Augmentations and Exact Lagrangian Cobordisms.

by

Yu Pan

Department of Mathematics
Duke University

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Approved:

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Lenhard Ng, Supervisor

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Dissertation submitted in partial fulfillment of the requirements for the degree of
Doctor of Philosophy in the Department of Mathematics
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Abstract

To a Legendrian knot, one can associate an A_∞ category, the augmentation category. An exact Lagrangian cobordism between two Legendrian knots gives a functor of the augmentation categories of the two knots. We study the functor and establish a long exact sequence relating the corresponding cohomology of morphisms of the two ends. As applications, we prove that the functor between augmentation categories is injective on the level of equivalence classes of objects and find new obstructions to the existence of exact Lagrangian cobordisms in terms of linearized contact homology and ruling polynomials.

As a related project, we study exact Lagrangian fillings of Legendrian $(2, n)$ links. For a Legendrian $(2, n)$ torus knot or link with maximal Thurston–Bennequin number, Ekholm, Honda, and Kálmán [EHK16] constructed C_n exact Lagrangian fillings, where C_n is the n–th Catalan number. We show that these exact Lagrangian fillings are pairwise non–isotopic through exact Lagrangian isotopy. To do that, we compute the augmentations induced by the exact Lagrangian fillings L to $\mathbb{Z}_2[H_1(L)]$ and distinguish the resulting augmentations.

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1

Introduction

Low–dimensional topology studies global properties of three– and four–dimensional geometric spaces. As powerful tools from algebra, analysis, geometry and physics were poured into the field, low–dimensional topology was rapidly developed in the last century. Currently, one of the standard approaches to low–dimensional topology is to apply tools from symplectic geometry. Originating from physics, symplectic geometry uses geometric and analytic methods to study symplectic manifolds, which are even–dimensional manifolds endowed with a closed non–degenerate 2–form. The analogous odd–dimensional manifolds are called contact manifolds. What I am interested in is called symplectic and contact topology, which studies global behaviors of symplectic and contact manifolds. Thanks to Gromov's introduction of the techniques of holomorphic curves in symplectic and algebraic geometry in the 1980's, symplectic and contact topology has undergone a revolution over the past three decades. One of the main effects is the development of Symplectic Field Theory [EGH00], which uses the Floer theory of holomorphic curves to find invariants of contact and symplectic manifolds as well as special knots in contact manifolds, the Legendrian knots.

In 2009, Nadler and Zaslow [NZ09] found a remarkable correspondence between the category of sheaves on a manifold and the derived Fukaya category of Lagrangians in the cotangent bundle of the manifold. This Nadler–Zaslow correspondence relates symplectic geometry to algebraic geometry (especially homological mirror symmetry) through sheaf theory. In 2014, Legendrian knots was introduced into the picture through the construction of a category of constructible sheaves for Legendrian knots [STZ]. Analogous to the Nadler–Zaslow correspondence, a correspondence between the STZ sheaves category and the augmentation category, a category built on the holomorphic curves theory, was proven by Ng, Rutherford, Shende, Sivek and Zaslow [NRS$^+$15]. This surprising connection between sheaf theory and contact topology allows one to solve questions in one field through techniques from the other field, which is a fresh perspective. Both sheaf theory and contact topology are relatively well developed but the techniques in the two fields are quite different from each other. I am interested in understanding how the topological techniques on the contact topology side and the algebraic techniques on the sheaf theory side correspond to each other as well as how they may interact with each other. Along this direction, I did two related projects as in Section 1.1 and Section 1.2.

1.1 Exact Lagrangian cobordism and the augmentation category

Specifically, a contact structure on $\mathbb{R}^3$ is a completely non–integrable 2–plane field. The contact structure we use is called the **standard contact structure** ξ on $\mathbb{R}^3$, which is the the kernel of the 1–form $\alpha = dz - ydx$ as shown in Figure 1.1. A **Legendrian submanifold** in $(\mathbb{R}^3, \xi)$ is a 1–dimensional submanifold in $\mathbb{R}^3$ that is tangent to the 2–plane field ξ everywhere.

In geometric topology, given two knots in $\mathbb{R}^3$, an interesting object to study is a surface in $\mathbb{R}^3 \times [0, 1]$ connecting the two knots, called a cobordism. There is an analogous object in contact topology. Let Λ_+ and Λ_- be Legendrian submanifolds

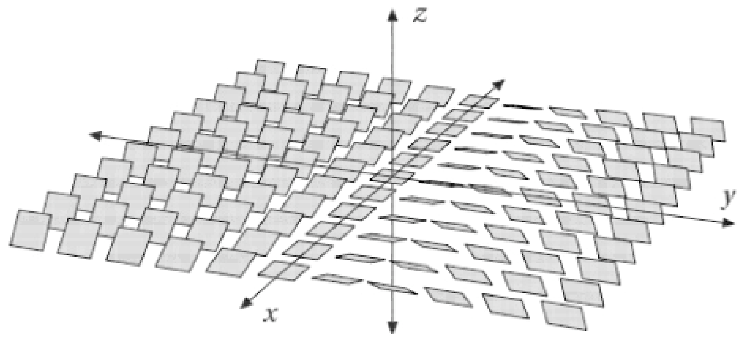


FIGURE 1.1: The standard contact structure in $\mathbb{R}^3$.

in the standard contact manifold $(\mathbb{R}^3, \xi = \ker \alpha)$. An exact Lagrangian cobordism Σ from Λ_- to Λ_+ is a 2–dimensional surface in the symplectization of $\mathbb{R}^3$ that has cylindrical ends over Λ_+ and Λ_- with some properties. See Figure 1.2 for a schematic picture and Definition 2.3.1 for a detailed description.

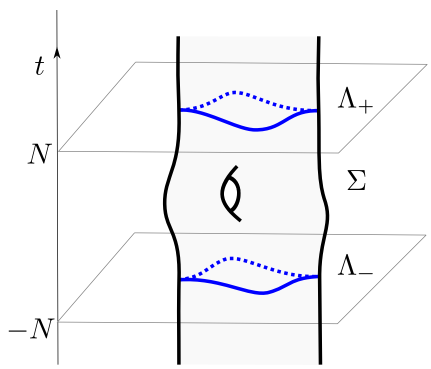


FIGURE 1.2: A Lagrangian cobordism Σ from Λ_- to Λ_+ lies in the symplectization of $\mathbb{R}^3$, which is $\big(\mathbb{R}_t \times \mathbb{R}^3, d(e^t\alpha)\big)$. The vertical direction is the t direction and each horizontal plane is $\mathbb{R}^3$. Two Legendrian submanifolds Λ_+ and Λ_- sit inside different $\mathbb{R}^3$ with t coordinates N and $-N$, respectively.

A natural question to ask is given two Legendrian knots Λ_+ and Λ_- in $(\mathbb{R}^3, \ker \alpha)$, is there an exact Lagrangian cobordism Σ in $\big(\mathbb{R} \times \mathbb{R}^3, d(e^t\alpha)\big)$ from Λ_- to Λ_+? If we forget about the exact Lagrangian condition and consider topological cobordisms, this question is easy. Any pair of knots can be connected by a cobordism. Because of the additional geometric structure exact Lagrangian cobordism has, the question above is hard to answer in general. Instead of answering it directly, we try to find relations between Legendrian knots that are connected by an exact Lagrangian cobordism. If two given Legendrian knots do not satisfy such relations, there does not exist an exact Lagrangian cobordism between them. In this way, we can give obstructions to the existence of exact Lagrangian cobordisms.

Several obstructions have been made. Chantraine [Cha10] first gave an obstruction in terms of the Thurston–Bennequin number (see Section 2.1 for definition). Thurston–Bennequin number is a canonical invariant of Legendrian knots, denoted by $tb(\Lambda)$. Chantraine gave a relationship as following:

$$tb(\Lambda_+) - tb(\Lambda_-) = -\chi(\Sigma), \tag{1.1}$$

where $\chi(\Sigma)$ is the Euler characteristic of the surface Σ. If Λ_+ and Λ_- are single component knots, note that $-\chi(\Sigma)$ is non–negative. Therefore, if two Legendrian knots Λ_+ and Λ_- satisfies $tb(\Lambda_+) < tb(\Lambda_-)$, there does not exist an exact Lagrangian cobordism from Λ_- to Λ_+.

Another obstruction is given by Cornwell, Ng and Sivek [CNS16] based on a key property of exact Lagrangian cobordisms from the work of [EHK16]. Chekanov–Eliashberg differential graded algebra (DGA) is one of the most useful structural invariants of Legendrian knots. First introduced by Chekanov [Che02b] and Eliashberg [Eli98] in the spirit of Symplectic Field Theory [EGH00], the DGA associated to a Legendrian knot is a graded algebra over a field $\mathbb{F}$ equipped with a differential (See Section 2.2 for a detailed definition). Ekholm, Honda and Kálmán [EHK16] showed that an exact Lagrangian Σ from Λ_- to Λ_+ induces a DGA map from the DGA of Λ_+ to the DGA of Λ_-. This gives a category functor from the category whose objects are Legendrian knots and morphisms are exact Lagrangian cobordisms to a category whose objects are DGAs and morphisms are DGA maps.

$$\mathcal{C}(\text{Legendrian knots}, \text{Exact Lagrangian cobordisms}) \to \mathcal{C}(\text{DGAs}, \text{DGA maps})$$

When Λ_- is empty, Σ is an exact Lagrangian filling of Λ_+. It gives a DGA map from the DGA of Λ_+ to the trivial DGA $(\mathbb{F}, 0)$, which is an augmentation of $DGA(\Lambda_+)$

(see Section 2.2.3 for a detailed definition). Moreover, suppose that Σ is an exact Lagrangian cobordism from Λ_- to Λ_+ and Λ_- has an augmentation ϵ_-. We can compose the DGA map $DGA(\Lambda_+) \to DGA(\Lambda_-)$ induced by Σ and $\epsilon_- : DGA(\Lambda_-) \to (\mathbb{F}, 0)$ to get an augmentation ϵ_+ of Λ_+. This gives an obstruction of the existence of exact Lagrangian cobordisms as follows. If Λ_- has an augmentation but Λ_+ does not have an augmentation, then there is no exact Lagrangian cobordism from Λ_- to Λ_+.

Further obstructions were explored in a lot of work including [BST15], [ST13], [BS] and [CDRGG15] using generating families and Floer theory.

We will give new obstructions using the augmentation category (See Chapter 3). Inspired by the derived Fukaya category [NZ09] for symplectic 4–manifolds whose objects are exact Lagrangian submanifolds, Ng, Rutherford, Shende, Sivek and Zaslow [NRS$^+$15] constructed an A_∞ category, the augmentation category $\mathcal{A}ug_+(\Lambda)$, for a Legendrian knot Λ based on the work of Bourgeois and Chantraine [BC14]. Objects of $\mathcal{A}ug_+(\Lambda)$ are augmentations of the DGA of Λ. An exact Lagrangian cobordism Σ from a Legendrian knot Λ_- to a Legendrian knot Λ_+ induces an augmentation category map from $\mathcal{A}ug_+(\Lambda_-)$ to $\mathcal{A}ug_+(\Lambda_+)$. This gives a category functor:

$$\mathcal{C}(\text{Legendrian knots}, \text{Exact Lagrangian cobordisms}) \to \mathcal{C}(\mathcal{A}ug_+(\Lambda), \mathcal{A}ug_+ \text{ maps}).$$

One of my main results shows that this augmentation category map has a surprisingly nice algebraic property.

Theorem 1.1.1 ([Pan16a]). *The augmentation category map induced by an exact Lagrangian cobordism is injective on objects level. Moreover, its induced map on cohomology categories is faithful and is fully faithful when the Euler characteristic of the surface is* 0. *(see Chapter 3 for definitions of cohomology category, faithful and fully faithful.)*

Not all of possible algebraic augmentation category maps have this injective property. Theorem 1.1.1 is true for the augmentation category map f_Σ induced by an exact Lagrangian cobordism Σ because of the geometric information it carries. We approach this theorem by understanding the augmentation category map f_Σ through symplectic geometry. Focusing on the augmentation category map restricted on morphisms, denoted by f_Σ as well, we find a geometric interpretation of f_Σ by studying the wrapped Floer homology of the 2–copy of Σ. The wrapped Floer homology for Lagrangian cobordisms was recently introduced by Chantraine, Dimitroglou Rizell, Ghiggini and Golovko [CDRGG15] in the spirit of Symplectic Field Theory. They associated to a pair of exact Lagrangian cobordisms a chain complex, called the Cthulhu chain complex, whose differential counts holomorphic disks with boundary on the union of cobordisms (see Chapter 4). We construct a special pair of exact Lagrangian cobordisms, which is the 2–copy of Σ, and investigate the differential of its Cthulhu chain complex. Inspired by [EES09], through studying the degeneration of holomorphic disks with boundary on the 2–copy of Σ into generalized disks on Σ, which are given by holomorphic disks and gradient flows on Σ, we prove that a part of the differential of the Cthulhu chain complex (a geometric map) is the same as f_Σ (an algebraic map). Applying the work of [CDRGG15] to the 2–copy of Σ, we obtain a long exact sequence involving morphisms in $\mathcal{A}ug_+(\Lambda_+)$, morphisms in $\mathcal{A}ug_+(\Lambda_-)$ and the relative cohomologies $H^*(\Sigma, \Lambda_-)$. Due to the special pair of cobordisms constructed, we attain more information about the morphisms of the corresponding augmentation categories than a long exact sequence. Combining with the algebraic information of the augmentation category, we prove Theorem 1.1.1.

If Σ is an exact Lagrangian cobordism from Λ_- to Λ_+, Theorem 1.1.1 shows that Λ_+ can not have less augmentations than Λ_-. This obstruction is a generalization of the obstruction given in [CNS16]. However, the amount of augmentations is relatively hard to compute. Alternately, we combine Theorem 1.1.1 with the notation of homotopy cardinality [NRSS] and give the following inequality in terms of ruling polynomial (Section 5.5.3).

Theorem 1.1.2 ([Pan16a]). *If there is an exact Lagrangian cobordism from Λ_- to Λ_+, their ruling polynomials $R_{\Lambda_-}(x)$ and $R_{\Lambda_+}(x)$ satisfy that*

$$R_{\Lambda_-}(q^{1/2} - q^{-1/2}) \leqslant q^{-\chi(\Sigma)/2} R_{\Lambda_+}(q^{1/2} - q^{-1/2}),$$

for any q that is a power of a prime number.

In addition, we obtain a strict relation between Λ_+ and Λ_-, which is a generalization of Seidel's Isomorphism [Sei08, DR16]. This gives a strong obstruction to the existence of exact Lagrangian cobordisms that recovers Chantraine's obstruction (the relation (1.1)).

Theorem 1.1.3 ([Pan16a]). *If there is an exact Lagrangian cobordism Σ from Λ_- to Λ_+, their linearized contact homologies (see Section 2.2.3) $LCH^{\epsilon_-}_k(\Lambda_-)$ and $LCH^{\epsilon_+}_k(\Lambda_+)$ over a field $\mathbb{F}$ satisfy that*

$$LCH^{\epsilon_+}_k(\Lambda_+) \cong LCH^{\epsilon_-}_k(\Lambda_-) \oplus \mathbb{F}^{-\chi(\Sigma)}[0],$$

where $\mathbb{F}^{-\chi(\Sigma)}[0]$ denotes a vector space over $\mathbb{F}$ in degree 0.

Both the ruling polynomial and the linearized contact homology are very easy to compute. Calculations of these invariants for small crossing Legendrian knots have been done in the atlas of [CN13]. Examples below show that Theorem 1.1.2 and Theorem 1.1.3 give new effective obstructions to the existence of exact Lagrangian cobordisms.

As an example for Theorem 1.1.2, Figure 1.3 gives two Legendrian knots Λ and Λ_0 of the knot type 9_{46} and unknot, respectively. They both have Thurston–Bennequin number -1 and there is an exact Lagrangian cobordism from Λ_0 to Λ. By Theorem

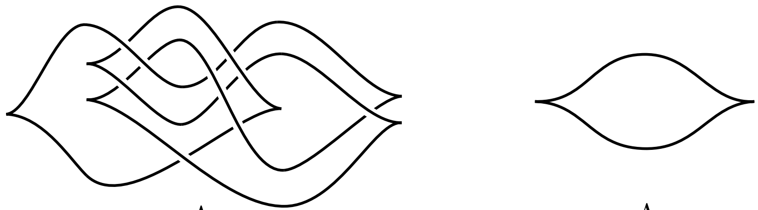


FIGURE 1.3: Legendrian knots of knot type 9_{46} and unknot.

1.1.2, there does not exist an exact Lagrangian cobordism from Λ to Λ_0 because the ruling polynomials for them are 2 and 1, respectively. It leads to the fact that Lagrangian concordance is not a symmetric relation. This fact was first proven by Chantraine [Cha15] using the A_∞ structure built in [BC14]. Cornwell, Ng and Sivek [CNS16] gave another proof using the existence of normal rulings. Here we give a new and simpler proof to Chantraine's result.

As an example for Theorem 1.1.3, Figure 1.4 shows two Legendrian knots Λ_1 and Λ_2 of smooth knot type 4_1 and 6_1, respectively. There is a topological cobordism between 4_1 and 6_1 with genus 1. The Thurston–Bennequin numbers of Λ_1 and Λ_2 are -3 and -5, respectively, and thus the Thurston–Bennequin number relation given in (1.1) does not give an obstruction. Therefore, there is a possibility to exist an exact Lagrangian cobordism from Λ_2 to Λ_1 with genus 1. However, the Poincaré polynomials of linearized contact homology for Λ_1 and Λ_2 are $t^{-1} + 2t$ and $2t^{-1} + 3t$, respectively. According to Theorem 1.1.3, the Poincaré polynomials of linearized contact homology of Λ_+ and Λ_- agree on all degrees except 0. Thus, we have the following proposition.

Proposition 1.1.4 ([Pan16a]). *There does not exist an exact Lagrangian cobordism with Maslov number* 0 *from* Λ_2 *to* Λ_1*, where* Λ_1 *and* Λ_2 *are shown in Figure 1.4.*

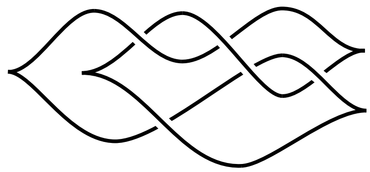


FIGURE 1.4: Legendrian knots Λ_1 and Λ_2 of knot type 4_1 and 6_1, respectively.

1.2 Exact Lagrangian fillings of Legendrian $(2, n)$ torus links

When the negative end Λ_- of an exact Lagrangian cobordism Σ is empty, we get an exact Lagrangian filling of Λ_+ as shown in Figure 1.5. By the functoriality of the Chekanov–Eliashberg DGA with respect to exact Lagrangian cobordisms [EHK16] as stated before, an exact Lagrangian filling of Λ induces an augmentation of the DGA of Λ. Moreover, isotopic exact Lagrangian fillings give homotopic augmentations. This illustrates an exciting connection between exact Lagrangian fillings (geometric objects) and augmentations (algebraic objects). Indeed, there is a map from the Fukaya category to the augmentation category that maps objects (exact Lagrangian fillings) to objects (augmentations) and morphisms to morphisms.

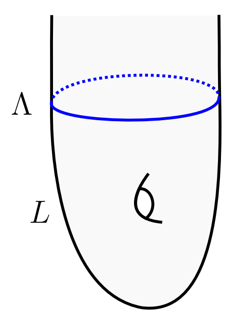


FIGURE 1.5: A picture of an exact Lagrangian filling.

In general, it is hard to construct exact Lagrangian fillings for Legendrian knots. However, Ekholm, Honda and Kálmán [EHK16] exhibited a way to construct exact Lagrangian fillings for Legendrian positive braid closures. They introduced a sequence of moves to resolve the diagram of a Legendrian positive braid closure, and showed that each move corresponds to an exact Lagrangian cobordism. Concatenating the cobordisms corresponding to each move, they constructed an exact Lagrangian filling for the Legendrian knot.

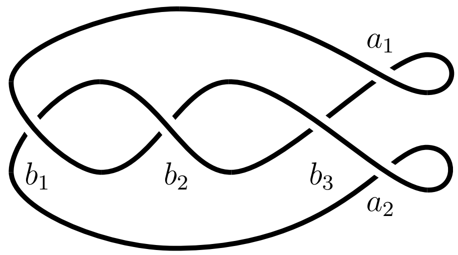


FIGURE 1.6: The Lagrangian projection of the Legendrian trefoil.

We will illustrate this construction in an example of the Legendrian trefoil, which is the $(2, 3)$ torus knot. First project it to the xy–plane and get a Lagrangian projection as shown in Figure 1.6. There are two types of moves in [EHK16]: the pinch move (corresponding to a saddle cobordism) and the minimum cobordism (corresponding to a disk). These two moves are shown in Figure 1.7 in terms of Lagrangian projections. Starting with the diagram in Figure 1.6, we first do three pinch moves at

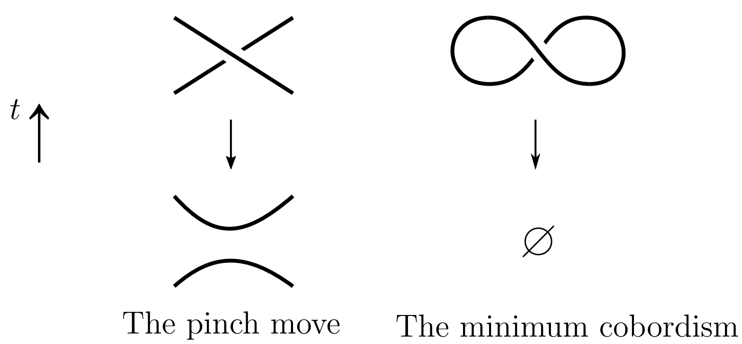


FIGURE 1.7: The pinch move and the minimum cobordism between Lagrangian projections of links.

b_1, b_2 and b_3 getting two unknots and then do two minimum cobordisms to close it up (as shown on the left side of Figure 1.8). Concatenating the corresponding exact Lagrangian surfaces, we obtain an exact Lagrangian filling of the Legendrian trefoil (as shown on the right side of Figure 1.8).

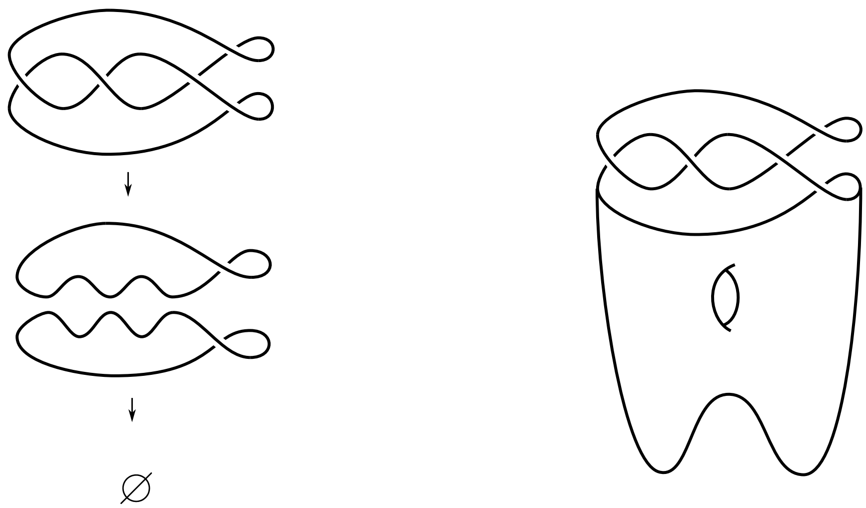


FIGURE 1.8: The EHK construction of exact Lagrangian fillings of the Legendrian trefoil.

Different orders of pinch moves may give different exact Lagrangian fillings. Ekholm, Honda and Kálmán gave 5 exact Lagrangian fillings in this construction, and distinguished them by showing the corresponding augmentations to $\mathbb{Z}_2$ were not homotopic.

A natural question is whether we can extend this result to general Legendrian knots. However, the induced augmentations to $\mathbb{Z}_2$ are not effective enough to distinguish all the exact Lagrangian fillings even for Legendrian $(2, n)$ torus knots where $n > 3$. Instead we study augmentations of the DGA of a Legendrian $(2, n)$ knot to $\mathbb{Z}_2[H_1(L)]$, which is the group ring over $\mathbb{Z}_2$ of the first homology of a Lagrangian filling L. We give a combinatorial formula of the augmentations to $\mathbb{Z}_2[H_1(L)]$ induced by exact Lagrangian fillings from the EHK construction and find a combinatorial invariant to distinguish these augmentations. In this way, we generalize the result in

[EHK16] to Legendrian $(2, n)$ torus links.

Theorem 1.2.1 ([Pan16b]). *The Legendrian $(2, n)$ torus link has at least C_n exact Lagrangian fillings up to exact Lagrangian isotopy, where C_n is the n–th Catalan number*

$$C_n = \frac{1}{n+1}\binom{2n}{n}.$$

1.3 Outline

In Chapter 2 we review the background in contact topology including the Chekanov–Eliashberg DGA of a Legendrian submanifold and the DGA map induced by an exact Lagrangian cobordism. Then we talk about two projects separately. Chapter 3, 4 and 5 are about the project in Section 1.1. In Chapter 3, we introduce the augmentation category for a Legendrian submanifold and describe the A_∞ category map induced by an exact Lagrangian cobordism. In Chapter 4, we review the Floer theory of Lagrangian cobordisms. Finally, using the techniques in Chapter 4, we prove Theorem 1.1.1 in Chapter 5 and discuss its applications. Chapter 6 is about the project in Section 1.2. We first talk about the construction of [EHK16] in Section 6.1 and then distinguish the fillings in Sections 6.2 and 6.3.

2

Background

2.1 Legendrian submanifolds

We start with reviewing the background of Legendrian submanifolds. Let Λ be a Legendrian submanifold in $\mathbb{R}^3$ with the standard contact structure $\xi = \ker \alpha$, where $\alpha = dz - ydx$. We can visualize Λ through two projection diagrams: the **Lagrangian projection** $\Pi_{xy} : \mathbb{R}^3 \to \mathbb{R}^2,\ (x, y, z) \mapsto (x, y)$ and the **front projection** $\Pi_{xz} : \mathbb{R}^3 \to \mathbb{R}^2,\ (x, y, z) \mapsto (x, z)$ respectively. As an example, a front projection and a Lagrangian projection of the Legendrian trefoil are shown in Figure 3.2.

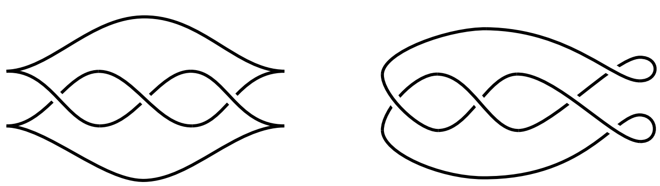


FIGURE 2.1: A front projection (left) and a Lagrangian projection (right) of the Legendrian trefoil.

Moreover, starting from a front projection of Λ, Ng [Ng03a] gave an algorithm to obtain a Lagrangian projection of Λ by smoothing the cusps of the front projection in a way shown in Figure 2.2.

FIGURE 2.2: Ng's algorithm to transfer a front projection to a Lagrangian projection by smoothing the left cusp directly and smoothing the right cusp with an additional crossing.

There are two canonical invariants of oriented Legendrian submanifolds: the Thurston–Bennequin number and the rotation number. Both of them are easy to compute. We start with a front projection $\Pi_{xz}(\Lambda)$ of an oriented Legendrian submanifold Λ. The Thurston–Bennequin number $tb(\Lambda)$ is given by

$$tb(\Lambda) = \text{writhe} - \frac{1}{2}\#\{\text{cusps}\},$$

where the writhe is a count of crossings with signs as given in Figure 2.3.

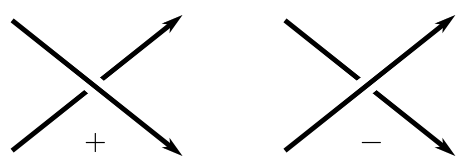


FIGURE 2.3: Sign of crossings.

The orientation of Λ divides the cusps into two sets, one with up orientation and one with down orientation. The rotation number $rot(\Lambda)$ is given by

$$rot(\Lambda) = \frac{1}{2}\big(\#\{\text{down cusps}\} - \#\{\text{up cusps}\}\big).$$

As an example, for the Legendrian trefoil Λ as shown in Figure 2.4, we have that $tb(\Lambda) = 1$ and $rot(\Lambda) = 0$.

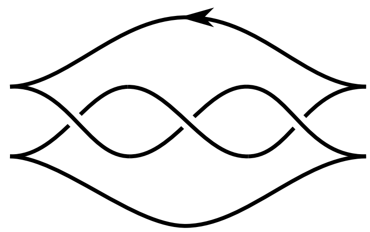


FIGURE 2.4: The front projection of the Legendrian trefoil.

As an important object in contact topology, a **Reeb vector field** of a contact manifold $(M, \xi = \ker \tilde{\alpha})$ is a unique vector field $R_{\tilde{\alpha}}$ such that

- $\tilde{\alpha}(R_{\tilde{\alpha}}) = 1$; and
- $d\tilde{\alpha}(R_{\tilde{\alpha}}, \cdot\,) = 0$.

A **Reeb chord** of a Legendrian knot Λ in (M, ξ) is a trajectory of the flow of the Reeb vector field with both ends on Λ. For example in $\mathbb{R}^3$ with the standard contact structure $\ker \alpha$, the Reeb vector field is $\dfrac{\partial}{\partial z}$. Thus Reeb chords of Λ are $\left(\mathbb{R}^3, \ker \alpha\right)$ are vertical line segments. After a perturbation of the $\Pi_{xy}(\Lambda)$ if necessary, we can assume that these Reeb chords are in $1-1$ correspondence to double points of $\Pi_{xy}(\Lambda)$, which by Ng's algorithm correspond to the crossings and right cusps of $\Pi_{xz}(\Lambda)$.

2.2 The Legendrian contact homology DGA

In this section, we review the Legendrian contact homology DGA from the geometric perspective of [EHK16] in Section 2.2.1 and the combinatorial perspective of [NRS$^+$15, Section 2.2.1] in Section 2.2.2. We refer readers to [Che02b, ENS02, Ng03b] for a more detailed introduction. In addition, we will have a brief introduction on augmentation in Section 2.2.3.

Let $\Lambda = \Lambda_1 \cup \Lambda_2 \cup \cdots \cup \Lambda_M$ be an oriented Legendrian link with M connected components. For simplicity when defining the degree, we assume throughout this dissertation that each component of Λ has rotation number 0.

2.2.1 Geometric perspective

Let $(\mathcal{A}(\Lambda; \mathbb{F}[H_1(\Lambda)]), \partial)$ denote the **Legendrian contact homology differential graded algebra (DGA) of** Λ, which is also called **Chekanov–Eliashberg DGA**. In order to describe it, we will talk about the underlying algebra, gradings and the differential of the DGA.

The **underlying algebra** $\mathcal{A}(\Lambda; \mathbb{F}[H_1(\Lambda)])$ is a non–commutative unital graded algebra over a field $\mathbb{F}$ generated by

$$\{c_1, \dots, c_m, t_1, t_1^{-1}, \dots, t_M, t_M^{-1}\}$$

with relations $t_i t_i^{-1} = 1$ for $i = 1, \dots, M$. Here $c_1, \dots, c_m$ are Reeb chords of Λ and $\{t_1, \dots, t_M\}$ is a basis of the singular homology $H_1(\Lambda)$.

The **grading** of a Reeb chord c is defined as

$$|c| = CZ(\gamma_c) - 1,$$

where γ_c is a capping path for c and CZ is the Conley–Zehnder index as described in [EES05a]. See [DR16, Section 4.1] for a way to choose a capping path γ_c for a Reeb chord of a Legendrian link. The grading of a Reeb chord depends on the choice of capping paths. But under our assumption that each component of Λ has rotation number 0, the grading is well defined.

We will introduce a combinatorial way to compute the grading of Reeb chords. Starting with the front projection $\Pi_{xz}(\Lambda)$, we write $C(\Pi_{xz}(\Lambda))$ for the set of cusps of $\Pi_{xz}(\Lambda)$, which divides $\Pi_{xz}(\Lambda) \backslash C(\Pi_{xz}(\Lambda))$ into strands (ignoring double points). The **Maslov potential** is a function μ that assigns an integer to each strand such that around each cusp, the Maslov potential of the lower strand is one less than that of the upper strand. This is well defined up to a global shift on each component of Λ. Once the Maslov potential is fixed, the grading of a Reeb chord c that corresponds to a crossing of $\Pi_{xz}(\Lambda)$ can be computed by

$$|c| = \mu(u) - \mu(l),$$

where u is the upper strand of the crossing and l is the lower strand of the crossing. The grading of Reeb chords that correspond to right cusps of $\Pi_{xz}(\Lambda)$ are 1.

Furthermore, set the grading of t_i to be zero for $i = 1, \ldots, M$, and extend the definition of degree to $\mathcal{A}(\Lambda; \mathbb{F}[H_1(\Lambda)])$ through the relation $|ab| = |a| + |b|$.

Now we move to the most important component of DGA, the differential ∂. To define it, we need to a **cylindrical almost complex structure** on $\left(\mathbb{R}\times\mathbb{R}^3, d(e^t\alpha)\right)$, which is defined as an almost complex structure J satisfying the following conditions:

- J is compatible with the symplectic form $d(e^t\alpha)$, i.e. $d(e^t\alpha)(\cdot, J\cdot)$ is a metric on ξ;
- J is invariant under the action of $\mathbb{R}_t$;
- $J(\partial_t) = R_\alpha$ and $J(\xi) = \xi$.

