

# ON THE CONFORMAL VOLUME OF 2-TORI

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ABSTRACT. This note provides a proof of a 1985 conjecture of Montiel and Ros about the conformal volume of tori.

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## 1. INTRODUCTION

**1.1. Conformal volume.** In a celebrated paper [3], Li and Yau introduced and studied the notion of *conformal volume* of a compact conformal manifold  $(M, [g])$ .

To recall this definition, let  $G_n$  be the group of conformal automorphisms of the standard unit  $n$ -sphere  $(S^n, g_0)$ ; it is known to be isomorphic to  $O(n+1, 1)$ . Given a smooth mapping  $f : M^m \rightarrow S^n$ , let  $V(f)$  denote the volume of the (possibly degenerate) metric  $f^*(g_0)$  on  $M$ . One then defines the *conformal volume* of  $f$  to be

$$V_c(f) = \sup_{\psi \in G_n} V(\psi \circ f).$$

(This is also what Gromov [2] has called the *visual volume* of  $f$ .) If  $f$  is an immersion at some point, then  $V_c(f) \geq V(S^m, g_0) > 0$ . Indeed, if  $m > 1$  (or  $m \geq 1$  if  $M$  is compact), one has  $V_c(f) > V(S^m, g_0)$  unless  $f(M)$  lies in a totally umbilic  $m$ -sphere in  $S^n$ . See [1, Proposition 1].)

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Now, every conformal automorphism  $\psi : S^n \rightarrow S^n$  extends uniquely to a conformal automorphism of the  $(n+1)$ -ball  $B^{n+1} \subset \mathbb{E}^{n+1}$  of which  $S^n$  is the boundary. If  $dV_f$  is the volume form on  $M$  induced by  $f : M^m \rightarrow S^n$ , then one finds that

$$dV_{\psi \circ f} = \frac{(1 - \psi(0) \cdot \psi(0))^{m/2}}{(1 - \psi(0) \cdot f)^m} dV_f.$$

Thus, one has the alternative formula

$$V_c(f) = \sup_{a \in B^{n+1}} \int_M \frac{(1 - a \cdot a)^{m/2}}{(1 - a \cdot f)^m} dV_f.$$

When  $M^m$  is endowed with a conformal structure  $[g]$ , Li and Yau define the *conformal  $n$ -volume* of  $(M, [g])$  to be

$$V_c(M, [g], n) = \inf \{ V_c(f) \mid f : (M, [g]) \rightarrow (S^n, [g_0]) \text{ is branched conformal} \}.$$

Of course,  $V_c(M, [g], n)$  is only defined when  $n$  is large enough for there to exist a branched conformal immersion  $f : M^m \rightarrow S^n$ . Since one has  $V_c(M, [g], n) \geq V_c(M, [g], n+1)$  when both quantities are defined, one can define the *conformal volume* of  $(M, [g])$  to be

$$V_c(M, [g]) = \lim_{n \rightarrow \infty} V_c(M, [g], n).$$

Naturally, in the compact case, one has  $V_c(M^m, [g]) \geq V(S^m, g_0)$ .

In [3], Li and Yau give a number of applications of this notion of conformal volume, showing, in particular, how it is related to the Willmore problem, eigenvalue estimates, and a number of other interesting differential geometric issues. They developed methods of estimating the conformal volume, but were able explicitly to compute it only in a small number of cases.

**1.2. The case of 2-tori.** Montiel and Ros [5] carried out a careful study of the conformal volume in the case of a torus  $T_\tau = \mathbb{C}/\Lambda_\tau$  where  $\Lambda_\tau$  is the discrete lattice in  $\mathbb{C} \simeq \mathbb{R}^2$  generated by 1 and  $\tau = x + iy$ , where  $0 \leq x \leq \frac{1}{2}$  and  $y \geq \sqrt{1 - x^2}$ , and the metric on the torus is  $|dz|^2 = dz \odot d\bar{z} = du^2 + dv^2$ . (Every metric on the 2-dimensional torus is conformal to this metric on  $\mathbb{C}/\Lambda_\tau$  for some unique  $\tau = x + iy$  satisfying these inequalities.)

For this conformal torus, they constructed a conformal (in fact, homothetic) embedding  $f_\tau : T_\tau \rightarrow S^5$  with volume

$$(1.1) \quad V(f_\tau) = \frac{4\pi^2 y}{y^2 + x^2 - x + 1},$$

and they showed that

$$(1.2) \quad V_c(T_\tau, [|dz|^2]) \geq \frac{4\pi^2 y}{y^2 + x^2 - x + 1}.$$

Since, the right hand side of (1.1) tends to zero as  $y$  goes to  $\infty$ , the inequality (1.2) must become strict for  $y$  sufficiently large.

Montiel and Ros showed [5, Theorem 8] that, when  $x = 0$  and  $1 \leq y \leq \sqrt{5/3}$ , equality does, in fact, hold in (1.2). Moreover, they conjectured that equality holds in (1.2) as long as  $(x - \frac{1}{2})^2 + y^2 < \frac{9}{4}$ , i.e.,  $|\tau - \frac{1}{2}| < \frac{3}{2}$ . (See Remark 1 for their motivation.)

