Augmentation Rank of Satellites with Braid Pattern

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A thesis submitted to the Department of Mathematics for graduation with distinction

Duke University
Durham, North Carolina
2015

1. Introduction

Let S^n be the n-sphere, that is, the set of points $x \in \mathbb{R}^{n+1}$ such that $|x| = 1$. A *knot* K is an embedding $K\colon S^1 \to S^3$. For the purpose of intuition, one we may also think of a knot as an embedding $S^1 \to \mathbb{R}^3$. We commonly abuse notation and refer to the image of this embedding in S^3 as the knot itself. A knot is *oriented* if it has a preferred direction.

We give a rigorous definition in Section 2, but roughly speaking two knots K_1 and K_2 are considered to be equivalent if the image of K_1 can be continously transformed to the image of K_2 without passing through itself. In general it is a difficult to determine whether two knots are equivalent, so it is helpful to define *knot invariants*. Knot invariants are quantities that can be computed for a given knot that will give the same value for two equivalent knots. In this thesis we study a particular knot invariant called the augmentation rank, and use our results to say things about two more knot invariants, called the meridional rank and the bridge number.

Let K be an oriented knot in S^3 and denote by π_K the fundamental group of its complement $S^3 \setminus K$, with some basepoint (see Section 2 for a definition of the fundamental group).

Definition 1.1. We call an element of π_K a *meridian* if it is represented by the oriented boundary of a disc, embedded in S^3, whose interior intersects K positively once

Intuitively, we can think of a meridian as an element of the fundamental group represented by a loop that travels straight from the base point to a point on the knot, loops around the knot once, and then travels straight back. It is possible to show that the group π_K is generated by meridians.

Definition 1.2. The *meridional rank* of K, written $\mathrm{mr}(K)$, is the minimal size of a generating set containing only meridians.

As we see from the definition, the meridional rank of a knot is arrived at via fairly algebraic means. Compared to the meridional rank, the bridge number is a relatively geometric knot invariant:

Definition 1.3. Choose a height function $h : S^3 \to \mathbb{R}$. The *bridge number* of K, denoted $b(K)$, is the minimum of the number of local maxima of $h|_{\varphi(S^1)}$ among embeddings $\varphi : S^1 \to S^3$ which realize K.

Intuitively, we can think of the knot K as lying in $\mathbb{R}^3$ and the bridge number of K as the minimum number of local maxima relative to the z-axis that a knot equivalent to K can have. It is not too difficult to show that $\mathrm{mr}(K) \leq b(K)$ for any $K \subset S^3$ (those familiar with the Wirtinger presentation of π_K can use it to find a meridional generating set with $b(K)$ elements). Whether the bound is equality for all knots is a question attibuted to Cappell and Shaneson:

Question 1.4 ([Kir97, Prob. 1.11])**.** Is $\mathrm{mr}(K) = b(K)$ for all knots K?

The question is both interesting and difficult because of how differently we arrive at the two invariants, and it remains open in general. Equality is known to hold for some families of knots due to work of various authors ([BZ85, Cor14b, RZ87]). In particular, equality is known to hold in the case when K is a *torus knot*, i.e. when K is equivalent to a knot whose image lies in the standard torus ([RZ87]).

In this thesis we study *augmentations* of K, which are maps that arise in the study of knot contact homology. To each augmentation is associated a rank and there is a maximal rank of augmentations of a given K, which is a knot invariant called the *augmentation rank* $\mathrm{ar}(K)$. For any K the inequality $\mathrm{ar}(K) \leq \mathrm{mr}(K)$ holds (see Section 2.4), so as we will see later, we can use results about $\mathrm{ar}(K)$ to say things about $\mathrm{mr}(K)$ and $b(K)$. In particular, we will examine the behavior of $\mathrm{ar}(K)$ under satellite operations with a braid pattern.

To be precise, denote the group of braids on n strands by B_n and write $\hat{\beta}$ for the *braid closure* of a braid β (see Section 2, Figure 4). We write $\imath_n$ for the identity in B_n.

Throughout the paper we let $\alpha \in B_k$ and $\gamma \in B_p$ and set $K = \hat{\alpha}$. We assume our braid closures are a (connected) knot. Note that $\mathrm{ar}(K) \leq k$.

Definition 1.5. Let $\imath_p(\alpha)$ be the braid in B_{kp} obtained by replacing each strand of α by p parallel copies (in the blackboard framing). Let $\bar{\gamma}$ be the inclusion of γ into B_{kp} by the map $\sigma_i \mapsto \sigma_i, 1 \leq i \leq p-1$. Set $\gamma(\alpha) = \imath_p(\alpha)\bar{\gamma}$. The *braid satellite* of K associated to α, γ is defined as $K(\alpha, \gamma) = \widehat{\gamma(\alpha)}$.

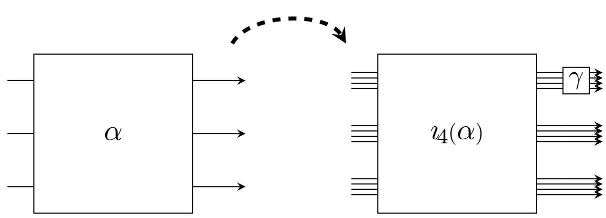

FIGURE 1. Constructing $\gamma(\alpha)$ from α; case $p = 4$.

As defined $K(\alpha, \gamma)$ depends on the choice of α. However, the construction is more intrinsic if we require the index k of α to be minimal among braid representatives of K (see [CH14, Section 2]).

Note that if $\hat{\alpha}$ and $\hat{\gamma}$ are each knots, $K(\alpha, \gamma)$ is also. It is then natural to ask how the augmentation ranks of $\hat{\alpha}$ and $\hat{\gamma}$ are related to the augmentation ransk of the new knot $K(\alpha, \gamma)$. Our main result describes this relationship in the case when the augmentation ranks of $\hat{\alpha}$ and $\hat{\gamma}$ are equal to the braid indices of α and γ, respectively.

Theorem 1.6. *If* $\alpha \in B_k$ *and* $\gamma \in B_p$ *are such that* $\mathrm{ar}(\hat{\alpha}) = k$ *and* $\mathrm{ar}(\hat{\gamma}) = p$, *then* $\mathrm{ar}(K(\alpha, \gamma)) = kp$.

A corollary of Theorem 1.6 involves Cappell and Shaneson's question for iterated torus knots. Let $\mathbf{p} = (p_1, \ldots, p_n)$ and $\mathbf{q} = (q_1, \ldots, q_n)$ be integral vectors. We write $T(\mathbf{p}, \mathbf{q})$ for the $(\mathbf{p}, \mathbf{q})$ *iterated torus knot*, defined as follows.

By convention take $T(\emptyset, \emptyset)$ as the unknot, then define $T(\mathbf{p}, \mathbf{q})$ inductively. Let $\hat{\mathbf{p}}, \hat{\mathbf{q}}$ be the truncated lists obtained from $\mathbf{p}, \mathbf{q}$ by removing the last integer in each. If α is a braid of minimal index such that $T(\hat{\mathbf{p}}, \hat{\mathbf{q}}) = \hat{\alpha}$ then define $T(\mathbf{p}, \mathbf{q}) = K(\alpha, (\sigma_1 \ldots \sigma_{p_n-1})^{q_n})$. Note that the braid closure of $(\sigma_1 \ldots \sigma_{p_n-1})^{q_n})$ is the (p_n, q_n)-torus knot, so in a sense this construction is embedding the (p_n, q_n)-torus knot into a neighborhood of the knot $T(\hat{p}, \hat{q})$.

We remark that $T(\mathbf{p}, \mathbf{q})$ is a cable of $T(\hat{\mathbf{p}}, \hat{\mathbf{q}})$, but not the (p_n, q_n)-cable in the traditional Seifert framing.

Corollary 1.7. *Given integral vectors* $\mathbf{p}$ *and* $\mathbf{q}$*, suppose that* $|p_i| < |q_i|$ *and* $gcd(p_i, q_i) = 1$ *for each* $1 \leq i \leq n$*. Then*

$$\operatorname{ar}(T(\mathbf{p}, \mathbf{q})) = \operatorname{mr}(T(\mathbf{p}, \mathbf{q})) = b(T(\mathbf{p}, \mathbf{q})) = p_1 p_2 \ldots p_n.$$

Note that since $T(p_1, q_1)$ is just the (p_1, q_1)-torus knot, this generalizes the result that $\operatorname{mr}(K) = b(K)$ when K is a torus knot. The assumption $|p_i| < |q_i|$ is needed for the hypothesis of Theorem 1.6 stipulating that the associated braids have closures with augmentation rank equal to the braid index. If instead we had some $|q_i| < |p_i|$, then $(\sigma_1 \ldots \sigma_{p_i-1})^{q_i}$ would have braid index $|p_i|$ but augmentation rank $|q_i|$ and thus the augmentation rank would not be large enough for Theorem 1.6 to apply. Furthermore, we cannot strengthen Theorem 1.6 to apply to all integral vectors $\mathbf{p}$ and $\mathbf{q}$ with $\gcd(p_i, q_i) = 1$, as there are cables of $(n, n+1)$ torus knots which do not attain the large augmentation rank in Corollary 1.7.

