RELATING INVARIANTS COMING FROM 3-COMPONENT TORUS LINKS

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Abstract. Given a 3-component torus link $T(p,q) \subset S^3$, we can construct a closed 3-manifold $\Sigma(p,q,2)$ called the double-branched cover of S^3 with branched set equal to $T(p,q)$. The aim of this thesis is to relate the Neumann-Siebenmann $\bar{\mu}$ invariant of $\Sigma(p,q,2)$ to the d-invariants coming from its Heegaard Floer homology.

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Contents

1. Introduction

Knots and links have been central objects of study in low-dimensional topology and knot theory for multiple decades. Their simple structure allows them to be manipulated in creative ways to construct new spaces that can have their own interesting features or are examples of more general spaces, and we study knots and links by constructing invariants for them. An example of this are the double-branched covers of links. These closed 3-manifolds carry information about the link itself but have extra geometric structure. In fact, in this case, they are Brieskorn manifolds $\Sigma(p, q, 2)$, which are also examples of Seifert fibered spaces. If these spaces are rational homology spheres ($\mathbb{Q}HS^3$ for short), i.e. their homology groups with $\mathbb{Q}$ coefficients are isomorphic to those of S^3, then they have special features that allow for an easier computation of certain invariants.

The invariants that we are interested in are the d-invariant of $\Sigma(p, q, 2)$, which comes from Heegaard Floer homology, and the Neumann-Siebenmann $\bar{\mu}$ invariant, which is related to the intersection form of 4-manifolds. The goal of this thesis was, initially, to understand for which p, q is $\Sigma(p, q, 2)$ a rational homology sphere, and then relate the d-invariants of $\Sigma(p, q, 2)$ to the signature of $T(p, q)$. However, along the way, we realized it was easier to do this using the $\bar{\mu}$ invariant, which is equivalent to the signature of certain types of links called plumbing links. To compute this invariant we will use a method arising from plumbing manifolds. It turns out that $\Sigma(p, q, 2)$ can be realized as the boundary of a plumbing 4-manifold X_Γ, which arises from a star-shaped graph Γ that can be constructed by using the Seifert invariants of $\Sigma(p, q, 2)$. The intersection form of X_Γ together with the graph Γ then give all the information necessary to compute $\bar{\mu}$.

In [12], Stipsicz proved that $\bar{\mu}(Y_\Gamma, \mathfrak{s}) = -4d(Y_\Gamma, \mathfrak{s})$ whenever $Y_\Gamma = \partial X_\Gamma$ is a rational surface singularity (which is equivalent to being an L-space, a condition related to the Heegaard Floer homology of Y_Γ) and $\mathfrak{s}$ is a spin-structure. In this thesis, we try to see if this relation between the $\bar{\mu}$-invariant and the d-invariant holds for $\Sigma(p, q, 2)$, which are not necessarily L-spaces.

2. Background

In this section, we will give all the background definitions, constructions, and theorems that will be useful for the computation and understanding of the invariants later on.

2.1. **Knots and Links.**

Definition 2.1. A *knot* is an embedding of S^1 in S^3. A *link* with n components is the disjoint union of n knots.

A special family of links is the *torus links*.

Definition 2.2. A *torus link* is a link that lies on the surface of a torus $T^2 = S^1 \times S^1$.

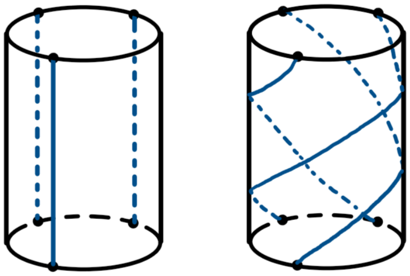

FIGURE 1. Example of torus link construction with $q = 3$ and $p = 2$

We can characterize all torus links by two integers p, q and write $T(p, q)$ to represent the torus link. We construct the torus link as follows.

Consider the cylinder $C = S^1 \times I/\sim$. Now draw the q vertical lines $l_k = (\exp(2\pi ik/q), t)$ where $k = 1, 2, ..., q - 1$ and $t \in I$.

Now, while fixing the base of the cylinder, twist the cylinder $2\pi p/q$ degrees, like in the example in the figure. Take the quotient $C/\sim$ where $(x, 0) \sim (x, 1)$ and we obtain a torus. The resulting closed curves given by the lines we drew earlier form the torus link $T(p, q)$.

Proposition 2.3. *The torus link $T(p, q)$ has $\gcd(p, q)$ components.*

Proof. Start with the line l_0. Since we twisted the cylinder by an angle of $2\pi p/q$, after taking the quotient by the equivalence relation, the finishing point of l_0 will be the starting point of l_p, so these two lines would be connected in the torus. If we keep following along the lines, we will eventually go back to the starting point of l_0, giving us one component of the torus link. To see the amount of lines we used for a single component we just need to find the smallest k such that $kp \equiv 0 \mod q$. This happens when $kp = lcm(p, q)$ so $k = lcm(p, q)/p$. If we have not covered all lines, repeat this process with the remaining lines. Then the total number of components is $q/k = \gcd(p, q)$. ☐

In this thesis, we are interested in 3-component torus links $T(p, q)$, so we want $\gcd(p, q) = 3$.

A useful characteristic of a knot or link is that it always bounds an orientable surface. Such a surface is called a *Seifert surface* of the knot or link. We introduce them as they will be useful in later constructions.

Proposition 2.4. *Every link is the boundary of an orientable surface.*

Proof. This proof follows [2]. Fix an oriented diagram of the link. Begin tracing around the diagram, starting at an arbitrary point in an arc. When we find a

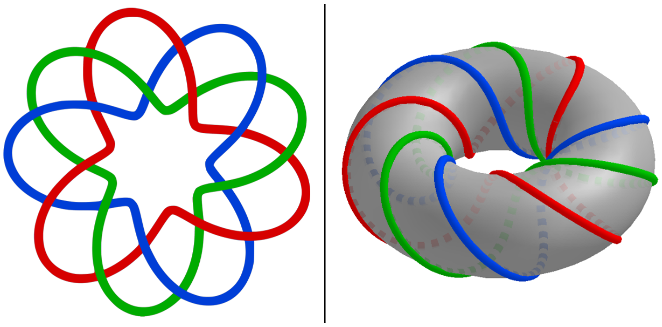

FIGURE 2. The torus link $T(3,9)$ [8]

crossing, we change arcs and follow the other arc according to its orientation. If at any point we come to our starting point, we can repeat this process with the untraced arcs of the diagram. This process results in a collection of circles called Seifert circles (see Figure 3).

Now, each of these circles is the boundary of a D^2 lying on the plane. If any circles are nested, then lift the inner circles to a higher plane (think about the outer circles as living in the plane $z = 0$ and the inner circles as living in $z = 1$ in $\mathbb{R}^3$).

To form the Seifert surface, attach twisted bands $I \times I$ at points corresponding to the crossing points in the original diagram. Twist the bands so that they correspond to the direction of the crossing in the link. This results in an orientable surface whose boundary is the link. □

Remark 2.5. It is important to note that one link can have many different Seifert surfaces.

2.2. **Branched covers and $\Sigma(p,q,2)$.** Now that we have our 3-component torus links, we will construct their double-branched covers. We begin with the general definition of a branched covering.

Definition 2.6. Let X, Y be n-manifolds. The proper map $f : X \to Y$ (i.e. such that $f^{-1}(\partial Y) = \partial(X)$) is a *d-fold branched covering* if:

(i) There exists a CW complex $B \subset Y$, called the *branched locus*, of dimension $n-2$, such that $f^{-1}(B) \subset X$ is also an $n-2$ dimensional CW complex.
(ii) $f|_{X-f^{-1}(B)} : (X - f^{-1}(B)) \to Y - B$ is a d-fold covering map.

Each point $x \in f^{-1}(B)$ has a *branching index* k, which means f is k-to-one near x (but not at x).

Example 2.7. For surfaces, i.e., 2-manifolds, branch loci are a collection of points. Locally, they are maps from $\mathbb{C} \to \mathbb{C}$ where $z \mapsto z^n$ and n is the branching index.