In this note (see Theorem 2), I prove (a slightly stronger version of) their conjecture, i.e.,

$$(1.3) \quad V_c(T_\tau, [dz^2]) = \frac{4\pi^2 y}{y^2 + x^2 - x + 1} \quad \text{when } |\tau - \frac{1}{2}| \leq \frac{3}{2}.$$

The proof proceeds in two steps: First, I show, as did Montiel and Ros, that, for all  $\tau$ , one has  $V_c(T_\tau, [dz^2]) \geq V(f_\tau)$ . Second, I show that  $V_c(f_\tau) = V(f_\tau)$  when  $|\tau - \frac{1}{2}| \leq \frac{3}{2}$ , i.e., that, for such  $\tau$ ,

$$(1.4) \quad \int_{T_\tau} \frac{(1 - a \cdot a)}{(1 - a \cdot f_\tau)^2} dV_{f_\tau} \leq \frac{4\pi^2 y}{y^2 + x^2 - x + 1} \quad \text{for all } a \in \mathbb{E}^6 \text{ with } a \cdot a < 1.$$

Since, by definition,

$$V_c(f_\tau) \geq V_c(T_\tau, [dz^2], n) \geq V_c(T_\tau, [dz^2])$$

for  $n \geq 5$ , the conjecture follows.

*Remark 1* (Motivation for the conjecture). By a Taylor series expansion, Montiel and Ros showed that the function of  $a$  on the left hand side of (1.4) attains a local maximum at  $a = 0 \in \mathbb{E}^6$  when  $|\tau - \frac{1}{2}| < \frac{3}{2}$  and that it does not have a local maximum there when  $|\tau - \frac{1}{2}| > \frac{3}{2}$ .

However, they were not able to estimate the integral accurately enough to show that  $a = 0$  actually is the maximum of the function on the unit ball in  $\mathbb{E}^6$  when  $|\tau - \frac{1}{2}| < \frac{3}{2}$ . My contribution to proving their conjecture is to show that, when  $|\tau - \frac{1}{2}| \leq \frac{3}{2}$ , the function on the left hand side of (1.4) does indeed attain its supremum on the unit ball in  $\mathbb{E}^6$  at  $a = 0$ .

## 2. BALANCED VOLUME

Throughout this section,  $\tau \in \mathbb{C}$  denotes  $x + iy$  where  $0 \leq x \leq \frac{1}{2}$  and  $y \geq \sqrt{1 - x^2}$ , and  $\Lambda \subset \mathbb{C}$  is the lattice generated by 1 and  $\tau$ . Let  $T = \mathbb{C}/\Lambda$  and let  $dA = \frac{i}{2} dz \wedge d\bar{z}$  be the area form on  $T$ . (I will write  $z = u + iv$  to denote the real and imaginary parts of the coordinate  $z$  on  $\mathbb{C}$ . While  $z$  is not a well-defined coordinate on  $T$ , the differential  $dz$  is well-defined on  $T$ , and, consequently, for any vector-valued function  $f$  on  $T$ , one has  $df = f_z dz + f_{\bar{z}} d\bar{z}$ , so that the quantities  $f_z$  and  $f_{\bar{z}}$  are also well-defined.)

A  $C^\infty$  map  $\phi : T \rightarrow S^n$  is said to be *dA-balanced* if

$$(2.1) \quad \int_T \phi \, dA = 0 \in \mathbb{E}^{n+1}.$$

The map  $\phi$  is said to be *weakly conformal* if  $\phi_z \cdot \phi_z = 0$ . When  $\phi$  is weakly conformal, set

$$(2.2) \quad V(\phi) = 2 \int_T \phi_z \cdot \phi_{\bar{z}} \, dA.$$

Note that, when  $\phi$  is weakly conformal, one has  $V(\phi) > 0$  unless  $\phi$  is constant.

*Definition 1.* The *balanced volume* of  $T$  is the infimum  $V_b(n, T)$  of the numbers  $V(\phi)$  where  $\phi$  ranges over the set of dA-balanced, weakly conformal smooth maps  $\phi : T \rightarrow S^n$ .

The following result was proved by Montiel and Ros [5, Proposition 5] (and, independently, by me) in 1985:

**Theorem 1.** *The following formulae hold*

- $V_b(2, T) = 8\pi$  for all  $\tau = x + iy$ .
- $V_b(n, T) = \frac{4\pi^2 y}{y^2 + 1}$  for all  $n \geq 3$  if  $x = 0$ .
- $V_b(n, T) = \frac{4\pi^2 y}{y^2 + x^2 - x + 1}$  for all  $n \geq 5$ .

Moreover, there is a dA-balanced, weakly conformal map  $f_\tau : T \rightarrow S^5 \subset S^n$ , unique up to rigid motions in  $S^n$ , for which  $V(f_\tau) = V_b(5, T)$ . This  $f_\tau$  is homothetic and, when  $x = 0$ , it is a linearly full immersion in an  $S^3 \subset S^5$  while, when  $x > 0$ , it is a linearly full immersion in  $S^5$ .

*Remark 2.* I do not know the values of  $V_b(3, T)$  or  $V_b(4, T)$  when  $x > 0$ , but, since  $V_b(n, T) \geq V_b(n+1, T)$ , one has the inequalities

$$V_b(3, T) \geq V_b(4, T) \geq V_b(5, T) = \frac{4\pi^2 y}{y^2 + x^2 - x + 1}.$$

*Proof.* The case  $n = 2$  is special because a dA-balanced, weakly conformal map  $\phi : T \rightarrow S^2$  is either holomorphic or anti-holomorphic and hence  $V(\phi) = 4|d|\pi$  where  $d$  is the degree of  $\phi$ , which necessarily satisfies  $|d| \geq 2$ . (The degree  $d$  cannot be zero, since, in this case,  $\phi$  is constant and hence cannot be dA-balanced. Moreover,  $d \neq \pm 1$  since such a map would necessarily be a biholomorphism between  $T$  and  $S^2$ .) Thus,  $V(\phi) \geq 8\pi$ , so  $V_b(2, T) \geq 8\pi$ . On the other hand, every torus  $T$  does admit a holomorphic mapping  $\phi : T \rightarrow S^2$  with degree  $d = 2$ . Moreover, it is possible to find such a  $\phi$  that is dA-balanced. Thus,  $V_b(2, T) = 8\pi$ .