For a generic choice of cylindrical almost complex structure J, the differential ∂ is defined by counting rigid J–holomorphic disks in $\left(\mathbb{R}_t \times \mathbb{R}^3, d(e^t\alpha)\right)$ with boundary on $\mathbb{R} \times \Lambda$. See Figure 2.2.1 for an example. For Reeb chords $a, b_1, \ldots, b_m$ of Λ, let $\mathcal{M}(a; b_1, \ldots, b_m)$ denote the moduli space of J–holomorphic disks:

$$u : (D_{m+1}, \partial D_{m+1}) \to (\mathbb{R} \times \mathbb{R}^3, \mathbb{R} \times \Lambda)$$

such that

- D_{m+1} is a 2–dimensional unit disk with $m + 1$ boundary points $p, q_1, \ldots, q_m$ removed and the points $p, q_1, \ldots, q_m$ are labeled in a counterclockwise order;

- u is asymptotic to $[0, \infty) \times a$ at p;
- u is asymptotic to $(-\infty, 0] \times b_i$ at q_i.

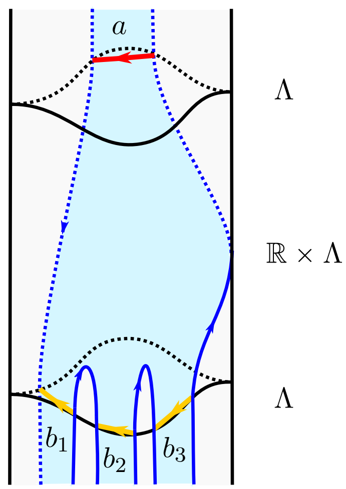


FIGURE 2.5: An example of a J–holomorphic disk with boundary on $\mathbb{R} \times \Lambda$. The arrows on the Reeb chords indicate the orientation of the Reeb chords, while the arrows on the disk boundary indicate the orientation inherited from the unit disk boundary with counterclockwise orientation through u.

An implicit condition for J–holomorphic disks is the positive energy constraint. For a Reeb chord a, define the **action** of c by

$$\mathfrak{a}(c) = \int_c \alpha,$$

which is the length of the Reeb chord c. The energy $E(u)$ of a J–holomorphic disks $u \in \mathcal{M}(a; b_1, \ldots, b_m)$ satisfies

$$E(u) = \mathfrak{a}(a) - \mathfrak{a}(b_1) - \cdots - \mathfrak{a}(b_m).$$

Therefore, to make each J–holomorphic disk endow positive energy, we have

$$\mathfrak{a}(b_1) + \cdots + \mathfrak{a}(b_m) < \mathfrak{a}(a).$$

Let $\widetilde{\mathcal{M}}(a; b_1, \ldots, b_m)$ denote the quotient of $\mathcal{M}(a; b_1, \ldots, b_m)$ by vertical translation of $\mathbb{R}_t$. When $\dim \widetilde{\mathcal{M}}(a; b_1, \ldots, b_m) = 0$, the disk $u \in \mathcal{M}(a; b_1, \ldots, b_m)$ is called **rigid**. The gradings of corresponding Reeb chords satisfy

$$|a| - |b_1| - \cdots - |b_m| = 1.$$

In this case, there are finitely many rigid holomorphic disks and hence we can count the number of rigid holomorphic disks.

For $i = 0, 1, \cdots, m$, denote β_i by the image of the boundary segment from q_i to q_{i+1} under u, where $q_0 = q_{m+1} = p$. In order to count the rigid holomorphic disks with $\mathbb{F}[H_1(\Lambda)]$ coefficients, we want to take the homology class of β_i in $H_1(\mathbb{R} \times \Lambda)$. However, these curves are not closed. Therefore, we introduce capping paths first to close β_i. Equip each connected component Λ_i with a **reference point** p_i, for $i = 1, \ldots, M$. For each $i \neq 1$, pick a path δ_{1i} in $\mathbb{R}^3 \backslash \Lambda$ that goes from p_1 to p_i. For each Reeb chord c of Λ from $c^- \in \Lambda_{i^-}$ to $c^+ \in \Lambda_{i^+}$, the **capping paths** $\gamma_c^\pm$ are defined by concatenating

- the chosen path $\delta_{1i^\pm}$ connecting p_1 to $p_{i^\pm}$ and
- a path on $\Lambda_{i^\pm}$ from $p_{i^\pm}$ to $c^\pm$,

respectively.

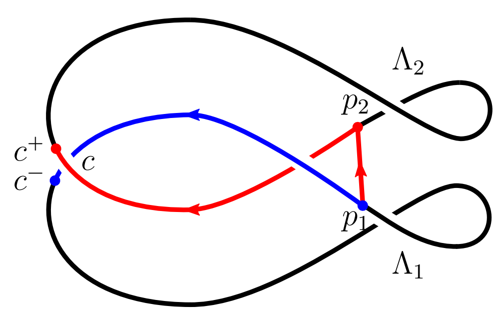


FIGURE 2.6: Consider the Legendrian Hopf link $\Lambda_1 \cup \Lambda_2$. For a Reeb chord c from $c^- \in \Lambda_1$ to $c^+ \in \Lambda_2$, the red curve is a capping path γ_c^+ and the blue curve is a capping path γ_c^-.

We can concatenate the curve β_i with appropriate capping paths (as listed below in formula (2.1)) to get a closed curve in $\mathbb{R} \times (\Lambda \cup \delta_{12} \cup \cdots \cup \delta_{1M})$, denoted by $\overline{\beta_i}$.

$$\overline{\beta_i} = \begin{cases} \gamma_i^+ \cup \beta_i \cup -\gamma_{i+1}^+, & \text{if } i = 0; \\ \gamma_i^- \cup \beta_i \cup -\gamma_{i+1}^+, & \text{if } i = 1, \cdots, m-1; \\ \gamma_i^- \cup \beta_i \cup -\gamma_{i+1}^-, & \text{if } i = m. \end{cases} \tag{2.1}$$

Notice that $H_1(\mathbb{R} \times (\Lambda \cup \delta_{12} \cup \cdots \cup \delta_{1M})) \cong H_1(\Lambda)$. Thus we can take the homology class of $\overline{\beta_i}$ in $H_1(\Lambda)$, denoted by $w(\overline{\beta_i})$.

Moreover, if Λ is spin, all the relevant moduli spaces of J–holomorphic disks admit a coherent orientation. Hence, one can associate a sign $s(u)$ to each rigid J–holomorphic disk u. In this way, we can associate the rigid J–holomorphic disk u with a monomial

$$w(u) = s(u) w(\overline{\beta_0}) b_1 w(\overline{\beta_1}) \cdots b_m w(\overline{\beta_m}).$$

We call the homology classes $w(\overline{\beta_i})$, for $i = 0, \dots, m$, the **coefficients** of $w(u)$.

The **differential** ∂ in the DGA on Reeb chords is defined by counting rigid J–holomorphic disks with coefficient in $\mathbb{F}[H_1(\Lambda)]$:

$$\partial a = \sum_{\dim \widetilde{\mathcal{M}}(a; b_1, \dots, b_m) = 0} \; \sum_{u \in \mathcal{M}(a; b_1, \dots, b_m)} w(u).$$

Let $\partial t_i = \partial t_i^{-1} = 0$ for $i = 1, \dots, M$ and extend the differential to $\mathcal{A}(\Lambda; \mathbb{F}[H_1(\Lambda)])$ through the Leibniz rule

$$\partial(xy) = (\partial x) y + (-1)^{|x|} x (\partial y).$$

It seems that the definition of differential is depend on the choice of capping paths. However, we have the following well–known proposition.

Proposition 2.2.1. *Let Λ be a Legendrian link and γ, γ' be two families of capping paths of Reeb chords of Λ. The corresponding DGAs $\big(\mathcal{A}^{\gamma}(\Lambda), \partial\big)$ and $\big(\mathcal{A}^{\gamma'}(\Lambda), \partial'\big)$ are isomorphic.*

Proof. First we write down the differential explicitly. For a Reeb chord a of Λ, we have that

$$\partial(a) = \sum_{\dim \mathcal{M}^{\Lambda}(a;b_1,\dots,b_m)=0} \sum_{u \in \mathcal{M}^{\Lambda}(a;b_1,\dots,b_m)} s(u) w(\gamma_0^+ \cup \beta_0 \cup -\gamma_1^+)\ b_1\ w(\gamma_1^- \cup \beta_1 \cup -\gamma_2^+)\ b_2 \cdots b_m\ w(\gamma_m^- \cup \beta_m \cup -\gamma_0^-).$$

One can write down the differential ∂' in a similar form. It is not hard to check that the map

$$\begin{aligned} f : (\mathcal{A}^{\gamma}(\Lambda), \partial) &\to (\mathcal{A}^{\gamma'}(\Lambda), \partial') \\ c &\mapsto w(\gamma_c^+ \cup -\gamma'^+_c)\ c\ w(\gamma'^-_c \cup -\gamma_c^-), \text{ if } c \text{ is an Reeb chord of } \Lambda; \\ t &\mapsto t\ \ , t \in H_1(\Lambda) \end{aligned}$$

is a chain map and is an isomorphism, where $w(\gamma_c^+ \cup -\gamma'^+_c)$ indicates the homology class in $H_1(\Lambda)$ of γ_c^+ concatenating γ'^+_c with the reverse orientation. □

Therefore, our DGA is independent of the choice of capping paths and thus is well defined. According to [Che02b], the map ∂ is a differential in degree -1. Moreover, up to stable tame isomorphism, the Chekanov–Eliashberg DGA $\big(\mathcal{A}(\Lambda; \mathbb{Z}_2[H_1(\Lambda)]), \partial\big)$ is an invariant of Λ under Legendrian isotopy.

Remark 2.2.2. In general, for any commutative ring R and a ring homomorphism $\mathbb{F}[H_1(\Lambda)] \to R$, we define the Chekanov–Eliashberg DGA $\big(\mathcal{A}(\Lambda; R), \partial\big)$ as a tensor product of the DGA $\mathcal{A}(\Lambda; \mathbb{F}[H_1(\Lambda)])$ with the ring R:

$$\mathcal{A}(\Lambda; R) = \mathcal{A}(\Lambda; \mathbb{F}[H_1(\Lambda)]) \otimes_{\mathbb{F}[H_1(\Lambda)]} R,$$

where the ring homomorphism gives R the structure of a module over $\mathbb{F}[H_1(\Lambda)]$.

2.2.2 Algebraic perpective

There is an equivalent definition from the combinatorial perspective. We work on the Lagrangian projection $\pi_{xy}(\Lambda)$ of Λ, where Λ is an M–component Legendrian link. After possibly perturbing Λ, we can assume that there is a 1–1 correspondence between the double points of $\pi_{xy}(\Lambda)$ and the Reeb chords of Λ.

First, by [ENS02], instead of counting J–holomorphic disks in $(\mathbb{R} \times \mathbb{R}^3, \mathbb{R} \times \Lambda)$, one can count holomorphic disks in $(\mathbb{R}^2_{xy}, \Pi_{xy}\Lambda)$ in the following way.

For any Reeb chords $a, b_1, \ldots, b_m$ of Λ, define $\Delta^{\Lambda}(a; b_1, \ldots, b_m)$ to be the moduli space of holomorphic disks:

$$u : (D_{m+1}, \partial D_{m+1}) \to \big(\mathbb{R}^2, \Pi_{xy}(\Lambda)\big)$$

with the following properties:

- D_{m+1} is a 2–dimensional unit disk with $m+1$ points $p, q_1, \ldots, q_m$ removed from the boundary and the points $p, q_1, \ldots, q_m$ are labeled in counterclockwise order.
- $\lim\limits_{r \to p} u(r) = a$ and the image of an neighborhood of p under u covers exactly one positive quadrant of the crossing a (see Figure 2.7).
- $\lim\limits_{r \to q_i} u(r) = b_i$, for $i = 1, \ldots, m$, and the image of an neighborhood of q_i under u covers exactly one negative quadrant of the crossing b_i (see Figure 2.7).

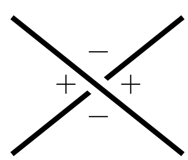


FIGURE 2.7: At each crossing, the quadrants labeled with + sign are **positive quadrants** and the ones labeled with − sign are **negative quadrants**.

Moreover, we can express elements in $H_1(\mathbb{R} \times \Lambda)$ in a combinatorial way. By Poincaré duality, we have $H^1(\mathbb{R} \times \Lambda) \cong H_1(\mathbb{R} \times \Lambda, \partial(\mathbb{R} \times \Lambda))$. In particular, for each oriented curve α in $\mathbb{R} \times \Lambda$ with ends on $\partial(\mathbb{R} \times \Lambda)$, which is an element in $H_1(\mathbb{R} \times \Lambda, \partial(\mathbb{R} \times \Lambda))$, there exists an element $\theta_\alpha \in H^1(\mathbb{R} \times \Lambda)$ such that for any oriented loop β in $\mathbb{R} \times \Lambda$, the intersection number of α and β is $\theta_\alpha(\beta)$. Thus, in order to know the homology class of a loop β in $H_1(\mathbb{R} \times \Lambda)$, we only need to count the intersection number of each generator curve of $H_1(\mathbb{R} \times \Lambda, \partial(\mathbb{R} \times \Lambda))$ with β. We choose the generators of $H_1(\mathbb{R} \times \Lambda, \partial(\mathbb{R} \times \Lambda))$ as $\{\mathbb{R}_t \times \{*_1\}, \cdots, \mathbb{R}_t \times \{*_M\}\}$ with a $-\mathbb{R}_t$ orientation, where $\{*_1, \ldots, *_M\}$ is a set of **minimal base points** of $\Pi_{xy}\Lambda$, i.e.,

- there is exactly one point in $\{*_1, \ldots, *_M\}$ on each component of Λ and
- the set $\{*_1, \ldots, *_M\}$ does not include any end points of Reeb chords of Λ and the reference points.

In particular, for a holomorphic disk u in $(\mathbb{R}^2_{xy}, \Pi_{xy}(\Lambda))$, the homology of corresponding $\overline{\beta_i}$ are in $H_1(\Lambda \cup \delta_{12} \cup \cdots \cup \delta_{1M}) \cong H_1(\Lambda)$. The homology of $\overline{\beta}$ can described specifically as follows. Assign the link Λ an orientation and let t_j be the element in $H_1(\mathbb{R} \times \Lambda)$ that corresponds to $\mathbb{R} \times \{*_j\}$, for $j = 1, \ldots, M$. The homology class of $\overline{\beta}$ counts the intersections of $\overline{\beta}$ and $*_j$, where $j = 1, \ldots, M$, with signs. The sign is positive the orientation of the curve agrees with the orientation of Λ around $*_j$ and is negative if the orientations are opposite to each other.

Moreover, by Preposition 2.2.1, the differential does not depend on the choice of capping paths. Benefitting from the minimal amount of base points, we can choose a particular family of capping paths $\{\gamma_b^\pm\}$ such that $\gamma_b^\pm$ for any Reeb chord b do not pass any base points. Therefore, in order to know the homology class of $\overline{\beta}$, we only need to count the intersection of β and the base points, denoted by $w(\beta)$.

Now we conclude the definition of DGA of Λ from the combinatorial viewpoint. Project Λ onto the xy–plane to get the Lagrangian projection $\pi_{xy}(\Lambda)$ of Λ. Decorate the Lagrangian diagram with an orientation and a set of minimal base points $\{*_1, \ldots, *_M\}$. The graded algebra $\mathcal{A}(\Lambda, *_1, \ldots, *_M)$ is a non-commutative unital graded algebra over a field $\mathbb{F}$ generated by $\{c_1, \ldots, c_m, t_1, t_1^{-1}, \ldots, t_M, t_M^{-1}\}$ with relations $\{t_i t_i^{-1} = 1 |\ i = 1, \ldots, M\}$, where $c_1, \ldots, c_m$ are double points of $\pi_{xy}(\Lambda)$ and

$t_1, \dots, t_M$ correspond to the base points $*_1, \dots, *_M$. The grading is defined the same as above.

The differential is defined by

$$\begin{aligned} \partial a &= \sum_{|a|-|b_1|-\cdots-|b_m|=1} \sum_{u \in \Delta(a;b_1,\dots,b_m)} w(u); \\ \partial t_j &= \partial t_j^{-1} = 0, \quad j = 1, \dots, M. \end{aligned}$$

In the formula, $w(u) = s(u)w(\beta_0)b_1w(\beta_1)b_2 \cdots b_m w(\beta_m)$, where $s(u)$ is the sign associated to the disk u induced from moduli space coherent orientation.

This can be extended to the whole DGA through the Leibniz rule.

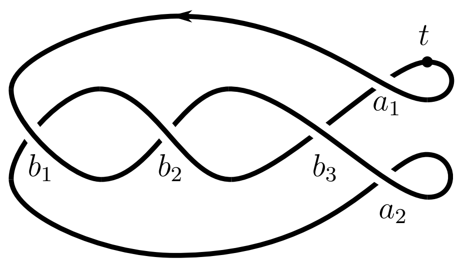


FIGURE 2.8: The Lagrangian projection of the Legendrian $(2, 3)$ torus knot with a single base point.

Example 2.2.3. For the Legendrian $(2, 3)$ torus knot Λ with a single base point t as shown in Figure 2.8. The underlying algebra $\mathcal{A}(\Lambda, t)$ is generated by Reeb chords a_1, a_2, b_1, b_2, b_3 over $\mathbb{Z}_2[t, t^{-1}]$. Reeb chords a_1 and a_2 are in degree 1 and the rest of Reeb chords are in degree 0. Since we are working on $\mathbb{Z}_2$, we can forget about the sign $s(u)$ of u. The differential is given by:

$$\begin{aligned} \partial(a_1) &= t^{-1} + b_1 + b_3 + b_1b_2b_3, \\ \partial(a_2) &= 1 + b_1 + b_3 + b_3b_2b_1, \\ \partial(b_i) &= 0, \quad i = 1, 2, 3, \\ \partial(t) &= \partial(t^{-1}) = 0. \end{aligned}$$

The differential ∂ has degree -1 and satisfies $\partial^2 = 0$ ([Che02b][ENS02]). Up to stable tame isomorphism, the Legendrian contact homology DGA is an invariant of Λ under Legendrian isotopy. In this sense of equivalence, the combinatorial definition does not depend on the choice of base points [NR13].

The definition of DGA of Legendrian link can be generalized to the case where there are more than one base point on some components of the link. Let Λ be an oriented Legendrian link and $\{t_1, \dots, t_l\}$ be a set of points on Λ such that each component of Λ has at least one point in the set and the set does not include any end of any Reeb chord of Λ. For a path β, associate it with a monomial $w(\beta) = \prod_{j=1}^{l} s_j^{n_j(\gamma)}$ in $\mathbb{F}[t_1^{\pm 1}, \dots, t_l^{\pm 1}]$, where n_j is defined similar as above. The DGA $\big(\mathcal{A}(\Lambda, \{t_1, \dots, t_l\}), \partial\big)$ is a unital graded algebra generated by Reeb chords of Λ and $t_1^{\pm 1}, \dots, t_l^{\pm 1}$ over $\mathbb{F}$ with relations $t_i t_i^{-1} = 1, i = 1, \dots, l$ endowed with a differential given by

$$\begin{aligned} \partial(a) &= \sum_{\dim \mathcal{M}^{\Lambda}(a; b_1, \dots, b_m) = 0} \ \sum_{u \in \mathcal{M}^{\Lambda}(a; b_1, \dots, b_m)} w(u), \\ \partial(s_i) &= 0, \quad i = 1, \dots, l. \end{aligned}$$

where $w(u) = s(u) w(\beta_0) b_1 w(\beta_1) b_2 \cdots b_m w(\beta_m)$ and $s(u)$ is the orientation of the disk in the moduli space. Note that this DGA may not be equivalent to the DGA $\mathcal{A}(\Lambda; \mathbb{F}[H_1(\Lambda)])$ any more. But according to [NR13], there is a DGA homomorphism from the DGA with minimal base points to this DGA.

2.2.3 Augmentations and Linearized Contact Homology

As one can see from the example 2.2.3, the homology of the DGA is hard to compute in general. Let us introduce augmentations of a DGA and use that to deduce linearized contact homology, which is much easier to compute. Let $(\mathcal{A}, \partial)$ be a DGA over a field $\mathbb{F}$ of a Legendrian link Λ with base points. A **graded augmentation** of

$\mathcal{A}$ is a DGA map

$$\epsilon : (\mathcal{A}, \partial) \to (\mathbb{F}, 0),$$

where $(\mathbb{F}, 0)$ is a chain complex that is $\mathbb{F}$ in degree 0 and is 0 in other degrees. In other words, a graded augmentation is an algebra map $\epsilon : \mathcal{A} \to \mathbb{F}$ such that $\epsilon(1) = 1$, $\epsilon \circ \partial = 0$ and $\epsilon(a) = 0$ if $|a| \neq 0$.

Given a graded augmentation ϵ, define $\mathcal{A}^\epsilon := \mathcal{A} \otimes \mathbb{F}/(t_i = \epsilon(t_i))$. Notice that the differential ∂ descends to $\mathcal{A}^\epsilon$ since $\partial(t_i) = 0$. Elements in $\mathcal{A}^\epsilon$ are summands of words of Reeb chords. Let C be a free $\mathbb{F}$-module generated by Reeb chords. We can decompose $\mathcal{A}^\epsilon$ in terms of word length as $\mathcal{A}^\epsilon = \bigoplus_{n \geqslant 0} C^{\otimes n}$. Let $\mathcal{A}^\epsilon_+$ be the part of $\mathcal{A}^\epsilon$ containing the words with length at least 1, i.e. $\mathcal{A}^\epsilon_+ = \bigoplus_{n \geqslant 1} C^{\otimes n}$. Consider a new differential $\partial^\epsilon : \mathcal{A}^\epsilon \to \mathcal{A}^\epsilon$:

$$\partial^\epsilon := \phi_\epsilon \circ \partial \circ \phi_\epsilon^{-1},$$

where $\phi_\epsilon : \mathcal{A}^\epsilon \to \mathcal{A}^\epsilon$ is an automorphism defined by $\phi_\epsilon(a) = a + \epsilon(a)$. Observe that ∂^ϵ preserves $\mathcal{A}^\epsilon_+$ and does not decrease the minimal length of a word. Thus, it descends to a differential on $\mathcal{A}^\epsilon_+/(\mathcal{A}^\epsilon_+)^2 \cong C$. The homology of (C, ∂^ϵ) is called **linearized contact homology** of Λ with respect to ϵ, denoted by $LCH^\epsilon_*(\Lambda)$. The chain complex (C, ∂^ϵ) is called **linearized contact homology chain complex**.

2.3 Exact Lagrangian cobordisms

Definition 2.3.1. Suppose $\Lambda_\pm$ are Legendrian submanifolds in $(\mathbb{R}^3, \ker \alpha)$, where $\alpha = dz - ydx$. An **exact Lagrangian cobordism** Σ from Λ_- to Λ_+ is a 2–dimensional surface in $\big(\mathbb{R} \times \mathbb{R}^3, \omega = d(e^t\alpha)\big)$ (see Figure 1.2) such that for some big number $N > 0$,

- $\Sigma \cap \big((N, \infty) \times \mathbb{R}^3\big) = (N, \infty) \times \Lambda_+$,
- $\Sigma \cap \big((-\infty, -N) \times \mathbb{R}^3\big) = (-\infty, -N) \times \Lambda_-$ and

- $\Sigma \cap \big([-N, N] \times \mathbb{R}^3\big)$ is compact.

Moreover, there exists a smooth function $g : \Sigma \to \mathbb{R}$ such that

$$e^t\alpha \mid_{T\Sigma} = dg$$

and g is constant when $t \leqslant -N$ and $t \geqslant N$. The function g is called a **primitive** of Σ.

When Λ_- is empty, the surface L satisfying the conditions above is called an **exact Lagrangian filling** of Λ_+.

Ekholm, Honda and Kálmán [EHK16] showed that an exact Lagrangian cobordism Σ from Λ_- to Λ_+ gives a DGA map

$$\phi_\Sigma : (\mathcal{A}(\Lambda_+; \mathbb{F}[H_1(\Sigma)]), \partial) \to (\mathcal{A}(\Lambda_-; \mathbb{F}[H_1(\Sigma)]), \partial),$$

which is defined by a count of rigid holomorphic disks in $\mathbb{R} \times \mathbb{R}^3$ as well, but with boundary on Σ. In the special case where Λ_- is empty, an exact Lagrangian filling Σ of Λ_+ gives a DGA map $(\mathcal{A}(\Lambda_+; \mathbb{F}[H_1(\Sigma)]), \partial) \to (\mathbb{F}, 0)$, which is an augmentation of $\mathcal{A}(\Lambda_+; \mathbb{F}[H_1(\Sigma)])$. Here $\mathbb{F}$ can be any field if the cobordism Σ is spin. If the condition is not satisfied, the field $\mathbb{F}$ is assumed to be $\mathbb{Z}_2$.

Remark 2.3.2. When Σ is spin, the boundary Legendrian knots Λ_+ and Λ_- get induced spin structure from the spin structure of Σ. This condition makes the moduli spaces of the holomorphic disks used in the DGA differentials and the DGA map equipped with a coherent orientation (following [EES05b]). In particular, when the dimension of a moduli space is 0, one can associate each rigid holomorphic disk in the moduli space with a sign. Therefore, we can count the disks with sign and get coefficients in any field $\mathbb{F}$. Otherwise, it is only reasonable to count the disks mod 2, which means ignoring the orientation. For the rest of the dissertation, we focus on the case where Σ is spin. If one is working on a non–spin cobordism, one can omit our description of orientation and get the corresponding statements for $\mathbb{F} = \mathbb{Z}_2$.

Moreover, Ekholm, Honda and Kálmán also shown that isotopic exact Lagrangian cobordisms give chain homotopic DGA maps. In particular, isotopic exact Lagrangian fillings give homotopic augmentations. This gives a way to distinguish exact Lagrangian fillings.

In Section 2.3.1, we review the DGA map induced by an exact Lagrangian cobordism given in [EHK16]. In Section 2.3.2 and Section 2.3.3 we introduce two DGA maps with modified coefficient rings that will be used in the two projects.

2.3.1 DGA map induced by an exact Lagrangian cobordism

For a spin exact Lagrangian cobordism Σ from Λ_- to Λ_+, the Legendrian submanifolds $\Lambda_\pm$ inherit induced spin structures. Hence $\Lambda_\pm$ have $\mathbb{F}[H_1(\Lambda_\pm)]$ coefficients DGAs $(\mathcal{A}(\Lambda_\pm;\mathbb{F}[H_1(\Lambda_\pm)]),\partial)$, respectively, as described in Section 2.2. Ekholm, Honda and Kálmán in [EHK16] showed that an exact Lagrangian cobordism Σ induces a DGA map from $\mathcal{A}(\Lambda_+)$ to $\mathcal{A}(\Lambda_-)$ with $\mathbb{F}[H_1(\Sigma)]$ coefficients. In order to see that, first, we need to view the DGAs of $\Lambda_\pm$ as DGAs with $\mathbb{F}[H_1(\Sigma)]$ coefficients. Notice that the inclusion $H_1(\Lambda_\pm) \hookrightarrow H_1(\Sigma)$ induces a canonical inclusion map $\mathbb{F}[H_1(\Lambda_\pm)] \hookrightarrow \mathbb{F}[H_1(\Sigma)]$ of the group ring coefficients, which makes it natural to consider the DGAs of $\Lambda_\pm$ with $\mathbb{F}[H_1(\Sigma)]$ coefficients in a way described in Remark 2.2.2. Specifically, the new DGA $\mathcal{A}\big(\Lambda_\pm;\mathbb{F}[H_1(\Sigma)]\big)$ is generated by Reeb chords of $\Lambda_\pm$ and elements in $H_1(\Sigma)$ over $\mathbb{F}$. The differential is defined by the original differential in $\mathcal{A}\big(\Lambda_\pm;\mathbb{F}[H_1(\Lambda_\pm)]\big)$ composed with the inclusion map $H_1(\Lambda_\pm) \hookrightarrow H_1(\Sigma)$.

Second, construct a DGA map with $\mathbb{F}[H_1(\Sigma)]$ coefficients. Consider an almost complex structure J that is compatible with the symplectic form ω and is cylindrical on both ends. In other words, J matches the cylindrical almost complex structures on both cylindrical ends. Fix a generic choice of such an almost complex structure J. For Reeb chords a of Λ_+ and $b_1,\dots,b_m$ of Λ_-, define $\mathcal{M}(a;b_1,\dots,b_m)$ to be the moduli space of the J-holomorphic disks:

$$u : (D_{m+1}, \partial D_{m+1}) \to (\mathbb{R} \times \mathbb{R}^3, \Sigma)$$

such that

- D_{m+1} is a 2-dimensional unit disk with $m+1$ boundary points $p, q_1, \ldots, q_m$ removed and the points $p, q_1, \ldots, q_m$ are arranged in a counterclockwise order;
- u is asymptotic to $[N, \infty) \times a$ at p;
- u is asymptotic to $(-\infty, -N] \times b_i$ at q_i.

When $\dim \mathcal{M}(a; b_1, \ldots, b_m) = 0$, the disk $u \in \mathcal{M}(a; b_1, \ldots, b_m)$ is called **rigid**. The gradings of corresponding Reeb chords satisfy

$$|a| - |b_1| - \cdots - |b_m| = 0.$$

For the image of the boundary segment from q_i to q_{i+1}, denoted by β_i, one can close up in a similar way as the one in the definition of the DGA differential (see formula (2.1)) to get $\overline{\beta}$. Denote its homology in $H_1(\Sigma)$ by $w(\overline{\beta})$. If Σ is spin, all the relevant moduli spaces of J–holomorphic disks admit a coherent orientation. In particular, each rigid J–holomorphic disk obtains a sign, denoted by $s(u)$. Associate a monomial $w(u)$ to the J–holomorphic disk u as

$$w(u) = s(u) w(\overline{\beta_0}) b_1 w(\overline{\beta_1}) \cdots b_m w(\overline{\beta_m}).$$

The homology classes $w(\overline{\beta_i})$ for $i = 0, \ldots, m$, are called the **coefficients** of $w(u)$. The DGA map is defined by counting rigid J–holomorphic disks with boundary on Σ:

$$\phi(a) = \sum_{\dim \mathcal{M}(a; b_1, \ldots, b_m) = 0} \ \sum_{u \in \mathcal{M}(a; b_1, \ldots, b_m)} w(u).$$

We can extend the morphism to $\mathcal{A}(\Lambda_+; \mathbb{F}[H_1(\Sigma)])$ by setting $\phi(t) = t$ for any generator t in $H_1(\Sigma)$ and applying the Leibniz rule.

2.3.2 DGA maps with $\mathbb{F}[V]$ coefficients

For the purpose of constructing an augmentation category map induced by Lagrangian cobordisms as in Section 3.2, we restrict to the case where Λ_+ and Λ_- are Legendrian knots with a single base point, denoted by $*_+$ and $*_-$, respectively. We modify the DGA map such that the coefficients only depend on the base points but not depend on the cobordism, i.e., we get a DGA map

$$\phi_\Sigma : \mathcal{A}(\Lambda_+, *_+) \to \mathcal{A}(\Lambda_-, *_-).$$

In order to modify the coefficients of the DGA map ϕ, let us consider $H_1(\Sigma)$ more precisely. To simplify the description, we restrict Σ to $[-N, N] \times \mathbb{R}^3$, denote by Σ as well. Similar as Section 2.3.1, according to Poincaré duality, $H^1(\Sigma) \cong H_1(\Sigma, \Lambda_+ \cup \Lambda_-)$. In particular, for any loop α in Σ with ends on $\Lambda_+ \cup \Lambda_-$, which is an element in $H_1(\Sigma, \Lambda_+ \cup \Lambda_-)$, there is an element θ_α in $H^1(\Sigma)$ such that for any oriented loop γ on Σ, the intersecting number of α and γ is $\theta_\alpha(\gamma)$. Thus, in order to know the homology class of a curve γ in $H_1(\Sigma)$, we only need to count the intersection number of each generator curve of $H_1(\Sigma, \Lambda_+ \cup \Lambda_-)$ with γ.

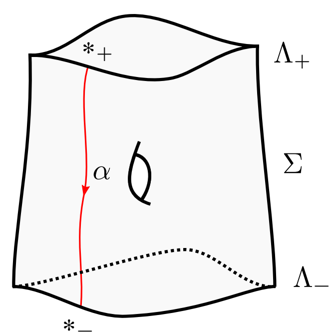


Figure 2.9: Curve α on a cobordism.

Consider a connected exact Lagrangian cobordism Σ from a Legendrian knot Λ_- to a Legendrian knot Λ_+ (see Remark 5.2.4 for the reason that we assume that Σ

is connected). Choose base points $*_+$ and $*_-$ for Λ_+ and Λ_-, respectively. There exists a curve α on Σ from $*_+$ to $*_-$ with exactly one intersection with Λ_+ and Λ_-, respectively. An example is shown in Figure 2.9. Let V^* denote the subgroup of $H^1(\Sigma)$ that is generated by the Poincaré dual of curve α. The dual space V in $H_1(\Sigma)$ is isomorphic to $\mathbb{Z}$.

Now we can modify the DGA map ϕ described above to be a map from $\mathcal{A}(\Lambda_+; \mathbb{F}[V])$ to $\mathcal{A}(\Lambda_-; \mathbb{F}[V])$. First, restrict the generators of $\mathcal{A}(\Lambda_\pm)$ to Reeb chords of $\Lambda_\pm$ and a basis of V. Second, project the coefficients $w(\overline{\beta_i})$ of the monomial $w(u)$ from $H_1(\Sigma)$ to V. Therefore, the DGA map works in $\mathbb{F}[V]$ coefficients. Indeed, the definitions of $\mathcal{A}(\Lambda_\pm; \mathbb{F}[V])$ match the definition of $\mathcal{A}(\Lambda_\pm, *_\pm)$, respectively. Hence a connected exact Lagrangian cobordism Σ induces a DGA map with $\mathbb{F}[V]$ coefficients from the DGA of Λ_+ with a single base point to the DGA of Λ_- with a single base point:

$$\phi : (\mathcal{A}(\Lambda_+, *_+), \partial) \to (\mathcal{A}(\Lambda_-, *_-), \partial).$$

Note that this DGA map does depend on the choice of the curve α that connecting the two base points.

