Towards a Characterization of the Complete Rotationally Symmetric Minimal Surfaces with Plateau Singularities

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Abstract

The problem of finding and characterizing the surfaces in $\mathbb{R}^3$ which locally minimize area is known as Plateau's problem. Although the catenoid and the plane were proven in the 1700s to minimize area, there has been little further study of rotationally symmetric minimal surfaces. In this study, we investigate the complete rotationally symmetric solutions to Plateau's problem, revealing surprising depth due to singularities that may appear in a broad class of minimal surfaces. Our analysis is structured around the topology of the surface's generating graph, and we first consider surfaces of a simple topological type. For these surfaces, we prove new statements about complexity and shape, relating the number of singularities to the Hausdorff distance from a canonical example. We then consider more complicated structures, producing a novel surface with a handle (in particular, whose generating graph contains a 4-cycle). We finally provide direction for future study.

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1 Introduction

1.1 Overview

The problem of finding and characterizing the surfaces of locally minimum area (alternatively, of critical area) with a given boundary is called *Plateau's problem*. Through his experiments in the 1840s, Joseph Plateau deduced a set of principles regarding soap films called *Plateau's Laws*, which may be formulated as follows [Ras92]:

1. Soap films are piecewise-smooth surfaces, in which each smooth piece has constant zero mean curvature;
2. Smooth surfaces on soap films meet at 3-fold singularities along arcs, and arcs meet at 4-fold points of singularity;
3. At arc singularities, surfaces meet three at a time at equal angles of 120°, and at point singularities, arcs meet at equal angles of roughly 109°28′.

Mean curvature refers to the arithmetic mean of the principle curvatures at a flat point on some surface embedded in $\mathbb{R}^3$; this is equivalent to the mean of the cross-sectional curvatures at this point. Any surface that has zero mean curvature at every point is called *minimal*.

The connection between soap films and minimal surfaces is due to the fact that the equation for mean curvature corresponds to the equation for signed surface tension. Interestingly, vanishing mean curvature merely indicates a critical point in surface area, not necessary a local minimum [CM23]. Thus, soap films may also form as "saddle-point surfaces," albeit in an unstable state (analogous to an upside-down pendulum). These surfaces are still called minimal.

Of course, the solution to Plateau's problem depends on the class of admissible surfaces. For example, the minimal smooth manifold with a given boundary will not have the singularities observed in many soap films. One might also consider minimal surfaces which are not embedded. Jean Taylor proved that a class of surfaces which are minimal among $(\mathbf{M}, \xi, \delta)$-*sets*, and can thus be called $(\mathbf{M}, \xi, \delta)$-*minimal*, adhere to Plateau's laws (and are embedded), and are therefore *soap-film-like* [Tay76].

Euler proved that the *catenoid*, the surface of rotation of a catenary curve (called the *generatrix* of the surface) about its directrix, is a minimal surface. The catenoid and the plane are the only rotationally symmetric minimal surfaces in $\mathbb{R}^3$ with no singularities. Thus, rotationally symmetric soap-film-like surfaces are unions of catenoids and planes with Plateau-like singularities.

We note that, while co-dimension-2 Plateau-like singularities may appear in soap films, they may not appear in rotationally symmetric soap films. We can

show this by considering two cases. First, suppose that such a singularity was present on the axis of rotation of our surface. No neighborhood of the singularity exhibits rotational symmetry about the axis, so this is not possible. In the second case, suppose that the singularity appeared on a point other than the axis of rotation. In this case, due to the rotational symmetry of our soap film, the point singularity must be rotated around the axis, giving us instead a co-dimension-1 singularity. Thus, it is sufficient in our construction to consider only 3-fold singularities along arcs.

While Plateau's problem classically deals with surfaces with boundary, interesting examples of *complete* minimal and soap-film-like surfaces have been constructed. Besides the catenoid and the plane, Costa's surface is an example of a complete minimal surface of finite genus; Schwartz's minimal surfaces are families of complete, triply-periodic minimal surfaces [Cos84, KP96].

The aim of the current project is to understand all soap films with rotational symmetry. We structure our analysis based on the topological structure of the generatrix, first providing a singly-periodic surface, and proceed to construct and describe a sequence of surfaces whose limit converges to the periodic example. We will then continue to more exotic surfaces, providing an example of a surface in which the generatrix contains a cycle, meaning the surface has non-trivial topological genus. Finally, we provide a number of directions for further study.

1.2 Definitions and Conventions

In this section, we make clear some terms which will be used throughout this paper.

Definition 1.1. We call a surface which is embedded in $\mathbb{R}^3$, connected, and adheres to each of Plateau's laws **soap-film-like**.

Definition 1.2. For $a, b \in \mathbb{R}$, $a \neq 0$, a **caternary** is a real-valued function

$$f(x) = a \cosh\left(\frac{x}{a} + b\right),$$

whose domain is a connected subset of $\mathbb{R}$, or the graph of such a function. The constant a is the **scaling factor** of the caternary. A **catenoid** is the surface of rotation of a catenary about its directrix (that is, about the line $x = 0$).

Note that all catenaries are *similar*, in the sense that they can be mapped onto each other by suitable translation and scaling. We do not allow for vertical translation in Definition 1.2 because we require that every catenary have the same directrix.

We will use the notation of the *anti-Gudermannian* function, as defined below.

Definition 1.3. The real-valued function

$$\mathrm{gd}^{-1}(x) = \int_0^x \sec(t)dt = \sinh^{-1}(\tan(x))$$

is called the **anti-Gudermannian**.

The anti-Gudermannian $\mathrm{gd}^{-1}(\theta)$ is the x coordinate at which $\cosh(x)$ has slope $\tan(\theta)$, and therefore makes an angle θ above the horizontal. We note that $\cosh(\sinh^{-1}(x)) = \sqrt{1+x^2}$, so $\cosh(\mathrm{gd}^{-1}(\theta)) = \sec(\theta)$.

Convention 1.4. We consider surfaces unique up to similarity.

Definition 1.5. We call a surface in $\mathbb{R}^3$ **rotationally symmetric** if it is invariant under rotation about a given axis.

Convention 1.6. Without loss of generality, we will take the axis of symmetry to be the x axis and the generatrix to by in the xy plane. Thus, we will be looking at unions of graphs of functions of x.

Convention 1.7. If f is a caternary and $\mathrm{Dom}\, f = [c, \infty)$ for some $c \in \mathbb{R}$, then f **opens to the right**. Similarly, if $\mathrm{Dom}\, f = (-\infty, c]$ for some $c \in \mathbb{R}$, then f **opens to the left**.

Convention 1.8. To examine its topological properties, will often consider the generatrix of a surface as an undirected graph, whose edges correspond to the smooth pieces of the generatrix and nodes correspond to intersections of these pieces. There are some edges of the generatrix which are unbounded; it is convenient to consider these edges connected to distinct *leaf nodes* of the graph (rather than, say, a single node at infinity). We will refer to the unique undirected graph with these properties as the graph *corresponding to* our surface of interest.

Definition 1.9. The **depth** of a node n in a graph G is

$$\mathrm{depth}(n) = \min_{l \in L} d(n, l),$$

where L is the set of leaf nodes (nodes of degree 1), and $d(n, l)$ is the length of the shortest path between n and l. The depth of the graph itself is

$$\mathrm{depth}(G) = \max_{n \in G} \mathrm{depth}(n).$$

See Figure 1 for an illustration of Convention 1.8 and Definition 1.9.

Definition 1.10. An **n-cycle** in a graph is a path from a node to itself of length n such that no edge is traversed twice.

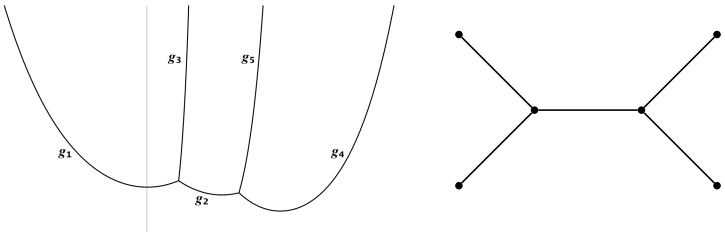

Figure 1: The generatrix of a complete soap-film-like surface (left), and its corresponding graph (right). The depth of the graph on the right is 1.