Theorem 1.8 ([CH14], Theorem 1.4)**.** *Given* $p > 1$ *and* $n > 1$, $\operatorname{ar}(T((n,p),(n+1,1))) < np$.

It is natural to wonder if the augmentation rank is multiplicative under weaker assumptions on α, γ than those in Theorem 1.6. We will not discuss it in this thesis, but the following is a possible generalization.

Conjecture 1.9. *Suppose* $K = \hat{\alpha}$ *for* $\alpha \in B_k$*, and that* α *has minimal index among braids with the same closure. Let* $\gamma \in B_p$*. Then* $\operatorname{ar}(K(\alpha, \gamma)) \geq \operatorname{ar}(\hat{\alpha}) \operatorname{ar}(\hat{\gamma})$.

Remark 1.10. There are examples when the inequality of Conjecture 1.9 is strict (see Section [CH14, Section 5]).

This thesis is organized as follows. In Section 2 we give the needed background in knot contact homology, specifically Ng's cord algebra, and discuss augmentation rank and the relationship to meridional rank. Section 2.5 reviews techniques used in the proof of Theorem 1.6. Section 3 is devoted to the proof of Theorem 1.6, its requisite supporting lemmas, and Corollary 1.7.

Acknowledgements

The author was supported in part by NSF grant DMS-0846346, as a fellow in the PRUV Fellowship program at Duke University, and thanks David Kraines and the Duke Math Department for organizing the PRUV program. This thesis was adapted from a paper the author coauthored (see [CH14]) with his PRUV advisor, Christopher Cornwell. In the course of writing the paper, Cornwell wrote large parts of what are now the Introduction and Background sections of this thesis, as well as making numerous edits to the proof of the main result. The author would like to thank Cornwell for his invaluable mentorship, guidance, and contributions. The author would also like to thank Lenhard Ng for his consultation and helpful comments.

2. Background

We first review definitions of the fundamental group and the braid group in Section 2.1. We describe in Section 2.2 the construction of $HC_0(K)$ from the viewpoint of the combinatorial knot differential graded algebra, which was first defined in [Ng08]; our conventions are those given in [Ng14]. In Section 2.3 we describe spanning arcs, which will be useful for making computations Section 3. In Section 2.4 we discuss augmentations in knot contact homology and their rank, which gives a lower bound on the meridional rank of the knot group. Section 2.5 contains a discussion of techniques from [Cor14a] that we use to calculate the augmentation rank.

2.1. **The fundamental group and the braid group.**

Let X and Y be topological spaces, and $f, g\colon X \to Y$ continuous functions. A *homotopy* from f to g is a continuous function $H\colon X \times [0,1] \to Y$ such that $H(x,0) = f(x)$ and $H(x,1) = g(x)$ for all $x \in X$. If f and g are also embeddings, an *isotopy* H from f to g is a homotopy from f to g such that $H(x,t)\colon X \to Y$ is an embedding for all $t \in [0,1]$. Again if f and g are embeddings, an *ambient isotopy* from f to g is a continuous map $H\colon Y \times [0,1] \to Y$ such that $H(t)\colon Y \to Y$ is a homeomorphism for all $t \in [0,1]$, $H(0)$ is the identity map, and $H(1) \circ f = g$. We say that f and g are *homotopic* if there exists a homotopy from one to the other, *isotopic* if there exists an isotopy from one to the other, and *ambient isotopic* if there exists an ambient isotopy from one to the other. We say that two knots are equivalent if they are ambient isotopic.

Let X be a topological space, and let x_0 be a point in X. When we have two continuous maps $f, g\colon [0,1] \to X$ such that $f(0) = f(1) = g(0) = g(1) = x_0$, we require that a homotopy H from f to g also satisfies $H(0,t) = H(1,t) = x_0$. Let $\pi_1(X)$ be the set of homotopy equivalence classes of continuous maps $f\colon [0,1] \to X$ such that $f(0) = f(1) = x_0$. Define $\circ\colon \pi_1(X) \times \pi_1(X) \to \pi_1(X)$ by

$$(f \circ g)(t) = \begin{cases} f(2t) & t < 1/2 \\ g(2t-1) & t \geq 1/2 \end{cases} \tag{1}$$

One can easily check that $\circ$ makes $\pi_1(X)$ into a group, which is called the *fundamental group* of X with base point x_0. Given a knot K in S^3 and a point $x_0 \in S^3 \setminus K$, the *knot group* π_K with base point x_0 is $\pi_1(S^3 \setminus K)$.

Let P_0 and P_1 be two parallel planes in $\mathbb{R}^3$ defined by $y = 0$ and $y = 1$, respectively. A *braid on n strands* is an embedding $B\colon \coprod_{i=1}^{n}[0,1] \to \mathbb{R}^3$ such that for each interval $[0,1]$ in the disjoint union, $B(0) \in P_0$, $B(1) \in P_1$, and $t_1 < t_2$ implies that $y(B(t_1)) < y(B(t_2))$. See Figure 2 for an example of a braid on three strands. We say that two braids are equivalent if there is an isotopy taking one braid to the other that fixes all points in P_0 and P_1. Let B_n be the set of equivalence classes of braids, and define a group operation $B_n \times B_n \to B_n$ by

$$(B_1 B_2)(t) = \begin{cases} B_1(2t) & t < 1/2 \\ B_2(2t-1) & t \geq 1/2 \end{cases} \tag{2}$$

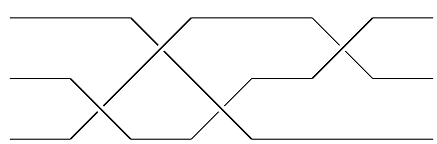

FIGURE 2. A braid in B_3

For each interval $[0,1]$ and point $t \in [0,1]$ in the disjoint union.

Throughout the paper we orient braids in B_n from left to right, labeling the strands $1, \ldots, n$, with 1 the topmost and n the bottommost strand. We work with Artin's generators $\{\sigma_i^{\pm}, i = 1, \ldots, n-1\}$ of B_n, where in σ_i only the i and $i{+}1$ strands interact, and they cross once in the manner depicted in Figure 3. Given a braid $\beta \in B_n$, the braid closure $\hat{\beta}$ of β is the link obtained

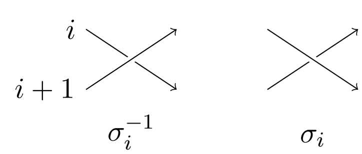

FIGURE 3. Generators of B_n

as shown in Figure 4. The *writhe* (or algebraic length) of β, denoted $\omega(\beta)$, is the sum of exponents of the Artin generators in a word representing β.

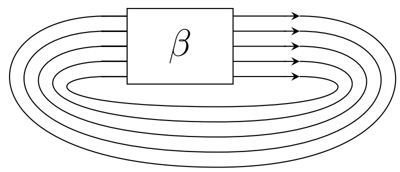

FIGURE 4. The braid closure of β

2.2. **Knot contact homology.** In the proof of our main result we will use the degree zero part of knot contact homology $HC_*(K)$, which is a knot invariant derived from Legendrian contact homology. Knot contact homology appears to be a knot invariant that is at the same time powerful and relatively easy to compute. Here we review the construction of the degree zero part of the combinatorial knot differential graded algebra of Ng. The combinatorial knot DGA was defined in order to be a calculation of knot contact homology and was shown to be so in [EENS13] (see [Ng14] for more details).

A *graded algebra* $\mathcal{A}$ is an algebra given by a direct sum of groups $\mathcal{A} = \bigoplus_{i\geq 0} A_i$, where $A_iA_j \subseteq A_{i+j}$ and $RA_i \subseteq A_i$ for all $i,j \geq 0$. We define $|a| = i$ when $a \in A_i$. A *differential graded algebra (DGA)* is a graded algebra equipped with a *differential* $\partial\colon \mathcal{A} \to \mathcal{A}$ satisfying $\partial(A_i) \subseteq A_{i-1}$, $\partial \circ \partial = 0$, and $\partial(ab) = \partial(a)b + (-1)^{|a|}a\partial(b)$ for all $i \geq 1$ and $a, b \in \mathcal{A}$.