2.3.3 DGA maps with $\mathbb{F}[H_1(\Sigma_-)]$ coefficients

For the purpose of computing augmentations of the Legendrian $(2, n)$ torus knots in Chapter 6, we revise the coefficient ring of the DGA map induced by exact Lagrangian cobordisms from [EHK16]. Instead of using $\mathbb{F}[H_1(\Sigma)]$ coefficients, we will show the following proposition:

Proposition 2.3.3. *Let Λ_+ and Λ_- be Legendrian submanifolds in $(\mathbb{R}^3, \ker \alpha)$ and Σ be a connected exact Lagrangian cobordism from Λ_- to Λ_+. Assume that $\overline{\Sigma}_+$ is a connected exact Lagrangian cobordism from Λ_+ to some other Legendrian link and $\overline{\Sigma}_-$ is the concatenation of $\overline{\Sigma}_+$ and Σ as shown in Figure 2.10. The exact Lagrangian* *cobordism* Σ *induces a DGA map*

$$\phi : \big(\mathcal{A}(\Lambda_+; \mathbb{F}[H_1(\overline{\Sigma}_+)]), \partial_+\big) \to \big(\mathcal{A}(\Lambda_-; \mathbb{F}[H_1(\overline{\Sigma}_-)]), \partial_-\big)$$

with $\mathbb{F}[H_1(\overline{\Sigma}_-)]$ *coefficients.*

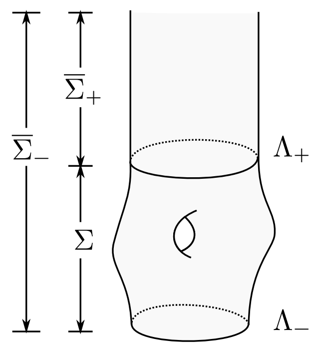


FIGURE 2.10: The relation among cobordisms $\overline{\Sigma}_+$, $\overline{\Sigma}_-$, and Σ.

Note that when $\overline{\Sigma}_+$ is an exact Lagrangian cylinder over Λ_+, this map agrees with the DGA map introduced by [EHK16]. This revision of the coefficient ring is based on a different choice of capping paths of Λ_+ and Λ_-. The capping paths are chosen on Σ in [EHK16] while we choose capping paths of Λ_+ on $\overline{\Sigma}_+$ and capping paths of Λ_- on $\overline{\Sigma}_-$. For the rest of the section, we will describe this DGA map.

Similar as what we did in Section 2.3.1, the inclusion map $\Lambda_+ \hookrightarrow \overline{\Sigma}_+$ makes it natural to define the DGA $\big(\mathcal{A}(\Lambda_+; \mathbb{F}[H_1(\overline{\Sigma}_+)]), \partial_+\big)$ in the sense of Remark 2.2.2. Specifically, the underlying algebra

$$\mathcal{A}(\Lambda_+; \mathbb{F}[H_1(\overline{\Sigma}_+)]) = \mathcal{A}(\Lambda_+; \mathbb{F}[H_1(\Lambda_+)]) \otimes_{\mathbb{F}[H_1(\Lambda_+)]} \mathbb{F}[H_1(\overline{\Sigma}_+)]$$

is generated by Reeb chords of Λ_+ and a basis of $H_1(\overline{\Sigma}_+)$ over $\mathbb{F}$. Given that $\overline{\Sigma}_+$ is connected, we can choose a family of capping paths for Λ_+ on $\overline{\Sigma}_+$. Therefore, for any closed curves $\overline{\beta}$ that are used in defining ∂_+, it is natural to take the homology class of $\overline{\beta}$ in $H_1(\overline{\Sigma}_+)$. Hence the differential coefficients of ∂_+ are in $\mathbb{F}[H_1(\overline{\Sigma}_+)]$. In addition, the DGA $\big(\mathcal{A}(\Lambda_+;\mathbb{F}[H_1(\overline{\Sigma}_+)]),\partial_+\big)$ does not depend on the choice of capping paths on $\overline{\Sigma}_+$ for a similar reason as Proposition 2.2.1. The DGA $\big(\mathcal{A}(\Lambda_-;\mathbb{F}[H_1(\overline{\Sigma}_-)]),\partial_-\big)$ is defined similarly.

The DGA map ϕ induced by Σ is a composition of two maps. The first map

$$\phi_1 : \big(\mathcal{A}(\Lambda_+;\mathbb{F}[H_1(\overline{\Sigma}_+)]),\partial_+\big) \to \big(\mathcal{A}(\Lambda_+;\mathbb{F}[H_1(\overline{\Sigma}_-)]),\partial_+\big)$$

is induced by the inclusion map $i : \overline{\Sigma}_+ \hookrightarrow \overline{\Sigma}_-$. As we stated in Section 2.3.1, the differential in $\mathcal{A}(\Lambda_+;\mathbb{F}[H_1(\overline{\Sigma}_-)])$ is a composition of the differential in $\mathcal{A}(\Lambda_+;\mathbb{F}[H_1(\overline{\Sigma}_+)])$ and the inclusion map i. Thus ϕ_1 is a DGA map. The second map

$$\phi_2 : \big(\mathcal{A}(\Lambda_+;\mathbb{F}[H_1(\overline{\Sigma}_-)]),\partial_+\big) \to \big(\mathcal{A}(\Lambda_-;\mathbb{F}[H_1(\overline{\Sigma}_-)]),\partial_-\big)$$

is defined by counting rigid holomorphic disks in $\mathbb{R}\times\mathbb{R}^3$ with boundary on Σ, which is almost the same as the DGA map defined in Section 2.3.1 except that we take the homology class of the corresponding curves in $H_1(\Sigma_-)$ instead of $H_1(\Sigma)$.

3

The Augmentation Category

3.1 A_∞ categories

In this section, we give a lightning review of A_∞ algebras and A_∞ categories following [NRS+15]. See [Kel01, GJ90] for a more detailed introduction.

Definition 3.1.1 ([Kel01, Section 3.1])**.** An A_∞ **algebra** over a field $\mathbb{F}$ is a $\mathbb{Z}$-graded vector space A endowed with degree $2-n$ maps $m_n : A^{\otimes n} \to A$ such that

$$\sum_{r+s+t=n} (-1)^{r+st} m_{r+1+t}(1^{\otimes r} \otimes m_s \otimes 1^{\otimes t}) = 0.$$

The most important things we need among these complicated relations are

- m_1 is a differential on A (i.e., $m_1^2 = 0$) and
- m_2 is associative after passing to the homology with respect to m_1.

An A_∞ algebra can be achieved nicely through the following construction. Let $\overline{T}(C) = \bigoplus_{n\geqslant 1} C^{\otimes n}$ be a graded vector space over $\mathbb{F}$ equipped with a **co-differential** b, i.e.,

- b has degree 1, $b^2 = 0$ and
- $b = \oplus b_n$, where b_n is a map $C^{\otimes n} \to C$, satisfies the co-Leibniz rule

$$\Delta b = (1 \otimes b + b \otimes 1)\Delta,$$

where $\Delta(a_1 \otimes \cdots \otimes a_n) = \sum_{i=1}^{n} (a_1 \otimes \cdots \otimes a_i) \otimes (a_{i+1} \otimes \cdots \otimes a_n)$.

Let $C^\vee := C[-1]$ and $s : C \to C^\vee$ be the canonical degree 1 identification map $a \mapsto a$. Define maps $m_n : (C^\vee)^{\otimes n} \to C^\vee$ such that the following diagram commutes for all n.

$$\begin{array}{ccc} C^{\otimes n} & \xrightarrow{b_n} & C \\ {\scriptstyle s^{\otimes n}}\downarrow & & \downarrow{\scriptstyle s} \\ (C^\vee)^{\otimes n} & \xrightarrow[m_n]{} & C^\vee \end{array}$$

Then $C^\vee$ is an A_∞ algebra with m_n as A_∞ operations [Sta63]. One can check that the degree of m_n is $2 - n$.

Example 3.1.2. If a Legendrian contact homology DGA $(\mathcal{A}(\Lambda), \partial)$ has an augmentation ϵ, the conjugated differential ∂^ϵ is a differential of $\mathcal{A}_+^\epsilon = \bigoplus_{n \geqslant 1} C^{\otimes n} = \overline{T}(C)$, where C is the vector space over a field $\mathbb{F}$ generated by Reeb chords of Λ. We define δ^ϵ to be the adjoint of ∂^ϵ on $\overline{T}(C^*) = \bigoplus_{n \geqslant 1} (C^*)^{\otimes n}$, where C^* is the dual of C. More specifically,

$$\delta^\epsilon(b_m^* \otimes \cdots \otimes b_1^*) = \sum_a \text{Coefficient}_{b_1 b_2 \cdots b_m}(\partial^\epsilon(a)).$$

It is not hard to check that δ^ϵ is a co-differential of $\overline{T}(C^*)$. Hence one can use the construction above to construct an A_∞ algebra $(C^*)^\vee$.

Definition 3.1.3. [Kel01] An A_∞ **category** over a field $\mathbb{F}$ is a category where, for any two objects ϵ_1 and ϵ_2, the morphism is a graded vector space $Hom(\epsilon_1, \epsilon_2)$. Moreover, for any objects $\epsilon_1, \epsilon_2, \ldots, \epsilon_{n+1}$, there exists a degree $2-n$ map

$$m_n : Hom(\epsilon_n, \epsilon_{n+1}) \otimes \cdots \otimes Hom(\epsilon_1, \epsilon_2) \to Hom(\epsilon_1, \epsilon_{n+1})$$

satisfying

$$\sum_{r+s+t=n} (-1)^{r+st} m_{r+1+t}(1^{\otimes r} \otimes m_s \otimes 1^{\otimes t}) = 0.$$

As noticed before, the first A_∞ operation m_1 is a differential for $Hom(\epsilon_1, \epsilon_2)$ with degree 1. Denote its cohomology by $H^*Hom(\epsilon_1, \epsilon_2)$. Moreover, we have that m_2 descends to an associative map on the cohomology level:

$$m_2 : H^*Hom(\epsilon_2, \epsilon_3) \otimes H^*Hom(\epsilon_1, \epsilon_2) \to H^*Hom(\epsilon_1, \epsilon_3)$$

for any objects $\epsilon_1, \epsilon_2, \epsilon_3$.

An A_∞ **morphism** between two A_∞ categories $f : \mathcal{A} \to \mathcal{B}$ maps the object ϵ of $\mathcal{A}$ to $f(\epsilon)$ of $\mathcal{B}$ and for any objects $\epsilon_1, \epsilon_2, \ldots, \epsilon_{n+1}$ of $\mathcal{A}$, there exists a map

$$f_n : Hom(\epsilon_n, \epsilon_{n+1}) \otimes \cdots \otimes Hom(\epsilon_1, \epsilon_2) \to Hom(f(\epsilon_1), f(\epsilon_{n+1}))$$

satisfying the A_∞ relations [Kel01]. In particular, the first map f_1, called **the category map on the level of morphisms**, maps the morphism $Hom(\epsilon_1, \epsilon_2)$ of $\mathcal{A}$ to the morphism $Hom(f(\epsilon_1), f(\epsilon_2))$ of $\mathcal{B}$. From the A_∞ relations, we know that

- the functor f_1, the category map on the level of morphisms, commutes with m_1 and thus f_1 descends to a map on cohomology:

$$f^* : H^*Hom(\epsilon_1, \epsilon_2) \to H^*Hom(f(\epsilon_1), f(\epsilon_2));$$

- for any $a \in Hom(\epsilon_2, \epsilon_3)$ and $b \in Hom(\epsilon_1, \epsilon_2)$, we have

$$f^*(m_2([a], [b])) = m_2(f^*[a], f^*[b]),$$

i.e, the composition map m_2 commutes with f^* when passing to the cohomology level.

An A_∞ morphism between two A_∞ categories $f : \mathcal{A} \to \mathcal{B}$ induces a functor on the cohomology categories, $\tilde{f} : H^*\mathcal{A} \to H^*\mathcal{B}$. It behaves the same as f on the object level. On the level of morphisms $\tilde{f} = f^*$. The functor $\tilde{f}$ is **faithful** if f^* is injective and is **fully faithful** if f^* is an isomorphism for any morphism in $H^*\mathcal{A}$.

3.2 The augmentation category

In this section, we briefly review the augmentation category $\mathcal{A}ug_+(\Lambda)$ following [NRS+15].

Let Λ be an oriented Legendrian knot in $(\mathbb{R}^3, \ker\alpha)$ endowed with a single base point $*$. Denote its Legendrian contact homology DGA by $(\mathcal{A}, \partial)$. Given a field $\mathbb{F}$, the objects of the augmentation category $\mathcal{A}ug_+(\Lambda)$ are augmentations of $(\mathcal{A}, \partial)$ to $\mathbb{F}$,

$$\epsilon : \mathcal{A} \to \mathbb{F}.$$

In order to describe the morphism $Hom_+(\epsilon_1, \epsilon_2)$ for any two objects ϵ_1 and ϵ_2, we need to study the DGA of a 2-copy of Λ, denoted by $\Lambda^{(2)}$.

By the Weinstein tubular neighborhood theorem, we can identify a neighborhood of Λ with a neighborhood of the zero section in the 1-jet space $J^1(\Lambda) = T^*(\Lambda) \times \mathbb{R}$ through a contactomorphism. The contact form in $J^1(\Lambda)$ is $\alpha = dz - pdq$, where q is the coordinate on Λ and p is the coordinate in the cotangent direction. For any C^1 small function $f : \Lambda \to \mathbb{R}$, the 1-jet $j^1 f = \{(q, f'(q), f(q)) | \, q \in \Lambda\}$ is a Legendrian knot in $J^1(\Lambda)$ and thus is a Legendrian knot in $\mathbb{R}^3$. Now choose a particular Morse function $f : \Lambda \to (0, \delta)$ such that

- δ is smaller than the minimum length of Reeb chords of Λ,

- the Morse function f has exactly 1 local maximum point at x and 1 local minimum point at y, and
- around the base point $*$, three points $*, x, y$ show up in order when traveling along the link (see Figure 3.1).

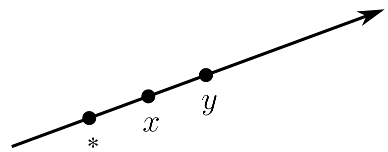


FIGURE 3.1: A neighborhood of the base point $*$ on Λ. The arrow indicates the orientation of Λ.

Decorate $j^1 f$ with a base point in the same location and with the same orientation as Λ. Now $\Lambda \cup j^1 f$ is a 2-copy of Λ, denoted by $\Lambda^{(2)}$. Label $\Lambda^{(2)}$ from top (higher z coordinate) to bottom (lower z coordinate) by Λ^1 and Λ^2. An example of the 2-copy of the trefoil with a single base point is shown in Figure 3.2.

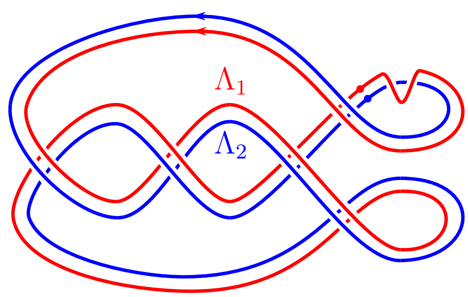


FIGURE 3.2: The Lagrangian projection of a 2-copy of the trefoil with a single base point.

The Legendrian contact homology DGA $(\mathcal{A}(\Lambda^{(2)}), \partial^{(2)})$ of $\Lambda^{(2)}$ can be recovered from the data carried by the DGA $(\mathcal{A}(\Lambda), \partial)$ of Λ. Recall that $\mathcal{A}(\Lambda)$ is generated the set $\mathcal{R}$ of Reeb chords $\{a_1, \dots, a_m\}$ and the set $\mathcal{T} = \{t, t^{-1}\}$ that corresponds to the base point as stated in Section 2.2.2. Similarly, divide the set of generators of

$\mathcal{A}(\Lambda^{(2)})$ into two parts $\mathcal{R}^{(2)} \cup \mathcal{T}^{(2)}$. It is obvious that $\Lambda^{(2)}$ has two base points and thus write $\mathcal{T}^{(2)}$ as $\{(t^1)^{\pm 1}, (t^2)^{\pm 1}\}$. As for the set of Reeb chords $\mathcal{R}^{(2)}$, we divide it into four parts $\mathcal{R}^{(2)} = \bigcup_{i,j=1,2} \mathcal{R}^{ij}$, where $\mathcal{R}^{ij}$ is the set of Reeb chords to Λ_i from Λ_j. Observe that Reeb chords of $\Lambda^{(2)}$ come from two sources.

- Each critical point x or y of the Morse function f gives one Reeb chord in $\mathcal{R}^{12}$, denoted by x^{12} and y^{12} respectively. We call this type of Reeb chords **Morse Reeb chords**.

- Each Reeb chord a_l of Λ gives four Reeb chords of $\Lambda^{(2)}$, denoted by $a_l^{ij} \in \mathcal{R}^{ij}$, where $i, j = 1, 2$ and $l = 1, \dots, m$. We call these Reeb chords **non-Morse Reeb chords**.

It is obvious that a^{ii} and t^i, for $i = 1, 2$, inherit the grading from a and t in $\mathcal{A}(\Lambda)$ respectively. We can choose a family of capping paths such that $|a^{ij}| = |a|$ for any Reeb chord a of Λ. Under this choice of capping paths γ, one can show that $CZ(\gamma_{x^{12}}) = Ind_f(x)$ for any Morse Reeb chord x^{12} through a similar computation as in [EES09]. Hence we have $|x^{12}| = 0$ and $|y^{12}| = -1$.

In order to describe the differential $\partial^{(2)}$, we encode the generators in matrices. Let A_l, X_k, Y_k, Δ_k, for $1 \leqslant l \leqslant m$, be 2×2 matrices:

$$A_l = \begin{pmatrix} a_l^{11} & a_l^{12} \\ a_l^{21} & a_l^{22} \end{pmatrix}, \quad X = \begin{pmatrix} 1 & x^{12} \\ 0 & 1 \end{pmatrix}, \quad Y = \begin{pmatrix} 0 & y^{12} \\ 0 & 0 \end{pmatrix}, \quad \Delta = \begin{pmatrix} t^1 & 0 \\ 0 & t^2 \end{pmatrix}.$$

The differential $\partial^{(2)}$ is defined on generators as follows by applying entry-by-entry to these matrices:

$$\begin{aligned} \partial^{(2)} A_l &= \Phi(\partial a_l) + Y A_l - (-1)^{|a_l|} A_l Y \\ \partial^{(2)} X &= \Delta^{-1} Y \Delta X - XY \\ \partial^{(2)} Y &= Y^2 \\ \partial^{(2)} \Delta &= 0, \end{aligned}$$

where $\Phi : \mathcal{A} \to Mat(2, \mathcal{A}(\Lambda^{(2)}))$ is a ring homomorphism given by $\Phi(a_l) = A_l$ and $\Phi(t) = \Delta X$.

Given two augmentations ϵ^1 and ϵ^2 of $(\mathcal{A}, \partial)$, we get an augmentation ϵ of $(\Lambda^{(2)}, \partial^{(2)})$ by sending $a_l^{ii} \mapsto \epsilon^i(a_l)$, $t^{ii} \mapsto \epsilon^i(t)$ and everything else to 0. Thus $\partial_\epsilon^{(2)} = \phi_\epsilon \circ \partial^{(2)} \circ \phi_\epsilon^{-1}$ is a differential of $\mathcal{A}^{(2)} = \mathcal{A}(\Lambda^{(2)})/(t^{ii} = \epsilon(t^{ii}))$. Both the morphism $Hom_+(\epsilon_1, \epsilon_2)$ and the first A_∞ operation m_1 are defined from $(\mathcal{A}^{(2)}, \partial_\epsilon^{(2)})$ through the construction stated in Section 3.1. For $i, j = 1, 2$, let C^{ij} denote the free graded $\mathbb{F}$ algebra generated by $\mathcal{R}^{ij}$, which is a sub algebra of $\mathcal{A}^{(2)}$. Notice that C^{12} and C^{21} are closed under $\partial_\epsilon^{(2)}$ since ϵ vanishes on the components in C^{11} and C^{22} of the image of $\partial_\epsilon^{(2)}$. Hence C^{12} and C^{21} are sub chain complexes of $(\mathcal{A}^{(2)}, \partial_\epsilon^{(2)})$. Define the morphism $Hom_+(\epsilon_1, \epsilon_2)$ between objects ϵ_1 and ϵ_2 to be $(C^{12})^\vee$. To simplify the notation, we write $(a_l^{12})^\vee$ as $a_l^\vee$, $(x^{12})^\vee$ as $x^\vee$ and $(y^{12})^\vee$ as $y^\vee$. Therefore, their gradings satisfy $|a_l^\vee| = |a_l| + 1$, $|x^\vee| = 1$ and $|y^\vee| = 0$. The first A_∞ operation m_1 is defined by the adjoint of $\partial_\epsilon^{(2)}$, i.e., for any Reeb chord $c \in \mathcal{R}$,

$$m_1(c^\vee) = \sum_{a \in \mathcal{R}} \text{Coefficient}_c(\partial_\epsilon^{(2)} a) a^\vee.$$

As we noted before, m_1 is a differential for $Hom_+(\epsilon_1, \epsilon_2)$. The corresponding cohomology is denoted by $H^* Hom_+(\epsilon_1, \epsilon_2)$. Similarly, define $(C^{21})^\vee$ to be $Hom_-(\epsilon_2, \epsilon_1)$. Take the cohomology of $Hom_-(\epsilon_2, \epsilon_1)$ with respect to m_1, denoted by $H^* Hom_-(\epsilon_2, \epsilon_1)$.

Remark 3.2.1. One may find the convention of $Hom_-(\epsilon_2, \epsilon_1)$ not natural. However, the notations are consistent in the sense that both $Hom_+(\epsilon, \epsilon')$ and $Hom_-(\epsilon, \epsilon')$ are generated by Reeb chords from the component with the augmentation ϵ' to the component with the augmentation ϵ.

The $Hom_+(\epsilon_1, \epsilon_2)$ space and the $Hom_-(\epsilon_1, \epsilon_2)$ space are closely related. Recall that the generators of $Hom_+(\epsilon_1, \epsilon_2)$ naturally correspond to the Reeb chords in $\mathcal{R}^{12}$, which consist of non-Morse Reeb chords and Morse Reeb chords. Note that the lengths of Morse Reeb chords are smaller than the lengths of non-Morse Reeb chords. Due to the positive energy constraint, there does not exist any holomorphic disk that has a positive puncture at a Morse Reeb chord and a negative puncture at a non-Morse Reeb chord. Therefore, the graded sub-vector space of $Hom_+(\epsilon_1, \epsilon_2)$ generated by non-Morse Reeb chords is closed under m_1, and thus is a sub-chain complex. Indeed, this sub-chain complex agrees with $(Hom_-(\epsilon_1, \epsilon_2), m_1)$. From [Lev], for a Legendrian knot Λ with a single base point, any two augmentations ϵ_1, and ϵ_2 agree on the generator t that corresponds to the base point. As a result, by [NRS$^+$15, Proposition 5.2], the quotient chain complex that is generated by $\{x^\vee, y^\vee\}$ is the Morse co-chain complex induced by the Morse function f. Therefore following long exact sequence holds:

$$\cdots \to H^{i-1}(\Lambda) \to H^i Hom_-(\epsilon_1, \epsilon_2) \to H^i Hom_+(\epsilon_1, \epsilon_2) \to H^i(\Lambda) \to \cdots . \qquad (3.1)$$

Furthermore, given that both $Hom_+(\epsilon_1, \epsilon_2)$ and $Hom_-(\epsilon_1, \epsilon_2)$ are vector spaces over the field $\mathbb{F}$, combining the Universal Coefficient Theorem with Sabloff Duality in [NRS$^+$15, Section 5.1.2], we have

$$H^k Hom_-(\epsilon_1, \epsilon_2) \cong H^{-k}(Hom_-(\epsilon_1, \epsilon_2)^\dagger) \cong H^{2-k} Hom_+(\epsilon_2, \epsilon_1). \qquad (3.2)$$

For a chain complex C, the chain complex $C^\dagger$ is obtained by dualizing the underlying vector space and differential of C and then negating the gradings.

For the other A_∞ operators m_n, one needs to consider an n-copy of Λ, denoted by $\Lambda^{(n)}$. Construct a DGA $(\mathcal{A}^n, \partial_\epsilon^{(n)})$ of $\Lambda^{(n)}$ that is analogous to $(\mathcal{A}^2, \partial_\epsilon^{(2)})$. Define m_n to be the adjoint of ∂_ϵ^n as in Example 3.1.2. See [NRS$^+$15] for more details.

By [NRS$^+$15], the augmentation category described above does not depend on the choice of the Morse function f. Moreover, up to A_∞ category equivalence, the augmentation category is invariant of Legendrian knot under Legendrian isotopy.

A key property of $\mathcal{A}ug_+(\Lambda)$ is that $\mathcal{A}ug_+(\Lambda)$ is a **strictly unital** A_∞ category, with the **units** given by

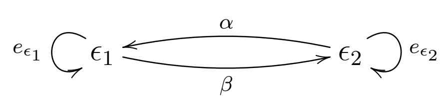
$$e_\epsilon = -y^\vee \in Hom_+(\epsilon, \epsilon),$$

i.e.,

- $m_1(e_\epsilon) = 0$;
- for any ϵ_1, ϵ_2 and any $c \in Hom_+(\epsilon_1, \epsilon_2)$, $m_2(c, e_{\epsilon_1}) = m_2(e_{\epsilon_2}, c) = c$;
- any higher composition involving e_ϵ is 0.

As a result, the corresponding cohomology category $H^*\mathcal{A}ug_+(\Lambda)$ is a unital category, which makes it natural to talk about the equivalence relation of objects in $\mathcal{A}ug_+(\Lambda)$.

Definition 3.2.2. Two objects ϵ_1 and ϵ_2 are **equivalent** in $\mathcal{A}ug_+(\Lambda)$ if they are isomorphic in $H^*\mathcal{A}ug_+(\Lambda)$, i.e. there exist $[\alpha] \in H^0Hom_+(\epsilon_1, \epsilon_2)$ and $[\beta] \in H^0Hom_+(\epsilon_2, \epsilon_1)$ such that $m_2([\alpha], [\beta]) = [e_{\epsilon_2}] \in H^0Hom_+(\epsilon_2, \epsilon_2)$ and $m_2([\beta], [\alpha]) = [e_{\epsilon_1}] \in H^0Hom_+(\epsilon_1, \epsilon_1)$.



By [NRS+15], for a Legendrian knot with a single base point, two augmentations are equivalent if and only if they are isomorphic as DGA maps.

Suppose Σ is a connected exact Lagrangian cobordism from a Legendrian knot Λ_- to a Legendrian knot Λ_+. It induces a DGA map ϕ from the DGA $(\mathcal{A}(\Lambda_+), \partial)$ with a single base point to a DGA $(\mathcal{A}(\Lambda_-), \partial)$ with a single base point as we described in Section 2.3.2. By [NRS+15, Proposition 3.29], this *DGA* map ϕ induces a unital A_∞ category morphism f from $\mathcal{A}ug_+(\Lambda_-)$ to $\mathcal{A}ug_+(\Lambda_+)$. The category map sends an augmentation ϵ_- of Λ_- to $\epsilon_+ = \epsilon_- \circ \phi$, which is an augmentation of Λ_+. The family of maps $\{f_n\}$ is constructed through a family of DGA morphisms of n-copies:

$$\begin{array}{rcl} f^{(n)} : (\mathcal{A}^{(n)}(\Lambda_+), \partial^{(n)}) & \mapsto & (\mathcal{A}^{(n)}(\Lambda_-), \partial^{(n)}) \\ \Delta & \mapsto & \Delta \\ Y & \mapsto & Y \\ X & \mapsto & \Delta^{-1} \cdot \Phi_- \circ f(t) \\ \Phi_+(a) & \mapsto & \Phi_- \circ f(a), \qquad\qquad a \in \mathcal{A}(\Lambda_+). \end{array}$$

Let ϵ_- be the augmentation of $(\mathcal{A}^{(n)}(\Lambda_+)), \partial^{(n)})$ that sends $a_l^{ii} \mapsto \epsilon_-^i(a_l)$, $t^{ii} \mapsto \epsilon_-^i(t)$ and everything else to 0. Define the map f_{n-1} to be the adjoint of $f_{\epsilon_-}^{(n)}$, where

$$f_{\epsilon_-}^{(n)} = \phi_{\epsilon_-} \circ f^{(n)} \circ \phi_{\epsilon_-}^{-1}.$$

In particular, f_1 can be written as

$$\begin{array}{rcl} f_1 : Hom_+(\epsilon_-^1, \epsilon_-^2) & \to & Hom_+(\epsilon_+^1, \epsilon_+^2) \\ y_-^\vee & \mapsto & y_+^\vee \\ c^\vee & \mapsto & \sum\limits_{a \in \mathcal{A}(\Lambda_+)} \text{Coeff}_c(f_{\epsilon_-}^{(2)}(a))a^\vee, \qquad\qquad c \in \mathcal{A}(\Lambda_-) \\ x_-^\vee & \mapsto & x_+^\vee + \sum\limits_{a \in \mathcal{A}(\Lambda_+)} \text{Coeff}_t(f_{\epsilon_-}^{(2)}(a))a^\vee. \end{array} \tag{3.3}$$

When computing $\text{Coeff}_b(f_{\epsilon_-}^{(2)}(a))$, where b is either a Reeb chord $c \in \mathcal{A}(\Lambda_-)$ or $t \in \mathcal{T}$, one consider all the terms of $f(a)$ including b. If a term of $f(a)$ including b can be written as $\mathbf{p}b\mathbf{q}$, where $\mathbf{p}$ and $\mathbf{q}$ are words of pure Reeb chords of Λ_-, this term contributes $\text{Coeff}_{\mathbf{p}b\mathbf{q}}(f(a))\epsilon_-^1(\mathbf{p})\epsilon_-^2(\mathbf{q})$ to $\text{Coeff}_b(f_{\epsilon_-}^{(2)}(a))$. Therefore we have

$$\text{Coeff}_b(f_{\epsilon_-}^{(2)}(a)) = \sum_{\mathbf{p}\ \mathbf{q}} \text{Coeff}_{\mathbf{p}b\mathbf{q}}(f(a))\epsilon_-^1(\mathbf{p})\epsilon_-^2(\mathbf{q}).$$

Remark 3.2.3. According to [NRS$^+$15, Proposition 3.29], the condition for a DGA map to induce a unital A_∞ category morphism is that the DGA map is compatible with the weak link gradings in the sense of [NRS$^+$15, Definition 3.19]. In our case where both Λ_+ and Λ_- are single component Legendrian knots with a single base point, this condition is trivially satisfied.

4

Floer theory for Lagrangian Cobordisms

In this chapter, we give a brief introduction to the Floer theory of a pair of exact Lagrangian cobordisms following [CDRGG15]. Let Σ^i, for $i = 1, 2$, be exact Lagrangian cobordisms from Λ^i_- to Λ^i_+ in $\big(\mathbb{R} \times \mathbb{R}^3, d(e^t\alpha)\big)$, where $\alpha = dz - ydx$. A schematic picture is shown in Figure 4.1. The union of the cobordisms $\Sigma^1 \cup \Sigma^2$ is cylindrical over $\Lambda^1_+ \cup \Lambda^2_+$ (resp. $\Lambda^1_- \cup \Lambda^2_-$) on the positive end (resp. negative end). If we view $\Sigma^1 \cup \Sigma^2$ as a Lagrangian cobordism from the Legendrian link $\Lambda^1_- \cup \Lambda^2_-$ to the Legendrian link $\Lambda^1_+ \cup \Lambda^2_+$, we obtain a chain complex generated by Reeb chords of $\Lambda^1_- \cup \Lambda^2_-$ and $\Lambda^1_+ \cup \Lambda^2_+$. On the other hand, if we lift the exact Lagrangian cobordism $\Sigma^1 \cup \Sigma^2$ to be a Legendrian manifold in $\mathbb{R} \times \mathbb{R}^3 \times \mathbb{R}$, we have its Legendrian contact homology DGA, which is generated by double points of $\Sigma^1 \cup \Sigma^2$. One can construct the Cthulhu chain complex $Cth(\Sigma^1, \Sigma^2)$ as a mix of the two chain complexes above. It is generated by some Reeb chords on the cylindrical ends and intersection points of Σ^1 and Σ^2. Moreover, this chain complex has trivial cohomology, i.e. $H^*Cth(\Sigma^1, \Sigma^2) = 0$.

For the simplicity in defining grading, we assume that Σ^i, for $i = 1, 2$, has trivial Maslov number throughout this dissertation.

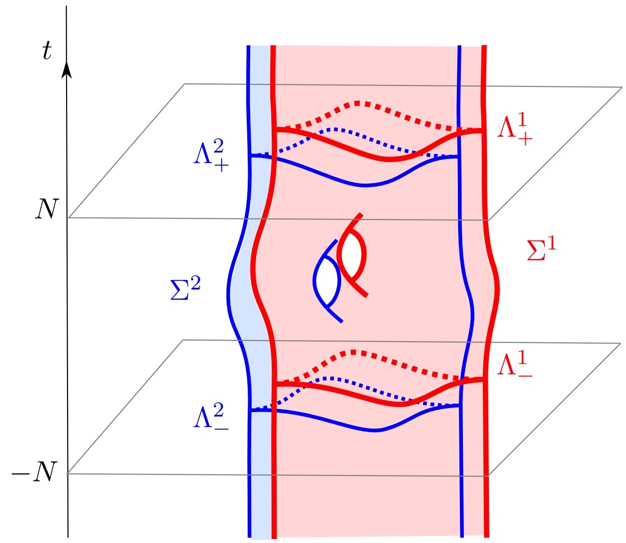


FIGURE 4.1: Pair of Lagrangian cobordisms in $\big(\mathbb{R} \times \mathbb{R}^3, d(e^t\alpha)\big)$.

