NOISE-INDUCED STABILIZATION OF PLANAR FLOWS II

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Abstract. We continue the work started in Part I [6], showing how the addition of noise can stabilize an otherwise unstable system. The analysis makes use of nearly optimal Lyapunov functions. In this continuation, we remove the main limiting assumption of Part I by an inductive procedure as well as establish a lower bound which shows that our construction is radially sharp. We also prove a version of Peskir's [7] generalized Tanaka formula adapted to patching together Lyapunov functions. This greatly simplifies the analysis used in previous works.

1. Introduction

In Part I of this work [6], we investigated the following complex-valued dynamics

$$\begin{cases} dz_t = (az_t^{n+1} + a_n z_t^n + \cdots + a_0)\, dt + \sigma\, dB_t \\ z_0 \in \mathbf{C} \end{cases} \tag{1.1}$$

where $n \geq 1$ is an integer, $a \in \mathbf{C} \setminus \{0\}$, $a_i \in \mathbf{C}$, $\sigma \geq 0$, and $B_t = B_t^{(1)} + iB_t^{(2)}$ is a complex Brownian motion defined on a probability space $(\Omega, \mathcal{F}, \mathbf{P})$. There, we studied how the presence of noise ($\sigma > 0$ in (1.1)) could stabilize the unstable underlying deterministic system ($\sigma = 0$ in (1.1)). To prove stability in the stochastic perturbation, we developed a framework for building Lyapunov functions and applied it to (1.1) assuming that the drift in equation (1.1) did not contain any "significant" lower-order terms; that is, we assumed that $a_j = 0$ for $\lfloor \frac{n}{2} \rfloor \leq j \leq n$. This was done in order to focus on the overarching elements of the construction of Lyapunov functions and to avoid any additional complexities caused by the presence of such lower-order terms. In this paper, we give an inductive asymptotic argument which shows how to remove this assumption, thereby proving the full version of Theorem 3.2 of Part I [6]. Here, we also provide a radially sharp lower bound on the decay rate of the invariant measure's density as stated in Theorem 5.5 of Part I [6]. This work extends and strengthens a stream of results on similar problems [1, 2, 4, 5, 8].

As first glance, it is surprising that the general case is substantially more complicated than those cases covered in Part I [6], as intuition suggests that the behavior of the process z_t at infinity is determined by the leading-order term z^{n+1} and the noise. We will see here, however, that there is a range in which each of the intermediate lower-order terms becomes dominant in the angular direction at infinity as one moves towards to regions where noise dominates. The scaling analysis of Section 7.1 of Part I [6] hinted at this possibility when we employed our simplifying assumption, for it implied that the dominant balance of terms transferred directly from the leading order term z^{n+1} to the angular diffusion term without any interference from the remaining lower-order terms. In this paper, we will perform the analogous analysis for the general case in Section 3, showing how to correctly study the process at infinity in the presence of the intermediate lower-order terms. We will see, in particular, that the analysis used in Section 7.1 of Part I [6] breaks down in "small" regions containing the explosive trajectories of the deterministic system ($\sigma = 0$ in equation (1.1)) and that the additional terms produce intermediate boundary layers which surround the inner most layer where noise dominates.

We begin in Section 2 by recalling the general setup of Part I [6]. There, we also state the main results we will prove in this paper. In Section 3, we perform the asymptotic analysis which guides and motivates the rest of the work. Specifically, we will use the asymptotically dominant operators yielded from it to define our Lyapunov functions in Section 4 by using a succession of associated PDEs based on these operators. In Section 5, we analyze boundary flux terms in order to show that the local Lyapunov functions can be patched together to produce a global Lyapunov function. Using these calculations, we verify the needed global Lyapunov structure in Section 6. In Section 7, we show that the family of Lyapunov functions we have constructed are radially optimal by establishing a matching lower bound at infinity of the invariant probability density function. In Section 8, we prove a version of Peskir's generalized Tanaka formula [7] which allows to avoid C^2-smoothing along the boundaries of the local Lyapunov functions. Being able to avoid such smoothing greatly simplifies former similar analyses [1, 4, 5]. In Section 9, we make some concluding remarks and suggestions for possible directions of future research.

2. Preliminaries

In this section, we will both recall the general setup of Part I [6] and state the main results to be proved in this paper. Throughout this remainder of this work, we will study more generally the complex-valued SDE

$$dz_t = [az_t^{n+1} + F(z_t, \bar{z}_t)]\, dt + \sigma\, dB_t \tag{2.1}$$

where $a \in \mathbf{C} \setminus \{0\}$, $n \geq 1$, $\sigma > 0$, $B_t = B_t^1 + iB_t^2$ is a complex Brownian motion and $F(z, \bar{z})$ is a complex polynomial in the variables $(z, \bar{z})$ with $F(z, \bar{z}) = \mathcal{O}(|z|^n)$ as $|z| \to \infty$. This is a slight generalization of the system (1.1) in that $F(z, \bar{z})$ need not be a complex polynomial in the variable z only.

The main goal of this work is to prove the following result.

Theorem 2.2. *The Markov process defined by (2.1) is non-explosive and possesses a unique stationary measure μ. In addition, μ satisfies:*

$$\int_{\mathbf{C}} (1 + |z|)^\gamma \, d\mu(z) < \infty \quad \textit{if and only if} \quad \gamma < 2n.$$

Furthermore, μ is ergodic and has a probability density function ρ with respect to Lebesgue measure on $\mathbf{R}^2$ which is smooth and everywhere positive.

In addition to proving Theorem 2.2, we will also characterize the convergence of the process z_t defined by (2.1) to the unique stationary measure μ. To state this result, for any measurable function $w\colon \mathbf{C} \to [1, \infty)$, let $\mathcal{M}_w(\mathbf{C})$ denote the set of probability measures ν on $\mathbf{C}$ satisfying $w \in L^1(\nu)$ and define the weighted total variation metric d_w on $\mathcal{M}_w(\mathbf{C})$ by

$$d_w(\nu_1, \nu_2) = \sup_{\substack{\phi : \mathbf{C} \to \mathbf{R} \\ |\phi(z)| \leq w(z)}} \left[\int \phi(z)\, \nu_1(dz) - \int \phi(z)\, \nu_2(dz) \right].$$

Theorem 2.3. *Let P_t denote the Markov semi-group corresponding to (2.1) and let $\alpha \in (0, n)$ be arbitrary. Then there exists a function $\Psi\colon \mathbf{C} \to [0, \infty)$ and positive constants c, d, K such that*

$$c|z|^\alpha \leq \Psi(z) \leq d|z|^{\alpha + \frac{n}{2} + 1}$$

for all $|z| \geq K$ and such that if $w(z) = 1 + \beta\Psi(z)$ for some $\beta > 0$, then $\nu P_t \in \mathcal{M}_w(\mathbf{C})$ for all $t > 0$ and any probability measure ν on $\mathbf{C}$. Moreover, with the same choice of w, there exist positive constants C, γ such that for any two probability measures ν_1, ν_2 on $\mathbf{C}$ and any $t \geq 1$

$$d_w(\nu_1 P_t, \nu_2 P_t) \leq Ce^{-\gamma t} \|\nu_1 - \nu_2\|_{TV}.$$

Most of the results stated above will be established by constructing certain types of Lyapunov functions. In Part I [6] of this work, however, we used a slightly more general formulation of a Lyapunov function than usually employed in existing literature. Therefore, we now recall what we mean by *Lyapunov pairs* as introduced in Section 4 of Part I [6].

Definition 2.4. Let ξ_t denote a time-homogeneous Itô diffusion on $\mathbf{R}^k$ with C^∞ coefficients and define stopping times $\tau_n = \inf\{t > 0 : |\xi_t| \geq n\}$ for $n \in \mathbf{N}$. Let $\Psi, \Phi : \mathbf{R}^k \to [0, \infty)$ be continuous. Then we call (Ψ, Φ) a LYAPUNOV PAIR CORRESPONDING TO ξ_t if:

a) $\Psi(\xi) \wedge \Phi(\xi) \to \infty$ as $|\xi| \to \infty$;
b) There exists a locally bounded and measurable function $g : \mathbf{R}^k \to \mathbf{R}$ such that the following equality holds for all $\xi_0 \in \mathbf{R}^k$, $n \in \mathbf{N}$ and all bounded stopping times υ:
$$\mathbf{E}_{\xi_0}\Psi(\xi_{\upsilon\wedge\tau_n}) = \Psi(\xi_0) + \mathbf{E}_{\xi_0}\int_0^{\upsilon\wedge\tau_n} g(\xi_s)\, ds + \mathrm{Flux}(\xi_0, \upsilon, n)$$
where $\mathrm{Flux}(\xi_0, \upsilon, n) \in (-\infty, 0]$ and $\mathrm{Flux}(\xi_0, t, l) \leq \mathrm{Flux}(\xi_0, s, n)$ for all $0 \leq s \leq t$, $n \leq l$, $\xi_0 \in \mathbf{R}^k$.
c) There exist constants $m, b > 0$ such that for all $\xi \in \mathbf{R}^k$
$$g(\xi) \leq -m\Phi(\xi) + b.$$

The function Ψ in a Lyapunov pair (Ψ, Φ) is called a LYAPUNOV FUNCTION.

For an explanation of the differences between the usual notion of a Lyapunov function and the notion used here, consult Remark 4.2 of Part I [6].

Most of the paper will be spent proving the following result giving the existence of certain types of Lyapunov pairs corresponding to the dynamics (2.1).

Theorem 2.5. *For each $\gamma \in (n, 2n)$ and $\delta = \delta_\gamma > 0$ sufficiently small, there exist Lyapunov pairs $(\Psi, \Psi^{1+\delta})$ and $(\Psi, |z|^\gamma)$ corresponding to the dynamics (2.1) such that the bound*
$$c|z|^{\gamma-n} \leq \Psi(z) \leq d|z|^{\gamma-n+\frac{n}{2}+1}$$
is satisfied for all $|z| \geq K$ for some positive constants c, d, K.

By the results of Section 4 of Part I [6], Theorem 2.5 implies almost all of the main results. In particular, all consequences of Theorem 2.2 and Theorem 2.3 follow except

$$\int_{\mathbf{C}} (1 + |z|)^\gamma\, d\mu(z) = \infty \quad \text{if} \quad \gamma \geq 2n. \tag{2.6}$$

To prove (2.6), we will show the following stronger result.

Theorem 2.7. *Let $\rho(x, y)$ denote the invariant probability density function of (2.1) with respect to Lebesgue measure on $\mathbf{R}^2$. Then there exist positive constants c, K such that*

$$|(x, y)|^{2n+2}\rho(x, y) \geq c \quad \text{for} \quad |(x, y)| \geq K \tag{2.8}$$

where $|(x, y)| = \sqrt{x^2 + y^2}$ denotes the standard Euclidean distance on $\mathbf{R}^2$.

Throughout, we will assume that the reader is familiar with Section 6 of Part I [6] which gives the general outline of the construction procedure used to produce Lyapunov pairs. These Lyapunov pairs will be constructed using this procedure in Sections 3-6, thus proving Theorem 2.5. In Section 7, we change our focus from constructing Lyapunov pairs to proving Theorem 2.7. Section 8 contains the proof of a version of Peskir's result [7].

Remark 2.9. Throughout the proofs of the main results, we will assume without loss of generality that $a = 1$ in equation (2.1). Indeed one can get from either system to the other by multiplying the solution by a non-zero complex constant and using the fact that $e^{i\theta}B_t$, $\theta \in \mathbf{R}$, is also a complex Brownian motion.

3. The Asymptotic operators and their associated regions

As in Part I [6], we will identify the asymptotically dominant terms in equation (2.1) at infinity by analyzing the time-changed Markov generator L of the process z_t as $r = |z| \to \infty$. Because doing this is substantially more involved than in Part I [6], we have provided a summary of the analysis that follows in Section 3.2 and added Figure 1 to help illustrate the regions and the corresponding deterministic flow.

By Remark 2.9, we may assume without loss of generality that $a = 1$ in equation (2.1) throughout the analysis. Hence, after making the time change $t \mapsto \tau = \int_0^t |z_s|^n ds$, the time-changed generator L has the following form when written in polar coordinates (r, θ):

$$L = r\cos(n\theta)\partial_r + \sin(n\theta)\partial_\theta + P(r,\theta)\partial_r + Q(r,\theta)\partial_\theta + \frac{\sigma^2}{2r^n}\partial_r^2 + \frac{\sigma^2}{2r^{n+2}}\partial_\theta^2 \tag{3.1}$$

where

$$P(r,\theta) = \sum_{k=0}^{n+2} r^{k-n-2} f_k(\theta) \quad \text{and} \quad Q(r,\theta) = \sum_{k=0}^{n+1} r^{k-n-2} g_k(\theta) \tag{3.2}$$

for some collection of smooth real-valued functions f_k, g_k which are 2π-periodic. As we recall, the inclusion of the $k = 0$ terms is not needed to encapsulate all terms in the generator of the process (2.1). However, their presence is required to deal with a secondary calculation needed in the proof of Theorem 2.7.

As in Section 6.2 and Section 6.3 of [6], we will build our Lyapunov function for all (r, θ) by restricting analysis of L to the principal wedge

$$\mathcal{R} = \{(r,\theta) \,:\, r \geq r^*,\, -\tfrac{\pi}{n} \leq \theta \leq \tfrac{\pi}{n}\}.$$

In [6], we recall that to do this construction, we divided $\mathcal{R}$ into four regions: a "priming" region $\mathcal{S}_0$ to initialize the construction, a transport region $\mathcal{S}_1$, a transition region $\mathcal{S}_2$ to blend between the transport region $\mathcal{S}_1$ and a region $\mathcal{S}_3$ where the noise still plays a role at infinity. Moreover, this division of the principal wedge $\mathcal{R}$ was implied by the asymptotic analysis of L carried out in Section 7.1 of [6]. Here, too, we will see that a division of $\mathcal{R}$ holds in the general case, but this time there are many more regions. Initially, the analysis in the general case will exactly coincide with the analysis done previously. Specifically, the first two regions, $\mathcal{S}_0$ and $\mathcal{S}_1$, will be of the same form as before. Afterwards, however, the dynamics at infinity undergoes further incremental changes, and this results in a significant increase in the number of regions and, consequently, the number of asymptotic operators.

As in Section 7.1 of [6], we introduce the family of scaling transformations

$$S_\alpha^\lambda : (r,\theta) \mapsto (\lambda r, \lambda^\alpha \theta)$$

where $\lambda > 0$ and $\alpha \geq 0$. This is done to facilitate the identification of the dominant balances in L as $r \to \infty$ with $(r,\theta) \in \mathcal{R}$. Operationally, we study the scaling properties of

$$\begin{aligned} L \circ S_\alpha^\lambda(r,\theta) &= r\cos(n\theta\lambda^{-\alpha})\partial_r + \lambda^\alpha \sin(n\theta\lambda^{-\alpha})\partial_\theta + \lambda^{-2-n}\frac{\sigma^2}{2r^n}\partial_r^2 \\ &\quad + \lambda^{2\alpha-n-2}\frac{\sigma^2}{2r^{n+2}}\partial_\theta^2 + \lambda^{-1}P(\lambda r, \lambda^{-\alpha}\theta)\partial_r + \lambda^\alpha Q(\lambda r, \lambda^{-\alpha}\theta)\partial_\theta. \end{aligned}$$

as $\lambda \to \infty$ for different choices of $\alpha \geq 0$.

As done in Section 7.1 of Part I [6], we begin by studying L as $r \to \infty$, $(r,\theta) \in \mathcal{R}$, in regions where $|\theta|$ is bounded away from zero; that is, we first consider $L \circ S_0^\lambda$ as $\lambda \to \infty$. Observing that

$$L \circ S_0^\lambda = r\cos(n\theta)\partial_r + \sin(n\theta)\partial_\theta + O(\lambda^{-1}) \text{ as } \lambda \to \infty,$$

we still expect

$$T_1 = r\cos(n\theta)\partial_r + \sin(n\theta)\partial_\theta \tag{3.3}$$

to satisfy $L \approx T_1$ for $r \gg 0$ with (r,θ) restricted to a region $\mathcal{S}_1$ of the form

$$\mathcal{S}_1 = \{(r,\theta) \in \mathcal{R} \,:\, 0 < \theta_1^* \le |\theta| \le \theta_0^* \le \tfrac{\pi}{n}\} \tag{3.4}$$

where θ_0^*, θ_1^* are any positive constants.

To see what happens in the remainder of $\mathcal{R}$, we now turn to analyzing $L \circ S_\alpha^\lambda$ as $\lambda \to \infty$ for $\alpha > 0$ fixed. By fixing the constant $\theta_1^* > 0$ from the definition of $\mathcal{S}_1$ above to be sufficiently small, it is reasonable to assume that the dominant behavior of $L \circ S_\alpha^\lambda$ in λ can be discovered in $\mathcal{R} \setminus \mathcal{S}_1$ by considering the power series expansion of the coefficients of L. Fixing a $J > \frac{n}{2} + 6$, we write

$$L = r\partial_r + n\theta\partial_\theta + \mathsf{P}(r,\theta)\partial_r + \mathsf{Q}(r,\theta)\partial_\theta + \frac{\sigma^2}{2r^n}\partial_r^2 + \frac{\sigma^2}{2r^{n+2}}\partial_\theta^2$$

with

$$\begin{aligned} \mathsf{P}(r,\theta) &= \sum_{i=1}^{J-1} \alpha_i r\theta^i + \sum_{i=0}^{n+1}\sum_{j=0}^{J-1} \beta_{ij} r^{-i}\theta^j + R_{\mathsf{P}}(r,\theta) \\ \mathsf{Q}(r,\theta) &= \sum_{i=1}^{n+1} \gamma_i r^{-i} + \sum_{i=2}^{J-1} \delta_i\theta^i + \sum_{i=1}^{n+2}\sum_{j=1}^{J-1} \epsilon_{ij} r^{-i}\theta^j + R_{\mathsf{Q}}(r,\theta) \end{aligned} \tag{3.5}$$

where $\alpha_i, \beta_{ij}, \gamma_i, \delta_i, \epsilon_{ij}$ are constants and the remainder functions R_{P} and R_{Q} satisfy

$$|R_{\mathsf{P}}(r,\theta)| \le C_{\mathsf{P}}(r+1)|\theta|^J, \quad |R_{\mathsf{Q}}(r,\theta)| \le C_{\mathsf{Q}}|\theta|^J, \quad J > \frac{n}{2} + 6, \tag{3.6}$$

for some positive constants $C_{\mathsf{P}}, C_{\mathsf{Q}}$. We have switched from P and Q to P and Q because, in P and Q, we include higher order terms from the power series expansion of $r\cos(n\theta)$ and $\sin(n\theta)$, respectively.