2 Acyclic Surfaces

In this section, we consider those surfaces whose corresponding graph is *acyclic*. Because each vertex has either 1 or 3 neighbors, these graphs are actually *(unrooted) binary trees*. We construct an illustrative example in Section 2.1. Then we characterize complete soap-film-like surfaces composed of exactly 3 catenaries, which tells us about the nature of triple-intersections. We expand this into "chains" of catenoids in which intersections are taken to be as far apart as possible. We prove that there exists a sequence of such surfaces that converges in Hausdorff distance to our surface from Section 2.1. We also show that a soap-film-like surface of revolution of a tree of depth 1 is either periodic on an unbounded interval or has finitely many intersections.

2.1 A Periodic Rotationally-Symmetric Minimal Surface

The following theorem constructs our periodic surface.

Theorem 2.1. *For $i \in \mathbb{Z}$, let*

$$f_i = \{(x, \cosh(x - 2ia)) \,|\, (2i-1)a \le x \le (2i+1)a\} \subset \mathbb{R}^2$$

and

$$g_i = \left\{ ((2i+1)a, y) \;\middle|\; \sec\left(\frac{\pi}{6}\right) \le y \right\} \subset \mathbb{R}^2,$$

where $a = \mathrm{gd}^{-1}(\pi/6)$. Then the surface of rotation of $\bigcup_i (f_i \cup g_i)$ about the x axis is a singly-periodic rotationally-symmetric soap-film-like surface.

Proof. Each f_i is a catenary centered about $2ia$, beginning and ending with a slope of $-\tan(\pi/6)$ and $\tan(\pi/6)$, respectively. Note that

$$f_i \cap f_j = \begin{cases} ((2i \pm 1)i, \sec(\pi/6)), & j = i \pm 1 \\ \emptyset & j \neq i \pm 1. \end{cases}$$

Thus, each f_i intersects $f_{i\pm1}$ at an angles of $2\pi/3$, and intersects no other f_j. Each g_i is a vertical ray stemming from the intersection of f_i and f_{i+1} (and intersects no other f_j). Thus, each f_i, f_{i+1}, g_i intersect at equal angles of $2\pi/3$,

and so do their surfaces of rotation about the x axis.

Because the surface of rotation about the x axis of each f_i is a smooth subset of a catenoid, and the surface of rotation of each g_i is a smooth subset of a plane, Plateau's laws are followed and our surface is soap-film-like.

It is clear from our construction that our surface is periodic in the x-variable with period $2a$. ■

A subset of this surface and its generatrix is shown in Figure 2.

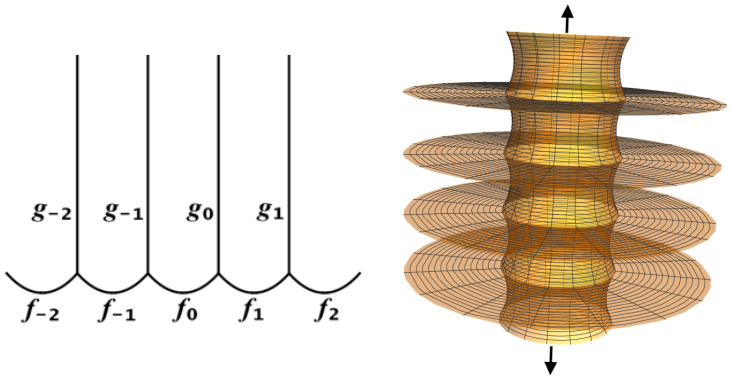

Figure 2: A subset of our periodic surface (right) and its generatrix (left).

2.2 Triple-Intersections of Catenaries

We state and prove the following lemma, which tells us when we are between the intersections of caternaries with positive, distinct scaling factors. This is important when dealing with complete surfaces, to ensure that there will be no unintentional intersections.

Lemma 2.2. *Let g_1 and g_2 be caternaries with $g_1 \neq g_2$, with respective scaling factors c_1 and c_2. Let $g = g_1 - g_2$.*

(a) If $c_1 = c_2$, then g has exactly 1 zero; call this x_0. We have $g_1'(x_0) = -g_2'(x_0)$.

(b) If $c_1 > c_2 > 0$, then g has exactly 2 zeros; call them x_1 and x_2, with $x_1 < x_2$. Then $g_1(x) > g_2(x) \iff x_1 < x < x_2$.

Proof. Without loss of generality, let

$$g_1(x) = c_1 \cosh\left(\frac{x}{c_1}\right),$$
$$g_2(x) = c_2 \cosh\left(\frac{x}{c_2} + c_3\right).$$

for for $c_1, c_2, c_3 \neq 0$.

(a) First, suppose $c_1 = c_2$. Then

$$g(x_0) = 0 \Rightarrow x_0 = \pm(x_0 + c_1 c_3) \Rightarrow x_0 = -\frac{c_1 c_3}{2}.$$

Thus, $g_1'(x_0) = -\sinh(c_3/2)$ and $g_2'(x_0) = \sinh(c_3/2) = -g_1'(x_0)$.

(b) Now, let $c_1 > c_2 > 0$. We observe that, because the global minima of $g_1(x)$ and $g_2(x)$ are their respective scaling factors c_1 and c_2, there exists some $x_0 \in \mathbb{R}$ such that $g(x_0) > 0$. Now, as $x \to \pm\infty$,

$$g_1 \sim ae^{\pm x/c_1},\ g_2 \sim be^{\pm x/c_2} \Rightarrow g(x) \to -\infty,$$

for some $a, b > 0$. Thus, there exists $x_1, x_2 \in \mathbb{R}$, $x_1 < x_2$, such that $g(x_1) = g(x_2) = 0$. Solving for the critical points of $g(x)$, we see that

$$g'(x) = 0 \Rightarrow \sinh\left(\frac{x}{c_1}\right) = \sinh\left(\frac{x}{c_2} + c_3\right) \Rightarrow x = \frac{c_1 c_2 c_3}{c_2 - c_1}.$$

There is only one critical point; it must be at the global maximum of $g(x)$, which is positive. Combining this fact with the asymptotic behavior of $g(x)$, we conclude that $g(x)$ has exactly 2 zeros. Thus, $g_1(x) > g_2(x) \iff x_1 < x < x_2$. ■

Lemma 2.2 simply proves the intuitive idea that a "fatter" catenary is only greater than a "skinnier" catenary between their 2 intersections; and that catenaries of the same width only intersect each other once.

Now, we proceed with the construction of three caternaries intersecting exactly once, at equal angles of $2\pi/3$. Call these caternaries f_1, f_2, f_3, and let them intersect at $x = x_0$. Let f_1 intersect from the positive x direction at an angle θ below the horizontal, and let f_2 intersect counterclockwise from f_1 (see Figure 3). Without loss of generality, we can take $\theta \in (-\pi/6, \pi/2)\backslash\{\pi/6\}$. Note that $\theta < \pi/6$ corresponds to f_2 opening to the left, and visa versa.

Let f_1 have scaling factor $c_1 > 0$. Then $f_1(x) = c_1 \cosh(x/c_1 + d_1)$ for some $d_1 \in \mathbb{R}$, and

$$f_1'(x_0) = \sinh\left(\frac{x_0}{c_1} + d_1\right) = -\tan(\theta) \Longrightarrow d_1 = -\frac{x_0}{c_1} - \mathrm{gd}^{-1}(\theta).$$

Because f_1 intersects from the positive x direction, it has the domain $[x_0, \infty)$. Therefore,

$$f_1(x) = c_1 \cosh\left(\frac{x - x_0}{c_1} - \mathrm{gd}^{-1}\theta\right), \quad x \in [x_0, \infty). \tag{1}$$

Now, say that $f_2(x) = c_2 \cosh(x/c_2 + d_2)$. First, assume the case where $\theta > \pi/6$, so f_2 has the domain $[x_0, \infty)$. Then

$$f_2'(x_0) = \sinh\left(\frac{x_0}{c_2} + d_2\right) = \tan\left(\frac{2\pi}{3} - \theta\right) \Rightarrow d_2 = -\frac{x_0}{c_2} + \mathrm{gd}^{-1}\left(\frac{2\pi}{3} - \theta\right).$$

The condition $f_1(x_0) = f_2(x_0)$ requires that

$$c_2 = \frac{c_1 \cos(2\pi/3 - \theta)}{\cos(\theta)}.$$

Therefore, setting c_2 this way, we have

$$f_2(x) = c_2 \cosh\left(\frac{x - x_0}{c_2} + \mathrm{gd}^{-1}\left(\frac{2\pi}{3} - \theta\right)\right), \quad x \in [x_0, \infty). \tag{2}$$

If instead $-\pi/6 < \theta < \pi/6$, then $f_2(x)$ intersects from the negative x direction with a slope of $-\tan(\pi/3 + \theta)$. In this case, f_2 has the domain $(-\infty, x_0]$, and computation similar to that of before yields

$$f_2(x) = c_2 \cosh\left(\frac{x - x_0}{c_2} - \mathrm{gd}^{-1}\left(\frac{\pi}{3} + \theta\right)\right), \tag{3}$$

where this time

$$c_2 = \frac{c_1 \cos(\pi/3 + \theta)}{\cos(\theta)}.$$

Finally, because f_3 intersects from the negative x direction with slope $\tan(\pi/3 - \theta)$, we arrive at

$$f_3(x) = c_3 \cosh\left(\frac{x - x_0}{c_3} + \mathrm{gd}^{-1}\Big(\pi/3 - \theta\Big)\right), \quad x \in (-\infty, x_0], \tag{4}$$

where

$$c_3 = \frac{c_1 \cos(\pi/3 - \theta)}{\cos(\theta)}.$$

Now, we will construct conditions on θ such that the caternaries intersect exactly once.