4.1 The graded vector space

Assume that both Σ^1 and Σ^2 are cylindrical outside of $[-N, N] \times \mathbb{R}^3$, where N is a positive number. The underlying vector space is a direct sum of three parts:

$$Cth(\Sigma^1, \Sigma^2) = C(\Lambda^1_+, \Lambda^2_+) \oplus CF(\Sigma^1, \Sigma^2) \oplus C(\Lambda^1_-, \Lambda^2_-).$$

The top level $C(\Lambda^1_+, \Lambda^2_+)$ (resp. bottom level $C(\Lambda^1_-, \Lambda^2_-)$) is an $\mathbb{F}$-module generated by Reeb chords to Λ^1_+ (resp. Λ^1_-) from Λ^2_+ (resp. Λ^2_-) that are lying on the slice of $t = N$ (resp. $t = -N$). The middle level $CF(\Sigma^1, \Sigma^2)$ is an $\mathbb{F}$-module generated by intersection points of Σ^1 and Σ^2, which are all contained in $(-N, N) \times \mathbb{R}^3$.

Grading. To define the degree, first fix a capping path γ_c for each generator c. For a Reeb chord a in $C(\Lambda^1_+, \Lambda^2_+)$ or $C(\Lambda^1_-, \Lambda^2_-)$, define the degree

$$|a| = CZ(\gamma_a) - 1,$$

which matches the definition of degree when viewing a as a generator in the Legendrian contact homology DGA of $\Lambda^1_+ \cup \Lambda^2_+$ or $\Lambda^1_- \cup \Lambda^2_-$. For an intersection point

$x \in CF(\Sigma^1, \Sigma^2)$, define the degree

$$|x| = CZ(\gamma_x),$$

following [Sei08]. One can also see [CDRGG15, Section 4.2] for details. Note that for a Reeb chord in $C(\Lambda^1_+, \Lambda^2_+)$, its degree in $Cth(\Sigma^1, \Sigma^2)$ will not be necessarily coincide with its degree in $C(\Lambda^1_+, \Lambda^2_+)$. It is shifted as we will see later.

Action. For $i = 1, 2$, suppose g_i is a primitive of the exact Lagrangian cobordism Σ^i, and hence g_i is constant when $t < -N$ or $t > N$. Note that primitive functions are well defined up to a overall shift by a constant number. Thus we may assume that the primitives g_i are both zero on $\Sigma^i \cup \big((-\infty, -N) \times \mathbb{R}^3\big)$ for $i = 1, 2$. The action of generators is defined under this choice of primitives.

For Reeb chords $a^+ \in C(\Lambda^1_+, \Lambda^2_+)$ and $a^- \in C(\Lambda^1_-, \Lambda^2_-)$, define the **action** $\mathfrak{a}$ by

$$\mathfrak{a}(a^+) = g_2(a^+) - g_1(a^+) + \int_{a^+} e^N \alpha$$

and

$$\mathfrak{a}(a^-) = g_2(a^-) - g_1(a^-) + \int_{a^-} e^{-N} \alpha = \int_{a^-} e^{-N} \alpha.$$

The last part is due to the special choice of primitives. For double points x of $\Sigma^1 \cup \Sigma^2$, the action $\mathfrak{a}(x)$ is defined by $\mathfrak{a}(x) = g_2(x) - g_1(x)$.

4.2 The differential

Remark 4.2.1. Throughout this dissertation, we restrict ourselves to the case where all intersection generators have positive actions since that is the case for the special pair of cobordisms constructed in Section 5.1. In general, the differential could include one more map from $CF(\Sigma^1, \Sigma^2)$ to $C(\Lambda^1_-, \Lambda^2_-)$, which is called the Nessie map. However, by [CDRGG15, Proposition 9.1], the positive energy condition of the holomorphic disks counted by the Nessie map requires the corresponding intersections in $CF(\Sigma^1, \Sigma^2)$ to have negative actions. Therefore, in our special case, we can exclude the Nessie map and get the differential as a upper triangle as below.

With the assumption in Remark 4.2.1, we define the differential under the decomposition

$$Cth(\Sigma^1, \Sigma^2) = C(\Lambda^1_+, \Lambda^2_+) \oplus CF(\Sigma^1, \Sigma^2) \oplus C(\Lambda^1_-, \Lambda^2_-)$$

by a degree 1 map of the form

$$d = \begin{pmatrix} d_{++} & d_{+0} & d_{+-} \\ 0 & d_{00} & d_{0-} \\ 0 & 0 & d_{--} \end{pmatrix}.$$

To describe the differential explicitly, we need to study the holomorphic disks with boundary on $\Sigma^1 \cup \Sigma^2$. Fix a generic domain dependent almost complex structure J that is compatible with the symplectic form on $\mathbb{R} \times \mathbb{R}^3$ and the cylindrical ends in the sense of [CDRGG15, Section 3.1.5]. Suppose that the induced cylindrical almost complex structure on the positive end $(\Sigma^1 \cup \Sigma^2) \cap \big([N, \infty) \times \mathbb{R}^3\big)$ (resp. the negative end $(\Sigma^1 \cup \Sigma^2) \cap \big((-\infty, -N] \times \mathbb{R}^3)\big)$ is J_+ (resp. J_-). The differential $d_{\pm\pm}$ of $C(\Lambda^1_\pm, \Lambda^2_\pm)$ counts rigid $J_\pm$-holomorphic disks with boundary on $\mathbb{R} \times (\Lambda^1_\pm \cup \Lambda^2_\pm)$, respectively, as described in Section 2.2. The corresponding moduli space is denoted by $\mathcal{M}_{J_\pm}(a^\pm; \mathbf{p}^\pm, b^\pm, \mathbf{q}^\pm)$, where $a^\pm$ and $b^\pm$ are Reeb chords to $\Lambda^1_\pm$ from $\Lambda^2_\pm$ while $\mathbf{p}^\pm$ and $\mathbf{q}^\pm$ are words of pure Reeb chords of $\Lambda^1_\pm$ and $\Lambda^2_\pm$, respectively. We also write $\widetilde{\mathcal{M}}_{J_\pm}(a^\pm; \mathbf{p}^\pm, b^\pm, \mathbf{q}^\pm)$ as the moduli space $\mathcal{M}_{J_\pm}(a^\pm; \mathbf{p}^\pm, b^\pm, \mathbf{q}^\pm)$ module the action of $\mathbb{R}$ in the t direction.

For the remaining maps in the differential, we need to describe J-holomorphic disks with boundary on $\Sigma^1 \cup \Sigma^2$, where J is the chosen domain dependent almost complex structure. The punctures of these J-holomorphic disks can be either Reeb chords or intersection points. For generators $c_0, c_1, \ldots, c_m$ in $Cth(\Sigma^1, \Sigma^2)$, let

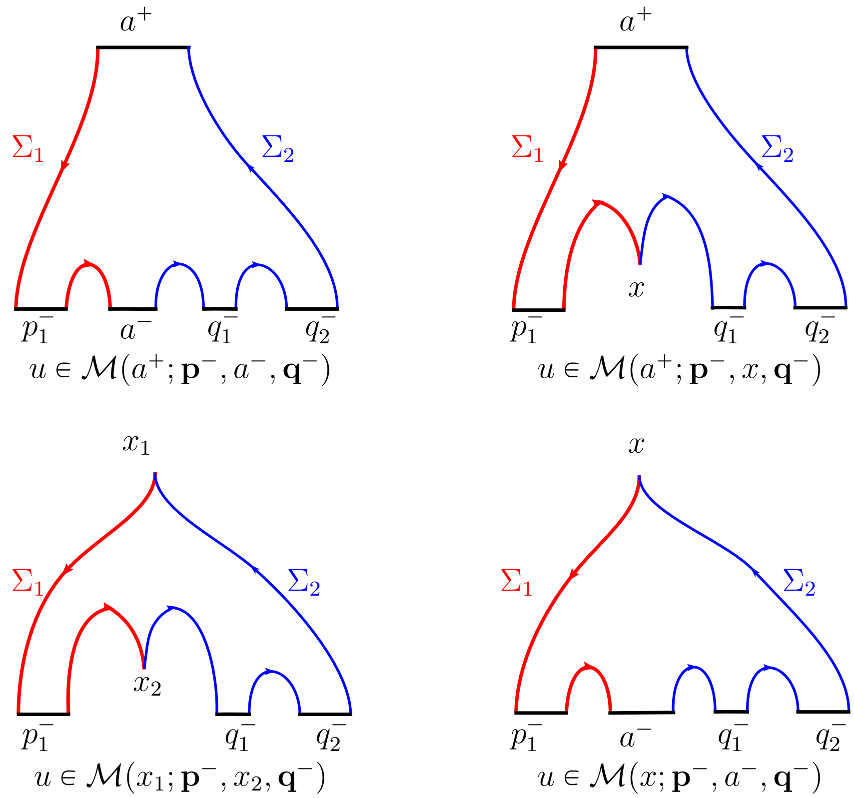


FIGURE 4.2: A sketch of the J-holomorphic disks in the differential d. Here $a^{\pm}$ are Reeb chords to $\Lambda^1_{\pm}$ from $\Lambda^2_{\pm}$, respectively, and x, x_1, x_2 are double points of $\Sigma^1 \cup \Sigma^2$. In these examples, $\mathbf{p}^-$ is a word of one pure Reeb chord p_1^- of Λ^1_- while $\mathbf{q}^-$ is a word of two pure Reeb chords $q_1^- q_2^-$ of Λ^2_-. The arrows denote the orientation inherited from the boundary of the unit disk.

$\mathcal{M}_J(c_0; c_1, \ldots, c_m)$ denote the moduli space of the J-holomorphic disks:

$$u : (D_{m+1}, \partial D_{m+1}) \to (\mathbb{R} \times \mathbb{R}^3, \Sigma^1 \cup \Sigma^2)$$

with the following properties:

- D_{m+1} is a 2-dimensional unit disk with $m+1$ boundary points $q_0, q_1, \ldots, q_m$ removed and the points $q_0, q_1, \ldots, q_m$ are arranged in a counterclockwise order.
- If c_0 is a Reeb chord, the image of u is asymptotic to $[N, \infty) \times c_0$ near q_0. If c_0

is an intersection point, then $\lim_{z \to q_0} u(z) = c_0$ and u maps the incoming segment (resp. outgoing segment) of the boundary to Σ^2 (resp. Σ^1).

- If, for $i > 0$, c_i is a Reeb chord, the image of u is asymptotic to $(-\infty, -N] \times c_i$ near q_i. If c_i is an intersection point, then $\lim_{z \to q_i} u(z) = c_i$ and u maps the incoming segment (resp. outgoing segment) of the boundary to Σ^1 (resp. Σ^2).

The four types of Moduli spaces used in the differential d of the Cthulhu chain complex are shown in Figure 4.2. For any one of these four moduli spaces $\mathcal{M}$, we say a disk $u \in \mathcal{M}$ is **rigid** if $\dim \mathcal{M} = 0$. All the moduli spaces of holomorphic disks introduced above admit a coherent orientation since both Σ^1 and Σ^2 are spin. Therefore, one can associate each rigid holomorphic disk $u \in \mathcal{M}$ with a sign and thus can count the number of rigid holomorphic disks in $\mathcal{M}$ with sign.

Let $a_i^{\pm}$'s be Reeb chords to $\Lambda^1_{\pm}$ from $\Lambda^2_{\pm}$ and x_i's be double points of $\Sigma^1 \cup \Sigma^2$. The bold letters $\mathbf{p}^{\pm}$, $\mathbf{q}^{\pm}$ are words of pure Reeb chords of $\Lambda^1_{\pm}$ and $\Lambda^2_{\pm}$, respectively. Here we assume that, for $i = 1, 2$, ϵ^i_- is an augmentation of $\mathcal{A}(\Lambda^i_-)$ and ϵ^i_+ is the augmentation of $\mathcal{A}(\Lambda^i_+)$ induced by Σ_i. The differential is defined as follows:

$$d_{++}(a_i^+) = \sum_{\dim \widetilde{\mathcal{M}}_{J_+}(a_j^+;\mathbf{p}^+,a_i^+,\mathbf{q}^+)=0} |\widetilde{\mathcal{M}}_{J_+}(a_j^+;\mathbf{p}^+,a_i^+,\mathbf{q}^+)|\epsilon_+^1(\mathbf{p}^+)\epsilon_+^2(\mathbf{q}^+)a_j^+;$$

$$d_{--}(a_i^-) = \sum_{\dim \widetilde{\mathcal{M}}_{J_-}(a_j^-;\mathbf{p}^-,a_i^-,\mathbf{q}^-)=0} |\widetilde{\mathcal{M}}_{J_-}(a_j^-;\mathbf{p}^-,a_i^-,\mathbf{q}^-)|\epsilon_-^1(\mathbf{p}^-)\epsilon_-^2(\mathbf{q}^-)a_j^-;$$

$$d_{00}(x_i) = \sum_{\dim \mathcal{M}_J(x_j;\mathbf{p}^-,x_i,\mathbf{q}^-)=0} |\mathcal{M}_J(x_j;\mathbf{p}^-,x_i,\mathbf{q}^-)|\epsilon_-^1(\mathbf{p}^-)\epsilon_-^2(\mathbf{q}^-)x_j;$$

$$d_{0-}(a_i^-) = \sum_{\dim \mathcal{M}_J(x_j;\mathbf{p}^-,a_i^-,\mathbf{q}^-)=0} |\mathcal{M}_J(x_j;\mathbf{p}^-,a_i^-,\mathbf{q}^-)|\epsilon_-^1(\mathbf{p}^-)\epsilon_-^2(\mathbf{q}^-)x_j;$$

$$d_{+0}(x_i) = \sum_{\dim \mathcal{M}_J(a_j^+;\mathbf{p}^-,x_i,\mathbf{q}^-)=0} |\mathcal{M}_J(a_j^+;\mathbf{p}^-,x_i,\mathbf{q}^-)|\epsilon_-^1(\mathbf{p}^-)\epsilon_-^2(\mathbf{q}^-)a_j^+;$$

$$d_{+-}(a_i^-) = \sum_{\dim \mathcal{M}_J(a_j^+;\mathbf{p}^-,a_i^-,\mathbf{q}^-)=0} |\mathcal{M}_J(a_j^+;\mathbf{p}^-,a_i^-,\mathbf{q}^-)|\epsilon_-^1(\mathbf{p}^-)\epsilon_-^2(\mathbf{q}^-)a_j^+,$$

where $|\mathcal{M}|$ denotes the number of rigid holomorphic disks in the moduli space $\mathcal{M}$ counted with sign. Note that the definition of differential depends on the choice of augmentations ϵ_1^- and ϵ_2^-, whose existence are essential to the Floer theory.

A holomorphic disk counted by the differential must satisfy the rigidity condition and the positive energy condition. We will describe these conditions in details.

The Rigidity Condition. Let us interpret the condition $\dim \mathcal{M}_J(c_1;\mathbf{p},c_2,\mathbf{q}) = 0$ in terms of $|c_i|$ for $i=1,2$, where c_i can be either a Reeb chord or an intersection point while $\mathbf{p}$ and $\mathbf{q}$ are words of pure Reeb chords in degree 0. Instead of deriving a formula for the dimension of a moduli space, we use the idea of the wrapped Floer homology to find the relation between $|c_1|$ and $|c_2|$. Recall that both Σ^1 and Σ^2 are cylindrical outside of $[-N,N]\times\mathbb{R}^3$. Consider a non-decreasing function $\sigma(t):\mathbb{R}_{\geqslant 0}\to\mathbb{R}_{\geqslant 0}$ such that $\sigma'(t)=0$ when $t\leqslant N$ and $\sigma'(t)=1$ when $t\geqslant N'$, where N' is a number bigger than N. Note that $X_H=-\sigma(|t|)\partial_z$ is a Hamiltonian vector field with its time-s flow denoted by Φ_H^s. Flow Σ^1 through X_H and get a new cobordism $\Phi^s_H(\Sigma^1)$, which is another exact Lagrangian cobordism according to Section 5.1. Observe that $\Phi^s_H(\Sigma^1)$ wraps Σ^1 on both ends in the negative Reed chord direction. Hence for a large enough number s, each Reeb chord c to Λ^1_+ (resp. Λ^1_-) from Λ^2_+ (resp. Λ^2_-) corresponds to a transversally double point $\check{c}$ of $\Phi^s_H(\Sigma^1) \cup \Sigma^2$ in $N < t < N'$ (resp. $-N' < t < -N$). Moreover, if c is a Reeb chord in $C(\Lambda^1_+, \Lambda^2_+)$, we have

$$|\check{c}| = CZ(\gamma_{\check{c}}) = CZ(\gamma_c) + 1 = |c| + 2.$$

If c is Reeb chord in $C(\Lambda^1_-, \Lambda^2_-)$,

$$|\check{c}| = CZ(\gamma_{\check{c}}) = CZ(\gamma_c) = |c| + 1.$$

Each double point x of $\Sigma^1 \cup \Sigma^2$ naturally corresponds to a double point $\check{x}$ of $\Phi^s_H(\Sigma^1) \cup \Sigma^2$ in $-N < t < N$ with gradings satisfying $|\check{x}| = |x|$.

Remark 4.2.2. The difference in grading correspondence between Reeb chords in $C(\Lambda^1_+, \Lambda^2_+)$ and Reeb chords in $C(\Lambda^1_-, \Lambda^2_-)$ can be understood better in a special case where Σ^1 is a push off of Σ^2 through a positive Morse function $F : \Sigma^2 \to \mathbb{R}_{>0}$. In other words, in a Weinstein neighborhood of Σ^2, the cobordism Σ^1 is the graph of dF for some positive Morse function $F : \Sigma^2 \to \mathbb{R}_{>0}$. In this case, the cobordism $\Phi^s_h(\Sigma^1)$ is a push off of Σ^2 through another Morse function $\tilde{F}$ as well. By the canonical Floer theory [Flo88], we can choose a family of capping paths so that $CZ(\gamma_x) = Ind_{\tilde{F}}(x)$ for any intersection point x of $\Phi^s_h(\Sigma^1)$ and Σ^2. Similarly, for any Morse Reeb chord c in $\Lambda^1_+ \cup \Lambda^2_+$, we can further require that $Ind_{f_+}(c) = CZ(\gamma_c)$, where $f_+ = F\big|_{\Lambda^2_+}$. Notice that $Ind_{\tilde{F}}(\check{c}) = Ind_{f_+}(c) + 1$. Therefore

$$|\check{c}| = CZ(\gamma_{\check{c}}) = Ind_{\tilde{F}}(\check{c}) = Ind_{f_+}(c) + 1 = CZ(\gamma_c) + 1 = |c| + 2.$$

For a Morse Reeb chord c in $\Lambda^1_- \cup \Lambda^2_-$, the indices satisfy $Ind_{\tilde{F}}(\check{c}) = Ind_{f_-}(c)$, where $f_- = F\big|_{\Lambda^2_-}$. Hence $|\check{c}| = CZ(\gamma_{\check{c}}) = Ind_{\tilde{F}}(\check{c}) = Ind_{f_-}(c) = CZ(\gamma_c) = |c| + 1$. A schematic figure is shown in Figure 4.3.

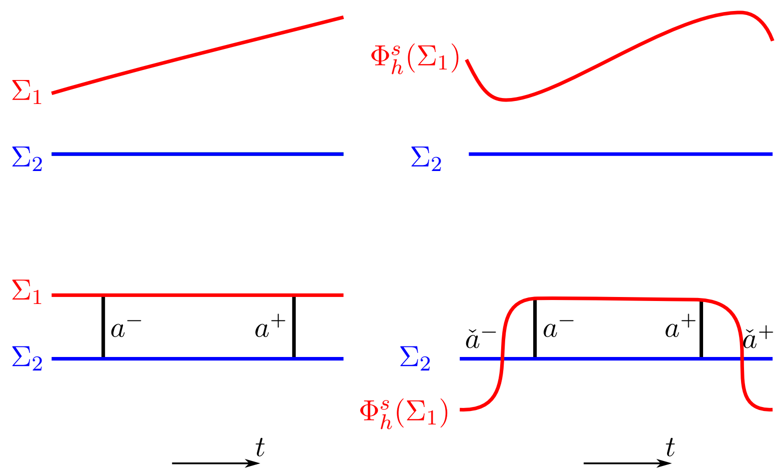


FIGURE 4.3: The top two are front projections while the bottom two are Lagrangian projections. The indices satisfy $|\check{a}^+| = |a^+| + 2$ and $|\check{a}^-| = |a^-| + 1$.

So far, we have shown that generators of $Cth(\Sigma^1, \Sigma^2)$ can be identified with intersection points of $\Phi_H^s(\Sigma^1)$ and Σ^2, which are generators of $Cth(\Phi_H^s(\Sigma^1), \Sigma^2)$. Moreover, by [CDRGG15, Proposition 8.2], the Cthulhu chain complexes $Cth(\Sigma^1, \Sigma^2)$ and $Cth(\Phi_H^s(\Sigma^1), \Sigma^2)$ are identified on the level of complexes as well. Note that the generators of $Cth(\Phi_H^s(\Sigma^1), \Sigma^2)$ do not contain any Reeb chords and hence we have $Cth(\Phi_H^s(\Sigma^1), \Sigma^2) = \big(CF(\Phi_H^s(\Sigma^1), \Sigma^2), d_{00}\big)$. Lift $\Phi_H^s(\Sigma^1) \cup \Sigma^2$ to be a Legendrian submanifold L in $\mathbb{R} \times \mathbb{R}^3 \times \mathbb{R}$. Note that $\big(CF(\Phi_H^s(\Sigma^1), \Sigma^2), d_{00}\big)$ is the dual of the linearized contact homology chain complex of L as introduced in Section 2.2.3. Hence if $\dim \mathcal{M}(c_1; \mathbf{p}, c_2, \mathbf{q}) = 0$, there are rigid holomorphic disks that have a positive puncture at $\check{c}_1$ and a negative puncture at $\check{c}_2$, which implies $|\check{c}_1| - |\check{c}_2| = 1$. We can get the corresponding grading relation between c_1 and c_2. In particular, let $a^{\pm}$ be Reeb chords in $C(\Lambda^1_{\pm}, \Lambda^2_{\pm})$ and x_1,x_2 be intersection points of Σ^1 and Σ^2 while $\mathbf{p}$ and $\mathbf{q}$ are words of pure Reeb chords in degree 0 of Λ^1_- and Λ^2_-, respectively. We have that

- if $\dim \mathcal{M}_J(x_1; \mathbf{p}, x_2, \mathbf{q}) = 0$, then $|c_1| - |c_2| = 1$;

- if $\dim \mathcal{M}_J(x_1; \mathbf{p}, a^-, \mathbf{q}) = 0$, then $|c_1| - |a^-| = 2$;
- if $\dim \mathcal{M}_J(a^+; \mathbf{p}, x_1, \mathbf{q}) = 0$, then $|a^+| - |x_1| = -1$;
- if $\dim \mathcal{M}_J(a^+; \mathbf{p}, a^-, \mathbf{q}) = 0$, then $|a^+| - |a^-| = 0$.

As a result, the Cthulhu chain complex can be written as

$$Cth^k(\Sigma^1, \Sigma^2) = C^{k-2}(\Lambda^1_+, \Lambda^2_+) \oplus CF^k(\Sigma^1, \Sigma^2) \oplus C^{k-1}(\Lambda^1_-, \Lambda^2_-).$$

Under this decomposition, the differential

$$d = \begin{pmatrix} d_{++} & d_{+0} & d_{+-} \\ 0 & d_{00} & d_{0-} \\ 0 & 0 & d_{--} \end{pmatrix}$$

has degree 1 as we expected.

The Positive Energy Condition. We interpret the positive energy condition of a holomorphic disk $u \in \mathcal{M}(c_0; c_1, \dots, c_m)$ in terms of the action of c_i, where $i = 0, \dots, m$. Following [Ekh08], we define the energy $E(u)$ of a holomorphic disk

$$u : (D^2, \partial D^2) \to (\mathbb{R} \times \mathbb{R}^3, \Sigma^1 \cup \Sigma^2),$$

by $E(u) = E_\omega(u) + E_\alpha(u)$, where the ω-energy

$$E_\omega(u) = \int_{u^{-1}([-N,N]\times\mathbb{R}^3)} u^*(\omega) + \int_{u^{-1}((-\infty,-N)\times\mathbb{R}^3)} u^*(e^{-N} d\alpha) + \int_{u^{-1}((N,\infty)\times\mathbb{R}^3)} u^*(e^N d\alpha).$$

Note that Write $u = (t, v)$, where $t : D^2 \to \mathbb{R}$ and $v : D^2 \to \mathbb{R}^3$. Define the α-energy by

$$E_\alpha(u) = \sup_{\phi_-} \left(\int_{u^{-1}((-\infty,-N)\times\mathbb{R}^3)} \phi_-(t)\ dt \wedge (v^*\alpha) \right) + \sup_{\phi_+} \left(\int_{u^{-1}((N,\infty)\times\mathbb{R}^3)} \phi_+(t)\ dt \wedge (v^*\alpha) \right),$$

where ϕ_+ and ϕ_- range over all compact supported smooth functions such that

$$\int_{-\infty}^{-N} \phi_-(t)\ dt = e^{-N} \text{ and } \int_N^\infty \phi_+(t)\ dt = e^N,$$

respectively. By Stokes' Theorem, for any holomorphic disk $u \in \mathcal{M}(c_0; c_1, \ldots, c_m)$, we have

$$E_\omega(u) = \mathfrak{a}(c_0) - \sum_{l=1}^{m} \mathfrak{a}(c_l).$$

A holomorphic disk has positive ω-energy, i.e. $E_\omega(u) > 0$, which implies that $\mathfrak{a}(c_0) > \sum_{l=1}^{m} \mathfrak{a}(c_l)$.

Under the assumption in Remark 4.2.1, the differential of the Cthulhu chain complex d is of the form of an upper triangle. By [CDRGG15, Section 6], we have $d^2 = 0$ and thus d is a differential map. Moreover, from [CDRGG15, Section 8], the induced cohomology $H^*Cth(\Sigma^1, \Sigma^2)$ is an invariant under compactly supported Hamiltonian isotopies. Push Σ^1 along the negative z direction until Σ^1 is far below Σ^2 and then, there is no Reeb chord to Σ^1 from Σ^2 nor intersection point between Σ^1 and Σ^2. It is obvious that the cohomology is trivial, i.e. $H^*Cth(\Sigma^1, \Sigma^2) = 0$.

5

The augmentation category map induced by exact Lagrangian cobordisms

In Section 5.1, we perturb an exact Lagrangian cobordism using a Morse function and obtain a pair of Lagrangian cobordisms. In Section 5.2, we apply the Floer theory to this pair of cobordisms and get a long exact sequence. In Section 5.3, we describe the rigid holomorphic disks counted by d_{+-}, which is a part of the differential map of the Cthulhu chain complex, in terms of holomorphic disks with boundary on Σ and Morse flow lines. This is useful when identifying f_1, i.e. the category map on the level of morphisms, with d_{+-}. In Section 5.4, we extend the method in Section 5.3 to describe the differential d of the Cthulhu chain complex and recover the long exact sequences in [CDRGG15]. Finally, we use the identification in Section 5.3 between f_1 and d_{+-} to prove Theorem 1.1.1 in Section 5.5.

5.1 Construction of the pair of cobordisms

First let us describe the neighborhood of a Lagrangian cobordism. Let Σ be an exact Lagrangian cobordism from Λ_- to Λ_+ in $\big(\mathbb{R} \times \mathbb{R}^3, d(e^t\alpha)\big)$, where Λ_- and Λ_+

are Legendrian links. By the Weinstein Lagrangian neighborhood theorem, there is a symplectomorphism

$$\psi : \text{nbhd}(\Sigma) \subset \big((\mathbb{R}\times\mathbb{R}^3), d(e^t\alpha)\big) \to \big(T^*\Sigma, d\theta\big),$$

where θ is the negative Liouville form $\theta = -\sum p_i dq_i$ of $T^*\Sigma$ with coordinates $((q_1, q_2), (p_1, p_2))$. Specifically, on the ($\pm\infty-$)boundary $\mathbb{R}\times\Lambda_\pm$, the symplectomorphism ψ is given by a composition of two symplectomorphisms $\psi_1\circ\psi_0$. As mentioned before, there is a contactomorphism from a tubular neighborhood of $\Lambda_\pm$ in $\mathbb{R}^3$ to a neighborhood of the zero section of $J^1(\Lambda_\pm)$. Composing with the identity map on $\mathbb{R}_t$, we get a symplectomorphism ψ_0 from the neighborhood of $\mathbb{R}\times\Lambda_\pm$ in $\mathbb{R}\times\mathbb{R}^3$ to $\mathbb{R}\times J^1(\Lambda_\pm)$. The second part ψ_1 is given by

$$\begin{aligned}\psi_1 : \text{nbhd}(\Sigma) \subset \big((\mathbb{R}\times J^1(\Lambda_\pm)), d(e^t\alpha)\big) &\to \big(T^*(\mathbb{R}_{>0}\times\Lambda_\pm), d\theta\big)\\ (t,(q,p,z)) &\mapsto \big((e^t,q),(z,e^tp)\big)\end{aligned}$$

For a Morse function $F : \Sigma \to \mathbb{R}_{\geqslant 0}$ such that the determinant of the Hessian matrix is small enough, the graph of dF is a Lagrangian submanifold in $T^*\Sigma$. Pull it back to $\mathbb{R}\times\mathbb{R}^3$ and denote $\psi^{-1}(\text{graph}(dF))$ by Σ'.

Now we show that Σ' is an exact Lagrangian submanifold as well. Notice that $V_{(\mathbf{q},\mathbf{p})} := dF|_{\mathbf{q}}$ is a Hamiltonian vector field in $T^*\Sigma$ since $\iota_V d\theta = -d\tilde{F}$, where $\tilde{F} = F\circ\pi$ and π is the natural projection $\pi : T^*(\Sigma) \to \Sigma$. In order to extend $\psi_*^{-1}(V)$ to be a Hamiltonian vector field in $\mathbb{R}\times\mathbb{R}^3$, we choose a smoothly cut off function $\gamma : T^*(\Sigma) \to \mathbb{R}$ such that $\gamma(\mathbf{q},\mathbf{p}) = 1$ in a tubular neighborhood of the zero section containing the graph of dF and $\gamma(\mathbf{q},\mathbf{p}) = 0$ outside of a slightly bigger tubular neighborhood of the zero section. Pull the Hamiltonian vector field of $\gamma\cdot\tilde{F}$ back through ψ and extend to a Hamiltonian vector field X_H in $\mathbb{R}\times\mathbb{R}^3$. For a suitable neighborhood of Σ in $\mathbb{R}\times\mathbb{R}^3$, we have

$$\iota_{X_H} d(e^t\alpha)\Big|_{nbhd(\Sigma)} = \psi^*(\iota_V d\theta) = \psi^*(-d\tilde{F}) = d(-\tilde{F}\circ\psi)\Big|_{nbhd(\Sigma)}.$$

Hence its Hamiltonian $H = -\tilde{F} \circ \psi$ around Σ. Denote the time s flow of X_H by ϕ_H^s and thus $\Sigma' = \phi_H^1(\Sigma)$. We can compute the 1-form on Σ':

$$\begin{aligned} {\phi_H^1}^* e^t\alpha &= e^t\alpha + \int_0^1 \frac{d}{ds}\, {\phi_H^s}^*(e^t\alpha)\; ds \\ &= e^t\alpha + \int_0^1 {\phi_H^s}^*(\iota_{X_H} d(e^t\alpha) + d(\iota_{X_H} e^t\alpha))\; ds \\ &= e^t\alpha + \int_0^1 {\phi_H^s}^*(dH + d(e^t\alpha(X_H)))\; ds \\ &= e^t\alpha + d(\int_0^1 (H + e^t\alpha(X_H)) \circ \phi_H^s\; ds). \end{aligned} \tag{5.1}$$

Thus Σ' is exact. Moreover, if Σ has a primitive g, then Σ' has a primitive

$$g + \int_0^1 (H + e^t\alpha(X_H)) \circ \phi_H^s\; ds.$$

We are going to construct a particular Morse function for Σ such that the image of the Morse function has cylindrical ends as well and thus is an exact Lagrangian cobordism. Suppose Σ is cylindrical outside of $[-N+\delta, N-\delta] \times \mathbb{R}^3$, where $0 < \delta < 1$. Choose Morse functions $g_\pm : \Lambda_\pm \to (0, 1/2)$ and $G : \Sigma \cap \big([-N, N] \times \mathbb{R}^3\big) \to (0, 1)$ such that

$$G\big|_{\Sigma \cap \{t \in [-N, -N+\delta] \cup [N-\delta, N]\}} = e^t.$$

Define a smooth non-decreasing function $\rho : \mathbb{R}_{>0} \to [0, 1]$ such that $\rho(s) = 0$ for $s \leqslant 1$ and $\rho(s) = 1$ for $s \geqslant e^\delta$.