We begin by considering the region just next to $\mathcal{S}_1$; that is, we analyze $L \circ S_\alpha^\lambda$ as $\lambda \to \infty$ when $\alpha > 0$ is fixed and small. Looking at $L \circ S_\alpha^\lambda$ for $\lambda > 0$ large, the following four terms are candidates for any dominant balance of L as $r \to \infty$:

$$r\partial_r + n\theta\partial_\theta + \sum_{i=1}^{\lfloor\frac{n}{2}\rfloor+1} \lambda^{\alpha-i}\gamma_i r^{-i}\partial_\theta + \lambda^{2\alpha-(n+2)}\frac{\sigma^2}{2r^{n+2}}\partial_\theta^2 = (I) + (II) + \sum_{i=1}^{\lfloor\frac{n}{2}\rfloor+1} (III_i) + (IV) \tag{3.7}$$

where $\lfloor x \rfloor$ is the greatest integer less than or equal to x. Note that we have neglected the $\delta_i\theta^i\partial_\theta$, $i \ge 2$, terms since for $|\theta|$ small they are dominated by $n\theta\partial_\theta$. Similarly, we have neglected all of the $\epsilon_{ij}r^{-i}\theta^j\partial_\theta$ terms since for θ small the corresponding $(III_i) = \gamma_i r^{-i}\partial_\theta$ term always dominates it. We have also neglected all of the $\alpha_i r\theta^i\partial_r$ and $\beta_{ij}r^{-i}\theta^j\partial_r$ terms since they are always dominated by the $r\partial_r$ term for r large and θ small. The terms (III_i) must be included since there is always a region, dictated by the value of $\alpha > 0$, where θ is small enough so that (II) is dominated by some collection of the (III_i) as $r \to \infty$.

It is also important to realize why we have truncated the sum $\sum (III_i)$ at $i = \lfloor\frac{n}{2}\rfloor + 1$. Comparing the diffusion term (IV) with the terms (III_i), observe that we need only consider indices i of (III_i) satisfying $\alpha - i \ge 2\alpha - (n+2)$. Rearranging this conditions produces the restriction $i \le n + 2 - \alpha$. To obtain the claimed condition $i \le \lfloor\frac{n}{2}\rfloor + 1$, we must first understand the relevant range of α. When $2\alpha - (n+2) = 0$ the term (II) balances the term (IV). Solving this condition to find $\alpha = \frac{n}{2} + 1$ and substituting this value of α into $i \le n + 2 - \alpha$, we obtain the claimed bound $i \le \lfloor\frac{n}{2}\rfloor + 1$.

Assumption 5.7 from [6] translated to this context implies that $\gamma_i = 0$ for $i \in \{1, \ldots \lfloor\frac{n}{2}\rfloor + 1\}$; and hence in the case considered previously, none of the (III_i) terms we have retained in (3.7) are present. To further illustrate the differences encountered here, we now analyze $L\circ S_\alpha^\lambda$ as $\lambda\to\infty$ for $\alpha>0$ fixed in the relevant range $(0,\lfloor\frac{n}{2}\rfloor+1]$.

For $0<\alpha<1$, observe that

$$(I)+(II)\gg\sum_{i=1}^{\lfloor\frac{n}{2}\rfloor+1}(III_i)+(IV)\quad\text{as}\quad\lambda\to\infty.$$

When $\alpha=1$, however, as $\lambda\to\infty$ we see that

$$(I)+(II)+(III)_1\gg\sum_{i=2}^{\lfloor n/2\rfloor+1}(III_i)+(IV)\,. \tag{3.8}$$

Hence we expect

$$T_2=r\partial_r+n\theta\partial_\theta \tag{3.9}$$

to satisfy $L\approx T_2$ as $r\to\infty$ when all paths to infinity are restricted to a region of the form

$$\{(r,\theta)\in\mathcal{R}\,:\,r\geq r^*,\,b(r)\leq|\theta|\leq\theta_1^*\}$$

where $\theta_1^*>0$ is small enough and

$$b(r)=cr^{-1}+o(r^{-1})\quad\text{as}\quad r\to\infty \tag{3.10}$$

for some large constant $c>0$. Note that $c>0$ is chosen to be large to assure that the term $(III)_1$ is not also dominant in the region defined above. We also leave open the choice of a specific curve b because what will happen in the remaining part of $\mathcal{R}$:

$$\{(r,\theta)\in\mathcal{R}\,:\,r\geq r^*,\,|\theta|\leq b(r)\}$$

will suggest its definition.

When $\alpha=1$, $(III)_1$ also becomes dominant suggesting that

$$r\partial_r+n\theta\partial_\theta+\gamma_1r^{-1}\partial_\theta \tag{3.11}$$

should be the asymptotic operator in the next region. However if $\gamma_1\neq0$, then on the curve $n\theta=-\gamma_1r^{-1}$

$$n\theta\partial_\theta+\gamma_1r^{-1}\partial_\theta=0.$$

and all of the ∂_θ terms in (3.11) vanish. Hence we must turn to the terms neglected above and do a further analysis of $L\circ S_\alpha^\lambda$ as $\lambda\to\infty$ to find the dominant ∂_θ term. In fact, it is likely that the dominant balance expressed in (3.11) will fail to hold before the terms above exactly cancel.

To help see which terms need to be included in a neighborhood of the curve defined by $n\theta=-\gamma_1r^{-1}$, we make a convenient change of coordinates. The basic idea is to remove the term $(III)_1$ by means of a coordinate transformation, returning us to a setting like that considered above when $\alpha\in(0,1)$. Introducing the mapping $(r,\theta)\mapsto(r,\phi_3)$ where ϕ_3 is defined by

$$\phi_3=r\theta+\frac{\gamma_1}{n+1}, \tag{3.12}$$

we see that the operator L transforms to

$$L_{(r,\phi_3)}=r\,\partial_r+(n+1)\phi_3\,\partial_{\phi_3}+\mathsf{P}_3\,\partial_r+\mathsf{Q}_3\,\partial_{\phi_3}+\frac{\sigma^2}{2r^n}\partial_{\phi_3}^2+\Big(\frac{\sigma^2}{2r^n}\partial_r^2\Big)_{(r,\phi_3)} \tag{3.13}$$

where

$$\mathsf{P}_3(r,\phi_3) = \sum_{i=0}^{n+J}\sum_{j=0}^{J-1}\alpha_{ij}^{(3)} r^{-i}\phi_3^j + R_{P_3} \tag{3.14}$$
$$\mathsf{Q}_3(r,\phi_3) = \sum_{i=1}^{n+J+1}\gamma_i^{(3)} r^{-i} + \sum_{i=1}^{n+J+1}\sum_{j=1}^{J}\beta_{ij}^{(3)} r^{-i}\phi_3^j + R_{Q_3},$$

$\alpha_{ij}^{(3)}, \gamma_i^{(3)}, \beta_{ij}^{(3)}$ are constants and, because $|\theta| \le \theta_1^* \le C$, the remainders R_{P_3}, R_{Q_3} satisfy

$$|R_{P_3}(r,\phi_3)| \le C_{P_3}(r+1)[|r^{-1}\phi_3|^J + r^{-J}]$$
$$|R_{Q_3}(r,\phi_3)| \le C_{Q_3}(r+1)[|r^{-1}\phi_3|^J + r^{-J}].$$

We have chosen not to write out the term $\frac{\sigma^2}{2r}\partial_r^2$ in (3.13) in the variables (r,ϕ_3) because it is too long of an expression and since it is always dominated, by considering the appropriate scaling transformation, by the other terms in $L_{(r,\phi_3)}$ as $r\to\infty$.

After this change of variables, note that $(II)+(III)_1$ has transformed into $(n+1)\phi_3\partial_{\phi_3}$, hence we have "removed" $(III)_1$. However, note that a new $\gamma_1^{(3)}r^{-1}$ term is generated, playing the same role as $(III)_1$ did in the previous coordinate system. While this may not seem like progress, notice that angular diffusion term (analogous to (IV)) now has a coefficient r^{-n} where it was r^{-n-2} in the old coordinates. Hence this term is more powerful. By a similar line of reasoning to the above, only the terms analogous to (III_i) with $i \in \{1,\dots,\lfloor \frac{n}{2}\rfloor\}$ are not dominated by the noise. This is one less than previously. Therefore by performing such substitutions iteratively, we will be able to remove enough terms so that in the final coordinate system, the angular diffusion term will dominate all analogous terms to the (III_i)'s . An important point which makes this iteration possible is that P_3 and Q_3 have that same forms as P and Q, respectively, in that the lower limits of the sums do not change. Even though the upper limits of the sums will increase, these added contributions are of lower order so they do not change the analysis.

To finish the analysis in the variables (r,ϕ_3), we need to complete our understanding of the boundary $|\theta| = b(r)$, extract the dominant operator which replaces T_2 after we cross this boundary, and determine the lower limit of the region where this new operator remains dominant.

To do this, we again consider $L_{(r,\phi_3)}$ under the scaling transformation $S_\alpha^\lambda(r,\phi_3) := (\lambda r, \lambda^{-\alpha}\phi_3)$. First, we note that when $\alpha = 0$

$$L_{(r,\phi_3)} \circ S_0^\lambda = r\partial_r + (n+1)\phi_3\partial_{\phi_3} + o(1) \quad \text{as} \quad \lambda\to\infty$$

implying that

$$T_3 = r\partial_r + (n+1)\phi_3\partial_{\phi_3} \tag{3.15}$$

satisfies $L_{(r,\phi_3)} \approx T_3$ as $r\to\infty$ when paths to infinity are restricted to a region where $|\phi_3|$ is bounded and bounded away from zero. Thus we choose the second region to be

$$\mathcal{S}_2 = \{(r,\theta)\in\mathcal{R} \,:\, r\ge r^*,\ |\phi_3|\ge\phi^*,\ |\theta|\le\theta_1^*\}. \tag{3.16}$$

Notice that this choice of boundary $|\phi_3| = \phi^*$ is consistent the previous requirement on the boundary function $b(r)$ in the (r,θ) variables given in (3.10) provided $\phi^* > \gamma_1/(n+1)$. In a subset of the region $|\phi_3| < \phi^*$, the approximation $L_{(r,\phi_3)} \approx T_3$ holds as $r\to\infty$. To discover the boundary of this region, we now study $L_{(r,\phi_3)} \circ S_\alpha^\lambda$ for $\alpha > 0$.

As before in the previous coordinate system, there are four terms which are potentially involved in any dominant balance of the terms in $L_{(r,\phi_3)} \circ S_\alpha^\lambda$ as $\lambda\to\infty$:

$$r\partial_r + (n+1)\phi_3\partial_{\phi_3} + \lambda^{\alpha-1}\gamma_1^{(3)} r^{-1}\partial_{\phi_3} + \lambda^{2\alpha-n}\frac{\sigma^2}{2r^n}\partial_{\phi_3}^2 = (I)_3 + (II)_3 + (III)_3 + (IV)_3.$$

Notice that the terms $(I)_3, (II)_3, (III)_3, (IV)_3$ are completely analogous the terms $(I), (II), (III)_1, (IV)$ from the preceding discussion.

As already noted, after the change to the (r, ϕ_3) coordinates, only terms $\gamma_i^{(3)} r^{-i}$ with $i \in \{1, \ldots, \lfloor \frac{n}{2} \rfloor\}$ could possibly dominate or balance the angular diffusion term $(IV)_3$, and the term $(III)_3$ is the leading order term of this form. Hence when $n = 1$ or $n = 2$, there is only one such dominant term of this form, namely $(III)_3$, and it is of either the same (case $n = 2$) or lesser (case $n = 1$) order as $(IV)_3$. In general ($n \geq 3$), we will have to perform additional transformations to remove all of the possibly dominant terms. Before considering general $n \geq 3$, we pause to finish the analysis in the cases when $n = 1$ and $n = 2$.

3.0.1. *Remaining operators and regions when* $n = 1$. For every $\alpha \geq 0$, we see that

$$(I)_3 + (II)_3 + (IV)_3 \gg (III)_3 \text{ as } \lambda \to \infty.$$

If $0 \leq \alpha < \frac{1}{2}$, then

$$(I)_3 + (II)_3 \gg (IV)_3 \text{ as } \lambda \to \infty.$$

When $\alpha = 1/2$, the term $(IV)_3$ also becomes dominant in λ. In particular, the region where we expect $L_{(r,\phi_3)} \approx T_3$ as $r \to \infty$ is precisely

$$\mathcal{S}_3 = \{(r, \theta) \in \mathcal{R} : r \geq r^*, \eta^* r^{-\frac{1}{2}} \leq |\phi_3| \leq \phi^*, |\theta| \leq \theta_1^*\}, \tag{3.17}$$

for some $\eta^* > 0$. Additionally, the operator

$$A = r\partial_r + 2\phi_3 \partial_{\phi_3} + \frac{\sigma^2}{2r} \partial_{\phi_3}^2 \tag{3.18}$$

contains the dominant part of $L_{(r,\phi_3)}$ in the region

$$\mathcal{S}_4 = \{(r, \theta) \in \mathcal{R} : r \geq r^*, |\phi_3| \leq \min(\eta^* r^{-1/2}, \phi^*), |\theta| \leq \theta_1^*\}. \tag{3.19}$$

Summing this up, we have seen that when $n = 1$, the approximating operators are T_1, T_2, T_3, A with corresponding regions $\mathcal{S}_1, \mathcal{S}_2, \mathcal{S}_3, \mathcal{S}_4$ where we expect the approximation to be valid for $r > 0$ large.

Remark 3.20. We have already introduced a number of parameters (e.g. θ_1^*, ϕ^*, η^*, r^*) thus far that will have to be chosen to satisfy a number of properties. Instead of writing these properties explicitly, we simply need to make sure that we vary the parameters in a consistent way to obtain them. That is, we will always choose $\theta_1^* > 0$ small enough, then pick $\phi^* = \phi^*(\theta_1^*) > 0$ large enough, then choose $\eta^* = \eta^*(\theta_1^*, \phi^*) > 0$ large enough, and then finally pick $r^* = r^*(\theta_1^*, \phi^*, \eta^*) > 0$ large enough. For example, to assure that $\mathcal{S}_3$ and $\mathcal{S}_4$ defined above are of the required form outlined in Section 6.2 of [6], we can choose the parameters θ_1^*, ϕ^*, η^*, and r^* in this way to see that in fact

$$\mathcal{S}_3 = \{(r, \theta) \in \mathcal{R} : r \geq r^*, \eta^* r^{-\frac{1}{2}} \leq |\phi_3| \leq \phi^*\},$$
$$\mathcal{S}_4 = \{(r, \theta) \in \mathcal{R} : r \geq r^*, |\phi_3| \leq \eta^* r^{-1/2}\}.$$

3.0.2. *Remaining operators and regions when* $n = 2$. Notice that for $0 \leq \alpha < 1$

$$(I)_3 + (II)_3 \gg (III)_3 + (IV)_3 \quad \text{as} \quad \lambda \to \infty.$$

When $\alpha = 1$, then $(III)_3 + (IV)_3$ also becomes dominant in λ. Therefore, this suggests that the region where $T_3 \approx L_{(r,\phi_3)}$ as $r \to \infty$ is of the form

$$\mathcal{S}_3 = \{(r, \theta) \in \mathcal{R} : r \geq r^*, \eta^* r^{-1} \leq |\phi_3| \leq \phi^*\} \tag{3.21}$$

for some $\eta^* > 0$. Here again, we have picked the parameters in the way discussed in Remark 3.20. Notice also that the operator

$$A = r\partial_r + 3\phi_3 \partial_{\phi_3} + \gamma_1^{(1)} r^{-1} \partial_{\phi_3} + \frac{\sigma^2}{2r^2} \partial_{\phi_3}^2 \tag{3.22}$$

contains the dominant part of $L_{(r,\phi_3)}$ as $r \to \infty$ in the region

$$\mathcal{S}_4 = \{(r,\theta) \in \mathcal{R} \,:\, r \geq r^*,\, |\phi_3| \leq \eta^* r^{-1}\} \tag{3.23}$$

where we have again picked ϕ^* and r^* according to Remark 3.20. Summing this up, we have seen that when $n = 2$, the asymptotic operators are T_1, T_2, T_3, A with corresponding regions $\mathcal{S}_1, \mathcal{S}_2, \mathcal{S}_3, \mathcal{S}_4$ where the approximation is expected to be valid.

3.1. **Remaining analysis when $n = 3, 4$.** If $0 \leq \alpha < 1$, then

$$(I)_3 + (II)_3 \gg (III)_3 + (IV)_3 \quad \text{as} \quad \lambda \to \infty.$$

Therefore, we have that $L_{(r,\phi_3)} \approx T_3$ as $r \to \infty$ in some region of the form

$$\{(r,\theta) \in \mathcal{R} \,:\, r \geq r^*,\, b(r) \leq |\phi_3| \leq \phi^*\} \tag{3.24}$$

where b satisfies

$$b(r) = cr^{-1} + o(r^{-1}) \quad \text{as} \quad r \to \infty$$

for some $c > 0$. If $\alpha \geq 1$ however, it is not clear if the terms in $L_{(r,\phi_3)}$ corresponding to $(I)_3 + (II)_3 + (III)_3 + (IV)_3$ contain the dominant part of the operator because

$$(n+1)\phi_3 \partial_{\phi_3} + \gamma_1^{(3)} r^{-1} \partial_{\phi_3} = 0$$

when $(n+1)\phi_3 = -\gamma_1^{(3)} r^{-1}$. Hence to analyze $L_{(r,\phi_3)}$ around this other potentially dangerous curve, we make another substitution $(r,\phi_3) \mapsto (r,\phi_4)$ where

$$\phi_4 = r\phi_3 + c_3.$$

and $c_3 = \frac{\gamma_1^{(3)}}{n+2}$. As before, we use the new variables (r,ϕ_4) to define the boundary curve b precisely by setting

$$\mathcal{S}_3 = \{(r,\theta) \in \mathcal{R} \,:\, r \geq r^*,\, |\phi_4| \geq \phi^*,\, |\phi_3| \leq \phi^*\}. \tag{3.25}$$

Now write $L_{(r,\phi_3)}$ in the variables (r,ϕ_4) to see that

$$L_{(r,\phi_4)} = r\partial_r + (n+2)\phi_4\partial_{\phi_4} + \mathsf{P}_4\partial_r + \mathsf{Q}_4\partial_{\phi_4} + \frac{\sigma^2}{2r^{n-2}}\partial_{\phi_4}^2 + \left(\frac{\sigma^2}{2r^n}\partial_r^2\right)_{(r,\phi_4)}$$

where

$$\mathsf{P}_4 = \sum_{i\geq 0, j\geq 0} \alpha_{ij}^{(4)} r^{-i}\phi_4^j + R_{P_4}$$
$$\mathsf{Q}_4 = \sum_{i\geq 1} \gamma_i^{(4)} r^{-i} + \sum_{i\geq 1, j\geq 1} \beta_{ij}^{(4)} r^{-i}\phi_4^j + R_{Q_4}$$

where $\alpha_{ij}^{(4)}, \gamma_i^{(4)}, \beta_{ij}^{(4)}$ are constants, all sums above are finite sums and, since ϕ_4 will be bounded by ϕ^* in any subsequent region, R_{P_4} and R_{Q_4} satisfy

$$|R_{P_4}| \leq C_{P_4}(r+1)[|r^{-2}\phi_4|^J + r^{-J}]$$
$$|R_{Q_4}| \leq C_{Q_4}(r^2+1)[|r^{-2}\phi_4|^J + r^{-J}]$$

for some positive constants C_{P_4}, C_{Q_4}. Here, note that both C_{P_4} and C_{Q_4} can be chosen independent of ϕ^* by picking $r^* > \phi_4^*$. Considering the effect of $L_{(r,\phi_4)}$ under $S_\alpha^\lambda(r,\phi_4) := (\lambda r, \lambda^{-\alpha}\phi_4)$, $\alpha \geq 0$, we again consider the following four terms in $L_{(r,\phi_4)} \circ S_\alpha^\lambda$ which could become dominant in λ as $\lambda \to \infty$:

$$\begin{aligned}&r\partial_r + (n+2)\phi_4\partial_{\phi_4} + \lambda^{\alpha-1}\gamma_1^{(4)} r^{-1}\partial_{\phi_4} + \lambda^{2\alpha-(n-2)}\frac{\sigma^2}{2r^{n-2}}\partial_{\phi_4}^2\\ &= (I)_4 + (II)_4 + (III)_4 + (IV)_4.\end{aligned}$$

Similarly, we can now uncover all asymptotic operators and their associated regions when $n = 3$ and $n = 4$ because the angular noise term $(IV)_4$ is of sufficient strength in λ.