Lemma 2.3. *The surface of revolution of the union of 3 caternaries is a complete soap-film-like surface if and only if it is the surface of revolution of $f_1 \cup f_2 \cup f_3$, where each f_i is defined as in (1), (3), and (4), for some value of x_0, c_1, and $\theta \in [0, \pi/3] \backslash \{\pi/6\}$.*

Proof. The surface of revolution of the union of 3 caternaries is a complete soap-film-like surface if and only if those caternaries intersect exactly once, at equal angles of $2\pi/3$. Thus, due to our earlier construction, it must be the surface of revolution of $f_1 \cup f_2 \cup f_3$, with f_1, f_2, f_3 as constructed before. We begin with the case of $\theta \in (\pi/6, \pi/2)$. Because $f_2'(x_0) > f_1'(x_0)$, there exists some $\varepsilon > 0$ such that $f_2(x) > f_1(x)$ for all $x \in (x_0, x_0 + \varepsilon)$. Therefore, by Lemma 2.2, in order for f_1 and f_2 not to intersect again for some $x > x_0$, we need $c_2 \leq c_1$. Thus,

$$\frac{c_1 \cos(2\pi/3 - \theta)}{\cos(\theta)} \leq c_1 \Longrightarrow \frac{\cos(2\pi/3 - \theta)}{\cos(\theta)} \leq 1 \Longrightarrow \theta \leq \pi/3.$$

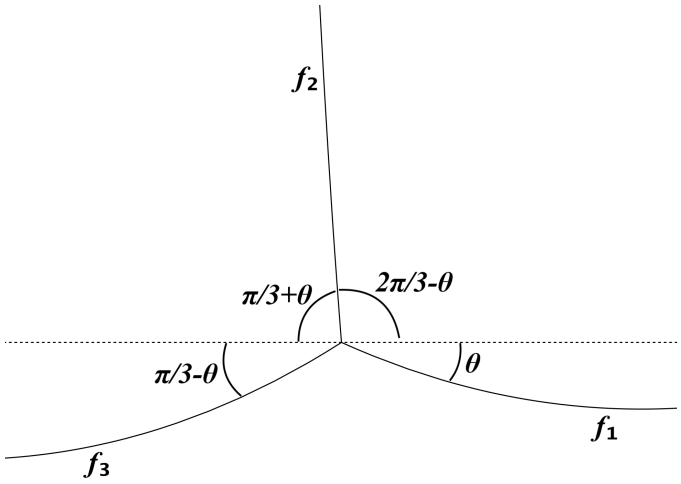

Figure 3: The construction of a caternary triple-intersection.

For $\theta \in (-\pi/6, \pi/6)$, we have $f_2'(x_0) < f_3'(x_0)$, so there exists some $\varepsilon > 0$ such that $f_2(x) > f_3(x)$ for all $x \in (x - \varepsilon, x_0)$. Therefore, by Lemma 2.2, in order for f_2 and f_3 not to intersect again, we need $c_2 \leq c_3$. Thus,

$$\frac{c_1 \cos(\pi/3 + \theta)}{\cos(\theta)} \leq \frac{c_1 \cos(\pi/3 - \theta)}{\cos(\theta)} \Longrightarrow \frac{\cos(\pi/3 + \theta)}{\cos(\pi/3 - \theta)} \leq 1 \Longrightarrow \theta \geq 0.$$

Therefore, the surface of revolution is minimal if and only if $\theta \in [0, \pi/3] \backslash \{\pi/6\}$. ■

Note that the angles $\theta = 0$ and $\theta = \pi/3$ are acceptable because catenaries with the same scaling factor intersect only once. These sorts of boundary cases will be of interest in the following investigation.

2.3 Surfaces Corresponding to Trees of Depth 1

In this section, we will generalize the case of the periodic surface constructed in Section 2.1, characterizing the surfaces whose corresponding graph is acyclic and of depth 1. That is, we will investigate the soap-film-like surfaces of revolution whose graphs have the property that each node is connected to at least one unbounded edge.

We will begin by constructing sequences of catenaries $\{f_i\}_{\theta_0}$ and $\{g_i\}_{\theta_0}$. We will omit the subscript θ_0 when it is understood. Without loss of generality, let

$$f_0(x) = c_0 \cosh\left(\frac{x}{c_0}\right). \tag{5}$$

Now let the first intersection occur when f_0 is making an angle θ_0 below the horizontal from the left, with $-\frac{\pi}{6} < \theta_0 < \frac{\pi}{6}$ so that the other two caternaries open to the right (this is without loss of generality; in any acceptable intersection of catenaries, two of them open in the same direction). Let g_1 be the caternary which has a greater slope at this intersection, and let f_1 be the third caternary (see Figure 4). We emphasize that this construction has one essential choice, that is, our choice of θ_0. The choice of c_0 is simply the scaling factor of our construction. We choose to write c_0 for consistency, because each f_i will have a different scaling factor c_i.

The first point of intersection is thus $T_1 = c_0 \operatorname{gd}^{-1}(\theta_0)$, and

$$f_1(x) = c_1 \cosh\Big(\frac{x - T_1}{c_1} - \operatorname{gd}^{-1}\Big(\frac{\pi}{3} - \theta_0\Big)\Big),$$

where

$$c_1 = \frac{c_0 \cos(\pi/3 - \theta_0)}{\cos(\theta_0)}.$$

Finally,

$$g_1(x) = d_1 \cosh\Big(\frac{x - T_1}{d_1} + \operatorname{gd}^{-1}\Big(\frac{\pi}{3} + \theta_0\Big)\Big),$$

where

$$d_1 = \frac{c_0 \cos(\pi/3 + \theta_0)}{\cos(\theta_0)}.$$

We computed the scaling factors c_1 and d_1 by imposing $f_0(T_1) = f_1(T_1)$ and $f_0(T_1) = g_1(T_1)$. Now, we can continue our sequence recursively.

If f_1 branches off at an angle of θ_1, then this would occur at

$$T_2 = T_1 + c_1\Big(\operatorname{gd}^{-1}(\theta_1) + \operatorname{gd}^{-1}\Big(\frac{\pi}{3} - \theta_0\Big)\Big).$$

In general,

$$T_{i+1} = T_i + c_i\Big(\operatorname{gd}^{-1}(\theta_i) + \operatorname{gd}^{-1}\Big(\frac{\pi}{3} - \theta_{i-1}\Big)\Big), \tag{6}$$

and we can write

$$f_i(x) = c_i \cosh\Big(\frac{x - T_i}{c_i} - \operatorname{gd}^{-1}\Big(\frac{\pi}{3} - \theta_{i-1}\Big)\Big). \tag{7}$$

The condition $f_i(T_{i+1}) = f_{i+1}(T_{i+1})$ implies that

$$c_{i+1} = \frac{c_i \cos(\pi/3 - \theta_i)}{\cos(\theta_i)} \tag{8}$$

for all $i \geq 0$. Because $-\frac{\pi}{6} < \theta_i < \frac{\pi}{6}$, the sequence $\{c_i\}$ is strictly decreasing.