Returning to general  $n$ , let  $\phi : T \rightarrow S^n$  be a dA-balanced, weakly conformal mapping. The proof will now proceed by expanding  $\phi$  in its Fourier series.

To do this, let  $\Lambda^* \subset \mathbb{C}$  be the lattice dual to  $\Lambda$  via the pairing

$$\langle \xi, z \rangle = \Re(\xi \bar{z}) = \frac{1}{2}(\xi \bar{z} + \bar{\xi} z).$$

Then  $\Lambda^*$  is generated as a lattice by  $\{i/y, 1 - ix/y\}$ .

The Fourier series expansion of  $\phi$  then takes the form

$$\phi = \sum_{\xi \in \Lambda^*} \phi_\xi e^{2\pi i \langle \xi, z \rangle}$$

where  $\phi_0 = 0$  (since  $\phi$  is dA-balanced) and  $\phi_{-\xi} = \overline{\phi_\xi}$  (since  $\phi$  is real-valued). Since  $\phi \cdot \phi \equiv 1$ , one has the identities

$$1 = \sum_{\xi \in \Lambda^*} |\phi_\xi|^2 \quad \text{and} \quad 0 = \sum_{\xi \in \Lambda^*} \phi_\xi \cdot \overline{\phi_{\xi-\eta}}$$

for all  $\eta \in \Lambda^* \setminus \{0\}$ . The equation  $\phi_z \cdot \phi_z \equiv 0$  (weak conformality) becomes

$$0 = \sum_{\xi \in \Lambda^*} \xi(\xi-\eta) \phi_\xi \cdot \overline{\phi_{\xi-\eta}}$$

for all  $\eta \in \Lambda^* \setminus \{0\}$ . In particular, setting  $\eta = 0$ , one finds<sup>1</sup>

$$(2.3) \quad 0 = \sum_{\xi \in \Lambda^*} \xi^2 |\phi_\xi|^2.$$

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<sup>1</sup>It will be seen that  $V(\phi) \geq V_b(5, T)$  holds for all dA-balanced maps  $\phi : T \rightarrow S^n$  that satisfy (2.3), which, of course, is equivalent to the very weak condition  $\int_T (\phi_z \cdot \phi_z) dA = 0$ .

Meanwhile, computation yields

$$V(\phi) = 2\pi^2 y \sum_{\xi \in \Lambda^*} |\xi|^2 |\phi_\xi|^2,$$

and it is this latter quantity for which I seek to determine a lower bound.

Let

$$\Lambda_+^* = \{\xi \in \Lambda^* \mid \Re(\xi) > 0 \text{ or } (\Re(\xi) = 0 \text{ and } \Im(\xi) > 0)\},$$

so that  $\Lambda^* = \Lambda_+^* \cup \{0\} \cup (-\Lambda_+^*)$ .

For  $\xi \in \Lambda_+^*$ , set  $p_\xi = 2|\phi_\xi|^2$ . Thus, one seeks the minimum value of

$$L = \sum_{\xi \in \Lambda_+^*} |\xi|^2 p_\xi$$

subject to the conditions that  $p_\xi \geq 0$  when  $\xi \in \Lambda_+^*$  while

$$1 = \sum_{\xi \in \Lambda_+^*} p_\xi \quad \text{and} \quad 0 = \sum_{\xi \in \Lambda_+^*} \xi^2 p_\xi.$$

Geometrically, this amounts to the following: For  $\xi \in \Lambda_+^*$ , define

$$\hat{\xi} = (\Re(\xi^2), \Im(\xi^2), |\xi|^2) \in N_+ \subset \mathbb{R}^3,$$

where  $N_+ = \{(a, b, \sqrt{a^2+b^2}) \in \mathbb{R}^3 \mid (a, b) \neq (0, 0)\}$  is the upper right circular cone in  $\mathbb{R}^3$ , and let  $C$  be the convex hull in  $\mathbb{R}^3$  of the set  $\{\hat{\xi} \mid \xi \in \Lambda_+^*\} \subset N_+$ . Thus,

$$C = \left\{ \sum_{\xi \in \Lambda_+^*} p_\xi \hat{\xi} \mid p_\xi \geq 0 \text{ and } \sum_{\xi \in \Lambda_+^*} p_\xi = 1 \right\}.$$

Then one seeks the infimum of the numbers  $\ell \in \mathbb{R}$  such that  $(0, 0, \ell)$  lies in  $C$ .

To determine this, let  $\xi_{m,n} = m(1 - i x/y) + n(i/y)$ . Thus,  $\xi_{m,n}$  lies in  $\Lambda_+^*$  if and only if either  $m > 0$  or  $m = 0$  and  $n > 0$ . Let  $P$  be the plane in  $\mathbb{R}^3$  that contains the three (non-collinear!) points  $\hat{\xi}_{0,1}$ ,  $\hat{\xi}_{1,0}$ , and  $\hat{\xi}_{1,1}$ .

I claim that  $\hat{\xi}_{m,n}$  lies strictly above  $P$  for all  $\xi_{m,n} \in \Lambda_+^*$  except when  $(m, n)$  is one of  $(0, 1)$ ,  $(1, 0)$ , or  $(1, 1)$ . (Here, ‘above’ means on the opposite side of  $P$  from  $(0, 0, 0) \in \mathbb{R}^3$ .) To see why, consider the determinant  $\Delta(m, n)$  of the matrix

$$\begin{pmatrix} \hat{\xi}_{0,1} & 1 \\ \hat{\xi}_{1,0} & 1 \\ \hat{\xi}_{1,1} & 1 \\ \hat{\xi}_{m,n} & 1 \end{pmatrix} = \begin{pmatrix} -1/y^2 & 0 & 1/y^2 & 1 \\ 1 - x^2/y^2 & -2x/y & 1 + x^2/y^2 & 1 \\ 1 - (x-1)^2/y^2 & -2(x-1)/y & 1 + (x-1)^2/y^2 & 1 \\ m^2 - (mx-n)^2/y^2 & -2m(mx-n)/y & m^2 + (mx-n)^2/y^2 & 1 \end{pmatrix}.$$

One finds that  $\Delta(m, n) = (4/y^3)(1 - (m^2 - mn + n^2))$ . The claim now follows by noting that  $\Delta(0, 0) > 0$ ; that  $\Delta(m, n) = 0$  if and only  $(m, n)$  is one of  $\pm(0, 1)$ ,  $\pm(1, 0)$ , or  $\pm(1, 1)$ ; and that  $\Delta(m, n) < 0$  otherwise.