3.1.1. *Remaining operators and regions when* $n = 3$. Analogous to the case when $n = 1$,

$$T_4 = r\partial_r + 5\phi_4\partial_{\phi_4} \tag{3.26}$$

satisfies $L_{(r,\phi_4)} \approx T_4$ as $r \to \infty$ when paths to infinity are restricted to a region of the form

$$\mathcal{S}_4 = \{(r,\theta) \in \mathcal{R} \,:\, r \geq r^*,\ \eta^* r^{-\frac{1}{2}} \leq |\phi_4| \leq \phi^*\} \tag{3.27}$$

where the parameters $\eta^*, \phi^* > 0$ have been chosen according to Remark 3.20. Also,

$$A = r\partial_r + 5\phi_4\partial_{\phi_4} + \frac{\sigma^2}{2r}\partial^2_{\phi_4} \tag{3.28}$$

can be used to approximate $L_{(r,\phi_4)}$ asymptotically for large r in a region of the form

$$\mathcal{S}_5 = \{(r,\theta) \in \mathcal{R} \,:\, r \geq r^*,\ |\phi_4| \leq \eta^* r^{-\frac{1}{2}}\} \tag{3.29}$$

where we have again picked the parameters as described in Remark 3.20. Thus, when $n = 3$, we obtain the approximating operators T_1, T_2, T_3, T_4, A with corresponding regions $\mathcal{S}_1, \mathcal{S}_2, \mathcal{S}_3, \mathcal{S}_4, \mathcal{S}_5$ where the approximation is expected to be valid.

3.1.2. *Remaining operators and regions when* $n = 4$. Similar to the case when $n = 2$, the region where

$$T_4 = r\partial_r + 6\phi_4\partial_{\phi_4}$$

is in good approximation to $L_{(r,\phi_4)}$ for $r > 0$ large is given by

$$\mathcal{S}_4 = \{(r,\theta) \in \mathcal{R} \,:\, r \geq r^*,\, \eta^* r^{-1} \leq |\phi_4| \leq \phi^*\}$$

where the parameters have been chosen appropriately. Also,

$$A = r\partial_r + 6\phi_4\partial_{\phi_4} + \gamma_1^{(4)} r^{-1}\partial_{\phi_4} + \frac{\sigma^2}{2r^2}\partial^2_{\phi_4}$$

contains the dominant, large r behavior corresponding to $L_{(r,\phi_4)}$ in

$$\mathcal{S}_5 = \{(r,\theta) \in \mathcal{R} \,:\, r \geq r^*,\, |\phi_4| \leq \eta^* r^{-1}\}.$$

Thus when $n = 4$, we obtain the asymptotic operators T_1, T_2, T_3, T_4, A and their regions $\mathcal{S}_1, \mathcal{S}_2, \mathcal{S}_3, \mathcal{S}_4, \mathcal{S}_5$ of approximation.

3.2. **All operators and regions for general** $n \geq 1$**:** We continue until this inductive procedure until it stops. More precisely, if $n = 2j + 1$ or $n = 2j + 2$ for some $j \geq 0$, then the analysis yields the asymptotic operators

$$T_1, T_2, \ldots, T_{j+3}, A$$

and respective regions

$$\mathcal{S}_1, \mathcal{S}_2, \ldots, \mathcal{S}_{j+3}, \mathcal{S}_{j+4}.$$

To write each of them explicitly, set $\phi_2 := \theta$ for $m \geq 3$ and let

$$\phi_m = r\phi_{m-1} + c_{m-1}. \tag{3.30}$$

where $c_2 = \frac{\gamma_1}{n+1}$ and $c_m = \frac{\gamma_1^{(m)}}{n+m-1}$ for $m \geq 3$. We see that $T_1, \ldots, T_{j+3}$ are given by

$$T_1 = r\cos(n\theta)\partial_r + \sin(n\theta)\partial_\theta$$
$$T_m = r\partial_r + (n+m-2)\phi_m\partial_{\phi_m},\ m = 2, 3, \ldots, j+3.$$

If $n = 2j + 1$, then the diffusive operator A satisfies

$$A = r\partial_r + (3j+2)\phi_{j+3}\partial_{\phi_{j+3}} + \frac{\sigma^2}{2r}\partial^2_{\phi_{j+3}}.$$

On the other hand if $n = 2j + 2$, we have

$$A = r\partial_r + (3j+3)\phi_{j+3}\partial_{\phi_{j+3}} + \gamma_1^{(j+3)} r^{-1}\partial_{\phi_{j+3}} + \frac{\sigma^2}{2r^2}\partial^2_{\phi_{j+3}}.$$

Choosing the parameters $\theta_1^*, \phi^*, \eta^*, r^*$ according to Remark 3.20, we may write all corresponding regions as follows. Note first that $\mathcal{S}_1, \mathcal{S}_2, \ldots, \mathcal{S}_{j+2}$ are given by

$$\mathcal{S}_1 = \{(r,\theta) \in \mathcal{R} \,:\, r \geq r^*,\, 0 < \theta_1^* \leq |\theta| \leq \theta_0^*\}$$
$$\mathcal{S}_2 = \{(r,\theta) \in \mathcal{R} \,:\, r \geq r^*,\, |\phi_3| \geq \phi^*,\, |\theta| \leq \theta_1^*\}$$
$$\mathcal{S}_m = \big\{(r,\theta) \in \mathcal{R} \,:\, r \geq r^*, |\phi_{m+1}| \geq \phi^*,\, |\phi_m| \leq \phi^*\big\}$$

for $m = 3, \ldots, j+2$. If $n = 2j+1$, the final two regions satisfy

$$\mathcal{S}_{j+3} = \big\{(r,\theta) \in \mathcal{R} \,:\, r \geq r^*,\, \eta^* r^{-\frac{1}{2}} \leq |\phi_{j+3}| \leq \phi^*\big\}$$
$$\mathcal{S}_{j+4} = \big\{(r,\theta) \in \mathcal{R} \,:\, r \geq r^*,\, |\phi_{j+3}| \leq \eta^* r^{-\frac{1}{2}}\big\}$$

On the other hand if $n = 2j+2$, then $\mathcal{S}_{j+3}$ and $\mathcal{S}_{j+4}$ are given by

$$\mathcal{S}_{j+3} = \big\{(r,\theta) \in \mathcal{R} \,:\, r \geq r^*,\, \eta^* r^{-1} \leq |\phi_{j+3}| \leq \phi^*\big\}$$
$$\mathcal{S}_{j+4} = \big\{(r,\theta) \in \mathcal{R} \,:\, r \geq r^*,\, |\phi_{j+3}| \leq \eta^* r^{-1}\big\}.$$

It is also important to notice that L, when written in the variables (r, ϕ_m) for $m = 3, \ldots, j+3$, satisfies

$$L_{(r,\phi_m)} = r\partial_r + (n+m-2)\phi_m\partial_{\phi_m} + \mathsf{P}_m\partial_r + \mathsf{Q}_m\partial_{\phi_m} + \frac{\sigma^2}{2r^{n-2m+6}}\partial^2_{\phi_m} + \left(\frac{\sigma^2}{2r^n}\partial_r^2\right)_{(r,\phi_m)} \tag{3.31}$$

where

$$\mathsf{P}_m = \sum_{i,j\geq 0} \alpha_{ij}^{(m)} r^{-i}\phi_m^j + R_{P_m}$$
$$\mathsf{Q}_m = \sum_{i\geq 1} \gamma_i^{(m)} r^{-i} + \sum_{i\geq 1, j\geq 1} \beta_{ij}^{(m)} r^{-i}\phi_m^j + R_{Q_m}$$

where $\alpha_{ij}^{(m)}, \gamma_i, \beta_{ij}^{(m)}$ are constants, all sums are finite sums, and by the choice of $J > \frac{n}{2} + 6$ the remainders satisfy the following bounds on $\mathcal{S}_m$

$$|R_{P_m}| \leq C_{P_m} r^{-2}, \qquad |R_{Q_m}| \leq C_{Q_m} r^{-2},$$

$m = 3, \ldots, j+4$. Note that the constants C_{P_m} and C_{Q_m}, $m \geq 3$, depend on ϕ^* but they do not depend on r^*.

4. The Construction of Ψ on $\mathcal{R}$ in the general case

Employing the asymptotic analysis of the previous section, we now define our candidate Lyapunov function Ψ on the principal wedge $\mathcal{R}$. Recalling Section 6.3 on Part I of this work [6], we break up the definition of Ψ on $\mathcal{R}$ as follows

$$\Psi(r,\theta) = \Big\{ \psi_i(r,\theta) \quad \text{if } (r,\theta) \in \mathcal{S}_i$$

where $i = 0, 1, \ldots, j+4$, $n = 2j+1$ or $n = 2j+2$.

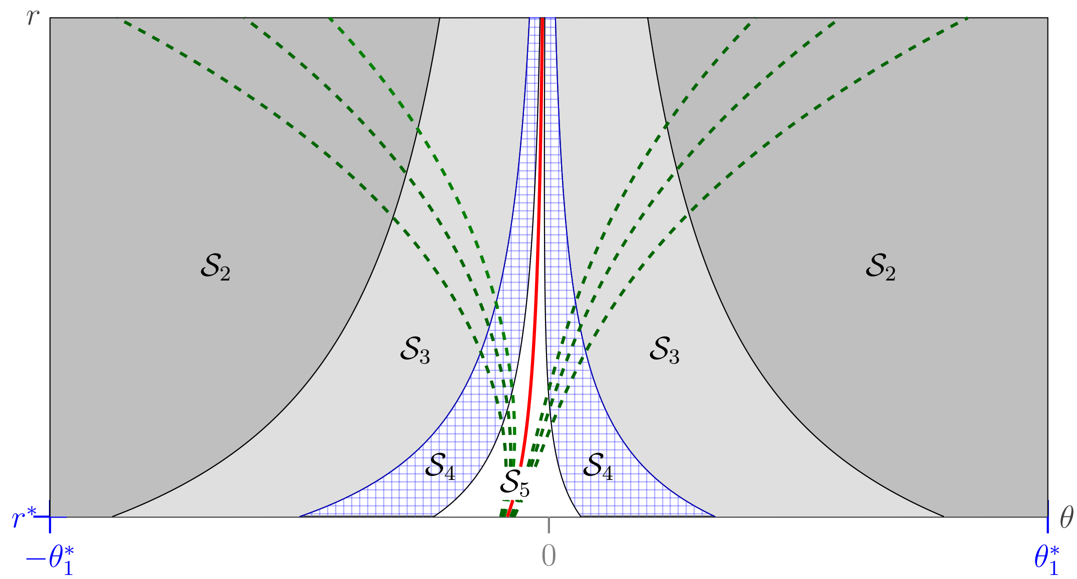

FIGURE 1. Recalling that the constants c_i were introduced below equation (3.30), a sketch of the regions $\mathcal{S}_i$, $i = 2, 3, 4, 5$, is plotted when $n = 3$, $c_2 = 1/2$, $c_3 = 1$, $\phi^* = 10$, and $\eta^* = 5$. A simple example of an operator which has this same region decomposition is $L = r\partial_r + (3\theta + 2r^{-1} + 5r^{-2})\partial_\theta + \frac{\sigma^2}{r^3}\partial_r^2 + \frac{\sigma^2}{r^5}\partial_\theta^2$. In the absence of noise, the dynamics defined by $r^3 L$ explodes in finite time along the solid trajectory splicing the interior of $\mathcal{S}_5$ in the figure above. The formula of this unstable trajectory is given by the equation $\phi_4 = r^2\theta + \frac{r}{2} + 1 = 0$. Moreover, away from this trajectory in the absence of noise, solutions along $r^3 L$ push away from this unstable trajectory, eventually exiting though one of the boundaries $\theta = \pm\theta_1^*$. The dashed curves plotted above are a few representative stable trajectories for the system corresponding to the operator $r^3 L$. The general formula for these stable trajectories is given by $\theta = \phi_4(0)r^3 - \frac{1}{2r} - \frac{1}{r^2}$, $\phi_4(0) \in \mathbf{R}_{\neq 0}$, .

As in Part I [6], to initialize the propagation procedure used to define all of the ψ_i's we need one more region $\mathcal{S}_0$ (hence the $i = 0$ above) defined by

$$\mathcal{S}_0 = \{(r, \theta) \in \mathcal{R} : r \geq r^*, \theta_0^* \leq |\theta| \leq \tfrac{\pi}{n}\}$$

where $\theta_0^* \in (\frac{\pi}{2n}, \frac{\pi}{n})$, and we define the initial function ψ_0 on $\mathcal{S}_0$ by

$$\psi_0(r, \theta) = r^p$$

for some $p \in (0, n)$. We recall that this choice is made because the radial dynamics along T_1 is decreasing in $\mathcal{S}_0$.

4.1. **The construction in the transport regions.** We first turn our attention to defining the functions $\psi_1, \psi_2, \ldots, \psi_{j+3}$ respectively on the regions $\mathcal{S}_1, \mathcal{S}_2, \ldots, \mathcal{S}_{j+3}$ as solutions to boundary-value problems involving the asymptotic operators $T_1, \ldots, T_{j+3}$. Because the number of indices will be daunting otherwise, we adopt the following conventions.

Convention 4.1. When it is clear which coordinate system in which we are working, (r, ϕ_m) will be written more simply as (r, ϕ). For example, $\psi_m(r, \phi_m)$ for $m = 3, \ldots, j + 3$ will often be written as $\psi_m(r, \phi)$.

Convention 4.2. In the expressions we will derive for $\psi_1, \ldots, \psi_{j+3}$, there will be several parameters with double indices, e.g. see $p_{l,m}$ and $q_{l,m}$ below in Lemma 4.11. The second index m simply corresponds to the function ψ_m. Thus when it is clear that we are working with ψ_m, we will often write $p_{l,m}$ and $q_{l,m}$ more compactly as p_l and q_l, respectively.

It is convenient in our analysis that the boundary conditions given for the Poisson equation defining ψ_{j+4} on the inner most region $\mathcal{S}_{j+4}$ (the one dominated by diffusion) are symmetric under reflection in the angular coordinate ϕ in $\mathcal{S}_{j+3}$. Thanks to this symmetry, the value of ψ_{j+4} at the time of exit of the diffusion from $\mathcal{S}_{j+4}$ depends only on the time of exit and not on which side of the boundary it exits (each being possible since the dynamics in $\mathcal{S}_{j+4}$ is diffusive). As we will see below, this allows us to define ψ_{j+4} more simply. Here, we will accomplish this desired symmetry by forcing the penultimate function ψ_{j+3} to satisfy

$$\psi_{j+3}(r,-\phi)=\psi_{j+3}(r,\phi)\,. \tag{4.3}$$

for $(r,\theta(r,\phi))\in\mathcal{S}_{j+3}$.

The cost of producing this symmetry in the penultimate region can be seen in the need to carefully choose the $h_i^{\pm}$ below in all of the preceding regions since both the regions and the corresponding dominant operators are inherently asymmetric in the angular coordinate. Since we will have to make different choices of defining problems above and below $\phi=0$ in each region to produce the symmetry, we will break up the definition of ψ_m in two pieces as follows

$$\psi_m(r,\phi)=\begin{cases}\psi_m^+(r,\phi), & (r,\theta(r,\phi))\in\mathcal{S}_m,\ \phi>0\\ \psi_m^-(r,\phi), & (r,\theta(r,\phi))\in\mathcal{S}_m,\ \phi<0.\end{cases}$$

The construction in $\mathcal{S}_1$. Let $\psi_1^{\pm}$ satisfy the following PDEs on $\mathcal{S}_1$:

$$\begin{cases}\big(T_1\psi_1^{\pm}\big)(r,\theta)=-h_1^{\pm}r^p|\theta|^{-q}\\ \quad\psi_1^{\pm}(r,\pm\theta_0^*)=\psi_0(r,\pm\theta_0^*).\end{cases} \tag{4.4}$$

where $q\in(\frac{p}{n},1)$ is fixed and $h_1^+,h_1^->0$ will be determined later (to produce the reflective symmetry).

Since $\theta_0^*>\frac{\pi}{2}$, we recall from Section 7.3 of Part I [6] that the equations above are not well-defined with the given boundary data because some characteristics along T_1 cross $r=r^*$ before reaching the lines $\theta=\pm\theta_0^*$. This can be easily remedied by enlarging the domain of definition of the equation (4.4) to

$$\widetilde{\mathcal{S}}_1=\Big\{(r,\theta)\in\mathcal{R}\,:\,0<\theta_1^*\le|\theta|\le\theta_0^*,\,r|\sin(n\theta_0^*)|^{\frac{1}{n}}\ge r^*\Big\}.$$

With this modification of the domain, solving (4.4) with the method of characteristics produces

$$\psi_1^{\pm}(r,\theta)=\frac{r^p}{|\sin(n\theta)|^{\frac{p}{n}}}\left(|\sin(n\theta_0^*)|+h_1^{\pm}\int_\theta^{\pm\theta_0^*}\frac{|\sin(n\alpha)|^{\frac{p}{n}}}{|\alpha|^q\sin(n\alpha)}\,d\alpha\right) \tag{4.5}$$

for $(r,\theta)\in\mathcal{S}_1$. In particular, we observe that ψ_1 is homogeneous under S_0^{λ}, $\psi_1(r,\theta)>0$ for all (r,θ) with $r>0$ and $|\theta|\in(0,\frac{\pi}{n})$, and $\psi_1(r,\theta)\to\infty$ as $r\to\infty$, $(r,\theta)\in\mathcal{S}_1$.