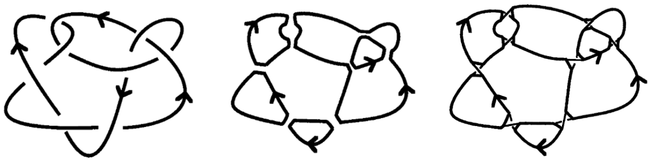

FIGURE 3. Constructing the Seifert surface of a knot ([2]).

Intuitively, a branched covering is like a regular covering map outside of its branched set, where it is one-to-one.

Example 2.8. A similar map $\mathbb{C} \to \mathbb{C}$ given by $z \mapsto z^n/|z^{n-1}|$ extends to $S^2 \to S^2$, with two branch points. We divide by $|z^{n-1}|$ to preserve unit length. In the figure, $n = 6$.

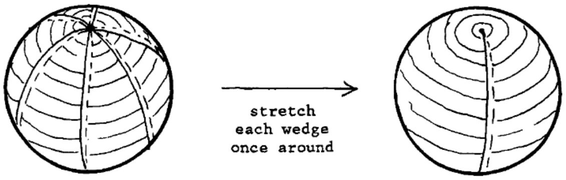

FIGURE 4. 6-fold branch cover of S^2. [1]

Example 2.9. It turns out $S^1 \times I$ is a 2-fold branched cover of D^2, the unit disk, with 2 branch points. It is easy to see from the picture that near the branch points, the map is locally $z \mapsto z^2$ for $z \in \mathbb{C}$.

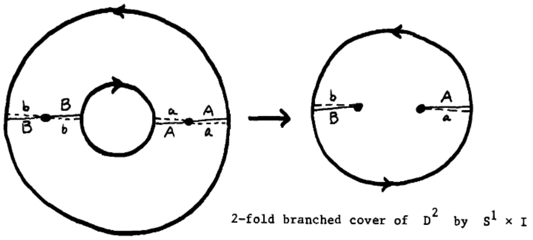

FIGURE 5. $S^1 \times I$ is a 2-fold cover of D^2 with 2 branch points. [1]

In this thesis we will consider 2-fold branched coverings (also known as double-branched coverings) of S^3. Then the branched locus is a 1-dimensional manifold, i.e., a curve.

We start by considering the general construction of a q-fold *cyclic branched cover* of S^3 over a link L (we will then take $q = 2$). Let $M \subset S^3$ be a Seifert surface of L. Let N be the map $N : int(M) \times (-1, 1) \to S^3$ where $N(x, 0) = x$ for all $x \in int(M)$. For simplicity, we write N to denote the image of the map N. Let

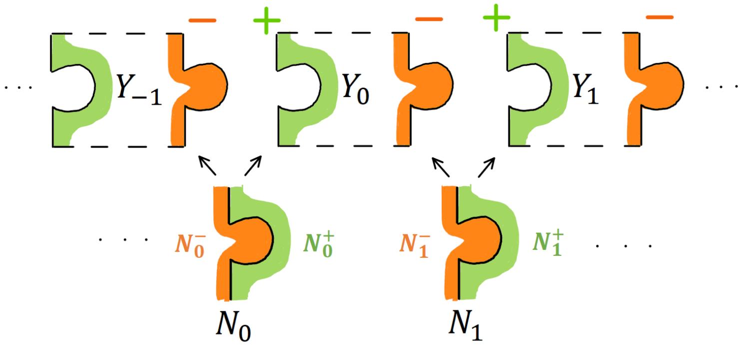

FIGURE 6. Constructing a cyclic branched cover of a link L

$N^- = N(int(M) \times (-1,0))$ and $N^+ = N(int(M) \times (0,1))$. Let $Y = S^3 - M$ and $X = S^3 - L$. Now, take identical copies of N and Y and index them by N_i, Y_i.

Note that since $N^+ \subset Y$ and $N^- \subset Y$ we can glue $N_i^+ \subset Y_i$ to $N_i^+ \subset N_i$ and $N_i^- \subset Y_i$ with $N_{i+1}^- \subset N_{i+1}$ as in figure 5.

If we take $\mathbb{Z}$ as the indexing set for i, then after the gluing process, we obtain a path connected, open 3-manifold $\tilde{X}$ which is called an *infinite cyclic cover* of $S^3 \setminus L$ with covering map $p : \tilde{X} \to S^3 \setminus L$. If we take finitely many i's, that is $i \in \{0, 1, ..., q-1\}$, and glue $N_{q-1}^- \subset Y_{q-1}$ to $N_0^- \subset N_0$ and $N_0^+ \subset Y_0$ to $N_0^+ \subset N_0$, then we get a *q-fold cyclic cover* of $S^3 \setminus L$.

Now that we have the construction of the q-fold cyclic cover of $S^3 \setminus L$, it is possible to compactify the cover such that it becomes a closed 3-manifold as follows. Let $N(L)$ be a closed tubular neighborhood of L (take a union of the tubular neighborhoods for each component of L). Construct the q-fold cyclic *unbranched* covering of $\overline{S^3 \setminus N(L)}$, the construction is equivalent to that of $S^3 \setminus L$. Moreover, for each component K of the link L, the preferred longitude of K, which "lives" in $\partial N(K)$, will be covered by q disjoint Seifert longitudes (also called 0-framed longitudes), i.e. loops upstairs. This is because the Seifert longitude in $\partial N(K)$ bounds a disc in $\overline{S^3 \setminus N(K)}$, so it lives in the induced subgroup of the covering map $p : \tilde{X} \to \overline{S^3 \setminus L}$, which means it lifts to q closed curves in $\tilde{X}$. Moreover, since the meridian intersects the longitude transversely only once in $\partial N(K)$, it intersects each of the q longitudes upstairs transversely once, which means the preimage of $\partial N(K)$ in the cover is also a torus, i.e. if we denote $\tilde{N}(K)$ as the preimage, then $\partial \tilde{N}(K) \cong T^2$. Thus, the meridian downstairs is covered q times by a single loop upstairs (see figure 7).

Now we can glue a solid torus $D^2 \times S^1$ to the unbranched cover we just described, identifying the boundaries in such a way that the meridian of the gluing torus matches with the preimage of the meridian of $\partial N(K)$ in $\partial \tilde{N}(K)$. This creates a

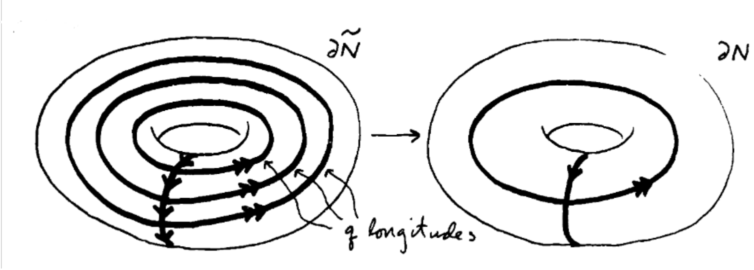

FIGURE 7. The preimage of $\partial N(K)$ is also a torus [1].

closed, connected, orientable 3-manifold upstairs, call it Σ_q, which consists of the cyclic covering $\tilde{X}$ of $\overline{S^3 - N(K)}$ and the glued solid torus. So we have a covering map from $\tilde{X} \to \overline{S^3 - N(K)}$, which we can extend to a branched covering map $\Sigma_q \to S^3$ by sending the solid torus upstairs $D^2 \times S^1$ onto $N(K)$ according to the map $D^2 \times S^1 \to N(K)$ where $(z, x) \mapsto (z^q/|z^{q-1}|, x)$. The branch set downstairs is given by the knot K (i.e. $z = 0$ and varying x). We can repeat this process for each component K of the link L, and so we have obtained the q-fold branched covering of S^3 over the link L denoted by $\Sigma_q(L)$.

In this thesis we will focus on double-branched coverings of 3-component torus links, so we take $q = 2$ and $L = T(p, q)$ with $\gcd(p, q) = 3$.

