

A CLASSICAL PROOF THAT THE ALGEBRAIC HOMOTOPY CLASS OF A RATIONAL FUNCTION IS THE RESIDUE PAIRING

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ABSTRACT. Cazanave has identified the algebraic homotopy class of a rational function of 1 variable with an explicit nondegenerate symmetric bilinear form. Here we show that Hurwitz’s proof of a classical result about real rational functions essentially gives an alternative proof of the stable part of Cazanave’s result. We also explain how this result can be interpreted in terms of the residue pairing and that this interpretation relates the result to the signature theorem of Eisenbud, Khimshiashvili, and Levine, showing that Cazanave’s result answers a question posed by Eisenbud for polynomial functions in 1 variable. Finally, we announce results answering this question for functions in an arbitrary number of variables.

1. INTRODUCTION

In this paper we show that a classical proof of Hurwitz can be modified to give a new proof of Cazanave’s description of the degree of a rational function in \mathbf{A}^1 -homotopy theory. In [Hur95], Hurwitz computed the topological degree of a real rational function $f/g \in \text{Frac } \mathbf{R}[x]$ where we assume f is monic and of degree $\mu > \deg(g)$. The real rational function f/g defines a self-map of the real projective line $\mathbf{P}_{\mathbf{R}}^1$, and the space $\mathbf{P}_{\mathbf{R}}^1(\mathbf{R})$ of real points is homeomorphic to the 1-sphere S^1 , so the self-map has a well-defined topological degree. Hurwitz’s theorem [Hur95, Section 3] is

Theorem 1 (Hurwitz). *The topological degree of $f/g: \mathbf{P}_{\mathbf{R}}^1(\mathbf{R}) \rightarrow \mathbf{P}_{\mathbf{R}}^1(\mathbf{R})$ equals the signature of the real symmetric matrix*

$$S(f/g) = \begin{pmatrix} s_1 & s_2 & \cdots & s_{\mu-1} & s_{\mu} \\ s_2 & s_3 & \cdots & s_{\mu} & s_{\mu-1} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ s_{\mu} & s_{\mu+1} & \cdots & s_{2\mu-2} & s_{2\mu-1} \end{pmatrix}$$

for s_1, s_2, \dots defined by $g(x)/f(x) = s_1/x + s_2/x^2 + s_3/x^3 + \dots$

Cazanave proved an enrichment of Hurwitz’s result in \mathbf{A}^1 -homotopy theory. The \mathbf{A}^1 - or motivic degree of a rational function over an arbitrary field k was constructed by Morel in celebrated work describing homotopy classes of self-maps of a motivic sphere. The \mathbf{A}^1 -degree of a rational function is valued in the Grothendieck–Witt group $\text{GW}(k)$ of stable isomorphism classes of nondegenerate symmetric bilinear forms. Morel’s construction is indirect, but Cazanave proved the \mathbf{A}^1 -degree has the following explicit description.

Date: March 15, 2018.

2010 Mathematics Subject Classification. Primary 14F42; Secondary 14B05, 55M25.

Theorem 2 (Cazanave). *The \mathbf{A}^1 -homotopical degree of $f/g: \mathbf{P}_k^1 \rightarrow \mathbf{P}_k^1$ equals the class of*

$$\text{Béz}(f/g) := \begin{pmatrix} b_{1,1} & \cdots & b_{1,\mu} \\ \vdots & \ddots & \vdots \\ b_{\mu,1} & \cdots & b_{\mu,\mu} \end{pmatrix}$$

for $b_{i,j}$ defined by

$$(1) \quad f(x)g(y) - f(y)g(x) = (x - y) \cdot \left(\sum_{i,j=1}^{\mu} b_{i,j} x^{i-1} y^{j-1} \right).$$

This is not explicitly stated in [Caz08, Caz12], but it follows from [Caz12, Theorem 1.2, Corollary 3.10], as we explain below. Cazanave also proved results stronger than Theorem 2, and we discuss them below as well.

The Bézout matrix $\text{Béz}(f/g)$ is conjugate to $S(f/g)$, so when $k = \mathbf{R}$, the real realization (in the sense of [MV99, Section 3.3]) of Cazanave's result is Hurwitz's result. Cazanave's proof is, however, different from Hurwitz's proof, and one goal of this paper is to show that Hurwitz's proof can be modified to prove Theorem 2.

We present the modification of Hurwitz's proof in a way to make the result accessible to readers without a background in \mathbf{A}^1 -homotopy theory. In proving the result, we only make use of some basic properties of the degree in \mathbf{A}^1 -homotopy theory that are straightforward analogues of familiar properties of the topological degree. We recall the properties we use at the beginning of Section 2, and a reader that knows no \mathbf{A}^1 -homotopy theory but is willing to accept that these properties hold should be able to understand the proof.

To explain the idea of Hurwitz's proof and contrast it to Cazanave's, let us recall Cazanave's proof. His proof makes fundamental use of ideas from homotopy theory. There are two important sets of homotopy classes of self-maps of \mathbf{P}_k^1 . The first set is the set $[\mathbf{P}_k^1, \mathbf{P}_k^1]^{\mathbf{A}^1}$ of (stable pointed) \mathbf{A}^1 -homotopy classes. This set is constructed using model categories, and it naturally has the structure of a \mathbf{Z} -graded group. Morel constructed the \mathbf{A}^1 -degree as an isomorphism $\text{deg}^{\mathbf{A}^1}: [\mathbf{P}_k^1, \mathbf{P}_k^1]^{\mathbf{A}^1} \rightarrow \text{GW}(k)$ of graded groups.

The second set of homotopy classes is the set $[\mathbf{P}_k^1, \mathbf{P}_k^1]^{\mathbf{N}}$ of (pointed) naive homotopy classes. This set is the quotient of the set of all self-maps of \mathbf{P}_k^1 by the equivalence relation defined by naively imitating the definition of homotopy in algebraic topology (with the unit interval replaced by the affine line). Cazanave endows $[\mathbf{P}_k^1, \mathbf{P}_k^1]^{\mathbf{N}}$ with the structure of a \mathbf{N} -graded monoid that makes the natural map $[\mathbf{P}_k^1, \mathbf{P}_k^1]^{\mathbf{N}} \rightarrow [\mathbf{P}_k^1, \mathbf{P}_k^1]^{\mathbf{A}^1}$ into a monoid homomorphism. Theorem 2 is then proven by showing that the rule sending f/g to $\text{Béz}(f/g)$ defines a homomorphism $[\mathbf{P}_k^1, \mathbf{P}_k^1]^{\mathbf{N}} \rightarrow \text{GW}(k)$. This proves the theorem as the rational functions x/A with $A \in k^*$ generate $[\mathbf{P}_k^1, \mathbf{P}_k^1]^{\mathbf{N}}$, and for these functions, both $\text{Béz}(f/g)$ and $\text{deg}^{\mathbf{A}^1}(f/g)$ equal $\langle A \rangle$, the class of the bilinear form with Gram matrix (A) .