Let $\mathcal{A}_n$ be the noncommutative unital algebra over $\mathbb{Z}$ freely generated by a_{ij}, $1 \leq i \neq j \leq n$. We define a homomorphism $\phi : B_n \to \operatorname{Aut}\mathcal{A}_n$ by defining it on the generators of B_n:

$$\phi_{\sigma_k}\colon \begin{cases} a_{ij} \mapsto a_{ij} & i,j \neq k, k+1 \\ a_{k+1,i} \mapsto a_{ki} & i \neq k, k+1 \\ a_{i,k+1} \mapsto a_{ik} & i \neq k, k+1 \\ a_{k,k+1} \mapsto -a_{k+1,k} & \\ a_{k+1,k} \mapsto -a_{k,k+1} & \\ a_{ki} \mapsto a_{k+1,i} - a_{k+1,k}a_{ki} & i \neq k, k+1 \\ a_{ik} \mapsto a_{i,k+1} - a_{ik}a_{k,k+1} & i \neq k, k+1 \end{cases} \tag{3}$$

Let $\iota\colon B_n \to B_{n+1}$ be the inclusion $\sigma_i \mapsto \sigma_i$ so that the $(n+1)$ strand does not interact with those from $\beta \in B_n$, and define $\phi^*_\beta \in \operatorname{Aut}\mathcal{A}_{n+1}$ by $\phi^*_\beta = \phi_{\iota(\beta)}$. We then define the $n \times n$ matrices Φ^L_β and Φ^R_β with entries in $\mathcal{A}_n$ by

$$\phi^*_\beta(a_{i,n+1}) = \sum_{j=1}^{n} (\Phi^L_\beta)_{ij} a_{j,n+1}$$

$$\phi^*_\beta(a_{n+1,i}) = \sum_{j=1}^{n} a_{n+1,j} (\Phi^R_\beta)_{ji}$$

Finally, let R_0 be the Laurent polynomial ring $\mathbb{Z}[\lambda^{\pm 1}, \mu^{\pm 1}]$ and define matrices $\mathbf{A}$ and $\mathbf{\Lambda}$ over R_0 by

$$\mathbf{A}_{ij} = \begin{cases} a_{ij} & i < j \\ -\mu a_{ij} & i > j \\ 1 - \mu & i = j \end{cases} \tag{4}$$

$$\mathbf{\Lambda} = \mathrm{diag}[\lambda \mu^{\omega(\beta)}, 1, \ldots, 1]. \tag{5}$$

Definition 2.1. Suppose that K is the closure of $\beta \in B_n$. Define $\mathcal{I} \subset \mathcal{A}_n \otimes R_0$ to be the ideal generated by the entries of $\mathbf{A} - \mathbf{\Lambda} \cdot \mathbf{\Phi}^{\mathbf{L}}_{\beta} \cdot \mathbf{A}$ and $\mathbf{A} - \mathbf{A} \cdot \mathbf{\Phi}^{\mathbf{R}}_{\beta} \cdot \mathbf{\Lambda}^{-1}$. The *degree zero homology of the combinatorial knot DGA* is $\mathrm{HC}_0(K) = (\mathcal{A}_n \otimes R_0)/\mathcal{I}$.

2.3. **Spanning arcs.** The proofs in Sections 3 require a number of computations of ϕ_β (and of ϕ^*_β, for computing Φ^L_β) for particular braids. Such computations are benefited by an alternate description of the automorphism, which we now explain.

Definition 2.2. Given $n > 0$, let D_n be a disk in $\mathbb{C}$ containing points $P = \{1, 2, \ldots, n\} \subset \mathbb{R}$ in its interior. A *spanning arc* of D_n is the isotopy class relative to P of an oriented embedded path in D which begins and ends in P and otherwise does not intersect P. We define $\mathscr{S}_n$ as the associative ring freely generated by spanning arcs of D_n modulo the ideal generated by the relation in Figure 5. Denote by $c_{ij} \in \mathscr{S}_n$ the element represented by a spanning arc contained in the upper half-disk beginning at i and ending at j.

We understand the spanning arcs in Figure 5 to agree outside of a neighborhood of the depicted point in P.

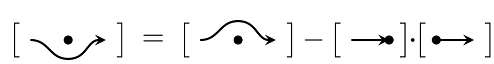

Figure 5. Relation in $\mathscr{S}_n$

We consider β as an isotopy of (D, P) and denote by $\beta \cdot c$ the image of the spanning arc c. By convention σ_k acts by rotating k and $k+1$ about their midpoint in counter-clockwise fashion. It was shown in [Ng05b, Section 2] that there is a unique, well-defined map χ which sends each spanning arc of D_n to an element of $\mathcal{A}_n$ such that

(i) $\chi(\beta \cdot c) = \phi_\beta(\chi(c))$ for any spanning arc c and $\beta \in B_n$;
(ii) $\chi(c_{ij}) = a_{ij}$ if $i < j$, $\chi(c_{ij}) = -a_{ij}$ if $i > j$.

Furthermore, χ factors through $\mathscr{S}_n$, is injective, and by the relation in Figure 5 the value of $\phi_\beta(a_{ij})$ can be determined from (i) and (ii). This constitutes an essential technique for our calculations of ϕ_β.

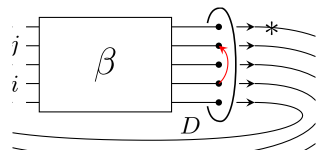

FIGURE 6. Cord c_{ij} of $K = \hat{\beta}$

Computations of Φ^L_β are carried out in likewise manner, including β into B_{n+1} and considering spanning arcs $c_{j,n+1}$, $1 \le j \le n$ of D_{n+1}. We will distinguish this situation by relabeling $n+1$ (and corresponding indices) with the symbol $*$. In figures, we put the point $*$ at the boundary of D.

It will be convenient for us in Section 3 to consider the free left $\mathcal{A}_n$-module $\mathcal{A}^L_n = \mathcal{A}_n\langle a_{1*}, \ldots, a_{n*}\rangle$ and right $\mathcal{A}_n$-module $\mathcal{A}^R_n = \langle a_{*1}, \ldots, a_{*n}\rangle\mathcal{A}_n$, which are each contained in $\mathcal{A}_{n+1}$. By definition, Φ^L_β (respectively Φ^R_β) is the matrix in the above basis for the $\mathcal{A}_n$-automorphism of $\mathcal{A}^L_n$ (respectively $\mathcal{A}^R_n$) determined by the image of the basis under ϕ^*_β.

Finally, as we are considering braid satellites $K(\alpha, \gamma)$ with $\gamma \in B_p$ our perspective often considers the points in D_{kp} as k groups of p points each. We find it convenient in figures of spanning arcs in $\mathscr{S}_{kp}$ to reflect this point of view. To do so, for each $i = 0, \ldots, k-1$, we depict the points $\{ip + 1, \ldots, (i+1)p\}$ by a horizontal segment, and if a spanning arc ends at $ip+s$ for $1 \le s \le p$, it is depicted ending on the $(i+1)^{st}$ segment with a label s (see example in Figure 7).

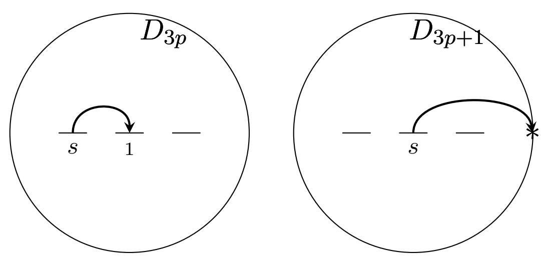

FIGURE 7. Spanning arcs $c_{s,p+1}$ and $c_{p+s,*}$, $1 \le s \le p$.

Let perm : $B_n \to S_n$ denote the homomorphism from B_n to the symmetric group sending σ_k to the simple transposition interchanging $k, k+1$.

Lemma 2.3. *For some $\beta \in B_n$ and $1 \le i \ne j \le n$, consider $(\Phi^L_\beta)_{ij} \in \mathcal{A}_n$ as a polynomial expression in the (non-commuting) variables $\{a_{kl}, 1 \le k \ne l \le n\}$. Writing $i_0 = perm(\beta)(i)$, every monomial in $(\Phi^L_\beta)_{ij}$ is a constant times $a_{i_0i_1}a_{i_1i_2}\ldots a_{i_{l-1},j}$ for some $l \ge 0$, the monomial being a constant if $l = 0$ and only if $i_0 = j$.*

Proof. We consider the spanning arc $\beta \cdot c_{i,*}$ which begins at i_0 and ends at $*$. Applying the relation in Figure 5 to the path equates it with a sum (or difference) of another path with the same endpoints and a product of two paths, the first beginning at i_0 and the other ending at $*$. A finite number of applications of this relation allows one to express the path as a polynomial in the $c_{kl}, 1 \leq k \neq l \leq n$ where each monomial has the form $c_{i_0i_1} \dots c_{i_{l-1},j}c_{j,*}$ for some j (see [Ng05b, Lemma 2.6]).