2.3. **Brieskorn manifolds and Seifert fibered spaces.** In this section, we will show that $\Sigma_r(T(p, q))$ is an example of what is known as a *Brieskorn manifold* $\Sigma(p, q, r)$. Moreover, a Brieskorn manifold is an example of a more general space called a *Seifert fibered spaces* or a *Seifert manifolds.*

First, we state an equivalent way of defining torus links [3].

Definition 2.10. The torus link $T(p, q)$ is the set of points $(z_1, z_2) \in S^3 \subset \mathbb{C}^2$ such that $z_1^p + z_2^q = 0$. The n^{th} component can be parametrized by setting $z_1 = \exp(2\pi i t/p)$ and $z_2 = \exp(2\pi i(t + n + \frac{1}{2})/q)$.

Now we state and prove the result, following [3].

Proposition 2.11. *The Brieskorn manifold* $\Sigma(p, q, r) = \{z_1^p + z_2^q + z_3^r = 0\} \cap S^5$ *is a compact* 3*-manifold that is homeomorphic to the* r*-fold cyclic branched covering of* S^3 *branched along the torus link* $T(p, q)$.

Proof. Let $V \subset \mathbb{C}^3$ be the variety $z_1^p + z_2^q + z_3^r = 0$, which is non-singular except at the origin. Consider the projection map $V \setminus \{0\} \to \mathbb{C}^2 \setminus \{0\}$ onto the first two coordinates $(z_1, z_2, z_3 \mapsto (z_1, z_2)$. The branch locus is given by $z_1^p + z_2^q = 0$. Away from this locus, each point of $\mathbb{C}^2 \setminus \{0\}$ has r pre-images in V. If $\xi \in \Omega \subset \mathbb{C}$ is an r^{th} root of unity, where $\Omega = \{z \in S^1 \mid z^r = 1\}$, then Ω permutes the pre-images cyclically by acting on $V \setminus \{0\}$ by the rule

$$\Omega : (z_1, z_2, z_3) \mapsto (z_1, z_2, z_3\xi)$$

Therefore we obtain a quotient space $(V \setminus \{0\})/\Omega$, and it maps homeomorphically onto $\mathbb{C}^2 \setminus \{0\}$. So, based on the action of Ω on $V \setminus \{0\}$, we can see that $V \setminus \{0\}$ is an r-fold cyclic branched covering of $\mathbb{C}^2 \setminus \{0\}$ branched along the link $T(p, q)$, i.e. the curve $z_1^p + z_2^q = 0$.

Now the group $\mathbb{R}^+$ acts freely on $V \setminus \{0\}$ by the rule

$$t : (z_1, z_2, z_3) \mapsto (t^{1/p}z_1, t^{1/q}z_2, t^{1/r}z_3)$$

for $t \in \mathbb{R}^+$. Each orbit of this $\mathbb{R}^+$ action intersects $S^5 \subset \mathbb{C}^3$ transversely and only once. Therefore, it follows that $V \setminus \{0\} \cong \mathbb{R}^+ \times \Sigma(p, q, r)$.

Similarly, we can let $\mathbb{R}^+$ act on $\mathbb{C}^2 \setminus \{0\}$ by the rule $(z_1, z_2) \mapsto (t^{1/p}z_1, t^{1/q}z_2)$, and by the same argument, it follows that $\mathbb{C}^2 \setminus \{0\} \cong \mathbb{R}^+ \times S^3$.

Using the following diagram, we get a map $\varphi : \mathbb{R}^+ \times \Sigma(p, q, r) \to \mathbb{R}^+ \times S^3$ by following the map in the square. Since the projection map we defined earlier $\pi : V - \{0\} \to \mathbb{C}^2 - \{0\}$ is $\mathbb{R}^+$-equivariant, so is φ (where the action of $\mathbb{R}^+$ on $\mathbb{R}^+ \times \Sigma(p, q, r)$ and $\mathbb{R}^+ \times S^3$ is given by multiplication on the first factor).

$$\begin{tikzcd}
V \setminus \{0\} \arrow[r, "\cong"] \arrow[d, "\pi"'] & \mathbb{R}^+ \times \Sigma(p, q, r) \arrow[d] \\
\mathbb{C}^2 \setminus \{0\} \arrow[r, "\cong"'] & \mathbb{R}^+ \times S^3
\end{tikzcd}$$

From this, we can form quotient spaces of the right terms under the action of $\mathbb{R}^+$, and this gives us that $\Sigma(p, q, r)$ is the r-fold cyclic branched cover of S^3 with branch locus $T(p, q)$. □

It turns out Brieskorn manifolds are examples of *Seifert fibered spaces.* One can generalize the definition of $\Sigma(p, q, r)$ as follows. Let

$$\Sigma(a_1, ..., a_n) = \{z_1^{a_1} + ... + z_n^{a_n} = 0\} \cap S^{2n-1} \subset \mathbb{C}^n$$

This space, which is usually called a *Brieskorn-Hamm complete intersection*, can be described topologically. Following [4], let $F = S^2 \setminus int(D_1^2 \cup ... \cup D_n^2)$ be a 2-sphere with n holes given by the discs D_i^2. Consider the S^1-bundle $W \to F$ with Euler number b and fixed trivialization over the boundary of F. Since the boundary of F consists of n circles ∂D_i^2, we have that the boundary of W consists of n tori $(\partial D_i^2) \times S^1$. Now let (α_k, ω_k) be n pairs of relatively prime integers, and paste n solid tori $D_k^2 \times S^1$ into W along the boundaries in such a way that any curve in the homology class $\alpha_k((\partial D_k^2)\times\{1\})+\omega_k(\{1\}\times S^1)$ in the k^{th} boundary component of W is null-homologous in the solid torus $D_k^2\times S^1$ after the pasting.

This construction results in a closed manifold called a *Seifert manifold* or *Seifert fibered space*, and it is denoted by $M(b;(\alpha_1,\omega_1),...,(\alpha_n,\omega_n))$ where the (α_i,ω_i) are called the *Seifert invariants.* Sometimes the invariants are also written as $(b;g;(\alpha_1,\omega_1),...,(\alpha_n,\omega_n))$, where g represents the genus of base manifold used in the construction (in the case above $g=0$ because the base is S^2). The *fibered* in the name comes from the local model for this space, which is a fibered torus where the core circle $\{0\}\times S^1\subset D_k^2\times S^1$ is called a *singular fiber* of degree a_k, and all other points in the D_k^2 determine *non-singular* fibers, which are "twisted" circles inside the torus.

It turns out that $\Sigma(a_1,...,a_n)$ is a Seifert fibered space with Seifert invariants given by the following proposition.

Proposition 2.12. *The Seifert invariants of* $\Sigma(a_1,...,a_n)$ *are*

$$(b;g;s_1(\alpha_1,\omega_1);...;s_n(\alpha_n,\omega_n))$$

where $s_k(\alpha_k,\omega_k)$ *means that the invariant* (α_k,ω_k) *is repeated* s_k *times, and*

$$a=lcm\{a_i\mid 1\le i\le n\};\quad \alpha_i=\frac{a}{lcm\{a_j\mid j\neq i\}};\quad s_i=\frac{a_1\cdots\hat{a_i}\cdots a_n}{lcm\{a_j\mid j\neq i\}}$$

$$2g-2=(n-2)\frac{a_1\cdots a_n}{a}-\sum_i s_i;\qquad \frac{\omega_i\cdot a}{a_i}\equiv -1 \mod \alpha_i;\quad -e=\frac{a_1\cdots a_n}{a^2}=b-\sum_{i=1}^n \omega_i/\alpha_i$$

Proof. For a proof, see [7]. □

This result will allow us to easily compute the Seifert invariants of $\Sigma(a_1,...,a_n)$, which in turn will help us not only compute the $\bar{\mu}$-invariant, but also check when $\Sigma(a_1,...,a_n)$ is an L-space (see section on Heegaard Floer homology).