By contrast, Hurwitz's proof uses matrix theory as follows. The topological degree of f/g equals the sum, over the real zeros of f/g , of the local topological degrees. To identify this sum with the signature of $S(f/g)$, we pass to the complex numbers. Over the complex

numbers, $S(f/g)$ is conjugate to a block diagonal matrix $\text{New}(f/g)$ with blocks indexed by the complex roots of $f(x)$. The proof is completed by computing the signature of $S(f/g)$ in terms of the blocks, computing that a pair of blocks corresponding to a complex conjugate pair of roots contributes 0 to the signature and a block corresponding to a real root contributes the local degree at the root.

We show that the same argument, suitably modified, gives an independent proof that $\deg^{\mathbf{A}^1}(f/g)$ is the class of $S(f/g)$ or equivalently $\text{Béz}(f/g)$. This argument, however, proves a result that is strictly weaker than the result proven by Cazanave. The \mathbf{A}^1 -degree of f/g determines the *stable* homotopy class of f/g , and Cazanave proves that in fact $\text{Béz}(f/g)$ represents the *unstable* homotopy class of f/g . Recall Morel proved that the unstable homotopy class of f/g is determined by $\deg^{\mathbf{A}^1}(f/g)$ together with a scalar $d(f/g) \in k^*$ representing the class of the discriminant of $\deg^{\mathbf{A}^1}(f/g)$. Cazanave proves that $d(f/g)$ equals the determinant of $\text{Béz}(f/g)$. This second result is not established by our modification of Hurwitz's proof, as we explain at the end of Section 2. We also do not study the monoid structure on $[\mathbf{P}_k^1, \mathbf{P}_k^1]^{\mathbf{N}}$. While the monoid structure plays a fundamental role in Cazanave's proof, it does not play a visible role in Hurwitz's proof.

A secondary goal of this paper is to explain the relation of the work of Hurwitz and Cazanave to the beautiful signature theorem of Eisenbud–Levine and Khimshiashvili. The signature formula identifies the local degree of a real polynomial function (possibly in many variables) as the signature of the residue pairing. In Section 3, we explain that the matrices $\text{Béz}(f/g)$ and $S(f/g)$ are Gram matrices for the residue pairing and that the signature formula generalizes Hurwitz's theorem. We announce the following theorem which generalizes the signature theorem and answers a question posed by Eisenbud in [Eis78]:

Theorem 3 (The main result of [KW]). *The local \mathbf{A}^1 -degree of $f: \mathbf{A}_k^n \rightarrow \mathbf{A}_k^n$ is the class of the residue pairing $\beta_0(\text{Res}_0)$.*

As we explain in Section 3, $\beta_0(\text{Res}_0)$ is a bilinear pairing on the localization of the algebra $Q_0(f/g) = k[x, g^{-1}]/(fg^{-1})$ at the ideal (x) . The reader familiar with Cazanave's work may recall that he identifies $\text{Béz}(f/g)$ as a Gram matrix of a bilinear pairing under suitable hypotheses, but his bilinear pairing is not $\beta_0(\text{Res}_0)$. We explain the relation between the pairings in Section 3.

In this paper we will assume $\text{char } k \neq 2$, primarily to simplify exposition. Both Cazanave's theorem and Theorem 3 remain valid in characteristic 2, but this assumption simplifies the proof of Theorem 10. At one step in the proof, we reduce to the case where the preimage $(f/g)^{-1}(0)$ of the origin is étale so that we can apply a formula (Equation (2) below) relating the global degree to the local degree, and the reduction makes use of the assumption $\text{char } k \neq 2$ (specifically the consequence that a purely inseparable extension has odd degree).

CONVENTIONS

k is a field of characteristic $\neq 2$.

A rational function is a nonconstant k -morphism $f/g: \mathbf{P}_k^1 \rightarrow \mathbf{P}_k^1$ satisfying $f/g(\infty) = \infty$. We always represent such a morphism by a fraction $f(x)/g(x) \in \text{Frac } k[x]$ with f monic, f relatively prime to g , and $\deg(f) > \deg(g)$. Observe that our rational functions are more properly called pointed rational functions, but we omit the term “pointed.”

The Gram matrix of a symmetric bilinear form β on a finite dimensional k -vector space V with respect to a basis e_1, e_2, \dots, e_μ is the matrix $(\beta(e_i, e_j))$.

We write $\langle A \rangle \in \text{GW}(k)$ for the class of the symmetric bilinear form with Gram matrix (A) .

2. A CLASSICAL PROOF

Here we adapt Hurwitz’s proof of Theorem 1 to prove Cazanave’s result, Theorem 2. The proof is closely modeled on the Kreĭn–Naĭmark’s exposition of Hurwitz’s result in [KN81], and after giving the proof, we closely compare our argument with the argument in loc. cit. We then explain why Hurwitz’s argument does not prove the stronger result that $\text{Béz}(f/g)$ represents the unstable homotopy class of f/g .

We begin by summarizing the basic properties of the \mathbf{A}^1 -degree that we use. The properties are all analogues of familiar properties of the usual topological degree of a real rational function $f/g: \mathbf{P}_{\mathbf{R}}^1 \rightarrow \mathbf{P}_{\mathbf{R}}^1$. The reader unfamiliar with the degree map in \mathbf{A}^1 -homotopy theory should be able to understand the proof of Theorem 10 using just the properties we summarize below (up to Lemma 4).

The degree map in \mathbf{A}^1 -homotopy theory takes values in the the Grothendieck–Witt $\text{GW}(k)$. This is the group completion of the monoid of nondegenerate quadratic forms. The theory supports both a global degree $\deg^{\mathbf{A}^1}(f/g)$ of a rational function f/g and a local degree $\deg_r^{\mathbf{A}^1}(f/g)$ at a closed point $r \in \mathbf{P}_k^1$ with residue field k . These degrees satisfy the following properties:

- the degree of a composition is the product of the degrees,
- the global degree can be expressed as a sum of traces of local degrees,
- both the local and global degrees are invariant under naive \mathbf{A}^1 -homotopies,
- the global degree is normalized so that $\deg^{\mathbf{A}^1}(x/A) = \langle A \rangle$.