The construction in $\mathcal{S}_2$. In a similar fashion, let $\psi_2^{\pm}$ solve

$$\begin{cases}\big(T_2\psi_2^{\pm}\big)(r,\theta)=-h_2^{\pm}r^p|\theta|^{-q}\\ \quad\psi_2^{\pm}(r,\pm\theta_1^*)=\psi_1^{\pm}(r,\pm\theta_1^*)\end{cases} \tag{4.6}$$

on $\mathcal{S}_2$ where h_2^+, $h_2^->0$. Note that we may solve (4.6) explicitly using the method of characteristics to obtain the following expression for $\psi_2^{\pm}$:

$$\psi_2^{\pm}(r,\theta)=d_{12}^{\pm}\frac{r^p}{|\theta|^{\frac{p}{n}}}+d_{22}^{\pm}\frac{r^p}{|\theta|^q} \tag{4.7}$$

where

$$d_{12}^{\pm} = |\theta_1^*|^{\frac{p}{n}} \psi_1^{\pm}(1, \pm\theta_1^*) - h_2^{\pm} \frac{|\theta_1^*|^{\frac{p}{n}-q}}{qn-p} \qquad \text{and} \qquad d_{22}^{\pm} = \frac{h_2^{\pm}}{qn-p}.$$

Observe that $\psi_2^{\pm}$ consists of two terms, each of which is homogeneous under S_α^λ for every $\alpha \geq 0$. Moreover,

$$\psi_2^{\pm}(r,\theta) \geq |\theta_1^*|^{\frac{p}{n}} \psi_1^{\pm}(1,\pm\theta_1^*) \frac{r^p}{|\theta|^{\frac{p}{n}}} \tag{4.8}$$

on $\mathcal{S}_2$. Thus we see that $\psi_2 > 0$ on $\mathcal{S}_2$ and $\psi_2(r,\theta) \to \infty$ as $r \to \infty$, $(r,\theta) \in \mathcal{S}_2$.

The inductive construction in the remaining transport regions. We can now use the same idea employed above for ψ_2 to define $\psi_3, \ldots, \psi_{j+3}$ inductively. Thus for $m = 3, \ldots, j+3$, define ψ_m as the solution of

$$\begin{cases} (T_m \psi_m^{\pm})(r,\phi) = -h_m^{\pm} r^{p_m} |\phi|^{-q_m} \\ \quad \psi_m^{\pm}(r, \pm\phi^*) = \psi_{m-1}(r, \phi_{m-1}(\pm\phi^*)) \end{cases} \tag{4.9}$$

for all $(r, \theta(r,\phi)) \in \mathcal{S}_m$ where $p_m, q_m, h_m^{\pm} > 0$. We again recall that we have suppressed the index m in (r,ϕ) using our convention. We have also suppressed the second index m in p_m and q_m above; that is, $p_{m,m} = p_m$ and $q_{m,m} = q_m$.

Inductively, p_m and q_m are chosen to satisfy

$$\begin{aligned} &p_2 = p, && q_2 = q, \\ &p_m = p_{m-1} + q_{m-1}, && m = 3, \ldots, j+3 \\ &q_m \in \left(q_{m-1} \vee \tfrac{p_m}{n+m-2}, 1\right) && m = 3, \ldots, j+3. \end{aligned} \tag{4.10}$$

While these choices at the outset may seem mysterious, they are all determined by the exit distribution of the diffusion generated by A from $\mathcal{S}_{j+4}$ and by the scaling relationships of the ψ_i's along common boundaries. For further information, see the discussion in Section 7.4 of Part I [6].

We now prove a lemma which gives an expression for ψ_m which is convenient for further analysis. Although we will need them, at first glance it is important to ignore the many relations that the constants in the statement of the result satisfy. The basic form of $\psi_m^{\pm}$ is what is most important.

Lemma 4.11. *For each $m = 3, \ldots, j+3$ we may write*

$$\psi_m^{\pm}(r,\phi) = \sum_{l=1}^{m} d_l^{\pm} \frac{r^{p_l}}{|\phi|^{q_l}} \tag{4.12}$$

where the positive constants $p_l = p_{l,m}$ and $q_l = q_{l,m}$ satisfy

$$\begin{aligned} &p_{1,2} = p_{2,2} = p && q_{1,2} = \frac{p}{n},\ q_{2,2} = q, \\ &p_{l,m} = p_{l,m-1} + q_{l,m-1} && l < m,\ l, m = 3, \ldots, j+3 \\ &p_{m,m} = p_{m-1,m} = p_m && m = 3, \ldots, j+3 \\ &q_{l,m} = \frac{p_{l,m}}{n+m-2}, && l < m,\ l, m = 3, \ldots, j+3 \\ &q_{m,m} = q_m, && m = 3, \ldots, j+3. \end{aligned} \tag{4.13}$$

Moreover, the constants $d_l^{\pm} = d_{l,m}^{\pm}$ are such that $\psi_m > 0$ on $\mathcal{S}_m$ and

$$\psi_m(r,\phi) \to \infty \tag{4.14}$$

as $r \to \infty$, $(r, \theta(r,\phi)) \in \mathcal{S}_m$.

Remark 4.15. By inducting on m, notice that (4.13) and (4.10) imply the following ordering of the constants $p_{l,m}$, $q_{l,m}$ for $m > 2$

$$\begin{aligned} p_{1,m} &< p_{2,m} < \cdots < p_{m-1,m} = p_{m,m} \\ q_{1,m} &< q_{2,m} < \cdots < q_{m-1,m} < q_{m,m} < 1. \end{aligned} \tag{4.16}$$

The above relations will be especially helpful later when we do asymptotic analysis of ψ_m.

Remark 4.17. In the proof of Lemma 4.11 we will, in addition, derive some properties of the constants $d_l^{\pm} = d_{l,m}^{\pm}$. These will be collected in the statement of Corollary 4.18 below, and they will be used, in particular, to show that we can choose the constants $h_m^{\pm} > 0$ in a natural way so that the symmetry property (4.3) is satisfied.

Corollary 4.18. *For $l < m$ with $l, m \in \{3, \dots, j+3\}$, define the following constants*

$$b_{l,m}^{\pm} = \frac{|\phi^*|^{q_{l,m}}}{|c_{m-1} \mp \phi^*|^{q_{l,m-1}}}, \qquad e_m = |\phi^*|^{q_{m-1,m} - q_{m,m}}.$$

Then for $l < m-1$, $l, m \in \{3, \dots, j+3\}$, we have

$$d_{l,m}^{\pm} = d_{l,m-1}^{\pm} b_{l,m}^{\pm}$$

and for $m = 3, \dots, j+3$

$$\begin{aligned} d_{m,m}^{\pm} &= \frac{h_m^{\pm}}{q_m(n+m-2) - p_m} \\ d_{m-1,m}^{\pm} &= d_{m-1,m-1}^{\pm} b_{m-1,m}^{\pm} - d_{m,m}^{\pm} e_m. \end{aligned}$$

Before proving the lemma and corollary above, we state another lemma which shows that, assuming the conclusions of Lemma 4.11 and Corollary 4.18, we can pick the constants $h_m^{\pm}$ in a reasonable way so as to have (4.3).

Lemma 4.19. *Fixing a constant $K_0 > 0$, for all $\epsilon > 0$ there exists a constant $K_1 > 0$ so that the following holds. If $h_1^+, h_2^+, \dots, h_{j+3}^+$ is a collection of positive parameters with $h_i^+ \le K_0$ for all i then for any $\phi^* \ge K_1$ there exist a unique choice of positive $h_1^-, h_2^-, \dots, h_{j+3}^-$ so that*

$$\psi_{j+3}^+(r, -\phi) = \psi_{j+3}^-(r, \phi)$$

for all (r, ϕ) with $(r, \theta(r, \phi)) \in \mathcal{S}_{j+3}$ and

$$|h_m^+ - h_m^-| \le \epsilon$$

for all $m = 1, 2, \dots, j+3$.

Remark 4.20. Later, we will use the parameters h_i^+ to ensure that the fluxes across the boundaries between osculating regions where $\theta > 0$ have the desired sign just as we did in Section 8.1 of Part I [6]. We will then need to choose the h_i^- to both satisfy the boundary flux condition and make the ψ_{j+3} have the desired symmetry. Note that since as $\phi^* \to \infty$ the regions become increasingly symmetric in the angular variable ϕ, it is intuitively clear that the h_i^- which produce a symmetric ϕ are close the h_i^+ which were already chosen. Hence, the h_i^- which produce symmetry also satisfy the needed boundary flux condition.

Remark 4.21. Notice that the choice of ϕ^* determined by the lemma above is consistent with our process of picking parameters as outlined in Remark 3.20.

We first give the proof of Lemma 4.11 and Corollary 4.18 together and then prove Lemma 4.19 immediately afterwards.

Proof of Lemma 4.11 and Corollary 4.18. The proof will be done by induction on $m \geq 3$. Suppose first that $m = 3$. Using the method of characteristics, one can easily derive the desired expression for ψ_3 and all claimed relations in Lemma 4.11 and Corollary 4.18. To check (4.14) is valid for $\psi_3^{\pm}$, consider the dynamics along T_3

$$\dot{r} = r \qquad \text{and} \qquad \dot{\phi} = (n+1)\phi,$$

and let $\tau^* = \inf_{t>0}\{|\phi_t| = \phi^*\}$. Using the inequality (4.8), notice for all (r, ϕ) such that $(r, \theta(r, \phi)) \in \mathcal{S}_3$

$$\begin{aligned}\psi_3^{\pm}(r,\phi) &= \psi_2^{\pm}\Big(r_{\tau^*}, \theta(r_{\tau^*}, \phi_{\tau^*})\Big) + h_3^{\pm}\int_0^{\tau^*} \frac{r_t^{p_3}}{|\phi_t|^{q_3}}\, dt \\ &\geq \psi_2^{\pm}\Big(r_{\tau^*}, \theta(r_{\tau^*}, \phi_{\tau^*})\Big) = \psi_2^{\pm}(r_{\tau^*}, r_{\tau^*}^{-1}(\phi^* - \frac{\gamma_1}{n+1})) \\ &\geq c\frac{r^{p_{1,3}}}{|\phi|^{q_{1,3}}}\end{aligned}$$

for some constant $c > 0$. Hence we now see that $\psi_3 > 0$ on $\mathcal{S}_3$ and $\psi_3 \to \infty$ as $r \to \infty$, $(r, \theta(r, \phi)) \in \mathcal{S}_3$.

Now assume all conclusions are valid for some $m - 1 \geq 3$. Using the method of characteristics and the inductive hypothesis, we can obtain the claimed expression for ψ_m as well as all relationships between constants in the statements of Lemma 4.11 and Corollary 4.18. To obtain (4.14), we may assume inductively that

$$\psi_{m-1}^{\pm}(r,\phi) \geq c\frac{r^{p_{1,m-1}}}{|\phi|^{q_{1,m-1}}}$$

for all (r, ϕ) with $(r, \theta(r, \phi)) \in \mathcal{S}_{m-1}$ where $c > 0$ is a constant (which is in general different from the one used above). As before, consider the dynamics along T_m:

$$\dot{r} = r \qquad \text{and} \qquad \dot{\phi} = (n+m-2)\phi$$

and let, recycling notation, $\tau^* = \inf_{t>0}\{|\phi_t| = \phi^*\}$. Then we similarly obtain

$$\begin{aligned}\psi_m^{\pm}(r,\phi) &= \psi_{m-1}^{\pm}\Big(r_{\tau^*}, \phi_{m-1}(r_{\tau^*}, \phi_{\tau^*})\Big) + h_m^{\pm}\int_0^{\tau^*} \frac{r_t^{p_m}}{|\phi_t|^{q_m}}\, dt \\ &\geq \psi_{m-1}^{\pm}\Big(r_{\tau^*}, \phi_{m-1}(r_{\tau^*}, \phi_{\tau^*})\Big) = \psi_{m-1}^{\pm}(r_{\tau^*}, r_{\tau^*}^{-1}(\phi^* - c_{m-1})) \\ &\geq c\frac{r^{p_{1,m}}}{|\phi|^{q_{1,m}}}\end{aligned}$$

for some $c > 0$ which is different from the c used above. This now finishes the proof of the result. □

Proof of Lemma 4.19. Let $h_1^+, h_2^+, \ldots, h_{j+3}^+$ be a bounded collection of positive parameters and fix $\epsilon > 0$. We will see that there is a unique choice of $h_1^-, h_2^-, \ldots, h_{j+3}^-$ which gives

$$d_{m,j+3}^{+} = d_{m,j+3}^{-} \tag{4.22}$$

for all $m = 1, \ldots, j+3$. By Corollary 4.18 and Lemma 4.11, the first conclusion of the lemma will then follow immediately since the $\psi_{j+3}^{\pm}$ are a linear combination functions with coefficients $d_{m,j+3}^{\pm}$ respectively. The closeness of the h's will follow for all ϕ^* large enough by inspection of the choice of the h_j^-'s giving (4.22) for all $m = 1, \ldots, j+3$.

We proceed inductively and begin by analyzing the equality

$$d_{j+3-m,j+3}^{+} = d_{j+3-m,j+3}^{-}$$

for $m = 0$. Note that Corollary 4.18 implies that

$$d^+_{j+3,j+3} = d^-_{j+3,j+3} \iff h^+_{j+3} = h^-_{j+3}.$$

This, in particular, forces us to choose $h^-_{j+3} = h^+_{j+3}$. Now consider the equality $d^+_{j+2,j+3} = d^-_{j+2,j+3}$. By Corollary 4.18 again and the fact that $d^+_{j+3,j+3} = d^-_{j+3,j+3}$, notice

$$d^{\pm}_{j+2,j+3} = d^{\pm}_{j+2,j+2} b^{\pm}_{j+2,j+3} - d^+_{j+3,j+3} e_{j+3}.$$

Hence

$$d^+_{j+2,j+3} = d^-_{j+2,j+3} \iff h^-_{j+2} = h^+_{j+2} \frac{b^+_{j+2,j+3}}{b^-_{j+2,j+3}}$$

implying that we must pick

$$h^-_{j+2} = h^+_{j+2} \frac{b^+_{j+2,j+3}}{b^-_{j+2,j+3}}.$$

Using the expressions given in Corollary 4.18 for the b's, a simple argument employing Taylor's theorem gives the following asymptotic formula as $\phi^* \to \infty$:

$$\frac{b^+_{j+2,j+3}}{b^-_{j+2,j+3}} = 1 + O\big((\phi^*)^{-1}\big).$$

Therefore

$$h^-_{j+2} = h^+_{j+2} + O\big((\phi^*)^{-1}\big) \tag{4.23}$$

as $\phi^* \to \infty$. In particular, this implies that the unique choice of h^+_{j+2} (which is positive for ϕ^* large enough) determined by the relation $d^+_{j+2,j+3} = d^-_{j+2,j+3}$ has the desired closeness property $|h^+_{j+2} - h^-_{j+2}| < \epsilon$ for all ϕ^* large enough. To continue by induction, we need one more step to see how to proceed in general. Notice that this is only necessary if $j \geq 1$ where $n = 2j+1$ or $n = 2j+2$. By Corollary 4.18, observe that

$$d^{\pm}_{j+1,j+3} = d^{\pm}_{j+1,j+2} b^{\pm}_{j+1,j+3} = (d^{\pm}_{j+1,j+1} b^{\pm}_{j+1,j+2} - d^{\pm}_{j+2,j+2} e_{j+2}) b^{\pm}_{j+1,j+3}$$

and, by the right most equality, $d^+_{j+1,j+3} = d^-_{j+1,j+3}$ is equivalent to

$$d^-_{j+1,j+1} = \left(d^+_{j+1,j+1} \frac{b^+_{j+1,j+2}}{b^-_{j+1,j+2}} - d^+_{j+2,j+2} \frac{e_{j+2}}{b^-_{j+1,j+2}} \right) \frac{b^+_{j+1,j+3}}{b^-_{j+1,j+3}} + d^-_{j+2,j+2} \frac{e_{j+2}}{b^-_{j+1,j+2}}.$$

By (4.23) and Corollary 4.18, we have

$$\begin{aligned} d^+_{j+2,j+2} &= d^-_{j+2,j+2} + O\big((\phi^*)^{-1}\big) \\ \frac{e_{j+2}}{b^-_{j+1,j+2}} &= (\phi^*)^{q_{j+1} - q_{j+2}} \Big(1 + O\big((\phi^*)^{-1}\big) \Big) \end{aligned}$$

as $\phi^* \to \infty$. Again, by Taylor's theorem we also have

$$\begin{aligned} \frac{b^+_{j+1,j+2}}{b^-_{j+1,j+2}} &= 1 + O\big((\phi^*)^{-1}\big) \\ \frac{b^+_{j+1,j+3}}{b^-_{j+1,j+3}} &= 1 + O\big((\phi^*)^{-1}\big) \end{aligned}$$

as $\phi^* \to \infty$. Putting these formulas together, since $h_1^+, h_2^+, \ldots, h_{j+3}^+$ were assumed to be bounded and $q_{j+1} - q_{j+2} < 0$ by (4.10) we obtain

$$\begin{aligned} d^-_{j+1,j+1} &= d^+_{j+1,j+1} + O\big((\phi^*)^{-1}\big) \\ h^-_{j+1} &= h^+_{j+1} + O\big((\phi^*)^{-1}\big) \end{aligned}$$

as $\phi^* \to \infty$. This finishes the result in this case. To see in general when $d^+_{m,j+3} = d^-_{m,j+3}$ for general $m = 2, \ldots, j$, assume by induction that

$$d^-_{m+1,m+1} = d^+_{m+1,m+1} + O\Big((\phi^*)^{-1}\Big)$$

and note by successively applying $d^\pm_{l,m} = d^\pm_{l,m-1} b^\pm_{l,m}$ we obtain

$$\begin{aligned} d^\pm_{m,j+3} &= d^\pm_{m,j+2} b^\pm_{m,j+3} = d^\pm_{m,m+1} b^\pm_{m,m+2} b^\pm_{m,m+3} \cdots b^\pm_{m,j+3} \\ &= (d^\pm_{m,m} b^\pm_{m,m+1} - d^\pm_{m+1,m+1} e_{m+1}) b^\pm_{m,m+2} b^\pm_{m,m+3} \cdots b^\pm_{m,j+3}. \end{aligned}$$

Therefore, $d^+_{m,j+3} = d^-_{m,j+3}$ is equivalent to

$$d^-_{m,m} = \left(d^+_{m,m} \frac{b^+_{m,m+1}}{b^-_{m,m+1}} - d^+_{m+1,m+1} \frac{e_{m+1}}{b^-_{m,m+1}} \right) \frac{b^+_{m,m+2} \cdots b^+_{m,j+3}}{b^-_{m,m+2} \cdots b^-_{m,j+3}} + d^-_{m+1,m+1} \frac{e_{m+1}}{b^-_{m,m+1}}.$$

Similarly, using Taylor's theorem and the asymptotic formulas above, we see that

$$\begin{aligned} d^-_{m,m} &= d^+_{m,m} + O\big((\phi^*)^{-1}\big) \\ h^-_m &= h^+_m + O\big((\phi^*)^{-1}\big). \end{aligned}$$

Thus we have established the result for $h_2^\pm, h_3^\pm, \ldots, h^\pm_{j+3}$. Finally, to obtain the equality

$$d^+_{1,j+3} = d^-_{1,j+3}$$

realize that it is equivalent to the relation

$$d^-_{1,2} = d^+_{1,2} \frac{b^+_{1,3} \cdots b^+_{1,j+3}}{b^-_{1,3} \cdots b^-_{1,j+3}}. \tag{4.24}$$

Since

$$d^\pm_{1,2} = |\theta_1^*|^{\frac{p}{n}} \psi_1^\pm(1, \pm\theta_1^*) - h_2^\pm \frac{|\theta_1^*|^{\frac{p}{n}-q}}{qn - p}$$

and

$$\frac{b^-_{1,3} \cdots b^-_{1,j+3}}{b^+_{1,3} \cdots b^+_{1,j+3}} = 1 + O\big((\phi^*)^{-1}\big)$$

as $\phi^* \to \infty$, one can easily deduce from (4.5) that for fixed θ_1^*, as $\phi^* \to \infty$ the choice of h_1^- determined by the symmetry condition (4.24) approaches h_1^+. Note that this finishes the proof of the result. □

Now that we have the desired symmetry we turn to defining the final function ψ_{j+4} in the region $\mathcal{S}_{j+4}$ where noise does play a role.