2.4. **Plumbing manifolds and rational homology spheres.**

We saw that the double-branched covers of 3-component torus links $\Sigma(p,q,2)$ are examples of Seifert fibered spaces. In this section, we will see that if a Seifert fibered space is a *rational homology sphere*, then it can be constructed as a *plumbing manifold.* We give the definitions below.

Definition 2.13. A manifold M is called a *rational homology n-sphere* (denoted $\mathbb{Q}HS^n$) if its homology groups with $\mathbb{Q}$ coefficients are isomorphic to those of S^n, i.e. $H_*(M;\mathbb{Q})\cong H_*(S^n;\mathbb{Q})$.

In our case, we will consider $\mathbb{Q}HS^3$. The following characterization of rational homology 3-spheres is useful.

Proposition 2.14. *A closed, connected, orientable 3-manifold M is a $\mathbb{Q}HS^3$ if and only if $H_1(M;\mathbb{Z})$ is finite.*

Proof. For the forward direction, we want to show that if $H_*(M;\mathbb{Q}) \cong H_*(S^3;\mathbb{Q})$ then $H_1(M;\mathbb{Z})$ is finite. First, by the universal coefficient theorem in homology, we have that $H_n(M;\mathbb{Q}) \cong H_n(M;\mathbb{Z}) \otimes \mathbb{Q}$ so since $H_1(M;\mathbb{Q}) = 0$ then it must be that $H_1(M;\mathbb{Z})$ is a finite abelian group of the form $\mathbb{Z}/d_1 \oplus ... \oplus \mathbb{Z}/d_k$.

Conversely, suppose $H_1(M;\mathbb{Z})$ is finite. Then by the universal coefficient theorem $H_1(M;\mathbb{Q}) \cong H_1(M;\mathbb{Z}) \otimes \mathbb{Q} = 0 = H_1(S^3;\mathbb{Q})$. Moreover, by Poincaré duality, $H_1(M;\mathbb{Z}) \cong H^2(M;\mathbb{Z})$, so $H^2(M;\mathbb{Z})$ is finite. By the universal coefficient theorem for cohomology we have

$$H^2(M;\mathbb{Z}) \cong Hom(H_2(M;\mathbb{Z}),\mathbb{Z}) \oplus Ext(H_1(M;\mathbb{Z}),\mathbb{Z})$$

So since $H^2(M;\mathbb{Z})$ and $H_1(M;\mathbb{Z})$ are finite, it follows that $H_2(M;\mathbb{Z})$ must also be finite, and this implies $H_2(M;\mathbb{Q}) \cong H_2(M;\mathbb{Z}) \otimes \mathbb{Q} = 0 = H_2(S^3;\mathbb{Q})$.

To conclude, the assumption that M is connected gives $H_0(M;\mathbb{Q}) \cong \mathbb{Q}$ and $H_3(M;\mathbb{Q}) \cong H_3(M;\mathbb{Z}) \otimes \mathbb{Q} \cong H^0(M;\mathbb{Z}) \otimes \mathbb{Q} \cong \mathbb{Q}$. □

Now, we describe the construction of a *plumbing manifold* following [4]. A *plumbing graph* Γ is a graph with no cycles, not necessarily connected, and with integer vertex weights e_i. To each vertex x_i we can associate a D^2-bundle $Y(e_i)$ over S^2 with Euler number e_i, the weight of x_i. If the vertex x_i has d_i edges connected to it in Γ, then choose d_i disjoint disks D^2 in the base of $Y(e_i)$ and call the disk bundle over the jth disk B_{ij} so that we have $B_{ij} = D^2 \times D^2$. Now for any two adjacent vertices x_i, x_k in Γ, glue B_{ij} with B_{kl} by identifying the base coordinates of $B_{ij} = D^2 \times D^2$ (given by the first D^2) with the fiber coordinates of $B_{kl} = D^2 \times D^2$ (given by the second D^2). This gluing operation is called *plumbing*, and after plumbing each vertex of Γ we are left with a smooth 4-manifold $P(\Gamma)$, which is said to be obtained by *plumbing according to* Γ.

Since $P(\Gamma)$ is a 4-manifold, it has an intersection form on its second homology. A basis of $H_2(P(\Gamma),\mathbb{Z})$ is given by the zero sections of the D^2-bundles over S^2 that we are plumbing. Thus, we can regard these spheres as embedded in $P(\Gamma)$ and oriented in such a way that the intersection form of $P(\Gamma)$ will be given by the adjacency matrix of Γ denoted by $A(\Gamma) = (a_{ij})$ (sometimes called the *plumbing matrix*) where

$$a_{ij} = \begin{cases} e_i, & \text{if } i = j \\ 1, & \text{if } x_i \text{ is connected to } x_j \text{ by an edge} \\ 0, & \text{otherwise} \end{cases}$$

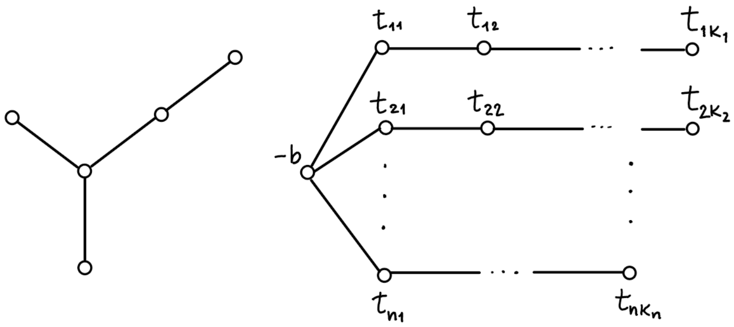

FIGURE 8. Examples of star-shaped graphs

The 3-manifold $Y_\Gamma = \partial P(\Gamma)$ is usually called the *plumbed manifold*, but this term may also refer to $P(\Gamma)$ itself. In this thesis, we are interested in Y_Γ. In fact, every Seifert fibered space can be seen as the plumbed 3-manifold of a star-shaped graph. (A star-shaped graph is a tree graph with one vertex of valence greater than or equal to 3 and all other vertices with valences less than or equal to 2, or a tree with all vertices of valence ≤ 2, i.e. a path graph).

The plumbing graph for the Seifert manifold $M(b; (\alpha_1, \omega_1), ..., (\alpha_n, \omega_n))$ is given by Figure 8 on the right, and the weights t_{ij} are defined by the continuous fraction expansion

$$\alpha_i/\omega_i = t_{i1} - \cfrac{1}{t_{i2} - \cfrac{1}{\cdots - \cfrac{1}{t_{ik_i}}}}$$

Plumbing across the above graph Γ will result in the 3-manifold Y_Γ, which is homeomorphic to the Seifert manifold $M(b; (\alpha_1, \omega_1), ..., (\alpha_n, \omega_n))$.

A natural question arises: when is $\Sigma(a_1, ..., a_n)$ a rational homology sphere? Fortunately, the answer completely depends on the Seifert invariants (α_i, ω_i). The following proposition is from [6] Proposition 6.3.

Proposition 2.15. $\Sigma(a_1, ..., a_n) = M(b; q_1(\alpha_1, \omega_1), ..., q_n(\alpha_n, \omega_n))$ *is a rational homology sphere if and only if the tuple* $(a_1, ..., a_n)$*, after possible permutations, has one of the following forms:*

(i) $(a_1, ..., a_n) = (kp_1, kp_2, p_3, ..., p_n)$ *where* $k \geq 1$*, the integers* $\{p_j\}_{j=1}^n$ *are pairwise coprime, and* $\gcd(k, p_j) = 1$ *for any* $j \geq 3$.

(ii) $(a_1, ..., a_n) = (2^c p_1, 2p_2, 2p_3, p_4, ..., p_n)$ *where the integers* $\{p_j\}_{j=1}^n$ *are pairwise coprime and* $c \geq 1$.

Proof. For a proof, see [6] Proposition 6.3. □

This proposition will be useful for the computations we will carry in the case discussed by this thesis, i.e. $\Sigma(p, q, 2)$ where $\gcd(p, q) = 3$.