The expression of the global degree as a sum of local degrees is the following one. Suppose $f/g: \mathbf{P}_k^1 \rightarrow \mathbf{P}_k^1$ is a rational function with the property that preimage of the origin is a union $\{r_1, \dots, r_m\}$ of closed points with residue fields that are separable extensions of k . Then the global \mathbf{A}^1 -degree satisfies

$$(2) \quad \deg^{\mathbf{A}^1}(f/g) = \text{Tr}_{k[r_1]/k}(\deg_{r_1}^{\mathbf{A}^1}(f/g \otimes k[r_1])) + \dots + \text{Tr}_{k[r_m]/k}(\deg_{r_m}^{\mathbf{A}^1}(f/g \otimes k[r_m])).$$

Here $\text{Tr}: \text{GW}(L) \rightarrow \text{GW}(k)$ is the trace function, which is defined by sending a bilinear pairing $\beta: V \times V \rightarrow L$ to the composition $\text{Tr}_{L/k} \circ \beta$ of β with the field trace. Under additional hypotheses, the formula is described in [Mor04, Section 2], [Mor06, page 1036],

[Lev08, Section 5.2]. The formula can be deduced from [KW, Proposition 13] and [KW, Proposition 32].

The invariance property of the global degree we use is that

$$\deg^{\mathbf{A}^1}(f_0/g_0) = \deg^{\mathbf{A}^1}(f_1/g_1)$$

when there exists a morphism $H: \mathbf{P}_k^1 \times_k \mathbf{A}_k^1 \rightarrow \mathbf{P}_k^1$ such that $f_0/g_0(x) = H(x, 0)$, $f_1/g_1(x) = H(x, 1)$, and $H(\infty, t) = \infty$ (i.e. when there exist a naive \mathbf{A}^1 -homotopy). The local analogue is the following lemma.

Lemma 4. *Let $f_0/g_0, f_1/g_1: \mathbf{P}_k^1 \rightarrow \mathbf{P}_k^1$ be rational functions satisfying $f_0(0) = f_1(0) = 0$. If there exists a morphism $H: \mathbf{P}_k^1 \times_k \mathbf{A}_k^1 \rightarrow \mathbf{P}_k^1$ such that*

$$H(x, 0) = f_0/g_0(x), H(x, 1) = f_1/g_1(x), \text{ and} \\ \{0\} \times_k \mathbf{A}_k^1 \text{ is a connected component of } H^{-1}(\{0\} \times \mathbf{A}_k^1),$$

then

$$\deg_0^{\mathbf{A}^1}(f_0/g_0) = \deg_0^{\mathbf{A}^1}(f_1/g_1).$$

Proof. The local degree $\deg_0^{\mathbf{A}^1}(f/g)$ of a rational function f/g with an isolated zero at the origin is defined using the purity theorem [MV99, Section 3, Lemma 2.1]. If $U \subset \mathbf{P}_k^1$ is the complement of the zeros distinct from the origin, then $f/g: \mathbf{P}_k^1 \rightarrow \mathbf{P}_k^1$ induces a morphism of quotient spaces

$$(3) \quad \frac{U}{U - \{0\}} \rightarrow \frac{\mathbf{P}_k^1}{\mathbf{P}_k^1 - \{0\}}.$$

The purity theorem identifies both the target and source quotient space with \mathbf{P}_k^1 , so the morphism (3) has a well-defined global \mathbf{A}^1 -degree that is defined to be $\deg_0^{\mathbf{A}^1}(f/g)$.

Now suppose that we are given H as in the statement of the lemma. We use H to construct a global naive \mathbf{A}^1 -homotopy as follows. Let Z be the union of the connected components of $H^{-1}(\{0\} \times_k \mathbf{A}_k^1)$ that are distinct from $\{0\} \times_k \mathbf{A}_k^1$. We have the identification

$$\begin{aligned} \frac{\mathbf{P}_k^1 \times_k \mathbf{A}_k^1}{\mathbf{P}_k^1 \times_k \mathbf{A}_k^1 - H^{-1}(0)} &= \frac{\mathbf{P}_k^1 \times_k \mathbf{A}_k^1}{\mathbf{P}_k^1 \times_k \mathbf{A}_k^1 - (\{x=0\} \cup Z)} \\ &= \frac{\mathbf{P}_k^1 \times_k \mathbf{A}_k^1}{\mathbf{P}_k^1 \times_k \mathbf{A}_k^1 - \{x=0\}} \vee \frac{\mathbf{P}_k^1 \times_k \mathbf{A}_k^1}{\mathbf{P}_k^1 \times_k \mathbf{A}_k^1 - Z} \end{aligned}$$

Thus the morphism $\frac{\mathbf{P}_k^1 \times_k \mathbf{A}_k^1}{\mathbf{P}_k^1 \times_k \mathbf{A}_k^1 - H^{-1}(0)} \rightarrow \frac{\mathbf{P}_k^1}{\mathbf{P}_k^1 - 0}$ induced by H can be identified with a morphism $\frac{\mathbf{P}_k^1 \times_k \mathbf{A}_k^1}{\mathbf{P}_k^1 \times_k \mathbf{A}_k^1 - \{x=0\}} \vee \frac{\mathbf{P}_k^1 \times_k \mathbf{A}_k^1}{\mathbf{P}_k^1 \times_k \mathbf{A}_k^1 - Z} \rightarrow \frac{\mathbf{P}_k^1}{\mathbf{P}_k^1 - 0}$. Pre-composing this morphism with the natural morphism $\frac{\mathbf{P}_k^1}{\mathbf{P}_k^1 - \{0\}} \times_k \mathbf{A}_k^1 \rightarrow \frac{\mathbf{P}_k^1 \times_k \mathbf{A}_k^1}{\mathbf{P}_k^1 \times_k \mathbf{A}_k^1 - \{x=0\}} \rightarrow \frac{\mathbf{P}_k^1 \times_k \mathbf{A}_k^1}{\mathbf{P}_k^1 \times_k \mathbf{A}_k^1 - \{x=0\}} \vee \frac{\mathbf{P}_k^1 \times_k \mathbf{A}_k^1}{\mathbf{P}_k^1 \times_k \mathbf{A}_k^1 - Z}$, we obtain a naive \mathbf{A}^1 -homotopy from the morphism induced by f_0/g_0 to the morphism induced by f_1/g_1 . \square

We first use these properties to compute the degree of a power map.

