcal{B}_3} \Psi_t \rangle \rangle \right|, \quad (25)$$

where $g_{\beta_{1,1}}$ is defined in Lemma 3.9. This term, which appears in (A.49) in [44] is the only term in $\gamma_x(\Psi_t, \varphi_t)$, $x \in \{a, b, c, d, e, f\}$ where the estimate given in [44] needs to be modified, using only the assumptions given in the Lemma above. By a general inequality (see Lemma 4.3. in [44] and (A.50)-(A.52) in [44]), it can be verified that

$$(25) \leq N^{-1-\epsilon} \|\mathbf{1}_{\overline{\mathcal{B}}_1} V_1(x_1 - x_2) \Psi_t\|^2 \quad (26)$$

$$+ CN^{6+\epsilon} \|g_{\beta_{1,1}}(x_1 - x_3) \mathbf{1}_{V_1}(x_1 - x_2) \widehat{m}^{a^{\varphi_t}} p_1^{\varphi_t} \mathbf{1}_{\mathcal{B}_3} \Psi_t\|^2 \quad (27)$$

$$+ CN^{7+\epsilon} \left| \langle \langle \Psi_t, \mathbf{1}_{\mathcal{B}_3} p_1^{\varphi_t} \widehat{m}^{a^{\varphi_t}} g_{\beta_{1,1}}(x_1 - x_3) \right. \\ \left. \mathbf{1}_{V_1}(x_1 - x_2) g_{\beta_{1,1}}(x_1 - x_4) \widehat{m}^{a^{\varphi_t}} p_1^{\varphi_t} \mathbf{1}_{\mathcal{B}_4} \Psi_t \rangle \rangle \right| \quad (28)$$

for all $\epsilon \in \mathbb{R}$. For nonnegative V and $\epsilon = 0$, it was possible to control (26) using a specific energy estimate, see Lemma 5.2.(3) in [44]. We do not expect this estimate to hold for potentials V which are not nonnegative. For an interaction potential V , fulfilling condition (20), we can however bound

$$(26) \leq CN^{-\epsilon}.$$

The estimate (27) $\leq CN^{-1+2\xi+\epsilon}$ given in (A.51) [44] is valid under conditions (19)-(23). Note that condition (23) implies $\|g_{\beta_{1,1}}(x_1 - x_2) \Omega\| \leq C \|\nabla_1 \Omega\|$ for $\Omega \in L^2(\mathbb{R}^{3N}, \mathbb{C})$, see Lemma 3.9. This is one key estimate in order to bound (27). Under the some conditions, it has been shown (c.f. (A.52) in [44]) that

$$(28) \leq CN^{-\frac{26}{9}+3\xi+\epsilon}.$$

Therefore, it follows for some $\eta > 0$ that

$$(25) \leq CN^{-\eta} \quad (29)$$

holds by choosing $\xi > 0$ and $\epsilon > 0$ small enough³.

□

Proof of Theorem 2.5: In the following, we will prove the inequalities (20), (21) and (22) for interaction potentials which fulfill assumption 2.2. Theorem 2.5, part (a) then follows from Lemma 3.3, together with the estimates given in Section 3.1. Part (b) of Theorem 2.5 follows from part (a) and the estimates given in [44].

□

³ Note that the factors $N^{2\xi}$ and $N^{3\xi}$ are due to the definition of $m(k)$. A factor of the form $N^{s\xi}$, $s \in \{1, 2, 3\}$ also appears in the other functionals $\gamma_x(\Psi_t, \varphi_t)$, $x \in \{b, c, e, f\}$. It therefore follows that the the respective bounds $|\gamma_x(\Psi_t, \varphi_t)| \leq CN^{-\eta}$, $\eta > 0$ given in [44] are valid choosing $\xi > 0$ small enough. We like to remark that one cannot choose $\xi = 0$, since the convergence of the reduced density matrices stated in Lemma 3.3 does only follow for $0 < \xi < 1/2$, see [44] for the precise argument.

3.1 The scattering state

In this section we analyze the microscopic structure which is induced by V_1 . While the principle estimates are the same as in [18, 44], we need to modify the proofs given there which relied on the nonnegativity of V .

Definition 3.5 *Let $V \in L_c^\infty(\mathbb{R}^3, \mathbb{R})$ fulfill assumption 2.2. Define the zero energy scattering state j by*

$$\begin{cases} (-\Delta_x + \frac{1}{2}V(x))j(x) = 0, \\ \lim_{|x| \rightarrow \infty} j(x) = 1. \end{cases} \quad (30)$$

Furthermore define the scattering length a by

$$a = \text{scat}\left(\frac{1}{2}V\right) = \frac{1}{4\pi} \int \frac{1}{2}V(x)j(x)d^3x. \quad (31)$$

We want to recall some important properties of the scattering state j , see also Appendix C of [32].

Lemma 3.6 *For the scattering state defined previously the following relations hold:*

(a) *j is a nonnegative, monotone nondecreasing function which is spherically symmetric in $|x|$. For $|x| \geq R$, j is given by*

$$j(x) = 1 - \frac{a}{|x|}.$$

(b) *The scattering length a fulfills $a \geq 0$.*

Proof:

(a)+(b) Since we assume $-\Delta + \frac{1}{2}V \geq 0$, one can define the scattering state j by a variational principle. Theorem C.1 in [32] then implies that j is a nonnegative, spherically symmetric function in $|x|$ such that $j(x) = 1 - \frac{a}{|x|}$ holds for $|x| \geq R$ with $a \in \mathbb{R}$ defined as above. By condition (10) it follows $a \geq 0$, see Theorem C.1., (C.8.) in [32]. It is only left to show that j is monotone nondecreasing in $|x|$. Let $t(|x|) = j(x)$ and define

$$a_r = \frac{1}{4\pi} \int_0^r \frac{1}{2}V(r'e_{r'})t(r')(r')^2 dr',$$

where $e_{r'}$ denotes the radial unit vector. Note that $a = \lim_{r \rightarrow \infty} a_r = a_R$. By Gauß-theorem and the scattering equation (30), it then follows for $r > 0$

$$\frac{d}{dr}t(r) = \frac{a_r}{r^2}.$$

Since $t(r) \geq 0$ holds for all $r \geq 0$, it follows $a_r > 0$ for all $r \in]0, r_2[$. If it were now that j is not monotone nondecreasing, there must exist a $\tilde{r} \geq r_2$, such that $a_{\tilde{r}} < 0$. $V(x) \leq 0$ and $t(r) \geq 0$ for all $|x| \in]r_2, R[$ then imply $a_r \leq a_{\tilde{r}}$ for all $r \geq \tilde{r}$. This, however, contradicts $a = a_R \geq 0$. Thus, it follows that j is monotone nondecreasing. □

Using a general idea, we will define a potential W_{β_1} with $0 < \beta_1 < 1$, such that $\frac{1}{2}(V_1 - W_{\beta_1})$ has scattering length zero. This allows us to “replace” V_1 by W_{β_1} , which has better scaling behavior and is easier to control.

Definition 3.7 *Let $V \in L_c^\infty(\mathbb{R}^3, \mathbb{R})$ satisfy assumption 2.2. Let a_N denote the scattering length of $\frac{1}{2}V_1(x) = \frac{1}{2}N^2V(Nx)$. For any $0 < \beta_1 < 1$ and any $R_{\beta_1} \geq N^{-\beta_1}$ we define the potential W_{β_1} via*

$$W_{\beta_1}(x) = \begin{cases} a_N N^{3\beta_1} & \text{if } N^{-\beta_1} < |x| \leq R_{\beta_1}, \\ 0 & \text{else.} \end{cases} \quad (32)$$

Furthermore, we define the zero energy scattering state $f_{\beta_1,1}$ of the potential $\frac{1}{2}(V_1 - W_{\beta_1})$, that is

$$\begin{cases} (-\Delta_x + \frac{1}{2}(V_1(x) - W_{\beta_1}(x))) f_{\beta_1,1}(x) = 0, \\ f_{\beta_1,1}(x) = 1 \text{ for } |x| = R_{\beta_1}. \end{cases} \quad (33)$$

Remark 3.8 *Note, by scaling, that $a_N = N^{-1}a$. Furthermore $j_N(x) := j(Nx)$ solves*

$$\begin{cases} (-\Delta_x + \frac{1}{2}V_1(x)) j_N(x) = 0, \\ \lim_{|x| \rightarrow \infty} j_N(x) = 1. \end{cases}$$

In the following Lemma we show that there exists a minimal value R_{β_1} such that the scattering length of the potential $\frac{1}{2}(V_1 - W_{\beta_1})$ is zero. In the two dimensional case, the analog of Lemma 3.9 is, except part (i), also valid in two dimensions (one has to replace the bounds below by the respective bounds given in [18]. Furthermore, W_{β_1} is defined differently.). Part (i) needs not to be proven in two dimensions, see Remark 3.12.

Lemma 3.9 *For the scattering state $f_{\beta_1,1}$, defined by (33), the following relations hold :*

(a) *There exists a minimal value $R_{\beta_1} \in \mathbb{R}$ such that $\int (V_1(x) - W_{\beta_1}(x)) f_{\beta_1,1}(x) d^3x = 0$.*

For the rest of the paper we assume that R_{β_1} is chosen such that (a) holds.