Set  $\xi_1 = \xi_{0,1}$ ,  $\xi_2 = \xi_{1,0}$ , and  $\xi_3 = \xi_{1,1}$ . It now follows that the triangle  $T$  with vertices  $\hat{\xi}_1$ ,  $\hat{\xi}_2$ ,  $\hat{\xi}_3$  is a face of  $C$ . Moreover, one finds that the ray  $\{(0, 0, \ell) \mid \ell \geq 0\}$  meets  $T$  uniquely in the point

$$r_1 \hat{\xi}_1 + r_2 \hat{\xi}_2 + r_3 \hat{\xi}_3 = (0, 0, S(x, y))$$

where  $r_1 \geq r_2 \geq r_3 \geq 0$  satisfy  $r_1 + r_2 + r_3 = 1$  and are given by

$$r_1 = \frac{y^2 + x^2 - x}{(y^2 + x^2 - x + 1)}, \quad r_2 = \frac{1 - x}{(y^2 + x^2 - x + 1)}, \quad r_3 = \frac{x}{(y^2 + x^2 - x + 1)},$$

while

$$S(x, y) = \frac{2}{(y^2 + x^2 - x + 1)}.$$

Thus, it follows that

$$V(\phi) \geq \frac{4\pi^2 y}{(y^2 + x^2 - x + 1)},$$

and that, moreover, equality holds only if  $\phi_\xi = 0$  for  $\xi \notin \{\pm\xi_1, \pm\xi_2, \pm\xi_3\}$  while  $|\phi_{\xi_i}|^2 = \frac{1}{2}r_i$  for  $i = 1, 2, 3$ .

To show that this lower bound is actually achieved by a dA-balanced weakly conformal immersion, one must show that it is possible to choose the  $\phi_{\xi_i}$  for  $i = 1, 2, 3$  so that the resulting  $\phi$  is actually weakly conformal, i.e., so that  $\phi_z \cdot \phi_z \equiv 0$ . It is easy to see that necessary and sufficient conditions for this are the relations

$$\phi_{\xi_i} \cdot \phi_{\xi_j} = 0 \text{ for all } i \text{ and } j$$

while

$$\phi_{\xi_i} \cdot \overline{\phi_{\xi_j}} = 0 \text{ for all } i \neq j.$$

Of course, one must also require  $|\phi_{\xi_i}|^2 = \frac{1}{2}r_i$  for  $i = 1, 2, 3$ . These equations can be satisfied (uniquely up to orthogonal transformation in  $\mathbb{E}^{n+1}$  because all of the inner products have been specified) by the vectors

$$\begin{aligned} \phi_{\xi_1} &= \frac{1}{2}\sqrt{r_1}(1, i, 0, 0, 0, 0), \\ \phi_{\xi_2} &= \frac{1}{2}\sqrt{r_2}(0, 0, 1, i, 0, 0), \\ \phi_{\xi_3} &= \frac{1}{2}\sqrt{r_3}(0, 0, 0, 0, 1, i). \end{aligned}$$

Using these vectors, one finds that the corresponding map  $f_\tau : T \rightarrow S^5$  given by

$$f_\tau(z) = \sum_{j=1}^3 \left( \phi_{\xi_j} e^{2\pi i \langle \xi_j, z \rangle} + \overline{\phi_{\xi_j}} e^{-2\pi i \langle \xi_j, z \rangle} \right)$$

satisfies

$$|df_\tau|^2 = \frac{4\pi^2}{(y^2 + x^2 - x + 1)} |dz|^2,$$

so this  $f_\tau$  is a homothetic immersion of  $T = \mathbb{C}/\Lambda$  with its flat conformal metric, and its induced area has the minimal value

$$V(f_\tau) = \frac{4\pi^2 y}{(y^2 + x^2 - x + 1)} = V_b(5, T).$$

The uniqueness up to orthogonal transformation of this  $V_b$ -minimizer is now clear.  $\square$

*Remark 3* (Estimating the conformal volume). The map  $f_\tau$  constructed at the end of the proof of Theorem 1 is, of course, the map that Montiel and Ros denote as  $\psi_{xy}$  (where  $\tau = x + iy$ ) in [5]. An immediate consequence is their Corollary 6, i.e., that

$$V_c(T, [|dz|^2]) \geq V(f_\tau) = \frac{4\pi^2 y}{(y^2 + x^2 - x + 1)}.$$

## 3. CONFORMAL VOLUME OF THE FUNDAMENTAL IMMERSION

In this section, I study the conformal volume of the conformal immersion of minimal dA-balanced volume of the torus  $T = \mathbb{C}/\Lambda$ , where  $\Lambda \subset \mathbb{C}$  is the lattice generated by 1 and  $\tau = x + iy$  where  $0 \leq x \leq \frac{1}{2}$  and  $y \geq \sqrt{1-x^2}$ . The three shortest elements in the dual lattice  $\Lambda^*$  are  $\xi_1 = i/y$ ,  $\xi_2 = 1 - ix/y$ , and  $\xi_3 = \xi_1 + \xi_2 = 1 - i(x-1)/y$ .