Now, note that if we write

$$g_i(x) = d_i \cosh\left(\frac{x - T_i}{d_i} + \mathrm{gd}\left(\frac{\pi}{3} + \theta_{i-1}\right)\right), \tag{9}$$

then we must have

$$d_i = \frac{c_{i-1}\cos\left(\frac{\pi}{3} + \theta_{i-1}\right)}{\cos(\theta_{i-1})}. \tag{10}$$

We again computed c_i and d_i by imposing $f_{i-1}(T_i) = f_i(T_i) = g_i(T_i)$. Thus, the scaling factor for g_{i+1} is strictly decreasing in θ_i for fixed i. By Lemma 2.2, there consequently must be some maximum value for each θ_i such that the right-unbounded catenaries g_i and g_{i+1} do not intersect. This is the value of θ_i such that d_i and d_{i+1} are equal. Thus, we define our sequences as constructed above, additionally taking the maximum value of θ_i given θ_{i-1} for each i, as constructed in the following proposition. We additionally take our sequence to terminate at i if $\theta_{i+1} < -\frac{\pi}{6}$, causing f_{i+2} to open the wrong way (and to terminate at $i-1$ if $\theta_{i+1} < -\frac{\pi}{6}$ and $\theta_i < 0$, which would mean that f_{i+1} and g_{i+1} would intersect unacceptably).

Proposition 2.4. *Suppose the graphs $\{g_i\}$ given by (9) do not intersect. Then the greatest value that θ_i may take given θ_{i-1} is*

$$\theta_i = \tan^{-1}\left[\frac{\sqrt{3}}{3} - \frac{2\sqrt{3}}{3}\left(\cos(2\theta_{i-1}) - \sin(2\theta_{i-1})\tan\left(\frac{\pi}{3} - \theta_{i-1}\right)\right)\right]. \tag{11}$$

Proof. By Lemma 2.2, because $g_{i+1}(T_{i+1}) < g_i(T_{i+1})$ for all i, g_i and g_{i+1} intersect right of T_{i+1} if and only if $d_{i+1} < d_i$. The expression (10) for d_i is continuous and decreasing in θ_{i-1} for fixed i. Therefore, there exists a maximum value of θ_i given θ_{i-1} such that g_i and g_{i+1} do not intersect right of T_{i+1}, produced by setting $d_i = d_{i+1}$ as follows.

$$\begin{aligned}
d_i = d_{i+1} &\iff \frac{c_{i-1}\cos\left(\frac{\pi}{3} + \theta_{i-1}\right)}{\cos(\theta_{i-1})} = \frac{c_{i-1}\cos\left(\frac{\pi}{3} - \theta_{i-1}\right)}{\cos(\theta_{i-1})}\frac{\cos\left(\frac{\pi}{3} + \theta_i\right)}{\cos(\theta_i)} \\
&\iff \frac{\cos\left(\frac{\pi}{3} + \theta_{i-1}\right)}{\cos\left(\frac{\pi}{3} - \theta_{i-1}\right)} = \frac{\cos\left(\frac{\pi}{3} + \theta_i\right)}{\cos(\theta_i)} = \frac{1}{2} - \frac{\sqrt{3}}{2}\tan(\theta_i) \\
&\iff \theta_i = \tan^{-1}\left[\frac{\sqrt{3}}{3} - \frac{2\sqrt{3}}{3}\left(\cos(2\theta_{i-1}) - \sin(2\theta_{i-1})\tan\left(\frac{\pi}{3} - \theta_{i-1}\right)\right)\right].
\end{aligned}$$

Finally, we note that, if no g_i and g_{i+1} intersect, then no g_i and g_j intersect for $i \neq j$ by transitivity (because each $g_i > g_{i+1}$ where their domains intersect). ∎

We thus have a recursive sequence $\{\theta_i\}$ for the greatest θ_i given an initial θ_0 such that no g_i and g_j intersect. We will now begin to inspect this sequence, as it provides us the essential structure of our minimal surface.

Lemma 2.5. *The sequence $\{\theta_i\}$ defined by (11) is decreasing in i, and $|\theta_{i+1} - \theta_i|$ is decreasing in θ_i (so increasing in i), for $\theta \in (-\frac{\pi}{6}, \frac{\pi}{6})$.*

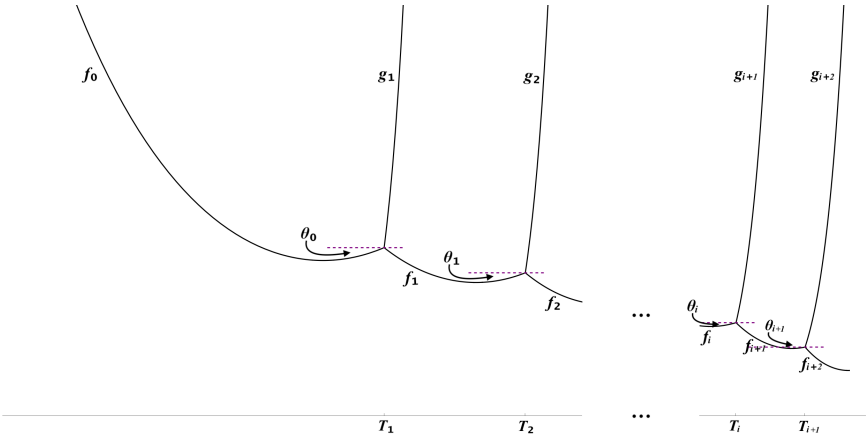

Figure 4: Our model.

Proof. Let $f(\theta_i) = \theta_{i+1} - \theta_i$. Then

$$f(\theta_i) = \tan^{-1}\left[\frac{\sqrt{3}}{3} - \frac{2\sqrt{3}}{3}\Big(\cos(2\theta_i) - \sin(2\theta_i)\tan\left(\frac{\pi}{3} - \theta_i\right)\Big)\right] - \theta_i.$$

This function $f(\theta_i)$ is negative and increasing for $\theta_i \in (-\frac{\pi}{6}, \frac{\pi}{6})$, which is the range of acceptable values for θ_i. ■

Lemma 2.6. *The sequence $\{\theta_i\}$ given by (11) and some initial $\theta_0 \in (-\frac{\pi}{6}, \frac{\pi}{6})$ is not bounded below by $-\frac{\pi}{6}$. In other words, there is some j such that $\theta_j < -\frac{\pi}{6}$.*

Proof. By Lemma 2.5, $\{\theta_i\}$ is monotone decreasing while $\theta_i \in (-\frac{\pi}{6}, \frac{\pi}{6})$. Thus, it either converges to a value in $[-\frac{\pi}{6}, \frac{\pi}{6})$ or it is not bounded below by $-\frac{\pi}{6}$, by the Monotone Convergence Theorem. But if $\{\theta_i\}$ converged to a value in $(-\frac{\pi}{6}, \frac{\pi}{6})$, its second difference would converge to 0 in that interval. By Lemma 2.5, this is not the case, so there must be some i such that $\theta_i < \frac{\pi}{6}$. ■

Theorem 2.7. *A soap-film-like surface of revolution whose corresponding graph is a tree of depth 1 is either periodic on an unbounded interval or has finitely many intersections.*

Proof. Consider an arbitrary soap-film-like surface of revolution whose corresponding graph has depth 1. From each node of the graph there must be at least one unbounded edge (that is, an edge connected to a leaf node). This unbounded edge corresponds to either a ray, a catenary opening the right, or a catenary opening to the left in the directrix of our surface.

Consider the generatrix of a such a surface. Suppose that the generatrix has a ray. Then pick a point of intersection between two catenaries and that ray. There are either no nodes right of this point of intersection (i.e., it is a point on a catenary opening to the right), or there is some node right of this point of

intersection connected to a catenary opening to the right, or there are infinitely many nodes right of this point of intersection, none of which are connected to a catenary opening to the right. In the third case, our surface is periodic on the unbounded interval from the x-coordinate of this point of intersection to infinity. In the second case, pick a node connected to a catenary opening to the right. Then the catenary connected to the left side of this node makes some angle $\theta_0 < \frac{\pi}{6}$ below the horizontal at the point of intersection. By Proposition 2.4, the recursive sequence $\{\theta_i\}$ given by (11) gives the maximum angle that the bounded catenary to the right of our vertex may make below the horizontal before splitting off again in our surface, given that the catenary to the right of the vertex makes an angle θ_{i-1} below the horizontal at this point of intersection. Therefore, by Lemma 2.6, there must a finite number of vertices to the right of a vertex connected to a catenary that opens to the right, because if $\theta_i < -\frac{\pi}{6}$ at a vertex, then two of the catenaries connected to that vertex lie to the left of that vertex, leading to an unacceptable intersection.