Lemma 5. For $A \in k^*$, we have

$$(4) \quad \deg^{\mathbf{A}^1}(x^\mu/A) = \begin{cases} \langle A \rangle + \frac{\mu-1}{2} \cdot \langle 1, -1 \rangle & \mu \text{ odd;} \\ \frac{\mu}{2} \cdot \langle 1, -1 \rangle & \mu \text{ even.} \end{cases}$$

Proof. First, we prove the case where $A = 1$ by induction on μ . The result holds for $\mu = 1$ by the construction of $\deg^{\mathbf{A}^1}$, so we assume the result holds for μ and then prove it holds for $\mu + 1$. Consider the auxiliary function $x^{\mu+1} + x^\mu: \mathbf{P}_k^1 \rightarrow \mathbf{P}_k^1$. We compute the degree in two different ways.

First, the expression $x^{\mu+1} + tx^\mu$ defines a naive \mathbf{A}^1 -homotopy from $x^{\mu+1} + x^\mu$ to $x^{\mu+1}$, so

$$(5) \quad \deg^{\mathbf{A}^1}(x^{\mu+1} + x^\mu) = \deg^{\mathbf{A}^1}(x^{\mu+1}).$$

Second, by Formula (2), we have that $\deg^{\mathbf{A}^1}(x^{\mu+1} + x^\mu) = \deg_0^{\mathbf{A}^1}(x^{\mu+1} + x^\mu) + \deg_{-1}^{\mathbf{A}^1}(x^{\mu+1} + x^\mu)$. To compute the local degrees, observe that $(x, t) \mapsto x^\mu(1 + tx)$ defines a naive local \mathbf{A}^1 -homotopy in the sense of Lemma 4, so

$$\deg_0^{\mathbf{A}^1}(x^{\mu+1} + x^\mu) = \deg_0^{\mathbf{A}^1}(x^\mu).$$

A similar argument shows

$$\deg_{-1}^{\mathbf{A}^1}(x^{\mu+1} + x^\mu) = \deg_{-1}^{\mathbf{A}^1}((-1)^\mu(x + 1)).$$

We now compute

$$\begin{aligned} \deg^{\mathbf{A}^1}(x^{\mu+1}) &= \deg^{\mathbf{A}^1}(x^{\mu+1} + x^\mu) \\ &= \deg_0^{\mathbf{A}^1}(x^\mu) + \deg_{-1}^{\mathbf{A}^1}((-1)^\mu(x + 1)) \\ &= \deg^{\mathbf{A}^1}(x^\mu) + \langle (-1)^\mu \rangle, \end{aligned}$$

completing induction.

We complete the proof by observing that $\deg^{\mathbf{A}^1}(x^\mu/A) = \deg^{\mathbf{A}^1}(x/A) \cdot \deg^{\mathbf{A}^1}(x^\mu)$ because $\deg^{\mathbf{A}^1}$ transforms composition into multiplication. □

The next lemma identifies the element of $\text{GW}(k)$ from the last lemma with an explicit matrix.

Lemma 6. If $A(1), A(2), \dots, A(\mu) \in k$ are scalars with $A(\mu)$ nonzero, then the nondegenerate symmetric matrix

$$\begin{pmatrix} A(1) & A(2) & \cdots & A(\mu-1) & A(\mu) \\ A(2) & A(3) & \cdots & A(\mu) & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ A(\mu-1) & A(\mu) & \cdots & 0 & 0 \\ A(\mu) & 0 & \cdots & 0 & 0 \end{pmatrix}$$

represents the Grothendieck–Witt class

$$w = \begin{cases} \langle A(\mu) \rangle + \frac{\mu-1}{2} \cdot \langle 1, -1 \rangle & \mu \text{ odd;} \\ \frac{\mu}{2} \cdot \langle 1, -1 \rangle & \mu \text{ even.} \end{cases}$$

Proof. Let x_1, \dots, x_μ denote the basis dual to the standard basis on $k^{\oplus \mu}$, so the bilinear form β defined by the given matrix is $\sum A(i+j-1)x_i \otimes x_j$. Rewrite the bilinear form as

(6) $x_1 \otimes \Psi_1 + \Psi_1 \otimes x_1 + x_2 \otimes \Psi_2 + \Psi_2 \otimes x_2 + \dots + x_{\mu/2} \otimes \Psi_{\mu/2} + \Psi_{\mu/2} \otimes x_{\mu/2}$ if μ even;
 $x_1 \otimes \Psi_1 + \Psi_1 \otimes x_1 + \dots + x_{(\mu-1)/2} \otimes \Psi_{(\mu-1)/2} + \Psi_{(\mu-1)/2} \otimes x_{(\mu-1)/2} + A(\mu)x_{(\mu+1)/2} \otimes x_{(\mu-3)/2}$ if μ odd
for Ψ_1, Ψ_2, \dots the linear functions

$$\begin{aligned} \Psi_1 &= \frac{A(1)}{2}x_1 + A(2)x_2 + A(3)x_3 + \dots + A(\mu-2)x_{\mu-2} + A(\mu-1)x_{\mu-1} + A(\mu)x_\mu, \\ \Psi_2 &= \frac{A(3)}{2}x_2 + A(4)x_3 + \dots + A(\mu-1)x_{\mu-2} + A(\mu)x_{\mu-1}, \\ \Psi_3 &= \frac{A(5)}{2}x_3 + \dots + A(\mu)x_{\mu-2}, \\ &\dots \end{aligned}$$

The elements $x_1, \Psi_1, x_2, \Psi_2, \dots$ form a dual basis, so (6) shows that β is the orthogonal sum of the hyperbolic planes $x_1 \otimes \Psi_1 + \Psi_1 \otimes x_1, x_2 \otimes \Psi_2 + \Psi_2 \otimes x_2, \dots$ and, when μ is odd, the 1-dimensional space $A(\mu)x_{(\mu-1)/2} \otimes x_{(\mu-1)/2}$. \square

We now introduce an auxiliary matrix, the Newton matrix $\text{New}(f/g)$, which will be used in the proof.

Definition 7. Given a root $r \in k$ of f of multiplicity μ_0 , let

$$(7) \quad g(x)/f(x) = \frac{A_r(\mu_0)}{(x-r)^{\mu_0}} + \frac{A_r(\mu_0-1)}{(x-r)^{\mu_0-1}} + \dots + \frac{A_r(2)}{(x-r)^2} + \frac{A_r(1)}{(x-r)} + \text{higher order terms}$$

be a partial fractions decomposition. Define the **local Newton matrix**

$$(8) \quad \text{New}(f/g, r) := \begin{pmatrix} A_r(1) & A_r(2) & \dots & A_r(\mu_0-1) & A_r(\mu_0) \\ A_r(2) & A_r(3) & \dots & A_r(\mu_0) & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ A_r(\mu_0-1) & A_r(\mu_0) & \dots & 0 & 0 \\ A_r(\mu_0) & 0 & \dots & 0 & 0 \end{pmatrix}.$$

If k contains all the roots $\{r_1, \dots, r_m\}$ of $f(x)$, then we define the **Newton matrix**

$$\text{New}(f/g) := \begin{pmatrix} \text{New}(f/g, r_1) & 0 & \dots & 0 & 0 \\ 0 & \text{New}(f/g, r_2) & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & \text{New}(f/g, r_{m-1}) & 0 \\ 0 & 0 & \dots & 0 & \text{New}(f/g, r_m) \end{pmatrix}.$$

The previous two lemmas shows that $\deg^{A^1}(f/g)$ equals the class of $\text{New}(f/g)$ when $f/g = x^\mu/A$. The following lemma establishes equality in greater generality.