I will use  $z = u + iv$  as the standard coordinate on  $\mathbb{C}$ , so that the differential  $dz$  is well-defined on  $T$ , with induced flat metric  $|dz|^2 = dz \odot d\bar{z}$ .

As was explained in §2, the fundamental immersion  $f_\tau : T \rightarrow S^5 \subset \mathbb{E}^6$  is given by

$$(3.1) \quad f_\tau(u + iv) = \begin{pmatrix} \sqrt{r_1} \cos(2\pi v/y) \\ \sqrt{r_1} \sin(2\pi v/y) \\ \sqrt{r_2} \cos(2\pi(u - vx/y)) \\ \sqrt{r_2} \sin(2\pi(u - vx/y)) \\ \sqrt{r_3} \cos(2\pi(u - v(x-1)/y)) \\ \sqrt{r_3} \sin(2\pi(u - v(x-1)/y)) \end{pmatrix},$$

where

$$(3.2) \quad r_1 = \frac{y^2 + x^2 - x}{(y^2 + x^2 - x + 1)}, \quad r_2 = \frac{1 - x}{(y^2 + x^2 - x + 1)}, \quad r_3 = \frac{x}{(y^2 + x^2 - x + 1)}.$$

These numbers satisfy  $1 > r_1 \geq r_2 \geq r_3 \geq 0$  and  $r_1 + r_2 + r_3 = 1$ . It is also worth noting that  $r_2 + r_3 \leq \frac{2}{3}$ , with equality if and only if  $(x, y) = (\frac{1}{2}, \frac{1}{2}\sqrt{3})$ , in which case  $r_i = \frac{1}{3}$  for all  $i$ .

The map  $f_\tau$  is a homothetic conformal immersion, satisfying

$$|df_\tau|^2 = \frac{4\pi^2}{(y^2 + x^2 - x + 1)} |dz|^2,$$

and so its induced area is

$$V(f_\tau) = \frac{4\pi^2 y}{(y^2 + x^2 - x + 1)}.$$

Note that, if  $x = 0$ , then  $r_3 = 0$  and the immersion  $f_\tau$  has image in  $S^3 \subset \mathbb{E}^4$ , otherwise, it is linearly full in  $S^5$ .

**Theorem 2.** *If  $r_1 \leq \frac{2}{3}$ , then  $V_c(f_\tau) = V(f_\tau)$ . In particular, in this range, one has the equality*

$$V_c(\mathbb{C}/\Lambda, [|dz|^2]) = V(f_\tau) = \frac{4\pi^2 y}{(y^2 + x^2 - x + 1)}$$

*Remark 4.* In [5, p. 165], Montiel and Ros had conjectured that this equality holds when  $r_1 < \frac{2}{3}$ . (Their condition  $(x - \frac{1}{2})^2 + y^2 < \frac{9}{4}$  is equivalent to  $r_1 < \frac{2}{3}$ .)

*Proof.* For simplicity, set

$$(s, t) = (2\pi(u - vx/y), 2\pi v/y)$$

so that  $0 \leq s, t \leq 2\pi$  is a fundamental rectangle for  $\Lambda$ , with  $y \, ds \wedge dt = 4\pi^2 \, du \wedge dv$ . In terms of  $s$  and  $t$ ,

$$f_\tau(s, t) = \begin{pmatrix} \sqrt{r_1} \cos t \\ \sqrt{r_1} \sin t \\ \sqrt{r_2} \cos s \\ \sqrt{r_2} \sin s \\ \sqrt{r_3} \cos(s+t) \\ \sqrt{r_3} \sin(s+t) \end{pmatrix} = \begin{pmatrix} \sqrt{r_1} e^{it} \\ \sqrt{r_2} e^{is} \\ \sqrt{r_3} e^{i(s+t)} \end{pmatrix}.$$

Let  $a = (a^1, a^2, \dots, a^6) \in \mathbb{E}^6$  satisfy  $a \cdot a < 1$  and define

$$V(a) = \frac{y}{(y^2 + x^2 - x + 1)} \int_0^{2\pi} \int_0^{2\pi} \frac{(1 - a \cdot a)}{(1 - a \cdot f_\tau(s, t))^2} \, ds \, dt.$$

Then

$$V_c(f_\tau) = \sup \{ V(a) \mid a \cdot a < 1 \}.$$

Because  $f_\tau$  is equivariant with respect to a 2-torus of rotations in  $\text{SO}(6)$ , one can apply a rotation from this torus to reduce to the cases in which  $a_1, a_3 \geq 0$  and  $a_2 = a_4 = 0$ , while  $a_5 = \cos \theta \sqrt{a_5^2 + a_6^2}$  and  $a_6 = -\sin \theta \sqrt{a_5^2 + a_6^2}$  for some angle  $\theta$ . Setting

$$\lambda = a_1 \sqrt{r_1}, \quad \mu = a_3 \sqrt{r_2}, \quad \nu = \sqrt{a_5^2 + a_6^2} \sqrt{r_3},$$

one has  $\lambda, \mu, \nu \geq 0$  and  $\lambda + \mu + \nu \leq (a_1^2 + a_3^2 + a_5^2 + a_6^2)^{1/2} (r_1 + r_2 + r_3)^{1/2} < 1$ .