For $0 < \eta < e^{-2}$, define a Morse function $F^\eta : \Sigma \to \mathbb{R}_{>0}$ to be

$$F^\eta(t, q) := \begin{cases} \eta^{2N} g_-(q) s, & \text{if } t < -N; \\ (\rho(e^N s)(\eta^N - \eta^{2N} g_-(q)) + \eta^{2N} g_-(q)) s, & \text{if } -N \leqslant t \leqslant -N+\delta; \\ \eta^N G, & \text{if } -N+\delta < t < N-\delta; \\ (\eta^N + \rho(e^{-N+\delta} s)\eta^N g_+(q)) s, & \text{if } N-\delta \leqslant t \leqslant N; \\ (\eta^N + \eta^N g_+(q)) s, & \text{if } t > N, \end{cases}$$

where $s = e^t$. One can check that F^η has the following properties:

- The Morse function F^η is increasing with respect to t when $t \leqslant -N + \delta$ or $t \geqslant N - \delta$. This implies that the critical points of F^η and the critical points of G are in $1-1$ correspondence and are all contained in $\Sigma \cap \big([-N, N] \times \mathbb{R}^3\big)$.
- The Morse function F^η is bounded by $2\eta^N e^N$ on $\Sigma \cap \big([-N, N] \times \mathbb{R}^3\big)$.
- Write $F^\eta\big|_{\{N\}\times\Lambda_+}$ as $f^\eta_+ e^N$ and $F^\eta\big|_{\{-N\}\times\Lambda_-}$ as $f^\eta_- e^{-N}$, respectively. The graph of dF^η on $(-\infty, -N) \times \Lambda_-$ is the same as $(-\infty, -N) \times graph(df^\eta_-)$ and the graph of dF^η on $(N, \infty) \times \Lambda_+$ is the same as $(N, \infty) \times graph(df^\eta_+)$.

Push $(\Sigma, \Lambda_+, \Lambda_-)$ off through F^η and obtain a copy of $(\Sigma, \Lambda_+, \Lambda_-)$, labeled by $(\Sigma^1, \Lambda^1_+, \Lambda^1_-)$. Label the original $(\Sigma, \Lambda_+, \Lambda_-)$ by $(\Sigma^2, \Lambda^2_+, \Lambda^2_-)$. Thus Σ^1 is a push off of Σ^2 through F^η and Λ^1_+ (resp. Λ^1_-) is a push off of Λ^2_+ (resp. Λ^2_-) through f^η_+ (resp. f^η_-).

5.2 The long exact sequence

Now we apply the Floer theory to the pair of cobordisms $\Sigma^1 \cup \Sigma^2$ constructed in Section 5.1 and get a long exact sequence. Combining the long exact sequence with the augmentation category map induced by the exact Lagrangian cobordism, we obtain an obstruction to the existence of the exact Lagrangian cobordisms.

Recall that the grading for generators in the Cthulhu chain complex depends on the choice of capping paths. According to the canonical Floer theory [Flo88], we can choose a family of capping paths such that the Conley-Zehnder index of any double point x of $\Sigma^1 \cup \Sigma^2$ satisfies $CZ(\Gamma_x) = Ind_{F^\eta(x)}$. Now we apply the Floer theory to the pair of Lagrangian cobordisms $\Sigma^1 \cup \Sigma^2$ and have the following theorem.

Theorem 5.2.1. *Let Σ^i, for $i = 1, 2$, be the cobordisms from Λ^i_- to Λ^i_+ as constructed in Section 5.1. Suppose ϵ^i_- is an augmentation of $\mathcal{A}(\Lambda^i_-)$ and ϵ^i_+ is the augmentation* *of $\mathcal{A}(\Lambda^i_+)$ induced by Σ^i. Fix a suitable domain dependent almost complex structure on $\mathbb{R} \times \mathbb{R}^3$ that is compatible with the symplectic form and cylindrical ends. For η small enough, the Cthulhu chain complex is*

$$Cth^k(\Sigma^1, \Sigma^2) = C^{k-2}(\Lambda^1_+, \Lambda^2_+) \oplus CF^k(\Sigma^1, \Sigma^2) \oplus C^{k-1}(\Lambda^1_-, \Lambda^2_-).$$

Under this decomposition, the differential is

$$d = \begin{pmatrix} d_{++} & d_{+0} & d_{+-} \\ 0 & d_{00} & d_{0-} \\ 0 & 0 & d_{--} \end{pmatrix}.$$

Moreover,

1. *the map d_{00} is the Morse co–differential induced by F^η, i.e., the chain complex $(CF^k(\Sigma^1, \Sigma^2), d_{00})$ is the Morse co–chain complex $(C^k_{Morse}F^\eta, d_{F^\eta})$ induced by F^η;*

2. *the chain complex $(C^{k-2}(\Lambda^1_+, \Lambda^2_+), d_{++}) = (Hom^{k-1}_+(\epsilon^1_+, \epsilon^2_+), m_1)$ while the chain complex $(C^{k-1}(\Lambda^1_-, \Lambda^2_-), d_{--}) = (Hom^k_+(\epsilon^1_-, \epsilon^2_-), m_1)$.*

Proof. First, we need to show that all the intersection points $x \in CF^k(\Sigma^1, \Sigma^2)$ have positive action, which is the condition for the differential to have the form above by Remark 4.2.1.

Let g_i be a primitive of Σ^i for $i = 1, 2$. According to the computation (5.1), we have

$$g_1 = g_2 + \int_0^1 (H + e^t \alpha(X_H)) \circ \phi^s_H \, ds,$$

where $H = -\tilde{F^\eta} \circ \psi$. It is not hard to check that $g_1 = g_2$ on $\Sigma \cap \big((-\infty, -N) \times \mathbb{R}^3\big)$. Therefore we can assume $g_1 = g_2 = 0$ on $\Sigma \cap \big((-\infty, -N) \times \mathbb{R}^3\big)$ and use g_1 and g_2 as primitives to define action. The action of each intersection point x is

$$\mathfrak{a}(x) = g_2(x) - g_1(x) = \int_0^1 (H + e^t \alpha(X_H)) \circ \phi^s_H \, ds.$$

Notice that the vector field X_H vanishes at the intersection point x. Hence

$$\mathfrak{a}(x) = -\int_0^1 H \circ \phi_H^s \, ds = -H = F^\eta \circ \psi(x) > 0.$$

Next, we are going to show that for η small enough, the rigid holomorphic disks that contribute to d_{00} do not include any pure Reeb chords as negative punctures. Let x and y be two double points of $\Sigma^1 \cup \Sigma^2$. For any rigid holomorphic disk $u \in \mathcal{M}(x; \mathbf{p}, y, \mathbf{q})$, where $\mathbf{p}$ and $\mathbf{q}$ are words of pure Reeb chords of $\Lambda_-^1 \cup \Lambda_-^2$, we have the energy estimate:

$$E_\omega(u) \leqslant \mathfrak{a}(x) - \mathfrak{a}(y) - \mathfrak{a}(\mathbf{p}) - \mathfrak{a}(\mathbf{q}).$$

Since u has positive energy, we have

$$\mathfrak{a}(\mathbf{p}) + \mathfrak{a}(\mathbf{q}) \leqslant F^\eta(\psi(x)) - F^\eta(\psi(y)) \leqslant F^\eta(\psi(x)).$$

Therefore, for η small enough such that the maximum of the Morse function F^η is smaller than the minimum action of pure Reeb chords of Λ_-^1 and Λ_-^2, the moduli space that contributes to d_{00} is of the form $\mathcal{M}(x; y)$. By [EES09, Lemma 6.11], the boundary of a rigid holomorphic disk with two punctures at intersection points converge to a rigid Morse flow line, which implies $d_{00} = d_{F^\eta}$. Furthermore, the gradings satisfy $|x| = CZ(\gamma_x) = Ind_{F^\eta(x)}$. Therefore $(CF^k(\Sigma^1, \Sigma^2), d_{00}) = (C^k_{Morse} F^\eta, d_{F^\eta})$.

Recall that there is a natural identity map with degree 1 from $C^{k-1}(\Lambda_\pm^1, \Lambda_\pm^2)$ to $Hom_+^k(\epsilon_\pm^1, \epsilon_\pm^2)$, respectively. Moreover, the definitions of $d_{\pm\pm}$ and m_1 match as well. Hence we have $(C^{k-1}(\Lambda_-^1, \Lambda_-^2), d_{--}) = (Hom_+^k(\epsilon_-^1, \epsilon_-^2), m_1)$ while $(C^{k-2}(\Lambda_+^1, \Lambda_+^2), d_{++}) = (Hom_+^{k-1}(\epsilon_+^1, \epsilon_+^2), m_1)$. □

For the rest of the paper, we fix a small enough η and write F^η, f_+^η, f_-^η as F, f_+, f_-, respectively. According to the Floer theory in Section 4, we have $H^k(Cth(\Sigma^1, \Sigma^2), d) = 0$, where

$$Cth^k(\Sigma^1, \Sigma^2) = Hom_+^{k-1}(\epsilon_+^1, \epsilon_+^2) \oplus C^k_{Morse} F \oplus Hom_+^k(\epsilon_-^1, \epsilon_-^2),$$

and

$$d = \begin{pmatrix} m_1 & d_{+0} & d_{+-} \\ 0 & d_F & d_{0-} \\ 0 & 0 & m_1 \end{pmatrix}.$$

Consider the chain map $\Psi = d_{+-} + d_{0-}$:

$$\Psi: \ (Hom^k_+(\epsilon^1_-, \epsilon^2_-), m_1) \to \left(Hom^k_+(\epsilon^1_+, \epsilon^2_+) \oplus C^{k+1}_{Morse} F, \ d' = \begin{pmatrix} m_1 & d_{+0} \\ 0 & d_F \end{pmatrix} \right).$$

Notice that the mapping cone of Ψ has trivial homology. Therefore,

$$H^k(Hom_+(\epsilon^1_-, \epsilon^2_-)) \cong H^k Cone(d_{+0}).$$

Hence we have the following long exact sequence:

$$\cdots \to H^k(C_{Morse}F, d_F) \to H^k Hom_+(\epsilon^1_+, \epsilon^2_+) \to H^k Hom_+(\epsilon^1_-, \epsilon^2_-) \to H^{k+1}(C_{Morse}F, d_F) \to \cdots$$

Moreover, notice that in the construction in Section 5.1, the gradient flows of F flow in from the bottom and out of the top. Hence we have

$$H^k(C_{Morse}F, d_F) = H^k(\Sigma, \Lambda_-).$$

Corollary 5.2.2. *Let Σ be an exact Lagrangian cobordism with Maslov number* 0 *from Λ_- to Λ_+. For $i = 1, 2$, if ϵ^i_- is an augmentation of $\mathcal{A}(\Lambda_-)$ and ϵ^i_+ is the augmentation of $\mathcal{A}(\Lambda_+)$ induced by Σ, then we have the following long exact sequence:*

$$\cdots \to H^k(\Sigma, \Lambda_-) \to H^k Hom_+(\epsilon^1_+, \epsilon^2_+) \to H^k Hom_+(\epsilon^1_-, \epsilon^2_-) \to H^{k+1}(\Sigma, \Lambda_-) \to \cdots \tag{5.2}$$

Remark 5.2.3. When the Maslov number of Σ is d, which is not 0, the method above works as well. The only difference is that the grading of generators in the Cthulhu chain complex is defined mod d. Thus, the long exact sequence (5.2) holds with gradings mod d.

If $\epsilon_-^1 = \epsilon_-^2 = \epsilon_-$, by [NRS+15, Section 5.2], we have the identification

$$H^k Hom_+(\epsilon, \epsilon) \cong LCH^{\epsilon}_{1-k}(\Lambda),$$

where $LCH^{\epsilon}_k(\Lambda)$ is the linearized contact homology of Λ. The long exact sequence (5.2) can be rewritten in terms of linearized contact homology:

$$\cdots \longrightarrow H^k(\Sigma, \Lambda_-) \longrightarrow LCH^{\epsilon_+}_{1-k}(\Lambda_+) \longrightarrow LCH^{\epsilon_-}_{1-k}(\Lambda_-) \longrightarrow H^{k+1}(\Sigma, \Lambda_-) \longrightarrow \cdots$$

Furthermore, if Λ_- is empty, then Σ is an exact Lagrangian filling of Λ_+ and ϵ_+ is an augmentation of $\mathcal{A}(\Lambda_+)$ induced by the Lagrangian filling. The long exact sequence (5.2) gives

$$H^k(\Sigma) \cong H^k Hom_+(\epsilon_+, \epsilon_+) \cong LCH^{\epsilon_+}_{1-k}(\Lambda_+),$$

which is the Seidel isomorphism (following [NRS+15]). This theorem was conjectured by Seidel [Sei08] and was proved by Dimitroglou Rizell [DR16].

If Λ_+, instead, is empty and $\mathcal{A}(\Lambda_-)$ has an augmentation ϵ_-, the long exact sequence (5.2) tells us that

$$H^k Hom_+(\epsilon_-, \epsilon_-) \cong H^{k+1}(\Sigma, \Lambda_-) = \begin{cases} \mathbb{F} & \text{if } k = 1, \\ \mathbb{F}^{1-\chi(\Sigma)} & \text{if } k = 0, \\ 0 & \text{otherwise.} \end{cases}$$

However, by Sabloff duality (3.2),

$$\dim H^0 Hom_+(\epsilon_-, \epsilon_-) = \dim H^2 Hom_-(\epsilon_-, \epsilon_-) = \dim H^2 Hom_+(\epsilon_-, \epsilon_-) = 0.$$

This is a contradiction since the unit $e_{\epsilon_-} = -y^\vee$ is always in $H^0 Hom_+(\epsilon_-, \epsilon_-)$ and is not 0. Thus if Λ_+ is empty, then Λ_- does not admit any augmentation. This result was previously known by [CDRGG15, DR15].

Remark 5.2.4. For the rest of the paper, we will focus on the case where Λ_+ and Λ_- are single component knots. Given the fact that there does not exist a compact Lagrangian manifold in $\mathbb{R} \times \mathbb{R}^3$ and Λ_- does not admit a cap (since Λ_- has an augmentation), we know that any cobordism Σ from Λ_- to Λ_+ must be connected.

Combining the long exact sequence (5.2) with the augmentation category map induced by the exact Lagrangian cobordism Σ, we have the following theorem.

Theorem 5.2.5. *Let Σ be an exact Lagrangian cobordism with Maslov number* 0 *from a Legendrian knot Λ_- to a Legendrian knot Λ_+. For $i = 1, 2$, assume ϵ_-^i is an augmentation of the $\mathcal{A}(\Lambda_-)$ with a single base point and ϵ_+^i is the augmentation of $\mathcal{A}(\Lambda_+)$ induced by Σ. Then the map*

$$i^0 : H^0 Hom_+(\epsilon_+^1, \epsilon_+^2) \to H^0 Hom_+(\epsilon_-^1, \epsilon_-^2)$$

in the long exact sequence (5.2) is an isomorphism. Moreover, we have that

$$H^* Hom_+(\epsilon_+^1, \epsilon_+^2) \cong H^* Hom_+(\epsilon_-^1, \epsilon_-^2) \oplus \mathbb{F}^{-\chi(\Sigma)}[1], \tag{5.3}$$

where $\mathbb{F}^{-\chi(\Sigma)}[1]$ denotes the vector space $\mathbb{F}^{-\chi(\Sigma)}$ in degree 1 *and $\chi(\Sigma)$ is the Euler characteristic of the surface Σ.*

Proof. By Remark 5.2.4, we have

$$H^k(\Sigma, \Lambda_-) = \begin{cases} \mathbb{F}^{-\chi(\Sigma)} & \text{if } k = 1 \\ 0 & \text{if else.} \end{cases}$$

The long exact sequence (5.2) shows that for $k > 1$ or $k < 0$, the map

$$i^k : H^k Hom_+(\epsilon_+^1, \epsilon_+^2) \to H^k Hom_+(\epsilon_-^1, \epsilon_-^2)$$

in the long exact sequence induces an isomorphism

$$H^k Hom_+(\epsilon_+^1, \epsilon_+^2) \cong H^k Hom_+(\epsilon_-^1, \epsilon_-^2).$$

When $k = 0$ or 1, we have

$$0 \to H^0 Hom_+(\epsilon^1_+, \epsilon^2_+) \xrightarrow{i^0} H^0 Hom_+(\epsilon^1_-, \epsilon^2_-) \to \mathbb{F}^{-\chi(\Sigma)} \to H^1 Hom_+(\epsilon^1_+, \epsilon^2_+) \to H^1 Hom_+(\epsilon^1_-, \epsilon^2_-) \to 0$$

For the rest of the proof, we need to show

$$\dim\left(H^0 Hom_+(\epsilon^1_+, \epsilon^2_+)\right) \geqslant \dim\left(H^0 Hom_+(\epsilon^1_-, \epsilon^2_-)\right).$$

Once this inequality holds, the fact that $i^0 : H^0 Hom_+(\epsilon^1_+, \epsilon^2_+) \to H^0 Hom_+(\epsilon^1_-, \epsilon^2_-)$ is injective implies that it is an isomorphism. Note that the long exact sequence (5.2) is over the field $\mathbb{F}$. It follows that

$$H^1 Hom_+(\epsilon^1_+, \epsilon^2_+) \cong H^1 Hom_+(\epsilon^1_-, \epsilon^2_-) \oplus \mathbb{F}^{-\chi(\Sigma)}.$$

To prove the inequality, we exchange the positions of ϵ^1 and ϵ^2 in the long exact sequence (5.2) and get

$$\cdots \longrightarrow H^k(\Sigma, \Lambda_-) \longrightarrow H^k Hom_+(\epsilon^2_+, \epsilon^1_+) \longrightarrow H^k Hom_+(\epsilon^2_-, \epsilon^1_-) \longrightarrow H^{k+1}(\Sigma, \Lambda_-) \longrightarrow \cdots,$$

which implies

$$H^2 Hom_+(\epsilon^2_+, \epsilon^1_+) \cong H^2 Hom_+(\epsilon^2_-, \epsilon^1_-).$$

By Sabloff duality (3.2), we have

$$\dim\left(H^0 Hom_-(\epsilon^1_\pm, \epsilon^2_\pm)\right) = \dim\left(H^2 Hom_+(\epsilon^2_\pm, \epsilon^1_\pm)\right).$$

Thus $\dim\left(H^0 Hom_-(\epsilon^1_+, \epsilon^2_+)\right) = \dim\left(H^0 Hom_-(\epsilon^1_-, \epsilon^2_-)\right)$.

Since Λ_+ and Λ_- are both Legendrian knots with a single base point, we have the long exact sequence (3.1) for Λ_+ and Λ_-:

$$0 \longrightarrow H^0 Hom_-(\epsilon^1_\pm, \epsilon^2_\pm) \longrightarrow H^0 Hom_+(\epsilon^1_\pm, \epsilon^2_\pm) \longrightarrow H^0(\Lambda_\pm) \xrightarrow{\delta_\pm} H^1 Hom_-(\epsilon^1_\pm, \epsilon^2_\pm) \longrightarrow \cdots.$$

From this long exact sequence, we have

$$\dim\left(H^0 Hom_+(\epsilon^1_\pm, \epsilon^2_\pm)\right) = \dim\left(H^0 Hom_-(\epsilon^1_\pm, \epsilon^2_\pm)\right) + \dim(\ker \delta_\pm).$$

Thus, to prove $\dim\left(H^0 Hom_+(\epsilon^1_+, \epsilon^2_+)\right) \geqslant \dim\left(H^0 Hom_+(\epsilon^1_-, \epsilon^2_-)\right)$, we only need to show $\dim(\ker \delta_+) \geqslant \dim(\ker \delta_-)$.

Recall that the cobordism Σ from Λ_- and Λ_+ induces an A_∞ category map

$$f : \mathcal{A}ug_+(\Lambda_-) \to \mathcal{A}ug_+(\Lambda_+)$$

in the way described in Section 3.2. In particular, we get the functor f_1 of augmentation categories on the level of morphisms:

$$f_1 : Hom_+(\epsilon_-^1, \epsilon_-^2) \to Hom_+(\epsilon_+^1, \epsilon_+^2).$$

This map descends to the cohomology level as $f^* : H^*Hom_+(\epsilon_-^1, \epsilon_-^2) \to H^*Hom_+(\epsilon_+^1, \epsilon_+^2)$. Notice that f_1 sends $Hom_-(\epsilon_-^1, \epsilon_-^2)$ to $Hom_-(\epsilon_+^1, \epsilon_+^2)$. Hence f_1 induces a map between the cohomology of the quotient chain complexes, denoted by f^* as well.

We have the following diagram commutes:

$$\begin{array}{ccccccccc} 0 \longrightarrow & H^0Hom_-(\epsilon_+^1, \epsilon_+^2) & \longrightarrow & H^0Hom_+(\epsilon_+^1, \epsilon_+^2) & \longrightarrow & H^0(\Lambda_+) & \xrightarrow{\delta_+} & H^1Hom_-(\epsilon_+^1, \epsilon_+^2) & \longrightarrow \cdots \\ & \uparrow f^* & & \uparrow f^* & & \uparrow f^* & & \uparrow f^* & \\ 0 \longrightarrow & H^0Hom_-(\epsilon_-^1, \epsilon_-^2) & \longrightarrow & H^0Hom_+(\epsilon_-^1, \epsilon_-^2) & \longrightarrow & H^0(\Lambda_-) & \xrightarrow{\delta_-} & H^1Hom_-(\epsilon_-^1, \epsilon_-^2) & \longrightarrow \cdots. \end{array}$$

Thus $f^*(\ker \delta_-) \subset \ker \delta_+$. Furthermore, notice that f_1 sends the generator $(y^-)^\vee \in C^0_{Morse}(\Lambda_-)$ to the corresponding $(y^+)^\vee \in C^0_{Morse}(\Lambda_+)$ and $C^{-1}_{Morse}(\Lambda_+) = 0$. Hence f^* is injective on $H^0(\Lambda_-)$, which implies

$$\dim(\ker \delta_+) \geqslant \dim(\ker \delta_-).$$

□

If $\epsilon_-^1 = \epsilon_-^2 = \epsilon_-$ and ϵ_- comes from a Lagrangian filling L_-, then ϵ_+ also comes from the filling L_+, which is a concatenation of Σ and L_-. By Seidel's isomorphism (following [NRS$^+$15]), we have $Hom_+^k(\epsilon_\pm, \epsilon_\pm) \cong H^k(L_\pm)$, which implies that

$$H^kHom_+(\epsilon_+, \epsilon_+) \cong H^kHom_+(\epsilon_-, \epsilon_-) \text{ for } k \neq 1$$

and when $k = 1$,

$$H^1 Hom_+(\epsilon_+, \epsilon_+) \cong H^1 Hom_+(\epsilon_-, \epsilon_-) \oplus \mathbb{F}^{-\chi(\Sigma)}.$$

Theorem 5.2.5 is a generalization of Seidel's isomorphism. Equation (5.3) holds even if ϵ_- does not come from a Lagrangian filling or ϵ_-^1 and ϵ_-^2 are not the same.

If the two augmentations are the same, we can identify the cohomology of Hom_+ space with the linearize contact homology by [NRS$^+$15]:

$$H^k Hom_+(\epsilon, \epsilon) \cong LCH^{\epsilon}_{1-k}(\Lambda).$$

Now we restate Theorem 5.2.5 in terms of linearized contact homology.

Corollary 5.2.6. *Let Σ be an exact Lagrangian cobordism with Maslov number* 0 *from a Legendrian knot Λ_- to a Legendrian knot Λ_+. Assume ϵ_- is an augmentation of $\mathcal{A}(\Lambda_-)$ and ϵ_+ is the augmentation of $\mathcal{A}(\Lambda_+)$ induced by Σ. Then*

$$LCH^{\epsilon_+}_*(\Lambda_+) \cong LCH^{\epsilon_-}_*(\Lambda_-) \oplus \mathbb{F}^{-\chi(\Sigma)}[0],$$

where $\mathbb{F}^{-\chi(\Sigma)}[0]$ denotes the vector space $\mathbb{F}^{-\chi(\Sigma)}$ in degree 0*.*

Therefore, if there exists an exact Lagrangian cobordism Σ from Λ_- to Λ_+, the Poincaré polynomial of linearized contact homology of Λ_+ agrees with that of Λ_- in all degrees except 0. In degree 0 their coefficients differ by $-\chi(\Sigma)$. This gives a strong and computable obstruction to the existence of exact Lagrangian cobordisms. One can check the Poincaré polynomials of linearize contact homology for any two Legendrian knots with small crossings through the atlas in [CN13]. If they do not satisfy the relation given in Corollary 5.2.6, there does not exist an exact Lagrangian cobordism between them.

For example, let Λ_1 and Λ_2 be the Legendrian knots with maximum Thurston–Bennequin number of smooth knot type 4_1 and 6_1, respectively (as shown in Figure

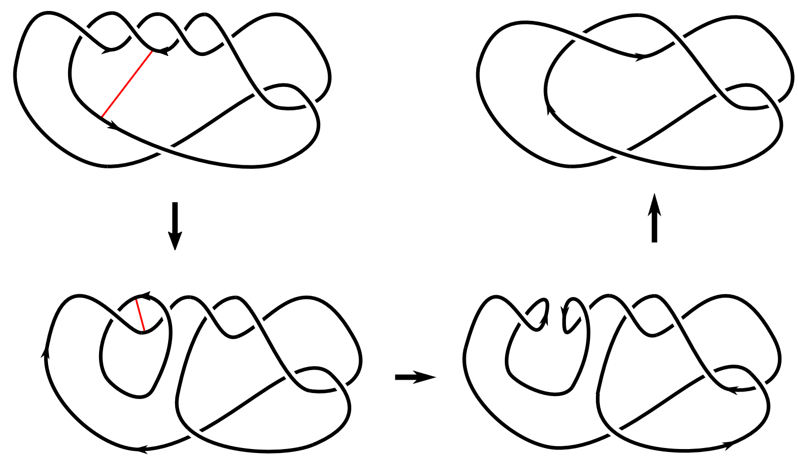


FIGURE 5.1: The topological cobordism between 6_1 and 4_1 can be achieved by two saddle moves along the red lines followed by an isotopy.

1.4). There is a topological cobordism between 4_1 and 6_1 with genus 1 as shown in Figure 5.1. Moreover, the Thurston–Bennequin numbers of Λ_1 and Λ_2 are -3 and -5, respectively, which satisfy the Thurston–Bennequin number relation (1.1). Thus there is a possibility to exist an exact Lagrangian cobordism from Λ_2 to Λ_1 with genus 1. However, we have the following proposition:

Proposition 5.2.7. *There does not exist an exact Lagrangian cobordism from* Λ_2 *to* Λ_1 *with Maslov number* 0*.*

Proof. The Poincaré polynomials of linearized contact homology for Λ_1 and Λ_2 are $t^{-1} + 2t$ and $2t^{-1} + 3t$, respectively. As a result of Corollary 5.2.6, there does not exist an exact Lagrangian cobordism from Λ_2 to Λ_1 with Maslov number 0. □

5.3 Geometric description of the differential map.

Let Σ be an exact Lagrangian cobordism from a Legendrian knot Λ_- to a Legendrian knot Λ_+. For $i = 1, 2$, assume ϵ_-^i is an augmentation of $\mathcal{A}(\Lambda_-)$ and ϵ_+^i is the augmentation induced by Σ. So far we have two maps from $Hom_+(\epsilon_-^1, \epsilon_-^2)$ to

$Hom_+(\epsilon^1_+, \epsilon^2_+)$. One is the geometric map d_{+-} in the differential of the Cthulhu chain complex $Cth(\Sigma^1, \Sigma^2)$ defined by counting rigid holomorphic disks with boundary on $\Sigma^1 \cup \Sigma^2$. The other map is the augmentation category map induced by Σ on the level of morphisms f_1, defined algebraically in Section 3.2. In this section, we will show that, with a choice of Morse function F on the cobordism Σ, the maps d_{+-} and f_1 are the same. To do that, we describe the two maps separately and then compare their images on each generator of $Hom_+(\epsilon^1_-, \epsilon^2_-)$.

In order to describe d_{+-}, we want to interpret rigid holomorphic disks with boundary on $\Sigma^1 \cup \Sigma^2$ in terms of rigid holomorphic disks with boundary on Σ together with negative gradient flows of a Morse function. This is analogous to a result in [EES09], which gives a correspondence between rigid holomorphic disks with boundary on a 2–copy of a Legendrian submanifold L and rigid holomorphic disks with boundary on L together with negative gradient flows of a Morse function. Now let us describe the result in [EES09] with details.

Let L be a Legendrian submanifold in the contact manifold $(P \times \mathbb{R}, \ker(dz - \theta))$, where $(P, d\theta)$ is an exact symplectic $2n$–dimensional manifold. Instead of considering holomorphic disks in the symplectization of $P \times \mathbb{R}$ with boundary on $\mathbb{R} \times L$, according to [DR16], we can consider holomorphic disks in P with boundary on $\pi(L)$, where π is the projection $P \times \mathbb{R} \to P$. See [EES09, Section 2.2.3] for the detailed definition of holomorphic disks with boundary on $\pi(L)$. As the points on $\pi(L)$ and points on L are naturally corresponded except that the double points of $\pi(L)$ correspond to the Reeb chords of L, we refer the holomorphic disks as J–**holomorphic disks with boundary on** L as in [EES09], where J is a generic almost complex structure on P. Choose a Morse–Smale pair (f, g), where f is a Morse function $L \to \mathbb{R}$ and g is a Riemannian metric on L, such that (f, g, J) is **adjusted to** L in the sense of [EES09, Section 6.3]. Push L off through the Morse function f and get a 2–copy of L, denoted by $2L$. In order to describe rigid holomorphic disks with boundary on $2L$, we need to introduce the generalized disks determined by (f, g, J). A **generalized disk** is a pair (u, γ), where

- $u \in \mathcal{M}$ is a J–holomorphic disk with boundary on L;
- γ is a negative gradient flow of f with one end on the boundary of u and the other end at a critical point p of the Morse function f;
- the boundary of u and γ intersect transversely.

The point p is called a **negative Morse puncture** if the flow line γ flows toward p, and is called a **positive Morse puncture** if γ flows away from p. The formal dimension $\dim(u, \gamma)$ is defined by

$$\dim(u,\gamma) = \begin{cases} \dim \mathcal{M} + 1 + Ind_f(p) - n, & \text{if } p \text{ is a positive Morse puncture,} \\ \dim \mathcal{M} + 1 - Ind_f(p), & \text{if } p \text{ is a negative Morse puncture.} \end{cases}$$

The generalized disk (u, γ) is called **rigid** if $\dim(u, \gamma) = 0$.

The rigid holomorphic disks with boundary on a 2–copy of L can be described as below in terms of whether their punctures are Morse Reeb chords or non–Morse Reeb chords (as defined in Section 3.2).

Lemma 5.3.1 ([EES09, Theorem 3.6])**.** *Let (f, g, J) be a pair described above that is adjusted to the Legendrian submanifold L. Push L off through the Morse function f and get a 2–copy $2L$. There are bijective correspondences below:*

- *Rigid holomorphic disks with boundary on $2L$ that have one positive puncture and one negative puncture at non–Morse mixed Reeb chords and the other punctures at pure Reeb chords are in $1-1$ correspondence with rigid holomorphic disks with boundary on L as shown in Figure 5.2 (a).*

- *Rigid holomorphic disks with boundary on* $2L$ *that have exactly one puncture at a Morse Reeb chord are in* $1-1$ *correspondence with rigid generalized disks* (u,γ) *determined by* (f,g,J) *as shown in Figure 5.2* (b).

- *Rigid holomorphic disks with boundary on* $2L$ *that have two punctures at Morse Reeb chords are in* $1-1$ *correspondence with rigid negative gradient flows of the Morse function* f.

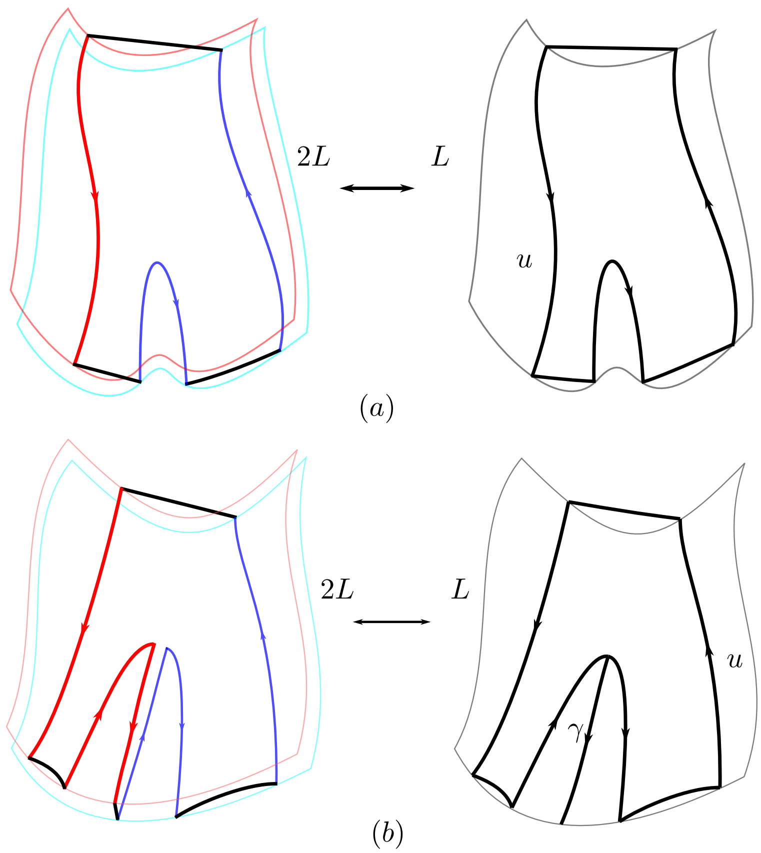


FIGURE 5.2: Schematic picture of the correspondences in Lemma 5.3.1. The arrows indicate the orientation of holomorphic disks and the negative gradient flow line.

In order to get an analogous description for a 2–copy of Σ, we need a result in [EHK16] to relate rigid holomorphic disks with boundary on a cobordism Σ to rigid holomorphic disks with boundary on some Legendrian submanifold L_Σ.