Corollary 8. *If $r \in k$ is a root of f , then $\text{New}(f/g, r)$ represents $\deg_r^{\mathbf{A}^1}(f/g)$.*

Proof. Say that the multiplicity of $r \in k$ in f is μ_0 and A equals $g(0)$. Write $f(x) = x^{\mu_0} f_0(x)$ with $f_0(0) \neq 0$ and $g(x) = xg_0(x) + A$. The expression

$$H(x, t) = (x^\mu + tf_0(x))/(txg_0(x) + A)$$

defines a naive local \mathbf{A}^1 -homotopy (in the sense of Lemma 4), showing $\deg_0^{\mathbf{A}^1}(f/g) = \deg_0^{\mathbf{A}^1}(x^{\mu_0}/A)$. We thus have reduced the claim to the special case $f/g = x^{\mu_0}/A$, which is Lemmas 5 and 6. \square

Corollary 9. *If k contains all the roots of $f(x)$, then $\text{New}(f/g)$ represents the class of $\deg^{\mathbf{A}^1}(f/g)$.*

Proof. This is immediate from Corollary 8 because $\deg^{\mathbf{A}^1}(f/g)$ is the sum of the local degrees $\deg_{r_i}^{\mathbf{A}^1}(f/g)$ by Formula (2) and $\text{New}(f/g)$ is the sum of the terms $\text{New}(f/g, r_i)$ by definition. \square

Theorem 10. *The matrix $S(f/g)$ represents $\deg^{\mathbf{A}^1}(f/g) \in \text{GW}(k)$.*

Proof. First, assume k contains all the roots of $f(x)$, so $\text{New}(f/g)$ is defined over k . We will show that $S(f/g)$ is conjugate to $\text{New}(f/g)$ by manipulating the bilinear polynomial $\sum s_{i+j-1} a_i b_j$ defined by $S(f/g)$. Consider the expression

$$(9) \quad g(x)/f(x) \cdot \Theta(a; x)\Theta(b; x)$$

for Θ the polynomial

$$\Theta(a; x) = a_0 + a_1 x + a_2 x^2 + \dots + a_{\mu-1} x^{\mu-1}.$$

The coefficient of $1/x$ in (9) is $\sum s_{i+j+1} a_i b_j$, as we see by substituting $s_1/x + s_2/x^2 + \dots$ for $g(x)/f(x)$ and multiplying.

Alternatively the coefficient of $1/x$ is the sum of the residues at the roots of f . Here we take the residue at a root r algebraically as the coefficient of $1/(x-r)$ in a series expansion. If r is a root of $f(x)$ with multiplicity μ_0 , write

$$\begin{aligned} \Theta(a; x) &= a_0 + a_1(x-r+r) + a_2(x-r+r)^2 + \dots + a_{\mu-1}(x-r+r)^{\mu-1} \\ &= (a_0 + a_1 r + a_2 r^2 + \dots + a_{\mu-1} r^{\mu-1}) + (a_1 + 2ra_2 + \dots)(x-r) + \dots + a_{\mu-1}(x-r)^{\mu-1} \\ &= \Theta_0(a; r) + \Theta_1(a; r)(x-r) + \dots + \Theta_{\mu-1}(a; r)(x-r)^{\mu-1} \end{aligned}$$

for some explicit polynomials $\Theta_0, \Theta_1, \dots, \Theta_{\mu-1}$ that are linear in the a_i 's. Substituting these expressions and $g(x)/f(x) = \frac{A_r(\mu_0)}{(x-r)^{\mu_0}} + \frac{A_r(\mu_0-1)}{(x-r)^{\mu_0-1}} + \dots$ into (9), we get that the residue at $x = r$ is

$$\sum A_r(i+j+1)\Theta_i(a; r)\Theta_j(b; r).$$

Summing over all roots, we get that the coefficient of $1/x$ is

$$\sum A_{r_1}(i+j+1)\Theta_i(a; r_1)\Theta_j(b; r_1) + \dots + \sum A_{r_m}(i+j+1)\Theta_i(a; r_m)\Theta_j(b; r_m).$$

As a bilinear polynomial in the Θ_i 's, this is the bilinear polynomial associated to $\text{New}(f/g)$. Thus if M is the change-of-basis matrix relating the a_i 's to the $\Theta_i(a; r_j)$'s, then

$$(10) \quad S(f/g) = M \cdot \text{New}(f/g) \cdot M^T.$$

The matrix M is invertible (M is the confluence Vandermonde matrix), so we conclude from Corollary 9 that $S(f/g)$ represents $\deg^{A^1}(f/g)$.

When $f(x)$ is arbitrary, we extend the above argument using the theory of descent (as in e.g. [Knu91, Chapter III]) as follows. Fix a splitting field L for $f(x)$. We can assume that L/k is Galois because the natural map from $\text{GW}(k)$ to the Grothendieck–Witt group of the separable closure is injective (recall $\text{char } k \neq 2$). Denote the irreducible factors of $f(x)$ by $\pi_1(x), \dots, \pi_m(x)$. For each factor $\pi(x)$, pick an embedding of the residue field $k[x]/\pi(x)$ into L and let $r_i \in L$ denote the image of x . By Formula (2) and Corollary 8, we have

$$(11) \quad \deg^{A^1}(f/g) = \text{Tr}_{k[r_1]/k}(\text{New}(f/g \otimes k[r_1], r_1)) + \dots + \text{Tr}_{k[r_m]/k}(\text{New}(f/g \otimes k[r_m], r_m)).$$