If  $r_3 \neq 0$ ,<sup>2</sup> set

$$(3.3) \quad V_\theta(\lambda, \mu, \nu) = \frac{y}{(y^2 + x^2 - x + 1)} \left( 1 - \frac{\lambda^2}{r_1} - \frac{\mu^2}{r_2} - \frac{\nu^2}{r_3} \right) \times \int_0^{2\pi} \int_0^{2\pi} \frac{ds \, dt}{(1 - \lambda \cos t - \mu \cos s - \nu \cos(s+t+\theta))^2}$$

Let

$$D_+ = \left\{ (\lambda, \mu, \nu) \mid \lambda, \mu, \nu \geq 0, \quad \frac{\lambda^2}{r_1} + \frac{\mu^2}{r_2} + \frac{\nu^2}{r_3} < 1 \right\} \subset \mathbb{R}^3$$

be the first octant of the ellipsoid. Then

$$V_c(f_\tau) = \sup \{ V_\theta(\lambda, \mu, \nu) \mid \theta \in \mathbb{R}, \quad (\lambda, \mu, \nu) \in D_+ \}.$$

Considering the series expansion of the integral near  $(\lambda, \mu, \nu) = (0, 0, 0)$ , namely

$$V_\theta(\lambda, \mu, \nu) = \frac{4\pi^2 y}{(y^2 + x^2 - x + 1)} \left( 1 - \frac{\lambda^2}{r_1} - \frac{\mu^2}{r_2} - \frac{\nu^2}{r_3} \right) \left( 1 + \frac{3}{2}(\lambda^2 + \mu^2 + \nu^2) + 6\lambda\mu\nu \cos \theta + \dots \right),$$

one sees that  $(0, 0, 0)$  can be a local maximum of  $V_\theta(\lambda, \mu, \nu)$  only if  $r_i \leq \frac{2}{3}$  for  $i = 1, 2, 3$ . Since  $0 \leq r_3 \leq r_2 \leq r_1$ , the only condition this imposes is  $r_1 \leq \frac{2}{3}$ .<sup>3</sup>

<sup>2</sup> When  $r_3 = 0$ , one can simply set  $\nu = 0$  and omit the term  $\nu^2/r_3$ . I will let the reader check that the following argument goes through *mutatis mutandis* in the simpler case in which  $r_3 = 0$ . Anyway, Montiel and Ros had already verified the conjecture when  $r_3 = 0$ .

<sup>3</sup>Meanwhile, if  $r_1 > \frac{2}{3}$ , then  $(\lambda, \mu, \nu) = (0, 0, 0)$  is *not* a local maximum of  $V_\theta(\lambda, \mu, \nu)$ .



Now, performing the  $s$ -integration<sup>4</sup> in (3.3) yields

$$V_\theta(\lambda, \mu, \nu) = \frac{2\pi y}{(y^2+x^2-x+1)} \left(1 - \frac{\lambda^2}{r_1} - \frac{\mu^2}{r_2} - \frac{\nu^2}{r_3}\right) \\ \times \int_0^{2\pi} \frac{(1 - \lambda \cos t) dt}{((1-\lambda \cos t)^2 - (\mu^2 + 2\mu\nu \cos(t+\theta) + \nu^2))^{3/2}}.$$

Since  $\mu^2 + 2\mu\nu \cos(t+\theta) + \nu^2 \leq (\mu+\nu)^2$ , one has the inequality

$$(3.4) \quad V_\theta(\lambda, \mu, \nu) \leq \frac{2\pi y}{(y^2+x^2-x+1)} \left(1 - \frac{\lambda^2}{r_1} - \frac{\mu^2}{r_2} - \frac{\nu^2}{r_3}\right) \\ \times \int_0^{2\pi} \frac{(1 - \lambda \cos t) dt}{((1-\lambda \cos t)^2 - (\mu+\nu)^2)^{3/2}}.$$

The remaining integral in (3.4) is not elementary, but it can be computed in terms of Legendre's complete elliptic integral of the second kind  $E$ , which is the function defined as follows for  $k \in [-1, 1]$

$$E(k) = \int_0^{\pi/2} \sqrt{1 - k^2 \sin^2 \theta} d\theta \simeq \frac{\pi}{2} \left(1 - \frac{1}{4}k^2 - \frac{3}{64}k^4 - \dots\right).$$

Because the coefficients of the convergent series expansion are obviously negative for all positive  $k$ -degrees,  $E$  satisfies

$$(3.5) \quad E(k) \leq \frac{\pi}{2}(1 - \frac{1}{4}k^2), \quad |k| \leq 1.$$

Now, when  $a, b \geq 0$  and  $a+b < 1$ , one has the formula<sup>5</sup>

$$(3.6) \quad \int_0^{2\pi} \frac{(1 - a \cos t) dt}{((1-a \cos t)^2 - b^2)^{3/2}} = \frac{4 E\left(\sqrt{4ab/(1-(a-b)^2)}\right)}{(1-(a-b)^2)^{1/2}(1-(a+b)^2)}.$$

Thus, (3.4) becomes

$$V_\theta(\lambda, \mu, \nu) \leq \frac{8\pi y}{(y^2+x^2-x+1)} \left(1 - \frac{\lambda^2}{r_1} - \frac{\mu^2}{r_2} - \frac{\nu^2}{r_3}\right) \\ \times \frac{E\left(\sqrt{4\lambda(\mu+\nu)/(1-(\lambda-\mu-\nu)^2)}\right)}{(1-(\lambda-\mu-\nu)^2)^{1/2}(1-(\lambda+\mu+\nu)^2)},$$

which, using the inequality (3.5), implies

$$(3.7) \quad V_\theta(\lambda, \mu, \nu) \leq \frac{4\pi^2 y}{(y^2+x^2-x+1)} \left(1 - \frac{\lambda^2}{r_1} - \frac{\mu^2}{r_2} - \frac{\nu^2}{r_3}\right) \\ \times \frac{1 - (\lambda+\mu+\nu)^2 + 3\lambda(\mu+\nu)}{(1-(\lambda-\mu-\nu)^2)^{3/2}(1-(\lambda+\mu+\nu)^2)}.$$

<sup>4</sup>This uses the formula (valid for  $a > |b|$ )

$$\int_0^{2\pi} \frac{ds}{(a - b \cos s)^2} = \frac{2\pi a}{(a^2 - b^2)^{3/2}}.$$

<sup>5</sup>This integral formula is not entirely standard, so I have included a derivation in Appendix A.