The result follows from $\phi_\beta^*(a_{i,*}) = \phi_\beta^*(\chi(c_{i,*})) = \chi(\beta \cdot c_{i,*})$. □

2.4. **Augmentations and augmentation rank.** A *graded map* $\varphi\colon M \to N$ of graded algebras is an algebra homomorphism satisfying $\varphi(M_i) \subseteq \varphi(N_i)$. Augmentations of a differential graded algebra $(\mathcal{A}, \partial)$ are graded maps $(\mathcal{A}, \partial) \to (\mathbb{C}, 0)$ that intertwine the differential (here $\mathbb{C}$ has grading zero). For our setting, if $\beta \in B_n$ is a braid representative of K, such a map corresponds precisely to a homomorphism $\epsilon : \mathcal{A}_n \otimes R_0 \to \mathbb{C}$ such that ϵ sends elements of $\mathcal{I}$ to zero (see Definition 2.1).

Definition 2.4. Suppose that K is the closure of $\beta \in B_n$. An *augmentation* of K is a homomorphism $\epsilon : \mathcal{A}_n \otimes R_0 \to \mathbb{C}$ such that each element of $\mathcal{I}$ is sent by ϵ to zero.

A correspondence between augmentations and certain representations of the knot group π_K were studied in [Cor14a]. Recall that π_K is generated by meridians, which for a knot are all conjugate. Fix some meridian m.

Definition 2.5. For any integer $r \geq 1$, a homomorphism $\rho : \pi_K \to \mathrm{GL}_r\mathbb{C}$ is a *KCH representation* if $\rho(m)$ is diagonalizable and has an eigenvalue of 1 with multiplicity $r - 1$. We call ρ a *KCH irrep* if it is irreducible.

In [Ng08], Ng describes an isomorphism between $HC_0(K)$ and an algebra constructed from elements of π_K. As discussed in [Ng14] a KCH representation $\rho : \pi_K \to \mathrm{GL}_r\mathbb{C}$ induces an augmentation ϵ_ρ of K. Given an augementation, the first author showed how to construct a KCH representation that induces it. In fact, we have the following rephrasing of results from [Cor14a].

Theorem 2.6 ([Cor14a])**.** *Let $\epsilon : \mathcal{A}_n \otimes R_0 \to \mathbb{C}$ be an augmentation with $\epsilon(\mu) \neq 1$. There is a KCH irrep $\rho : \pi_K \to GL_r\mathbb{C}$ such that $\epsilon_\rho = \epsilon$. Furthermore, for any KCH irrep $\rho : \pi_K \to GL_r\mathbb{C}$ such that $\epsilon_\rho = \epsilon$, the rank of $\epsilon(\mathbf{A})$ equals r.*

The abuse of notation $\epsilon(\mathbf{A})$, means that ϵ is applied to each entry of $\mathbf{A}$. Similar notation will be used in the remainder of the paper. Considering Theorem 2.6 we make the following definition.

Definition 2.7. The *rank* of an augmentation $\epsilon : \mathcal{A}_n \otimes R_0 \to \mathbb{C}$ with $\epsilon(\mu) \neq 1$ is the rank of $\epsilon(\mathbf{A})$. Given a knot K, the *augmentation rank of K*, denoted $\mathrm{ar}(K)$, is the maximum rank among augmentations of K.

Remark 2.8. By Theorem 2.6 the set of ranks of augmentations of a given K does not depend on choice of braid representative.

It is the case that $\mathrm{ar}(K)$ is well-defined. That is, given K there is a bound on the maximal rank of an augmentation of K.

Theorem 2.9 ([Cor14b]). *Given a knot $K \subset S^3$, if $g_1, \ldots, g_d$ are meridians that generate π_K and $\rho : \pi_K \to GL_r\mathbb{C}$ is a KCH irrep then $r \leq d$.*

As in the introduction, if we denote the meridional rank of π_K by $\mathrm{mr}(K)$, then Theorem 2.9 implies that $\mathrm{ar}(K) \leq \mathrm{mr}(K)$. In addition, the geometric quantity $b(K)$ called the bridge index of K is never less than $\mathrm{mr}(K)$. Thus we have the following corollary.

Corollary 2.10 ([Cor14b]). *Given a knot $K \subset S^3$,*

$$ar(K) \leq mr(K) \leq b(K)$$

Hence to verify that $\mathrm{mr}(K) = b(K)$ it suffices to find a rank $b(K)$ augmentation of K. Herein we concern ourselves with a setting where $\mathrm{ar}(K) = n$ and there is a braid $\beta \in B_n$ which closes to K. This is a special situation, since $b(K)$ is strictly less than the braid index for many knots.

2.5. **Finding augmentations.** The following theorem concerns the behavior of the matrices Φ^L_β and Φ^R_β under the product in B_n. It is an essential tool for studying $HC_0(K)$ and is central to our arguments.

Theorem 2.11 ([Ng05a], Chain Rule). *Let β_1, β_2 be braids in B_n. Then $\Phi^L_{\beta_1\beta_2} = \phi_{\beta_1}(\Phi^L_{\beta_2}) \cdot \Phi^L_{\beta_1}$ and $\Phi^R_{\beta_1\beta_2} = \Phi^R_{\beta_1} \cdot \phi_{\beta_1}(\Phi^R_{\beta_2})$.*

Another property of Φ^L_β and Φ^R_β that is important to us is the following symmetry. Define an involution $x \mapsto \overline{x}$ on $\mathcal{A}_n$ (termed *conjugation*) as follows: first set $\overline{a_{ij}} = a_{ji}$; then, for any $x, y \in \mathcal{A}_n$, define $\overline{xy} = \overline{y}\,\overline{x}$ and extend the operation linearly to $\mathcal{A}_n$.

Theorem 2.12 ([Ng05a], Prop. 6.2). *For a matrix of elements in $\mathcal{A}_n$, let $\overline{M}$ be the matrix such that $\left(\overline{M}\right)_{ij} = \overline{M_{ij}}$. Then for $\beta \in B_n$, Φ^R_β is the transpose of $\overline{\Phi^L_\beta}$.*

The main result of the paper concerns augmentations with rank equal to the braid index of K. Define the diagonal matrix $\Delta(\beta) = \mathrm{diag}[(-1)^{w(\beta)}, 1, \ldots, 1]$. From Section 5 of [Cor14a] we have the following.

Theorem 2.13 ([Cor14a]). *If K is the closure of $\beta \in B_n$ and has a rank n augmentation $\epsilon : \mathcal{A}_n \otimes R_0 \to \mathbb{C}$, then*

$$\epsilon(\Phi^L_\beta) = \Delta(\beta) = \epsilon(\Phi^R_\beta). \tag{6}$$

Furthermore, any homomorphism $\epsilon : \mathcal{A}_n \to \mathbb{C}$ which satisfies (6) can be extended to $\mathcal{A}_n \otimes R_0$ to produce a rank n augmentation of K.

3. Main Result

The proof of Theorem 1.6 relies heavily on the characterization presented in Theorem 2.13. We define a homomorphism $\psi : \mathcal{A}_{kp} \to \mathcal{A}_k \otimes \mathcal{A}_p$ which, for $\alpha \in B_k$, suitably simplifies $\Phi^L_{\imath_p(\alpha)}$ and $\Phi^R_{\imath_p(\alpha)}$ when applied to the entries. Given $\gamma \in B_p$, Theorem 2.11 then allows us to construct a map that satisfies (6) for $\beta = \gamma(\alpha)$. The map in question is "close to" the tensor product of an augmentation of $\hat{\alpha}$ and an augmentation of $\hat{\gamma}$, composed with ψ.

Section 3.1 begins with an intermediate result, Proposition 3.1, followed by the proofs of Theorem 1.6 and Corollary 1.7. In Section 3.2 we prove Lemmas 3.2 and 3.3, which are needed to prove Proposition 3.1.

3.1. **Proof of main result.**

We recall the statement of Theorem 1.6.

Theorem 1.6. If $\alpha \in B_k$ and $\gamma \in B_p$ are such that $\operatorname{ar}(\hat{\alpha}) = k$ and $\operatorname{ar}(\hat{\gamma}) = p$, then $\operatorname{ar}(K(\alpha, \gamma)) = kp$.

For $1 \le i \le kp$, write $i = (q_i - 1)p + r_i$, where $1 \le r_i \le p$ and $1 \le q_i \le k$. For each generator $a_{ij} \in \mathcal{A}_{kp}, 1 \le i \ne j \le kp$, define

$$\psi(a_{ij}) = \begin{cases} 1 \otimes a_{r_i r_j} & : q_i = q_j \\ a_{q_i q_j} \otimes 1 & : r_i = r_j \\ 0 & : (q_i - q_j)(r_i - r_j) < 0 \\ a_{q_i q_j} \otimes a_{r_i r_j} & : (q_i - q_j)(r_i - r_j) > 0 \end{cases}, \tag{7}$$

which determines an algebra map $\psi \colon \mathcal{A}_{kp} \to \mathcal{A}_k \otimes \mathcal{A}_p$. Extend ψ to a map $\psi^* : \mathcal{A}^L_{kp} \to \mathcal{A}^L_k \otimes \mathcal{A}^L_p$ that carries one basis to another: $\psi^*(a_{i*}) = a_{q_i,*} \otimes a_{r_i,*}$ for any $1 \le i \le kp$. Note, if we extend conjugation to $\mathcal{A}_k \otimes \mathcal{A}_p$ by applying it to each factor, then $\psi(\overline{a_{ij}}) = \overline{\psi(a_{ij})}$.