Now, by symmetry, there must be a finite number of intersections to the left of an intersection connected to a catenary opening to the left. Therefore, if there is a catenary opening to the left and a catenary opening to the right in the generatrix of our surface, then there are finitely many intersections in our surface. On the other hand, if there is no catenary opening to the right in the generatrix of our surface, then that surface must be periodic on a right-unbounded interval, and if there is no catenary opening to the left in the generatrix of our surface, then that surface must be periodic on a left-unbounded interval.

■

We will provide an explicit formula for the sequence induced by (11). First, we provide an explicit formula for (8), our sequence of scaling factors.

Lemma 2.8. *For a sequence catenaries $\{f_i\}$ defined by (7) following the maximum angle sequence (11),*

(a) the sequence of scaling factors $\{c_i\}$ is linear, and

(b) $c_i = c_0 - c_0 i(\frac{1}{2} - \frac{\sqrt{3}}{2}\tan(\theta_0))$.

Proof. Part (a) holds if $c_{i+1} - c_{i+2} = c_i - c_{i+1}$ for all i. Now,

$$\begin{aligned}
c_i - c_{i+1} &= c_{i+1} - c_{i+2} \\
\iff c_i\Big(1 - \frac{\cos(\frac{\pi}{3} - \theta_i)}{\cos(\theta_i)}\Big) &= c_i\Big(\frac{\cos(\frac{\pi}{3} - \theta_i)}{\cos(\theta_i)}\Big)\Big(1 - \frac{\cos(\frac{\pi}{3} - \theta_{i+1})}{\cos(\theta_{i+1})}\Big). \\
\iff 2 &= \frac{\cos(\theta_i)}{\cos(\frac{\pi}{3} - \theta_i)} + \frac{\cos(\frac{\pi}{3} - \theta_{i+1})}{\cos(\theta_{i+1})} \\
\iff 2 &= \frac{\cos(\theta_i)}{\cos(\frac{\pi}{3} - \theta_i)} + \frac{1}{2} + \frac{\sqrt{3}}{2}\tan(\theta_{i+1}) \\
\iff 2 &= \frac{\cos(\theta_i)}{\cos(\frac{\pi}{3} - \theta_i)} + \frac{1}{2} + \frac{\sqrt{3}}{2}\Big[\frac{\sqrt{3}}{3}\Big(1 - \frac{2\cos(\frac{\pi}{3} - \theta_i)}{\cos(\frac{\pi}{3} - \theta_i)}\Big)\Big] \\
\iff 2 &= \frac{\cos(\theta_i) + \cos(\frac{\pi}{3} - \theta_i) - \cos(\frac{\pi}{3} + \theta_i)}{\cos(\frac{\pi}{3} - \theta_i)} \\
\iff 2 &= \frac{2\cos(\frac{\pi}{3} - \theta_i)}{\cos(\frac{\pi}{3} - \theta_i)},
\end{aligned}$$

proving part (a). Part (b) immediately follows. ∎

This is a surprising result. After our complicated recursive formula (11) for θ_i, and complicated formula (8) for c_i in terms of c_{i-1} and θ_{i-1}, it is quite unusual that the sequence of scaling factors for f_i decreases linearly when we impose that the sequence of scaling factors for g_i remains constant.

We now combine the recursive and explicit formulas for c_i to produce an explicit formula for θ_i.

Lemma 2.9. *Given $\theta_0 \in (-\frac{\pi}{6}, \frac{\pi}{6})$, the sequence $\{\theta_i\}$ defined by the recursive formula (11) has the explicit form*

$$\theta_i = \tan^{-1}\left[\frac{\sqrt{3}}{3}\left(1 + \frac{2\sqrt{3}\tan(\theta_0) - 2}{2 + i(\sqrt{3}\tan(\theta_0) - 1)}\right)\right]. \tag{12}$$

Proof. Applying Lemma 2.8 to c_i in the recursive formula for c_{i+1},

$$c_{i+1} = \frac{c_i\cos(\frac{\pi}{3} - \theta_i)}{\cos(\theta_i)} = c_0\Big[1 - i\Big(\frac{1}{2} - \frac{\sqrt{3}}{2}\tan\theta_0\Big)\Big]\Big(\frac{1}{2} + \frac{\sqrt{3}}{2}\tan\theta_i\Big).$$

Once more invoking Lemma 2.8,

$$c_0\Big[1 - i\Big(\frac{1}{2} - \frac{\sqrt{3}}{2}\tan\theta_0\Big)\Big]\Big(\frac{1}{2} + \frac{\sqrt{3}}{2}\tan\theta_i\Big) = c_0 - c_0(i+1)\Big(\frac{1}{2} - \frac{\sqrt{3}}{2}\tan\theta_0\Big).$$

Solving for θ_i yields the desired result. ∎

Proposition 2.10. *For an acyclic soap-film-like surface of revolution whose corresponding graph has depth 1, and in which the catenary intersecting the*

leftmost catenary opening to the right from the left has vertex $x = 0$ and scaling factor c_0, the maximum $x-$coordinate of an intersection in the surface is

$$c_0 \,\mathrm{gd}^{-1}(\theta_0) + \sum_{i=1}^{\lfloor \frac{6\tan\theta_0}{\sqrt{3}-3\tan\theta_0} \rfloor} c_i \Big[\mathrm{gd}^{-1}(\theta_i) - \mathrm{gd}^{-1}\Big(\frac{\pi}{3} - \theta_{i-1}\Big)\Big],$$

where $c_i = c_0 - c_0 i\Big(\frac{1}{2} - \frac{\sqrt{3}}{2}\tan(\theta_0)\Big)$ *and* $\theta_i = \tan^{-1}\left[\frac{\sqrt{3}}{3}\left(1 + \frac{2\sqrt{3}\tan(\theta_0)-2}{2+i(\sqrt{3}\tan(\theta_0)-1)}\right)\right]$.

Proof. Suppose that the generatrix of our surface beginning with the catenary intersecting the leftmost catenary opening to the right from the left is the union of $\{f_i\}$ and $\{g_i\}$ defined by (7) and (9). If $\theta_i < 0$, then $c_{i+1} < d_{i+1}$, meaning f_{i+1} will intersect with g_{i+1} unless one of them splits. But if $\theta_{i+1} < 0$, then $\theta_{i+2} < -\frac{\pi}{6}$ according to (11), and so f_{i+2} opens to the left, which is not acceptable. Thus, we must stop branching while $\theta_i \geq 0$. Note that $\theta_i = 0$ is acceptable as a last branch, because then $c_{i+1} = d_{i+1}$. Thus, the maximum i that we may have is $\max_{0\leq\theta_i} i$. Setting

$$f(x) = \tan^{-1}\left[\frac{\sqrt{3}}{3}\left(1 + \frac{2\sqrt{3}\tan(\theta_0) - 2}{2 + x(\sqrt{3}\tan(\theta_0) - 1)}\right)\right],$$

we see that

$$f(x^*) = 0 \Rightarrow x^* = \frac{6\tan\theta_0}{\sqrt{3} - 3\tan\theta_0}.$$

Therefore, the largest acceptable value of i is the floor of that value, and the coordinate of the last intersection is $T_{\lceil x^* \rceil}$. The expression in the proposition then follows from the fact that $T_0 = c_0 \mathrm{gd}^{-1}(\theta_0)$ and

$$T_{i+1} = T_i + c_i\Big(\mathrm{gd}^{-1}(\theta_i) + \mathrm{gd}^{-1}\Big(\frac{\pi}{3} - \theta_{i-1}\Big)\Big).$$

■

Corollary 2.10.1. *For any $n \in \mathbb{N}$, there exists a soap-film-like surface of revolution, which is not periodic on any unbounded interval, with N intersections for some $N > n$.*

Proof. The result follows from Proposition 2.10 and the fact that

$$\lim_{\theta \to \frac{\pi}{6}^-} \frac{6\tan\theta}{\sqrt{3} - 3\tan\theta} = \infty,$$

so choosing θ_0 close enough to $\frac{\pi}{6}$ allows for enough intersections. ■

Now, let's extend our sequence as follows. Take f_0 as from (5), and for $i > 0$ take f_i as from (7) with domain $[T_i, T_i + 1]$, and take g_i as from (9) with domain $[T_i, \infty)$. Now, for $i < 0$ let $f_i(x) = f_{-i}(-x)$ with domain $[-T_{i+1}, -T_i]$, and let $g_i(x) = g_{-i}(-x)$ with domain $(-\infty, -T_i]$. Finally, only define these curves for

$|i| \leq \lceil \frac{6\tan\theta_0}{\sqrt{3}-3\tan\theta_0} \rceil$, the bound constructed in Proposition 2.10.