4.2. **The construction in the noise region $\mathcal{S}_{j+4}$.** Here, let $(r,\phi)=(r,\phi_{j+3})$ and define ψ_{j+4} on $\mathcal{S}_{j+4}$ as the solution of the following PDE

$$\begin{cases} A\psi_{j+4}(r,\phi) = -h_{j+4}\, r^{p_{j+4}} \\ \psi_{j+3}=\psi_{j+3} \text{ on } \partial(\mathcal{S}_{j+3}\cap\mathcal{S}_{j+4}) \end{cases} \tag{4.25}$$

for all (r,ϕ) such that $(r,\theta(r,\phi))\in\mathcal{S}_{j+4}$ where $h_{j+4}>0$ and

$$p_{j+4}=\begin{cases} p_{j+3}+\frac{q_{j+3}}{2} & \text{if } n=2j+1,\ j\geq 0\\ p_{j+3}+q_{j+3} & \text{if } n=2j+2,\ j\geq 0. \end{cases} \tag{4.26}$$

We assume that the reader is familiar with the content of Section 7.3 of [6] which outlines how one is able to solve the PDE above.

To solve for ψ_{j+4}, for simplicity let

$$p_{m,j+4}=\begin{cases} p_{m,j+3}+\frac{q_{m,j+3}}{2} & \text{if } n=2j+1,\ j\geq 0\\ p_{m,j+3}+q_{m,j+3} & \text{if } n=2j+2,\ j\geq 0 \end{cases}$$

for $m=1,2,\ldots,j+3$. Also, let (r_t,ϕ_t) denote the diffusion defined by A and $\tau=\inf_{t>0}\{(r_t,\phi_t)\notin\mathcal{S}_{j+4}\}$. Recalling the definition of $\partial(\mathcal{S}_{j+3}\cap\mathcal{S}_{j+4})$, we then see that

$$\begin{aligned}\psi_{j+4}(r,\phi) &= \mathbf{E}_{(r,\phi)}\psi_{j+3}(r_\tau,\phi_\tau)+h_{j+1}\mathbf{E}_{(r,\phi)}\int_0^\tau r_t^{p_{j+4}}\,dt\\ &= \sum_{m=1}^{j+3}\frac{d^+_{m,j+3}}{(\eta^*)^{q_m}}r^{p_m}\mathbf{E}_{(r,\phi)}e^{p_m\tau}+\frac{h_{j+4}}{p_{j+4}}r^{p_{j+4}}\mathbf{E}_{(r,\phi)}(e^{p_{j+4}\tau}-1)\end{aligned} \tag{4.27}$$

where we have concatenated $p_{m,j+4}$ and $q_{m,j+3}$ to p_m and q_m respectively.

To see that the maps $(r,\phi)\mapsto\mathbf{E}_{(r,\phi)}e^{p_m\tau}$ for $m=1,2,\ldots,j+4$ are well-defined and smooth on $\mathcal{S}_{j+4}$, first observe that the process

$$\eta_t=\begin{cases} r_t^{\frac{1}{2}}\phi_t & \text{if } n=2j+1\\ r_t\phi_t+c_{j+3} & \text{if } n=2j+2 \end{cases}$$

satisfies the Gaussian SDE

$$d\eta_t=\Big(\tfrac{3}{2}n+1\Big)\eta_t\,dt+\sigma\,dW_t. \tag{4.28}$$

Hence, we may write

$$\tau=\begin{cases} \inf\{t>0\,:\,\eta_t\notin[-\eta^*,\eta^*]\} & \text{if } n=2j+1\\ \inf\{t>0\,:\,\eta_t\notin[-\eta^*+c_{j+2},\eta^*+c_{j+2}]\} & \text{if } n=2j+2. \end{cases}$$

Applying Lemma 7.22 of Part I [6], by choosing $\eta^*>|c_{j+2}|$ large enough, it suffices to show that the constants $p_m=p_{m,j+4}$ satisfy

$$p_{1,j+4}<p_{2,j+4}<\cdots<p_{j+4,j+4}<\tfrac{3n+2}{2}.$$

The fact that

$$p_{1,j+4}<p_{2,j+4}<\cdots<p_{j+4,j+4}$$

follows by Remark 4.15 and the definition of the constants $p_{m,j+4}$, $m=1,2,\ldots,j+4$. The remaining bound can be obtained inductively in either case ($n=2j+1$ or $n=2j+2$) by using the definition of $p_{j+4,j+4}=p_{j+4}$, the relations (4.10), and the choice of $p\in(0,n)$.

To show that ψ_{j+4} is strictly positive on $\mathcal{S}_{j+4}$ and $\psi_{j+4}(r,\phi)\to\infty$ as $r\to\infty$ with (r,ϕ) such that $(r,\theta(r,\phi))\in\mathcal{S}_{j+4}$, using (4.27) we see that for some constant $c>0$

$$\begin{aligned}\psi_{j+4}(r,\phi) &\geq \mathbf{E}_{(r,\phi)}\psi_{j+3}(r_\tau,\phi_\tau)\\ &\geq c\mathbf{E}_{(r,\phi)}\frac{r_\tau^{p_{1,j+3}}}{|\phi_\tau|^{q_{1,j+3}}}\end{aligned}$$

where the last inequality follows by the inductive argument proving both Lemma 4.11 and Corollary 4.18. We thus obtain the desired bound

$$\begin{aligned}\psi_{j+4}(r,\phi) &\geq c\mathbf{E}_{(r,\phi)}\frac{r_\tau^{p_{1,j+3}}}{|\phi_\tau|^{q_{1,j+3}}}\\ &\geq \frac{c}{(\eta^*)^{q_{1,j+4}}}r^{p_{1,j+4}}\mathbf{E}_{(r,\phi)}e^{p_{1,j+4}\tau} > c' r^{p_{1,j+4}}\end{aligned}$$

for some $c'>0$.

4.3. **Summary of the construction.** Now that we have finished defining our Lyapunov function on each region $\mathcal{S}_i$, $i=1,\dots,j+4$, we pause for a moment to provide a summary of the construction up to this point. In the following sections, we will finish proving Theorem 2.5 by making sure the boundary-flux conditions of Corollary 6.8 of Part I [6] are satisfied and that each ψ_i is indeed a local Lyapunov function on its domain of definition $\mathcal{S}_i$.

4.3.1. *Regions and Asymptotic Operators.* Recalling that $n=2j+1$ or $n=2j+2$, the analysis of Section 3 uncovered the asymptotic operators

$$T_1,\dots,T_{j+3},A$$

and corresponding regions where we expect each to approximate well the time-changed Markov generator L as $r\to\infty$. The analysis of this section is summarized in the following three tables.

Remark 4.29. Recall that the constants c_i, $i=2,\dots,j+2$, were defined inductively and depend on the Taylor expansion of the coefficients of L at $\theta=0$. Also recall that $\theta_0^*\in(\frac{\pi}{2n},\frac{\pi}{n})$ is fixed and the constants θ_1^*, ϕ^* and η^* are chosen in the way outlined in Remark 3.20.

Region $\mathcal{S}_i$, $i=0,\dots,j+2$	Asymptotic Operator	Coordinates
$\mathcal{S}_0=\{r\geq r^*,\ \theta_0^*\leq\lvert\theta\rvert\leq\frac{\pi}{n}\}\cap\mathcal{R}$	$T_1=r\cos(n\theta)\partial_r+\sin(n\theta)\partial_\theta$	$r,\ \theta$
$\mathcal{S}_1=\{r\geq r^*,\ 0<\theta_1^*\leq\lvert\theta\rvert\leq\theta_0^*\}\cap\mathcal{R}$	$T_1=r\cos(n\theta)\partial_r+\sin(n\theta)\partial_\theta$	$r,\ \theta$
$\mathcal{S}_2=\{r\geq r^*,\ \lvert\phi_3\rvert\geq\phi^*,\ \lvert\theta\rvert\leq\theta_1^*\}\cap\mathcal{R}$	$T_2=r\partial_r+n\theta\partial_\theta$	$r,\ \theta$
$\mathcal{S}_3=\{r\geq r^*,\lvert\phi_4\rvert\geq\phi^*,\ \lvert\phi_3\rvert\leq\phi^*\}\cap\mathcal{R}$	$T_3=r\partial_r+(n+1)\phi_3\partial_{\phi_3}$	$r,\ \phi_3=r\theta+c_2$
$\mathcal{S}_4=\{r\geq r^*,\lvert\phi_5\rvert\geq\phi^*,\ \lvert\phi_4\rvert\leq\phi^*\}\cap\mathcal{R}$	$T_4=r\partial_r+(n+2)\phi_4\partial_{\phi_4}$	$r,\ \phi_4=r\phi_3+c_3$
$\vdots$	$\vdots$	$\vdots$
$\mathcal{S}_m=\{r\geq r^*,\lvert\phi_{m+1}\rvert\geq\phi^*,\ \lvert\phi_m\rvert\leq\phi^*\}\cap\mathcal{R}$	$T_m=r\partial_r+(n+m-2)\phi_m\partial_{\phi_m}$	$r,\ \phi_m=r\phi_{m-1}+c_{m-1}$
$\vdots$	$\vdots$	$\vdots$
$\mathcal{S}_{j+2}=\{r\geq r^*,\lvert\phi_{j+3}\rvert\geq\phi^*,\ \lvert\phi_{j+2}\rvert\leq\phi^*\}\cap\mathcal{R}$	$T_{j+2}=r\partial_r+(n+j)\phi_{j+2}\partial_{\phi_{j+2}}$	$r,\ \phi_{j+2}=r\phi_{j+1}+c_{j+1}$

Regions S_{j+3}, $\mathcal{S}_{j+4}$, $n=2j+1$	Asymptotic Operator	Coordinates
$\mathcal{S}_{j+3}=\{r\geq r^*,\ \eta^*r^{-\frac{1}{2}}\leq\lvert\phi_{j+3}\rvert\leq\phi^*\}\cap\mathcal{R}$	$T_{j+3}=r\partial_r+(3j+2)\phi_{j+3}\partial_{\phi_{j+3}}$	$r,\ \phi_{j+3}=r\phi_{j+2}+c_{j+2}$
$\mathcal{S}_{j+4}=\{r\geq r^*,\ \lvert\phi_{j+3}\rvert\leq\eta^*r^{-1/2}\}\cap\mathcal{R}$	$A=r\partial_r+(3j+2)\phi_{j+3}\partial_{\phi_{j+3}}+\frac{\sigma^2}{2r}\partial^2_{\phi_{j+3}}$	$r,\ \phi_{j+3}=r\phi_{j+2}+c_{j+2}$

Regions $\mathcal{S}_{j+3}$, $\mathcal{S}_{j+4}$, $n=2j+2$	Asymptotic Operator	Coordinates
$\mathcal{S}_{j+3}=\{r\geq r^*,\ \eta^*r^{-1}\leq\lvert\phi_{j+3}\rvert\leq\phi^*\}\cap\mathcal{R}$	$T_{j+3}=r\partial_r+(3j+3)\phi_{j+3}\partial_{\phi_{j+3}}$	$r,\ \phi_{j+3}=r\phi_{j+2}+c_{j+2}$
$\mathcal{S}_{j+4}=\{r\geq r^*,\ \lvert\phi_{j+3}\rvert\leq\eta^*r^{-1}\}\cap\mathcal{R}$	$A=r\partial_r+[(3j+3)\phi_{j+3}+\gamma_1^{(j+3)}r^{-1}]\partial_{\phi_{j+3}}+\frac{\sigma^2}{2r}\partial^2_{\phi_{j+3}}$	$r,\ \phi_{j+3}=r\phi_{j+2}+c_{j+2}$

4.3.2. *Properties of the Lyapunov Function in Each Region.* Below we give a summary of some of the basic properties of our Lyapunov function Ψ on the principal wedge $\mathcal{R}$ in each region $\mathcal{S}_i$. We recall that the constants p, q satisfy $p \in (0, n)$, $q \in (\frac{p}{n}, 1)$ and the constants $d^{\pm}_{l,m}$ are determined by the boundary conditions in each Poisson equation. As mentioned in Remark 4.20, the constants $h^{\pm}_i > 0$ will be chosen so that both the reflective symmetry (4.3) and the boundary-flux conditions are satisfied.

Region $\mathcal{S}$	Asymptotic Operator $\mathcal{O}$	$\Psi\vert_{\mathcal{S}}$	$\mathcal{O}(\Psi\vert_{\mathcal{S}})$ on $\mathcal{S}$
$\mathcal{S}_0$	T_1	r^p	$p\cos(n\theta)r^p$
$\mathcal{S}_1$	T_1	eqn. (4.5)	$-h^{\pm}_1 r^p\vert\theta\vert^{-q}$
$\mathcal{S}_2$	T_2	$d^{\pm}_{12}\frac{r^p}{\vert\theta\vert^{p/n}} + d^{\pm}_{22}\frac{r^p}{\vert\theta\vert^q}$	$-h^{\pm}_2 r^p\vert\theta\vert^{-q}$
$\mathcal{S}_3$	T_3	$\sum_{l=1}^{3} d^{\pm}_{l,3}\frac{r^{p_{l,3}}}{\vert\phi\vert^{q_{l,3}}}$	$-h^{\pm}_3 r^{p_3}\vert\phi_3\vert^{-q_3}$
$\vdots$	$\vdots$	$\vdots$	$\vdots$
$\mathcal{S}_m$	T_m	$\sum_{l=1}^{m} d^{\pm}_{l,m}\frac{r^{p_{l,m}}}{\vert\phi_m\vert^{q_{l,m}}}$	$-h^{\pm}_m r^{p_m}\vert\phi_m\vert^{-q_m}$
$\vdots$	$\vdots$	$\vdots$	$\vdots$
$\mathcal{S}_{j+3}$	T_{j+3}	$\sum_{l=1}^{j+3} d^{\pm}_{l,j+3}\frac{r^{p_{l,j+3}}}{\vert\phi_{j+3}\vert^{q_{l,j+3}}}$	$-h^{\pm}_{j+3} r^{p_{j+3}}\vert\phi_{j+3}\vert^{-q_{j+3}}$
$\mathcal{S}_{j+4}$	A	eqn. (4.27)	$-h_{j+4}r^{p_{j+4}}$

5. Boundary-flux calculations

Here show how one can choose the positive parameters h^+_i, θ^*_1, ϕ^*, η^* so that the jump conditions of Corollary 6.8 of Part I [6] are also satisfied. We must be careful to see that all choices are consistent with Remark 3.20 and Lemma 4.19. Each boundary has two disjoint parts, implying that we must check two, although very similar, flux conditions. We proceed from boundary to boundary, starting with the:

5.1. **Boundary between $\mathcal{S}_0$ and $\mathcal{S}_1$.** We begin on the side of the boundary where $\theta > 0$. We must pick the parameters so that

$$\left[\frac{\partial\psi_0}{\partial\theta} - \frac{\partial\psi_1^+}{\partial\theta}\right]_{\theta=\theta_0^*} \leq 0 \tag{5.1}$$

for $r \geq r^*$. By inspection of the formula (4.5), we first note that $\psi_1^+(r,\theta) = r^p\psi_1^+(1,\theta)$. Using this and the equation (4.4) defining ψ_1^+, observe also that

$$-h_1^+ r^p|\theta|^{-q} = \frac{\partial\psi_1^+}{\partial r} r\cos(n\theta) + \frac{\partial\psi_1^+}{\partial\theta}\sin(n\theta).$$

Rearranging this produces

$$\frac{\partial\psi_1^+}{\partial\theta} = -r^p\Big(\frac{p\cos(n\theta)\psi_1^+(1,\theta) + h_1^+|\theta|^{-q}}{\sin(n\theta)}\Big). \tag{5.2}$$

Therefore combining $\frac{\partial\psi_0}{\partial\theta} = 0$ with (5.2) gives

$$\left[\frac{\partial\psi_0}{\partial\theta} - \frac{\partial\psi_1^+}{\partial\theta}\right]_{\theta=\theta_0^*} = r^p\Big(\frac{p\cos(n\theta_0^*)\psi_1^+(1,\theta_0^*) + h_1^+|\theta_0^*|^{-q}}{\sin(n\theta_0^*)}\Big).$$

Because $\psi_1^+(1,\theta_0^*) = 1$, $\sin(n\theta_0^*) > 0$ and $\cos(n\theta_0^*) < 0$, picking

$$0 < h_1^+ < p(\theta_0^*)^q|\cos(n\theta_0^*)| \tag{5.3}$$

results in (5.1).