Let us first construct the Legendrian submanifold L_Σ. Suppose Σ is an exact Lagrangian submanifold in $\big(\mathbb{R}\times\mathbb{R}^3, d(e^t\alpha)\big)$ from Λ_- to Λ_+. Assume it is cylindrical outside of $[-N+\delta, N-\delta]\times\mathbb{R}^3$, where δ is a small positive number. Under the symplectomorphism

$$\begin{aligned}\psi : \big(\mathbb{R}\times\mathbb{R}^3, d(e^t\alpha)\big) &\to \big(T^*(\mathbb{R}_{>0}\times\mathbb{R}), d\theta\big)\\ (t,x,y,z) &\mapsto \big((e^t,x),(z,e^t y)\big),\end{aligned}$$

the cobordism Σ can be viewed as a cobordism in $\big(T^*(\mathbb{R}_{>0}\times\mathbb{R}), d\theta\big)$, where θ is the negative Liouville form of the cotangent bundle. Let $a_-=e^{-N}$ and $a_+=e^N$. There exists a small number $\epsilon>0$ such that Σ is cylindrical outside of $T^*\big([a_-+\epsilon, a_+-\epsilon]\times\mathbb{R}\big)$. Chopping off the ends of Σ, we get a cobordism in $T^*\big([a_-,a_+]\times\mathbb{R}\big)$ with the canonical symplectic form. Lift it to be a Legendrian submanifold $\overline{\Sigma}$ in the 1–jet space $J^1\big([a_-,a_+]\times\mathbb{R}\big)=T^*\big([a_-,a_+]\times\mathbb{R}\big)\times\mathbb{R}$. Near the positive boundary $J^1\big((a_+-\epsilon,a_+]\times\mathbb{R}\big)$, the Legendrian $\overline{\Sigma}$ can be parametrized as

$$(s,\ x_+(q),\ z_+(q),\ sy_+(q),\ sz_+(q)+B_+) = j^1(sz_+(q)+B_+)$$

for some constant number B_+, where $s=e^t$ and $(t,q)\in\Sigma\cap\big((\log(a_+-\epsilon),\log a_+]\times\mathbb{R}^3\big)=(\log(a_+-\epsilon),\log a_+]\times\Lambda_+$. Here $sz_+(q)+B_+$ may not be a function of (s,x). However, consider $\{(x_+(q),z_+(q))|q\in\Lambda_+\}$, which is the front projection of Λ to xz–plane. The cusps divide the front diagram of Λ_+ into pieces. Note that on each piece $z_+(p)$ is a perfect function of $x_+(p)$ and at each cusp, the two functions from different pieces match at the cusp. Therefore, we can write the parametrization as $j^1(sz_+(q)+B_+)$. Similarly, near the negative boundary $J^1\big([a_-,a_-+\epsilon)\times\mathbb{R}\big)$, the Legendrian $\overline{\Sigma}$ can be parametrized as

$$(s,\ x_-(q),\ z_-(q),\ sy_-(q),\ sz_-(q)+B_-) = j^1(sz_-(q)+B_-),$$

where $(s,q)\in[a_-,a_-+\epsilon)\times\Lambda_-$ and B_- is a constant number.

However, notice that $\overline{\Sigma}$ does not have any Reeb chords. Therefore, we consider the **Morse Legendrian** $\overline{\Sigma^{Mo}}$, which is a Legendrian submanifold in $J^1\big([a_-, a_+] \times \mathbb{R}\big)$ that agrees with $\overline{\Sigma}$ on $J^1\big((a_- + \epsilon, a_+ - \epsilon) \times \mathbb{R}\big)$. But near the $(\pm)$–boundary, the Morse Legendrian can be parametrized as $j^1\big((A_\pm \mp (s - a_\pm)^2)z_\pm(q)\big)$, i.e.

$$\big(s,\ x_\pm(q),\ \mp 2(s - a_\pm)z_\pm(q),\ (A_\pm \mp (s - a_\pm)^2)y_\pm(q),\ (A_\pm \mp (s - a_\pm)^2)z_\pm(q)\big), \quad (5.4)$$

where $A_\pm$ are positive numbers. The key property of the Morse Legendrian is that the Reeb chords of $\overline{\Sigma^{Mo}}$ on the $(\pm)$–boundary are in bijective correspondence with Reeb chords of $\Lambda_\pm$, respectively.

There are isotopies from $sz_\pm(q) + B_\pm$ to $(A_\pm \mp (s - a_\pm)^2)z_\pm(q)$, respectively, which induce a diffeomorphism from $\overline{\Sigma}$ to the Morse Legendrian $\overline{\Sigma^{Mo}}$. Extend $\overline{\Sigma^{Mo}}$ to be a Legendrian submanifold L_Σ in $J^1(\mathbb{R}_{>0} \times \mathbb{R})$ by adding

$$j^1\big((A_+ - (s - a_+)^2)z_+(q)\big)$$

with $(s, q) \in (a_+, \infty) \times \Lambda_+$ to the positive boundary and adding

$$j^1\big((A_- + (s - a_-)^2)z_-(q)\big)$$

with $(s, q) \in (0, a_-) \times \Lambda_-$ to the negative boundary. In other words, when $s < a_- + \epsilon$ or $s > a_+ - \epsilon$, we can parametrize L_Σ as (5.4). Note that

$$L_\Sigma \cap J^1\big((a_- + \epsilon, a_+ - \epsilon) \times \mathbb{R}\big) = \overline{\Sigma^{Mo}}.$$

Moreover, according to [EK08], there is a natural bijective correspondence between rigid holomorphic disks with boundary on L_Σ and rigid holomorphic disks with boundary on $\overline{\Sigma^{Mo}}$. Combining with a result in [EHK16], we know that rigid holomorphic disks with boundary on an exact Lagrangian cobordism Σ are in $1-1$ correspondence with rigid holomorphic disks with boundary on L_Σ that have positive (resp. negative) punctures at Reeb chords lying the slice $s = a_+$ (resp. $s = a_-$). The proof of this result can be applied directly to the case of immersed exact Lagrangian submanifolds with cylindrical ends, where we only consider the rigid holomorphic disks with punctures on Reeb chords but no double points. Hence we have the following result for a 2–copy of Σ, denoted by $\Sigma \cup \Sigma'$.

Lemma 5.3.2. *Let Σ and Σ' be exact Lagrangian cobordisms from Λ_- to Λ_+ and from Λ'_- to Λ'_+, respectively. The Morse Legendrian $L_{\Sigma\cup\Sigma'}$ constructed above is a union of L_Σ and $L_{\Sigma'}$. Moreover, rigid holomorphic disks with boundary on $\Sigma \cup \Sigma'$ that have positive (resp. negative) punctures at Reeb chords of $\Lambda_+ \cup \Lambda'_+$ (resp. $\Lambda_- \cup \Lambda'_-$) are in 1–1 correspondence with rigid holomorphic disks with boundary on $L_\Sigma \cup L_{\Sigma'}$ that have positive (resp. negative) punctures at the Reeb chords lying in the slice $s = a_+$ (resp. $s = a_-$).*

Note that the rigid holomorphic disks with boundary on $\Sigma \cup \Sigma'$ considered in Lemma 5.3.2 are not all the rigid holomorphic disks since we did not talk about holomorphic disks with punctures at double points. The disks we considered are the ones counted by d_{+-}.

In order to apply Lemma 5.3.1 to $L_{\Sigma\cup\Sigma'}$ and get the analog correspondences for exact Lagrangian cobordisms, we need to view $L_{\Sigma'}$ as the 1–jet of a function

$$\tilde{F} : L_\Sigma \to \mathbb{R}$$

in the neighborhood of L_Σ and show that $\tilde{F}$ is Morse. To describe the function easily, pull it back to be a function $\Sigma \to \mathbb{R}$, denoted by $\tilde{F}$ as well. Note that $\tilde{F} = F$ on $\Sigma \cap T^*\big([a_- + \epsilon, a_+ - \epsilon] \times \mathbb{R}\big)$.

Now let us focus on the part $s \in (a_+ - \epsilon, \infty)$. Denote $L_\Sigma \cap J^1\big((a_+ - \epsilon, \infty) \times \mathbb{R}\big)$ by $\partial_+(L_\Sigma)$ and denote $\Sigma \cap T^*\big((a_+ - \epsilon, \infty) \times \mathbb{R}\big)$ by $\partial_+(\Sigma)$. One can check that the Reeb chords from $\partial_+(L_\Sigma)$ to $\partial_+(L_{\Sigma'})$ are in bijective correspondence with the Reeb chords from Λ_+ to Λ'_+ by a property of Morse Legendrian. As a result, the only critical points of $\tilde{F}$ on $\partial_+(\Sigma)$ are (s, q), where $s = a_+$ and $f'_+(q) = 0$.

Let π_1 and π_2 be the natural projections as follows:

$$\begin{array}{ccc} & J^1(\mathbb{R}_{>0} \times \mathbb{R}_x) & \\ {\scriptstyle \pi_1} \swarrow & & \searrow {\scriptstyle \pi_2} \\ J^1(\mathbb{R}_{>0}) & & J^1(\mathbb{R}_x) \end{array}$$

First project $\partial_+(L_\Sigma)$ and $\partial_+(L_{\Sigma'})$ to $J^1(\mathbb{R}_x)$. We have

$$\pi_2(\partial_+(L_\Sigma)) = (x_+(q), (A_+ - (s - a_+)^2)y_+(q), (A_+ - (s - a_+)^2)z_+(q))$$

and

$$\pi_2(\partial_+(L_{\Sigma'})) = (x'_+(q), (A_+ - (s - a_+)^2)y'_+(q), (A_+ - (s - a_+)^2)z'_+(q)).$$

Thus for fixed $s \in (a_+ - \epsilon, \infty)$, we have $\tilde{F}(s, q) = (A_+ - (s - a_+)^2)f_+(q)$, where $f_+ = F\big|_{\{a_+\}\times\Lambda_+}$. Second, project $\partial_+(L_\Sigma)$ and $\partial_+(L_{\Sigma'})$ to $J^1(\mathbb{R}_{>0})$. We have

$$\pi_1(\partial_+(L_\Sigma)) = (s, -2(s - a_+)z_+(q), (A_+ - (s - a_+)^2)z_+(q)),$$

and

$$\pi_1(\partial_+(L_{\Sigma'})) = (s, -2(s - a_+)z'_+(q), (A_+ - (s - a_+)^2)z'_+(q)).$$

For a fixed $q \in \Lambda_+$, the only non-degenerate singularity of $\tilde{F}(s, q)$ is a_+. In particular, it is a local maximum since $z'_+(q) > z_+(q)$, which comes from the fact that $f_+ > 0$ as constructed in Section 5.1. Therefore, we have

$$Ind_{\tilde{F}}(a_+, q) = Ind_{f_+}(q) + 1.$$

Similarly, on the negative side, denote $F\big|_{\{a_-\}\times\Lambda_-}$ as f_-. The critical points of $\tilde{F}$ on $\Sigma \cap T^*\big((-\infty, a_- + \epsilon) \times \mathbb{R}\big)$ agree with the critical points of f_- that lie in the slice $s = a_-$. Moreover, the indices satisfy $Ind_{\tilde{F}}(a_-, q) = Ind_{f_-}(q)$. Hence $\tilde{F}$ is a Morse function.

Choose a Riemannian metric g on Σ and a generic almost complex structure J on $\mathbb{R}\times\mathbb{R}^3$ that is adjusted to cylindrical ends such that the pair $(\tilde{F}, g, J)$ is adjusted to L_Σ. Now we can apply Lemma 5.3.1 to the 2-copy $L_\Sigma \cup L_{\Sigma'}$.

Define a **generalized disk** to be a pair (u,γ) consisting of a J-holomorphic disk u with boundary on Σ as defined in Section 2.3 and a negative gradient flow line γ of $\tilde{F}$ with one end on the boundary of u and one end at a critical point p of $\tilde{F}$ such that the boundary of u intersect transversely with the negative gradient flow γ. The point p is called a **negative Morse puncture** if the flow line γ flows toward p, and is called a **positive Morse puncture** if γ flows away from p. The formal dimension $\dim(u,\gamma)$ is defined by

$$\dim(u,\gamma) = \begin{cases} \dim\mathcal{M} + 1 + Ind_f(p) - 2, & \text{if } p \text{ is a positive Morse puncture,} \\ \dim\mathcal{M} + 1 - Ind_f(p), & \text{if } p \text{ is a negative Morse puncture.} \end{cases} \tag{5.5}$$

The generalized disk (u,γ) is called **rigid** if $\dim(u,\gamma) = 0$. We have the following result that is analogous to Lemma 5.3.1.

Theorem 5.3.3. *Let Σ be an exact Lagrangian cobordism in $\big(\mathbb{R}\times\mathbb{R}^3, d(e^t\alpha)\big)$ from Λ_- to Λ_+ and is cylindrical outside of $[-N+\delta, N-\delta]\times\mathbb{R}^3$. Let $F:\Sigma\to\mathbb{R}_{>0}$ be a positive Morse function. Push Σ off through F and get a new cobordism Σ'.*

Denote $F\big|_{\{N\}\times\Lambda_+}$ by f_+ and $F\big|_{\{-N\}\times\Lambda_-}$ by f_-. Define a new Morse function $\tilde{F}:\Sigma\to\mathbb{R}$ satisfying the following properties:

- *The Morse function $\tilde{F} = F$ on $\Sigma\cap\big([-N+\delta, N-\delta]\times\mathbb{R}^3\big)$.*
- *On $\Sigma\cap\big((N-\delta,\infty)\times\mathbb{R}^3\big)$, all the critical points of $\tilde{F}$ lie in $\Sigma\cap(\{N\}\times\mathbb{R}^3) = \{N\}\times\Lambda_+$ and agree with the critical points of f_+. Moreover, at each critical point c, we have $Ind_{\tilde{F}}c = Ind_{f_+}c + 1$;*

- *On* $\Sigma \cap \big((-\infty, -N+\delta)\times\mathbb{R}^3\big)$*, all the critical points of* $\tilde{F}$ *lie in* $\Sigma\cap(\{-N\}\times\mathbb{R}^3) = \{-N\}\times\Lambda_-$ *and agree with the critical points of* f_-*. Moreover, at each critical point* c*, we have* $Ind_{\tilde{F}}c = Ind_{f_-}c$*.*

The Riemannian metric g *and almost complex structure* J *are chosen as above. Then we can describe the rigid holomorphic disks with boundary on* $\Sigma\cup\Sigma'$ *that have punctures on Reeb chords of* $\Lambda_+\cup\Lambda'_+$ *and* $\Lambda_-\cup\Lambda'_-$ *in terms of whether the Reeb chords are Morse or non–Morse as defined in Section 3.2.*

1. *Rigid holomorphic disks with boundary on* $\Sigma\cup\Sigma'$ *that have two punctures at non–Morse mixed Reeb chords are in* $1-1$ *correspondence with rigid holomorphic disks with boundary on* Σ*. See Figure 5.3* (*a*)*.*

2. *Rigid holomorphic disks with boundary on* $\Sigma\cup\Sigma'$ *that have exactly one puncture at a Morse Reeb chord are in* $1-1$ *correspondence with rigid generalized disks* (u,γ) *determined by* $(\tilde{F}, g, J)$*. See Figure 5.3* (*b*)*.*

3. *Rigid holomorphic disks with boundary on* $\Sigma\cup\Sigma'$ *with two punctures at Morse Reeb chords are in* $1-1$ *correspondence with rigid negative gradient flows of the Morse function* $\tilde{F}$ *from a critical point on* Λ_+ *to a critical point on* Λ_-*.*

Proof. According to Lemma 5.3.2, rigid holomorphic disks with boundary on $\Sigma\cup\Sigma'$ that have two punctures at mixed Reeb chords correspond to rigid holomorphic disks with boundary on $L_\Sigma\cup L_{\Sigma'}$ that have positive (resp. negative) boundary at mixed Reeb chords lying in the slice $s=a_+$ (resp. $s=a_-$). By Lemma 5.3.1, these disks with boundary on $L_\Sigma\cup L_{\Sigma'}$ are in 1–1 correspondence with holomorphic disks with boundary on L_Σ that have positive (resp. negative) boundary at the Reeb chords lying in the slice $s=a_+$ (resp. $s=a_-$) together with Morse flow lines of $\tilde{F}$. If it is a rigid Morse flow line of $\tilde{F}$ on L_Σ, it flows from a critical point on Λ_+ to a

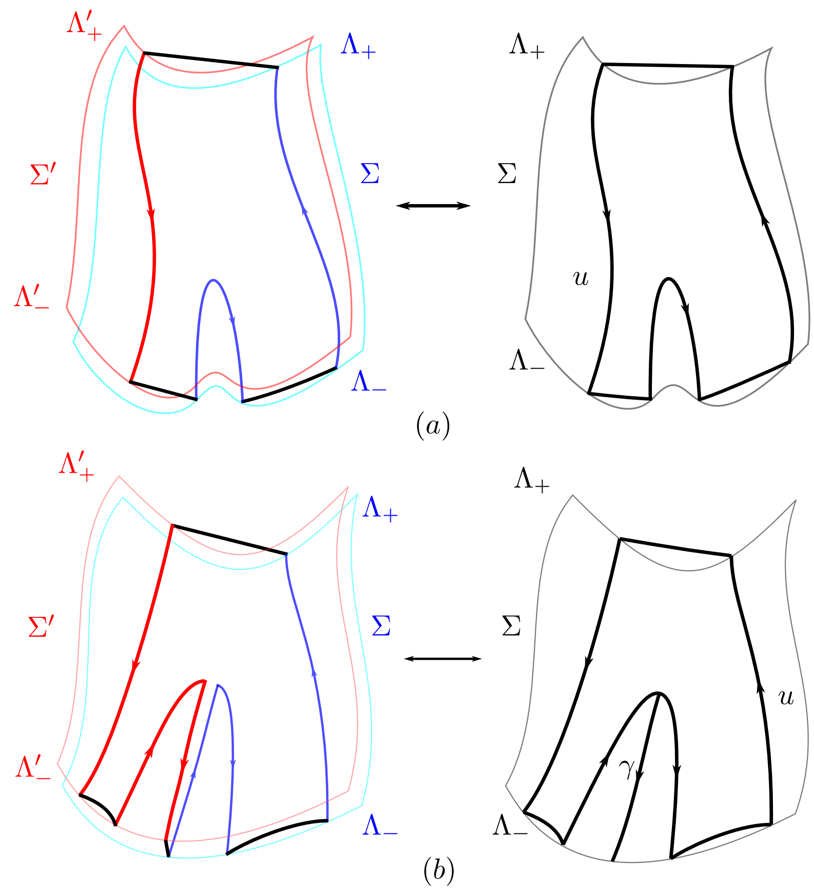


FIGURE 5.3: Schematic picture of the correspondences in Theorem 5.3.3. The arrows denote the orientation of holomorphic disks and the negative gradient flow line.

critical point on Λ_-. Pull it back to be a flow line Σ and get the correspondence (3). If it is a rigid holomorphic disk with boundary on L_Σ that has positive (resp. negative) boundary at the Reeb chords lying in the slice $s = a_+$ (resp. $s = a_-$), by [EHK16], it corresponds to a rigid holomorphic disk with boundary on Σ, which is the respondence (1). Otherwise, it it is a rigid generalized disk (u, γ) determined by $(\tilde{F}, g, J)$. From the construction of $\tilde{F}$, one can note that all the critical points of $\tilde{F}$ on $\Sigma \cap (\{N\} \times \mathbb{R}^3)$ are of index 1 or 2 while all the critical points of $\tilde{F}$ on $\Sigma \cap (\{-N\} \times \mathbb{R}^3)$ are of index 0 or 1. By the dimension formula (5.5), the generalized disk (u, γ) is rigid if and only if u is a rigid holomorphic disk. Each rigid holomorphic disk u with boundary on L_Σ that has positive (resp. negative) boundary at the Reeb chords lying in the slice $s = a_+$ (resp. $s = a_-$) in turn corresponds to a rigid holomorphic disk with boundary on Σ. Pulling γ back to Σ and get a rigid generalized disk on Σ determined by $(\tilde{F}, g, J)$. Hence we get the correspondence (2). □

Recall that f_1 is defined algebraically as follows. The exact Lagrangian cobordism Σ from a Legendrian knot Λ_- to a Legendrian knot Λ_+ induces a DGA map ϕ between the DGA's with a single base point by counting rigid holomorphic disks with boundary on Σ:

$$\phi : (\mathcal{A}(\Lambda_+), \partial) \to (\mathcal{A}(\Lambda_-), \partial),$$

as described in Section 2.3. This DGA map ϕ induces an A_∞ category map

$$f : \mathcal{A}ug_+(\Lambda_-) \to \mathcal{A}ug_+(\Lambda_+)$$

in the way described in Section 3.2. Restrict the category map on the level of morphisms, we have

$$f_1 : Hom_+(\epsilon_-^1, \epsilon_-^2) \to Hom_+(\epsilon_+^1, \epsilon_+^2).$$

See calculation (3.3) for the explicit formula.

Theorem 5.3.4. *With a choice of Morse function $F : \Sigma \to \mathbb{R}$, we have $d_{+-} = f_1$.*

Proof. We show $d_{+-} = f_1$ by checking their images on generators of $Hom_+(\epsilon_-^1, \epsilon_-^2)$. Recall that $Hom_+(\epsilon_-^1, \epsilon_-^2)$ are generated by the elements in $Hom_-(\epsilon_-^1, \epsilon_-^2)$ that correspond to non–Morse Reeb chords and the elements in $T = \{x_-^\vee, y_-^\vee\}$ that correspond to Morse Reeb chords, respectively.

First consider the element $b^\vee$ in $Hom_-(\epsilon_-^1, \epsilon_-^2)$. Notice that Morse Reeb chords are much shorter than non–Morse Reeb chords. The energy restriction ensures that $d_{+-}(b^\vee)$ does not include any element in T. Therefore d_{+-} sends $b^\vee$ to $a^\vee \in$

$Hom_-(\epsilon_+^1, \epsilon_+^2)$ by counting rigid holomorphic disks $u \in \mathcal{M}(a^{12}; \mathbf{p^{11}}, b^{12}, \mathbf{q^{22}})$ with boundary on $\Sigma^1 \cup \Sigma^2$, where $\mathbf{p^{11}}$ and $\mathbf{q^{22}}$ are words of pure Reeb chords of Λ_-^1 and Λ_-^2, respectively. According to the correspondence (1) in Theorem 5.3.3, these disks correspond to rigid holomorphic disks $u \in \mathcal{M}(a; \mathbf{p}, b, \mathbf{q})$ with boundary on Σ (as shown in Figure 5.4), which are the disks counted in f_1. Notice that both d_{+-} and f_1 sends $b^\vee$ to $|\mathcal{M}(a; \mathbf{p}, b, \mathbf{q})| \epsilon_-^1(\mathbf{p}) \epsilon_-^2(\mathbf{q}) a^\vee$, where $|\mathcal{M}(a; \mathbf{p}, b, \mathbf{q})|$ is the number of rigid disks in $\mathcal{M}$ counted with sign. Hence the definition of d_{+-} matches the definition of f_1 on $Hom_-(\epsilon_-^1, \epsilon_-^2)$.

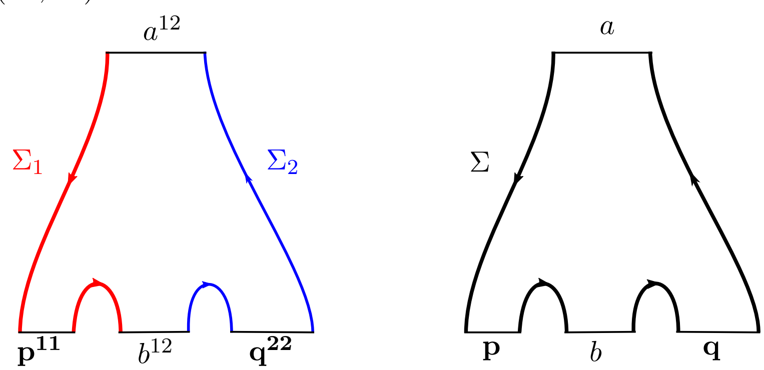


FIGURE 5.4: The disk on the left is counted in d_{+-} while the disk on the right is counted in ϕ.

In order to simplify the map d_{+-}, we can choose a Morse function F such that the negative gradient flow of F flows from $*_+$ directly to $*_-$ without going through any critical points. We can further require that the negative gradient flow of F behave the same in a collar neighborhood of the flow line from $*_+$ to $*_-$ as shown in Figure 5.5. As $x_\pm$ and $y_\pm$ sit right besides $*_\pm$, the negative gradient flow lines of $\tilde{F}$ flow from x_+ and y_+ directly to x_- and y_-, respectively.

For the element $c^\vee \in T$, the map d_{+-} counts the rigid holomorphic disks in $\Sigma_1 \cup \Sigma_2$ that have a negative puncture at the Morse Reeb chord c. For the rigid disk that has a positive puncture at a Morse Reeb chord as well, according to the correspondence (3) in Theorem 5.3.3, it corresponds to a rigid Morse flow line of $\tilde{F}$. The indices of

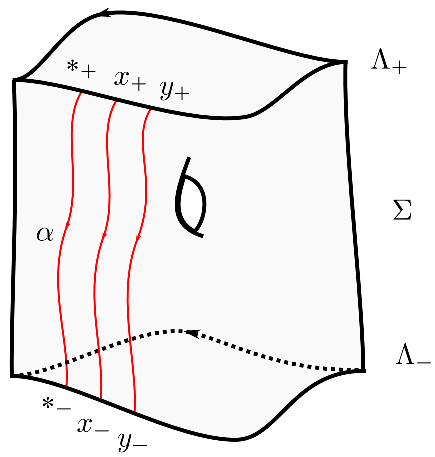


FIGURE 5.5: An example of Morse flows of F.

$\tilde{F}$ on y_-, x_-, y_+ and x_+ are 0, 1, 1 and 2 respectively. Therefore $d_{+-}(x_-^\vee)$ has $x_+^\vee$ as a term and $d_{+-}(y_-^\vee)$ has $y_+^\vee$ as a term. If the rigid disk that has a positive puncture at a non–Morse mixed Reeb chord a^{12}, we denote it by $u \in \mathcal{M}(a^{12}; \mathbf{p^{11}}, c^{12}, \mathbf{q^{22}})$. By the correspondence (2) in Theorem 5.3.3, it corresponds to a rigid generalized disk (u, γ), where $u \in \mathcal{M}(a; \mathbf{p}, \mathbf{q})$ is a holomorphic disk with boundary on Σ and γ is a Morse flow of $\tilde{F}$ that flows towards c (see Figure 5.6). Due to the dimension formula (5.5) of generalized disks, no rigid disk has a negative puncture at y_- since $Ind_{\tilde{F}} y_- = 0$ but $\dim \mathcal{M} \geqslant 0$. Hence $d_{+-}(y_-^\vee) = y_+^\vee$, which matches the definition of f_1 on y_k^-.

For the element $x_-^\vee$, we know that $f_1(x_+^\vee)$ counts the element $a^\vee$ if t shows up in the image of the DGA map $\phi(a)$. In other words, there exists a rigid holomorphic disk $u \in \mathcal{M}(a; \mathbf{p}, \mathbf{q})$ with boundary on Σ, where $\mathbf{p}$ and $\mathbf{q}$ are words of pure Reeb chords of Λ_-, such that u has a nontrivial intersection number with α, where α is the curve from the base point $*_+$ to $*_-$. Each rigid holomorphic disk u contributes to $f_1(x_+^\vee)$ a term of $a^\vee$ with coefficient $s(u, \alpha)\epsilon_-^1(\mathbf{p})\epsilon_-^2(\mathbf{q})\mathbf{a}^\vee$, where $s(u, \alpha)$ is the intersection number of the boundary of u and α. We can make the Morse function F satisfy the

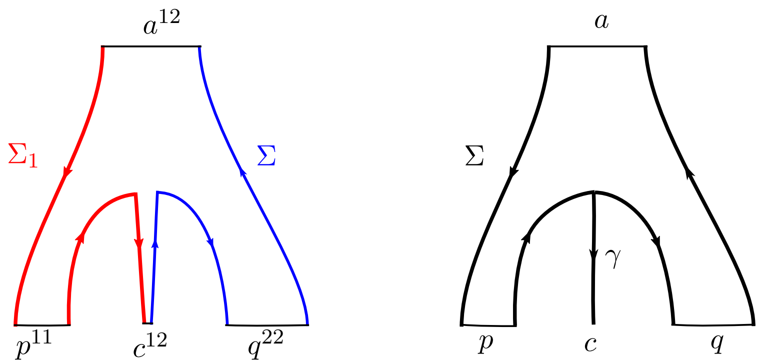


FIGURE 5.6: The disk on the left is counted in d_{+-} while the disk on the right is counted by ϕ.

property that the negative gradient flow line γ of F from x_+ to x_- is parallel to α and of the same orientation. For each intersection point p_i of the boundary of u and γ, denote the part of γ from p_i to c by γ_i. By the correspondence (3) in Theorem 5.3.3, the rigid generalized disk (u, γ_i) corresponds to a rigid holomorphic disk in $\mathcal{M}(a^{12}; \mathbf{p^{11}}, c^{12}, \mathbf{q^{22}})$ with boundary on $\Sigma^1 \cup \Sigma^2$, and hence contributes to $d_{+-}(x_-^\vee)$ with a term $s(u,\gamma_i)\epsilon_-^1(\mathbf{p^{11}})\epsilon_-^2(\mathbf{q^{22}})\mathbf{a}^\vee$, where $s(u,\gamma_i)$ is the sign of the intersection. Sum over all the intersections of the boundary of u and γ, the rigid holomorphic disk u contributes $s(u,\gamma)\epsilon_-^1(\mathbf{p})\epsilon_-^2(\mathbf{q})\mathbf{a}^\vee$ to $d_{+-}(x_-^\vee)$, where $s(u,\gamma)$ is the intersection number of the boundary of u and γ. Therefore, we have $d_{+-} = f_1$ on $x_-^\vee$. □

5.4 Aside.

In this section, we describe the differential map of the Cthulhu chain complex in terms of holomorphic disks with boundary on Σ and Morse flow lines. This allows us to recover the long exact sequences in [CDRGG15]. The theorem in this section is stated without rigorous proof. But it will not be used in the other part of the dissertation.

In Section 5.3, we only need to describe the rigid disks with boundary on $\Sigma^1 \cup \Sigma^2$ that have punctures at Reeb chords. Hence we only have correspondences for those types of disks. However, the method should work for all the rigid holomorphic disks with boundary on $\Sigma^1 \cup \Sigma^2$ including the disks counted by d_{+0} and d_{0-}. We state the following theorem without proof.

Theorem 5.4.1. *Let Σ be an exact Lagrangian cobordism from Λ_- to Λ_+ and $\Sigma^1 \cup \Sigma^2$ be a 2–copy of Σ as constructed in Section 5.1. For $i = 1, 2$, assume ϵ^i_- is an augmentation of $\mathcal{A}(\Lambda_-)$ and ϵ^i_+ is the augmentation of $\mathcal{A}(\Lambda_+)$ induced by Σ. For η small enough, the Cthulhu chain complex can be decomposed into five parts:*

$$Cth^k(\Sigma^1, \Sigma^2) = Hom_-^{k-1}(\epsilon^1_+, \epsilon^2_+) \oplus C^{k-1}_{Morse} f_+ \oplus C^k_{Morse} F \oplus Hom^k_-(\epsilon^1_-, \epsilon^2_-) \oplus C^k_{Morse} f_-.$$

Under this decomposition, the differential can be written as

$$d = \begin{pmatrix} m_1 & d_{+f_+} & d_{+F} & d_{+-} & d_{+f_-} \\ 0 & d_{f_+} & d_{f_+F} & 0 & d_{f_+f_-} \\ 0 & 0 & d_F & 0 & d_{Ff_-} \\ 0 & 0 & 0 & m_1 & d_{-f_-} \\ 0 & 0 & 0 & 0 & d_{f_-} \end{pmatrix}$$

Moreover,

1. *The holomorphic disks counted by d_{+F} and d_{+f_-} are in $1-1$ correspondence with rigid generalized disks on Σ determined by $(\tilde{F}, g, J)$.*
2. *The holomorphic disks counted by d_{+-} are in $1-1$ correspondence with rigid holomorphic disks with boundary on Σ.*
3. *The holomorphic disks counted by d_{f_+F}, $d_{f_+f_-}$ and d_{Ff_-} are in $1-1$ correspondence with rigid Morse flow lines of $\tilde{F}$.*

This theorem is similar to the conjectural analytic Lemma in [Ekh12, Lemma 4.11], which describes the correspondence in the case of exact Lagrangian fillings.

We next discuss how to recover the three long exact sequences in [CDRGG15] from this chain complex.

1. Decompose the Cthulhu chain complex as

$$Hom_-^{k-1}(\epsilon_+^1, \epsilon_+^2) \oplus (C_{Morse}^{k-1} f_+ \oplus C_{Morse}^k F) \oplus Hom_+^k(\epsilon_-^1, \epsilon_-^2).$$

Notice that the chain complex

$$\left(C_{Morse}^{k-1} f_+ \oplus C_{Morse}^k F, \begin{pmatrix} d_{f_+} & d_{f_+F} \\ 0 & d_F \end{pmatrix} \right)$$

can be identified with the Morse co-chain complex $(C_{Morse}^k \bar{F}, d_{\bar{F}})$ induced by a Morse function $\bar{F}$, where $\bar{F}$ agrees with $\tilde{F}$ near Λ_+ and agrees with F for the rest part. Hence

$$H^*(C_{Morse}^k \bar{F}, d_{\bar{F}}) = H^k(\Sigma, \Lambda_+ \cup \Lambda_-).$$

Therefore, we have the following long exact sequence:

$$\cdots \to H^k(\Sigma, \Lambda_+ \cup \Lambda_-) \to H^k Hom_-(\epsilon_+^1, \epsilon_+^2) \to H^k Hom_+(\epsilon_-^1, \epsilon_-^2) \to H^{k+1}(\Sigma, \Lambda_+ \cup \Lambda_-) \to \cdots,$$

which is Theorem 1.5 of [CDRGG15].