(b) *There exists $K_{\beta_1} \in \mathbb{R}$, $K_{\beta_1} > 0$ such that $K_{\beta_1} f_{\beta_1,1}(x) = j(Nx) \forall |x| \leq N^{-\beta_1}$.*

(c) *$f_{\beta_1,1}$ is a nonnegative, monotone nondecreasing function in $|x|$. Furthermore,*

$$f_{\beta_1,1}(x) = 1 \text{ for } |x| \geq R_{\beta_1}. \quad (34)$$

(d)

$$1 \geq K_{\beta_1} \geq 1 - \frac{a}{N^{1-\beta_1}}. \quad (35)$$

(e) *$R_{\beta_1} \leq CN^{-\beta_1}$.*

For any fixed $0 < \beta_1$, N sufficiently large such that V_1 and W_{β_1} do not overlap, we obtain

(f)

$$|N\|V_1 f_{\beta_1,1}\|_1 - 8\pi a| = |N\|W_{\beta_1} f_{\beta_1,1}\|_1 - 8\pi a| \leq CN^{-1-\beta_1} .$$

(g) Define

$$g_{\beta_1,1}(x) = 1 - f_{\beta_1,1}(x) .$$

Then,

$$\|g_{\beta_1,1}\|_1 \leq CN^{-1-2\beta_1} , \|g_{\beta_1,1}\|_{3/2} \leq CN^{-1-\beta_1} , \|g_{\beta_1,1}\| \leq CN^{-1-\beta_1/2} , \|g_{\beta_1,1}\|_\infty \leq 1 .$$

(h)

$$|N\|W_{\beta_1}\|_1 - 8\pi a| \leq CN^{-1+\beta_1} .$$

(i) For any $\Omega \in H^1(\mathbb{R}^{3N}, \mathbb{C})$, we have

$$\|g_{\beta_1,1}(x_1 - x_2)\Omega\| \leq CN^{-1}\|\nabla_1\Omega\| .$$

Proof:

(a) In the following, we will sometimes denote, with a slight abuse of notation, $f_{\beta_1,1}(x) = f_{\beta_1,1}(r)$ and $j(x) = j(r)$ for $r = |x|$ (for this, recall that $f_{\beta_1,1}$ and j are radially symmetric). We further denote by $f'_{\beta_1,1}(r)$ the derivative of $f_{\beta_1,1}$ with respect to the radial coordinate r . We first show by contradiction that $f_{\beta_1,1}(N^{-\beta_1}) \neq 0$. For this, assume that $f_{\beta_1,1}(x) = 0$ for all $|x| \leq N^{-\beta_1}$. Since $f_{\beta_1,1}$ is continuous, there exists a maximal value $r_0 \geq N^{-\beta_1}$ such that the scattering equation (33) is equivalent to

$$\begin{cases} (-\Delta_x - \frac{1}{2}W_{\beta_1}(x)) f_{\beta_1,1}(x) = 0, \\ f_{\beta_1,1}(x) = 1 \text{ for } |x| = R_{\beta_1}, \\ f_{\beta_1,1}(x) = 0 \text{ for } |x| \leq r_0 . \end{cases} \quad (36)$$

Using (33) and Gauss'-theorem, we further obtain

$$f'_{\beta_1,1}(r) = \frac{1}{8\pi r^2} \int_{B_r(0)} d^3x (V_1(x) - W_{\beta_1}(x)) f_{\beta_1,1}(x) . \quad (37)$$

(36) and (37) then imply for $r > r_0$

$$\begin{aligned} |f'_{\beta_1,1}(r)| &= \frac{1}{8\pi r^2} \left| \int_{B_r(0)} d^3x W_{\beta_1}(x) f_{\beta_1,1}(x) \right| = \frac{aN^{-1+3\beta_1}}{2r^2} \left| \int_{r_0}^r dr' r'^2 f_{\beta_1,1}(r') \right| \\ &\leq \frac{aN^{-1+3\beta_1}}{2r^2} \left| \int_{r_0}^r dr' r'^2 (r' - r_0) \sup_{r_0 \leq s \leq r} |f'_{\beta_1,1}(s)| \right| . \end{aligned}$$

Taking the supreme over the interval $[r_0, r]$, the inequality above then implies that there exists a constant $C(r, r_0) \neq 0$, $\lim_{r \rightarrow r_0} C(r, r_0) = 0$ such that $\sup_{r_0 \leq s \leq r} |f'_{\beta_1,1}(s)| \leq$

$C(r, r_0)N^{-1+3\beta_1} \sup_{r_0 \leq s \leq r} |f'_{\beta_1,1}(s)|$. Thus, for r close enough to r_0 , the inequality above can only hold if $f'_{\beta_1,1}(s) = 0$ for $s \in [r_0, r]$, yielding a contradiction to the choice of r_0 . Consequently, there exists a $x_0 \in \mathbb{R}^3$, $|x_0| \leq N^{-\beta_1}$, such that $f_{\beta_1,1}(x_0) \neq 0$. We can thus define

$$h(x) = f_{\beta_1,1}(x) \frac{j(Nx_0)}{f_{\beta_1,1}(x_0)}$$

on the compact set $\overline{B_{x_0}(0)}$. One easily sees that $h(x) = j(Nx)$ on $\partial \overline{B_{x_0}(0)}$ and satisfies the zero energy scattering equation (30) for $x \in \overline{B_{N^{-\beta_1}}(0)}$. Note that the scattering equations (30) and (33) have a unique solution on any compact set. It then follows that $h(x) = j(Nx) \forall x \in \overline{B_{N^{-\beta_1}}(0)}$. Since $j(NN^{-\beta_1}) \neq 0$, we then obtain $f_{\beta_1,1}(N^{-\beta_1}) \neq 0$.

Thus, $f_{\beta_1,1}(x) = j(Nx) \frac{f_{\beta_1,1}(x_0)}{j(Nx_0)}$ holds for all $|x| \leq N^{-\beta_1}$ and for all $x_0 \in]0, N^{-\beta_1}]$. Lemma 3.6 further implies that either $f_{\beta_1,1}$ or $-f_{\beta_1,1}$ is a nonnegative, spherically symmetric and monotone nondecreasing function in $|x|$ for all $|x| \leq N^{-\beta_1}$.

Recall that W_{β_1} and hence $f_{\beta_1,1}(x)$ depend on $R_{\beta_1} \in [N^{-\beta_1}, \infty[$. For conceptual clarity, we denote $W_{\beta_1}^{(R_{\beta_1})}(x) = W_{\beta_1}(x)$ and $f_{\beta_1,1}^{(R_{\beta_1})}(x) = f_{\beta_1,1}(x)$ for the rest of the proof of part (a). For β_1 fixed, consider the function

$$s : [N^{-\beta_1}, \infty[\rightarrow \mathbb{R} \quad (38)$$

$$R_{\beta_1} \mapsto \int_{B_{R_{\beta_1}}(0)} d^3x (V_1(x) - W_{\beta_1}^{(R_{\beta_1})}(x)) f_{\beta_1,1}^{(R_{\beta_1})}(x). \quad (39)$$

We show by contradiction that the function s has at least one zero. Assume $s \neq 0$ were to hold. We can assume w.l.o.g. $s > 0$. It then follows from Gauss'-theorem that $f_{\beta_1,1}^{(R_{\beta_1})}(R_{\beta_1}) > 0$ for all $R_{\beta_1} \geq N^{-\beta_1}$. By uniqueness of the solution of the scattering equation (33), for $\tilde{R}_{\beta_1} < R_{\beta_1}$ there exists a constant $K_{\tilde{R}_{\beta_1}, R_{\beta_1}} \neq 0$, such that for all $|x| \leq \tilde{R}_{\beta_1}$ we have $f_{\beta_1,1}^{(\tilde{R}_{\beta_1})}(x) = K_{\tilde{R}_{\beta_1}, R_{\beta_1}} f_{\beta_1,1}^{(R_{\beta_1})}(x)$. If $K_{\tilde{R}_{\beta_1}, R_{\beta_1}} < 0$ were to hold, we could conclude from

$$0 < s(\tilde{R}_{\beta_1}) = 8\pi(\tilde{R}_{\beta_1})^2 f_{\beta_1,1}^{(\tilde{R}_{\beta_1})}(\tilde{R}_{\beta_1}) = 8\pi(\tilde{R}_{\beta_1})^2 K_{\tilde{R}_{\beta_1}, R_{\beta_1}} f_{\beta_1,1}^{(R_{\beta_1})}(\tilde{R}_{\beta_1})$$

that $f_{\beta_1,1}^{(R_{\beta_1})}(\tilde{R}_{\beta_1}) < 0$. By continuity of $f_{\beta_1,1}^{(R_{\beta_1})}$ and $f_{\beta_1,1}^{(R_{\beta_1})}(R_{\beta_1}) > 0$, there exists $r \in]\tilde{R}_{\beta_1}, R_{\beta_1}[$, such that $0 = f_{\beta_1,1}^{(R_{\beta_1})}(r) = K_{R_{\beta_1}, r} f_{\beta_1,1}^{(r)}(r)$, yielding to a contradiction to $s > 0$.

We can therefore conclude $K_{\tilde{R}_{\beta_1}, R_{\beta_1}} > 0$. From Lemma 3.6, the assumption $s(N^{-\beta_1}) > 0$ and $K_{\tilde{R}_{\beta_1}, R_{\beta_1}} > 0$, we obtain, for all $r \in [0, N^{-\beta_1}]$ and for all $R_{\beta_1} \in [N^{-\beta_1}, \infty[$, that $f_{\beta_1,1}^{(R_{\beta_1})}(r) \geq 0$ holds. From $s \neq 0$, it then follows that, for all $r \in [N^{-\beta_1}, \infty[$ and for all $R_{\beta_1} \in [N^{-\beta_1}, \infty[$, $f_{\beta_1,1}^{(R_{\beta_1})}(r) \neq 0$. Thus, for all $r \in [N^{-\beta_1}, \infty[$ and for all $R_{\beta_1} \in [N^{-\beta_1}, \infty[$, the function $f_{\beta_1,1}^{(R_{\beta_1})}(r)$ doesn't change sign. This, however, implies $\lim_{R_{\beta_1} \rightarrow \infty} s(R_{\beta_1}) = -\infty$ yielding to a contradiction. By continuity of s , there exists thus a minimal value $R_{\beta_1} \geq N^{-\beta_1}$ such that $s(R_{\beta_1}) = 0$.

Remark 3.10 *As mentioned, we will from now on fix $R_{\beta_1} \in [N^{-\beta_1}, \infty[$ as the minimal value such that $s(R_{\beta_1}) = 0$. Furthermore, we may assume $a > 0$ and $R_{\beta_1} > N^{-\beta_1}$ in the following. For $a = 0$, we can choose $R_{\beta_1} = N^{-\beta_1}$, such that $f_{\beta_1,1}(x) = j(Nx)/j(NN^{-\beta_1})$. It is then easy to verify that the Lemma stated is valid.*

(b) From $j(Nx) = f_{\beta_1,1}(x) \frac{j(NN^{-\beta_1})}{f_{\beta_1,1}(N^{-\beta_1})}$, for all $|x| \leq N^{-\beta_1}$, we can conclude that

$$K_{\beta_1} = \frac{j(NN^{-\beta_1})}{f_{\beta_1,1}(N^{-\beta_1})}. \quad (40)$$

Next, we show that the constant K_{β_1} is positive. Since $j(NN^{-\beta_1})$ is positive, it follows from Eq. (40) that K_{β_1} and $f_{\beta_1,1}(N^{-\beta_1})$ have equal sign. By (a), the sign of $f_{\beta_1,1}$ is constant for $|x| \leq R_{\beta_1}$. Furthermore, from Gauss'-theorem and the scattering equation (33) we have

$$f'_{\beta_1,1}(r) = \frac{1}{8\pi r^2 K_{\beta_1}} \int_{B_r(0)} V_1(x) j(Nx) d^3x \quad (41)$$

for all $0 < r \leq N^{-\beta_1}$. Since $\int_{B_r(0)} V_1(x) j(Nx) d^3x$ is nonnegative for all $0 < r \leq N^{-\beta_1}$ (see the proof of Lemma 3.6), we then conclude

$$\text{sgn} \left(f'_{\beta_1,1}(N^{-\beta_1}) \right) = \text{sgn}(K_{\beta_1}). \quad (42)$$

Recall that $f'_{\beta_1,1}(R_{\beta_1}) = 0$. If it were now that K_{β_1} is negative, we could conclude from (40) and (42) that $f'_{\beta_1,1}(N^{-\beta_1}) < 0$ and $f_{\beta_1,1}(N^{-\beta_1}) < 0$. Since R_{β_1} is by definition the smallest value where $f'_{\beta_1,1}$ vanishes, we were able to conclude from the continuity of the derivative that $f'_{\beta_1,1}(r) < 0$ for all $r < R_{\beta_1}$ and hence $f(R_{\beta_1}) < 0$. However, this were in contradiction to the boundary condition of the zero energy scattering state (see (33)) and thus $K_{\beta_1} > 0$ follows.