On the side of the boundary where $\theta < 0$, we must see that this choice of h_1^+ also implies

$$\left[\frac{\partial \psi_1^-}{\partial \theta} - \frac{\partial \psi_0}{\partial \theta}\right]_{\theta=-\theta_0^*} \leq 0 \tag{5.4}$$

for $r \geq r^*$. By Lemma 4.19, we have already picked h_1^- and we recall that as $\phi^* \to \infty$, $h_1^- \to h_1^+$. Using the very same process as above, (5.4) is satisfied provided

$$0 < h_1^- < p(\theta_0^*)^q |\cos(n\theta_0^*)|. \tag{5.5}$$

Therefore, both quantities can be seen to be negative by first picking

$$0 < h_1^+ < p(\theta_0^*)^q |\cos(n\theta_0^*)| \tag{5.6}$$

and then taking $\phi^* > 0$ sufficiently large. Note that this is consistent with the flow of choices outlined in Remark 3.20.

5.2. **Boundary between $\mathcal{S}_1$ and $\mathcal{S}_2$.** We proceed in a similar fashion by first doing the computation on the side of the boundary where $\theta > 0$. We first show that

$$\left[\frac{\partial \psi_1^+}{\partial \theta} - \frac{\partial \psi_2^+}{\partial \theta}\right]_{\theta=\theta_1^*} \leq 0 \tag{5.7}$$

for $r \geq r^*$ whenever $\theta_1^* > 0$ is small enough. Using $\psi_2^+(r,\theta) = r^p \psi_2^+(1,\theta)$ and the equation ψ_2^+ satisfies, we obtain

$$\frac{\partial \psi_2^+}{\partial \theta} = -r^p \left[\frac{p\psi_2^+(1,\theta) + h_2^+|\theta|^{-q}}{n\theta}\right].$$

Since $\psi_1^+(1,\theta_1^*) = \psi_2^+(1,\theta_1^*)$, notice

$$\begin{aligned}
&\left[\frac{\partial \psi_1^+}{\partial \theta} - \frac{\partial \psi_2^+}{\partial \theta}\right]_{\theta=\theta_1^*} \\
&= -r^p \left[-\frac{p\psi_1^+(1,\theta_1^*) + h_2^+|\theta_1^*|^{-q}}{n\theta_1^*} + \frac{p\cos(n\theta_1^*)\psi_1^+(1,\theta_1^*) + h_1^+|\theta_1^*|^{-q}}{\sin(n\theta_1^*)}\right] \\
&= -\frac{r^p}{|\theta_1^*|^{q+1}}\left[\left(\frac{p\cos(n\theta_1^*)}{\sin(n\theta_1^*)} - \frac{p}{n\theta_1^*}\right)\psi_1^+(1,\theta_1^*)|\theta_1^*|^{q+1} + \left(\frac{h_1^+}{\sin(n\theta_1^*)} - \frac{h_2^+}{n\theta_1^*}\right)|\theta_1^*|\right].
\end{aligned}$$

The expression (4.5) implies that $\psi_1^+(1,\theta_1^*)|\theta_1^*|^q \to 0$ as $\theta_1^* \downarrow 0$. Using this fact and expanding $\sin(n\theta_1^*)$ and $\cos(n\theta_1^*)$ in power series about $\theta_1^* = 0$, we arrive at the asymptotic formula

$$\begin{aligned}
&\left[\left(\frac{p\cos(n\theta_1^*)}{\sin(n\theta_1^*)} - \frac{p}{n\theta_1^*}\right)\psi_1^+(1,\theta_1^*)|\theta_1^*|^{q+1} + \left(\frac{h_1^+}{\sin(n\theta_1^*)} - \frac{h_2^+}{n\theta_1^*}\right)|\theta_1^*|\right] \\
&= \left(\frac{h_1^+}{n} - \frac{h_2^+}{n}\right) + o(1)
\end{aligned}$$

as $\theta_1^* \downarrow 0$. Therefore, for every choice of

$$h_2^+ < h_1^+ \tag{5.8}$$

we may pick $\theta_1^* > 0$ sufficiently small so that the flux across the boundary where $\theta > 0$ is negative. On the side of the boundary where $\theta < 0$, a similar line of reasoning shows that the choice

$$h_2^- < h_1^- \tag{5.9}$$

results in a negative flux for all $\theta_1^* > 0$ small. Recall, also, that this is consistent with both Remark 3.20 and Lemma 4.19 by, after choosing $\theta_1^* > 0$ small, choosing $\phi^* > 0$ large.

5.3. **Boundary between $\mathcal{S}_2$ and $\mathcal{S}_3$.** For illustrative purposes, we perform one more boundary-flux estimate before proceeding on to the general, inductive calculation in the remaining transport regions. We begin on the side of the boundary where $\phi_3 > 0$. Note that for $\phi^* > 0$ large, it is also true that $\theta > 0$ on this side.

As opposed to the previous cases, it is more convenient to use the explicit expressions obtained for $\psi_2^\pm$ and $\psi_3^\pm$. In doing this, we first note that

$$\left[\frac{\partial \psi_2^+}{\partial \theta} - \frac{\partial \psi_3^+}{\partial \theta}\right]_{\phi_3=\phi^*} \leq C_1(\phi^*) r^{p+q+1} + C_2(\phi^*) r^{p+\frac{p}{n}+1} \tag{5.10}$$

where

$$C_1(\phi^*) = -\frac{q_2 d_{2,2}^+}{|\phi^* - c_1|^{q_2+1}} + \frac{q_{2,3} d_{2,3}^+}{|\phi^*|^{q_{2,3}+1}} + \frac{q_3 d_{3,3}^+}{|\phi^*|^{q_3+1}}$$

and $C_2(\phi^*)$ is a constant depending on ϕ^*. Our goal is to see that for ϕ^* large enough, $C_1(\phi^*) < 0$. Hence for $r^* > 0$ large enough, the quantity (5.10) will also be negative. Recalling the dependence of $d_{2,3}^+$ on ϕ^* in Corollary 4.18 and that $q_3 > q_2$, we note that

$$C_1(\phi^*) = -(\phi^*)^{-q_2-1}\big((q_2 - q_{2,3}) d_{2,2}^+ + o(1)\big)$$

as $\phi^* \to \infty$. Since $d_{2,2}^+$ is positive and independent of ϕ^* and

$$q_{2,3} = \frac{p_{2,3}}{n+1} = \frac{p_{2,2} + q_{2,2}}{n+1} = \frac{p+q}{n+1}$$

where $q_2 = q \in (\frac{p}{n}, 1)$, we find that $q_2 > q_{2,3}$. Hence, choosing $\phi^* > 0$ large enough, $C_1(\phi^*)$ is negative. Thus for $r^* > 0$, the quantity on the left-hand side of (5.10) is also negative. A nearly identical computation will yield the desired result on the other side of the boundary.

5.4. **The boundaries between the remaining transport regions.** We now consider the flux across the two boundaries between $\mathcal{S}_m$ and $\mathcal{S}_{m+1}$ where $k = 3, \ldots, j+2$. As done in the previous case, we focus on the side of the boundary where $\phi_{m+1} > 0$. Note, too, with $\phi^* > 0$ large enough, ϕ_m is also positive on that side of the boundary. Using the expressions derived in Lemma 4.11, realize that

$$\left[\frac{\partial \psi_m^+}{\partial \theta} - \frac{\partial \psi_{m+1}^+}{\partial \theta}\right]_{\phi_{m+1}=\phi^*} \leq C_1(\phi^*) r^{p_{m+1}+m-1} + C_2(\phi^*) r^c, \tag{5.11}$$

where $c < p_{m+1} + m - 1$,

$$C_1(\phi^*) = -q_m \frac{d_{m,m}^+}{(\phi^* - c_m)^{q_m+1}} + q_{m,m+1} \frac{d_{m,m+1}^+}{(\phi^*)^{q_{m,m+1}+1}} + q_{m+1} \frac{d_{m+1,m+1}^+}{(\phi^*)^{q_{m+1}+1}}$$

and $C_2(\phi^*)$ is a constant that depends on ϕ^*. Using Corollary 4.18 to write out $d_{m,m+1}^+$ and recalling that $q_{m+1} > q_m$, note that as $\phi^* \to \infty$

$$C_1(\phi^*) = -(\phi^*)^{-q_m-1}\Bigg((q_m - q_{m,m+1}) d_{m,m}^+ + o(1)\Bigg).$$

Recalling that

$$q_{m,m+1} = \frac{p_{m,m+1}}{n+m-1} = \frac{p_m + q_m}{n+m-1}$$

and $q_m \in (\frac{p_m}{n+m-2}, 1)$ we see that $q_m > q_{m,m+1}$ giving that $C_1(\phi^*)$ is negative for ϕ^* large enough. Thus for $r^* > 0$ large enough the quantity on the left-hand side of (5.11) is negative. A nearly identical result holds on the other side of the boundary.

5.5. **Boundary between $\mathcal{S}_{j+3}$ and $\mathcal{S}_{j+4}$.** In the following computation, we will need to employ Lemma 7.22 of Part I [6] since the expression for ψ_{j+4} in (4.27) is not explicit. Also, we only show the case when $n = 2j+1$, $j \geq 0$, as the other case is similar.

Consider the side of the boundary where $\phi_{j+3} > 0$. Recalling the notation $G_{a,c}$ introduced in Section 8.1 of Part I [6], observe that

$$\left[\frac{\partial \psi_{j+3}}{\partial \theta} - \frac{\partial \psi_{j+4}}{\partial \theta}\right]_{\eta=\eta^*} \leq C_1(\eta^*) r^{p_{j+4}+j+\frac{3}{2}} + C_2(\eta^*) r^c \tag{5.12}$$

for some $c < p_{j+4} + j + \frac{3}{2}$ where

$$C_1(\eta^*) = -\left[\frac{d^+_{j+3,j+3}}{(\eta^*)^{q_{j+3}}} + \frac{h_{j+4}}{p_{j+4}}\right] G'_{p_{j+4},0}(\eta^*) - \frac{d^+_{j+3,j+3}\, q_{j+3}}{(\eta^*)^{q_{j+3}+1}}$$

and $C_2(\eta^*)$ is a constant which depends on η^*. Choosing

$$h_{j+4} = h p_{j+4} (\eta^*)^{-q_{j+3}}$$

for some $h > 0$ and applying the Lemma 7.22 of Part I [6], realize that as $\eta^* \to \infty$

$$C_1(\eta^*) = (\eta^*)^{-q_{j+3}-1}\left(d^+_{j+3,j+3}\frac{2p_{j+4}}{3n+2} - d^+_{j+3,j+3} q_{j+3} + \frac{2h}{3n+2} + o(1)\right).$$

Using (4.26) and the relations $q_{j+3} > p_{j+3}/(n+j+1)$ and $n = 2j+1$, we see that

$$\frac{2p_{j+4}}{3n+2} < q_{j+3}$$

Picking h small enough implies that $C_1(\eta^*) < 0$ for $\eta^* > 0$ large enough. Therefore choosing $r^* > 0$ large enough implies that the quantity on the left-hand side of (5.12) is negative. A similar result is easily seen to hold on the other side of the boundary.

6. Checking the Global Lyapunov Bounds

6.1. **Checking the Local Lyapunov Property.** Here we check that the approximating operators $T_1, T_2, \ldots, T_{j+3}, A$ were chosen correctly so that $\psi_0, \psi_1, \ldots, \psi_{j+4}$ are actually locally Lyapunov functions on their respective domains $\mathcal{S}_0, \mathcal{S}_1, \ldots, \mathcal{S}_{j+4}$. This simply involves replacing each asymptotic operator with L and estimating the remainder locally on each region. Factoring in the time change, the required bound for $\mathcal{L}\psi_i$ on $\mathcal{S}_i$ will then follow easily.

Region $\mathcal{S}_0$. Since $\psi_0(r,\theta) = r^p$, it is not hard to see that as $r \to \infty$, $(r,\theta) \in \mathcal{S}_0$,

$$L\psi_0(r,\theta) = pr^p \cos(n\theta) + o(r^p). \tag{6.1}$$

Since $\cos(n\theta) \leq -c < 0$ for $(r,\theta) \in \mathcal{S}_0$, the relation (6.1) implies that there exist positive constants c_0, d_0 such that

$$L\psi_0(r,\theta) \leq -c_0 r^p + d_0$$

for all $(r,\theta) \in \mathcal{S}_0$. Undoing the time change, we see that there exist positive constants C_0, D_0 such that on $\mathcal{S}_0$

$$\mathcal{L}\psi_0(r,\theta) \leq -C_0 r^{p+n} + D_0. \tag{6.2}$$

Region $\mathcal{S}_1$. First observe that by definition of $\psi_1^\pm$, we see that

$$\begin{aligned} L\psi_1^\pm(r,\theta) &= T_1\psi_1^\pm(r,\theta) + (L-T_1)\psi_1^\pm(r,\theta) \\ &= -h_1^\pm \frac{r^p}{|\theta|^q} + (L-T_1)\psi_1^\pm(r,\theta) \end{aligned}$$

on $\mathcal{S}_1$ where the $\pm$ indicates the values of the functions above when θ is, respectively, positive or negative. To bound the remainder term $(L-T_1)\psi_1^\pm(r,\theta)$, recall by (4.5) we may write $\psi_1^\pm(r,\theta) = r^p\psi_1^\pm(1,\theta)$ where the mapping $\theta \mapsto \psi_1^\pm(1,\theta)$ is a smooth and positive function in θ for all $0<\theta_1^*\leq |\theta| \leq \theta_0^*$. In particular, since $0<\theta_1^* \leq |\theta| \leq \theta_0^*$ for $(r,\theta)\in\mathcal{S}_1$, we see that as $r\to\infty$, $(r,\theta)\in\mathcal{S}_1$,

$$L\psi_1^\pm(r,\theta) = -h_1^\pm\frac{r^p}{|\theta|^q} + o(r^p).$$

From this, we obtain the inequality

$$L\psi_1^\pm(r,\theta) \leq -c_1\frac{r^p}{|\theta|^q} + d_1$$

for some constants $c_1, d_1>0$, for all $(r,\theta)\in\mathcal{S}_1$. Undoing the time change, we see that there exist constants $C_1, D_1 > 0$ such that on $\mathcal{S}_1$

$$\mathcal{L}\psi_1^\pm(r,\theta) \leq -C_1\frac{r^{p+n}}{|\theta|^q} + D_1. \tag{6.3}$$

Region $\mathcal{S}_2$. By definition of $\psi_2^\pm$, first observe that on $\mathcal{S}_2$

$$\begin{aligned} L\psi_2^\pm(r,\theta) &= T_2\psi_2^\pm(r,\theta) + (T_1-T_2)\psi_2^\pm(r,\theta) + (L-T_1)\psi_2^\pm(r,\theta) \\ &= -h_2^\pm\frac{r^p}{|\theta|^q} + (T_1-T_2)\psi_2^\pm(r,\theta) + (L-T_1)\psi_2^\pm(r,\theta). \end{aligned}$$

Using the Taylor expansions for $\sin(n\theta)$ and $\cos(n\theta)$ notice that there exists a constant $C>0$ so that

$$\begin{aligned} &(T_1-T_2)\psi_2^\pm(r,\theta) \\ &\leq C\theta^2\left[\left(|\theta_1^*|^{\frac{p}{n}}\psi_1^\pm(1,\pm\theta_1^*) + h_2^\pm\frac{|\theta_1^*|^{\frac{p}{n}-q}}{qn-p}\right)\frac{r^p}{|\theta|^{p/n}} + \frac{h_2}{qn-p}\frac{r^p}{|\theta|^q}\right] \\ &\leq C(\theta_1^*)^2\left[\left(|\theta_1^*|^{\frac{p}{n}}\psi_1^\pm(1,\pm\theta_1^*) + h_2^\pm\frac{|\theta_1^*|^{\frac{p}{n}-q}}{qn-p}\right)\frac{r^p}{|\theta|^{p/n}} + \frac{h_2^\pm}{qn-p}\frac{r^p}{|\theta|^q}\right] \end{aligned}$$

for all $(r,\theta)\in\mathcal{S}_2$. Since $\psi_1^\pm(1,\pm\theta_1^*) = O((\theta_1^*)^{-1})$ as $\theta_1^*\downarrow 0$, it follows that for all $\theta_1^*>0$ sufficiently small

$$(T_1-T_2)\psi_2^\pm(r,\theta) \leq \frac{h_2^+\wedge h_2^-}{2}\frac{r^p}{|\theta|^q}$$

for all $(r,\theta)\in\mathcal{S}_2$. Therefore, for all $\theta_1^*>0$ small enough we have the bound

$$L\psi_2^\pm(r,\theta) \leq -\frac{h_2^\pm}{2}\frac{r^p}{|\theta|^q} + (L-T_1)\psi_2^\pm(r,\theta)$$

on $\mathcal{S}_2$.

To control the remaining term, first recall the definition of the region $\mathcal{S}_2$. Notice then that there exists a positive constant $C=C(\phi^*,r^*)$ such that on $\mathcal{S}_2$

$$(L-T_1)\psi_2^\pm(r,\theta) \leq C(r^*,\phi^*)\frac{r^p}{|\theta|^q}$$

where $C(r^*, \phi^*) > 0$ satisfies the following property: For every $\epsilon > 0$, there exists $K > 0$ such that for $\phi^* \wedge r^* \geq K$

$$C(r^*, \phi^*) \leq \epsilon.$$

Hence we may pick $K > 0$ large enough so that for $\phi^* \wedge r^* \geq K$

$$L\psi_2^\pm(r, \theta) \leq -c_2 \frac{r^p}{|\theta|^q} + d_2$$

for all $(r, \theta) \in \mathcal{S}_2$. Undoing the time change, we then determine the existence of positive constants C_2, D_2 such that on $\mathcal{S}_2$

$$\mathcal{L}\psi_2^\pm \leq -C_2 \frac{r^{p+n}}{|\theta|^q} + D_2 \tag{6.4}$$

Remark 6.5. Before proceeding onto the remaining regions, it is important to note that Corollary 4.18 and the relations (4.16) imply that for $m \in \{3, 4, \ldots, j+3\}$ and $l \in \{1, 2, \ldots, m-1\}$:

$$d_{l,m}^\pm = O((\phi^*)^{q_{l,m} - q_l}) \text{ as } \phi^* \to \infty \tag{6.6}$$

where the constant in the asymptotic formula above is independent of θ_1^*, η^* and r^*. The above fact will be helpful when controlling remainder terms in what follows.