With this definition, let $S(\theta_0)$ be the surface of revolution of

$$\left[\bigcup_{0 \neq i \in [-i^*, i^*]} f_i \cup g_i \right] \cup f_0, \tag{13}$$

where each f_i and g_i are constructed according to θ_0, and $i^* = \lceil \frac{6\tan\theta_0}{\sqrt{3}-3\tan\theta_0} \rceil$.

We prove that $S(\theta_0) \to S$ as $\theta_0 \to \pi/6^-$, where S indicates our periodic surface from Section 2.1.

Theorem 2.11. *Take S as constructed in Theorem 2.1 and $S(\theta_0)$ as in (13). Then for any $r \in \mathbb{R}$ and $\varepsilon > 0$, there exists some $\phi < \frac{\pi}{6}$ such that $d_H(S_r, (S_r(\phi^*)) < \varepsilon$ for all $\phi < \phi^* < \frac{\pi}{6}$, where d_H indicates the Hausdorff distance, and S_r and $S_r(\phi^*)$ indicate S and $S(\phi^*)$ restricted to the cylinder $\{(x,y,z) \in \mathbb{R}^3 \,|\, (x,y) \in [-r,r] \times B_r\}$.*

Proof. Due to the rotational symmetry of each surface, and the mirror symmetry of each surface about the line $x = 0$, it suffices to show that the part of the generatrix of $S(\theta_0)$ to the right of the x-axis converges to the part of the generatrix of S to the right of the x axis, in Hausdorff distance.

First, we show that $\theta_i \to \frac{\pi}{6}$ as $\theta_0 \to \frac{\pi}{6}^-$ for all $i > 0$. Now,

$$\lim_{\theta_0 \to \pi/6^-} \theta_i = \frac{\pi}{6} \iff \lim_{\theta_0 \to \pi/6^-} \frac{2\sqrt{3}\tan(\theta_0) - 2}{2 + i(\sqrt{3}\tan(\theta_0) - 1)} = 0.$$

The right equality holds because the function in the limit is continuous around $\theta_0 = \frac{\pi}{6}$ and evaluates to 0 at that point for all $i > 0$; therefore, the left equality holds as well.

Thus,

$$\lim_{\theta_0 \to \pi/6^-} c_i = c_0 - c_0 i \left(\frac{1}{2} - \frac{\sqrt{3}}{2} \frac{\sqrt{3}}{3} \right) = c_0.$$

Finally,

$$\lim_{\theta_0 \to \pi/6^-} T_i = c_0 \mathrm{gd}^{-1}\left(\frac{\pi}{6}\right) + 2c_0 i \mathrm{gd}^{-1}\left(\frac{\pi}{6}\right).$$

Note that we can take all of these limits because, by Corollary 2.10.1, for all i there exists some $\delta > 0$ such that θ_i, c_i, and T_i all exist for all $\frac{\pi}{6} - \delta < \theta_0 < \frac{\pi}{6}$.

Now, recall that the generatrix of S is composed of

$$\bigcup_i f_i^* \cup g_i^*,$$

where

$$f_i^* = \left\{ (x, c_0 \cosh\left(\frac{x - 2ia}{c_0}\right)) \;\middle|\; c_0(2i-1)a \le x \le c_0(2i+1)a \right\}$$

and

$$g_i^* = \left\{ ((2i+1)a, y) \;\middle|\; c_0 \sec\left(\frac{\pi}{6}\right) \le y \right\},$$

and $a = \mathrm{gd}^{-1}(\pi/6)$. We will write $f_i^*(x) = c_0 \cosh((x-2ia)/c_0)$. Here, we use f_i^* and g_i^* for the components of S to reserve f_i and g_i for the components of $S(\theta_0)$.

Now, a helper lemma before finishing our proof.

Lemma 2.12. *Let A and B be subsets of some Euclidean space, let $\{A_1, \ldots, A_n\}$ be a covering of A, and let $\{B_1, \ldots, B_n\}$ be a covering of B. If $d_H(A_i, B_i) < \varepsilon$ for each i, then $d_H(A, B) < \varepsilon$.*

Proof. A point in A can be no closer to a subset of B than to B itself. ■

By Lemma 2.12, it suffices to show that the Hausdorff distance between each component f_i of $S(\theta_0)$ and f_i^* of S, and each component g_i of $S(\theta_0)$ and g_i^* of S, goes to 0 as $\theta_0 \to \frac{\pi}{6}$.

Now, because of the convergences of θ_i, c_i, and T_i shown above and the continuity of $\cosh(x)$, for every $\varepsilon_1 > 0$ there exists some $\delta_1 > 0$ such that $|f_i(x) - f_i^*(x)| < \varepsilon_1$ for all x and $\theta_0 \in (\frac{\pi}{6} - \delta_1, \frac{\pi}{6})$. Note that the domain of $f_i(x)$ as a component of $S(\theta_0)$ is $[T_i, T_{i+1}]$, while the domain of $f_i^*(x)$ as a component of S is $[c_0(2i-1)a, c_0(2i+1)a]$. Due to the smoothness of $\cosh(x)$ and the convergence of T_i to $c_0(2i-1)a$ from the left, for all $\varepsilon_2 > 0$ there exists some $\delta_2 > 0$ such that $|\nu - c_0(2i-1)a|^2 + |f_i(\nu) - f_i^*(c_0(2i-1)a)|^2 < \varepsilon_2^2$ for all $T_0 \le \nu < c_0(2i-1)a$, for all $\theta_0 \in (\frac{\pi}{6} - \delta_2, \frac{\pi}{6})$.

Therefore, all of the components f_i of $S(\theta_0)$ converge to the components f_i^* of S in Hausdorff distance on any finite interval $[-r, r]$ (we need just take the greatest i such that the domain of f_i or f_i^* has a nonempty intersection with this interval).

Now, we turn out attention to each g_i. Recall that

$$g_i(x) = d_i \cosh\left(\frac{x - T_i}{d_i} + \mathrm{gd}\left(\frac{\pi}{3} + \theta_{i-1}\right)\right),$$

where

$$d_i = \frac{c_{i-1}\cos(\frac{\pi}{3} + \theta_{i-1})}{\cos(\theta_{i-1}}.$$

Note that $d_i \to 0$ and $g_i(T_i) \to c_0 \sec(\frac{\pi}{6})$ as $\theta_0 \to \frac{\pi}{6}$. Also note that, because $d_i \mathrm{gd}(\frac{\pi}{3} + \theta_{i-1}) \to 0$ as $\theta_0 \to \frac{\pi}{6}$, we have

$$g_i^{-1}(r) \to c_0(2i-1)a + d_i \cosh^{-1}\left(\frac{r}{d_i}\right) \to c_0(2i-1)a$$

as $\theta_0 \to \frac{\pi}{6}$. Therefore, for all $r \in \mathbb{R}$ and $\varepsilon > 0$, there exists $\delta > 0$ such that the maximum distance between g_i^* and $g_i(x)$, restricted to $y < r$, is ε, for all $\theta_0 \in (\frac{\pi}{6} - \delta, \frac{\pi}{6})$.

Therefore, for all $i \in \mathbb{Z}$, $r \in \mathbb{R}$, and $\varepsilon > 0$, there exists $\delta > 0$ such that $d_H(f_i, f_i^*) < \varepsilon$ and $d_H(g_i, g_i^*) < \varepsilon$ for f_i, f_i^*, g_i, g_i^* restricted to $[-r, r] \times [-r, r]$, for all $\theta_0 \in (\frac{\pi}{6} - \delta, \frac{\pi}{6})$. Hence, by rotational symmetry, the same holds for the surface of rotation of each f_i, f_i^*, g_i, g_i^* about the x axis, restricted to the cylinder $[-r, r] \times B_r$; thus, for all $r \in \mathbb{R}$, $S_r(\theta_0) \to S_r$ in Hausdorff distance as $\theta_0 \to \frac{\pi}{6}^-$. ■

This gif shows the right half of the generatrix of $S(\theta_0)$ appearing to converge according to Theorem 2.11.