To show that (11) and $S(f/g)$ define isomorphic bilinear spaces, it is enough to extend scalars to L and construct an isomorphism over L that respects the natural descent data. The space obtained from (11) is the space defined by

$$\begin{aligned} & (\oplus_{\sigma \in \text{Gal}(L/k[r_1])} \text{New}(f/g \otimes L, \sigma(r_1))) \oplus \dots \oplus (\oplus_{\sigma \in \text{Gal}(L/k[r_m])} \text{New}(f/g \otimes L, \sigma(r_m))) \\ & = \text{New}(f/g \otimes L). \end{aligned}$$

by [Lam05, Theorem 6.1]. The descent data on this space is the descent data associated to the permutation action of $\text{Gal}(L/k)$ on the roots of $f(x)$, and the isomorphism defined by the matrix M from (10) transforms this descent data into the tautological descent data on the bilinear space defined by $S(f/g)$. \square

Corollary 11. *The matrix $\text{Béz}(f/g)$ represents $\deg^{A^1}(f/g)$.*

Proof. It is enough to show that $\text{Béz}(f/g)$ is congruent to $S(f/g)$. Writing $f(x) = x^\mu + a_1x^{\mu-1} + \dots + a_\mu$, we compute

$$\begin{aligned} \frac{f(x)g(y) - f(y)g(x)}{x - y} &= f(x)f(y) \frac{\frac{g(y)}{f(y)} - \frac{g(x)}{f(x)}}{x - y} \\ &= f(x)f(y) \frac{\sum_{i=1}^{\infty} s_i(y^{-i} - x^{-i})}{x - y} \\ &= f(x)f(y) \frac{\sum_{i=1}^{\infty} s_i(x^i - y^i)x^{-i}y^{-i}}{x - y} \\ &= f(x)f(y) \sum_{i,j=1}^{\infty} s_{i+j-1}x^{-i}y^{-j} \\ (12) \quad &= \sum_{i,j=1}^{\infty} s_{i+j-1}(x^{\mu-i} + a_1x^{\mu-i-1} + \dots + a_\mu x^{-i})(y^{\mu-j} + a_1y^{\mu-j-1} + \dots + a_\mu y^{-j}). \end{aligned}$$

The above expression equals $\sum b_{i,j}x^{i-1}y^{j-1}$, so the terms in (12) where x or y appears with negative degree must cancel out, and we deduce

$$\sum_{i,j=1}^{\mu} b_{i,j}x^{i-1}y^{j-1} = \sum_{i,j=1}^{\mu} s_{i+j-1}(x^{\mu-i} + a_1x^{\mu-i-1} + \dots + a_{\mu-i})(y^{\mu-j} + a_1y^{\mu-j-1} + \dots + a_{\mu-j}).$$

This shows that the relevant change-of-basis matrix conjugates $\text{Béz}(f/g)$ to $S(f/g)$. \square

Let us compare the argument just given with Kreĭn–Naĭmark’s treatment of Hurwitz’s result in [KN81]. We deduced the result that $\text{Béz}(f/g)$ represents $\deg^{\mathbf{A}^1}(f/g)$ (Corollary 11) from the analogous result about $S(f/g)$ (Theorem 10) in the same way that Kreĭn–Naĭmark do on [KN81, page 277]. Our proof of the result about $S(f/g)$ is similar to the argument on [KN81, pages 280–282] but with two significant differences. First, both we and Kreĭn–Naĭmark compute the class of $S(f/g)$ by passing to a splitting field L for $f(x)$ and working with $\text{New}(f/g \otimes L)$, but the details are different. Kreĭn–Naĭmark only consider $k = \mathbf{R}$, and they compute the signature of $S(f/g)$ from $\text{New}(f/g \otimes L)$ by observing that a complex conjugate pair of roots correspond to a summand of $S(f/g)$ with signature zero. We allow k to be arbitrary (with $\text{char } k \neq 2$), and in contrast to the case $k = \mathbf{R}$, a Galois orbit of roots can contribute to the class of $S(f/g)$ in a complicated way, so instead of directly computing we use descent theory.

The second difference occurs in our proof of Corollary 8, which identifies the local degree $\deg_r^{\mathbf{A}^1}(f/g)$ with the class of $\text{New}(f/g, r)$. The corollary is deduced from Lemmas 5 and 6. Lemma 6 is the computation of the class of $\text{New}(f/g, r)$, and the computation is the same as the one appearing on [KN81, page 281]. Lemma 5, however, does not have a direct analogue in [KN81]. That lemma computes the local degree as the expression in Lemma 6, and for Kreĭn–Naĭmark, the analogous fact does not need a proof since it is their definition of the local degree (or rather local Cauchy index; see Section 2.1 below).

The proof just given provides a second proof of Cazanave’s result that $\text{Béz}(f/g)$ represents $\deg^{\mathbf{A}^1}(f/g)$. Cazanave in fact proves the stronger result that $\text{Béz}(f/g)$ represents the unstable homotopy class of f/g , and our proof does not yield this stronger result. The reason for this is clearly demonstrated by the special case where $f(x)$ has μ distinct roots defined over k . We deduced the result about $\text{Béz}(f/g)$ from the analogous result about $\text{New}(f/g)$, and $\text{New}(f/g)$ does not represent the unstable homotopy class of f/g . Indeed, the unstable homotopy class is determined by $\deg^{\mathbf{A}^1}(f/g)$ and a scalar $d(f/g) \in k^*$ that represents the discriminant of $\deg^{\mathbf{A}^1}(f/g)$. The Newton matrix is a diagonal matrix with diagonal entries $g(r_1)/f'(r_1), \dots, g(r_\mu)/f'(r_\mu)$, so

$$(13) \quad \begin{aligned} \det(\text{New}(f/g)) &= g(r_1)/f'(r_1) \cdots g(r_\mu)/f'(r_\mu) \text{ but} \\ \det(\text{Béz}(f/g)) &= (-1)^{\mu(\mu-1)/2} \text{Resultant}(f, g) \\ &= \text{Disc}(f) \cdot g(r_1)/f'(r_1) \cdots g(r_\mu)/f'(r_\mu). \end{aligned}$$

Here Resultant is the resultant, and Disc the discriminant.

Equation (13) shows that, unlike $\deg^{\mathbf{A}^1}(f/g)$, $d(f/g)$ is not determined by the derivatives of $f(x)/g(x)$ at the roots. For example, both $f_1(x)/g_1(x) = x^2 - x$ and $f_2(x)/g_2(x) = (x^2 - 1)/2$ have two simple zeros at which the values of the derivative are $+1$ and -1 respectively, but the functions are not unstably homotopic since $d(f_1/g_1) = -1$ but $d(f_2/g_2) = -4$. This should not be surprising as the formula relating the global degree to a sum of local degrees is proven using stable techniques.