Region $\mathcal{S}_m$, $m = 3, \ldots, j+2$. In the following computations, it is helpful to consult (3.31) and the remainder estimate immediately below it. For lack of better notation, we will also use $\psi_m^\pm$ to denote the function of (r, θ) determined by $\psi_m^\pm = \psi_m^\pm(r, \phi)$. Let

$$N = \left(\frac{\sigma^2}{2r^n} \partial_r^2\right)_{(r,\phi)} \tag{6.7}$$

and write

$$\begin{aligned} L\psi_m^\pm(r, \theta) &= L_{(r,\phi)} \psi_m^\pm(r, \phi) \\ &= T_m \psi_m^\pm(r, \phi) + (L_{(r,\phi)} - T_m - N)\psi_m^\pm(r, \phi) + N\psi_m^\pm(r, \phi) \\ &= -h_m^\pm \frac{r^{p_m}}{|\phi|^{q_m}} + (L_{(r,\phi)} - T_m - N)\psi_m^\pm(r, \phi) + N\psi_m^\pm(r, \phi) \end{aligned}$$

where each equality above is valid on $\mathcal{S}_m$. We first focus on estimating $(L_{(r,\phi)} - T_m - N)\psi_m^\pm(r, \phi)$ for all (r, ϕ) such that $(r, \theta(r, \phi)) \in \mathcal{S}_m$. Using the simple nature of the expression derived for $\psi_m^\pm$ as well as Remark 6.5, we note that the bound

$$|(L_{(s,\phi)} - T_m - N)\psi_m^\pm(r, \phi)| \leq C_1(r^*, \phi^*) \frac{r^{p_m}}{|\phi|^{q_m}}$$

holds on $\mathcal{S}_m$ where $C_1(r^*, \phi^*)$ is a constant which can be chosen to be a small as we wish by first picking $\phi^* > 0$ large and then picking $r^* > 0$ large. Therefore, making such choices we see that

$$L\psi_m^\pm(r, \theta) \leq -\frac{h_m^\pm}{2} \frac{r^{p_m}}{|\phi|^{q_m}} + N\psi_m^\pm(r, \phi).$$

To estimate the remaining term $N\psi_m^\pm(r, \phi)$, first recall that

$$\phi = r^{m-2}\theta + r^{m-3}c_2 + \cdots + c_{m-1}$$

and so we may write

$$N = \frac{\sigma^2}{2r^n} \left(\partial_r + [(m-2)r^{-1}\phi + r^{-1}Y(r)]\partial_\phi\right)^2$$

where Y is a polynomial in r of degree at most $m-3$. Using this allows us to obtain a similar bound

$$|N\psi_m^{\pm}(s,\phi)| \leq C_2(r^*,\phi^*)\frac{s^{p_m}}{|\phi|^{q_m}}$$

where $C_2(r^*,\phi^*)$ is a constant satisfying the same property as $C_1(r^*,\phi^*)$ above. Hence, we may choose $\phi^*>0$ large enough and then $r^*>0$ large enough so that for some positive constants c_m, d_m

$$L\psi_m^{\pm}(r,\theta) \leq -c_m\frac{r^{p_m}}{|\phi|^{q_m}} + d_m.$$

for all (s,ϕ) with $(r,\theta(r,\phi)) \in \mathcal{R}_m$. Undoing the time change, we see that there exist positive constants C_m, D_m such that on $\mathcal{S}_m$

$$\mathcal{L}\psi_m^{\pm}(r,\theta) \leq -C_m\frac{r^{p_m+n}}{|\phi|^{q_m}} + D_m. \tag{6.8}$$

Region $\mathcal{S}_{j+3}$. Here, we again use $\psi_{j+3}^{\pm}$ to also denote the function of (r,θ) defined by $\psi_{j+3}^{\pm}(r,\phi)$. The estimate in this region is nearly identical to the one that precedes it, except that the lower bound in the definition of $\mathcal{S}_{j+3}$ is slightly different depending on the parity of n. Nevertheless, we may essentially trace through the inequalities in the previous case to see that the same estimates hold, except that the constants C_1 and C_2 in this case depend on, in addition to ϕ^* and r^*, η^*. We may, however, still pick the parameters according to Remark 3.20 to arrive at the desired estimate on $\mathcal{S}_{j+3}$

$$\mathcal{L}\psi_{j+3}^{\pm}(r,\theta) \leq -C_{j+3}\frac{r^{p_{j+3}+n}}{|\phi|^{q_{j+3}}} + D_{j+3} \tag{6.9}$$

for some positive constants C_{j+3}, D_{j+3}.

Region $\mathcal{S}_{j+4}$. The estimates in this case are also very similar to the previous ones. In fact, fact they are a little easier since we include more terms of L in the approximating operator A. In what follows, we again use ψ_{j+4} to denote the function of (r,θ) determined by $\psi_{j+4}(r,\phi)$ where $(r,\phi)=(r,\phi_{j+3})$. Notice that for all (r,ϕ) with $(r,\theta(r,\phi)) \in \mathcal{S}_{j+4}$

$$\begin{aligned}
L\psi_{j+4}(r,\theta) &= L_{(r,\phi)}\psi_{j+4}(r,\phi)\\
&= A\psi_{j+4}(r,\phi) + (L_{(r,\phi)}-A-N)\psi_{j+4}(r,\phi) + N\psi_{j+4}(r,\phi)\\
&= -h_{j+4}r^{p_{j+4}} + (L_{(r,\phi)}-A-N)\psi_{j+4}(r,\phi) + N\psi_{j+4}(r,\phi)
\end{aligned}$$

where the operator N was defined in (6.7). Using the very same ideas in the previous two regions and recalling that $\gamma_1^{(j+3)}r^{-1}\partial_\phi$ is included in A when $n=2j+2$, we note that for all (r,ϕ) with $(r,\theta(r,\phi)) \in \mathcal{S}_{j+4}$

$$|(L_{(r,\phi)}-A-N)\psi_{j+4}(r,\phi)| + |N\psi_{j+4}(r,\phi)| \leq C(r^*,\eta^*,\phi^*)r^{p_{j+4}}$$

where $C(r^*,\eta^*,\phi^*)$ is a constant which can be made arbitrarily small by picking r^*,η^*,ϕ^* according to Remark 3.20. Thus choosing these parameters accordingly, we see that there exist constants $c_{j+4}, d_{j+4}>0$ such that

$$L\psi_{j+4}(r,\theta) \leq -c_{j+4}r^{p_{j+4}} + d_{j+4}$$

for all (s,ϕ) with $(r(s,\phi),\theta(s,\phi)) \in \mathcal{S}_{j+4}$. Undoing the time change, we determine the existence of positive constants C_{j+4}, D_{j+4} such that on $\mathcal{S}_{j+4}$

$$\mathcal{L}\psi_{j+4}(r,\theta) \leq -C_{j+4}r^{p_{j+4}+n} + D_{j+4}. \tag{6.10}$$

6.2. **Checking the specific Lyapunov bounds.** Here we show that we can pick parameters so that the conditions of Proposition 6.12 of Part I [6] are satisfied when $\gamma \in (n, 2n)$ is arbitrary. By the estimates of the previous section, all we need is the following proposition.

Proposition 6.11. *There exist positive constants l_i, u_i such that*

$$\begin{aligned} \psi_0(r,\theta) &= r^p && (r,\theta) \in \mathcal{S}_0 \\ l_1 r^p \leq & \psi_1(r,\theta) \leq u_1 r^p && (r,\theta) \in \mathcal{S}_1 \\ l_2 \frac{r^p}{|\theta|^{\frac{p}{n}}} \leq & \psi_2(r,\theta) \leq u_2 \frac{r^p}{|\theta|^q} && (r,\theta) \in \mathcal{S}_2 \\ l_m \frac{r^{p_{1,m}}}{|\phi|^{q_{1,m}}} \leq & \psi_m(r,\phi) \leq u_m \frac{r^{p_m}}{|\phi|^{q_m}} && (r,\theta(r,\phi)) \in \mathcal{S}_m \\ l_{j+4} r^{p_{1,j+4}} \leq & \psi_{j+4}(r,\phi) \leq u_{j+4} r^{p_{j+4}} && (r,\theta(r,\phi)) \in \mathcal{S}_{j+4} \end{aligned}$$

where $m = 3, \ldots, j+3$ and $(r,\phi) = (r,\phi_{j+3})$ in the last inequality.

Proof of Proposition 6.11. The lower bounds have already been established and the upper bounds follow directly from the expressions derived for each ψ_i. □

7. Optimality

Recalling that μ denotes the invariant measure of (2.1) and ρ its density with respect to Lebesgue measure on $\mathbf{R}^2$, in this section we prove Theorem 2.7. Before giving the precise details, let us give the intuitive idea behind the proof. To study the process z_t defined by (2.1) is a neighborhood of the point at infinity, it is convenient to make a substitution which maps the point at infinity to 0 and 0 to the point at infinity. There are many changes of variables which accomplish precisely this, but only one gives the desired bound on the invariant density: $w_t = 1/z_t^n$. The reason for this choice is that the drift of the process w_t is non-zero and bounded at $w = 0$. In particular, the invariant measure for the process w_t cannot possibly vanish nor can it blow up at $w = 0$. By construction, $|z|^{2n+2}\rho(z,\bar{z})$ when written in the variables $(w,\bar{w})$ is this invariant measure; hence, by positivity of this quantity as $|z| \to \infty$, Theorem 2.7 would then follow. However in the proof of Theorem 2.7, we will never actually make the substitution $w_t = 1/z_t^n$ described above because the inversion of the mapping $w = 1/z^n$ is multi-valued and this leads to unnecessary complications. Nonetheless, this transformation can be seen to motivate many of the manipulations performed.

In the proof of Theorem 2.7 we will need the following result which is a corollary of the proof of Theorem 2.5. The result gives uniform bounds in the initial condition on return times to large compact sets of the process, time-changed to accommodate the "substitution" $w_t = 1/z_t^n$, determined by the adjoint $\mathscr{L}^*$.

Corollary 7.1. *Consider the stochastic differential equation on $\mathbf{C} \setminus \{0\}$*

$$dz_t^* = -|z_t^*|^{-(n-1)} \left(\mathcal{P}(z_t^*, \overline{z_t^*}) + \frac{\sigma^2(n+1)}{\overline{z_t^*}} \right) dt + \sigma |z_t^*|^{-\frac{n-1}{2}}\, dB_t$$

where $\mathcal{P}(z,\bar{z}) = z^{n+1} + F(z,\bar{z})$, and F, n, σ and B_t are as in equation (2.1). For $\gamma > 0$, let $S_\gamma = \inf\{t > 0 : |z_t^*| \leq \gamma\}$. Then the stopped process $z_{t\wedge S_\gamma}^*$ is non-explosive and for each $\gamma > 0$ sufficiently large we have:

$$\sup_{z\in\mathbf{C}\setminus\{0\}} \mathbf{P}_z[S_\gamma = \infty] = 0.$$

Additionally, for each $t, \epsilon > 0$ there exists $\gamma > 0$ large enough so that

$$\inf_{z\in\mathbf{C}\setminus\{0\}} \mathbf{P}_z[S_\gamma \leq t] \geq 1 - \epsilon.$$

Proof of Corollary 7.1. Let $a \in \mathbf{C}$ be such that $a^n = -1$ and consider the process $az^*_{t\wedge S_\gamma}$. Our goal is to show that $az_{t\wedge S_\gamma}$ has a Lyapunov pair $(\Psi, \Psi^{1+\delta})$ for some $\delta > 0$. Non-explosivity will follow from Lemma 4.5 of Part I [6], and the remaining conclusions concerning the entrance times S_γ, $\gamma > 0$, will follow from Proposition 4.8 of [6]. Note first that the generator M of $az^*_{t\wedge S_\gamma}$ is of the following form when written in polar coordinates (r, θ):

$$M = rL$$

where L is of the form (3.1). Hence, because of the form of M, our Lyapunov function Ψ will be the same one constructed in Section 4. Upon replacing n by 1 in the inequalities (6.2), (6.3), (6.4), (6.8), (6.9), and (6.10), and then applying Proposition 6.8, we see that the chosen Ψ has the required local Lyapunov estimate for $(r, \theta) \in S_m$, $m = 0, 1, \ldots, j+4$:

$$(M\Psi)(r,\theta) \leq -C\Psi(r,\theta)^{1+\delta} + D$$

where C, D and δ are positive constants. Since the boundary flux contributions will have the appropriate sign, the result now follows. $\square$

Proof of Theorem 2.7. First note that the generator $\mathscr{L}$ of the process z_t has the following form when written in the variables $(z, \bar{z})$:

$$\mathscr{L} = \mathcal{P}(z,\bar{z})\partial_z + \overline{\mathcal{P}(z,\bar{z})}\partial_{\bar{z}} + \sigma^2\partial_z\partial_{\bar{z}}$$

where $\mathcal{P}(z,\bar{z}) = z^{n+1} + F(z,\bar{z})$. Let $\mathscr{L}^*$ denote the formal adjoint of $\mathscr{L}$. Motivated by the discussion of the substitution $w_t = 1/z_t^n$ at the beginning of the section, define $c(z,\bar{z}) = |z|^{2n+2}\rho(z,\bar{z})$ where ρ is the invariant probability density function with respect to Lebesgue measure on $\mathbf{R}^2$. Since $\mathscr{L}^*$ is elliptic and $\mathscr{L}^*\rho = 0$, observe that c is a smooth function everywhere since ρ is smooth everywhere. To see which equation c satisfies, write $\rho = c|z|^{-2n-2}$ and use the fact that $\mathscr{L}^*\rho = 0$ to see that

$$(|z|^{-2n-2}\mathcal{M}c)(z,\bar{z}) = 0 \;\text{ for } z \neq 0,$$

where $\mathcal{M}$ is of the form

$$\mathcal{M} = \mathscr{L}^* - \frac{\sigma^2(n+1)}{\bar{z}}\partial_z - \frac{\sigma^2(n+1)}{z}\partial_{\bar{z}} + f(z,\bar{z})$$

and the potential f satisfies

$$\begin{aligned} f(z,\bar{z}) = &-\partial_z(\mathcal{P}(z,\bar{z})) - \partial_{\bar{z}}(\overline{\mathcal{P}(z,\bar{z})}) \\ &+ \frac{(n+1)}{z}\mathcal{P}(z,\bar{z}) + \frac{(n+1)}{\bar{z}}\overline{\mathcal{P}(z,\bar{z})} + \frac{\sigma^2(n+1)^2}{|z|^2}. \end{aligned}$$

In particular, we also note that c solves the equation for $z \neq 0$

$$(|z|^{-n-1}\mathcal{M}c)(z,\bar{z}) = 0. \tag{7.2}$$

Using (7.2), we will now apply Feynman-Kac to obtain an expression for $c(z,\bar{z})$ that can be analyzed as $|z| \to \infty$.

Now consider the time-changed process $z^*_{t\wedge S_\gamma}$, $\gamma > 2$, introduced in Corollary 7.1. Observe that the generator of $z^*_{t\wedge S_\gamma}$ constitutes every term in $|z|^{-(n-1)}\mathcal{M}$ except for multiplication by the potential function $|z|^{-(n-1)}f(z,\bar{z})$ which is smooth in $(z,\bar{z})$ for $z \neq 0$ and satisfies

$$|z|^{-(n-1)}f(z,\bar{z}) = \mathcal{O}(1) \text{ as } |z| \to \infty.$$

Hence, in particular, $|z|^{-(n-1)}f(z,\bar{z})$ is bounded on the set $\{z \in \mathbf{C} : |z| \geq \gamma\}$ for all $\gamma > 0$. Let $S_{\gamma,n}$ be the first exit time of z^*_t from the annulus $A_{\gamma,n} = \{\gamma < |z| < n\}$. By Feynman-Kac, we have for $\gamma \geq 2$

$$c(z,\bar{z}) = \mathbf{E}_{(z,\bar{z})}c(z^*_{t\wedge S_{\gamma,n}}, \overline{z^*_{t\wedge S_{\gamma,n}}})e^{\int_0^{t\wedge S_{\gamma,n}} |z^*_s|^{-(n-1)}f(z^*_s,\overline{z^*_s})\,ds}, \;\; z \in A_{\gamma,n}.$$

By Corollary 7.1, we have that $z^*_{t\wedge S_\gamma}$ is non-explosive. Thus by Fatou's lemma, taking the $\liminf_{n\to\infty}$ of both sides of the above we obtain for $|z|\geq \gamma \geq 2$

$$c(z,\bar{z}) \geq \mathbf{E}_{(z,\bar{z})} c(z^*_{t\wedge S_\gamma}, \overline{z^*_{t\wedge S_\gamma}}) e^{\int_0^{t\wedge S_\gamma} |z_s^*|^{-(n-1)} f(z_s^*, \overline{z_s^*})\, ds}.$$

Applying Corollary 7.1 again, for $\gamma > 0$ large enough $S_\gamma < \infty$ almost surely. Hence, applying Fatou's lemma and taking the $\liminf_{t\to\infty}$ of both sides of the previous inequality we see that

$$c(z,\bar{z}) \geq \mathbf{E}_{(z,\bar{z})} c(z^*_{S_\gamma}, \overline{z^*_{S_\gamma}}) e^{\int_0^{S_\gamma} |z_s^*|^{-(n-1)} f(z_s^*, \overline{z_s^*})\, ds}. \tag{7.3}$$

We now bound the right-hand side of (7.3) from below. Since

$$(|z|^{-n-1}\mathcal{M}c)(z,\bar{z}) = 0 \text{ for } z\neq 0$$

and the operator $|z|^{-n-1}\mathcal{M}$ is elliptic for $z\neq 0$, there exists a constant $C(\gamma)>0$ such that

$$c(z,\bar{z}) \geq C(\gamma) > 0, \qquad |z| = \gamma.$$

Moreover, since $|z|^{-(n-1)}f$ is bounded for $|z|\geq\gamma$, there exists a constant $D(\gamma)>0$ such that

$$||z|^{-(n-1)} f(z,\bar{z})| \leq D(\gamma), \qquad |z|\geq \gamma.$$

Hence, we obtain

$$\begin{aligned} c(z,\bar{z}) &\geq C(\gamma)\mathbf{E}_{(z,\bar{z})} e^{-S_\gamma D(\gamma)} \\ &\geq C(\gamma)\mathbf{E}_{(z,\bar{z})} e^{-S_\gamma D(\gamma)} 1_{\{S_\gamma \leq t\}} \\ &\geq C(\gamma) e^{-tD(\gamma)} \mathbf{P}_{(z,\bar{z})}[S_\gamma \leq t] \end{aligned}$$

where the inequality above holds for all $\gamma, t>0$. Applying Corollary 7.1 once more, we see that for each $t>0$ there exists $\gamma>0$ such that

$$\inf_{|z|\geq\gamma} c(z,\bar{z}) > 0$$

finishing the result. □

8. Generalized Itô's Formula

In this section, we give a differently packaged proof of a weaker version of Peskir's extension of Tanaka's formula [7] which still affords the structure needed to build the Lyapunov functions contained in this paper and Part I [6]. Instead of making use of Tanaka's formula as in [7], we opt to mollify along interfaces where the function is not C^2 and then take limits. For convenience, we deal solely with the case of a time-homogeneous diffusion process ξ_t on $\mathbf{R}^m$ with generator

$$\mathscr{L} = \sum_{j=1}^d f^j(\xi)\partial_{\xi^j} + \sum_{i,j=1}^d \frac{1}{2} g^{ij}(\xi)\partial^2_{\xi^i\xi^j}$$

where f^i, g^{ij} are locally Lipschitz and the matrix (g^{ij}) is non-negative definite. Furthermore, assume that $\varphi \in C(\mathbf{R}^m : \mathbf{R})$ is such that

$$\varphi(x) = \begin{cases} \varphi_1(x) & x^m \leq b(x^1, x^2, \dots, x^{m-1}) \\ \varphi_2(x) & x^m \geq b(x^1, x^2, \dots, x^{m-1}) \end{cases}$$

where the φ_i's are C^2 on the domains above and $b\in C^2(\mathbf{R}^{m-1} : \mathbf{R})$. The case of finitely many non-intersecting boundaries is a simple consequence of the following result.