What we have shown is intuitive as soon as we know that each θ_i exists and goes to $\pi/6$ as θ_0 goes to $\pi/6$. We broke S and $S(\theta_0)$ into covers and showed that the distance between S_i and $S(\theta_0)$ goes to 0 for each S_i in the cover of S, and visa versa. This approach was convenient because S and $S(\theta_0)$ are stratified manifolds in which the strata make convenient coverings. Finally, as S and $S(\theta_0)$ are both unbounded, showing convergence in a cylinder of length and radius r for any $r \in \mathbb{R}$ is the best we can do. The approximation of S by $S(\theta_0)$ for θ_0 close to $\frac{\pi}{6}$ resembles the nth-degree Taylor approximation of an analytic function for large n, in the sense that it can hold arbitrarily well in an arbitrarily large neighborhood, but fails outside of that neighborhood.

We leave as a conjecture that the graph corresponding to a soap-film-like surface of revolution has depth 1 if it is acyclic (see Conjecture 4.1).

3 Cyclic Surfaces

Our investigation into soap-film-like surfaces of revolution is not complete without inspecting surfaces corresponding to cyclic graphs. The surface of revolution of a cyclic graph is topologically interesting, as it is a surface containing a handle.

Proposition 3.1. *There may be no 3-cycles in the graph of a soap-film-like surface of revolution.*

Proof. If there were a 3-cycle, then the interior angles would all have to be $\frac{2\pi}{3}$. The graph of the 3-cycle consists of catenaries, which are all concave-up. By the pigeonhole principle, two catenaries must intersect either above or below the relative interior of a third.

First, we consider with the case where the intersection occurs above the third catenary (see Figure 5, left). Say the third catenary begins at an angle θ_0 below the horizontal and ends at an angle θ_1 below the horizontal. Also suppose that the upper-left catenary ends at an angle of φ_0 left of vertical, and that the

upper-right catenary begins at an angle φ_1 right of vertical (see Figure 5 below).

Then $\varphi_0 + \varphi_1 = \frac{2\pi}{3}$, and due to the concavity of the two upper catenaries, $\theta_0 - \frac{\pi}{6} > \varphi_0$ and $\theta_1 - \frac{\pi}{6} > \varphi_1$. These three inequalities imply that $\theta_0 + \theta_1 - \frac{\pi}{3} > \frac{2\pi}{3}$ and hence $\theta_0 + \theta_1 > \pi$. But $\theta_0, \theta_1 < \frac{\pi}{2}$, a contradiction.

In the case of two catenaries intersecting below the third (see Figure 5, right), then due to the concavity of each curve, either the interior angle formed by the lower-left and upper catenaries (θ_0 below) or the angle formed by the lower-right and upper catenaries (θ_1 below) must be acute, and therefore not equal to $\frac{2\pi}{3}$. ■

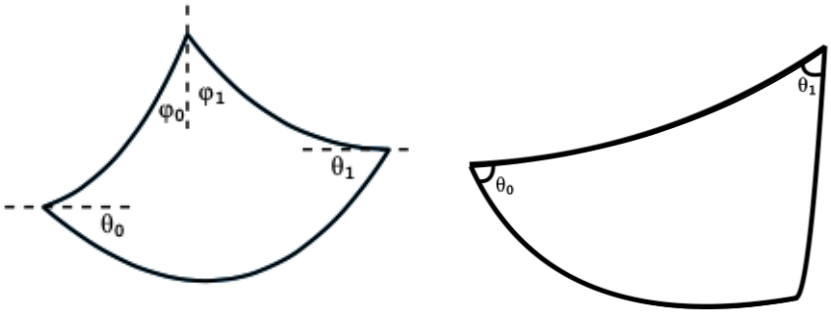

Figure 5: 3-cycles of concave-up curves where the intersection of two curves is above (left) and below (right) the interior of the third curve.

However, there may be 4-cycles, as proven below.

Theorem 3.2. *There exists a soap-film-like surface of revolution whose corresponding graph contains a 4-cycle.*

Proof. We will construct an example of a rotationally symmetric soap-film-like surface with a 4-cycle.

For simplicity of construction, we will impose our 4-cycle to be symmetric about the line $x = 0$ (see Figure 6).

Let $f_1(x) = \cosh(x)$. Let f_1 split into

$$f_2(x) = c_2 \cosh\left(\frac{x - \mathrm{gd}^{-1}(\theta_0)}{c_2} - \mathrm{gd}^{-1}\left(\frac{2\pi}{3} - \theta_0\right)\right)$$

and

$$f_4(x) = c_4 \cosh\left(\frac{x - \mathrm{gd}^{-1}(\theta_0)}{c_4} + \mathrm{gd}^{-1}\left(\theta_0 - \frac{\pi}{3}\right)\right)$$

at $x = \mathrm{gd}^{-1}(\theta_0)$, where

$$c_2 = \frac{\cos\left(\frac{2\pi}{3} - \theta_0\right)}{\cos(\theta_0)}$$

and

$$c_4 = \frac{\cos\left(\theta_0 - \frac{\pi}{3}\right)}{\cos(\theta_0)}.$$

Then f_1, f_2, and f_4 meet at equal angles of $\frac{2\pi}{3}$ as required.

Also let $f_1(x)$ split into $f_6(x) = f_2(-x)$ and $f_7(x) = f_4(-x)$ at $x = -\mathrm{gd}(\theta_0)$, leading to symmetry so far about $x = 0$.

Now, let $f_2(x)$ split into

$$f_3(x) = c_3 \cosh\left(\frac{x - x_0}{c_3} - \mathrm{gd}^{-1}\left(\frac{\pi}{3} - \theta_1\right)\right)$$

and

$$f_5(x) = c_5 \cosh\left(\frac{x - x_0}{c_5} + \mathrm{gd}^{-1}\left(\frac{2\pi}{3} - \theta_1\right)\right)$$

at

$$x_0 = c_2\left(\mathrm{gd}^{-1}\left(\frac{2\pi}{3} - \theta_0\right) - \mathrm{gd}^{-1}\theta_1\right) + \mathrm{gd}^{-1}\theta_0,$$

where

$$c_3 = \frac{c_2 \cos\left(\frac{\pi}{3} - \theta_1\right)}{\cos(\theta_1)}$$

and

$$c_5 = \frac{c_2 \cos\left(\frac{2\pi}{3} - \theta_1\right)}{\cos(\theta_1)}.$$

Note that f_2 makes an angle of θ_1 below the horizontal at $x = x_0$ to the right of the intersection with f_3 and f_5. Also, f_2, f_3, and f_5 meet at equal angles of $\frac{2\pi}{3}$ at $x = x_0$. Finally, note that there are precisely two degrees of freedom in our construction: the choices of θ_0 and θ_1.

Now, we need $f_6(x)$ to split into $f_8(x) = f_5(-x)$ and $f_3(x)$ at $x = -x_0$ at equal angles of $\frac{2\pi}{3}$. It is necessary and sufficient for $f_3(x)$ to be symmetric about the line $x = 0$. This requires that

$$x_0 = c_3 \mathrm{gd}^{-1}\left(\frac{\pi}{3} - \theta_1\right). \tag{14}$$

We further constrain that $0 < x_0 < \mathrm{gd}^{-1}(\theta_0)$, so that f_2 and f_6 face the correct direction do not intersect each other. Solving (14) is equivalent to finding a zero to the function

$$\frac{\mathrm{gd}^{-1}\left(\frac{\pi}{3} - \theta_1\right)\cos\left(\frac{\pi}{3} - \theta_1\right)}{\cos(\theta_1)} + \mathrm{gd}^{-1}(\theta_1) - \mathrm{gd}^{-1}\left(\frac{2\pi}{3} - \theta_0\right) - \frac{\mathrm{gd}^{-1}(\theta_0)\cos(\theta_0)}{\cos\left(\frac{2\pi}{3} - \theta_0\right)}. \tag{15}$$

There exists a suitable solution to θ_1 for $\theta_a < \theta_0 < \theta_b$, where the lower bound is the solution to

$$\mathrm{gd}^{-1}\left(\frac{\pi}{3}\right) = \mathrm{gd}^{-1}(\theta_a)\cos(\theta_a)\sec\left(\frac{2\pi}{3} - \theta_a\right) - \mathrm{gd}^{-1}\left(\frac{2\pi}{3} - \theta_a\right),$$

and the upper bound is the solution to

$$\mathrm{gd}^{-1}\left(\theta_b - \frac{\pi}{3}\right) = \mathrm{gd}^{-1}(\theta_b)\cos(\theta_b)\sec\left(\theta_b - \frac{\pi}{3}\right),$$

for $\pi/6 < \theta_a, \theta_b < \pi/2$. Numerical methods produce $\theta_a \approx 1.3958$ and $\theta_b \approx 1.4332$. This gif shows the surface generated as θ_0 varies between these values. It is easy to verify that $c_4 > c_5$ in these cases.