2.5. **Heegaard Floer Homology, L-spaces and spin^c structures.** We will first give a definition of a Spin^c structure. These structures are then used to split the Heegaard Floer homology of a 3-manifold into direct summands. The d-invariant is a collection of rational numbers, one for each of the summands of the Heegaard Floer homology.

For this definition of a Spin^c structure, I am following [9].

Definition 2.16. Let M be a closed, oriented 3-manifold. Let g be a Riemannian metric on M and consider the associated principal $SO(3)$-bundle of oriented orthonormal frames $f : F \to M$. A *Spin^c-structure* on M is a lift of f to a principal $U(2)$-bundle. More precisely, a Spin^c structure is an isomorphism class of the pair $(P \to M, \alpha)$, where $P \to M$ is a principal $U(2)$-bundle, and α is an isomorphism of the principal $SO(3)$-bundle $P/U(1) \to M$ onto $f : F \to M$.

An equivalent definition, according to [9], is that a Spin^c structure on M is an element of $H^2(F)$ whose reduction to every fiber is a nonzero element of $H^2(SO(3)) = \mathbb{Z}_2$.

Remark 2.17. An important fact is that the set of Spin^c structures on a 3-manifold M is in (non-canonical) bijection with $H^2(M; \mathbb{Z}) \cong H_1(M; \mathbb{Z})$. This means that if M is a $\mathbb{Q}HS^3$, then $H_1(M; \mathbb{Z})$ is finite, and so there are only finitely many Spin^c structures on M.

A related structure is the *Spin* structure of a 3-manifold Y. We are interested in computing the d-invariants of Y at the *Spin* structures of Y.

Since $Spin^c(3) = Spin(3) \times S^1/\{\pm 1\}$, we define conjugation action on $Spin^c$ to be the map that sends $[w, z] \in Spin^c$ to $[w, \overline{z}]$ where $w \in Spin(3)$ and $z \in S^1 \subset \mathbb{C}$. We can define the *Spin* structures on a manifold M as the *Spin^c* structures that are self-conjugate, i.e. equal to themselves under the conjugation action mentioned above.

Now, we can review the basics of Heegaard Floer homology. Instead of defining everything from scratch, I will give a short summary of the main ideas (following [10]).

Given a 3-manifold Y, one can obtain a *Heegaard diagram* H of Y. Intuitively, a Heegaard diagram is obtained by looking at a splitting of a 3-manifold into 2-handlebodies. From this splitting we can build a finitely generated, graded chain complex over $\mathbb{Z}_2[U]$ denoted by $CF^-(H)$, where U is a formal variable of degree -2. One can also build other complexes denoted $CF^+(H)$ and $\widehat{CF}(H)$. The chain homotopy type of $CF^-(H)$ is an invariant of Y. We denote the homology of the complex by $HF^-(Y)$, which is a finitely generated, graded module over $\mathbb{Z}_2[U]$.

The homology $HF^-(Y)$ splits as a sum over the Spin^c structures on Y, that is

$$HF^-(Y) = \bigoplus_{\mathfrak{s}\in Spin^c(Y)} HF^-(Y, \mathfrak{s}).$$

Note that the homology $HF^-(Y)$ is a finitely generated module over a PID, and the only homogeneously graded polynomials over $\mathbb{Z}_2[U]$ are of the form U^n. Then, by the classification theorem of finitely generated modules over a PID, for each $\mathfrak{s} \in spin^c$, we can write

$$HF^-(Y, \mathfrak{s}) \cong \bigoplus_i \mathbb{Z}_2[U]_{(d)} \oplus \bigoplus_j \mathbb{Z}_2[U]_{c_j}/(U^{n_j}) \tag{2.18}$$

We will be interested in the case where Y is a $\mathbb{Q}HS^3$. Then there is only one free summand in Equation 2.18.

Note that $\mathbb{Z}_2[U]_{(d)}$ denotes the ring $\mathbb{Z}_2[U]$ where $1 \in \mathbb{Z}_2[U]$ has grading d.

Definition 2.19. We define the *d-invariant* of $(Y, \mathfrak{s})$ to be

$$d(Y, \mathfrak{s}) = \max\{gr(x) \mid x \in HF^-(Y, \mathfrak{s}), U^n x \neq 0 \ \forall n > 0\}$$

It follows from this definition that, in equation 2.18, $d(Y, \mathfrak{s}) = d$. The torsion part of equation 2.18 is usually denoted by $HF_{red}(Y, \mathfrak{s})$.

Now we can state the definition of an L-space, which is a a space with the simplest nontrivial Heegaard Floer homology.

Definition 2.20. A rational homology sphere Y with $HF_{red}(Y, \mathfrak{s}) = 0$ for all $\mathfrak{s} \in Spin^c$ is called an L-space.

It turns out being an L-space is a sufficient condition to relate the Neumann-Siebenmann $\bar{\mu}$-invariant of a 4-manifold to the d-invariant of its boundary, but we will see this later.

2.6. **The Neumann-Siebenmann $\bar{\mu}$-invariant.**

Now we will introduce the $\bar{\mu}$-invariant of a plumbing manifold $Y_\Gamma = \partial X_\Gamma$ following [12]. Suppose Γ is the weighted tree from which we construct the plumbing manifold X_Γ (see section 2.4). The following algorithm determines $H_1(Y_\Gamma, \mathbb{Z}_2)$ and, along the way, we obtain a graph $\Gamma' \subset \Gamma$ with no edges.

Consider an arbitrary leaf v in Γ, i.e., a vertex that only has one adjacent vertex, and let w be the adjacent vertex.

- **Move 1**: If the weight of v is *even*, then erase v and w from Γ along with any adjacent edges.

- **Move 2**: If the weight of v is *odd*, then erase v and any adjacent wedges of v from Γ and change the parity of w.

Continue performing move 1 or move 2 until we reach a graph with no edges. Call this graph Γ', and suppose it contains p vertices with odd weight and q edges with even weight.

Proposition 2.21. *(Lemma 2.1* [12]*) The dimension of the vector space $H_1(Y_\Gamma, \mathbb{Z}_2)$ is equal to q.*

Given that the *spin* structures on Y_Γ are indexed by $H_1(Y_\Gamma; \mathbb{Z}_2)$, [12] defines a convenient parametrization of the set of *spin* structures depending only on Γ.

Start with $\Gamma' \subset \Gamma$, the subset of the original graph with no edges defined above, and consider the subsets of Γ' which contain all the vertices of odd weight. From every such subset of Γ', we will construct a subset of vertices $S \subset \Gamma$ in the original graph inductively as follows. At any given step of the process of obtaining Γ' from Γ, we will have performed **move 1** or **move 2**.

- Suppose Γ' was obtained by **move 1** from Γ (via erasing v and w) and let $S' \subset \Gamma'$ be the specified subset of vertices (in the first step, this is a subset containing all odd-weight vertices). Now, if the number of vertices adjacent to w in Γ has the same parity as the weight of w, then set $S = S'$. Otherwise, set $S = S' \cup \{v\}$.
- Instead, if Γ' is derived from Γ by **move 2** (via erasing v), then if w was in S' we set $S = S'$. Otherwise, we set $S = S' \cup \{v\}$.

It turns out every subset S constructed in this way gives rise to a spin structure $\mathfrak{s}_S$ on Y_Γ. The set S is called a *Wu set* of the corresponding spin structure. It is not hard to believe that two different Wu sets give rise to two different spin structures on Y_Γ. Furthermore, every *Spin* structure on Y_Γ can be obtained from a Wu set of Γ. Hence, there is a bijection between the set of Wu sets of Γ and the set of *Spin* structures on Y_Γ.

Now let $\Sigma_S \subset X_\Gamma$ be a disjoint union of embedded spheres corresponding to the vertices in S (since S doesn't contain any adjacent vertices, the union is disjoint).

We are almost ready to define the $\bar{\mu}$-invariant of Y_Γ. The last ingredient we need is the signature of X_Γ and the intersection form of Σ_S.