Theorem 8.1. *Let* $\tau_n = \inf\{t > 0 : |\xi_t| \geq n\}$*. Then for all* $\xi \in \mathbf{R}^m$*,* $n \in \mathbf{N}$ *and all bounded stopping times* υ*:*

(8.2)
$$\mathbf{E}_\xi \varphi(\xi_{\upsilon\wedge\tau_n}) = \varphi(\xi) + \mathbf{E}_\xi \int_0^{\upsilon\wedge\tau_n} \left[\tfrac{1}{2}\mathscr{L}\varphi(\xi_s^1, \dots, (\xi_s^m)^+) + \tfrac{1}{2}\mathscr{L}\varphi(\xi_s^1, \dots, (\xi_s^m)^-)\right] ds + \mathrm{Flux}(\xi, \upsilon, n)$$

where

$$(\mathscr{L}\varphi)(\xi^1, \dots, (\xi^m)^+) = \lim_{x^m \downarrow \xi^m} (\mathscr{L}\varphi)(\xi^1, \dots, \xi^{m-1}, x^m)$$
$$(\mathscr{L}\varphi)(\xi^1, \dots, (\xi^m)^-) = \lim_{x^m \uparrow \xi^m} (\mathscr{L}\varphi)(\xi^1, \dots, \xi^{m-1}, x^m)$$

and $\mathrm{Flux}(\xi, t, n)$ *satisfies the following properties:*

- *If* $\partial_{x^m}\varphi_2(x) - \partial_{x^m}\varphi_1(x) \leq 0$ *for all* $x \in \mathbf{R}^m$ *with* $x^m = b(x^1, \dots, x^{m-1})$*, then* $\mathrm{Flux}(\xi, \upsilon, n) \in (-\infty, 0]$ *and* $\mathrm{Flux}(\xi, t, k) \leq \mathrm{Flux}(\xi, s, n)$ *for* $s \leq t$ *and* $n \leq k$*.*
- *If* $\partial_{x^m}\varphi_2(x) - \partial_{x^m}\varphi_1(x) \geq 0$ *for all* $x \in \mathbf{R}^m$ *with* $x^m = b(x^1, \dots, x^{m-1})$*, then* $\mathrm{Flux}(\xi, \upsilon, n) \geq 0$ *and the maps* $\upsilon \mapsto \mathrm{Flux}(\xi, \upsilon, n)$*,* $n \mapsto \mathrm{Flux}(\xi, \upsilon, n)$ *are increasing.*

Proof of Theorem 8.1. Because we will stop the process ξ_t at time τ_n, without loss of generality we may assume that φ has compact support (e.g. in a ball centered at the origin with radius much larger than n). Let $\chi : \mathbf{R}^m \to \mathbf{R}$ be a smooth mollifier, set $\chi_\epsilon(\xi) = \epsilon^{-m}\chi(\epsilon^{-1}\xi)$ and define

$$\varphi_\epsilon(\xi) = \int_{\mathbf{R}^m} \chi_\epsilon(\xi - x)\varphi(x)\, dx.$$

Applying Dynkin's formula we have

$$\mathbf{E}_\xi \varphi_\epsilon(\xi_{\upsilon\wedge\tau_n}) - \varphi_\epsilon(\xi) = \mathbf{E}_\xi \int_0^{\upsilon\wedge\tau_n} (\mathscr{L}\varphi_\epsilon)(\xi_s)\, ds. \tag{8.3}$$

To obtain the desired formula, we begin computing partial derivatives of φ_ϵ. To keep expressions compact, we will use ∂_{ξ^j} to denote $\frac{\partial}{\partial \xi^j}$. Write

$$\varphi_\epsilon(\xi) = \int_{U^-} \chi_\epsilon(\xi - x)\varphi_1(x)\, dx + \int_{U^+} \chi_\epsilon(\xi - x)\varphi_2(x)\, dx.$$

where $U^- = \{x^m < b(x^1, \dots, x^{m-1})\}$ and $U^+ = \{x^m \geq b(x^1, \dots, x^{m-1})\}$. Integrate by parts once and use the fact that φ_1 and φ_2 agree on the boundary $\Gamma = \{x \in \mathbf{R}^m : x^m = b(x^1, \dots, x^{m-1})\}$ to see that

$$\begin{aligned}\partial_{\xi^j}\varphi_\epsilon(\xi) &= -\int_{U^-} \partial_{x^j}\chi_\epsilon(\xi - x)\varphi_1(x)\, dx - \int_{U^+} \partial_{x^j}\chi_\epsilon(\xi - x)\varphi_2(x)\, dx \\ &= \int_{U^-} \chi_\epsilon(\xi - x)\partial_{x^j}\varphi_1(x)\, dy + \int_{U^+} \chi_\epsilon(\xi - x)\partial_{x^j}\varphi_2(x)\, dx.\end{aligned}$$

Using the equality on the previous line, apply ∂_{ξ^i} to both sides and then integrate by parts in the same fashion to obtain

$$\begin{aligned}\partial^2_{\xi^i\xi^j}\varphi_\epsilon(\xi) &= \int_{U^-} \chi_\epsilon(\xi - x)\partial^2_{x^ix^j}\varphi_1(x)\, dx + \int_{U^+} \chi_\epsilon(\xi - x)\partial^2_{x^ix^j}\varphi_2(x)\, dx \\ &\quad + \int_\Gamma \chi_\epsilon(\xi - x)\big(\partial_{x^j}\varphi_2 - \partial_{x^j}\varphi_1\big)(x)\boldsymbol{\sigma}^i\, dS_\Gamma(x)\end{aligned} \tag{8.4}$$

where $\Gamma = \{x : x^m = b(x^1, \dots, x^{m-1})\}$ and $\boldsymbol{\sigma}^i$ is the i-th component of the unit surface normal vector $\boldsymbol{\sigma} = (-\nabla b(x), 1)/\sqrt{1 + |\nabla b(x)|^2}$ of Γ. We now claim that for $x \in \Gamma$ and $j = 1, \dots, m-1$

$$(\partial_{x^j}\varphi_2 - \partial_{x^j}\varphi_1)(x) = \big(\partial_{x^m}\varphi_2 - \partial_{x^m}\varphi_1\big)(x)\boldsymbol{\sigma}^j\sqrt{1 + |\nabla b(x)|^2}.$$

To prove this claim, for $x \in \mathbf{R}^m$ define

$$h(x^1, \dots, x^{m-1}) = \varphi(x^1, \dots, x^{m-1}, b(x^1, \dots, x^{m-1})).$$

Since $b \in C^1(\mathbf{R}^{m-1} : \mathbf{R})$, $h \in C^1(\mathbf{R}^{m-1} : \mathbf{R})$. Moreover, φ_i are C^2 on their closed domains of definition, each of which include the boundary Γ. Hence, computing derivatives we see that for $j = 1, \dots, m-1$ and $i = 1, 2$

$$\begin{aligned}\partial_{x^j} h(x^1, \dots, x^{m-1}) &= (\partial_{x^j}\varphi_i)(x^1, \dots, x^{m-1}, b(x^1, \dots, x^{m-1})) \\ &\quad + (\partial_{x^m}\varphi_i)(x^1, \dots, x^{m-1}, b(x^1, \dots, x^{m-1}))\partial_{x^j} b(x^1, \dots, x^{m-1})\end{aligned}$$

for $i = 1, 2$. Therefore

$$\begin{aligned}0 &= \partial_{x^j} h(x^1, \dots, x^{m-1}) - \partial_{x^j} h(x^1, \dots, x^{m-1}) \\ &= (\partial_{x^j}\varphi_2)(x^1, \dots, x^{m-1}, b(x^1, \dots, x^{m-1})) - (\partial_{x^j}\varphi_1)(x^1, \dots, x^{m-1}, b(x^1, \dots, x^{m-1})) \\ &\quad + (\partial_{x^m}\varphi_2)(x^1, \dots, x^{m-1}, b(x^1, \dots, x^{m-1}))\partial_{x^j} b(x^1, \dots, x^{m-1}) \\ &\quad - (\partial_{x^m}\varphi_1)(x^1, \dots, x^{m-1}, b(x^1, \dots, x^{m-1}))\partial_{x^j} b(x^1, \dots, x^{m-1}),\end{aligned}$$

from which the claim now follows. Since $\boldsymbol{\sigma}^m = 1/\sqrt{1 + |\nabla b|^2}$, the claim in particular allows us to write

$$\begin{aligned}\partial^2_{\xi^i\xi^j}\varphi_\epsilon(\xi) &= \int_{U^-} \chi_\epsilon(\xi - x)\partial^2_{x^i x^j}\varphi_1(x)\, dx + \int_{U^+} \chi_\epsilon(\xi - x)\partial^2_{x^i x^j}\varphi_2(x)\, dx \\ &\quad + \int_\Gamma \chi_\epsilon(\xi - x)\big(\partial_{x^m}\varphi_2 - \partial_{x^m}\varphi_1\big)(x)\boldsymbol{\sigma}^i\boldsymbol{\sigma}^j\sqrt{1 + |\nabla b(x)|^2}\, dS_\Gamma(x).\end{aligned} \tag{8.5}$$

for $i, j = 1, 2, \dots, m$.

Let us now see what the computations above tell us. Letting $*$ denote convolution, we can now write (8.3) as

$$\begin{aligned}&\mathbf{E}_\xi\varphi_\epsilon(\xi_{\upsilon\wedge\tau_n}) - \varphi_\epsilon(\xi) \\ &\quad - \mathbf{E}_\xi \sum_{j=1}^m \int_0^{\upsilon\wedge\tau_n} f^j(\xi_s)(\chi_\epsilon * 1_{U^-}\partial_{\xi^j}\varphi_1)(\xi_s) + f^j(\xi_s)(\chi_\epsilon * 1_{U^+}\partial_{\xi^j}\varphi_2)(\xi_s)\, ds \\ &\quad - \frac{1}{2}\mathbf{E}_\xi \sum_{i,j=1}^m \int_0^{\upsilon\wedge\tau_n} g^{ij}(\xi_s)(\chi_\epsilon * 1_{U^-}\partial_{\xi^i\xi^j}\varphi_1)(\xi_s) + g^{ij}(\xi_s)(\chi_\epsilon * 1_{U^+}\partial_{\xi^i\xi^j}\varphi_2)(\xi_s)\, ds \\ &\quad = \frac{1}{2}\mathbf{E}_\xi \int_0^{\upsilon\wedge\tau_n}\int_\Gamma \big(\partial_{x^m}\varphi_2 - \partial_{x^m}\varphi_1\big)(x)\chi_\epsilon(\xi_s - x)\sqrt{1 + |\nabla b(x)|^2}\sum_{i,j=1}^m g^{ij}(\xi_s)\boldsymbol{\sigma}^i\boldsymbol{\sigma}^j\, dS_\Gamma(x) ds\end{aligned} \tag{8.6}$$

Since f^i, g^{ij} are locally bounded, by dominated convergence we may pass the the limit as $\epsilon \downarrow 0$ through all integrals on the lefthand side to see that

$$\begin{aligned}&\mathbf{E}_\xi\varphi(\xi_{\upsilon\wedge\tau_n}) - \varphi(\xi) - \tfrac{1}{2}\mathbf{E}_\xi \int_0^{\upsilon\wedge\tau_n}[(\mathscr{L}\varphi)(\xi_s^1, \dots, (\xi_s^d)^+)]\, ds - \tfrac{1}{2}\mathbf{E}_\xi \int_0^{\upsilon\wedge\tau_n}[(\mathscr{L}\varphi)(\xi_s^1, \dots, (\xi_s^d)^-)]\, ds \\ &= \lim_{\epsilon\downarrow 0}\frac{1}{2}\mathbf{E}_\xi \int_0^{\upsilon\wedge\tau_n}\int_\Gamma \big(\partial_{x^m}\varphi_2 - \partial_{x^m}\varphi_1\big)(x)\chi_\epsilon(\xi_s - x)\sqrt{1 + |\nabla b(x)|^2}\sum_{i,j=1}^m g^{ij}(\xi_s)\boldsymbol{\sigma}^i\boldsymbol{\sigma}^j\, dS_\Gamma(x) ds \\ &:= \mathrm{Flux}(\xi, \upsilon, n).\end{aligned}$$

To see that $\mathrm{Flux}(\xi, \upsilon, n)$ has the claimed properties, note that since the matrix (g^{ij}) is non-negative we have that

$$\chi_\epsilon(\xi_s - x)\sqrt{1 + |\nabla b(x)|^2}\sum_{i,j} g^{ij}(\xi_s)\boldsymbol{\sigma}^i\boldsymbol{\sigma}^j \geq 0.$$

Also, the surface measure dS_Γ is a nonnegative measure. In particular, $\text{Flux}(\xi, \upsilon, n)$ satisfies all claimed properties of the result. ☐

9. Conclusions

The techniques developed in this and its accompanying work provide a general framework for constructing a Lyapunov function well adapted to the dynamics of a particular problem. This systematic approach began in [1]. Here, however, a significant number of advances have been made, allowing us to both cover a much larger class of problems and simplify many details in the analysis. In particular, the use of a generalized Tanaka formula [7] greatly simplifies the patching together of the piecewise-defined Lyapunov functions when compared to the treatments of the similar situations in [1, 2, 4].

A few considerations remain incomplete in this work. Section 2.4 in Part I [6] makes a compelling argument supported by numerical simulations for the scaling of large excursions. It would be interesting to add the missing details, producing a rigorous argument. Related to this, it would be also interesting to see if one could scale time and space so that in the limit of vanishing noise, the system converges to a random distribution on the loops of the underlying deterministic system. The limiting loop system would be in the spirit of the random spider/graph processes considered in [3] and subsequent works. A possible path one could take to achieve this (and also some of the results of this paper) is to work in the coordinate variable $w = 1/z^n$ as done in Section 7. One could then obtain path properties of the diffusion $w_t = 1/z_t^n$ near the origin in the w-plane by controlling the martingale part using the exponential martingale inequality. If present, the lower-order terms in the drift would then have to be dealt with, perhaps by using time-changes and Girsanov transformations and/or further substitutions inspired by those made in Section 3 of this paper.

There also is a number of possible directions for generalization. Here we have only considered complex polynomials whose highest order term is the monomial az^{n+1}. More generally, one could consider leading-order monomial terms of the form $az^k\bar{z}^j$ where $k+j = n+1$. If $k > j+1$, then the system with noise added can be proven to be stable by essentially the same arguments used in this and its companion paper [6]. In this case, the invariant measure will again have polynomial decay at infinity. If $k = j$, then the system is trivially stable if $a < 0$ and trivially unstable if $a > 0$. In this case, the norm-squared $|z|^2$ is easily shown to be a Lyapunov function if $a < 0$. If $j < k$ then the deterministic flow rotates towards the unstable directions and not away from them as was the case when $k < j$. Here one expects to be able to prove that they system blows-up with probability one. More interesting is the case when the leading order monomial is replaced by a polynomial made up of terms which all have homogeneity $n + 1$ under the radial scaling $z \mapsto \lambda z$. This will produce a richer collection of possibilities. Nonetheless, we expect the ideas contained in these notes to be very useful in determining and proving stability properties when noise is added.

Another possible direction of generalization would be to consider state-dependent noise, e.g. $\sigma\, dB_t \mapsto \sigma(z, \bar{z})\, dB_t$ where $\sigma(z, \bar{z})$ is a suitable polynomial. In many cases, analysis of the resulting stochastic system should be possible so long as $\sigma(z, \bar{z})$ does not grow too fast at infinity (relative to the leading-order drift term az^{n+1}). Also, the analysis may be greatly simplified in some cases by transforming to an equation with additive noise by a time change and/or substitution. A more difficult direction of generalization would be to consider higher dimensional unstable ODEs under the addition of noise. Here the geometry of the underlying, deterministic dynamics can be quite complicated, if not chaotic. In this work, we relied on the simplicity of the underling dynamics in our analysis. Understanding how different regions patch together could be much more complicated, if not intractable, in higher dimensions. The most interesting and wide-open direction to pursue would be to consider an unstable deterministic PDE and show that it stabilizes under the addition of noise.

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References

[1] Avanti Athreya, Tiffany Kolba, and Jonathan C. Mattingly. Propagating Lyapunov functions to prove noise-induced stabilization. *Electron. J. Probab.*, 17:no. 96, 38, 2012.

[2] Katarína Bod̆ová and Charles R. Doering. Noise-induced statistically stable oscillations in a deterministically divergent nonlinear dynamical system. *Commun. Math. Sci.*, 10(1):137–157, 2012.

[3] Mark I. Freidlin and Alexander D. Wentzell. *Random perturbations of dynamical systems*, volume 260 of *Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]*. Springer, Heidelberg, third edition, 2012. Translated from the 1979 Russian original by Joseph Szücs.

[4] Krzysztof Gawędzki, David P. Herzog, and Jan Wehr. Ergodic properties of a model for turbulent dispersion of inertial particles. *Comm. Math. Phys.*, 308(1):49–80, 2011.

[5] David P. Herzog. *Geometry's fundamental role in the stability of stochastic differential equations*. ProQuest LLC, Ann Arbor, MI, 2011. Thesis (Ph.D.)–The University of Arizona.

[6] David P. Herzog and Jonathan C. Mattingly. Noise–induced stabilization of planar flows I. *Submitted*, 2013.

[7] Goran Peskir. A change-of-variable formula with local time on surfaces. In *Séminaire de Probabilités XL*, volume 1899 of *Lecture Notes in Math.*, pages 69–96. Springer, Berlin, 2007.

[8] M. Scheutzow. Stabilization and destabilization by noise in the plane. *Stochastic Anal. Appl.*, 11(1):97–113, 1993.