2.1. References. The authors used Kreĭn–Naĭmark’s survey [KN81] as the primarily source on Hurwitz’s theorem. The proof in *loc. cit.* is essentially the same as Hurwitz’s proof in [Hur95, Section 3]. Other treatments of the result include [Gan64, page 210, Theorem 9],

[RS02, Theorem 10.6.5], [BPR06, Theorem 9.4], and [Fuh12, Theorem 8.59]. All of these references do not discuss the topological degree and instead discuss the Cauchy index, an equivalent invariant. The local Cauchy index $\text{ind}_r(f/g)$ at a pole $r \in \mathbf{R}$ of f/g (i.e. at a root of g) is

$$\text{ind}_r(f/g) := \begin{cases} +1 & \text{if } f/g \text{ jumps from } -\infty \text{ to } +\infty \text{ at } r; \\ -1 & \text{if } f/g \text{ jumps from } +\infty \text{ to } -\infty \text{ at } r; \\ 0 & \text{otherwise.} \end{cases}$$

The global Cauchy index $\text{ind}(f/g)$ is the sum of the local Cauchy indices. The local Cauchy index of f/g at r equals the local topological of the *reciprocal* function g/f and similarly with the global degree.

3. CONNECTION WITH WORK OF EISENBUD–LEVINE AND KHIMSHIASHVILI

Here we explain how Cazanave’s description of $\text{deg}^{\mathbf{A}^1}(f/g)$ as the Bézout matrix is related to the signature formula of Eisenbud–Levine and Khimshiashvili, as well as results of Palamodov and the present authors.

The signature formula computes the local topological degree of the germ $h_0: (\mathbf{R}^n, 0) \rightarrow (\mathbf{R}^n, 0)$ of a C^∞ -function as the signature of a bilinear form. The formula applies when h_0 has the property that the local algebra

$$(14) \quad Q_0(h_0) := C_0^\infty(\mathbf{R}^n)/(h_0).$$

has finite length. Here $C_0^\infty(\mathbf{R}^n)$ is the ring of germs of smooth real-valued functions on \mathbf{R}^n based at 0 and (h_0) is the ideal generated by the components of h_0 .

The bilinear form appearing in the signature formula is defined in terms of the Jacobian element $J := \det\left(\frac{\partial f_i}{\partial x_j}\right) \in Q_0(h_0)$. The formula states that if $\eta: Q_0(h_0) \rightarrow \mathbf{R}$ is a \mathbf{R} -linear function satisfying $\eta(J) > 0$, then the symmetric bilinear pairing $\beta_0(\eta)$ on $Q_0(h_0)$ defined by

$$\beta_0(\eta)(a, b) = \eta(ab)$$

satisfies

$$(15) \quad \text{deg}_0^{\mathbf{R}}(h_0) = \text{signature of } \beta_0(\eta).$$

This is [EL77, Theorem 1.2] and [Khi77]; see [Khi01] and [AGZV12, Chapter 5] for recent expositions.

Earlier Palamodov [Pal67, Corollary 4] proved a parallel result that computes the topological degree $\text{deg}^{\mathbf{C}}(h_0)$ of the complexification $h_0 \otimes \mathbf{C}: \mathbf{C}^n \rightarrow \mathbf{C}^n$ as

$$(16) \quad \text{deg}^{\mathbf{C}}(h_0) = \text{rank of } \beta_0(\eta)$$

under the assumption that h_0 is analytic (so a complexification exists).

In the special case where $n = 1$ and h_0 is the germ of a rational function f/g , the signature formula is closely related to Hurwitz’s theorem. For such a h_0 , the natural map $\mathbf{R}[x, 1/g] \rightarrow C_0^\infty(\mathbf{R})$ induces an isomorphism

$$Q_0(f/g) \rightarrow Q_0(h_0)$$

between $Q_0(\mathfrak{h}_0)$ and the localization $Q_0(f/g)$ of the algebra $Q(f/g) := \mathbf{R}[x, g^{-1}]/(fg^{-1})$ at the maximal ideal (x) of the origin, by the finiteness of $Q_0(\mathfrak{h}_0)$.

The algebra $Q(f/g)$ admits a distinguished functional $\text{Res}: Q(f/g) \rightarrow \mathbf{R}$ called the residue functional (or generalized trace) that is constructed using local duality. The functional is related to the residues which are defined in complex analysis. Extending scalars to \mathbf{C} , the residue functional satisfies

$$\text{Res}(\mathfrak{a}) = \sum_{f(r)=0} \text{Res}_{z=r} (g(z)/f(z) \cdot \mathfrak{a}(z)),$$

where $\text{Res}_{z=r}(p(z))$ is the residue of $p(z)$ at $z = r$ in the sense of complex analysis.

The matrix $S(f/g)$ is nothing other than a Gram matrix for $\beta(\text{Res})$. This should not be too surprising. We essentially showed this in the proof of Theorem 10, where we computed the bilinear polynomial associated to $S(f/g)$ as the coefficient of $1/x$ in $g(x)/f(x) \cdot \Theta(\mathfrak{a}; x)\Theta(\mathfrak{b}; x)$, i.e. as the sum of the residues of $g(x)/f(x) \cdot \Theta(\mathfrak{a}; x)\Theta(\mathfrak{b}; x)$. The identification of $S(f/g)$ as a Gram matrix is naturally derived from general results of Scheja–Storch, as we now explain.

Scheja–Storch described Res more generally over an arbitrary field using commutative algebra in [SS75]. In the present context, the key result is their description of the bilinear pairing $\beta(\text{Res})$ in terms of the element

$$\Delta := \frac{f/g(x) - f/g(y)}{x - y}.$$

They show that if v_1, \dots, v_μ is a k -basis for $Q(f/g)$ and

$$(17) \quad \Delta = \sum b_{i,j} v_i(x) v_j(y) \text{ modulo } f(x)/g(x) \text{ and } f(y)/g(y), \text{ then}$$

$$(18) \quad (b_{i,j}) = \text{the Gram matrix of } \beta \text{ with respect to the basis dual to } v_1, \dots, v_\mu.$$

This is not explicitly stated by Scheja–Storch, but it can be derived from the formula on [SS75, page 182, bottom]. That formula states that if v_1^*, \dots, v_μ^* is the basis dual to v_1, \dots, v_μ with respect to $\beta(\text{Res})$, then $\Delta = v_1(x)v_1^*(y) + \dots + v_\mu(x)v_\mu^*(y)$. The two bases are related by $v_i = \sum b_{i,j} v_j^*$, and the equality (18) follows from substituting $\sum b_{i,j} v_j^*(x)$ for $v_i(x)$ in $\sum v_i(x)v_i^*(y)$. (Strictly speaking Scheja–Storch only work with $Q(f/g)$ when $g = 1$ because they work with quotients of $k[x]$, but their argument applies to the localization $k[x, 1/g]$ with only notational changes.)