In particular, numerical computations with accuracy up to 10 bits show that setting $\theta_0 = 1.4$ and $\theta_1 = 1.1$, (15) evaluates to approximately $0.00975343 > 0$. Setting $\theta_0 = 1.4$ and $\theta_1 = 1.2$, (15) evaluates to approximately $-0.046418 < 0$. Because, fixing $\theta_0 = 1.4$, (15) is continuous in θ_1 between $\theta_1 = 1.1$ and $\theta_1 = 1.2$, there must be a zero to (15) with $\theta_0 = 1.4$ and $1.1 < \theta_1 < 1.2$, which also satisfies the constraint $0 < x_0 < \mathrm{gd}^{-1}(\theta_0)$. Indeed, numerical methods with high accuracy produce the approximate solution $\theta_1 = 1.13151$ in this case.

Therefore, there exists a minimal surface which is the surface of rotation of a graph with a 4-cycle. ■

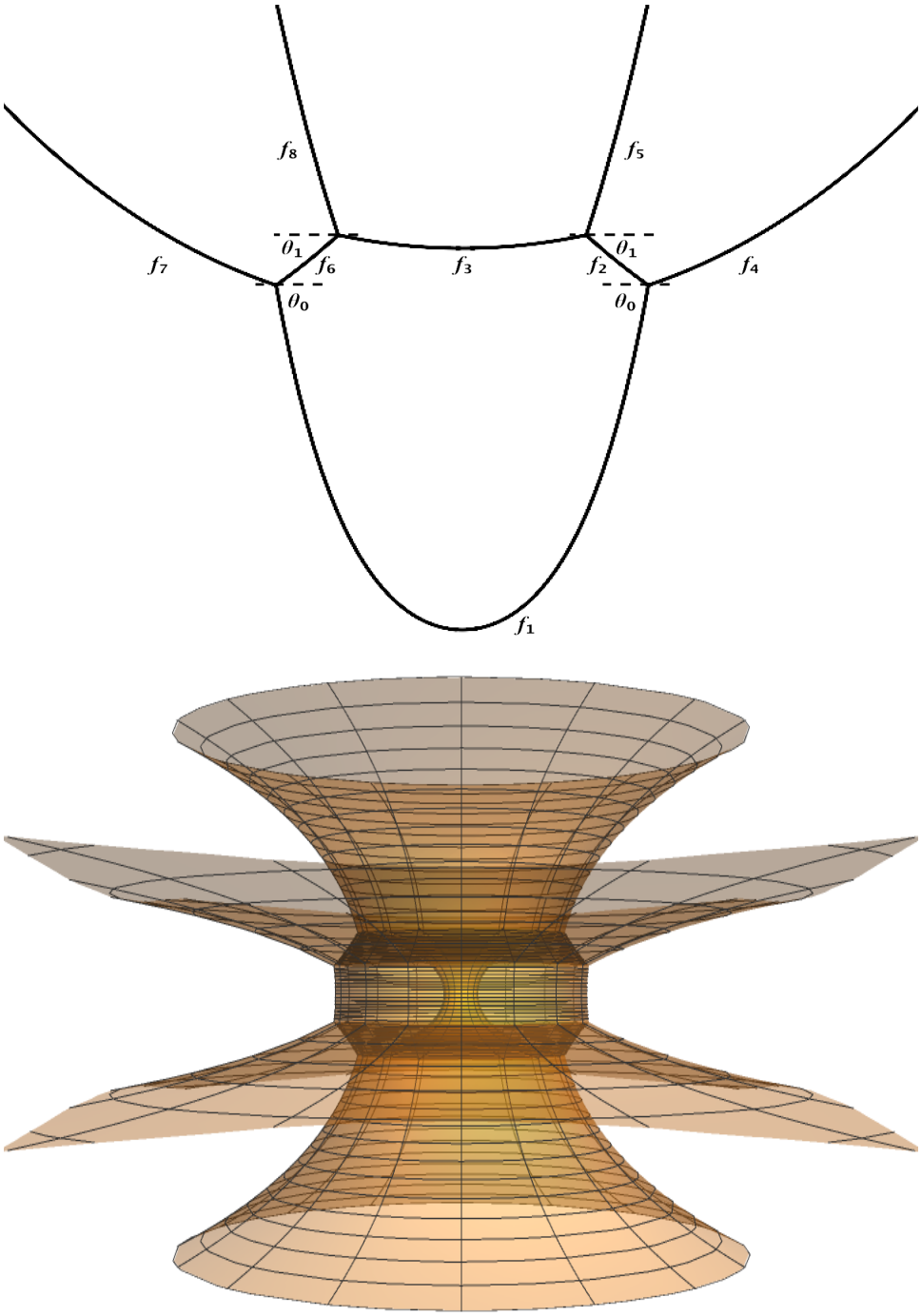

Figure 6: A 4-cycle of catenaries (top), and a minimal surface generated by these catenaries (bottom). Topologically, this is a torus glued to four annuli.

4 Summary and Future Direction

Until now, there has been little investigation into soap-film-like surfaces of revolution with singularities. The results of this paper reveal a rich structure in these surfaces. We constructed a periodic surface and a one-parameter family of surfaces converging to this periodic surface. We also constructed a surface with a handle. There is considerable interest in complete minimal surfaces with finite, non-zero topological genus, with examples produced such as Costa's surface, the Costa-Hoffman-Meeks surfaces, and "helicoidal" minimal surfaces of prescribed genus [MP12, HTW16]. While one must be cautious assigning a topological genus to a stratified surface like that constructed in Theorem 3.2, it seems that this is the first rotationally-symmetric example of a minimal surface with a handle to be produced. Just as the investigation of minimal surfaces began with rotationally symmetric examples, we hope that this paper will spur an deeper investigation into soap-film-like surfaces with singularities.

Now, we present a number open questions raised by this investigation.

Conjecture 4.1. *The graph corresponding to a soap-film-like surface of revolution has depth 1 if it is acyclic.*

Figure 7 below sketches a canonical vertex of depth 2 or greater. In the figure, we have $\alpha_1 = \frac{\pi}{6} - x < \frac{\pi}{6}$ for some $x > 0$, and $\alpha_2 = \theta_1 + \frac{\pi}{3} > \frac{\pi}{6}$ for some $\theta_1 > -\frac{\pi}{6}$. Thus, $\alpha_1 < \alpha_2$, and so a and b will intersect unless one of them splits again. It remains to be shown whether further splitting from catenary a can prevent the formation of a cycle.

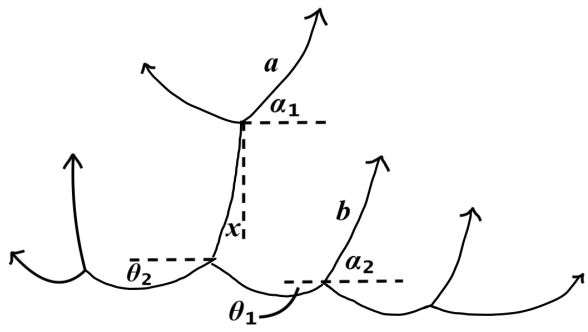

Figure 7: There can be no acyclic graphs of depth 2 or greater.

Next, we move on to conjectures regarding soap films with cyclic graphs.

Conjecture 4.2. *There exists a soap-film-like surface of revolution whose generatrix contains an n-cycle for some $n \neq 4$.*

Conjecture 4.3. *There exists a soap-film-like surface of revolution whose generatrix contains a cycle that does not exhibit mirror symmetry.*

Constructing an asymmetric cycle would likely require a deeper and more systematic understanding of the way cycles behave in these surfaces. Also, as 3-cycles were proven not to exist but a 4-cycle was constructed, there is still a possibility that odd-cycles do not exist. The following is a major question.

Conjecture 4.4. *There exists a cyclic, periodic, soap-film-like surface of revolution.*

It seems clear that this surface constructed in Section 2.1 is the only *acyclic* soap-film-like surface of revolution. However, it is unclear to us how to construct a periodic surface with cycles. An example might take advantage of symmetry, alternating n-cycles and m-cycles for odd n and even m. If Conjecture 4.4 turned out to be true, then the following statement would be somewhat analogous to Corollary 2.10.1.

Conjecture 4.5. *For any $n \in \mathbb{N}$, there exists a minimal surface of revolution whose generatrix has precisely n cycles.*

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