(c) From the proof of property (b), we see that $f_{\beta_1,1}$ and its derivative is positive at $N^{-\beta_1}$. From (37), we obtain $f'_{\beta_1,1}(r) = 0$ for all $r > R_{\beta_1}$. Thus $f_{\beta_1,1}(x) = 1$ for all $|x| \geq R_{\beta_1}$. Due to continuity $f'_{\beta_1,1}(r) > 0$ for all $r < R_{\beta_1}$. Since $f_{\beta_1,1}$ is continuous, positive at $N^{-\beta_1}$, and its derivative is a nonnegative function, it follows that $f_{\beta_1,1}$ is a nonnegative, monotone nondecreasing function in $|x|$.

(d) Since $f_{\beta_1,1}$ is a positive monotone nondecreasing function in $|x|$, we obtain

$$1 \geq f_{\beta_1,1}(N^{-\beta_1}) = j(NN^{-\beta_1})/K_{\beta_1} = \left(1 - \frac{a}{N^{1-\beta_1}}\right) / K_{\beta_1}.$$

We obtain the lower bound

$$K_{\beta_1} \geq 1 - \frac{a}{N^{1-\beta_1}}.$$

For the upper bound, we first prove that $f_{\beta_1}(x) \geq j(Nx)/j(NR_{\beta_1})$ holds for all $|x| \leq N^{-\beta_1}$. Define $m(x) = j(Nx)/j(NR_{\beta_1}) - f_{\beta_1,1}(x)$. Using the scattering equations (30) and (33), we obtain

$$\begin{cases} \Delta_x m(x) = \frac{1}{2} V_1(x) m(x) + \frac{1}{2} W_{\beta_1}(x) f_{\beta_1,1}(x), \\ m(R_{\beta_1}) = 0. \end{cases} \quad (43)$$

Since $W_{\beta_1}(x)f_{\beta_1,1}(x) \geq 0$, we obtain that $\Delta_x m(x) \geq 0$ for $N^{-1}R \leq |x| \leq R_{\beta_1}$. That is, $m(x)$ is subharmonic for $N^{-1}R < |x| < R_{\beta_1}$. Using the maximum principle, we obtain, using that $m(x)$ is spherically symmetric

$$\max_{N^{-1}R \leq |x| \leq R_{\beta_1}} (m(x)) = \max_{|x| \in \{N^{-1}R, R_{\beta_1}\}} (m(x)). \quad (44)$$

If it were now that $\max_{|x| \in \{N^{-1}R, R_{\beta_1}\}} (m(x)) = m(N^{-1}R) \geq m(R_{\beta_1}) = 0$, we could assume $m(x) > 0$ for all $N^{-1}R \leq |x| \leq N^{-\beta_1}$ (otherwise we would have $m(N^{-\beta_1}) = 0$, which implies $K_{\beta_1} = j(NR_{\beta_1}) = 1 - \frac{a}{NR_{\beta_1}} \leq 1$). Note that $m(x)$ then solves

$$\begin{cases} -\Delta_x m(x) + \frac{1}{2}V_1(x)m(x) = 0 & \text{for } |x| \leq N^{-\beta_1}, \\ m(N^{-1}R) > 0. \end{cases}$$

By Theorem C.1 in [32] (note that we can assume $a > 0$), m is strictly increasing for $N^{-1}R \leq |x| \leq N^{-\beta_1}$. This, however, contradicts $\max_{|x| \in \{N^{-1}R, R_{\beta_1}\}} (m(x)) = m(N^{-1}R)$.

Therefore, we can conclude in (44) that $\max_{|x| \in \{N^{-1}R, R_{\beta_1}\}} (m(x)) = m(R_{\beta_1}) = 0$ holds. Then, it follows that $f_{\beta}(x) - j_{N, R_{\beta}}(x) \geq 0$ for all $N^{-1}R \leq |x| \leq N^{-\beta_1}$. Using the zero energy scattering equation

$$-\Delta(f_{\beta_1,1}(x) - j(Nx)/j(NR_{\beta_1})) + \frac{1}{2}V_1(x)(f_{\beta_1,1}(x) - j(Nx)/j(NR_{\beta_1})) = 0$$

for $|x| \leq N^{-\beta_1}$, we can, together with $f_{\beta_1,1}(N^{-\beta_1}) - j(NN^{-\beta_1})/j(NR_{\beta_1}) \geq 0$, conclude that $f_{\beta_1,1}(x) - j(Nx)/j(NR_{\beta_1}) \geq 0$ for all $|x| \leq R_{\beta_1}$.

As a consequence, we obtain the desired bound $K_{\beta} = \frac{j(NN^{-\beta_1})}{f_{\beta_1,1}(N^{-\beta_1})} \leq j(NR_{\beta_1}) \leq 1$.

(e) Since $f_{\beta_1,1}$ is a nonnegative, monotone nondecreasing function in $|x|$, it follows that

$$\begin{aligned} N^{-1}f_{\beta_1,1}(N^{-\beta_1}) \int V(x)d^3x &= f_{\beta_1,1}(N^{-\beta_1}) \int V_1(x)d^3x \geq \int V_1(x)f_{\beta_1,1}(x)d^3x \\ &= \int W_{\beta_1}(x)f_{\beta_1,1}(x)d^3x \geq f_{\beta_1,1}(N^{-\beta_1}) \int W_{\beta_1}(x)d^3x. \end{aligned}$$

Therefore, $\int W_{\beta_1}(x)d^3x \leq CN^{-1}$ holds, which implies that $R_{\beta_1} \leq CN^{-\beta_1}$.

Remark 3.11 *We will now prove the the two-dimensional analog of (e) which requires a more refined estimate. This is due to the fact that $\int_{\mathbb{R}^2} e^{2N}V(e^N x)d^2x = \mathcal{O}(1)$ does not decay like N^{-1} . We refer to [18] for the precise definition and notation we use in the following.*

Proof of part (e) for the two-dimensional system:

Since f_{β} is a nonnegative, monotone nondecreasing function in $|x|$ with $f_{\beta}(x) = 1 \forall |x| \geq R_{\beta}$, it follows that

$$\begin{aligned} f_{\beta}(N^{-\beta}) \int_{\mathbb{R}^2} d^2x V(x) &= f_{\beta}(N^{-\beta}) \int_{\mathbb{R}^2} d^2x V_N(x) \geq \int V_N(x) f_{\beta}(x) d^2x \\ &= \int_{\mathbb{R}^2} d^2x W_{\beta}(x) f_{\beta}(x) \geq f_{\beta}(N^{-\beta}) \int_{\mathbb{R}^2} d^2x W_{\beta}(x). \end{aligned}$$

Therefore, $\int_{\mathbb{R}^2} d^2x W_\beta(x) \leq C$ holds, which implies that $R_\beta \leq CN^{1/2-\beta}$. From

$$\begin{aligned} \frac{1}{K_\beta} \frac{4\pi}{N + \ln\left(\frac{R_\beta}{a}\right)} &= \frac{1}{K_\beta} \int_{\mathbb{R}^2} d^2x V_N(x) j_{N,R_\beta}(x) = \int_{\mathbb{R}^2} d^2x V_N(x) f_\beta(x) \\ &= \int_{\mathbb{R}^2} d^2x M_\beta(x) f_\beta(x) = 8\pi^2 N^{-1+2\beta} \int_{N^{-\beta}}^{R_\beta} dr r f_\beta(r) \end{aligned}$$

we conclude that

$$\int_{N^{-\beta}}^{R_\beta} dr r f_\beta(r) = \frac{N^{1-2\beta}}{2\pi K_\beta \left(N + \ln\left(\frac{R_\beta}{a}\right)\right)}.$$

Since f_β is a nonnegative, monotone nondecreasing function in $|x|$,

$$\frac{1}{2}(R_\beta^2 - N^{-2\beta}) \frac{j_{N,R_\beta}(N^{-\beta})}{K_\beta} = \frac{1}{2}(R_\beta^2 - N^{-2\beta}) f_\beta(N^{-\beta}) \leq \int_{N^{-\beta}}^{R_\beta} dr r f_\beta(r)$$

which implies

$$R_\beta^2 N^{2\beta} \leq \frac{N}{\pi \left(N + \ln\left(\frac{R_\beta}{a}\right)\right) j_{N,R_\beta}(N^{-\beta})} + 1.$$

Using $R_\beta \leq CN^{1/2-\beta}$, it then follows

$$j_{N,R_\beta}(N^{-\beta}) = 1 + \frac{1}{N + \ln\left(\frac{R_\beta}{a}\right)} \ln\left(\frac{N^{-\beta}}{R_\beta}\right) \geq 1 - \frac{C}{N},$$

which implies $R_\beta \leq CN^{-\beta}$.