Definition 2.22. Let M be a 4-dimensional manifold with non-empty boundary. The cup product gives rise to a symmetric, bilinear, nondegenerate quadratic form

$$Q : H_2(M; \mathbb{Z}) \times H_2(M; \mathbb{Z}) \to \mathbb{Z}$$

called the *intersection form* of M. The *signature* of M, denoted by $\sigma(M)$, is defined as the signature of the matrix Q, that is, $\sigma(M) = n_+ - n_-$ where n_+, n_- are the number of positive and negative entries, respectively, of a diagonalization of Q.

For a plumbing manifold, the intersection form matrix is given by the plumbing matrix itself. In this case, for any surface $\Sigma \subset X_\Gamma$ we write $[\Sigma]^2$ to mean $Q([\Sigma],[\Sigma])$ where $[\Sigma] \in H_2(X_\Gamma;\mathbb{Z})$. We are finally ready to define the $\bar{\mu}$-invariant.

Definition 2.23. For a spin structure $\mathfrak{s}$ on $Y_\Gamma = \partial X_\Gamma$, consider the corresponding Wu set S and the embedded surface $\Sigma_S \subset X_\Gamma$. Then define

$$\bar{\mu}(Y_\Gamma, \mathfrak{s}) = \sigma(X_\Gamma) - [\Sigma_S]^2$$

The well-definedness of this invariant is proven in ([12] Proposition 2.3).

Remark 2.24. This definition is useful since once we have Σ_S, it is easy to compute $[\Sigma_S]^2$: this is just the sum of the weights of the vertices in S.

3. Results/Computations

3.1. **When is $\Sigma(p,q,2)$ a rational homology sphere?** We are interested in the case when $M = \Sigma(p,q,2)$ is a rational homology sphere, since we want to study its d-invariants. Using Proposition 2.15, the following result is not hard to prove.

Proposition 3.1. *The 3-component torus links L which have their 2-fold branch cover a rational homology 3-sphere are of the form $L = T(3n,3m)$ where n,m are odd and coprime.*

Proof. Following Proposition 2.15, first note that $(p,q,2)$ cannot have the form of (ii) since this would mean $gcd(p,q) = 2 \cdot gcd(p_i,p_j) \neq 3$. Thus we have $(p,q,2) = (kn,km,r)$. If $r = 2$, then without loss of generality, we have $p = kn$ and $q = km$. Our condition is that $\gcd(p,q) = \gcd(kn,km) = k \cdot \gcd(n,m) = 3$, which since by (i) we must have $\gcd(n,m) = 1$ then $k = 3$. Moreover, by (i) the integers $n,m,2$ are pairwise coprime, and $\gcd(k,2) = 1$ (which is always true since $k = 3$). Therefore $\gcd(m,2) = 1$ and $\gcd(n,2) = 1$, meaning m,n are odd.

In the case where $r \neq 2$ we must have, say, $kn = 2$, which implies $k = 2$ and $n = 1$ or $k = 1$ and $n = 2$. Suppose $k = 1$. This would imply that in our case, $p = m$ and $q = r$, so $\gcd(p,r) = \gcd(m,r) = 3$, which violates the condition in (i) that requires the p_i's to be pairwise coprime. Thus we must have $k = 2$. So we must have $\gcd(p,q) = \gcd(2m,r) = 3$ from our condition (3 component torus link), and $\gcd(2,r) = 1$ and $\gcd(m,r) = 1$ by (i). But since $\gcd(m,r) = 1$ then $\gcd(2m,r) \neq 3$, a contradiction. Therefore, we conclude that the above configuration is the only possible one, i.e. $(p,q,2) = (3n,3m,2)$. $\square$

Given this result, we can now compute the Seifert invariants of $\Sigma(3n,3m,2)$ using Proposition 2.12.

FIGURE 9. The Torus link $T(3,3)$

Proposition 3.2. *The rational homology 3-sphere double branched cover of a 3-component torus link* $\Sigma(3n,3m,2)$ *has normalized Seifert invariants*

$$(b_0;0;(n,b_1);(m,b_2);(2,b_3),(2,b_3),(2,b_3))$$

where $b_3=1,\ 0\leq b_1<n,\ 0\leq b_2<m$ *and*

$$2mb_1\equiv -1 \ (mod\ n)$$

$$2nb_2\equiv -1 \ (mod\ m)$$

$$b_0=(2mb_1+2nb_2+3nm+1)/2nm$$

Proof. One obtains the invariants and the equations by plugging in the values $a_1=3n, a_2=3m, a_3=2$ into the equations given in proposition 2.12. $\square$

Remark 3.3. One must first compute e to find b_0. The conditions above give the *normalized* Seifert invariants. To compute the *unnormalized* Seifert invariants, one can set $b_0=0$ and find solutions to the equation $2mb_1+2nb_2+3nmb_3=-1$, disregarding the bounds on the b_i. However, one must always remember to check that the pairs (a_i,b_i) are coprime, i.e. $\gcd(a_i,b_i)=1$ for all i.

Example. The Torus link $T(3,3)$ is an example of a 3-component torus link that has $\Sigma(3,3,2)$ a rational homology 3-sphere (see Figure 9).

3.2. *Spin*^{c} structures on $\Sigma(3n,3m,2)$.

Recall that the *Spin*^{c} structures on a manifold M are indexed by $H^2(M;\mathbb{Z})$, that is, there is a (non-canonical) bijection between the two sets. Moreover, the number of *Spin* structures on the double-branched cover of a q-component link is given by the number of quasi orientations of the strands.

We can use the following well-known theorem to compute the size of $H^2(M;\mathbb{Z})\cong H_1(M;\mathbb{Z})$.

Proposition 3.4. *If* $M = \Sigma_2(L)$ *is the double-branched cover of a link* L*, then* $|H_1(M;\mathbb{Z})| = |\det(L)|$.

Proof. Let V be a Seifert matrix of L. It is known that $H_1(\Sigma_2(L),\mathbb{Z})$ is presented by the matrix $V+V^T$. Suppose $Q(V+V^T)P^{-1} = D = diag(d_1,..,d_k,0,...,0)$ is a diagonalization of $V+V^T$. Then $\det(L) = \det(V+V^T) \neq 0 \iff |\det(L)| = |d_1 \cdots d_k|$ and since $H_1(\Sigma_2(L),\mathbb{Z}) \cong \mathbb{Z}/d_1 \oplus \cdots \oplus Z/d_k$ then $|H_1(\Sigma_2(L),\mathbb{Z})| = |d_1 \cdots d_k|$. $\square$

Computing the determinant of the link L is an easy task for torus links thanks to the following theorem.

Theorem 3.5. *(Murasugi 7.3.2) Let* $\Delta(t)$ *be the Alexander polynomial of a torus link* $T(p,q)$ *where* $\gcd(p,q) = d \geq 1$*. Then*

$$t^{\frac{(p-1)(q-1)}{2}}\Delta(t) = (-1)^{d-1}\frac{(1-t)(1-t^{\frac{pq}{d}})^d}{(1-t^p)(1-t^q)}$$

This leads us to a result relating the *spin* structures of $M = \Sigma(3n,3m,2)$ with the Spin^c structures, which will be useful in relating the $\bar{\mu}$-invariant and the d-invariant related to $\Sigma(3n,3m,2)$. First, we briefly introduce *spin* structures.

Following [11], let M be an oriented, connected, smooth m-dimensional manifold, and let $q : E \to M$ be the principal $GL_m(\mathbb{R})$-bundle associated with TM, the tangent bundle of M.

Definition 3.6. A *spin* structure on M is an arbitrary element of $H^1(M;\mathbb{Z}_2)$ whose restriction to the fiber $GL_m(\mathbb{R})$ of the bundle $q : E \to M$ is the nonzero element of $H^1(GL_m(\mathbb{R});\mathbb{Z}_2) = \mathbb{Z}_2$.

Remark 3.7. It follows that the set of *spin* structures on a 3-manifold Y is indexed by $H^1(Y;\mathbb{Z}_2)$, i.e. there is a bijection between the two sets.