2. View the Cthulhu chain complex as a direct sum of $Hom_-^{k-1}(\epsilon_+^1, \epsilon_+^2)$, $C_{Morse}^{k-1} f_+ \oplus C_{Morse}^k F \oplus Hom_-^k(\epsilon_-^1, \epsilon_-^2)$ and the Morse co–chain complex $C_{Morse}^k f_-$. We have the following long exact sequence:

$$\cdots \to H^{k-1}(\Lambda_-) \to H^k(\Sigma, \Lambda_+ \cup \Lambda_-) \oplus H^k Hom_-(\epsilon_-^1, \epsilon_-^2) \to H^k Hom_-(\epsilon_+^1, \epsilon_+^2) \to H^k(\Lambda_-) \to \cdots,$$

which is Theorem 1.6 in [CDRGG15].

3. Rewrite the Cthulhu chain complex as

$$\begin{aligned} Cth^k(\Sigma^1, \Sigma^2) &= Hom_-^{k-1}(\epsilon_+^1, \epsilon_+^2) \oplus Hom_-^k(\epsilon_-^1, \epsilon_-^2) \oplus C_{Morse}^{k-1} f_+ \oplus C_{Morse}^k F \oplus C_{Morse}^k f_- \\ &= Hom_-^{k-1}(\epsilon_+^1, \epsilon_+^2) \oplus Hom_-^k(\epsilon_-^1, \epsilon_-^2) \oplus C_{Morse}^k \tilde{F} \end{aligned}$$

with the differential

$$d = \begin{pmatrix} m_1 & * & * \\ 0 & m_1 & * \\ 0 & 0 & d_{\tilde{F}} \end{pmatrix}.$$

We have the long exact sequence:

$$\cdots \to H^k Hom_-(\epsilon^1_-, \epsilon^2_-) \to H^k Hom_-(\epsilon^1_+, \epsilon^2_+) \to H^{k+1}(\Sigma, \Lambda_+) \to H^{k+1} Hom_-(\epsilon^1_-, \epsilon^2_-) \to \cdots ,$$

which is Theorem 1.4 in [CDRGG15].

One may get other long exact sequences from the Cthulhu chain complex above. One example is :

$$\cdots \to H^k Hom_-(\epsilon^1_-, \epsilon^2_-) \to H^k Hom_+(\epsilon^1_+, \epsilon^2_+) \to H^k(\Sigma) \to H^{k+1} Hom_-(\epsilon^1_-, \epsilon^2_-) \to \cdots ,$$

which is obtained by decomposing the Cthulhu chain complex as a direct sum of $Hom^{k-1}_+(\epsilon^1_+, \epsilon^2_+)$, $Hom^k_-(\epsilon^1_-, \epsilon^2_-)$ and $C^k_{Morse}F \oplus C^k_{Morse}f_-$.

5.5 Injectivity

Theorem 5.3.4 implies that d_{+-} is a chain map and hence it gives the Cthulhu chain complex a stronger algebraic structure. In this section, we use these algebraic information to deduce that the augmentation category map induced by the exact Lagrangian cobordism Σ is injective on the level of equivalence classes of objects. And its induced map on the cohomology category $H^*\mathcal{A}ug_+$ is faithful.

Notice that $d_{+-} = f_1$ implies that d_{+-} is a chain map and thus induces maps $d^k_{+-} : H^k Hom_+(\epsilon^1_-, \epsilon^2_-) \to H^k Hom_+(\epsilon^1_+, \epsilon^2_+)$ for $k \in \mathbb{Z}$. Then we have the following theorem deduced from the double cone structure of the Cthulhu chain complex. We would like to thank the referee for pointing out this theorem.

Theorem 5.5.1. *Let Σ be an exact Lagrangian cobordism from a Legendrian knot Λ_- to a Legendrian knot Λ_+ with Maslov number 0. For $i = 1, 2$, assume ϵ^i_- is an augmentation of $\mathcal{A}(\Lambda_-)$ and ϵ^i_+ is the augmentation of $\mathcal{A}(\Lambda_+)$ induced by Σ. With the same choice of Morse function as in Theorem 5.3.4, we have the following statement.*

For fixed $k \in \mathbb{Z}$, the map

$$i^k : H^k Hom_+(\epsilon^1_+, \epsilon^2_+) \to H^k Hom_+(\epsilon^1_-, \epsilon^2_-)$$

in the long exact sequence (5.2) is injective (resp. surjective) if and only if the map

$$d^k_{+-} : H^k Hom_+(\epsilon^1_-, \epsilon^2_-) \to H^k Hom_+(\epsilon^1_+, \epsilon^2_+)$$

is surjective (resp. injective).

Proof. We will first prove that i^k is surjective if and only if d^k_{+-} is injective for fixed k.

Consider the Cthulhu chain complex as a mapping cone of $\Phi : Hom_+(\epsilon^1_-, \epsilon^2_-) \to Cone(d_{+0})$, where $\Phi = d_{+-} + d_{0-}$. The trivial cohomology of the Cthulhu chain complex implies that Φ induces isomorphisms

$$\Phi^k : H^k Hom_+(\epsilon^1_-, \epsilon^2_-) \to H^k Cone(d_{+0}) \text{ for } k \in \mathbb{Z}.$$

We have the following diagram commutes.

$$\begin{array}{ccccccc} \cdots \longrightarrow H^k Hom_+(\epsilon^1_+, \epsilon^2_+) & \xrightarrow{i_k} & H^k Cone(d_{+0}) & \xrightarrow{j_k} & H^{k+1}(C_{\text{Morse}} F) \longrightarrow \cdots \\ & & \Phi^k \uparrow \cong & \nearrow d^k_{0-} & \\ & & H^k Hom_+(\epsilon^1_-, \epsilon^2_-) & & \end{array}$$

Comparing to the long exact sequence (5.2), we know that $i^k = (\Phi^k)^{-1} \circ i_k$. Notice that d_{+-} is a chain map and thus $d_{+0} \circ d_{0-} = 0$. It is not hard to show that $\Phi^k = d^k_{+-} + d^k_{0-}$. Thus

$$H^k Cone(d_{+0}) \cong d^k_{+-}\big(H^k Hom_+(\epsilon^1_-, \epsilon^2_-)\big) \oplus d^k_{0-}\big(H^k Hom_+(\epsilon^1_-, \epsilon^2_-)\big).$$

Since we are working over the field $\mathbb{F}$, we have the following relation on dimensions:

$$\begin{aligned} \dim\big(H^k Hom_+(\epsilon^1_-, \epsilon^2_-)\big) &= \dim\big(H^k Cone(d_{+0})\big) \\ &= \dim\big(d^k_{+-}\big(H^k Hom_+(\epsilon^1_-, \epsilon^2_-)\big)\big) + \dim\big(d^k_{0-}\big(H^k Hom_+(\epsilon^1_-, \epsilon^2_-)\big)\big). \end{aligned}$$

Thus $\dim\big(d^k_{+-}\big(H^k Hom_+(\epsilon^1_-, \epsilon^2_-)\big)\big) \leqslant \dim\big(H^k Hom_+(\epsilon^1_-, \epsilon^2_-)\big)$ and the equality holds if and only if $d^k_{0-} = 0$, which is equivalent to the condition that i^k is surjective. Hence d^k_{+-} is injective if and only if i^k is surjective.

The proof of the statement that i^k is injective if and only if d_{+-}^k is surjective is basically the same if we consider the Cthulhu chain complex as a mapping cone of $\Psi : Cone(d_{0-}) \to Hom_+(\epsilon_+^1, \epsilon_+^2)$, where $\Psi = d_{+0} + d_{+-}$. □

Thanks to Theorem 5.3.4, we know that d_{+-} agrees with f_1. Theorem 5.2.5 shows that i^0 is both injective and surjective. Therefore we have the following corollary.

Corollary 5.5.2. *Let f^* denote the induced map of f_1 on cohomology. Then f^* restricted on 0 degree cohomology*

$$f^* : H^0 Hom_+(\epsilon_-^1, \epsilon_-^2) \to H^0 Hom_+(\epsilon_+^1, \epsilon_+^2)$$

is an isomorphism.

Theorem 5.5.3. *Let Σ be an exact Lagrangian cobordism from a Legendrian knot Λ_- to a Legendrian knot Λ_+ with Maslov number 0. The A_∞ category map f : $\mathcal{A}ug_+(\Lambda_-) \to \mathcal{A}ug_+(\Lambda_+)$ induced by the exact Lagrangian cobordism Σ is injective on the level of equivalence classes of objects.*

In other words, for $i = 1, 2$, assume ϵ_-^i is an augmentation of $\mathcal{A}(\Lambda_-)$ with a single base point and ϵ_+^i is the augmentation of $\mathcal{A}(\Lambda_+)$ with a single base point induced by Σ. If ϵ_+^1 and ϵ_+^2 are equivalent in $\mathcal{A}ug_+(\Lambda_+)$, then ϵ_-^1 and ϵ_-^2 are equivalent in $\mathcal{A}ug_+(\Lambda_-)$.

Proof. Since ϵ_+^1 and ϵ_+^2 are equivalent in $\mathcal{A}ug_+(\Lambda_+)$, there exist $[\alpha_+] \in H^0 Hom_+(\epsilon_+^1, \epsilon_+^2)$ and $[\beta_+] \in H^0 Hom_+(\epsilon_+^2, \epsilon_+^1)$ such that

$$[m_2(\alpha_+, \beta_+)] = [e_{\epsilon_+^2}] \in H^0 Hom_+(\epsilon_+^2, \epsilon_+^2)$$

and

$$[m_2(\beta_+, \alpha_+)] = [e_{\epsilon_+^1}] \in H^0 Hom_+(\epsilon_+^1, \epsilon_+^1).$$

Corollary 5.5.2 shows that $f^* : H^0 Hom_+(\epsilon_-^1, \epsilon_-^2) \to H^0 Hom_+(\epsilon_+^1, \epsilon_+^2)$ is an isomorphism. Hence there exists $[\alpha_-] \in H^0 Hom_+(\epsilon_-^1, \epsilon_-^2)$ such that

$$f^*([\alpha_-]) = [\alpha_+] \in H^0 Hom_+(\epsilon_+^1, \epsilon_+^2).$$

Similarly, there exists $[\beta_-] \in H^0 Hom_+(\epsilon_-^2, \epsilon_-^1)$ such that

$$f^*([\beta_-]) = [\beta_+] \in H^0 Hom_+(\epsilon_+^2, \epsilon_+^1).$$

Moreover, we have

$$f^*[m_2(\alpha_-, \beta_-)] = m_2(f^*([\alpha_-]), f^*([\beta_-])) = m_2([\alpha_+], [\beta_+]) = [e_{\epsilon_+^2}] \in H^0 Hom_+(\epsilon_+^2, \epsilon_+^2).$$

Notice that f sends $y_-^\vee \in Hom_+^0(\epsilon_-^2, \epsilon_-^2)$ to $y_+^\vee \in Hom_+^0(\epsilon_+^2, \epsilon_+^2)$ and hence $f^*[e_{\epsilon_-^2}] = [e_{\epsilon_+^2}]$. By Corollart 5.5.2, the map $f^* : H^0 Hom_+(\epsilon_-^2, \epsilon_-^2) \to H^0 Hom_+(\epsilon_+^2, \epsilon_+^2)$ is an isomorphism. Hence $[m_2(\alpha_-, \beta_-)] = [e_{\epsilon_-^2}] \in H^0 Hom_+(\epsilon_-^2, \epsilon_-^2)$. Similarly, we have $[m_2(\beta_-, \alpha_-)] = [e_{\epsilon_-^1}] \in H^0 Hom_+(\epsilon_-^1, \epsilon_-^1)$. Therefore ϵ_-^1 and ϵ_-^2 are equivalent in $\mathcal{A}ug_+(\Lambda_-)$. □

In addition, the exact Lagrangian cobordism Σ described above also induces a category functor on the cohomology category as described in Section 3.1

$$\tilde{f} : H^* \mathcal{A}ug_+(\Lambda_-) \to H^* \mathcal{A}ug_+(\Lambda_+).$$

We have the following statement.

Theorem 5.5.4. *Let Σ be an exact Lagrangian cobordism from a Legendrian knot Λ_- to a Legendrian knot Λ_+ with Maslov number 0. The corresponding cohomology category map $\tilde{f} : H^* \mathcal{A}ug_+(\Lambda_-) \to H^* \mathcal{A}ug_+(\Lambda_+)$ induced by Σ is faithful. Moreover, if $\chi(\Sigma) = 0$, this functor is fully faithful.*

Proof. Notice that the category map $\tilde{f}$ restricted on the level of morphisms is

$$f^* : H^* Hom_+(\epsilon_-^1, \epsilon_-^2) \to H^* Hom_+(\epsilon_+^1, \epsilon_+^2).$$

The long exact sequence (5.2) tells us that i^k are surjective for all the $k \in \mathbb{Z}$. By Theorem 5.5.1, we know that f^* is injective. Therefore $\tilde{f}$ is faithful.

In particular, if $\chi(\Sigma) = 0$, Theorem 5.2.5 implies that i^k are isomorphisms for all the $k \in \mathbb{Z}$. Therefore, by Theorem 5.5.1, the map f^* is an isomorphism, which implies that $\tilde{f}$ is fully faithful. □

As a result of Theorem 5.5.3, there is an induced map from the equivalence classes of augmentations of Λ_- to the equivalence classes of augmentations of $\mathcal{A}ug_+(\Lambda_+)$. Thus the number of equivalence classes of augmentations of $\mathcal{A}(\Lambda_-)$ is less than or equal to the number of equivalence classes of augmentations of $\mathcal{A}(\Lambda_+)$. However, the equivalence classes of augmentations is difficult to count in general. Ng, Rutherford, Shende and Sivek [NRSS] introduced another way to count objects: the homotopy cardinality of $\pi_{\geqslant 0}\mathcal{A}ug_+(\Lambda; \mathbb{F}_q)^*$, where $\mathbb{F}_q$ is a finite field. This can be computed using ruling polynomials.

The **homotopy cardinality** is defined by

$$\pi_{\geqslant 0}\mathcal{A}ug_+(\Lambda; \mathbb{F}_q)^* = \sum_{[\epsilon] \in \mathcal{A}ug_+(\Lambda; \mathbb{F}_q)/\sim} \frac{1}{|Aut(\epsilon)|} \cdot \frac{|H^{-1}Hom_+(\epsilon,\epsilon)| \cdot |H^{-3}Hom_+(\epsilon,\epsilon)| \cdots}{|H^{-2}Hom_+(\epsilon,\epsilon)| \cdot |H^{-4}Hom_+(\epsilon,\epsilon)| \cdots},$$

where $[\epsilon]$ is the equivalence class of ϵ in the augmentation category $\mathcal{A}ug_+(\Lambda)$ and $|Aut(\epsilon)|$ is the number of invertible elements in $H^0Hom_+(\epsilon, \epsilon)$.

Corollary 5.5.5. *Let Σ be a spin exact Lagrangian cobordism from a Legendrian knot Λ_- to a Legendrian knot Λ_+ with Maslov number 0. Then for any finite field $\mathbb{F}_q$, we have*

$$\pi_{\geqslant 0}\mathcal{A}ug_+(\Lambda_-; \mathbb{F}_q)^* \leqslant \pi_{\geqslant 0}\mathcal{A}ug_+(\Lambda_+; \mathbb{F}_q)^*.$$

Proof. Assume $[\epsilon_-]$ is an equivalence class in $\mathcal{A}ug_+(\Lambda_-; \mathbb{F}_q)$ and $[\epsilon_+]$ is the induced equivalence class in $\mathcal{A}ug_+(\Lambda_+; \mathbb{F}_q)$. Theorem 5.2.5 implies

$$H^kHom_+(\epsilon_-, \epsilon_-) \cong H^kHom_+(\epsilon_+, \epsilon_+) \text{ for } k < 1.$$

In particular, we have $H^0Hom_+(\epsilon_-, \epsilon_-) \cong H^0Hom_+(\epsilon_+, \epsilon_+)$, which implies $|Aut(\epsilon_-)| = |Aut(\epsilon_+)|$. Notice that $\mathcal{A}ug_+(\Lambda_+; \mathbb{F}_q)$ may have more equivalence classes than $\mathcal{A}ug_+(\Lambda_-; \mathbb{F}_q)$.

Therefore, we have

$$\pi_{\geqslant 0}\mathcal{A}ug_{+}(\Lambda_{-};\mathbb{F}_q)^{*} \leqslant \pi_{\geqslant 0}\mathcal{A}ug_{+}(\Lambda_{+};\mathbb{F}_q)^{*}.$$

□

From [NRSS, Corollary 2], this cardinality can be related to the ruling polynomial in the following way:

$$\pi_{\geqslant 0}\mathcal{A}ug_{+}(\Lambda;\mathbb{F}_q)^{*} = q^{tb(\Lambda)/2}R_{\Lambda}(q^{1/2}-q^{-1/2}).$$

Recall that a **normal ruling** R is a decomposition of the front projection of Λ into embedded disks connected by switches that satisfy some requirements (see details in [Che02a]). The **ruling polynomial** is defined by

$$R_{\Lambda}(z) = \sum_{R} z^{\#(\text{switches})-\#(\text{diskes})}.$$

Corollary 5.5.6. *Suppose there is a spin exact Lagrangian cobordism from a Legendrian knot Λ_- to a Legendrian knot Λ_+ with Maslov number* 0*. Then the ruling polynomials satisfy:*

$$R_{\Lambda_-}(q^{1/2}-q^{-1/2}) \leqslant q^{-\chi(\Sigma)/2}R_{\Lambda_+}(q^{1/2}-q^{-1/2}),$$

for any q that is a power of a prime number.

When Σ is decomposable, i.e. consists of pinch moves and minimum cobordisms [EHK16], there is a map from the rulings of Λ_- to rulings of Λ_+. For each pinch move or minimal cobordism, any normal ruling of the bottom knot gives a normal ruling of the top knot, as shown in Figure 5.7. Moreover, different rulings of the bottom knot give different rulings of the top knot. Therefore the ruling polynomials of Λ_+ and Λ_- satisfy the relation in Corollary 5.5.6. This corollary shows that the result is true even if the cobordism is not decomposable.

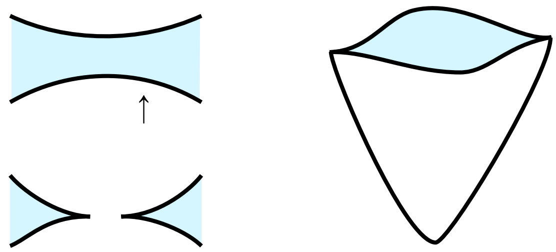


FIGURE 5.7: The relation between rulings of Legendrian submanifolds that are related by a pinch move or a minimum cobordism.

One can check the atlas in [CN13] for the ruling polynomials of small crossing Legendrian knots. This corollary gives a new and easily computable obstruction to the existence of exact Lagrangian cobordisms. We can use Corollary 5.5.6 to give a new proof of the follow theorem which is a result in [Cha15] and was reproved by [CNS16].

Theorem 5.5.7 ([Cha15])**.** *Lagrangian concordance is not a symmetric relation.*

Proof. Consider the Legendrian knot Λ of smooth knot type $m(9_{46})$ with maximum Thurston–Bennequin number and the Legendrian unknot Λ_0 as shown in Figure 5.8. There is an exact Lagrangian concordance from the Λ_0 to Λ by doing a pinch move at the red line in Figure 5.8 and Legendrian isotopy. However, there does not exist an exact Lagrangian concordance from Λ to Λ_0 since the ruling polynomial of Λ is 2 while the ruling polynomial of Λ_0 is 1. $\square$

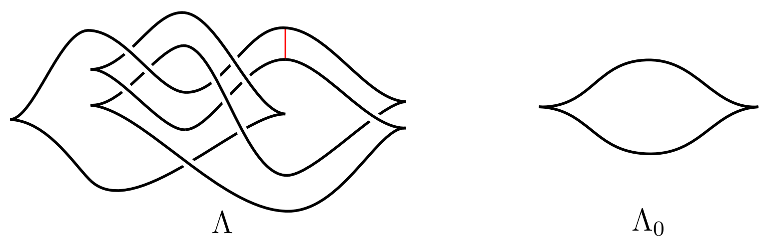


FIGURE 5.8: Front projections of Legendrian knot Λ of knot type $m(9_{46})$ and the Legendrian unknot Λ_0.

6

Exact Lagrangian fillings of Legendrian $2, n$ torus links

6.1 Introduction and results

In this section, we move to the second project, which studies oriented exact Lagrangian fillings of the Legendrian $(2, n)$ torus links Λ with maximal Thurston–Bennequin number ($n > 0$) (see Figure 6.1). When n is even, we also require the link to have the right Maslov potential such that Reeb chords $b_1, \dots, b_n$ in Figure 6.1 are in degree 0 (see Section 2.2.1 for detailed definitions).

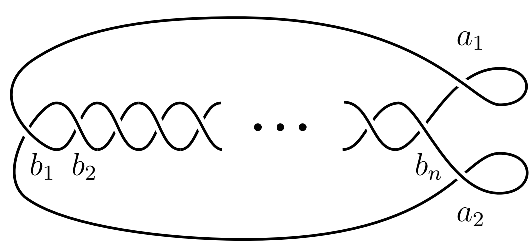
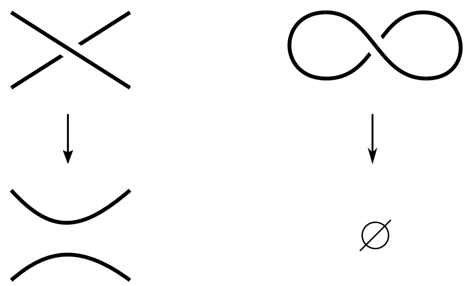


FIGURE 6.1: The Lagrangian projection of the Legendrian $(2, n)$ torus knot.

Ekholm, Honda, and Kálmán [EHK16] gave an algorithm to construct exact Lagrangian fillings of the Legendrian $(2, n)$ torus link Λ as follows. Starting with a Lagrangian projection of Λ as shown in Figure 6.1, we can successively resolve crossings b_i in any order through pinch moves (see Figure 6.2), which correspond to saddle cobordisms. As a result, we get two Legendrian unknots, which admit minimum cobordisms as shown in Figure 6.2. Concatenating the n saddle cobordisms with these two minimum cobordisms, we get an exact Lagrangian filling of Λ.



The pinch move   The minimum cobordism

FIGURE 6.2: The pinch move and the minimum cobordism between Lagrangian projections of links.

Different orders of resolving crossings $b_1, \ldots, b_n$ may give different exact Lagrangian fillings of Λ up to exact Lagrangian isotopy. Given a permutation $\sigma = \big(\sigma(1), \sigma(2), \ldots, \sigma(n)\big)$ of $\{1, \ldots, n\}$, write L_σ for the exact Lagrangian filling achieved by doing n successive pinch moves at $b_{\sigma(1)}$, $b_{\sigma(2)}$, $\ldots$, $b_{\sigma(n)}$, respectively, and then concatenating with the two minimum cobordisms. Observe that two permutations may give isotopic exact Lagrangian fillings. For instance, let Λ be the Legendrian $(2,3)$ torus knot and consider the exact Lagrangian fillings of Λ that correspond to permutations $(1,3,2)$ and $(3,1,2)$, respectively. Since the saddles corresponding to the pinch moves at b_1 and b_3 are disjoint when projected to $\mathbb{R}^3$, one can use a Hamiltonian vector field in the t direction to exchange the heights of these two saddles. Therefore, the two fillings $L_{(1,3,2)}$ and $L_{(3,1,2)}$ are Hamiltonian isotopic and thus are exact Lagrangian isotopic. In general, for the Legendrian $(2,n)$ torus link Λ, given any numbers i, j, k such that $i < k < j$, two permutations $(\ldots, i, j, \ldots, k, \ldots)$ and

$(\dots, j, i, \dots, k, \dots)$, where only i and j are interchanged, give the same exact Lagrangian fillings of Λ up to exact Lagrangian isotopy. Taking all the permutations of $\{1, \dots, n\}$ modded out by this relation, we obtain C_n exact Lagrangian fillings of Λ, where

$$C_n = \frac{1}{n+1}\binom{2n}{n}$$

is the n–th Catalan number. In this section, we prove the following theorem.

Theorem 6.1.1 (see Theorem 6.3.7 and Corollary 6.3.8)**.** *The C_n exact Lagrangian fillings that come from the algorithm in [EHK16] are all of different exact Lagrangian isotopy classes. In other words, the Legendrian $(2, n)$ torus link has at least C_n exact Lagrangian fillings up to exact Lagrangian isotopy.*

Shende, Treumann, Williams and Zaslow [STWZ15] have also constructed C_n exact Lagrangian fillings of the Legendrian $(2, n)$ torus knot using cluster varieties and shown that they are distinct up to Hamiltonian isotopy. They remarked that these are presumably the same as fillings obtained by [EHK16]. But we do not resolve this issue here.

Remark 6.1.2. We will see from Corollary 6.3.8 that the conclusion of Theorem 6.1.1 for the case n even can be derived from the result for the case when n is odd. Therefore, for most of the dissertation, we focus on the case when n is odd, which means Λ is a knot.

Inspired by [EHK16], we use augmentations to distinguish the C_n exact Lagrangian fillings of the Legendrian $(2, n)$ torus knot Λ. As we mentioned in Section 2.3, exact Lagrangian isotopic fillings give homotopic augmentations. Therefore, in order to distinguish two fillings, we only need to show their induced augmentations are not chain homotopic.

In [EHK16], the authors distinguished all the exact Lagrangian fillings from the algorithm when $n = 3$ by computing all the augmentations of the Legendrian $(2, 3)$ torus knot to $\mathbb{Z}_2$ and finding that they are pairwise non–chain homotopic. However, when $n \geqslant 5$, a computation shows that the number of augmentations of the DGA to $\mathbb{Z}_2$ is much less than the Catalan number C_n.

In this section, for an exact Lagrangian filling L of the Legendrian $(2, n)$ torus knot Λ, we consider its induced augmentation of $\mathcal{A}(\Lambda)$ to $\mathbb{Z}_2[H_1(L)]$, where $H_1(L)$ is the singular homology of L. Note that $H_1(L) \cong H_2(\mathbb{R} \times \mathbb{R}^3, L)$ and thus it is natural to count the rigid holomorphic disks in $\mathbb{R} \times \mathbb{R}^3$ with boundary on L with $\mathbb{Z}_2[H_1(L)]$ coefficients. However, the computation of augmentations is not as easy as the case with $\mathbb{Z}_2$ coefficients. For each exact Lagrangian filling L from the [EHK16] algorithm, we give a combinatorial formula of the induced augmentation of $\mathcal{A}(\Lambda)$ to $\mathbb{Z}_2[H_1(L)]$. Observing from the formula, we find a combinatorial invariant to show that the augmentations are pairwise non–chain homotopic. In this way, we distinguish all the C_n exact Lagrangian fillings of the Legendrian $(2, n)$ torus knot Λ up to exact Lagrangian isotopy.

6.2 Computation of augmentations

In this section, we combine results in [EHK16] and Proposition 2.3.3 to write down combinatorial formulas for the DGA maps induced by a pinch move and a minimum cobordism. Composing all the DGA maps induced by n ordered pinch moves and the two minimum cobordisms, we obtain a combinatorial formula for augmentations of $\mathcal{A}(\Lambda)$ to $\mathbb{Z}_2[H_1(L)]$ induced by exact Lagrangian fillings L.

First, we describe the EHK construction of exact Lagrangian fillings with more details. Consider the Lagrangian projection of the Legendrian $(2, n)$ torus knot Λ with a base point $\tilde{s}_0$ and label the n crossings in degree 0 from left to right by $b_1, \ldots, b_n$ as shown in Figure 6.3.

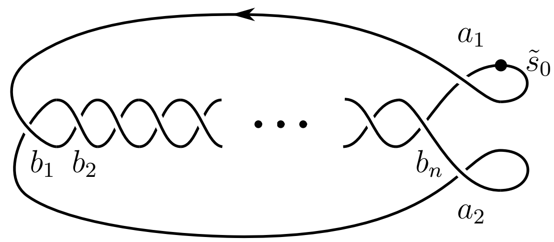


FIGURE 6.3: The Lagrangian projection of the Legendrian $(2, n)$ torus knot with a single base point.

For each permutation σ of $\{1, \ldots, n\}$, the corresponding exact Lagrangian filling L_σ of the Legendrian $(2, n)$ torus knot Λ is achieved in the following way:

- Start with an exact Lagrangian cylinder over Λ, denoted by $\overline{\Sigma}_0$. Label Λ as Λ_0.
- For $i = 1, \ldots, n$, concatenate $\overline{\Sigma}_{i-1}$ from the bottom with a saddle cobordism Σ_i corresponding to the pinch move at crossing $b_{\sigma(i)}$ and get a new exact Lagrangian cobordism $\overline{\Sigma}_i$. Label the new Legendrian submanifold after pinch move as Λ_i.
- Finally, use two minimal cobordisms, denoted by Σ_{n+1}, to close up $\overline{\Sigma}_n$ from the bottom and get the exact Lagrangian filling L_σ. To be consistent, let Λ_{n+1} be the empty set.

Remark 6.2.1. For the purpose of computing augmentations, we slightly change the definition of DGA in Section 2.2, where the underlying algebra is completely non–commutative. In our definition, we allow elements in the coefficient ring to commute with the elements corresponding to Reeb chords. This is a generalization of the definition of Chekanov–Eliashberg DGA from [ENS02]. See [EENS13, Section 2.3.2] for further discussions.

By Proposition 2.3.3, for $i = 1, \ldots, n + 1$, each exact Lagrangian cobordism Σ_i

induces a DGA map:

$$\Phi_i : \big(\mathcal{A}(\Lambda_{i-1}; \mathbb{Z}_2[H_1(\overline{\Sigma}_{i-1})]), \partial_{i-1}\big) \to \big(\mathcal{A}(\Lambda_i; \mathbb{Z}_2[H_1(\overline{\Sigma}_i)]), \partial_i\big).$$

Under the revision made in Remark 6.2.1, this map can be easily described by

$$\Phi_i(a) = \sum_{\dim \mathcal{M}(a;b_1,\ldots,b_m)=0} \; \sum_{u \in \mathcal{M}(a;b_1,\ldots,b_m)} w(\tilde{u}) b_1 \cdots b_m,$$

where $\tilde{u} = \overline{\beta_1} \cup \cdots \cup \overline{\beta_m}$ by the notations in Section 2.3.1.

The map Φ_{n+1} that is induced by minimum cobordisms is well understood while the maps Φ_i for $i = 1, \ldots, n$ that correspond to pinch moves are not. We will first study $H_1(\overline{\Sigma}_n)$ and give a geometric description of the DGA map that corresponds to a pinch move. Combining with [EHK16], we will write down an explicit combinatorial formula for each Φ_i, for $i = 1, \ldots, n+1$.

Similar as in Section 2.3.2, to describe $H_1(\overline{\Sigma}_n)$ easily, we chop off the cylindrical ends of $\overline{\Sigma}_n$ and view it as a surface with boundary $\Lambda \cup \Lambda_n$, denoted by $\overline{\Sigma}_n$ as well. By Poincaré duality, we have $H^1(\overline{\Sigma}_n) \cong H_1(\overline{\Sigma}_n, \Lambda \cup \Lambda_n)$. Thus, in order to know the homology class of a loop β in $H_1(\overline{\Sigma}_n)$, we only need to count the intersection number of each generator curve of $H_1(\overline{\Sigma}_n, \Lambda \cup \Lambda_n)$ with β.

We choose the set of generator curves of $H_1(\overline{\Sigma}_n, \Lambda \cup \Lambda_n)$ as follows. Use t coordinate to slice $\overline{\Sigma}_n$ into a movie of diagrams (some of them may not be Legendrian diagrams). We study the trace of points on the diagram when t is decreasing. For $i = 1, \ldots, n$, the saddle cobordism Σ_i flows all the points directly downward except ends of the Reeb chord $b_{\sigma(i)}$. According to [Lin16], the ends of the Reeb chord $b_{\sigma(i)}$ merge to a point $r_{\sigma(i)}$, and then split into two points, labeled as $\tilde{s}_{\sigma(i)}$ and $\tilde{s}^{-1}_{\sigma(i)}$ respectively. Now for $i = 1, \ldots, n$, consider the trace of $\tilde{s}$ in $\overline{\Sigma}_n$, which is a flow line from r_i to the bottom of $\overline{\Sigma}_n$. Concatenating it with the inverse trace of $\tilde{s}^{-1}_i$ in $\overline{\Sigma}_n$, we get a curve α_i in $\overline{\Sigma}_n$ as shown in Figure 6.4. In addition, denote the trace of the

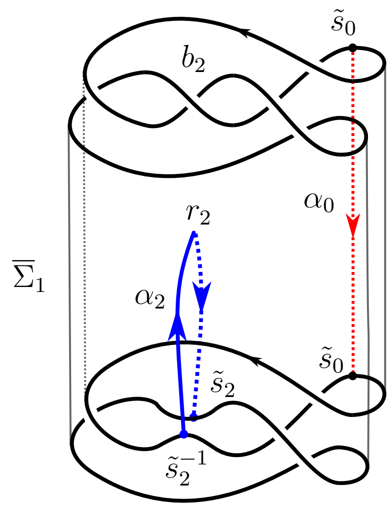


FIGURE 6.4: As an example, assume Λ is the Legendrian $(2,3)$ torus knot and the first pinch move is taken at b_2. The blue curve and the red curve are α_2 and α_0 restricted on $\overline{\Sigma}_1$, respectively.

base point $\tilde{s}_0$ in $\overline{\Sigma}_n$ by α_0. In this way, we have that $\alpha = \{\alpha_0, \alpha_1, \ldots, \alpha_n\}$ is a set of generator curves of $H_1(\overline{\Sigma}_n, \Lambda \cup \Lambda_n) \cong \mathbb{Z}^{n+1}$.

For each curve α_i, where $i = 0, \ldots, n$, the Poincaré duality gives an element $\theta_{\alpha_i} \in H^1(\overline{\Sigma}_n)$. Denote its dual in $H_1(\Sigma_n)$ by $\tilde{s}_i$. Therefore, for any union of paths β in $\overline{\Sigma}_n$, the monomial $w(\beta)$ associated to β in $\mathbb{Z}_2[H_1(\overline{\Sigma}_n)]$ is

$$w(\beta) = \prod_{i=0}^{n} \tilde{s}_i^{n_i(\beta)},$$

where $n_i(\beta)$ is intersection number of α_i and β counted with signs.