(f) Using

$$\|W_{\beta_1} f_{\beta_1,1}\|_1 = \|V_1 f_{\beta_1,1}\|_1 = K_{\beta_1}^{-1} \|V_1 j(N \cdot)\|_1 = K_{\beta_1}^{-1} 8\pi \frac{a}{N},$$

we obtain

$$|N \|V_1 f_{\beta_1,1}\|_1 - 8\pi a| = |N \|W_{\beta_1} f_{\beta_1,1}\|_1 - 8\pi a| = 8\pi \left| K_{\beta_1}^{-1} - 1 \right| \leq CN^{-1+\beta_1}.$$

(g) Using for $|x| \leq R_{\beta_1}$ the inequality $1 \geq f_{\beta_1,1}(x) \geq j(Nx)/j(NR_{\beta_1})$, it follows for $|x| \leq R_{\beta_1}$

$$0 \leq g_{\beta_1,1}(x) = 1 - f_{\beta_1,1}(x) \leq 1 - j(Nx)/j(NR_{\beta_1}).$$

Let \tilde{j} solve

$$\begin{cases} (-\Delta_x + \frac{1}{2}V(x)\mathbf{1}_{|x| \leq r_2}) \tilde{j}(x) = 0, \\ \tilde{j}(2R) = j(2R). \end{cases}$$

It then follows that $\tilde{a} = \text{scat}(\frac{1}{2}V(x)\mathbb{1}_{|x|\leq r_2}) > 0$. Furthermore, it follows from Theorem C.1 and Lemma C.2 in [32] that

$$\tilde{j}(x) \geq \frac{1 - \frac{\tilde{a}}{|x|}}{1 - \frac{\tilde{a}}{2R}} j(2R) = \left(1 - \frac{\tilde{a}}{|x|}\right) \frac{1 - \frac{a}{2R}}{1 - \frac{\tilde{a}}{2R}}$$

holds for all $x \in \mathbb{R}^3$. Consider $n(x) = \tilde{j}(x) - j(x)$. n then solves

$$\begin{cases} \Delta_x n(x) = \frac{1}{2}V(x)n(x) + \frac{1}{2}V(x)\mathbb{1}_{|x|\leq r_2}\tilde{j}(x), \\ n(2R) = 0. \end{cases}$$

As before (see (43)), we can conclude $n(x) \leq 0$ for all $|x| \leq 2R$, which implies $j(x) \geq \tilde{j}(x)$, for $|x| \leq 2R$. Therefore,

$$j(Nx) \geq \begin{cases} \left(1 - \frac{\tilde{a}}{N|x|}\right) \frac{1 - \frac{a}{2NR}}{1 - \frac{\tilde{a}}{2NR}} \text{ for } N|x| \leq R, \\ 1 - \frac{a}{N|x|} \text{ else.} \end{cases}$$

This implies, using part (d),

$$\begin{aligned} g_{\beta_1,1}(x) &\leq 1 - \begin{cases} \left(1 - \frac{\tilde{a}}{N|x|}\right) \frac{1 - \frac{a}{2NR}}{(1 - \frac{\tilde{a}}{2NR})(1 - \frac{a}{NR\beta_1})} \text{ for } N|x| \leq R, \\ \frac{1 - \frac{a}{N|x|}}{(1 - \frac{a}{NR\beta_1})} \text{ else.} \end{cases} \\ &\leq \begin{cases} \frac{\tilde{a}}{N|x|} + CN^{-1} \text{ for } N|x| \leq R, \\ \frac{a}{N|x|} + CN^{-1+\beta_1} \text{ else.} \end{cases} \end{aligned} \quad (45)$$

Since $g_{\beta_1,1}(x) = 0$ for $|x| > R_\beta$, we conclude with $R_{\beta_1} \leq CN^{-\beta_1}$ that

$$\|g_{\beta_1,1}\|_1 \leq N^{-1-2\beta_1},$$

as well as

$$\|g_{\beta_1,1}\|_{3/2} \leq CN^{-1-\beta_1}, \quad \|g_{\beta_1,1}\| \leq CN^{-1-\beta_1/2}.$$

Furthermore, $\|g_{\beta_1,1}\|_\infty = \|1 - f_{\beta_1,1}\|_\infty \leq 1$, since $f_{\beta_1,1}$ is a nonnegative, monotone nondecreasing function with $f_{\beta_1,1}(x) \leq 1$.

(h) Using (f) and (g), we obtain with $\|W_{\beta_1}\|_1 \leq CN^{-1}$

$$\begin{aligned} |N\|W_{\beta_1}\|_1 - 8\pi a| &\leq |N\|W_{\beta_1}f_{\beta_1,1}\|_1 - 8\pi a| + N\|W_{\beta_1}g_{\beta_1,1}\|_1 \\ &\leq C \left(N^{-1+\beta_1} + \|\mathbb{1}_{|\cdot|\geq N^{-\beta_1}}g_{\beta_1,1}\|_\infty \right). \end{aligned}$$

Since $g_{\beta_1,1}(x)$ is a nonnegative, monotone nonincreasing function, it follows with $K_{\beta_1} \leq 1$

$$\|\mathbb{1}_{|\cdot|\geq N^{-\beta_1}}g_{\beta_1,1}\|_\infty = g_{\beta_1,1}(N^{-\beta_1}) = 1 - f_{\beta_1,1}(N^{-\beta_1}) = 1 - \frac{j(NN^{-\beta_1})}{K_{\beta_1}} \leq aN^{-1+\beta_1}.$$

and (h) follows.

(i) Using the pointwise estimate (45), we obtain for any $\Omega \in H^1(\mathbb{R}^{3N}, \mathbb{C})$

$$\|g_{\beta_1,1}(x_1 - x_2)\Omega\| \leq C(N^{-1+\beta_1}\|\mathbb{1}_{B_{CN^{-\beta_1}}(0)}(x_1 - x_2)\Omega\| + N^{-1}\||x_1 - x_2|^{-1}\Omega\|).$$

Since $\||x_1 - x_2|^{-1}\Omega\| \leq 2\|\nabla_1\Omega\|$ as well as $\|\mathbb{1}_{B_{CN^{-\beta_1}}(0)}(x_1 - x_2)\Omega\| \leq CN^{-3\beta_1/2}\|\nabla_1\Omega\|$ holds, we obtain part (i).

Remark 3.12 *Part (i) is not valid in two dimensions. However, this specific inequality is only used in the three dimensional case to control (25). It can be verified (see equations (92)-(97) in [18]) that it is not necessary to control (25) in two dimensions.*

□

3.2 Nonnegativity of the Hamiltonian H_U

Next, we prove two important operator inequalities related to the Hamiltonian H , see Corollary 3.19. These inequalities will be used in order to show the inequalities (20), (21) and (22).

Lemma 3.13 *Let $U \in L_c^\infty(\mathbb{R}^3, \mathbb{R})$ fulfill assumption 2.2 and define*

$$H_U = -\sum_{k=1}^N \Delta_k + \sum_{i<j=1}^N U(x_i - x_j).$$

Then

$$H_U \geq 0.$$

In order to prove this Lemma, we first define

Definition 3.14 *For $\tilde{R} \geq 2R$, where R is defined as in assumption 2.2, let for any $j, k = 1, \dots, N$ with $j \neq k$*

$$b_{j,k} := \{(x_1, x_2, \dots, x_N) \in \mathbb{R}^{3N} : |x_j - x_k| \leq \tilde{R}\} \quad (46)$$

$$\bar{c}_l := \bigcup_{j,k \neq l} b_{j,k}, \quad c_l := \mathbb{R}^{3N} \setminus \bar{c}_l.$$

Proof: Let

$$H_{\bar{c}} = \sum_{k=1}^N -\Delta_k \mathbb{1}_{\bar{c}_k} + \sum_{i \neq j} \mathbb{1}_{\bar{c}_j} \frac{1}{2} U(x_i - x_j),$$

$$H_c = \sum_{k=1}^N -\Delta_k \mathbb{1}_{c_k} + \sum_{i \neq j} \mathbb{1}_{c_j} \frac{1}{2} U(x_i - x_j).$$

Note that

$$H_c = \sum_{k=1}^N -\Delta_k \mathbb{1}_{c_k} + \frac{1}{4} \sum_{i \neq j} (\mathbb{1}_{c_j} + \mathbb{1}_{c_i}) \frac{1}{2} U(x_i - x_j)$$

is a symmetric operator w.r.t. to exchange of coordinates x_1, \dots, x_N . Therefore, it suffices to prove $\langle \Psi, H_C \Psi \rangle \geq 0$ for $\Psi \in L_s^2(\mathbb{R}^{3N}, \mathbb{C})$, since

$$\inf_{\Psi \in L_s^2(\mathbb{R}^{3N}, \mathbb{C}), \|\Psi\|=1} \langle \Psi, H_C \Psi \rangle = \inf_{\Psi \in L_s^2(\mathbb{R}^{3N}, \mathbb{C}), \|\Psi\|=1} \langle \Psi, H_C \Psi \rangle.$$

In order to prove $H_C \geq 0$, we show $K_1 = -\Delta_1 \mathbb{1}_{\mathcal{C}_1} + \frac{1}{2} \sum_{j=2}^N \mathbb{1}_{\mathcal{C}_1} \frac{1}{2} U(x_1 - x_j) \geq 0$ on $L_s^2(\mathbb{R}^{3N}, \mathbb{C})$. Since

$$\begin{aligned} \inf_{\Psi \in L_s^2(\mathbb{R}^3, \mathbb{C}), \|\Psi\|=1} \langle \Psi, H_C \Psi \rangle &= \inf_{\Psi \in L_s^2(\mathbb{R}^3, \mathbb{C}), \|\Psi\|=1} \sum_{i=1}^N \langle \Psi, K_i \Psi \rangle \\ &= N \inf_{\Psi \in L_s^2(\mathbb{R}^3, \mathbb{C}), \|\Psi\|=1} \langle \Psi, K_1 \Psi \rangle \end{aligned}$$

holds, it then follows $H_C \geq 0$.

The next Lemmata prove that $K_1 \geq 0$ and $H_{\overline{\mathcal{C}}} \geq 0$. Since $H_U = \sum_{i=1}^N K_i + H_{\overline{\mathcal{C}}}$, it then follows $H_U \geq 0$. □

Remark 3.15 *The reason to split the Hamiltonian as done above is the following: The interaction $\mathbb{1}_{\overline{\mathcal{C}}_j} \frac{1}{2} U(x_i - x_j)$ is only nonzero, if, for fixed configurations (x_1, \dots, x_N) , x_i is closer than R to x_j , but no other particles are closer than R to neither x_i nor x_j . Therefore, the set $\overline{\mathcal{C}}$ excludes those configurations, where three-particle interactions occur. The strategy to separate the configurations of possible three-particle interactions is well known within the literature, see e.g. [32, 49] and references therein.*

Let us restate an important Lemma.

Lemma 3.16

- (a) *Let R_{β_1} and W_{β_1} be defined as in Lemma 3.7. Let V fulfill assumption 2.2. Then, for any $\Psi \in H^1(\mathbb{R}^{3N}, \mathbb{C})$*

$$\|\mathbb{1}_{|x_1 - x_2| \leq R_{\beta_1}} \nabla_1 \Psi\|^2 + \frac{1}{2} \langle \Psi, (V_1 - W_{\beta_1})(x_1 - x_2) \Psi \rangle \geq 0.$$

- (b) *Let W_{β_1} be defined as in Lemma 3.7. Let V fulfill assumption 2.2 and let $\Psi \in L_s^2(\mathbb{R}^{3N}, \mathbb{C}) \cap H^1(\mathbb{R}^{3N}, \mathbb{C})$. Then, for sufficiently large N*

$$\|\mathbb{1}_{\mathcal{B}_1} \mathbb{1}_{\overline{\mathcal{A}}_1} \nabla_1 \Psi\|^2 + \frac{1}{2} \langle \Psi, \sum_{j \neq 1} \mathbb{1}_{\mathcal{B}_1} (V_1 - W_{\beta_1})(x_1 - x_j) \Psi \rangle \geq 0.$$

For nonnegative V , the proof has been given in [44] for the three-dimensional case (see Lemma 5.1. (3)) and in [18] for the two-dimensional analog (see Lemma 7.10). The proof given in these works is not using the nonnegativity of V directly, but is based on the fact that $f_{\beta_1,1}$ is a nonnegative function. Therefore, the proof is also applicable in our setting, using Lemma 3.9.