Now, the right hand side of (3.7) will have its maximum in  $D_+$  at  $(\lambda, \mu, \nu) = (0, 0, 0)$  if  $R \geq 0$  in  $D_+$ , where

$$R(\lambda, \mu, \nu) = (1 - (\lambda + \mu + \nu)^2)(1 - (\lambda - \mu - \nu)^2)^{3/2} - \left(1 - \frac{\lambda^2}{r_1} - \frac{\mu^2}{r_2} - \frac{\nu^2}{r_3}\right) (1 - (\lambda + \mu + \nu)^2 + 3\lambda(\mu + \nu)).$$

Since  $(1 - c)^{3/2} \geq (1 - \frac{3}{2}c)$  when  $0 \leq c \leq 1$ , one has  $R \geq Q$  on  $D_+$ , where

$$\begin{aligned} Q(\lambda, \mu, \nu) &= (1 - (\lambda + \mu + \nu)^2)(1 - \frac{3}{2}(\lambda - \mu - \nu)^2) \\ &\quad - \left(1 - \frac{\lambda^2}{r_1} - \frac{\mu^2}{r_2} - \frac{\nu^2}{r_3}\right) (1 - (\lambda + \mu + \nu)^2 + 3\lambda(\mu + \nu)) \\ &= Q_2(\lambda, \mu, \nu) + Q_4(\lambda, \mu, \nu), \\ &= \frac{\lambda^2}{r_1} + \frac{\mu^2}{r_2} + \frac{\nu^2}{r_3} - \frac{3}{2}(\lambda^2 + (\mu + \nu)^2) + Q_4(\lambda, \mu, \nu), \end{aligned}$$

and where  $Q_j(\lambda, \mu, \nu)$  is homogeneous of degree  $j$ . Thus, it suffices to show that  $Q \geq 0$  on  $D_+$ .

Since  $r_1 \leq \frac{2}{3}$  and  $r_2 + r_3 \leq \frac{2}{3}$ , the quadratic form  $Q_2$  is nonnegative on  $\mathbb{R}^3$ . Thus, when one considers, for  $(\lambda, \mu, \nu) \in D_+$ , the quartic polynomial in  $t$  defined by

$$q(t) = Q(t\lambda, t\mu, t\nu) = Q_2(\lambda, \mu, \nu)t^2 + Q_4(\lambda, \mu, \nu)t^4,$$

one has  $Q_2(\lambda, \mu, \nu) \geq 0$ . Meanwhile, assuming  $(\lambda, \mu, \nu) \neq (0, 0, 0)$  and setting  $T = (\lambda + \mu + \nu)^{-1} > 1$ , one finds, directly from the defining formula of  $Q$ , that

$$q(T) = T^4 \left( \frac{\lambda^2}{r_1} + \frac{\mu^2}{r_2} + \frac{\nu^2}{r_3} - (\lambda + \mu + \nu)^2 \right) (3\lambda(\mu + \nu)).$$

Since  $r_i < 1$  while  $r_1 + r_2 + r_3 = 1$ , the quadratic form

$$G(\lambda, \mu, \nu) = \frac{\lambda^2}{r_1} + \frac{\mu^2}{r_2} + \frac{\nu^2}{r_3} - (\lambda + \mu + \nu)^2$$

is nonnegative. Thus,  $q(T) \geq 0$ .

Thus, writing  $q(t) = t^2(a + bt^2)$ , one has

$$a = Q_2(\lambda, \mu, \nu) \geq 0 \quad \text{and} \quad a + bT^2 = q(T)T^{-2} \geq 0.$$

Since  $T > 1$ , it follows that

$$Q(\lambda, \mu, \nu) = q(1) = a + b \geq 0.$$

Thus,  $R \geq Q \geq 0$  on  $D_+$ , implying that  $V_\theta(\lambda, \mu, \nu)$  attains its supremum on  $D_+$  at  $(\lambda, \mu, \nu) = (0, 0, 0)$ .

Thus, when  $r_1 \leq \frac{2}{3}$ , one has

$$V_c(f_\tau) = \frac{4\pi^2 y}{(y^2 + x^2 - x + 1)}$$

as was to be shown.  $\square$

*Remark 5* (An upper bound on the conformal volume). When  $r_1 > \frac{2}{3}$ , one can show (by a rather involved calculation) that the maximum value of  $V_\theta(\lambda, \mu, \nu)$  occurs at  $(\lambda, \mu, \nu) = (\sqrt{3r_1 - 2}, 0, 0)$ , so

$$V_c(f_\tau) = V_\theta(\sqrt{3r_1 - 2}, 0, 0) = \frac{8\pi^2 y \sqrt{(y^2 + x^2 - x + 1)}}{3\sqrt{3}(y^2 + x^2 - x)},$$

and this gives an upper bound for the conformal volume of  $T = \mathbb{C}/\Lambda$ . For large  $y$ , this upper bound is about 20% above the obvious lower bound of  $4\pi$ .

*Remark 6* (Relation with Willmore inequalities). One of the original motivations for introducing the conformal volume was to shed light on the Willmore Conjecture concerning the infimum of the integral

$$W(\phi) = \int_T |H_\phi|^2 dA_\phi$$

as  $\phi$  ranges over the immersions of a torus  $T$  into  $\mathbb{E}^n$ . The conjecture (now proved [4]) was that  $W(\phi) \geq 2\pi^2$ .