In the same paper ([11]), Turaev defines a bijection between the quasi-orientations of a link $L \subset S^3$ and the *spin* structures on $\Sigma_2(L)$. Recall that a *quasi-orientation* of a link L is simply a choice of orientation for each component of the link, and we say two orientations are equivalent as quasi-orientations if one is obtained by reversing the orientation of each component of the link. This means that a link of k components will have 2^{k-1} different quasi-orientations.

Now we can state the result relating the *spin* structures of $\Sigma(3n,3m,2)$ to the Spin^c structures. This result will be useful in relating the $\bar{\mu}$-invariant to the d-invariant of $\Sigma(3n,3m,2)$.

Proposition 3.8. *For* $M = \Sigma(3n,3m,2)$*, we have* $|H_1(M,\mathbb{Z})| = 4$*. This means that the spin structures on* M *are in bijective correspondence with the Spin*^{c} *structures on* M*.*

Proof. Let $L = T(3n, 3m)$ be the torus link. Recall that $|H_1(M, \mathbb{Z})| = |\det(L)| = |\Delta(-1)|$. Plugging in -1 in the equation from theorem 3.5 and $p = 3m, q = 3m, d = 3$, and recalling that m, n are odd, we obtain the following

$$(-1)^{\frac{(3n-1)(3m-1)}{2}}\Delta(-1) = (-1)^{3-1}\frac{(1-(-1))(1-(-1)^{9mn})^3}{(1-(-1)^{3n})(1-(-1)^{3m})}$$

$$\implies \Delta(-1) = \frac{2\cdot 2^3}{2\cdot 2} = 4$$

This tells us that the number of $Spin^c$ structures on M is 4. Since the number of *Spin* structures is given by the number of quasi-orientations on the link, which in our case is $2^{d-1} = 2^{3-1} = 4$, and there is an inclusion from *Spin* into $Spin^c$, this implies that in this case there is a bijection, i.e. every $Spin^c$ structure on M comes from a *Spin* structure on M. ☐

3.3. **A program to compute the Seifert invariants of $\Sigma(3n, 3m, 2)$.** Given that the equations to compute the Seifert invariants of $\Sigma(3n, 3m, 2)$ are simple, here is a Python program that helped me compute multiple examples very quickly.

The `seif_invariants` function computes the Seifert invariants of $\Sigma(3n, 3m, 2)$. The `continued_fraction` function computes the continued fraction expansion of a rational number as

$$\frac{a}{b} = t_1 - \frac{1}{t_2 - \frac{1}{t_3 - \cdots}}$$

The `tree_weights` function computes the weights of the star-shaped plumbing graph that gives rise to $\Sigma(3n, 3m, 2)$ as a plumbing manifold.

```python
import math

def seif_invariants(p,q):
    a_1 = p//3  # n
    a_2 = q//3  # m
    if (a_1%2==0) or (a_2%2==0):
        print("n␣or␣m␣are␣not␣odd")
        return
    if math.gcd(a_1,a_2) != 1:
        print("gcd(m,n)␣is␣not␣1")
        return
    for j in range(0, a_1):  # (n, b_1)
        if ((2*a_2*j)%a_1 == (a_1-1)):
            b_1 = j
            break
    for i in range(0, a_2):  # (m, b_2)
        if (2*a_1*i)%a_2 == (a_2-1):
            b_2 = i
```

```python
            break
    b_0 = (2*a_2*b_1 + 2*a_1*b_2 + 3*a_1*a_2 + 1) // (2*a_1*a_2)
    return [-b_0, b_1, a_1, b_2, a_2, 1, 2, 1, 2, 1, 2]

def continued_fraction(x):
    key = 0
    r = []
    tol = 1e-12
    i = 1
    if x > 0:
        while key == 0:
            if math.isclose(x, math.ceil(x), abs_tol=tol):
                key = 1
                r.append(math.ceil(x))
            else:
                r.append(math.ceil(x))
                x = 1 / (math.ceil(x) - x)
            i += 1
        return r

def tree_weights(p,q):
    S = seif_invariants(p,q)
    f_1 = S[2] / S[1]
    f_2 = S[4] / S[3]
    t1 = continued_fraction(f_1)
    t2 = continued_fraction(f_2)
    return t1, t2, 2, 2, 2
```

Listing 1. Python program for Seifert invariants

This program was useful to quickly obtain the Seifert invariants of different examples of $\Sigma(3n, 3m, 2)$. Moreover, thanks to a program developed by Karakurt ([13]) based on the work of Nemethi in [14], one can compute the d-invariants of any Seifert manifold that is a rational homology sphere and negative definite.

3.4. ***L*-spaces and a relation between $\bar{\mu}(Y_\Gamma, \mathfrak{s})$ and $d(Y_\Gamma, \mathfrak{s})$.** Thanks to the two programs mentioned in the previous section, it was easy to compute some cases and try to see if any relationship between the $\bar{\mu}$-invariant and the d-invariant of $\Sigma(3n, 3m, 2)$ emerged. In [12] Theorem 1.4, Stipsicz establishes the following relation between the two invariants in a more general setting.

Theorem 3.9. *Suppose that* Γ *is a negative definite plumbing tree giving rise to a rational surface singularity. Let* $\mathfrak{s}$ *be a given spin structure on the associated* 3*-manifold* Y_Γ*. Then* $\bar{\mu}(Y_\Gamma, \mathfrak{s}) = -4d(Y_\Gamma, \mathfrak{s})$*.*

Thanks to the work of Nemethi in [15], we know that the link of a normal surface singularity is an L-space if and only if the singularity is rational. Therefore, we can replace "rational surface singularity" in Theorem 3.9 with "L-space".

One of the driving questions of this thesis was if the relationship $\bar{\mu}(Y_\Gamma, \mathfrak{s}) = -4d(Y_\Gamma, \mathfrak{s})$ still holds in the case where $Y_\Gamma = \Sigma(3n, 3m, 2)$, which might not necessarily be an L-space. The computation of some examples disproved this hypothesis.

Example 3.10. Consider the torus link $T(3,9)$ and its double-branched cover $\Sigma(3,9,2)$. Using proposition 3.2 we obtain the following Seifert invariants

$$\Sigma(3,9,2) = M(2; 0; (1,0), (3,1), (2,1), (2,1), (2,1))$$

Using these invariants and Karakurt's program, we compute all the possible d-invariants for each Spin^c structure (recall that we showed that the Spin^c structures were in bijective correspondence with the spin structures, and we showed there are 4 spin structures). We obtain $d(\Sigma(3,9,2)) \in \{0, 0, -1, 0\}$. Now we can construct the graph Γ that will give rise to $Y_\Gamma = \Sigma(3,9,2)$ as a plumbing manifold (see Figure 10) by using the python program above, which computes the tree weights. We obtain the following graph, where the weights are $w(t_{11}) = 3$ and $w(t_{21}) = w(t_{31}) = w(t_{41}) = 2$ and $w(v) = b = 2$:

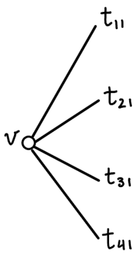

FIGURE 10. The star shaped graph giving rise to $Y_\Gamma = \Sigma(3,9,2)$

This graph gives us the following plumbing matrix, which is also the intersection form of X_Γ:

$$\begin{pmatrix} 2 & 1 & 1 & 1 & 1 \\ 1 & 3 & 0 & 0 & 0 \\ 1 & 0 & 2 & 0 & 0 \\ 1 & 0 & 0 & 2 & 0 \\ 1 & 0 & 0 & 0 & 2 \end{pmatrix}$$

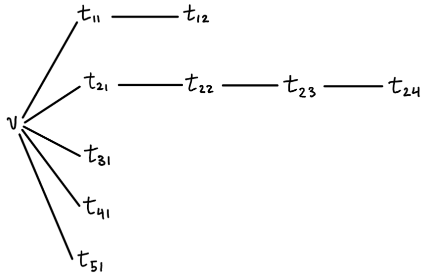

FIGURE 11. Star-shaped graph giving rise to $Y_\Gamma = \Sigma(9,15,2)$

We can easily compute the signature of this matrix and obtain $\sigma(X_\Gamma) = 5$. Now to compute the $\bar{\mu}$-invariant, we must also compute the possible Wu sets for this graph. Following the algorithm described in section 2.6, we obtain 4 different Wu sets (as expected) which are

$$S_1 = \{t_{11}, t_{21}\}, S_2 = \{t_{11}, t_{31}\}, S_3 = \{t_{11}, t_{41}\}, S_4 = \{t_{11}, t_{21}, t_{31}, t_{41}\}$$