Lemma 3.17 *Let K_1 and $H_{\bar{\mathcal{C}}}$ be defined as above. Under the assumptions of Lemma 3.13, we have*

(a)

$$K_1 \geq 0 \text{ on } L_s^2(\mathbb{R}^{3N}, \mathbb{C}).$$

(b)

$$H_{\bar{\mathcal{C}}} \geq 0 \text{ on } L^2(\mathbb{R}^{3N}, \mathbb{C}).$$

Proof:

(a) The proof of Lemma 3.16, part (b) can be straightforwardly applied to prove part (a), see Lemma 5.1. (3) in [44] and Lemma 7.10 in [18]. Note for the proof to be valid, it is important that $\mathbb{1}_{\mathcal{C}_k}(x_1, \dots, x_N)$ excludes those configurations where the distance of two distinct particles x_i and x_j , $i, j \neq k$ to x_k is smaller than R , which is the radius of the support of U . We refer the reader to [18, 44] for the details of the proof.

(b) **Remark 3.18** *The proof of part (b) originates from Lemma 10. in [49]. The author, however, does not introduce the set \mathcal{C}_k , but uses a slightly different technique to exclude three particle interactions. For conceptual clarity, we adapt the proof of Lemma 10. in [49] to our definition of $H_{\bar{\mathcal{C}}}$. Since the proof given by Jun Yin is very elegant in our opinion, parts of the following are taken verbatim from [49].*

Recall that

$$H_{\bar{\mathcal{C}}} = \sum_{k=1}^N -\Delta_k \mathbb{1}_{\bar{\mathcal{C}}_k} + \sum_{i \neq j} \mathbb{1}_{\bar{\mathcal{C}}_j} \frac{1}{2} U(x_i - x_j).$$

Assume first that N is even, i.e., $N = 2N_1$ with $N_1 \in \mathbb{N}$. Let $P = (\pi_1, \pi_2)$ be a partition of $1, \dots, N$ into two disjoint sets with N_1 integers in π_1 and π_2 , respectively. Let

$$U_{1,1} = U_{2,2} = U^+ \geq 0, \quad U_{1,2} = 2U_1^+ - 4U^-, \quad (47)$$

with $U_{1,2}^- = -4U^-$, $U_{1,2}^+ = 2U^+$. It then follows

$$\frac{1}{4}(U_{1,1} + U_{2,1} + U_{2,2}) = U.$$

For each P , we define (for shorter notation, we will implicitly assume $i \neq j$ in the following)

$$\begin{aligned} H_P = H_{(\pi_1, \pi_2)} &\equiv \sum_{j \in \pi_1} -2\Delta_j \mathbb{1}_{\bar{\mathcal{C}}_j} + \sum_{i, j \in \pi_1} \mathbb{1}_{\bar{\mathcal{C}}_j} \frac{1}{2} U_{1,1}(x_i - x_j) \\ &+ \sum_{i \in \pi_2, j \in \pi_1} \mathbb{1}_{\bar{\mathcal{C}}_j} \frac{1}{2} U_{1,2}(x_i - x_j) + \sum_{i, j \in \pi_2} \mathbb{1}_{\bar{\mathcal{C}}_j} \frac{1}{2} U_{2,2}(x_i - x_j). \end{aligned}$$

Consequently, $U_{\alpha,\beta}$ denotes the interaction potential between particles in π_α and π_β . Note that

$$\begin{aligned}
& - \sum_P \sum_{j \in \pi_1} \Delta_j \mathbf{1}_{\bar{c}_j} = - \sum_{j=1}^N \Delta_j \mathbf{1}_{\bar{c}_j} \frac{1}{2} \sum_P, \\
& \sum_P \sum_{i,j \in \pi_1} \mathbf{1}_{\bar{c}_j} U_{1,1}(x_i - x_j) = \sum_P \sum_{i,j \in \pi_2} \mathbf{1}_{\bar{c}_j} U_{2,2}(x_i - x_j) \\
& = \sum_{i \neq j=1}^N \mathbf{1}_{\bar{c}_j} U^+(x_i - x_j) \frac{1}{4} \sum_P, \\
& \sum_P \sum_{i \in \pi_1, j \in \pi_2} \mathbf{1}_{\bar{c}_j} U_{1,2}(x_i - x_j) \\
& = \sum_{i \neq j=1}^N \mathbf{1}_{\bar{c}_j} (2U^+(x_i - x_j) - 4U^-(x_i - x_j)) \frac{1}{4} \sum_P.
\end{aligned}$$

Therefore,

$$H_{\bar{c}} = \sum_P H_P / \sum_P 1. \quad (48)$$

Hence, for N even, to obtain $H_{\bar{c}} \geq 0$, it is sufficient to prove that for $\forall P$, $H_P \geq 0$.

If N is odd, we divide $P = (\pi_2, \pi_2)$, with $N_1 = (N-1)/2$ integers in π_1 and $(N+1)/2$ integers in π_2 .

Let A_j be a one-particle operator and define, for any partition $P = (\pi_1, \pi_2)$, $\delta_{j \in \pi_1}$ such that $\delta_{j \in \pi_1} = 1$ if $j \in \pi_1$, otherwise 0. Then $\sum_P \sum_{j \in \pi_1} A_j = \sum_{j=1}^N A_j \sum_P \delta_{j \in \pi_1}$. Note that

$$\sum_P \delta_{j \in \pi_1} = \frac{\sum_P \delta_{j \in \pi_1}}{\sum_P} \sum_P = \frac{\binom{N-1}{\frac{N-3}{2}}}{\binom{N}{\frac{N-1}{2}}} \sum_P = \frac{1 - \frac{1}{N}}{2} \sum_P.$$

Furthermore, for any two-particle operator $A_{i,j}$, we obtain, for $a, b \in \{1, 2\}$,

$$\sum_P \sum_{i \in \pi_a, j \in \pi_b, i \neq j} A_{i,j} = \sum_{i \neq j=1}^N A_{i,j} \sum_P \delta_{i \in \pi_a} \delta_{j \in \pi_b}.$$

Let $i \neq j$. With

$$\begin{aligned}
\frac{1}{\sum_P} \sum_P \delta_{i \in \pi_1} \delta_{j \in \pi_1} &= \frac{\binom{N-2}{\frac{N-5}{2}}}{\binom{N}{\frac{N-1}{2}}} = \frac{1}{4} \left(1 - \frac{3}{N} \right), & \frac{1}{\sum_P} \sum_P \delta_{i \in \pi_1} \delta_{j \in \pi_2} &= \frac{\binom{N-2}{\frac{N-3}{2}}}{\binom{N}{\frac{N-1}{2}}} = \frac{1}{4} \left(1 + \frac{1}{N} \right), \\
\frac{1}{\sum_P} \sum_P \delta_{i \in \pi_2} \delta_{j \in \pi_1} &= \frac{\binom{N-2}{\frac{N-3}{2}}}{\binom{N}{\frac{N-1}{2}}} = \frac{1}{4} \left(1 + \frac{1}{N} \right), & \frac{1}{\sum_P} \sum_P \delta_{i \in \pi_2} \delta_{j \in \pi_2} &= \frac{\binom{N-2}{\frac{N-1}{2}}}{\binom{N}{\frac{N-1}{2}}} = \frac{1}{4} \left(1 + \frac{1}{N} \right),
\end{aligned}$$

it follows that

$$\begin{aligned}
-\sum_P \sum_{j \in \pi_1} \Delta_j \mathbb{1}_{\bar{\mathcal{C}}_j} &= -\frac{1 - \frac{1}{N}}{2} \sum_{j=1}^N \Delta_j \mathbb{1}_{\bar{\mathcal{C}}_j} \sum_P, \\
\sum_P \sum_{i, j \in \pi_1} \mathbb{1}_{\bar{\mathcal{C}}_j} U_{1,1}(x_i - x_j) &= \frac{1}{4} \left(1 - \frac{3}{N}\right) \sum_{i \neq j=1}^N \mathbb{1}_{\bar{\mathcal{C}}_j} U^+(x_i - x_j) \sum_P, \\
\sum_P \sum_{i, j \in \pi_2} \mathbb{1}_{\bar{\mathcal{C}}_j} U_{2,2}(x_i - x_j) &= \frac{1}{4} \left(1 + \frac{1}{N}\right) \sum_{i \neq j=1}^N \mathbb{1}_{\bar{\mathcal{C}}_j} U^+(x_i - x_j) \sum_P, \\
\sum_P \sum_{i \in \pi_1, j \in \pi_2} \mathbb{1}_{\bar{\mathcal{C}}_j} U_{1,2}(x_i - x_j) &= \frac{1}{4} \left(1 + \frac{1}{N}\right) \sum_{i \neq j=1}^N \mathbb{1}_{\bar{\mathcal{C}}_j} U_{1,2}(x_i - x_j) \sum_P.
\end{aligned}$$

For N odd and N large enough, the bound of $H_P \geq 0, \forall P$ then implies, together with the assumption 2.2 on U , that $H_{\bar{\mathcal{C}}} \geq 0$.

We will now prove $H_P \geq 0, \forall P$. The advantage to consider H_P instead of $H_{\bar{\mathcal{C}}}$ is that we can analyze $H_P \geq 0$ for fixed configurations of x_i 's with $i \in \pi_2$. This pointwise estimate is sufficient, since there is no kinetic energy of the π_2 -particles. Since permutation of the labels in π_1 and π_2 is irrelevant, we can further assume that $\pi_1 = \{1, \dots, N_1\}$, $\pi_2 = \{N_1 + 1, \dots, N\}$.