Proposition 3.1. *$\psi\left(\Phi^L_{\imath_p(\alpha)}\right) = \Phi^L_\alpha \otimes I_p$ and $\psi\left(\Phi^R_{\imath_p(\alpha)}\right) = \Phi^R_\alpha \otimes I_p$ for any braid α.*

A comment on notation is in order. The tensor product (over $\mathbb{Z}$) of $\mathcal{A}^L_k$ and $\mathcal{A}^L_p$ is a left $(\mathcal{A}_k \otimes \mathcal{A}_p)$-module with basis $\{a_{i*} \otimes a_{j*}\}$. By $\Phi^L_\alpha \otimes I_p$ we mean the matrix in this basis for the $(\mathcal{A}_k \otimes \mathcal{A}_p)$-linear map equal to the tensor product of the map corresponding to Φ^L_α with the identity on $\mathcal{A}^L_p$. Similarly for $\mathcal{A}^R_k$ and $\mathcal{A}^R_p$.

Proposition 3.1 hinges on the following lemma, proved in Section 3.2.

Lemma 3.2. *For $\alpha \in B_k$ the following diagram commutes.*

$$\begin{array}{ccc} \mathcal{A}^L_{kp} & \xrightarrow{\ \phi^*_{\imath_p(\alpha)}\ } & \mathcal{A}^L_{kp} \\ \psi^* \downarrow & & \downarrow \psi^* \\ \mathcal{A}^L_k \otimes \mathcal{A}^L_p & \xrightarrow[\ \phi^*_\alpha \otimes \mathrm{id}\ ]{} & \mathcal{A}^L_k \otimes \mathcal{A}^L_p \end{array}$$

In particular, $\psi^*(\phi^*_{\imath_p(\alpha)}(a_{i,*})) = (\phi^*_\alpha \otimes \mathrm{id})(\psi^*(a_{i,*}))$ *for any* $1 \leq i \leq kp$.

Proof of Proposition 3.1. The proposition readily follows from Lemma 3.2. Fixing $\alpha \in B_k$ and $1 \leq i \leq kp$, we have

$$\begin{aligned}\left(\sum_{l=1}^{k} \left(\Phi^L_\alpha\right)_{q_i l} a_{l*}\right) \otimes a_{r_i *} &= \left(\phi^*_\alpha \otimes \mathrm{id}\right) \psi^* \left(a_{i*}\right) \\ &= \psi^* \left(\phi^*_{\imath_p(\alpha)} \left(a_{i*}\right)\right) \\ &= \sum_{j=1}^{kp} \psi \left(\left(\Phi^L_{\imath_p(\alpha)}\right)_{ij}\right) \left(a_{q_j *} \otimes a_{r_j *}\right).\end{aligned}$$

Hence $\psi((\Phi^L_{\imath_p(\alpha)})_{ij}) = 0$ if $r_i \neq r_j$ and $\psi((\Phi^L_{\imath_p(\alpha)})_{ij}) = (\Phi^L_\alpha)_{q_i q_j} \otimes 1$ if $r_i = r_j$, since for each $1 \leq l \leq k$ exactly one j satisfies both $r_j = r_i$ and $q_j = l$. We conclude $\psi(\Phi^L_{\imath_p(\alpha)}) = \Phi^L_\alpha \otimes I_p$. That $\psi(\Phi^R_{\imath_p(\alpha)}) = \Phi^R_\alpha \otimes I_p$ follows from $\Phi^R_\alpha = \overline{\Phi^L_\alpha}^t$ and $\psi(\overline{a_{ij}}) = \overline{\psi(a_{ij})}$. □

Proof of Theorem 1.6. By Theorem 2.13 there exist augmentations $\epsilon_k \colon \mathcal{A}_k \otimes R_0 \to \mathbb{C}$ and $\epsilon_p \colon \mathcal{A}_p \otimes R_0 \to \mathbb{C}$, for the closures of α, γ respectively, such that $\epsilon_k\left(\Phi^L_\alpha\right) = \epsilon_k\left(\Phi^R_\alpha\right) = \Delta(\alpha)$ and $\epsilon_p\left(\Phi^L_\gamma\right) = \epsilon_p\left(\Phi^R_\gamma\right) = \Delta(\gamma)$. Theorem 2.13 also implies that it suffices to prove that there exists an augmentation $\epsilon \colon \mathcal{A}_{kp} \otimes R_0 \to \mathbb{C}$ such that $\epsilon\left(\Phi^L_{\gamma(\alpha)}\right) = \epsilon\left(\Phi^R_{\gamma(\alpha)}\right) = \Delta(\gamma(\alpha))$.

Below we will define a homomorphism $\delta \colon \mathcal{A}_p \to \mathbb{C}$ such that for each generator a_{ij} we have $\delta(a_{ij}) = \pm\epsilon_p(a_{ij})$, the sign depending on the parity of $w(\alpha)$ and p. Let $\pi \colon \mathbb{C} \otimes \mathbb{C} \to \mathbb{C}$ be the multiplication $a \otimes b \mapsto ab$. Our desired map is defined by $\epsilon = \pi \circ (\epsilon_k \otimes \delta) \circ \psi$.

The Chain Rule theorem gives that

$$\pi \circ (\epsilon_k \otimes \delta) \circ \psi\left(\Phi^L_{\gamma(\alpha)}\right) = \pi \circ (\epsilon_k \otimes \delta) \psi\left(\phi_{\imath_p(\alpha)}\left(\Phi^L_{\bar{\gamma}}\right)\right) \psi\left(\Phi^L_{\imath_p(\alpha)}\right) \tag{8}$$

Consider how the homomorphism $\phi_{\imath_p(\alpha)}$ acts on spanning arcs. For $1 \leq i \neq j \leq p$, since the points $\{1, \ldots, p\} \in D_{kp}$ are moved as one block by the action of $\imath_p(\alpha)$, there is an $0 \leq m < k$ so that $\phi_{\imath_p(\alpha)}(a_{ij}) = a_{i+mp,j+mp}$. As $\psi(a_{i+mp,j+mp}) = 1 \otimes a_{ij}$,

$$\psi\left(\phi_{\imath_p(\alpha)}\left(\Phi^L_{\bar{\gamma}}\right)\right) = \left(1 \otimes \left(\Phi^L_{\bar{\gamma}}\right)_{ij}\right).$$

Note that while the entries of $\Phi^L_{\bar{\gamma}}$ are elements of $\mathcal{A}_{kp}$, all of them lie in the image of the natural inclusion of $\mathcal{A}_p$ into $\mathcal{A}_{kp}$, so we regard the entries of the matrix on the right hand side as elements of $\mathcal{A}_k \otimes \mathcal{A}_p$. Returning to the right hand side of (8), by Proposition 3.1 we have

$$\pi\circ(\epsilon_k\otimes\delta)\left(\psi\left(\phi_{\imath_p(\alpha)}\left(\Phi^L_{\bar\gamma}\right)\right)\psi\left(\Phi^L_{\imath_p(\alpha)}\right)\right)=\pi\circ(\epsilon_k\otimes\delta)\left(\left(1\otimes\left(\Phi^L_{\bar\gamma}\right)_{ij}\right)\left(\Phi^L_\alpha\otimes I_p\right)\right)$$
$$=\delta\left(\Phi^L_{\bar\gamma}\right)\pi\left(\Delta(\alpha)\otimes I_p\right).$$

We are done if we define δ so that $\delta\left(\Phi^L_{\bar\gamma}\right)\pi\left(\Delta(\alpha)\otimes I_p\right)=\Delta(\gamma(\alpha))$. When $w(\alpha)$ is even $w(\imath_p(\alpha))$ is also, and further $\Delta(\alpha)=I_k$. Letting $\delta=\epsilon_p$ makes

$$\delta\left(\Phi^L_{\bar\gamma}\right)\pi\left(\Delta(\alpha)\otimes I_p\right)=\epsilon_p\left(\Phi^L_{\bar\gamma}\right)=\Delta(\bar\gamma)=\Delta(\gamma(\alpha)).$$

Suppose $w(\alpha)$ is odd. Define $g\colon\{1,\dots,p\}\to\{\pm1\}$ as follows. Let $x_1=1$, and $x_l=\mathrm{perm}(\bar\gamma)(x_{l-1})$ for $1<l\le p$. Since the first p strands of $\bar\gamma$ close to a knot, $\mathrm{perm}(\bar\gamma)$ is given by the p-cycle $(x_1x_2\dots x_p)$. If p is even, we let $g(x_1)=1$, and $g(x_l)=-g(x_{l-1})$ for $1<l\le p$. If p is odd, let $g(x_1)=g(x_2)=1$ and $g(x_l)=-g(x_{l-1})$ for $2<l\le p$.