For $i < n$, the map $H_1(\overline{\Sigma}_i) \to H_1(\overline{\Sigma}_n)$ induced by the inclusion map is injective. A similar argument shows that for a closed curve β in $\overline{\Sigma}_i$, the monomial associated to β in $\mathbb{Z}_2[H_1(\overline{\Sigma}_i)]$ counts intersections of $\alpha_0, \alpha_{\sigma(1)}, \ldots, \alpha_{\sigma(i)}$ with β. Notice that the curves $\alpha_{\sigma(i+1)}, \ldots, \alpha_{\sigma(n)}$ do not intersect $\overline{\Sigma}_i$. Hence the monomial in $\mathbb{Z}_2[H_1(\overline{\Sigma}_i)]$ agrees with $w(\beta)$ in $\mathbb{Z}_2[H_1(\overline{\Sigma}_n)]$.

Choose a family of capping paths for Λ_i on $\overline{\Sigma}_i$ for $i = 0, \ldots, n$. By Proposition

2.3.3, for $i = 1, \ldots, n+1$, each exact Lagrangian cobordism Σ_i gives a DGA map Φ_i:

$$\Phi_i : \big(\mathcal{A}(\Lambda_{i-1}; \mathbb{Z}_2[H_1(\overline{\Sigma}_{i-1})]), \partial_{i-1}\big) \to \big(\mathcal{A}(\Lambda_i; \mathbb{Z}_2[H_1(\overline{\Sigma}_i)]), \partial_i\big),$$

which maps any Reeb chord a of Λ_{i-1} to

$$\begin{aligned} & \sum_{\dim \mathcal{M}^{\Sigma_i}(a; b_1, \ldots, b_m) = 0} \ \sum_{u \in \mathcal{M}^{\Sigma_i}(a; b_1, \ldots, b_m)} w(\tilde{u})\ b_1 \cdots b_m \\ = & \sum_{\dim \mathcal{M}^{\Sigma_i}(a; b_1, \ldots, b_m) = 0} \ \sum_{u \in \mathcal{M}^{\Sigma_i}(a; b_1, \ldots, b_m)} w(\overline{\beta_1}) \cdots w(\overline{\beta_m})\ b_1 \cdots b_m. \end{aligned}$$

Note that the $s(u)$ is omitted in the formula because we are working on $\mathbb{Z}_2$.

Similar to Proposition 2.2.1, we have that the DGA map induced by an exact Lagrangian cobordisms is independent of the choice of capping paths.

Theorem 6.2.2. *Let γ and γ' be two families of capping paths of Λ_i on $\overline{\Sigma}_i$ for $i = 0, \ldots, n$. Denote the corresponding DGAs by $\big(\mathcal{A}^{\gamma}(\Lambda_i;\ \mathbb{Z}_2[H_1(\overline{\Sigma}_i)]), \partial_i^{\gamma}\big)$ and $\big(\mathcal{A}^{\gamma'}(\Lambda_i;\ \mathbb{Z}_2[H_1(\overline{\Sigma}_i)]), \partial_i^{\gamma'}\big)$. Assume Φ_i^{γ} and $\Phi_i^{\gamma'}$ are the corresponding the DGA maps induced by Σ_i. Then the maps*

$$\begin{aligned} f_i : \big(\mathcal{A}^{\gamma}(\Lambda_i; \mathbb{Z}_2[H_1(\overline{\Sigma}_i)]), \partial_i^{\gamma}\big) &\to \big(\mathcal{A}^{\gamma'}(\Lambda_i; \mathbb{Z}_2[H_1(\overline{\Sigma}_i)]), \partial_i^{\gamma'}\big) \\ c &\mapsto w(\gamma_c^+ \cup -\gamma_c'^+) w(\gamma_c'^- \cup -\gamma_c^-)\ c \end{aligned}$$

are DGA isomorphisms for $i = 0, \ldots, n$. Moreover, the following diagram commutes:

$$\begin{array}{ccc} \big(\mathcal{A}^{\gamma}(\Lambda_{i-1}; \mathbb{Z}_2[H_1(\overline{\Sigma}_{i-1})]), \partial_{i-1}^{\gamma}\big) & \xrightarrow{f_{i-1}} & \big(\mathcal{A}^{\gamma'}(\Lambda_{i-1}; \mathbb{Z}_2[H_1(\overline{\Sigma}_{i-1})]), \partial_{i-1}^{\gamma'}\big) \\ \Phi_i^{\gamma} \downarrow & & \downarrow \Phi_i^{\gamma'} \\ \big(\mathcal{A}^{\gamma}(\Lambda_i; \mathbb{Z}_2[H_1(\overline{\Sigma}_i)]), \partial_i^{\gamma}\big) & \xrightarrow{f_i} & \big(\mathcal{A}^{\gamma'}(\Lambda_i; \mathbb{Z}_2[H_1(\overline{\Sigma}_i)]), \partial_i^{\gamma'}\big). \end{array}$$

Observe that, if we cut $\overline{\Sigma}_i$ along the curves $\alpha_0, \alpha_{\sigma(1)}, \ldots, \alpha_{\sigma(i)}$, the resulting surface is connected. Therefore, we can choose a family γ of capping paths for Λ_i on $\overline{\Sigma}_i$ such that none of them intersect the curves $\alpha_0, \alpha_{\sigma(1)}, \dots, \alpha_{\sigma(i)}$. Choose families of capping paths for $\Lambda_0, \dots, \Lambda_n$ in a similar way. As a result, for any rigid holomorphic disk u used in differentials of DGAs and DGA maps, we only need to count the intersections of curves in α with the disk boundary.

With this choice of capping paths, we can write down the DGA $\big(\mathcal{A}(\Lambda_i; \mathbb{Z}_2[H_1(\overline{\Sigma}_i)]), \partial_i\big)$ combinatorially, for $i = 1, \dots, n$. There are $2i + 1$ points on Λ_i given by the intersection of α_0 and Λ_i, labeled by $\tilde{s}_0$, along with the two intersections of $\alpha_{\sigma(j)}$ and Λ_i, labeled by $\tilde{s}_{\sigma(j)}$ (positive intersection) and $\tilde{s}'_{\sigma(j)}$ (negative intersection), for $j = 1, \dots, i$. One then takes the DGA of Λ_i with these $2i + 1$ base points, which has coefficients $\mathbb{Z}_2[\tilde{s}_0^{\pm 1}, \tilde{s}_{\sigma(1)}^{\pm 1}, \tilde{s}'^{\pm 1}_{\sigma(1)}, \dots, \tilde{s}_{\sigma(i)}^{\pm 1}, \tilde{s}'^{\pm 1}_{\sigma(i)}]$, and quotients by the relations $\tilde{s}'_{\sigma(j)} = \tilde{s}_{\sigma(j)}^{-1}$ for $j = 1, \dots, i$, to get the DGA $\big(\mathcal{A}(\Lambda_i; \mathbb{Z}_2[H_1(\overline{\Sigma}_i)]), \partial_i\big)$, which a DGA is over $\mathbb{Z}_2[\tilde{s}_0^{\pm 1}, \tilde{s}_{\sigma(1)}^{\pm 1}, \dots, \tilde{s}_{\sigma(i)}^{\pm 1}]$ and $\{\tilde{s}_0, \tilde{s}_{\sigma(1)}, \dots, \tilde{s}_{\sigma(i)}\}$ is a basis of $H_1(\overline{\Sigma}_i)$ that correspond to the curves $\alpha_0, \alpha_{\sigma(1)}, \dots, \alpha_{\sigma(i)}$.

Now we are ready to describe the DGA map Φ_i induced by the exact Lagrangian cobordism Σ_i, for $i = 1, \dots, n$, which corresponds to a pinch move at crossing $b_{\sigma(i)}$. Combining [EHK16, Section 6.5] with Proposition 2.3.3, we find that the DGA map

$$\Phi_i : \big(\mathcal{A}(\Lambda_{i-1}; \mathbb{Z}_2[H_1(\overline{\Sigma}_{i-1})]), \partial_{i-1}\big) \to \big(\mathcal{A}(\Lambda_i; \mathbb{Z}_2[H_1(\overline{\Sigma}_i)]), \partial_i\big)$$

maps the Reeb chord $b_{\sigma(i)}$ to $\tilde{s}_{\sigma(i)}$ and any other Reeb chord c to

$$c + \sum_{\dim \mathcal{M}(c, b_{\sigma(i)}; c_1, \dots, c_m) = 1} \; \sum_{u \in \mathcal{M}(c, b_{\sigma(i)}; c_1, \dots, c_m)} w(\tilde{u}) \tilde{s}_{\sigma(i)}^{-1} \, c_1 \cdots c_m,$$

where $\mathcal{M}(c, b_{\sigma(i)}; c_1, \dots, c_m)$ is the moduli space of holomorphic disks in $\mathbb{R}^2_{xy}$ with boundary on $\Pi_{xy}(\Lambda_{i-1})$ that covers a positive quadrant around c and $b_{\sigma(i)}$ and a negative quadrant around $c_1, \dots, c_m$. Please see [EHK16, Section 6.5] for a detailed definition.

Here we are going to talk about why do the formulas make sense. The pinch move at $b_{\sigma(i)}$ pinches the Reeb chord $b_{\sigma(i)}$ down, which gives a holomorphic disk (as

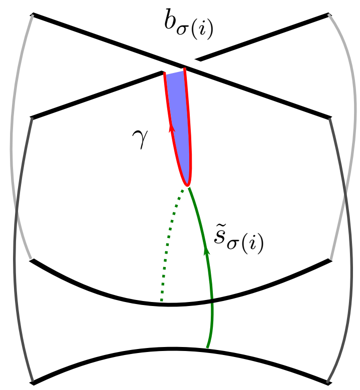


FIGURE 6.5: In a cobordism that corresponding to a pinch move, the purple disk represents a holomorphic disk with a positive puncture at $b_{\sigma(i)}$.

shown in Figure 6.5) with a positive puncture at $b_{\sigma(i)}$ and intersect $\tilde{s}_{\sigma(i)}$ exactly once. For a holomorphic disk $u \in \mathcal{M}(c, b_{\sigma(i)}; c_1, \dots, c_m)$, one can close the puncture of u at $b_{\sigma(i)}$ using the disk in Figure 6.5, which gives a holomorphic disk that contributes to $\Phi_i(c)$. Note that the boundary of this disk consists of the boundary of u and γ^{-1}. Thus the homology class of the boundary is $w(\tilde{u})\tilde{s}_{\sigma(i)}^{-1}$, which matches the formula above.

In our case, in order to describe Φ_i combinatorially, we introduce two notations first.

Definition 6.2.3. Let σ be a permutation of $\{1, \dots, n\}$. For $i \in \{1, \dots, n\}$, we define

$$\begin{aligned} T_\sigma^i := \ & \{j \in \{1, \dots, n\} \mid \sigma^{-1}(j) > \sigma^{-1}(i) \quad \text{and if } i < k < j \text{ or } j < k < i, \\ & \qquad \text{then } \sigma^{-1}(k) < \sigma^{-1}(i)\}, \\ S_\sigma^i := \ & \{j \in \{1, \dots, n\} \mid i \in T_\sigma^j\} \\ = \ & \{j \in \{1, \dots, n\} \mid \sigma^{-1}(j) < \sigma^{-1}(i) \quad \text{and if } i < k < j \text{ or } j < k < i, \\ & \qquad \text{then } \sigma^{-1}(k) < \sigma^{-1}(j)\}. \end{aligned}$$

Here T_σ^i captures all the Reeb chords b_j with the property that, before doing a pinch move at b_i, one can find a holomorphic disk with exactly two positive punctures at b_i and b_j. In other words, it gathers all the Reeb chords on which the DGA map induced by the pinch move at b_i acts non–trivially. The other set S^i_σ, on the other hand, detects all the Reeb chords b_j where a pinch move at b_j gives a DGA map that act non–trivially on b_i.

If $j \in T^{\sigma(i)}_\sigma$ (an example is shown in Figure 6.6), the map Φ_i sends b_j to

$$\Phi_i(b_j) = b_j + \tilde{s}^{-1}_{\sigma(i)} \prod_{\substack{j<k<\sigma(i) \text{ or} \\ \sigma(i)<k<j}} \tilde{s}^{-2}_k.$$

For a_1, a_2 and the rest of b_j's, the map Φ_i is identity.

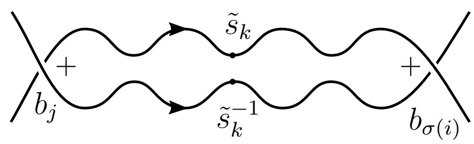


FIGURE 6.6: A part of the Lagrangian projection of Λ_{i-1}.

Composing all the maps Φ_i for $i = 1, \ldots, n$ together, we get a DGA map

$$\overline{\Phi}_n : \big(\mathcal{A}(\Lambda; \mathbb{Z}_2[H_1(\Lambda)]), \partial\big) \to \big(\mathcal{A}(\Lambda_n; \mathbb{Z}_2[H_1(\overline{\Sigma}_n)]), \partial_n\big).$$

that is identity on the Reeb chords a_1, a_2. For $i = 1, \ldots, n$, in order to know $\overline{\Phi_n}(b_i)$, we consider pinch moves at b_j such that $j \in S^i_\sigma$ together with the pinch move at b_i. These pinch moves correspond to all the DGA maps that contribute to $\overline{\Phi_n}$. Composing all these maps together, we have that

$$\overline{\Phi}_n(b_i) = \Phi_1 \circ \cdots \circ \Phi_{\sigma^{-1}(i)}(b_i) = \tilde{s}_i + \sum_{j \in S^i_\sigma} \left(\tilde{s}^{-1}_j \prod_{\substack{j<k<i \text{ or} \\ i<k<j}} \tilde{s}^{-2}_k \right).$$

Now we describe the last DGA map

$$\Phi_{n+1} : \big(\mathcal{A}(\Lambda_n; \mathbb{Z}_2[H_1(\overline{\Sigma}_n)]), \partial_n\big) \to \big(\mathbb{Z}_2[H_1(L_\sigma)], 0\big).$$

As shown in Figure 6.7, the underlying algebra of Λ_n is generated by a_1 and a_2 and the differential is given by

$$\begin{aligned}\partial_n(a_1) &= \tilde{s}_1\tilde{s}_2 \cdots \tilde{s}_n + \tilde{s}_0^{-1},\\ \partial_n(a_2) &= \tilde{s}_n\tilde{s}_{n-1} \cdots \tilde{s}_1 + 1.\end{aligned}$$

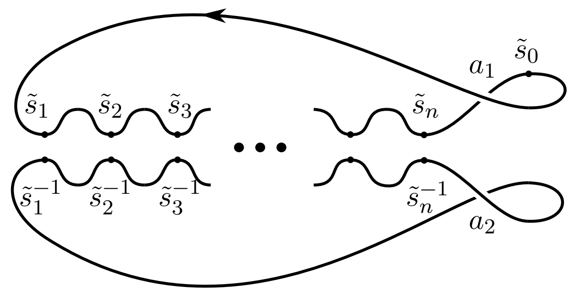


FIGURE 6.7: The Lagrangian projection of Λ_n.

Consider the map $\psi : H_1(\overline{\Sigma}_n) \to H_1(L_\sigma)$ induced by the inclusion map $\overline{\Sigma}_n \hookrightarrow L_\sigma$. Since the DGA map

$$\Phi_{n+1} : \big(\mathcal{A}(\Lambda_n; \mathbb{Z}_2[H_1(\overline{\Sigma}_n)]), \partial_n\big) \to \big(\mathbb{Z}_2[H_1(L_\sigma)], 0\big)$$

satisfies that $\Phi_{n+1} \circ \partial_n = 0 \circ \Phi_{n+1} = 0$, we have $\psi(\tilde{s}_0) = 1$ and $\psi(\tilde{s}_1)\psi(\tilde{s}_2)\cdots\psi(\tilde{s}_n) = 1$. Given that the map ψ is surjective, we assume a basis of $H_1(L_\sigma)$ is $\{s_1, \ldots, s_{n-1}\}$, where $s_i = \tilde{s}_i$, for $i = 1, \ldots, n-1$. The DGA map Φ_{n+1} is given by

$$\begin{aligned}a_1 &\mapsto 0,\\ a_2 &\mapsto 0,\\ \tilde{s}_0 &\mapsto 1,\\ \tilde{s}_i &\mapsto s_i, \quad i = 1, \ldots, n-1,\\ \tilde{s}_n &\mapsto (s_1 s_2 \cdots s_{n-1})^{-1}.\end{aligned}$$

Composing Φ_{n+1} with $\overline{\Phi}_n$, we get the augmentation ϵ_σ induced by L_σ as follows.

Theorem 6.2.4. *Given a permutation σ of $\{1, \ldots, n\}$, let L_σ be the exact Lagrangian filling of the Legendrian $(2, n)$ torus knot Λ constructed from the [EHK16] algorithm. If we write*

$$\begin{aligned} \mathbb{Z}_2[H_1(\Lambda)] &= \mathbb{Z}_2[\tilde{s}_0, \tilde{s}_0^{-1}], \\ \mathbb{Z}_2[H_1(L_\sigma)] &= \mathbb{Z}_2[s_1^{\pm 1}, \ldots, s_{n-1}^{\pm 1}], \end{aligned}$$

and set $s_n = (s_1 s_2 \cdots s_{n-1})^{-1}$, then the augmentation

$$\epsilon_\sigma : \mathcal{A}(\Lambda; \mathbb{Z}_2[H_1(\Lambda)]) \to \mathbb{Z}_2[H_1(L_\sigma)]$$

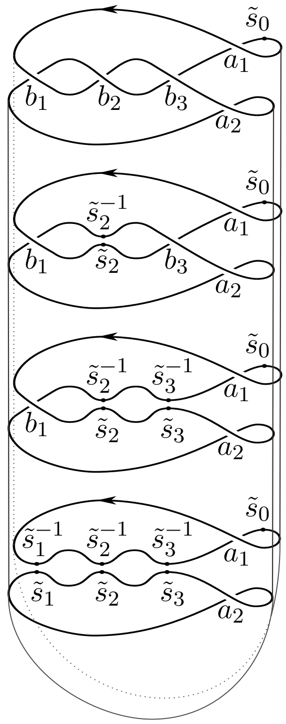
induced by L_σ is given by

$$\begin{aligned} \epsilon_\sigma(a_j) &= 0, \quad j = 1, 2; \\ \epsilon_\sigma(b_i) &= s_i + \sum_{j \in S_\sigma^i} \left(s_j^{-1} \prod_{\substack{j<k<i \text{ or} \\ i<k<j}} s_k^{-2} \right), \quad i = 1, \ldots, n; \\ \epsilon_\sigma(\tilde{s}_0) &= 1. \end{aligned}$$

Example 6.2.5. As an example, we compute the augmentation $\epsilon_{(2,3,1)}$ of the Legendrian $(2, 3)$ torus knot induced by the exact Lagrangian filling $L_{(2,3,1)}$.

Similarly, one can compute the augmentation for each permutation of $\{1, 2, 3\}$ and get the following result.

ϵ	$\epsilon(b_1)$	$\epsilon(b_2)$	$\epsilon(b_3)$
$\epsilon_{(1,2,3)}$	s_1	$s_2 + s_1^{-1}$	$s_1^{-1}s_2^{-1} + s_2^{-1}$
$\epsilon_{(1,3,2)} = \epsilon_{(3,1,2)}$	s_1	$s_2 + s_1^{-1} + s_1 s_2$	$s_1^{-1}s_2^{-1}$
$\epsilon_{(2,1,3)}$	$s_1 + s_2^{-1}$	s_2	$s_1^{-1}s_2^{-1} + s_2^{-1} + s_1^{-1}s_2^{-2}$
$\epsilon_{(2,3,1)}$	$s_1 + s_2^{-1} + s_1 s_2^{-1}$	s_2	$s_1^{-1}s_2^{-1} + s_2^{-1}$
$\epsilon_{(3,2,1)}$	$s_1 + s_2^{-1}$	$s_2 + s_1 s_2$	$s_1^{-1}s_2^{-1}$



	b_1	b_2	b_3
Φ_1	$\downarrow$	$\downarrow$	$\downarrow$
	$b_1 + \tilde{s}_2^{-1}$	$\tilde{s}_2$	$b_3 + \tilde{s}_2^{-1}$
Φ_2	$\downarrow$	$\downarrow$	$\downarrow$
	$b_1 + \tilde{s}_2^{-1} + \tilde{s}_2^{-2}\tilde{s}_3^{-1}$	$\tilde{s}_2$	$\tilde{s}_3 + \tilde{s}_2^{-1}$
Φ_3	$\downarrow$	$\downarrow$	$\downarrow$
	$\tilde{s}_1 + \tilde{s}_2^{-1} + \tilde{s}_2^{-2}\tilde{s}_3^{-1}$	$\tilde{s}_2$	$\tilde{s}_3 + \tilde{s}_2^{-1}$
Φ_4	$\downarrow$	$\downarrow$	$\downarrow$
	$s_1 + s_2^{-1} + s_1 s_2^{-1}$	s_2	$s_1^{-1}s_2^{-1} + s_2^{-1}$

FIGURE 6.8: A computation of the augmentation induced by an exact Lagrangian filling of the Legendrian $(2,3)$ torus knot.We keep track of the image of b_1, b_2, b_3 underthe composition of Φ_1, Φ_2, Φ_3 and Φ_4. The last line is the image of b_1, b_2, b_3 under the augmentation $\epsilon_{(2,3,1)}$.

6.3 Proof of the main theorem

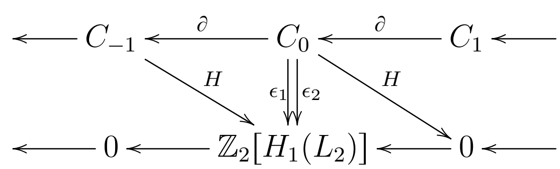
In this section, we use Theorem 6.2.4 to find an combinatorial invariant of augmentations induced from the exact Lagrangian fillings obtained from the [EHK16] algorithm. As a result, we distinguish all the augmentations in Theorem 6.2.4 and thus prove Theorem 6.1.1.

Lemma 6.3.1. *Let L_1 and L_2 be two exact Lagrangian fillings of the Legendrian $(2,n)$ torus knot Λ constructed from the [EHK16] algorithm. If L_1 and L_2 are exact Lagrangian isotopic, then there exists an invertible map $g : H_1(L_1) \to H_1(L_2)$ such* *that the following diagram commutes:*

$$\begin{array}{ccc} (\mathcal{A}(\Lambda), \partial) & \xrightarrow{Id} & (\mathcal{A}(\Lambda), \partial) \\ \downarrow \epsilon_{L_1} & & \downarrow \epsilon_{L_2} \\ \mathbb{Z}_2[H_1(L_1)] & \xrightarrow[g]{} & \mathbb{Z}_2[H_1(L_2)], \end{array} \tag{6.1}$$

where ϵ_{L_1} and ϵ_{L_2} are augmentations induced by L_1 and L_2 respectively.

Proof. The isotopy between L_1 and L_2 induces an invertible map $g : H_1(L_1) \to H_1(L_2)$. If we identify both $H_1(L_1)$ and $H_1(L_2)$ with $\mathbb{Z}^{n-1}$, then $g \in GL(n-1, \mathbb{Z})$. This map induces a natural map on the corresponding group rings $\mathbb{Z}_2[H_1(L_1)] \to \mathbb{Z}_2[H_1(L_2)]$, denoted by g as well. Thus, we have two augmentations of $\mathcal{A}(\Lambda)$ to $\mathbb{Z}_2[H_1(L_2)]$: $\epsilon_1 = g \circ \epsilon_{L_1}$ and $\epsilon_2 = \epsilon_{L_2}$. Since the two fillings L_1 and L_2 are isotopic through a family of exact Lagrangian fillings, according to [EHK16, Theorem 1.3], we know that ϵ_1 and ϵ_2 are chain homotopic. In other words, there exists a degree 1 map $H : \mathcal{A}(\Lambda) \to \mathbb{Z}_2[H_1(L_2)]$ such that $H \circ \partial = \epsilon_1 - \epsilon_2$ as one can see from following diagram, where C_i denotes degree i part of $\mathcal{A}(\Lambda)$.



Note that Λ has a Lagrangian projection (as shown in Figure 6.3) such that no Reeb chords are in negative degree. Hence $C_{-1} = 0$ and $\epsilon_1 - \epsilon_2 = H \circ \partial = 0$. Therefore $\epsilon_1 = \epsilon_2$, i.e. the diagram (6.1) commutes. □

Remark 6.3.2. For any DGA $\mathcal{A}$ that vanishes on degree -1 part, by the same argument, we have that two augmentations ϵ_1 and ϵ_2 of $\mathcal{A}$ are chain homotopic if and only if they are identically the same. For a more general criteria of two augmentations to be chain homotopic, one can check [NRS+15, Proposition 5.16].

Therefore, in order to distinguish exact Lagrangian fillings, we only need to distinguish their induced augmentations up to a $GL(n-1,\mathbb{Z})$ action. Observing the formula of the augmentation ϵ_σ in Theorem 6.2.4, we get a combinatorial way to define the number of terms in $\epsilon_\sigma(b_i)$ for $i=1,\dots,n$ as follows.

Definition 6.3.3. For each permutation σ of $\{1,\dots,n\}$ and any number $i \in \{1,\dots,n\}$, we define $C_\sigma := (C_\sigma^1, C_\sigma^2, \dots, C_\sigma^n)$, where $C_\sigma^i = |S_\sigma^i| + 1$.

Example 6.3.4. We compute the vector C_σ for all the permutations σ of $\{1,2,3\}$ as follows.

σ	$(1,2,3)$	$(1,3,2) \sim (3,1,2)$	$(2,1,3)$	$(2,3,1)$	$(3,2,1)$
C_σ	$(1,2,2)$	$(1,3,1)$	$(2,1,3)$	$(3,1,2)$	$(2,2,1)$

Proposition 6.3.5. *If two exact Lagrangian fillings L_{σ_1} and L_{σ_2} are exact Lagrangian isotopic, then $C_{\sigma_1} = C_{\sigma_2}$. In other words, the vector C_σ is an invariant of the exact Lagrangian filling L_σ up to exact Lagrangian isotopy.*

Proof. Using the formula in Theorem 6.2.4, we first show that C_σ^i is the number of terms in $\epsilon_\sigma(b_i)$. In order to do that, we need to prove that $\epsilon_\sigma(b_i)$ as a sum of monomials can not be shorter, i.e, no terms in $\epsilon_\sigma(b_i)$ can be canceled by another term. First, replace s_n with $(s_1 \cdots s_{n-1})^{-1}$. If $i \neq n$, then each term of $\epsilon_\sigma(b_i)$ is one of the following forms:

1. s_i;
2. $s_k^{-1} \prod_{j \in S} s_j^{-2}$ for some $k \neq i \in \{1,\dots,n-1\}$ and a subset $S \subset \{1,\dots,n-1\}$ that does not contain i,k (can be a empty set);
3. $\prod_{j\in T} s_j^{-1} \prod_{k \notin T} s_k$ for some subset $T \subset \{1,\dots,n-1\}$ that does not contain i (can be a empty set).

If $i = n$, each term of $\epsilon_\sigma(b_n)$ can be either $s_1^{-1} \cdots s_{n-1}^{-1}$ or the form (2) above. Comparing degrees of $s_1, \ldots, s_{n-1}$ of each term, we know that no terms can be canceled.

If L_{σ_1} and L_{σ_2} are exact Lagrangian isotopic, by Lemma 6.3.1, there is a map $g : \mathbb{Z}_2[H_1(L_1)] \to \mathbb{Z}_2[H_1(L_2)]$ such that $g \circ \epsilon_{L_1} = \epsilon_{L_2}$. Note that the map g on the group rings $\mathbb{Z}_2[H_1(L_1)] \to \mathbb{Z}_2[H_1(L_2)]$ is induced from an invertible map $H_1(L_1) \to H_1(L_2)$ and thus g maps a monomial to a monomial. Therefore $\epsilon_{\sigma_1}(b_i)$ and $\epsilon_{\sigma_2}(b_i)$ have the same number of terms, i.e., $C_{\sigma_1} = C_{\sigma_2}$.

□

We say that two permutations σ_1 and σ_2 of $\{1, \ldots, n\}$ are **isotopy equivalent** if they are equivalent via a sequence of relations of the form

$$(\ldots, i, j, \ldots, k, \ldots) \sim (\ldots, j, i, \ldots, k, \ldots) \text{ where } i < k < j. \tag{6.2}$$

By [EHK16], if σ_1 and σ_2 are isotopy equivalent, the corresponding exact Lagrangian fillings L_{ϵ_1} and L_{ϵ_2} are exact Lagrangian isotopic and hence $C_{\sigma_1} = C_{\sigma_2}$. Conversely, we have the following Lemma.

Lemma 6.3.6. *If $C_{\sigma_1} = C_{\sigma_2}$, then σ_1 and σ_2 are isotopy equivalent.*

Proof. If $\sigma_1(1) = k$, then $C^k_{\sigma_1} = 1$. So $C^k_{\sigma_2} = 1$, i.e., we have that $S^k_{\sigma_2} = \varnothing$. If $\sigma_2(1) \neq k$, assume the element in σ_2 right before k is l, i.e. $\sigma_2\big(\sigma_2^{-1}(k) - 1\big) = l$. Note that $l \notin S^k_{\sigma_2}$, i.e. there exists i such that $l < i < k$ or $k < i < l$ and $\sigma_2^{-1}(i) > \sigma_2^{-1}(l)$. Note that $i \neq k$ and hence $\sigma_2^{-1}(i) > \sigma_2^{-1}(k) = \sigma_2^{-1}(l) + 1$. Thus we can use the relation (6.2) to switch l and k. In this way we can switch k to the first position in σ_2, i.e. $\sigma_2(1) = k = \sigma_1(1)$.

By induction, assume $\sigma_2(i) = \sigma_1(i)$ for $i < l$ and $\sigma_1(l) = k$. We have $S^k_{\sigma_1} \subset S^k_{\sigma_2}$. The assumption $C^k_{\sigma_2} = C^k_{\sigma_1}$ implies that $|S^k_{\sigma_1}| = |S^k_{\sigma_2}|$ and hence $S^k_{\sigma_1} = S^k_{\sigma_2}$. If $\sigma_2(l) \neq k$, for a similar reason as above, one can switch k to the l-th position and get $\sigma_2(l) = \sigma_1(l)$. Therefore, we have that σ_1 and σ_2 are isotopy equivalent. □

Theorem 6.3.7. *If n is odd, the C_n exact Lagrangian fillings of the Legendrian $(2,n)$ torus knot Λ from the algorithm in [EHK16] are all of different exact Lagrangian isotopy classes.*

Proof. If two augmentations σ_1 and σ_2 are not isotopy equivalent, by Lemma 6.3.6, we have $C_{\sigma_1} \neq C_{\sigma_2}$. According to Proposition 6.3.5, the corresponding exact Lagrangian fillings L_{σ_1} and L_{σ_2} are not exact Lagrangian isotopic. Therefore, the Legendrian $(2,n)$ torus knot has at least C_n exact Lagrangian fillings up to exact Lagrangian isotopy. $\square$

Corollary 6.3.8. *When n is even, the Legendrian $(2,n)$ torus link Λ has at least C_n exact Lagrangian fillings.*

Proof. Start with the Legendrian $(2,n+1)$-knot Λ_0 and label its degree 0 Reeb chords from left to right by $b_1,\ldots,b_{n+1}$ as usual. Let Σ be the exact Lagrangian cobordism from Λ to Λ_0 that corresponds to a pinch move of Λ_0 at b_{n+1}. For any permutation σ of $\{1,\ldots,n\}$, the exact Lagrangian filling L_σ of Λ gives an exact Lagrangian filling of Λ_0 by concatenating with Σ on the top. This new exact Lagrangian filling of Λ_0 corresponds to the permutation $\tilde{\sigma} = (n+1, \sigma(1), \ldots, \sigma(n))$ of $\{1,2,\ldots,n+1\}$, i.e., it is the filling $L_{\tilde{\sigma}}$ of Λ_0. Note that $C_{\tilde{\sigma}}^{n+1} = 1$. Moreover, we have that $C_{\tilde{\sigma}}^i = C_\sigma^i$ for $i = 1,\ldots,n-1$ and $C_{\tilde{\sigma}}^n = C_\sigma^n + 1$. Thus $C_{\tilde{\sigma}}$ is determined by C_σ. Therefore, by Proposition 6.3.5 and Lemma 6.3.6, if two permutations σ_1 and σ_2 of $\{1,\ldots,n\}$ are not isotopy equivalent, their induced permutations $\tilde{\sigma}_1$ and $\tilde{\sigma}_2$ of $\{1,\ldots,n+1\}$ are not isotopy equivalent. According to Theorem 6.3.7, the corresponding exact Lagrangian fillings $L_{\tilde{\sigma}_1}$ and $L_{\tilde{\sigma}_2}$ of Λ_0 are not exact Lagrangian isotopic. Hence L_{σ_1} and L_{σ_2} are not exact Lagrangian isotopic. $\square$

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Biography

Yu Pan was born on October 21, 1989 in Shangzhi, China. She attended the S. S. Chern elite class of Nankai University in 2008 and earned a B.S. in Mathematics in 2012. In Fall 2012, Pan entered the graduate program of Mathematics department in Duke University. She obtained a Ph.D. in 2017, under the supervision of Prof. Lenny Ng. During her time at Duke, Pan won a Duke Summer Research Fellowship, an AMS travel funding and an AWM travel funding. In the meantime, she had two papers [Pan16a] [Pan16b] accepted. Pan will continue her academic career as a CLE Moore Instructor at Massachusetts Institute of Technology beginning in Fall 2017.