Following the idea of [49], for any fixed configuration (x_{N_1+1}, \dots, x_N) , we consider two cases:

- If there are more than m_1 π_2 -particles in a sphere of radius R with $m_1 \geq 2n_1$, the positive interaction $U_{2,2}$, together with $U_{1,1}$ cancels the negative part of $U_{1,2}$. Recall that n_1 is the number of cubes of side length $r_1/\sqrt{3}$ which are needed to cover a sphere of radius R . Therefore, if m_2 π_2 -particles are located in such a sphere, it is possible to derive that at least $\mathcal{O}(m_2^2/n_1)$ π_2 -particles are closer than r_1 to each other. Therefore, if m_1 π_1 -particles and m_2 π_2 -particles are close to each other, the potential energy is of order $\mathcal{O}(m_1^2) + \mathcal{O}(m_2^2) - \mathcal{O}(m_1 m_2)$. This energy is positive, if the negative part of U is small enough.
- If there are less than $2n_1$ π_2 -particles in a sphere of radius R , it is possible to use assumption 2.2, (9), that is

$$-\mathbb{1}_{|x| \leq R} \Delta_x + n_1(2U^+(x) - 4U^-(x)) \geq 0.$$

As in Definition 2.1, we divide \mathbb{R}^3 into cubes C_n ($n \in \mathbb{N}$) of side length $\frac{1}{\sqrt{3}}r_1$, such that the distance between to points $x_i, x_j \in C_n$ is not greater than r_1 . Therefore, for $x_i, x_j \in C_n$ we have by assumption $U(x_i - x_j) \geq \lambda^+$. Next, for fixed $x_i, i \in \pi_2$, for any $x \in \mathbb{R}^3$, we define $G(x)$ as the set of i 's which satisfy $i \in \pi_2$ and $|x_i - x| \leq R$, i.e.,

$$G(x) \equiv \{i \in \pi_2 : |x_i - x| \leq R\}. \quad (49)$$

We denote $|G(x)|$ as the number of the elements of $G(x)$. Note that for $i, j \in G(x)$, it follows that $|x_i - x_j| \leq 2R$.

We denote $d(x, C_n)$ as the distance between the cube $C_n \subset \mathbb{R}^3$ and $x \in \mathbb{R}^3$. Since $|G(y)|$ is uniformly bounded ($|G(y)| \leq N_1$), there must exist a point $X(C_n) \in \mathbb{R}^3$ satisfying $d(X(C_n), C_n) \leq 2R$ and

$$|G(X(C_n))| = \max\{|G(y)| : d(y, C_n) \leq 2R\}. \quad (50)$$

We define $G(C_n) \equiv G(X(C_n))$. Let $\mathbb{1}_{C_n}(x_j)$ denote the projection onto C_n in the coordinate x_j . Furthermore, let Θ denote the usual Heaviside step function. We prove

$$\begin{aligned} \mathcal{H}_1 &= \sum_{i,j \in \pi_2} \mathbb{1}_{\bar{C}_j} U_{2,2}(x_i - x_j) + \sum_{i,j \in \pi_1} \mathbb{1}_{\bar{C}_j} U_{1,1}(x_i - x_j) \\ &\quad - \sum_{n \in \mathbb{N}} \Theta(|G(C_n)| - 2n_1) \sum_{j \in \pi_1, i \in \pi_2} \mathbb{1}_{C_n}(x_j) \mathbb{1}_{\bar{C}_j} U_{1,2}^-(x_i - x_j) \geq 0 \\ \mathcal{H}_{2,j} &= -2\Delta_j \mathbb{1}_{\bar{C}_j} + \sum_{i \in \pi_2} \mathbb{1}_{\bar{C}_j} \frac{1}{2} U_{1,2}^+(x_i - x_j) \\ &\quad - \sum_{n \in \mathbb{N}} \Theta(2n_1 - |G(C_n)|) \sum_{i \in \pi_2} \mathbb{1}_{C_n}(x_j) \mathbb{1}_{\bar{C}_j} \frac{1}{2} U_{1,2}^-(x_i - x_j) \geq 0. \end{aligned}$$

Note that this implies $H_p \geq 0$, since $H_p = \frac{1}{2}\mathcal{H}_1 + \sum_{j \in \pi_1} \mathcal{H}_{2,j}$.

Proof of $\mathcal{H}_1 \geq 0$:

First, we derive the lower bound on the total energy of $U_{2,2}$. With the definition of $G(C_n) = G(X(C_n))$, we know that the set $\{x_k : k \in G(C_n)\}$ can be covered by a sphere of radius R . So the number of the cubes which one need to cover this set is less than n_1 . We denote these cubes as $C_{n_1} \cdots C_{n_m}$ ($m \leq n_1$) and assume the number of i 's satisfying $i \in G(C_n)$ and $x_i \in C_{n_k}$ is a_{n_k} . Because the side length of C_{n_k} is equal to $r_1/\sqrt{3}$, the distance between the two particles in the same cube is no more than r_1 . Hence, we obtain, for $i \neq j$,

$$\sum_{i,j \in G(C_n)} \theta_{r_1}(x_i - x_j) \geq \sum_{k=1}^m \sum_{i,j \in C_{n_k}} = \sum_{k=1}^m [(a_{n_k})^2 - (a_{n_k})] \quad \text{and} \quad \sum_{k=1}^m a_{n_k} = |G(C_n)|.$$

Using Jensen's inequality, together with $m \leq n_1$,

$$\sum_{i,j \in G(C_n)} \theta_{r_1}(x_i - x_j) \geq \frac{1}{2n_1} |G(C_n)|^2.$$

Note that for fixed $i \in \pi_2$, the number of cubes C_n , which satisfy $i \in G(C_n)$ is less than n_2 . Since $U_{2,2}$ is nonnegative, we then obtain

$$\begin{aligned} \sum_{i,j \in \pi_2} \mathbb{1}_{\bar{C}_j} U_{2,2}(x_i - x_j) &= \sum_{n \in \mathbb{N}} \sum_{i,j \in \pi_2} \mathbb{1}_{C_n}(x_i) \mathbb{1}_{\bar{C}_j} U_{2,2}(x_i - x_j) \\ &\geq \frac{1}{n_2} \sum_{n \in \mathbb{N}} \sum_{i,j \in \pi_2, i \in G(C_n)} \mathbb{1}_{\bar{C}_j} U_{2,2}(x_i - x_j) \\ &\geq \frac{1}{n_2} \sum_{n \in \mathbb{N}} \Theta(|G(C_n)| - 2n_1) \sum_{i,j \in G(C_n)} \mathbb{1}_{\bar{C}_j} U_{2,2}(x_i - x_j). \end{aligned}$$

Since $r_1 < R$, it also follows that $n_1 \geq 2$. We then obtain $\mathbb{1}_{\bar{C}_j} U_{2,2}(x_i - x_j) = U_{2,2}(x_i - x_j)$, whenever $i, j \in G(C_n)$ with $|G(C_n)| \geq 2n_1$. Using $U_{2,2}(x) \geq \lambda^+ \Theta_{r_1}(x_i - x_j)$, we have with the estimates above

$$\sum_{i,j \in \pi_2} \mathbb{1}_{\bar{C}_j} U_{2,2}(x_i - x_j) \geq \sum_{n \in \mathbb{N}} \Theta(|G(C_n)| - 2n_1) \frac{\lambda^+}{2n_1 n_2} |G(C_n)|^2.$$

Next, we derive the lower bound on the interaction potential between particles in π_1 . Let $\Pi_1(C_n)$ be defined as the set of i 's such that $i \in \pi_1$ and $x_i \in C_n$. Let $|\Pi_1(C_n)|$ denote the number of the elements of $\Pi_1(C_n)$. If $x_i \in C_n$ and $|G(C_n)| \geq 1$, there must be a $k \in \pi_2$ satisfying $|x_i - x_k| \leq 2R$. Thus, for any C_n we have that

$$\begin{aligned} \sum_{i,j \in \pi_1} \mathbb{1}_{\bar{C}_j} U_{1,1}(x_i - x_j) &= \sum_{n \in \mathbb{N}} \sum_{i,j \in \pi_1} \mathbb{1}_{C_n}(x_i) \mathbb{1}_{\bar{C}_j} U_{1,1}(x_i - x_j) \\ &\geq \sum_{n \in \mathbb{N}} \Theta(|G(C_n)| - 2n_1) \sum_{i,j \in \Pi_1(C_n)} U_{1,1}(x_i - x_j). \end{aligned}$$

For $i, j \in \Pi_1(C_n)$, $i \neq j$, the distance between x_i and x_j is not more than r_1 . Hence,

$$\sum_{i,j \in \Pi_1(C_n)} U_{1,1}(x_i - x_j) \geq \lambda^+ \left(|\Pi_1(C_n)|^2 - |\Pi_1(C_n)| \right). \quad (51)$$

At last, we derive the lower bound on $U_{1,2}^-$.

By the definitions of $|G(C_n)|$ and $U_{1,2}$, we have that $\forall x \in C_n$,

$$- \sum_{i \in \pi_2} U_{1,2}^-(x - x_i) \geq -4\lambda^- |G(C_n)|.$$

This yields to

$$- \sum_{j \in \Pi_1(C_n), i \in \pi_2} \mathbb{1}_{\bar{C}_j} U_{2,1}^-(x_i - x_j) \geq -4\lambda^- |\Pi_1(C_n)| |G(C_n)|. \quad (52)$$

We now consider

$$\begin{aligned} \sum_{i,j \in \Pi_1(C_n)} U_{1,1}(x_i - x_j) - \sum_{j \in \Pi_1(C_n), i \in \pi_2} \mathbb{1}_{\bar{C}_j} U_{1,2}^-(x_i - x_j) \\ \geq \lambda^+ \left(|\Pi_1(C_n)|^2 - |\Pi_1(C_n)| \right) - 4\lambda^- |\Pi_1(C_n)| |G(C_n)|. \end{aligned} \quad (53)$$

Using $\lambda^- \leq \frac{1}{8n_2} \lambda^+$, we then obtain for $|G(C_n)| \geq n_1$

$$(53) \geq \lambda^+ \left(|\Pi_1(C_n)|^2 - |\Pi_1(C_n)| - \frac{1}{2n_2} |\Pi_1(C_n)| |G(C_n)| \right).$$

If $|\Pi_1(C_n)| = 1$, we obtain for $|G(C_n)| \geq 2n_1$

$$(53) \geq -\lambda^+ \frac{|G(C_n)|^2}{4n_1 n_2}.$$

For $|\Pi_1(C_n)| \geq 2$, we have $|\Pi_1(C_n)|^2 - |\Pi_1(C_n)| \geq \frac{1}{2}|\Pi_1(C_n)|^2$ and therefore, for $|G(C_n)| \geq 2n_1$

$$(53) \geq \frac{\lambda^+}{2} \left(|\Pi_1(C_n)|^2 - 2|\Pi_1(C_n)| \frac{1}{2n_2} |G(C_n)| \right) \geq -\frac{\lambda^+}{2} \frac{1}{4(n_2)^2} |G(C_n)|^2.$$

Since $n_2 \geq n_1$ holds, we then obtain for $|G(C_n)| \geq 2n_1$ and for all $|\Pi_1(C_n)| \in \mathbb{N}$

$$(53) \geq -\lambda^+ \frac{|G(C_n)|^2}{4n_1n_2}.$$

Therefore, we obtain

$$\mathcal{H}_1 \geq \sum_{n \in \mathbb{N}} \Theta(|G(C_n)| - 2n_1) \left(\frac{\lambda^+}{2n_1n_2} |G(C_n)|^2 - \frac{\lambda^+}{4n_1n_2} |G(C_n)|^2 \right) \geq 0.$$

Proof of $\mathcal{H}_{2,j} \geq 0$:

Since there is no kinetic energy for the π_2 particles, we prove $\mathcal{H}_{2,j} \geq 0$ for fixed x_i , $i \in \pi_2$. Define

$$\tilde{\mathcal{H}}_{2,j} = -2\Delta_j + \sum_{i \in \pi_2} \frac{1}{2} U_{1,2}^+(x_i - x_j) - \sum_{n \in \mathbb{N}} \Theta(2n_1 - |G(C_n)|) \sum_{i \in \pi_2} \mathbf{1}_{C_n}(x_j) \frac{1}{2} U_{1,2}^-(x_i - x_j) \quad (54)$$

Note that

$$\mathcal{H}_{2,j} = \mathbf{1}_{\bar{c}_j} \tilde{\mathcal{H}}_{2,j}$$

and $\mathbf{1}_{\bar{c}_j}$ commutes with $-\Delta_j$. Hence, it suffices to prove $\tilde{\mathcal{H}}_{2,j} \geq 0$. Let

$$\pi'_2 = \{i \in \pi_2 : \exists C_n, D(x_i, C_n) \leq R, |G(C_n)| \leq 2n_1\}.$$

For fixed x_i and x_j , if

$$\Theta(2n_1 - |G(C_n)|) \mathbf{1}_{C_n}(x_j) \frac{1}{2} U_{1,2}^-(x_i - x_j) \neq 0,$$

it then follows $i \in \pi'_2$. Therefore,

$$\sum_{n \in \mathbb{N}} \Theta(2n_1 - |G(C_n)|) \sum_{i \in \pi_2} \frac{1}{2} \mathbf{1}_{C_n}(x_j) U_{1,2}^-(x_i - x_j) \leq \sum_{i \in \pi'_2} \frac{1}{2} U_{1,2}^-(x_i - x_j).$$

Since $\pi'_2 \subset \pi_2$, it follows that

$$(54) \geq -2\Delta_j + \sum_{i \in \pi'_2} \frac{1}{2} \left(U_{1,2}^+(x_i - x_j) - U_{1,2}^-(x_i - x_j) \right).$$

By the definition of π'_2 , it follows that for any $x \in \mathbb{R}^3$

$$\sum_{i \in \pi'_2} \mathbf{1}_{|x_i - x| \leq R} \leq 2n_1.$$

Under the assumptions on U , we obtain

$$(54) \geq \frac{1}{n_1} \sum_{i \in \pi'_2} \left(-\mathbf{1}_{|x_i - x_j| \leq R} \Delta_j + \frac{n_1}{2} U_{1,2}(x_i - x_j) \right) \geq 0.$$

□

Corollary 3.19 *Let V fulfill assumption 2.2. Then, there exists $0 < \epsilon < 1$ such that*

$$-\sum_{k=1}^N \Delta_k + \sum_{i < j=1}^N (V_1^+(x_i - x_j) - (1 + \epsilon)V_1^-(x_i - x_j)) \geq 0, \quad (55)$$

$$(1 - \epsilon) \sum_{k=1}^N -\Delta_k \mathbb{1}_{\overline{\mathcal{B}}_k} + \sum_{i \neq j} \mathbb{1}_{\overline{\mathcal{B}}_j} \frac{1}{2} V_1(x_i - x_j) \geq 0. \quad (56)$$

Remark 3.20 *These operator inequalities are crucial in order to prove conditions (20), (21) and (22), see below. We do not except the persistence of condensation if (55) and (56) were not true. In that case, one would rather expect the condensate to collapse in the limit $N \rightarrow \infty$ in finite time.*

Proof: By rescaling $Nx \rightarrow x$, the first inequality (55) is equivalent to $-\sum_{k=1}^N \Delta_k + \sum_{i < j=1}^N (V^+(x_i - x_j) - (1 + \epsilon)V^-(x_i - x_j)) \geq 0$. Setting $U(x) = V^+(x) - (1 + \epsilon)V^-(x)$, U then fulfills the conditions of Lemma 3.13 which implies the inequality above.

Setting $\overline{\mathcal{D}}_j := \bigcup_{k, l \neq j} \{(x_1, x_2, \dots, x_N) \in \mathbb{R}^{3N} : |x_l - x_k| < NN^{-26/27}\}$, the second inequality is equivalent to

$$(1 - \epsilon) \sum_{k=1}^N -\Delta_k \mathbb{1}_{\overline{\mathcal{D}}_k} + \sum_{i \neq j} \mathbb{1}_{\overline{\mathcal{D}}_j} \frac{1}{2} V(x_i - x_j) \geq 0.$$

Note that the set $\overline{\mathcal{D}}_j$ defined above fulfills $\tilde{R} = N^{1/27} > 2R$. Hence, Lemma 3.17, part (b) implies the second inequality (56), setting $U = \frac{1}{1-\epsilon}V$.

□

3.3 Proof of condition (20) and (21)

Lemma 3.21 *Let V fulfill assumption 2.2 and let $A_t \in L^\infty(\mathbb{R}^3, \mathbb{R})$. Then, for all $\Psi \in L_s^2(\mathbb{R}^{3N}, \mathbb{C}) \cap H^2(\mathbb{R}^{3N}, \mathbb{C})$*

(a)

$$\|V_1(x_1 - x_3)\Psi\|^2 \leq C\langle\Psi, H\Psi\rangle + CN. \quad (57)$$

(b)

$$\|\nabla_1\Psi\|^2 \leq \frac{C}{N}(\langle\Psi, H\Psi\rangle + 1). \quad (58)$$

Proof:

(a) Let, for $0 < \epsilon < 1$,

$$H^{(\epsilon)} = -\sum_{k=1}^N \Delta_k + \sum_{i < j} (V_1^+(x_i - x_j) - (1 + \epsilon)V_1^-(x_i - x_j)) + \sum_{k=1}^N A_t(x_k).$$

Since V fulfills assumption 2.2, Corollary 3.19 then implies together with $A_t \in L^\infty(\mathbb{R}^3, \mathbb{R})$, $H^{(\epsilon)} \geq -CN$. We then obtain

$$\epsilon \sum_{i < j=1}^N V_1^-(x_i - x_j) \leq H + CN.$$

Furthermore

$$\sum_{i < j=1}^N V_1^+(x_i - x_j) \leq H + \sum_{i < j=1}^N V_1^-(x_i - x_j) + N\|A_t\|_\infty \leq \left(1 + \frac{1}{\epsilon}\right)H + CN.$$

Thus,

$$\begin{aligned} \|V_1(x_1 - x_3)\Psi\|^2 &\leq \|V_1\|_\infty (\langle \Psi, V_1^+(x_1 - x_3)\Psi \rangle + \langle \Psi, V_1^-(x_1 - x_3)\Psi \rangle) \\ &\leq C \left(\langle \Psi, \sum_{i < j=1}^N V_1^+(x_i - x_j)\Psi \rangle + \langle \Psi, \sum_{i < j=1}^N V_1^-(x_i - x_j)\Psi \rangle \right) \\ &\leq C \langle \Psi, H\Psi \rangle + CN. \end{aligned}$$

(b) We use

$$-CN \leq H^{(\epsilon)} \leq (1 + \epsilon) \left(\frac{-1}{1 + \epsilon} \sum_{k=1}^N \Delta_k + \sum_{i < j} V_1(x_i - x_j) + \sum_{k=1}^N \frac{1}{1 + \epsilon} A_t(x_k) \right).$$

Let $\mu = 1 - \frac{1}{1 + \epsilon} > 0$. Using $A_t \in L^\infty(\mathbb{R}^3, \mathbb{R})$, we then obtain

$$-\mu \sum_{k=1}^N \Delta_k \leq H + CN.$$

□

Using Lemma 3.21 together with $\frac{\langle \Psi_t, H\Psi_t \rangle}{N} \leq C$, we then obtain condition (20) and (21).

3.4 Proof of condition (22)

We will first restate a Lemma which we will need in the following.

Proposition 3.22 *Let $\Omega \in H^1(\mathbb{R}^{3N}, \mathbb{C})$. Then, for all $j \neq k$*

$$\|\mathbf{1}_{\bar{\mathcal{B}}_j} \Omega\| \leq CN^{-7/54} \|\nabla_j \Omega\|.$$

Proof: The proof of this Lemma, which is a direct consequence of Sobolev's inequality, can be found in [44], Proposition A.1. for the three dimensional case and in [18], Lemma 7.4. for the two dimensional analog (note that the set $\bar{\mathcal{B}}_j$ and the respective N -dependent bound are different in two dimensions.).

□

Lemma 3.23 *Assume V fulfills assumption 2.2. Then, for any $\Psi \in L_s^2(\mathbb{R}^{3N}, \mathbb{C}) \cap H^2(\mathbb{R}^{3N}, \mathbb{C})$ and any $\varphi \in H^2(\mathbb{R}^3, \mathbb{C})$ there exists a $\eta > 0$ such that*

(a)

$$\|\mathbb{1}_{\mathcal{A}_1} \nabla_1 q_1^\varphi \Psi\|^2 \leq C (\langle \Psi, \widehat{n}^\varphi \Psi \rangle + N^{-\eta}) + |\mathcal{E}(\Psi) - \mathcal{E}^{GP}(\varphi)|.$$

(b)

$$\|\mathbb{1}_{\overline{\mathcal{B}}_1} \nabla_1 \Psi\|^2 \leq C (\langle \Psi, \widehat{n}^\varphi \Psi \rangle + N^{-\eta}) + |\mathcal{E}(\Psi) - \mathcal{E}^{GP}(\varphi)|.$$

Remark 3.24 *For nonnegative potentials, the proof of Lemma 3.23 was given in Lemma 5.2 in [44] for the three dimensional case and in Lemma 7.9 in [18] for the two dimensional case. For potentials which fulfill assumption 2.2 we use Corollary 3.19 in order to obtain the same bound.*

Proof: Let us first split up the energy difference. Since $\Psi \in L_s^2(\mathbb{R}^{3N}, \mathbb{C})$ is symmetric,

$$\begin{aligned} \mathcal{E}(\Psi) - \mathcal{E}^{GP}(\varphi) &= \|\nabla_1 \Psi\|^2 + (N-1) \langle \Psi, V_1(x_1 - x_2) \Psi \rangle \\ &\quad - \|\nabla \varphi\|^2 - 2a \|\varphi^2\|^2 + \langle \Psi, A_t \Psi \rangle - \langle \varphi, A_t \varphi \rangle. \end{aligned}$$