Define $\delta:\mathcal{A}_p\to\mathbb{C}$ by setting $\delta(a_{ij})=g(i)g(j)\epsilon_p(a_{ij})$ for $1\le i\ne j\le p$. Fix i,j and consider a monomial M of $\left(\Phi^L_{\bar\gamma}\right)_{ij}$, which is constant if $i>p$ or $j>p$. For $i,j\le p$, writing $i_0=\mathrm{perm}(\bar\gamma)(i)$, Proposition 2.3 implies $M=c_{ij}a_{i_0,j_1}a_{j_1,j_2}\dots a_{j_m,j}$ for some $j_1,\dots j_m\in\{1,\dots,p\}$, possibly being constant if $i_0=j$, implying that

$$\delta(M)=g(i_0)g(j)\left(\prod_{k=1}^m g(j_k)^2\right)\epsilon_p(M)=g(i_0)g(j)\epsilon_p(M).$$

For M a constant, $\delta(M)=M=g(i_0)g(j)\epsilon_p(M)$ since $i_0=j$. This holds for each monomial, thus

$$\delta\left(\left(\Phi^L_{\bar\gamma}\right)_{ij}\right)=g(i_0)g(j)\epsilon_p\left(\left(\Phi^L_{\bar\gamma}\right)_{ij}\right).$$

When p is even, $w(\imath_p(\alpha))$ is also even and so the opposite parity of $w(\alpha)$. Our definition of g gives $\delta\left(\left(\Phi^L_{\bar\gamma}\right)_{ii}\right)=-\epsilon\left(\left(\Phi^L_{\bar\gamma}\right)_{ii}\right)$ for $i\le p$. Thus

$$\delta\left(\Phi^L_{\bar\gamma}\right)=\begin{pmatrix}(-1)^{w(\bar\gamma)+1}&0&0\\0&-I_{p-1}&0\\0&0&I_{(k-1)p}\end{pmatrix}$$

and therefore

$$\delta\left(\Phi^L_{\bar\gamma}\right)\left(\Delta(\alpha)\otimes I_p\right)=\mathrm{diag}[(-1)^{w(\alpha)+w(\bar\gamma)+1},1\dots1]=\Delta(\gamma(\alpha))$$

as desired.

When p is odd, $w(\imath_p(\alpha))$ is odd and therefore the same parity of $w(\alpha)$. Our definition of g gives that $\delta\left(\left(\Phi^L_{\bar\gamma}\right)_{11}\right)=\epsilon\left(\left(\Phi^L_{\bar\gamma}\right)_{11}\right)$ and $\delta\left(\left(\Phi^L_{\bar\gamma}\right)_{ii}\right)=-\epsilon\left(\left(\Phi^L_{\bar\gamma}\right)_{ii}\right)$ for $1<i\le p$, so

$$\delta\left(\Phi^L_{\bar\gamma}\right)=\begin{pmatrix}(-1)^{w(\bar\gamma)}&0&0\\0&-I_{p-1}&0\\0&0&I_{(k-1)p}\end{pmatrix}$$

and therefore

$$\delta\left(\Phi^L_{\bar\gamma}\right)(\Delta(\alpha)\otimes I_p) = \mathrm{diag}[(-1)^{w(\alpha)+w(\bar\gamma)}, 1\ldots 1] = \Delta(\gamma(\alpha))$$

as desired.

There is little difference in the proof that $\epsilon(\Phi^R_{\gamma(\alpha)}) = \Delta(\gamma(\alpha))$, except that monomials in $(\Phi^R_{\bar\gamma})_{ij}$ are of the form $c_{ij}a_{i,j_1}a_{j_1,j_2}\cdots a_{j_k,j'}$ where $j' = \mathrm{perm}(\bar\gamma)(j)$. Applying Theorem 2.13 now completes the proof. □

Proof of Corollary 1.7. We prove the corollary by induction on the dimensions of the vectors $\mathbf{p}$ and $\mathbf{q}$. If $\mathbf{p}$ and $\mathbf{q}$ have one entry, then $T(\mathbf{p},\mathbf{q})$ is simply the (p_1, q_1)-torus knot, and by Theorem 1.3 from [Cor14b] we have $\mathrm{ar}(T(\mathbf{p},\mathbf{q})) = p_1$.

Suppose that $\mathbf{p}$ and $\mathbf{q}$ have n entries and $\mathrm{ar}(T(\hat{\mathbf{p}},\hat{\mathbf{q}})) = p_1p_2\cdots p_{n-1}$. Choose a braid $\alpha \in B_{p_1p_2\cdots p_{n-1}}$ such that $\hat\alpha = T(\hat{\mathbf{p}},\hat{\mathbf{q}})$, and let $\gamma = (\sigma_1\ldots\sigma_{p_n-1})^{q_n}$. Theorem 1.3 from [Cor14b] implies that $\mathrm{ar}(\hat\gamma) = p_n$, and since $T(\mathbf{p},\mathbf{q}) = K(\alpha,\gamma)$, Theorem 1.6 gives the desired result. □

3.2. **Supporting Lemmas.** In this section we prove Lemma 3.2, for which we make some definitions. Set $X_{m,l} = \{m, m+1, \ldots, m+l-1\}$ for any $m, l > 0$. Given $Y \subseteq X_{m,l}$ of cardinality υ, denote elements of Y by $\{y_1,\ldots,y_\upsilon\}$ such that $y_1 < \ldots < y_\upsilon$. Suppose $1 \le i \ne j \le kp+1$. If $i, j \notin X_{m,l}$ we define

$$A(i,j,X_{m,l}) = \sum_{Y\subseteq X_{m,l}} (-1)^{|Y|} a_{iy_1}a_{y_1y_2}\cdots a_{y_\upsilon j};$$
$$A'(i,j,X_{m,l}) = \sum_{Y\subseteq X_{m,l}} (-1)^{|Y|} a_{iy_\upsilon}a_{y_\upsilon y_{\upsilon-1}}\cdots a_{y_1 j}.$$

If $j \in X_{m,l}$ and $i \notin X_{m,l}$ define

$$B'(i,j,X_{m,l}) = \sum_{Y\subseteq X_{m,l},\ y_1\ne j} c_Y a_{iy_\upsilon}a_{y_\upsilon y_{\upsilon-1}}\cdots a_{y_1 j}$$

where $c_Y = (-1)^{|Y|+1}$ if $Y \cap X_{m,j-m+1} = \emptyset$, and $c_Y = (-1)^{|Y|}$ if $Y \cap X_{m,j-m} \ne \emptyset$ (the $y_1 \ne j$ condition makes this the complement of the first condition). To prove Lemma 3.2 we use two lemmas. In the proof of Lemma 3.2 below we are able to focus on generators a_{ij}, $i < j$. Also, we write $*$ for $j = kp+1$. Recall the definition of the spanning arc c_{ij} and the map $\chi : \mathscr{S}_{kp+1} \to \mathcal{A}^L_{kp}$ from Section 2.3.

Lemma 3.3. *Given $1 \le n \le k-1$ let $X_n^{(p)} = X_{(n-1)p+1,p}$. For $1 \le i < j \le kp+1$ we have*

$$\phi_{\imath_p(\sigma_n)}(a_{ij}) = \begin{cases} a_{i+p,j+p} & : i,j \in X_n^{(p)} \\ a_{i-p,j-p} & : i,j \in X_{n+1}^{(p)} \\ B'(i+p, j-p, X_n^{(p)}) & : i \in X_n^{(p)}, j \in X_{n+1}^{(p)} \\ a_{i-p,j} & : j > (n+1)p, i \in X_{n+1}^{(p)} \\ a_{i,j-p} & : i \le (n-1)p, j \in X_{n+1}^{(p)} \\ A(i, j+p, X_n^{(p)}) & : i \le (n-1)p, j \in X_n^{(p)} \\ A'(i+p, j, X_n^{(p)}) & : j > (n+1)p, i \in X_n^{(p)} \\ a_{ij} & : \text{otherwise} \end{cases}.$$

Proof. Define $\tau_{m,l} = \sigma_m \sigma_{m+1} \cdots \sigma_{m+l-1}$ and let $\kappa_{m,l} = \tau_{m+l-1,p} \tau_{m+l-2,p} \cdots \tau_{m,p}$. Note that $\kappa_{m,p} = \imath_p(\sigma_n)$ if $m = (n-1)p+1$. We may prove the result, therefore, by showing that for $i < j$ if $l \le p$ then

$$\phi_{\kappa_{m,l}}(a_{ij}) = \begin{cases} a_{i+l,j+l} & : i,j \in X_{m,p} \\ a_{i-p,j-p} & : i,j \in X_{m+p,l} \\ B'(i+l, j-p, X_{m,l}) & : i \in X_{m,p}, j \in X_{m+p,l} \\ a_{i-p,j} & : j \ge m+l+p, i \in X_{m+p,l} \\ a_{i,j-p} & : i < m, j \in X_{m+p,l} \\ A(i, j+l, X_{m,l}) & : i < m, j \in X_{m,p} \\ A'(i+l, j, X_{m,l}) & : j \ge m+p+l, i \in X_{m,p} \\ a_{ij} & : \text{otherwise} \end{cases}. \tag{9}$$