The connection with conformal volume is that conformal volume can be used to give a lower bound for the Willmore integral, which is, itself, conformally invariant. In fact, as Li and Yau showed [3], when  $\phi : \mathbb{C}/\Lambda \rightarrow \mathbb{E}^n$  is a conformal immersion, one has

$$W(\phi) \geq V_c(\mathbb{C}/\Lambda, [|\mathrm{d}z|^2]).$$

Note that this alone, coupled with Theorem 2, is enough to prove the Willmore conjecture for tori conformally equivalent to  $\mathbb{C}/\Lambda$  when  $(y-1)^2 + (x-\frac{1}{2})^2 < \frac{1}{4}$ , as pointed out by Montiel and Ros [5, Corollary 7].

Now, one could ask for a refined version of this lower bound, one that fixes a conformal structure on the torus and allows  $\phi$  to range only over conformal immersions rather than all immersions.

While a sharp form of such an inequality does not seem to be within reach, by using arguments similar to those used to prove Theorem 1, one can prove, for any *isometric* immersion  $\phi$  of  $\mathbb{C}/\Lambda$  endowed with the metric  $|\mathrm{d}z|^2$  into  $\mathbb{E}^n$ , that one has

$$W(\phi) \geq \frac{\pi^2(y^4 + (1+2x-2x^2)y^2 - 3x^2(1-x)^2)}{y^3}$$

and that equality holds for such an isometric  $\phi$  if and only if  $\phi = f_\tau$  up to translation and rotation.

Note that this lower bound is at least  $2\pi^2$ , with equality for  $\tau = x + iy$  in the allowable hyperbolic triangle<sup>6</sup> if and only if  $(x, y) = (0, 1)$ , i.e., for the square torus. This lower bound also goes to infinity as  $y \rightarrow \infty$ . In particular, for large  $y$ , it exceeds  $8\pi$ , which is the value of  $W(\phi)$  when  $\phi$  is a holomorphic branched double cover of the standard 2-sphere.

For (unbranched) conformal embeddings  $\phi$ , I am not aware of any violations of the above inequality. While it is, perhaps, it is too much to hope that the above inequality holds for all conformal immersions (rather than just isometric ones), it is an interesting question as to what the lower bound for conformal immersions might be.

#### APPENDIX A. AN ELLIPTIC INTEGRAL

Since the integral formula (3.6) is not entirely standard (in fact, none of the symbolic integration packages to which I have access seem to be able to demonstrate it), I include a proof here for the benefit of the reader.

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<sup>6</sup>I.e.,  $0 \leq x \leq \frac{1}{2}$  and  $y \geq \sqrt{1-x^2}$ .

Thus, assume that  $a$  and  $b$  are constants satisfying  $a, b \geq 0$  and  $a+b < 1$ . To evaluate the integral

$$I(a, b) = \int_0^{2\pi} \frac{(1 - a \cos t) dt}{((1-a \cos t)^2 - b^2)^{3/2}} = 2 \int_0^\pi \frac{(1 - a \cos t) dt}{((1-a \cos t)^2 - b^2)^{3/2}},$$

first, set

$$\tau = \sqrt{\frac{1-b+a}{1-b-a}} \quad \text{and} \quad k = \sqrt{\frac{4ab}{1-(b-a)^2}}$$

and note that  $0 \leq k \leq 1$ . Now, make the substitution

$$t = 2 \tan^{-1} \left( \frac{\tan \theta}{\tau} \right),$$

in the integral, noting that

$$\cos t = \frac{\tau^2 - \tan^2 \theta}{\tau^2 + \tan^2 \theta} \quad \text{and} \quad dt = \frac{2\tau \sec \theta d\theta}{\tau^2 + \tan^2 \theta}.$$

One then obtains

$$\begin{aligned} I(a, b) &= \frac{4\tau^2}{(1-(b-a)^2)^{3/2}(1-b+a)^2} \\ &\times \int_0^{\pi/2} \frac{(1-a)(1-b+a)^2 - 2a(1-(b-a)^2) \sin^2 \theta + 4a^2b \sin^4 \theta}{(1-k^2 \sin^2 \theta)^{3/2}} d\theta. \end{aligned}$$

Now, let

$$\omega_1 = d \left( \frac{\sin \theta \cos \theta}{(1-k^2 \sin^2 \theta)^{1/2}} \right) = \frac{1-2\sin^2 \theta + k^2 \sin^4 \theta}{(1-k^2 \sin^2 \theta)^{3/2}} d\theta$$

and

$$\omega_2 = (1-k^2 \sin^2 \theta)^{1/2} d\theta = \frac{1-2k^2 \sin^2 \theta + k^4 \sin^4 \theta}{(1-k^2 \sin^2 \theta)^{3/2}} d\theta$$

and note that, by the fundamental theorem of calculus and by definition, one has

$$\int_0^{\pi/2} \omega_1 = 0 \quad \text{and} \quad \int_0^{\pi/2} \omega_2 = E(k),$$

where  $E$  is Legendre's complete elliptic integral of the second kind.

Now, note the algebraic identity

$$\frac{(1-a)(1-b+a)^2 - 2a(1-(b-a)^2) \sin^2 \theta + 4a^2b \sin^4 \theta}{(1-k^2 \sin^2 \theta)^{3/2}} d\theta = c_1 \omega_1 + c_2 \omega_2,$$

where

$$c_1 = \frac{a((1-b)^2 - a^2)(1-(b-a)^2)}{(1+b)^2 - a^2} \quad \text{and} \quad c_2 = \frac{(1-(b-a)^2)^2}{(1+b)^2 - a^2}.$$

Thus,

$$I(a, b) = \frac{4\tau^2}{(1-(b-a)^2)^{3/2}(1-b+a)^2} \left( \frac{(1-(b-a)^2)^2}{(1+b)^2 - a^2} \right) E(k),$$

which, given the formulae for  $\tau$  and  $k$ , simplifies to

$$I(a, b) = \frac{4 E \left( \sqrt{4ab/(1-(a-b)^2)} \right)}{(1-(a-b)^2)^{1/2}(1-(a+b)^2)},$$

which is (3.6).

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