Let W_{β_1} be defined as in Lemma 3.7 for some β_1 . Then,

$$\begin{aligned} \mathcal{E}(\Psi) - \mathcal{E}^{GP}(\varphi) &= \|\mathbb{1}_{\mathcal{A}_1} \nabla_1 \Psi\|^2 + \|\mathbb{1}_{\overline{\mathcal{B}}_1} \mathbb{1}_{\overline{\mathcal{A}}_1} \nabla_1 \Psi\|^2 + \|\mathbb{1}_{\mathcal{B}_1} \mathbb{1}_{\overline{\mathcal{A}}_1} \nabla_1 \Psi\|^2 \\ &\quad + (N-1) \langle \Psi, \mathbb{1}_{\overline{\mathcal{B}}_1} V_1(x_1 - x_2) \Psi \rangle \\ &\quad + \langle \Psi, \sum_{j \neq 1} \mathbb{1}_{\mathcal{B}_1} (V_1 - W_{\beta_1})(x_1 - x_j) \Psi \rangle \\ &\quad + \langle \Psi, \sum_{j \neq 1} \mathbb{1}_{\mathcal{B}_1} W_{\beta_1}(x_1 - x_j) \Psi \rangle - \|\nabla \varphi\|^2 - 2a \|\varphi^2\|^2 \\ &\quad + \langle \Psi A_t \Psi \rangle - \langle \varphi A_t \varphi \rangle. \end{aligned}$$

Using that $q_1 = 1 - p_1$, we obtain for $0 < \epsilon < 1$,

$$\mathcal{E}(\Psi) - \mathcal{E}^{GP}(\varphi) \tag{59}$$

$$= \epsilon \left(\|\mathbb{1}_{\mathcal{A}_1} \nabla_1 q_1 \Psi\|^2 + \|\mathbb{1}_{\overline{\mathcal{B}}_1} \mathbb{1}_{\overline{\mathcal{A}}_1} \nabla_1 \Psi\|^2 \right) \tag{60}$$

$$+ 2\Re \left(\langle \nabla_1 q_1 \Psi, \mathbb{1}_{\mathcal{A}_1} \nabla_1 p_1 \Psi \rangle \right) \tag{61}$$

$$+ \|\mathbb{1}_{\mathcal{B}_1} \mathbb{1}_{\overline{\mathcal{A}}_1} \nabla_1 \Psi\|^2 + \frac{1}{2} \langle \Psi, \sum_{j=2}^N \mathbb{1}_{\mathcal{B}_1} (V_1 - W_{\beta_1})(x_1 - x_j) \Psi \rangle \tag{62}$$

$$+ \frac{N-1}{2} \langle \Psi, \mathbb{1}_{\mathcal{B}_1} p_1 p_2 W_{\beta_1}(x_1 - x_2) p_1 p_2 \mathbb{1}_{\mathcal{B}_1} \Psi \rangle - \frac{a}{2} \|\varphi^2\|^2 \tag{63}$$

$$+ (N-1) \Re \langle \Psi, \mathbb{1}_{\mathcal{B}_1} (1 - p_1 p_2) W_{\beta_1}(x_1 - x_2) p_1 p_2 \mathbb{1}_{\mathcal{B}_1} \Psi \rangle \tag{64}$$

$$+ \frac{N-1}{2} \langle \Psi, \mathbb{1}_{\mathcal{B}_1} (1 - p_1 p_2) W_{\beta_1}(x_1 - x_2) (1 - p_1 p_2) \mathbb{1}_{\mathcal{B}_1} \Psi \rangle \tag{65}$$

$$+ \|\mathbb{1}_{\mathcal{A}_1} \nabla_1 p_1 \Psi\|^2 - \|\nabla \varphi\|^2 \tag{66}$$

$$+ \langle \Psi, A_t(x_1) \Psi \rangle - \langle \varphi, A_t \varphi \rangle \tag{67}$$

$$+ (1 - \epsilon) \left(\|\mathbb{1}_{\mathcal{A}_1} \nabla_1 q_1 \Psi\|^2 + \|\mathbb{1}_{\overline{\mathcal{B}}_1} \mathbb{1}_{\overline{\mathcal{A}}_1} \nabla_1 \Psi\|^2 \right) \tag{68}$$

$$+ \frac{N-1}{2} \langle \Psi, \mathbb{1}_{\overline{\mathcal{B}}_1} V_1(x_1 - x_2) \Psi \rangle. \tag{69}$$

It has been shown in [44] that for some suitable chosen $0 < \beta_1 < 1$ there exists an $\eta > 0$ such that

$$|(59)| + |(61)| + |(63)| + |(66)| + |(67)| \leq C (\langle \Psi, \widehat{n}^\varphi \Psi \rangle + N^{-\eta}) + |\mathcal{E}(\Psi) - \mathcal{E}^{GP}(\varphi)|.$$

Since (62) ≥ 0 , (64) ≥ 0 , we are left to control (68) and (69) in order to show

$$\epsilon \left(\|\mathbb{1}_{\mathcal{A}_1} \nabla_1 q_1 \Psi\|^2 + \|\mathbb{1}_{\overline{\mathcal{B}}_1} \mathbb{1}_{\overline{\mathcal{A}}_1} \nabla_1 \Psi\|^2 \right) \leq C (\langle \Psi, \widehat{n}^\varphi \Psi \rangle + N^{-\eta}) + |\mathcal{E}(\Psi) - \mathcal{E}^{GP}(\varphi)|.$$

For nonnegative potentials, the trivial bound (68) + (69) ≥ 0 is sufficient in order to prove Lemma 3.23. For potentials fulfilling assumption 2.2, we use

$$\begin{aligned} (68) + (69) &= (1 - \epsilon) \left(\|\mathbb{1}_{\mathcal{A}_1} \mathbb{1}_{\overline{\mathcal{B}}_1} \nabla_1 \Psi\|^2 + \|\mathbb{1}_{\overline{\mathcal{B}}_1} \mathbb{1}_{\overline{\mathcal{A}}_1} \nabla_1 \Psi\|^2 \right) + \frac{N-1}{2} \langle \Psi, \mathbb{1}_{\overline{\mathcal{B}}_1} V_1(x_1 - x_2) \Psi \rangle \\ &\quad - (1 - \epsilon) 2\Re \left(\langle \nabla_1 \Psi, \mathbb{1}_{\mathcal{A}_1} \mathbb{1}_{\overline{\mathcal{B}}_1} \nabla_1 p_1 \Psi \rangle \right) \\ &\quad + (1 - \epsilon) \left(\|\mathbb{1}_{\mathcal{A}_1} \mathbb{1}_{\mathcal{B}_1} \nabla_1 q_1 \Psi\|^2 + \|\mathbb{1}_{\mathcal{A}_1} \mathbb{1}_{\overline{\mathcal{B}}_1} \nabla_1 p_1 \Psi\|^2 \right). \end{aligned}$$

We will estimate each line separately. The third line is positive. Using Proposition 3.22, we obtain

$$\|\mathbb{1}_{\mathcal{A}_1} \mathbb{1}_{\overline{\mathcal{B}}_1} \nabla_1 p_1 \Psi\| \leq \|\mathbb{1}_{\overline{\mathcal{B}}_1} \nabla_1 p_1 \Psi\| \leq CN^{-7/54} \|\Delta_1 p_1 \Psi\|.$$

This implies for the second line

$$|2\Re \left(\langle \nabla_1 \Psi, \mathbb{1}_{\overline{\mathcal{B}}_1} \mathbb{1}_{\mathcal{A}_1} \nabla_1 p_1 \Psi \rangle \right)| \leq CN^{-7/54}.$$

Focusing on the first term, we obtain with Corollary 3.19

$$\begin{aligned} & (1 - \epsilon) \left(\|\mathbb{1}_{\mathcal{A}_1} \mathbb{1}_{\overline{\mathcal{B}}_1} \nabla_1 \Psi\|^2 + \|\mathbb{1}_{\overline{\mathcal{B}}_1} \mathbb{1}_{\overline{\mathcal{A}}_1} \nabla_1 \Psi\|^2 \right) + \frac{N-1}{2} \langle \langle \Psi, \mathbb{1}_{\overline{\mathcal{B}}_1} V_1(x_1 - x_2) \Psi \rangle \rangle \\ &= \frac{1}{N} \langle \langle \Psi, \left((1 - \epsilon) \sum_{k=1}^N -\Delta_k \mathbb{1}_{\overline{\mathcal{B}}_k} + \sum_{i \neq j} \mathbb{1}_{\overline{\mathcal{B}}_j} \frac{1}{2} V_1(x_i - x_j) \Psi \right) \rangle \rangle \geq 0. \end{aligned}$$

We have therefore shown

$$\|\mathbb{1}_{\mathcal{A}_1} \nabla_1 q_1 \Psi\|^2 + \|\mathbb{1}_{\overline{\mathcal{B}}_1} \mathbb{1}_{\overline{\mathcal{A}}_1} \nabla_1 \Psi\|^2 \leq C \left(\langle \Psi, \widehat{n}^\varphi \Psi \rangle + N^{-\eta} + |\mathcal{E}(\Psi) - \mathcal{E}^{GP}(\varphi)| \right).$$

Note that

$$\begin{aligned} \|\mathbb{1}_{\overline{\mathcal{B}}_1} \nabla_1 q_1 \Psi\|^2 &= \|\mathbb{1}_{\overline{\mathcal{A}}_1} \mathbb{1}_{\overline{\mathcal{B}}_1} \nabla_1 q_1 \Psi\|^2 + \|\mathbb{1}_{\mathcal{A}_1} \mathbb{1}_{\overline{\mathcal{B}}_1} \nabla_1 q_1 \Psi\|^2 \\ &\leq \|\mathbb{1}_{\overline{\mathcal{A}}_1} \mathbb{1}_{\overline{\mathcal{B}}_1} \nabla_1 (1 - p_1) \Psi\|^2 + \|\mathbb{1}_{\mathcal{A}_1} \nabla_1 q_1 \Psi\|^2 \\ &\leq 2 \|\mathbb{1}_{\overline{\mathcal{A}}_1} \mathbb{1}_{\overline{\mathcal{B}}_1} \nabla_1 \Psi\|^2 + 2 \|\mathbb{1}_{\overline{\mathcal{A}}_1} \mathbb{1}_{\overline{\mathcal{B}}_1} \nabla_1 p_1 \Psi\|^2 + \|\mathbb{1}_{\mathcal{A}_1} \nabla_1 q_1 \Psi\|^2. \end{aligned}$$

Using $\|\mathbb{1}_{\overline{\mathcal{B}}_1} \mathbb{1}_{\overline{\mathcal{A}}_1} \nabla_1 p_1 \Psi\| \leq \|\mathbb{1}_{\overline{\mathcal{B}}_1} \nabla_1 p_1 \Psi\| \leq CN^{-7/54} \|\Delta_1 p_1 \Psi\|$, we then obtain the Lemma. □

Acknowledgments

We are grateful to Nikolai Leopold and Robert Seiringer for pointing out to us the results of [49]. We also thank Phillip Grass for helpful remarks. M.J. gratefully acknowledges financial support by the German National Academic Foundation.

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