The proof of (9) is by induction on l. For the case $l = 1$, note that $\kappa_{m,1} = \tau_{m,p}$. It is relatively straightforward to calculate, for $1 \le m \le (k-1)p$ and $i < j$, that

$$\phi_{\tau_{m,p}}(a_{ij}) = \begin{cases} a_{i+1,j+1} & : m \le i < j < m+p \\ -a_{i+1,m} & : m \le i < j = m+p \\ a_{mj} & : m+p = i < j \\ a_{im} & : i < m < m+p = j \\ a_{i,j+1} - a_{i,m} a_{m,j+1} & : i < m \le j < m+p \\ a_{i+1,j} - a_{i+1,m} a_{m,j} & : m \le i < m+p < j \\ a_{ij} & : \text{otherwise} \end{cases}. \tag{10}$$

Indeed, the effect of $\tau_{m,p}$ is to move points $\{m, \ldots, m+p-1\}$ in (D,P) one to the right and the point at $m+p$ is carried through the upper half-disk to m. Figure 8 shows $\tau_{m,p} \cdot c_{ij}$ for two interesting cases in (10). Use the relation in Figure 5 at the point m, we get $\tau_{m,p} \cdot c_{ij} = c_{i,j+1} - c_{im} c_{m,j+1}$ if $i < m \le j < m+p$, and $\tau_{m,p} \cdot c_{ij} = c_{i+1,j} + c_{i+1,m} c_{mj}$ if $m \le i < m+p < j$.

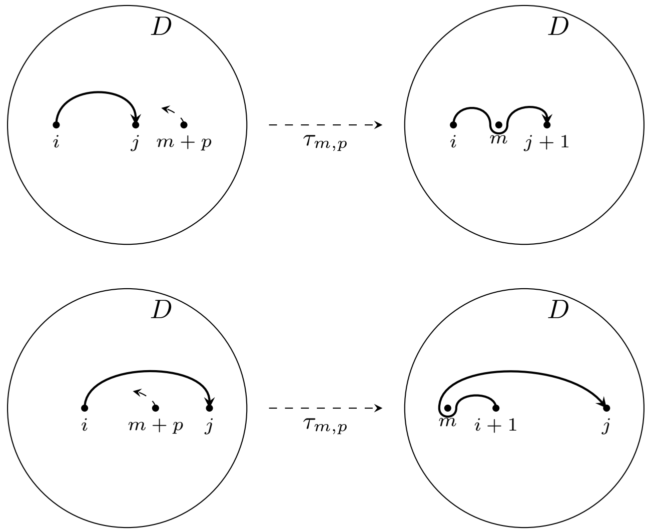

FIGURE 8. $\tau_{m,p} \cdot c_{ij}$, two possible cases

Applying the map χ gives the calculation in (10) for these cases. Verification of the other cases are left to the reader.

Since $X_{m,1} = \{m\}$, we have $A(i, j+1, X_{m,1}) = a_{i,j+1} - a_{im}a_{m,j+1}$ and $A'(i+1, j, X_{m,1}) = a_{i+1,j} - a_{i+1,m}a_{mj}$. Also, when $j = m+p$ the subsets considered for $B'(i+1, j-p, X_{m,1})$ must be empty, so it is $-a_{i+1,m}$. The other cases clearly agree with (9) for $l = 1$, proving the base case.

The argument for $l > 1$ is handled in each case appearing in (9). We present the argument in the cases $i < m, j \in X_{m,p}$ and $j \geq m+p+l, i \in X_{m,p}$ and when $i \in X_{m,p}, j \in X_{m+p,l}$.

If $i < m, j \in X_{m,p}$ then

$$\begin{aligned}
\phi_{\kappa_{m,l}}(a_{ij}) &= \phi_{\tau_{m+l-1,p}}(\phi_{\kappa_{m,l-1}}(a_{ij})) \\
&= \sum_{Y \subseteq \{m,\ldots,m+l-2\}} (-1)^{|Y|} \phi_{\tau_{m+l-1,p}}(a_{i,y_1}a_{y_1y_2}\cdots a_{y_v,j+l-1}) \\
&= \sum_{Y \subseteq \{m,\ldots,m+l-2\}} (-1)^{|Y|} a_{iy_1}a_{y_1y_2}\cdots a_{y_{v-1}y_v}(a_{y_v,j+l} - a_{y_v,m+l-1}a_{m+l-1,j+l}) \\
&= \sum_{Y \subseteq \{m,\ldots,m+l-1\}} (-1)^{|Y|} a_{i,y_1}a_{y_1y_2}\cdots a_{y_v,j+l} \\
&= A(i, j+l, X_{m,l}),
\end{aligned}$$

The third equality uses (10) and holds because $l \leq p$.

The case $j \geq m+p+l, i \in X_{m,p}$ is very similar, but that the indices of generators appearing in the sum are descending, so we also use that $\phi_{\tau_{m+l-1,p}}$ commutes with conjugation.

Finally, suppose $i \in X_{m,p}, j \in X_{m+p,l}$. Note $j-(m+p) \le l-1$. If $j-m-p = l-1$, then by the preceding case

$$\phi_{\kappa_{m,l-1}}(a_{ij}) = A'(i+j-m-p, j, X_{m,j-m-p}).$$

We then have

$$\begin{aligned}
&\phi_{\tau_{m+(j-m-p),p}}(A'(i+j-m-p,j,X_{m,j-m-p}))\\
&= \sum_{Y\subseteq\{m,\dots,j-p-1\}} (-1)^{|Y|}\phi_{\tau_{j-p,p}}(a_{i+j-m-p,y_\upsilon}a_{y_\upsilon y_{\upsilon-1}}\cdots a_{y_1,j})\\
&= -a_{i+j-m-p+1,j-p}\\
&+ \sum_{\substack{Y\subseteq\{m,\dots,j-p-1\}\\ Y\neq\emptyset}} (-1)^{|Y|}(a_{i+j-m-p+1,y_\upsilon} - a_{i+j+m-p+1,j-p}a_{j-p,y_\upsilon})a_{y_\upsilon y_{\upsilon-1}}\cdots a_{y_2y_1}a_{y_1,j-p}\\
&= B'(i+l, j-p, X_{m,l}).
\end{aligned}$$

If instead $j-m-p < l-1$, and $l \le p$, we conclude the proof by checking

$$\begin{aligned}
&\phi_{\tau_{m+l-1,p}}(B'(i+l-1,j-p,X_{m,l-1}))\\
&= \sum_{\substack{Y\subseteq\{m,\dots,m+l-2\}\\ Y\cap X_{m,j-m-p}\neq\emptyset}} (-1)^{|Y|}\phi_{\tau_{m+l-1,p}}(a_{i+l-1,y_\upsilon}a_{y_\upsilon y_{\upsilon-1}}\cdots a_{y_1,j-p})\\
&\quad - \sum_{\substack{Y\subseteq\{m,\dots,m+l-2\}\\ Y\cap X_{m,j-m-p+1}=\emptyset}} (-1)^{|Y|}\phi_{\tau_{m+l-1,p}}(a_{i+l-1,y_\upsilon}a_{y_\upsilon y_{\upsilon-1}}\cdots a_{y_1,j-p})\\
&= \sum_{\substack{Y\subseteq\{m,\dots,m+l-2\}\\ Y\cap X_{m,j-m-p}\neq\emptyset}} (-1)^{|Y|}(a_{i+l,y_\upsilon} - a_{i+l,m+l-1}a_{m+l-1,y_\upsilon})a_{y_\upsilon y_{\upsilon-1}}\cdots a_{y_1,j-p}\\
&\quad - \sum_{\substack{Y\subseteq\{m,\dots,m+l-2\}\\ Y\cap X_{m,j-m-p+1}=\emptyset}} (-1)^{|Y|}(a_{i+l,y_\upsilon} - a_{i+l,m+l-1}a_{m+l-1,y_\upsilon})a_{y_\upsilon y_{\upsilon-1}}\cdots a_{y_1,j-p}\\
&= B'(i+l, j-p, X_{m,l}).
\end{aligned}$$

□

Lemma 3.4. *Fix $1 \le i < j \le kp+1$ and define $\alpha_i = (n-1)p + r_i \in X_n^{(p)}$. We have the following equalities.*

$$\begin{aligned}
\psi(A(i,j+p,X_n^{(p)})) &= \psi(a_{i,j+p} - a_{i\alpha_i}a_{\alpha_i,j+p}) &&: i \le (n-1)p, j \in X_n^{(p)}\\
\psi^*(A'(i+p,j,X_n^{(p)})) &= \psi^*(a_{i+p,j} - a_{i+p,i}a_{i,j}) &&: i \in X_n^{(p)}, j > (n+1)p\\
\psi(B'(i+p,j-p,X_n^{(p)})) &= \psi(-a_{i+p,j-p} + \delta a_{i+p,i}a_{i,j-p}) &&: i \in X_n^{(p)}, j \in X_{n+1}^{(p)},
\end{aligned}$$

where $\delta \in \{-1,0,1\}$ is 0 if $i = j-p$, and is the sign of $i-(j-p)$ otherwise.