Therefore, we can compute $\bar{\mu}(Y_\Gamma, \mathfrak{s}_{S_i}) = \sigma(X_\Gamma) - [\Sigma_{S_i}]^2$ corresponding to each Wu set and obtain

$$\bar{\mu}(Y_\Gamma, \mathfrak{s}_{S_1}) = \bar{\mu}(Y_\Gamma, \mathfrak{s}_{S_2}) = \bar{\mu}(Y_\Gamma, \mathfrak{s}_{S_3}) = \sigma(X_\Gamma) - (3+2) = 0$$

$$\bar{\mu}(Y_\Gamma, \mathfrak{s}_{S_4}) = \sigma(X_\Gamma) - [\Sigma_{S_4}]^2 = 5 - (3+2+2+2) = -4$$

Therefore, in this case, we see that the relationship $\bar{\mu}(Y_\Gamma, \mathfrak{s}) = 4d(Y_\Gamma, \mathfrak{s})$ holds instead of $\bar{\mu}(Y_\Gamma, \mathfrak{s}) = -4d(Y_\Gamma, \mathfrak{s})$, which is the one in Theorem 3.9. This means that, by the contrapositive of Theorem 3.9, we can conclude that $\Sigma(3,9,2)$ is not an L-space.

Example 3.11. Consider the torus link $T(9,15)$ and its double-branched cover $\Sigma(9,15,2)$. Applying proposition 3.2 we obtain the following Seifert invariants

$$\Sigma(9,15,2) = M(3;0;(3,2),(5,4),(2,1),(2,1),(2,1))$$

Moreover, using Karakurt's program we compute that the d-invariants are $d(\Sigma(9,15,2)) \in \{-2,-1,-2,-2\}$. Now we can construct the graph Γ (see Figure 11) that will give rise to $Y_\Gamma = \Sigma(9,15,2)$ as a plumbing manifold as earlier (computing the continuous fraction expansion of each α_i/ω_i). We obtain the following star-shaped graph, where $w(v) = b = 3$ and $w(t_{ij}) = 2$ for all i, j:

which has the following plumbing matrix (which is the intersection form of X_Γ)

$$\begin{pmatrix} 3 & 1 & 0 & 1 & 0 & 0 & 0 & 1 & 1 & 1 \\ 1 & 2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 2 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 2 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 2 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 2 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 \end{pmatrix}$$

We easily compute that the signature is $\sigma(X_\Gamma) = 10$. Now, we can compute the possible Wu sets of Γ following section 2.6 to obtain

$$S_1 = \{t_{31}\}, S_2 = \{t_{41}\}, S_3 = \{t_{51}\}, S_4 = \{t_{31}, t_{41}, t_{51}\}$$

Therefore, the $\bar{\mu}$-invariant corresponding to each Wu set (equivalently, each spin structure) is

$$\bar{\mu}(Y_\Gamma, \mathfrak{s}_{S_1}) = \bar{\mu}(Y_\Gamma, \mathfrak{s}_{S_2}) = \bar{\mu}(Y_\Gamma, \mathfrak{s}_{S_3}) = \sigma(X_\Gamma) - [\Sigma_{S_1}]^2 = 10 - 2 = 8$$

$$\bar{\mu}(Y_\Gamma, \mathfrak{s}_{S_1}) = \sigma(X_\Gamma) - [\Sigma_{S^4}]^2 = 10 - (2 + 2 + 2) = 4$$

Therefore, we see that in this case we have $\bar{\mu}(Y_\Gamma, \mathfrak{s}) = -4d(Y_\Gamma, \mathfrak{s})$.

However, in this example, this does not show that Y_Γ is an L-space. Fortunately, there is a theorem of Rasmussen and Rasmussen that lets us check if Y_Γ is an L-space completely from the Seifert invariants.

Theorem 3.12. *(*[16] *Theorem 5.1) If $M(b; 0; (\alpha_1, \omega_1), ..., (\alpha_n, \omega_n))$ is the Seifert fibered space over S^2, then $M(b; 0; (\alpha_1, \omega_1), ..., (\alpha_n, \omega_n))$ is not an L-space if and only if at least one of the following conditions holds*

(i) $-b + \sum_{i=1}^{n} \frac{\omega_i}{\alpha_i} = 0$

(ii) $b + \min_{0<x<s} -\frac{1}{x}\left(-1 + \sum_{i=1}^{n} \left\lceil \frac{\omega_i x}{\alpha_i} \right\rceil\right) < 0 < b + \max_{0<x<s} -\frac{1}{x}\left(1 + \sum_{i=1}^{n} \left\lfloor \frac{\omega_i x}{\alpha_i} \right\rfloor\right)$

where $s = lcm(s_1, ..., s_n)$.

Thanks to this theorem, we can check that $Y_\Gamma = \Sigma(9, 15, 2)$ from Example 3.11 is not an L-space. We first see that condition (i) fails for every n, m

$$-b + \sum_{i=1}^{n} \frac{\omega_i}{\alpha_i} = -\frac{2mb_1 + 2nb_2 + 3nm + 1}{2nm} + \frac{b_1}{n} + \frac{b_2}{m} + \frac{1}{2} + \frac{1}{2} + \frac{1}{2} = -\frac{1}{2nm} \neq 0$$

Now, for condition (ii), we substitute our values for n, m, b_1 and b_2 to obtain $b = 3$ and $s = lcm(2,3,5) = 30$. We can compute by hand that

$$\frac{-1}{x}\left(-1+\left\lceil\frac{2x}{3}\right\rceil+\left\lceil\frac{4x}{5}\right\rceil+3\left\lceil\frac{x}{2}\right\rceil\right)$$

has a minimum when $x \to 2$ from the right. Computing this value gives

$$\min_{0<x<30}\frac{-1}{x}\left(-1+\left\lceil\frac{2x}{3}\right\rceil+\left\lceil\frac{4x}{5}\right\rceil+3\left\lceil\frac{x}{2}\right\rceil\right)=-\frac{9}{2}$$

Similarly, we can compute the maximum value of

$$\frac{-1}{x}\left(1+\left\lfloor\frac{2x}{3}\right\rfloor+\left\lfloor\frac{4x}{5}\right\rfloor+3\left\lfloor\frac{x}{2}\right\rfloor\right)$$

which occurs when $x \to 5/4$ from the left, so

$$\max_{0<x<30}\frac{-1}{x}\left(1+\left\lfloor\frac{2x}{3}\right\rfloor+\left\lfloor\frac{4x}{5}\right\rfloor+3\left\lfloor\frac{x}{2}\right\rfloor\right)=-\frac{8}{5}$$

So condition (ii) translates to checking $3 - 9/2 < 0 < 3 - 8/5$ which is true. Therefore, we conclude that $\Sigma(9,15,2)$ is not an L-space.

However, the relation $\bar{\mu}(Y_\Gamma, \mathfrak{s}) = -4d(Y_\Gamma, \mathfrak{s})$ still holds for $Y_\Gamma = \Sigma(9,15,2)$. This means that Stipsicz's result can be proven for a larger set of spaces that are not necessarily L-spaces.

3.5. **Future Work.**

The next step in this project would be to find a sub-family of the torus links $T(3n, 3m)$ where $\bar{\mu}(Y_\Gamma, \mathfrak{s}) = -4d(Y_\Gamma, \mathfrak{s})$ and Y_Γ is not an L-space.

Another possible computation would be to give a characterization of when $\Sigma(3n, 3m, 3)$ are L-spaces, which would show when these spaces satisfy the $\bar{\mu}(Y_\Gamma, \mathfrak{s}) = -4d(Y_\Gamma, \mathfrak{s})$ relation.

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