The matrices $\text{Béz}(f/g)$, $S(f/g)$, and $\text{New}(f/g)$ are nothing other than Gram matrices for β with respect to distinguished bases that we now define.

Definition 12. Define the *monomial basis* of $Q(f/g)$ to be $1/g, x/g, \dots, x^{\mu-1}/g$.

Write $f(x) = x^\mu + a_1 x^{\mu-1} + \dots + a_{\mu-1} x + a_\mu$ and define the *Horner basis* of $Q(f/g)$ to be $(x^{\mu-1} + a_1 x^{\mu-2} + \dots + a_{\mu-1})/g, (x^{\mu-2} + a_1 x^{\mu-3} + \dots + a_{\mu-2})/g, \dots, (x + a_1)/g, 1/g$.

If f factors into linear factors $f(x) = (x-r_1)^{\mu_1} \dots (x-r_m)^{\mu_m}$ over k , then define the *Newton basis* to be $\frac{f(x)}{(x-r_1)g(x)}, \frac{f(x)}{(x-r_1)^2 g(x)}, \dots, \frac{f(x)}{(x-r_1)^{\mu_1-1} g(x)}, \frac{f(x)}{(x-r_2)g(x)}, \dots, \frac{f(x)}{(x-r_m)^{\mu_m-1} g(x)}$.

TABLE 1. Gram Matrices

Gram matrix	Dual basis
Béz(f/g)	Monomial basis
S(f/g)	Horner basis
New(f/g)	Newton basis

Table 1 identifies Béz(f/g), S(f/g), and New(f/g) as Gram matrices. These identifications follow from our computations in Section 2. We get the identification of the Bézout matrix by dividing (1) by $g(x)g(y)$. If we instead divide (12), we get the identification of S(f/g). Replacing the use of $g/f = \sum s_i/x^i$ with $g/f = \sum A_{r_i}(j)/(x - r_i)^j$ in (12) identifies New(f/g).

We point out that this identification of Béz(f/g) as a Gram matrix is different from the identification that appears in [Caz12]. The relevant text is Lemma C.2 of the preprint version (but not in the published version) of loc. cit. that is available on the online repository the arXiv as arXiv:0912.2227v2. There Cazanave identifies the inverse matrix $\text{Béz}(f/g)^{-1}$, and hence Béz(f/g), as a Gram matrix in an important special case, namely the case where $f(x)$ and $g(x)$ have indeterminant coefficients and k is a suitable algebraic closure. His discussion, however, applies more generally to the case where $f(x)$ has only simple roots (so $Q(f/g)$ is étale). He shows that $\text{Béz}(f/g)^{-1}$ is a Gram matrix for the bilinear pairing $\beta(\phi)$ on $Q(f/g)$ defined by functional

$$(19) \quad \phi(\mathfrak{a}) = \text{Tr} \left(\frac{\mathfrak{a}}{f'(x)g(x)} \right).$$

Here $\text{Tr}: Q(f/g) \rightarrow k$ is the algebra trace. Observe that $f'(x)$ is a unit in $Q(f/g)$ because we assumed that $f(x)$ has distinct roots, so (19) is well-defined.

The functional ϕ is not equal to Res. Indeed, when $f(x)$ has simple roots, Scheja–Storch show

$$\text{Res}(\mathfrak{a}) = \text{Tr} \left(\frac{\mathfrak{a}g(x)}{f'(x)} \right)$$

This is [SS75, (4.2) Satz] (specifically, it is equivalent to the equality $\text{Sp} = J \cdot \eta$ in loc. cit.). The two functionals do, however, define isomorphic symmetric bilinear forms since multiplication by $1/g(x)$ transforms $\beta(\text{Res})$ into $\beta(\phi)$.

Having explained the connection between the residue pairing $\beta(\text{Res})$ and Hurwitz’s theorem, we now explain the relation to the signature formula. Since $Q(f/g)$ is an artin algebra, $Q_0(f/g)$ is a direct summand of $Q(f/g)$. Scheja–Storch show that the restriction Res_0 of the residue functional satisfies $\text{Res}_0(J) = \text{length } Q_0(f/g)$, so in particular, the restriction $\beta_0(\text{Res}_0)$ of $\beta(\text{Res})$ to $Q_0(f/g)$ satisfies the hypothesis of the signature formula. We conclude that

$$\text{deg}_0^{\mathbf{R}}(f/g) = \text{the signature of } \beta_0(\text{Res}_0).$$

In this manner, applied to a rational function f/g , the signature formula is the local analogue of Hurwitz’s theorem.

Cazanave’s theorem is an enrichment of Hurwitz’s theorem, and it implies the following analogous enrichment of the signature formula:

$$(20) \quad \deg_0^{\mathbf{A}^1}(f/g) = \beta_0(\text{Res}_0) \text{ in } \text{GW}(k).$$

(This is Corollary 8.)

Equality (20) answers a question of Eisenbud for polynomial maps of 1 variable. Eisenbud, in a survey article on his work with Levine, observed that $\beta_0(\text{Res}_0)$ is defined for a polynomial map $f: \mathbf{A}_k^n \rightarrow \mathbf{A}_k^n$ with an isolated zero at the origin when k is an arbitrary field of odd characteristic. He then proposed β_0 as the definition of the degree and asked if this degree has an interpretation or usefulness, say in cohomology theory [Eis78, Some remaining questions (3)].

In this paper we have shown that, in odd characteristic, the stable isomorphism class of $\beta_0(\text{Res}_0)$ is the local degree in \mathbf{A}^1 -homotopy theory when f is a polynomial map of 1 variable. In the companion paper [KW], the present authors answer Eisenbud’s question in full generality: the main theorem of that paper (Theorem 3) states that (20) remains valid when f/g is replaced by a polynomial map $f: \mathbf{A}_k^n \rightarrow \mathbf{A}_k^n$ in any number of variables that has an isolated zero at the origin.

4. ACKNOWLEDGEMENTS

TO BE ADDED AFTER THE REFEREEING PROCESS.

Jesse Leo Kass was partially sponsored by the National Security Agency under Grant Number H98230-15-1-0264 and the Simons Foundation under Award Number 429929. The United States Government is authorized to reproduce and distribute reprints notwithstanding any copyright notation herein. This manuscript is submitted for publication with the understanding that the United States Government is authorized to reproduce and distribute reprints.

Kirsten Wickelgren was partially supported by National Science Foundation Awards DMS-1406380 and DMS-1552730.

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