Remark 3.5. It is possible to have $j = *$ only in the case that $j > (n+1)p$, hence the decoration ψ^*. This observation plays a role in Lemma 3.2.

Proof of Lemma 3.4. Each of the three cases involves a sum over subsets $Y \subseteq X_n^{(p)}$.

In the case $i \leq (n-1)p$, any $y_1 < \alpha_i$ satisfies $r_{y_1} < r_i$ and $q_i < q_{y_1}$. Hence $\psi(a_{iy_1}) = 0$. Thus we restrict to subsets $Y \subseteq \{\alpha_i, \dots, np\}$, i.e.

$$\psi(A(i, j+p, X_n^{(p)})) = \sum_{Y \subseteq \{\alpha_i, \dots, np\}} (-1)^{|Y|} \psi(a_{iy_1} a_{y_1 y_2} \cdots a_{y_v, j+p}).$$

For any $y_1 \in \{\alpha_i + 1, \dots, np\}$ we get

$$\psi(a_{iy_1} - a_{i\alpha_i} a_{\alpha_i y_1}) = a_{q_i, q_{y_1}} \otimes a_{r_i, r_{y_1}} - (a_{q_i, n} \otimes 1)(1 \otimes a_{r_i, r_{y_1}}) = 0,$$

and so

$$\begin{aligned} &\psi(A(i, j+p, X_n^{(p)}) \\ &= \psi(a_{i,j+p} - a_{i\alpha_i} a_{\alpha_i, j+p}) \quad + \sum_{\substack{Y \subseteq \{\alpha_i+1, \dots, np\} \\ Y \neq \emptyset}} (-1)^{|Y|} \psi(a_{iy_1} - a_{i\alpha_i} a_{\alpha_i y_1}) \psi(a_{y_1 y_2} \cdots a_{y_v, j+p}) \\ &= \psi(a_{i,j+p} - a_{i\alpha_i} a_{\alpha_i, j+p}). \end{aligned}$$

In the remaining cases $i \in X_n^{(p)}$, and so $\alpha_i = i$. If $y_v > i$ then $r_{y_v} > r_{i+p}$ and $q_{i+p} > q_{y_v}$ so that $\psi(a_{i+p, y_v}) = 0$. Thus in these cases we restrict to $Y \subseteq \{(n-1)p+1, \dots, i\}$. The argument for the second case then proceeds analogously to the first.

In the third case, with $j \in X_{n+1}^{(p)}$, we must account for the condition $y_1 \neq j-p$ in each summand. This causes the non-vanishing part of the sum to vary, depending on whether i is larger than, equal to, or smaller than $j-p$. The δ in the statement of the lemma incorporates the three situations. Noting that if $\emptyset \neq Y \subseteq \{(n-1)p+1, \dots, i-1\}$ then $c_Y = -c_{Y \cup \{\alpha_i\}}$ (recall $\alpha_i = i$), the argument then proceeds analogously to the first. □

Proof of Lemma 3.2. The statement holds when α is the identity braid. We prove for $1 \leq n < k$ that

$$\psi^* \circ \phi^*_{\imath_p(\sigma_n)} = \left(\phi^*_{\sigma_n} \otimes \mathrm{id}\right) \circ \psi^*.$$

As the maps $B_k \to \mathrm{Aut}(\mathcal{A}_k^L \otimes \mathcal{A}_p^L)$, given by $\alpha \mapsto \phi^*_\alpha \otimes \mathrm{id}$, and $B_k \to \mathrm{Aut}(\mathcal{A}_{kp}^L)$, given by $\alpha \mapsto \phi^*_{\imath_p(\alpha)}$, are homomorphisms, this suffices to prove the lemma.

Furthermore, for β any braid, ϕ_β and ψ both commute with conjugation, so we only need prove that

$$\psi^*(\phi^*_{\imath_p(\sigma_n)})(a_{ij}) = (\phi^*_\alpha \otimes \mathrm{id})\psi^*(a_{ij}) \tag{11}$$

for $i < j$, possibly $j = *$. We check (11) for each case in the statment of Lemma 3.3.

In the first two cases both sides of (11) equal $1 \otimes a_{r_i r_j}$.

When $j > (n+1)p, i \in X_{n+1}^{(p)}$, we could have $j = *$. Since $q_i = n+1$, we get $\psi^*(a_{i-p,*}) = a_{q_i-1,*} \otimes a_{r_i,*} = (\phi^*_{\sigma_n} \otimes \mathrm{id})\psi^*(a_{i,*})$. If $j \leq kp$ then $\psi(a_{i-p,j}) = a_{q_i-1,q_j} \otimes x$ where $x = a_{r_i r_j}, 1$ or 0 depending on the relation of r_i to r_j. Again $q_i - 1 = n$, and $q_j > n+1$, so $a_{q_i-1,q_j} = \phi_{\sigma_n}(a_{q_i q_j})$, proving the statement. The case $i \leq (n-1)p, j \in X_{n+1}^{(p)}$ is similar.

In the case that $\phi_{\iota_p(\sigma_n)}(a_{ij}) = A(i, j+p, X_n^{(p)})$ we have by Lemma 3.4 that

$$\psi(\phi_{\iota_p(\sigma_n)}(a_{ij})) = \psi(a_{i,j+p} - a_{i\alpha_i}a_{\alpha_i,j+p}).$$

But since $q_i < q_{j+p} = n+1$ and $q_{\alpha_i} = n$, we see that

$$\begin{aligned}\psi(a_{i,j+p} - a_{i\alpha_i}a_{\alpha_i,j+p}) &= (a_{q_i,n+1} - a_{q_i,n}a_{n,n+1}) \otimes x \\ &= (\phi_{\sigma_n} \otimes \mathrm{id})(\psi(a_{ij})),\end{aligned}$$

where $x = a_{r_ir_j}$ if $r_i < r_j$, $x = 1$ if $r_i = r_j$ and $x = 0$ if $r_i > r_j$.

In the case that $\psi^*(a_{ij}) = A'(i+p, j, X_n^{(p)})$ (here $j = *$ is possible),

$$\psi^*(\phi_{\iota_p(\sigma_n)}(a_{ij})) = \psi^*(a_{i+p,j} - a_{i+p,i}a_{ij}).$$

Then, as $q_{i+p} = n+1 < q_j$ we get (replace q_j with $*$ if $j = *$)

$$\psi^*(a_{i+p,j} - a_{i+p,i}a_{i,j}) = (a_{n+1,q_j} - a_{n+1,n}a_{n,q_j}) \otimes x = (\phi^*_{\sigma_n} \otimes \mathrm{id})(\psi^*(a_{ij})),$$

with x either as before, or $x = a_{r_i,*}$ if $j = *$.

Finally, suppose $i \in X_n^{(p)}, j \in X_{n+1}^{(p)}$. Then $q_{j-p} = q_i = n$ and $\alpha_i = i$. If $j - p < i$ then $r_j < r_i$. By Lemmas 3.3 and 3.4

$$\begin{aligned}\psi(\phi_{\iota_p(\sigma_n)}(a_{ij})) = \psi(-a_{i+p,j-p} + \delta a_{i+p,i}a_{i,j-p}) &= \psi(-a_{i+p,j-p} + a_{i+p,i}a_{i,j-p}) \\ &= (-a_{q_i+1,q_i} + a_{q_i+1,q_i}) \otimes a_{r_i,r_j} \\ &= 0,\end{aligned}$$

which equals $(\phi_{\sigma_n} \otimes \mathrm{id})\psi(a_{ij})$. If $j - p > i$ then

$$\psi(-a_{i+p,j-p} - a_{i+p,i}a_{i,j-p}) = -(a_{q_i+1,q_i} \otimes 1)(1 \otimes a_{r_i,r_j}) = (\phi_{\sigma_n} \otimes \mathrm{id})\psi(a_{ij}).$$

When $j - p = i$ then $r_i = r_j$ and $\psi(-a_{i+p,j-p}) = -a_{n+1,n} \otimes 1 = (\phi_{\sigma_n} \otimes \mathrm{id})\psi(a_{ij})$, and this finishes the proof. □

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